2.2 Intro to Bases

MATH 294 FALL 1981 PRELIM 1 # 3

2.2.1 a) Show that the set of vectors
\[ \{1 + t, 1 - t, 1 - t^2\} \]
is a basis for the vector space of all polynomials
\[ \vec{p} = a_0 + a_1 t + a_2 t^2 \]
of degree less than three.
b) Express the vector
\[ 2 + 3t + 4t^2 \]
in terms of the above basis.

MATH 294 SPRING 1982 PRELIM 1 # 2

2.2.2 Let V be the space of all solutions of
\[ \vec{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}. \]

Consider the vectors
\[ \vec{x}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}, \vec{x}_2(t) = \begin{pmatrix} e^t \\ 0 \\ e^t \end{pmatrix}. \]
a) Do \( \vec{x}_1(t), \vec{x}_2(t) \) belong to V?
b) Are \( \vec{x}_1(t), \vec{x}_2(t) \) linearly independent? Give reasons for your answer.
c) Do the vectors \( \vec{x}_1(t), \vec{x}_2(t) \) form a basis for V? Give reasons for your answer.
MATH 294  SPRING 1983  FINAL  # 10
2.2.3  a) Find a basis for the vector space of all $2 \times 2$ matrices.
   b) $A$ is the matrix given below, $\vec{v}$ is an eigenvector of $A$. Find any eigenvalue of $A$.
      
      \[
      A = \begin{bmatrix}
      3 & 0 & 4 & 2 \\
      8 & 5 & 1 & 3 \\
      4 & 0 & 9 & 8 \\
      2 & 0 & 1 & 6 \\
      \end{bmatrix}
      \]
      \[
      \vec{v} = \begin{bmatrix}
      0 \\
      2 \\
      0 \\
      0 \\
      \end{bmatrix}
      \]
      
      c) Find one solution to each system of equations below, if possible. If not possible, explain why not.
      
      \[
      \begin{bmatrix}
      1 & 1 & 1 & 1 \\
      2 & 2 & 2 & 2 \\
      3 & 3 & 3 & 3 \\
      4 & 4 & 4 & 4 \\
      \end{bmatrix} \cdot \vec{x} = \vec{b},
      \vec{b} = \begin{bmatrix}
      1 \\
      2 \\
      3 \\
      4 \\
      \end{bmatrix}
      \]
      \[
      \vec{b} = \begin{bmatrix}
      1 \\
      0 \\
      0 \\
      0 \\
      \end{bmatrix}
      \]
      
      d) Read carefully. Solve for $\vec{x}$ in the equation $A \cdot \vec{b} = \vec{x}$ with:
      \[
      A = \begin{bmatrix}
      1 & 2 & 3 \\
      3 & 2 & 1 \\
      1 & 0 & 1 \\
      \end{bmatrix}
      \]
      \[
      \vec{b} = \begin{bmatrix}
      1 \\
      -1 \\
      2 \\
      3 \\
      \end{bmatrix}
      \]
      
      e) Find the inverse of the matrix $A$.
      
      \[
      \begin{bmatrix}
      1 & 0 & 0 \\
      0 & 1 & 0 \\
      0 & 0 & 1 \\
      \end{bmatrix}
      \]

MATH 294  SPRING 1984  FINAL  # 2
2.2.4  Determine whether the given vectors form a basis for $S$, and find the dimension of the subspace. $S$ is the set of all vectors of the form $(a, b, 2a, 2b)$ in $\mathbb{R}^4$. The given set is $(1, 0, 2, 0), (0, 1, 0, 3), (1, -1, 2, -3)$.

MATH 294  FALL 1986  FINAL  # 1
2.2.5  The vectors $(1,0,2,-1,3),(0,1,-1,2,4),(-1,1,-2,1,-3),(0,1,1,-2,-4),(1,4,2,-1,3)$ span a subspace $S$ of $\mathbb{R}^5$.
   a) What is the dimension of $S$?
   b) Find a basis for $S$.

MATH 294  FALL 1986  FINAL  # 2
2.2.6  a) Solve the linear system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix}
      1 & 0 & -2 & 4 \\
      2 & 1 & -4 & 6 \\
      -1 & 2 & 5 & -3 \\
      3 & 3 & -5 & 4 \\
      \end{bmatrix}$ and $\vec{b} = \begin{bmatrix}
      4 \\
      9 \\
      9 \\
      15 \\
      \end{bmatrix}$.
   b) Solve the linear system $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix}
      -3 & -1 & 0 & 1 & -2 \\
      1 & 2 & -1 & 0 & 3 \\
      2 & 1 & 1 & -2 & 1 \\
      1 & 5 & 2 & -5 & 4 \\
      \end{bmatrix}$ Express your answer in vector form, and give a basis for the space of solutions.
2.2. INTRO TO BASES

MATH 294 FALL 1987 PRELIM 3  # 6
2.2.7 Find an orthonormal basis for the subspace of \( \mathbb{R}^3 \) consisting of all 3-vectors \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) such that \( x + y + z = 0 \).

MATH 294 FALL 1989 PRELIM 3  # 3
2.2.8 Let \( W \) be the following subspace of \( \mathbb{R}^3 \),
\[ W = \text{Comb} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} \]
a) Show that \( \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} \) is a basis for \( W \)

For b) and c) below, let \( T \) be the following linear transformation \( T : W \rightarrow \mathbb{R}^3 \).
\[ T \left( \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \]
for those \( \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \) in \( \mathbb{R}^3 \) which belong to \( W \). [You are allowed to use a) even if you did not solve it.]

b) What is the dimension of \( \text{Range}(T) \)? (Complete reasoning, please.)
c) What is the dimension of \( \text{Ker}(T) \)? (Complete reasoning, please.)

MATH 293 SPRING 1990 PRELIM 1  # 3
2.2.9 Find the dimension and a basis for the following spaces
a) The space spanned by \( \{(1,0,-2,1),(0,3,1,-1),(2,3,-3,1),(3,0,-6,-1)\} \)
b) The set of all polynomials \( p(t) \) in \( P^3 \) satisfying the two conditions
i) \( \frac{d}{dt}p(t) = 0 \) for all \( t \)
ii) \( p(t) + \frac{d}{dt}p(t) = 0 \) at \( t = 0 \)
c) The subspace of the space of functions of \( t \) spanned by \( \{e^{at}, e^{bt}\} \) if \( a \neq b \).
d) The space spanned by \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \) in \( W \), given that \( \{\vec{v}_2, \vec{v}_3, \vec{v}_4\} \) is a basis for \( W \).

MATH 293 SPRING 1990 PRELIM 1  # 4
2.2.10 a) Show that \( B = \{t^2 - 1, t^2 + 1, t\} \) is a basis for \( P^2 \)
b) Express the vectors in \( \{1, t, t^2\} \) in terms of those in \( B \) and find the components of \( p(t) = (1 + t)^2 \) with respect to \( B \).
c) Find the components of the vector \( \vec{x} = (1, 2, 3) \) with respect to the basis \( \{(1,0,0),(1,1,0),(1,1,1)\} \).
MATH 293 FALL 1990 PRELIM 2 # 1

2.2.11 a) Express the vectors \( \vec{u}, \vec{v} \) in terms of \( \vec{a}, \vec{b} \), given that

\[ 3\vec{u} + 2\vec{v} = \vec{a}, \quad \vec{u} - \vec{v} = \vec{b} \]

b) If \( \vec{u}, \vec{b} \) are linearly independent, find a basis for the span of \( \{ \vec{u}, \vec{v}, \vec{a}, \vec{b} \} \)

c) Find \( \vec{u}, \vec{v} \), if \( \vec{a} = (-1, 2, 8), \vec{b} = (-2, -1, 1) \)

MATH 293 FALL 1991 PRELIM 3 # 1

2.2.12 Consider the matrix

\[
A = \begin{pmatrix}
2 & -1 & 1 & 3 \\
-1 & 2 & -2 & -2 \\
2 & 5 & -4 & 1 \\
1 & 4 & -4 & 0
\end{pmatrix}
\]

a) Find a basis for the row space of \( A \).
b) Find a basis for the column space of \( A \).

MATH 293 SPRING 1992 PRELIM 3 # 6

2.2.13 Given \( A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 5 \\ 2 & -1 & 3 & 4 \end{pmatrix} \).

a) Find a basis for the null space of \( A \).
b) Find the rank of \( A \).

d) Are the two space \( W_1 \) and \( W_2 \) the same subspace of \( V_4 \)? Explain your answer carefully in order to get credit for this part.

MATH 293 SUMMER 1992 PRELIM 7/21 # 3

2.2.14 Given a matrix \( A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & -2 & 0 & -3 \end{pmatrix} \).

a) Find a basis for the row space \( W_1 \) of \( A \).
b) Find a basis for the range \( W_2 \) of \( A \).
c) Find the rank of \( A \).
d) Are the two space \( W_1 \) and \( W_2 \) the same subspace of \( V_4 \)? Explain your answer carefully in order to get credit for this part.
2.2. INTRO TO BASES

MATH 293 SPRING 1992 FINAL # 2

2.2.15 a) Find a basis for $V_4$ that contains at least two of the following vectors:

$$
\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}
$$

b) $A$ is a $3 \times 3$ matrix. If

$$
A \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}
$$

and

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
$$

is a basis for the nullspace of $A$, then find the general solution $\vec{x}$ of the equation $A\vec{x} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}$.

Find, also, the determinant of $A$.

MATH 293 SUMMER 1992 PRELIM 7/21 # 4

2.2.16 Given four vectors in $V_4$

$$
\vec{v}_1 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ -4 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 4 \\ 4 \\ 0 \\ -6 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix}
$$

a) Find the space $W$ spanned by the vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$

b) Find a basis for $W$.

c) Find a basis for $V_4$ that contains as many of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ as possible.

MATH 293 FALL 1992 PRELIM 3 # 2

2.2.17 Consider the matrix

$$
A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 2 & 0 & -6 & -2 \\ -1 & 1 & 5 & 3 \end{pmatrix}
$$

a) Find a basis for the column space of $A$ from among the set of column vectors.

b) Find a basis for the row space of $A$.

c) Find a basis for the null space of $A$.

d) What is the rank of $A$ and the dimension of the null space (the nullity)?
2.2.18 Let $C(-\pi, \pi)$ be the vector space of continuous functions on the interval $-\pi \leq x \leq \pi$. Which of the following subsets $S$ of $C(-\pi, \pi)$ are subspaces? If it is not a subspace say why. If it is, then say why and find a basis.

Note: You must show that the basis you choose consists of linearly independent vectors. In what follows $a_0, a_1$ and $a_2$ are arbitrary scalars unless otherwise stated.

a) $S$ is the set of functions of the form $f(x) = 1 + a_1 \sin(x) + a_2 \cos(x)$

b) $S$ is the set of functions of the form $f(x) = 1 + a_1 \sin(x) + a_2 \cos(x)$, subject to the condition $\int_{-\pi}^{\pi} f(x) \, dx = 2\pi$

c) $S$ is the set of functions of the form $f(x) = 1 + a_1 \sin(x) + a_2 \cos(x)$, subject to the condition $\int_{-\pi}^{\pi} f(x) \, dx = 0$

2.2.19 a) Let $A$ be an $n \times n$ nonsingular matrix. Prove that $\det(A^{-1}) = \frac{1}{\det(A)}$. Hint: You may use the fact that if $A$ and $B$ are $n \times n$ matrices $\det(AB) = \det(A) \det(B)$.

b) An $n \times n$ matrix $A$ has a nontrivial null space. Find $\det(A)$ and explain your answer.

c) Given two vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ in $V_3$. Find a vector (or vectors) $\vec{w}_1, \vec{w}_2, \ldots$ in $V_3$ such that the set $\{ \vec{v}_1, \vec{v}_2, \vec{w}_1, \ldots \}$ is a basis for $V_3$.

d) Let $S$ be the set of all vectors of the form $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the usual mutually perpendicular unit vectors. Let $W$ be the set of all vectors that are perpendicular to the vector $\vec{v} = \vec{i} + \vec{j} + \vec{k}$. Is $W$ a vector subspace of $V_3$? Explain your answer.
2.2. **INTRO TO BASES**

2.2.22 a) Consider the vector space $V$ whose elements are $3 \times 3$ matrices.
   i) Find a basis for the subspace $W_1$ of $V$ which consists of all upper-triangular $3 \times 3$ matrices.
   ii) Find a basis for the subspace $W_1$ of $V$ which consists of all upper-triangular $3 \times 3$ matrices with zero trace.
      The trace of a matrix is the sum of its diagonal elements.
   b) Consider the polynomial space $P^3$ of polynomials with degree $\leq 3$ on $0 \leq t \leq 1$.
      Find a basis for the subspace $W$ of $P^3$ which consists of polynomials of degree $\leq 3$ with the constraint
      \[ \frac{d^2p}{dt^2} + \frac{dp}{dt} \bigg|_{t=0} = 0. \]

2.2.23 Let $A$ be the matrix
\[
\begin{pmatrix}
1 & 2 & -1 & 3 \\
2 & 2 & -1 & 2 \\
1 & 0 & 0 & -1
\end{pmatrix}
\]
   a) Find a basis for the Null Space of $A$. What is the nullity of $A$?
   b) Find a basis for the Row Space of $A$. What is its dimension?
   c) Find a basis for the Column Space of $A$. What is its dimension?
   d) What is the rank of $A$?

2.2.24 a) Find a basis for the space spanned by: $\{(1,0,1),(1,1,0),(-1,-4,-3)\}$.
   b) Show that the functions $e^{2x} \cos(x)$ and $e^{2x} \sin(x)$ are linearly independent.

2.2.25 Let $P_3$ be the space of polynomials $p(t)$ of degree $\leq 3$. Consider the subspace
   $S \subset P_3$ of polynomials that satisfy
   \[ p(0) + \frac{dp}{dt} \bigg|_{t=0} = 0 \]
   a) Show that $S$ is a subspace of $P_3$.
   b) Find a basis for $S$.
   c) What is the dimension of $S$?

2.2.26 a) Find a basis for the plane $P \subset \mathbb{R}^3$ of equation
   \[ x + 2y + 3z = 0 \]
   b) Find an orthonormal basis for $P$. 
Let \( P_3 \) be the space of polynomials \( p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \) of degree \( \leq 3 \). Consider the subset \( S \) of polynomials that satisfy
\[
p''(0) = 4p(0) = 0
\]
Here \( p''(0) \) means, as usual, \( \left. \frac{d^2 p}{dt^2} \right|_{t=0} \).

(a) Show that \( S \) is a subspace of \( P_3 \). Give reasons.
(b) Find a basis for \( S \).
(c) What is the dimension of \( S \)? Give reasons for your answer.

Hint: What constraint, if any, does the given formula impose on the constants \( a_0, a_1, a_2, \) and \( a_3 \) of a general \( p(t) \)?

Consider the subspace \( W \) of \( \mathbb{R}^4 \) which is defined as
\[
W = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

(a) Find a basis for \( W \).
(b) What is the dimension of \( W \)?
(c) It is claimed that \( W \) is a “plane” in \( \mathbb{R}^4 \). Do you agree? Give reasons for your answer.
(d) It is claimed that the “plane” \( W \) can be described as the intersection of two 3-D regions \( S_1 \) and \( S_2 \) in \( \mathbb{R}^4 \). The equations of \( S_1 \) and \( S_2 \) are:
\[
S_1: \quad x - u = 0 \\
S_2: \quad ax + by + cz + du = 0
\]
where \( \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} \) is a generic point in \( \mathbb{R}^4 \) and \( a, b, c, d \) are real constants.

Find one possible set of values for the constants \( a, b, c, \) and \( d \).
2.2. INTRO TO BASES

MATH 293 SPRING 1996 PRELIM 3 # 1

2.2.29 The set \( W \) of vectors in \( \mathbb{R}^3 \) of the form \((a, b, c)\), where \( a + b + c = 0 \), is a subspace of \( \mathbb{R}^3 \).

a) Verify that the sum of any two vectors in \( W \) is again in \( W \).

b) The set of vectors

\[ S = (1, -1, 0), (1, 1, -2), (-1, 1, 0), (1, 2, -3) \]

is in \( W \). Show that \( S \) is linearly dependent.

c) Find a subset of \( S \) which is a basis for \( W \).

d) If the condition \( a + b + c = 0 \) above is replaced with \( a + b + c = 1 \), is \( W \) still a subspace? Why/why not?

MATH 293 SPRING 1996 PRELIM 3 # 2

2.2.30 Which of the following subsets are bases for \( \mathbb{R}^2 \)? Show any algebra involved or state a theorem to justify your answer.

\[ S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \right\}, S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}, S_3 = \left\{ \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \right\}. \]

MATH 293 SPRING 1996 FINAL # 22

2.2.31 Let

\[ W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \right\}. \]

Then an orthonormal basis for \( W \) is

a) \( \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \right\} \)

b) \( \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\} \)

c) \( \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\} \)

d) \( \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\} \)

e) none of the above
2.2.32 Consider the vector space $P_2$ of all polynomials of degree $\leq 2$. Consider two bases of $P_2$:
- $S : \{1, t, t^2\}$, the standard basis, and
- $H : \{1, 2t, -2 + 4t^2\}$, the Hermite basis.

a) Find the matrices $P_{S \leftarrow H}$ and $P_{H \leftarrow S}$.

b) Consider $p_1(t) = 1 + 2t + 3t^2$ in $P_2$, and $p_2(t) = \frac{d}{dt}p_1(t)$. Find

$$[p_1(t)]_S, [p_2(t)]_S, [p_1(t)]_H, [p_2(t)]_H,$$

i.e. the coordinates of $p_1$ and $p_2$ in the bases $S$ and $H$.

2.2.33 Let $W$ be the subspace of $\mathbb{R}^4$ defined as

$$W = \text{span} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 0 & -2 & -6 \\ 0 & 4 & 4 \end{pmatrix}.$$

a) Find a basis for $W$. What is the dimension of $W$?

b) It is claimed that $W$ can be described as the intersection of two linear spaces $S_1$ and $S_2$ in $\mathbb{R}^4$. The equations of $S_1$ and $S_2$ are

$$S_1 : x - y = 0,$$

and

$$S_2 : ax + by + cz + dw = 0,$$

where $a, b, c, d$ are real constants that must be determined. Find one possible set of values of $a, b, c$ and $d$.

2.2.34 Let $V$ be the vector space of $2 \times 2$ matrices.

a) Find a basis for $V$.

b) Determine whether the following subsets of $V$ are subspaces. If so, find a basis. If not, explain why not.

i) $\{ A \in V | \det A = 0 \}$

ii) $\{ A \in V | A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$.

c) Determine whether the following are linear transformations. Give a short justification for your answers.

i) $T : V \to V$, where $T(A) = A^T$.

ii) $T : V \to \mathbb{R}^1$, where $T(A) = \det A$. 
2.2. INTRO TO BASES

MATH 294 FALL 1998 FINAL # 4

2.2.35 Here we consider the vector spaces \( P_1, P_2, \) and \( P_3 \) (the spaces of polynomials of degree 1, 2 and 3).

a) Which of the following transformations are linear? (Justify your answer.)
   i) \( T : P_1 \to P_3, T(p) = t^2 p(t) + p(0) \)
   ii) \( T : P_1 \to P_1, T(p) = p(t) + t \)

b) Consider the linear transformation \( T : P_2 \to P_2 \) defined by \( T(a_0 + a_1 t + a_2 t^2) = (-a_1 + a_2) + (-a_0 + a_1) t + (a_2) t^2. \) With respect to the standard basis of \( P_2, \)
   \( B = \{1, t, t^2\}, \)
   \( A = \begin{bmatrix}
   0 & -1 & 1 \\
   -1 & 1 & 0 \\
   0 & 0 & 1 
   \end{bmatrix} \)
   Note that an eigenvalue/eigenvector pair of \( A \) is \( \lambda = 1, \vec{v} = \begin{bmatrix}
   0 \\
   1 \\
   1 
   \end{bmatrix} \). Find an eigenvalue/eigenvector (or eigenfunction) pair of \( T. \) That is, find \( \lambda \) and \( g(t) \) in \( P_2 \) such that \( T(g(t)) = \lambda g(t) \).

c) Is the set of vectors in \( P_2 \) \( \{3 + t, -2 + t, 1 + t^2\} \) a basis of \( P_2? \) (Justify your answer.)

MATH 293 SPRING # C

2.2.36 Give a definition for addition and for scalar multiplication which will turn the set of all pairs \((\vec{u}, \vec{v})\) of vectors, for \( \vec{u}, \vec{v} \) in \( V_2 \), into a vector space \( V \).

a) What is the zero vector of \( V? \)

b) What is the dimension of \( V? \)

c) What is a basis for \( V? \)

MATH 294 FALL 1987 PRELIM 3 MAKE-UP # 2

2.2.37 On parts (a) - (g), answer true or false.

a) \( \text{span} (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \mathbb{R}^3 \), where \( \vec{v}_1 = \begin{bmatrix}
   1 \\
   2 \\
   3 
   \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix}
   3 \\
   2 \\
   1 
   \end{bmatrix} \), \( \vec{v}_3 = \begin{bmatrix}
   1 \\
   0 \\
   -1 
   \end{bmatrix} \), \( \vec{v}_4 = \begin{bmatrix}
   0 \\
   1 \\
   1 
   \end{bmatrix} \).

b) The four vectors in (a) are independent.

c) Referring to a again, all vectors \( \vec{v} = \begin{bmatrix}
   x_1 \\
   x_2 \\
   x_3 
   \end{bmatrix} \) in \( \text{span} (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) \) satisfy a linear equation \( ax_1 + bx_2 + cx_3 = 0 \) for scalars \( a, b, c \) not all 0.

d) The rank of the matrix \( \begin{bmatrix}
   1 & 2 & 3 \\
   3 & 2 & 1 \\
   1 & 0 & -1 \\
   0 & 1 & 1 
   \end{bmatrix} \) is 3.

e) In \( \mathbb{R}^n \) \( n \) distinct vectors are independent.

f) \( n + 1 \) distinct vectors always span \( \mathbb{R}^n \), for \( n > 1. \)

g) If the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) span \( \mathbb{R}^n \), then they are a basis for \( \mathbb{R}^n. \)
MATH 293  UNKNOWN  PRACTICE  # 4a
2.2.38  a) Find a basis for the row space of the matrix
\[
A = \begin{bmatrix}
1 & 2 & -1 & 4 \\
3 & 6 & 1 & 12 \\
9 & 18 & 1 & 36 \\
\end{bmatrix}
\]

2.2.39  If \( A \) is an \( m \times n \) matrix show that \( B = A^T A \) and \( C = AA^T \) are both square. What are their sizes? Show that \( B = B^T, C = C^T \)

MATH 294  FALL  # 1  MAKE-UP
2.2.40  Consider the homogeneous system of equations \( B\vec{x} = \vec{0} \), where
\[
B = \begin{bmatrix}
0 & 1 & 0 & -3 & 1 \\
2 & -1 & 0 & 3 & 0 \\
2 & -3 & 0 & 0 & 4 \\
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}, \quad \text{and} \quad \vec{0} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

a) Find a basis for the subspace \( W \subset \mathbb{R}^5 \), where \( W = \text{set of all solutions of} \ B\vec{x} = \vec{0} \)
b) Is \( B \) 1-1 (as a transformation of \( \mathbb{R}^5 \to \mathbb{R}^3 \))? Why?
c) Is \( B: \mathbb{R}^5 \to \mathbb{R}^3 \) onto why?

d) Is the set of all solutions of \( B\vec{x} = \begin{bmatrix}
3 \\
0 \\
0 \\
\end{bmatrix} \) a subspace of \( \mathbb{R}^5 \)? Why?