2.11 Inner Product Spaces

MATH 294 SPRING 1983 FINAL # 3

2.11.1 Consider the vector space of functions over the interval $0 \leq x \leq 1$ and the inner product

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Find an orthogonal basis for the subspace spanned by $1, e^x, 2e^x, e^{-x}$.

MATH 294 SPRING 1987 PRELIM 3 # 6

2.11.2 Find an element of the vector space V which is functions of the form $ae^{-t} + be^{-2t}$ (where a and b are arbitrary constants) which is orthogonal to E^{-t}. Use the following inner product: $< f(t), g(t) >\equiv \int_0^\infty f(t)g(t)dt$.

MATH 294 FALL 1987 PRELIM 3 # 5 MAKEUP

2.11.3 Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an orthonormal basis for $\Re^n$, and $\vec{v}$ is a vector in $\Re^n$ such that $< \vec{v}, \vec{v}_i >= 0$ for all $i = 1, 2, \ldots, n$. (Here $< v, w >$ denotes the standard inner product in $\Re^n$.) Then what can you conclude about $\vec{v}$? Why? (Hint: write $\vec{v}$ as a linear combination of the basis elements $\vec{v}_1, \ldots, \vec{v}_n$, then apply the condition $< \vec{v}, \vec{v}_i >= 0$.)

MATH 294 FALL 1987 PRELIM 3 # 6 MAKE-UP

2.11.4 With the inner product $< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt$ in the vector space of continuous functions defined on $[-\pi, \pi]$ what is $||a \sin x + b \cos x||$?

MATH 294 FALL 1987 FINAL # 8

2.11.5 Let $< f, g >= \int_0^1 f(t)g(t)dt$ be an inner product on $C_\infty[0, 1]$. Determine:

a) the component of $f(x) = 1 + x$ along $g(x) = x^2$.

b) the component of $f(x)$ perpendicular to $g(x)$.

MATH 294 SUMMER 1989 PRELIM 2 # 3

2.11.6 Consider the vector space $C[0, 1]$ of continuous functions over the interval $0 \leq t \leq 1$ and inner product

$$< f, g >= \int_0^1 f(t)g(t)dt.$$

a) Show that $\left\{1, t, t^2\right\}$ is a set of linearly independent vectors in $C[0, 1]$.

b) Find an orthogonal basis for the subspace of $C[, 1]$ spanned by $1, t, 4t, t^2$.

MATH 294 FALL 1989 PRELIM 3 # 4

2.11.7 Let $C[-1,1]$ denote the continuous real-valued functions on $[-1,1]$, and let W be the following subspace thereof:

$$W = \left\{c_1 + c_2 t + c_3 t^4 : c_1, c_2, c_3 \text{ real numbers } \right\}.$$

a) Show that W is three dimensional.

b) For functions $p(t), q(t)$ i W, introduce the following inner product:

$$< p, q >= \int_{-1}^{1} p(t)q(t)dt.$$

Find an orthogonal basis for W which contains the function $p(t) \equiv 1$ as one element.

MATH 293 SPRING 1992 FINAL # 5

2.11.8 In V_4, $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

a) Using the standard inner product

$$(\vec{u}, \vec{v}) = \sum_{i=1}^{4} u_i v_i$$

find the length of $\vec{u}$ and determine whether the angle between $\vec{u}$and $\vec{v}$ is greater or less than 90 degrees.

b) Using the nonstandard inner product

$$(\vec{x}, \vec{y}) = \sum_{k=1}^{4} k x_k y_k = x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 + 4x_4 y_4$$

find the length of $\vec{u}$and determine whether the angle between $\vec{u}$ and $\vec{v}$ is more or less than 90 degrees.

MATH 293 SUMMER 1992 FINAL # 5

2.11.9 Given the set of functions

$$\left\{1, x, x^2, x^3\right\} \text{ with } -1 \le x \le 1$$

and the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$

a) Find an orthogonal basis $(w_1(x), w_2(x), w_3(x), w_4(x))$ of the space spanned by the functions $1, x, x^2$ and x^3. Use the Schmidt orthogonalization procedure.

b) Given $\phi(x) = 1 + 2x + 3x^2$ and $\phi(x) = c_1 w_1(x) + c_2 w_2(x) + c_3 w_3(x) + c_4 w_4(x)$, find c_2.

MATH 294 SPRING 1997 FINAL # 8

2.11.10 You are given a vector space V with an inner product $<,>$ and an orthogonal basis $B = \left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\right\}$ for which $||\vec{b}_|| = 2, i = 1, \dots, 5$. Suppose that $\vec{v}$ is in V and

$$< \vec{v}, \vec{b}_1 >=< \vec{v}, \vec{b}_2 >= 0$$

$$\text{and } < \vec{v}, \vec{b}_3 >= 3, < \vec{v}, \vec{b}_4 >= 4, < \vec{v}, \vec{b}_5 >= 5$$

Find the coordinates of $\vec{v}$ with respect to the basis B i.e. find c_1, c_2, c_3, c_4, c_5 such that

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 + c_5\vec{b}_5$$

MATH 294 FALL 1997 FINAL # 7

2.11.11 Consider the subspace

$$W = span\{1, t\}, \text{ for } 0 \leq t \leq 1,$$

equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Find the best approximation to the function $f(t) = e^t$ in W.

MATH 294 SPRING 1998 FINAL # 7

2.11.12 Regard P^2 as a subspace of $C[-1, 1]$ and construct an orthogonal basis from the standard basis $E = \left\{1, t, t^2\right\}$ using the inner product

$$\langle p, q\rangle = \int_{-1}^1 p(t)q(t)dt.$$

MATH 294 SPRING 1998 FINAL # 8

2.11.13 Consider P^2 to be a subspace of $C[-1, 1]$

a) Check that $\left\{1, t, t^2 - \frac{2}{3}\right\}$ is an orthogonal basis for this subspace with respect to the inner product

$$< f, g >\equiv f(-1)g(-1) + f(0)g(0) + f(1)g(1).$$

b) Determine the beat second order polynomial approximation, $a_0, a_1t + a_2t$, to the function e^t with respect to this inner product.

MATH 294 FALL 1998 PRELIM 3 # 2

2.11.14 For all the questions below use the vector space $C[-\pi,\pi]$, the set of all continuous functions between $-\pi$ and π, and the inner product:

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$$

[Hint $\int_{-\pi}^{\pi}\sin^2(t)dt = \int_{-\pi}^{\pi}\cos^2 dt = \pi$.]

a) What is the "distance" between the function t and the function 1?

b) Do the three functions $\{1, \sin(t), \cos(t)\}$ form an orthogonal basis for a subspace of $C[-\pi,\pi]$?

c) What value should A have if $A\sin(t)$ is to be the best possible fit to the function t?

MATH 293 SPRING ? PRELIM 2 # 5

2.11.15 Consider $C^0([1,3])$. Is the constant function $g(x) = 1(1 \le x \le 3)$ a unit vector? Find the orthogonal projection $w(x)$ of $f(x) = \frac{1}{x}(1 \le x \le 3)$ onto the span of $g(x)$.

$$[\text{Here } f\cdot g = \int_1^3 f(x)h(x)dx]$$

MATH 293 SPRING ? FINAL # 5

2.11.16 In the space of continuous functions of x in the interval $1 \le x \le 2$ one may define an inner product (other than the usual one) as follows: $f\cdot g = \int_1^2 \frac{1}{x}f(x)g(x)dx$.

a) Using this inner product of f and g find

$$||f|| \text{ (norm of } f\text{) if } f(x) = \sqrt{x} \text{ for } 1 \le x \le 2,$$

b) Determine the real constants a, b, c which make the set $\{a, b + cx\}$ orthonormal (leave $\ln 2$ as is in answers).

MATH 293 # 3 PRACTICE

2.11.17 Let $V = C^0[-1,1]$ with the inner product

$$f\cdot g = \int_{-1}^{1} f(x)g(x)dx.$$

a) Find an orthogonal basis for the space spanned by the functions $f_1(x) = 1, f_2(x) = x, f_3(x) = x^2$.

b) Find the orthogonal projection of x^3 onto the subspace spanned by the functions $1, x, x^2$.