Chapter 2

More Linear Algebra

2.1 Determinants

MATH 293 FALL 1981 PRELIM 1 # 2

2.1.1* Consider the matrices

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 6 \\ 1 & 3 & 4 & 6 \\ 1 & 4 & 5 & 8 \end{bmatrix}.$$

a) Find det A and det B.

b) Find A^{-1} and B^{-1} If they exist. If you think that either of the inverses does not exist, give a reason.

MATH 294 SPRING 1982 PRELIM 1 # 1

2.1.2* **a**) Write the system of equations $\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & -2 \\ 3x_1 & + & 4x_2 & + & 6x_3 & = & 0 \end{array}$ in the form $A\vec{x} = B$.

b) Find det A for A in part (a) above.

c) Does A^{-1} exist?

d) Solve the above system of equations for $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

e) Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$. Find $A \cdot B$ (i.e. calculate the product AB).

MATH 293 FALL 1991 PRELIM 3 # 4

2.1.3* Compute the determinants of the following matrices:

a) $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & -11 & -3 \\ 0 & 0 & 4 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & -5 & 17 & 31 \\ 0 & 3 & 9 & 14 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix}^{-1}$

MATH 293 FALL 1991 PRELIM 3 # 6

2.1.4* True/False

a) All three row operations preserve the absolute value of the determinant of a square matrix.

b) A singular $n \times n$ matrix has a zero determinant.

c) If A is a $n \times n$ matrix $\det(A^t) = \det(A)$.

d) If each entry in a square matrix A is replaced by its reciprocal (inverse), producing a new matrix B, then $\det(B) = (\det(\mathrm{A}))^{-1}$.

e) If a matrix A is nonsingular, then $\det(\mathrm{A}^{-1}) = (\det(\mathrm{A}))^{-1}$

f) For a square matrix A and a scalar k, $\det(kA) = k\det(A)$

g) Let A and B by $n \times n$ matrices

$$\det\begin{pmatrix} A & O_n \\ O_n & B \end{pmatrix} = \det(A)\det(B)$$

where O_n is a $n \times n$ matrix with all elements equal to zero.

MATH 293 SPRING 1992 PRELIM 3 # 1

2.1.5* Compute

a) $\det\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 5 & 0 & 8 \end{bmatrix}$

b) $\det\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

c) $\det\begin{bmatrix} 1 & 1 & 7 & 0 & 0 \\ 1 & 1 & 0 & 3 & 3 \\ 5 & 5 & 1 & 8 & 9 \\ 6 & 6 & 1 & 0 & 1 \\ 6 & 6 & 1 & 0 & 1 \end{bmatrix}$

MATH 293 SUMMER 1992 PRELIM 7_21 # 2

2.1.6* Compute the following determinants:

a) $\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 2 & 3 & 2 & 3 \\ -3 & -5 & 0 & -1 \end{vmatrix}$

b) $\begin{vmatrix} 3-\lambda & 0 & 1 \\ 2 & 1-\lambda & -4 \\ 1 & 0 & -1-\lambda \end{vmatrix}$

MATH 293 FALL 1992 PRELIM 3 # 1

2.1.7* Compute the determinant of the matrix

$$A = \begin{pmatrix} b & a & a & a & a \\ b & b & a & a & a \\ b & b & b & a & a \\ b & b & b & b & a \\ b & b & b & b & b \end{pmatrix}$$

MATH 293 FALL 1992 PRELIM 3 # 4

2.1.8* Let A be an $n \times n$ matrix. Assume that it is known that the equation $Ax = 0$ has nontrivial solutions if and only if $\det(A) = 0$

Let

$$A = \begin{pmatrix} 3-s & 0 & 1 \\ 2 & 1-s & -4 \\ 1 & 0 & -(1+s) \end{pmatrix}$$

where s is an arbitrary scalar.

a) Compute $\det(A)$

b) Find those values of s for which the equation $Ax = 0$ has nontrivial solutions.

MATH 293 FALL 1992 FINAL # 3

2.1.9*

a) Let A be an $n \times n$ nonsingular matrix. Prove that $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$. Hint: You may use the fact that if A and B are $n \times n$ matrices, $\det(AB) = \det(A)\det(B)$.

b) An $n \times n$ matrix A has a nontrivial null space. Find $\det(A)$ and explain your answer.

MATH 293 SPRING 1993 PRELIM 3 # 1

2.1.10* Given the matrix

$$A = \begin{pmatrix} -2 & 1 & 2 \\ -2 & 2 & 2 \\ -9 & 3 & 7 \end{pmatrix}$$

Find $\det A$.

MATH 293 SPRING 1994 PRELIM 2 # 6
2.1.11*

a) Compute the determinant of the matrix $A(\lambda) = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 2 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$, writing your result as a function of λ.

b) Partially check your result by computing the determinant of $A(0)$, and compare this value with the value of the function you found in a) when $\lambda = 0$

MATH 293 FALL 1994 PRELIM 2 # 3
2.1.12* Compute

$$\det \begin{bmatrix} 1 & -2 & 3 \\ -3 & 5 & -8 \\ 2 & 2 & 5 \end{bmatrix}$$

by the following two methods:

a) Use row ops to change the matrix into an upperright triangular matrix with the same det

b) Use cofactors of entries in the first row

MATH 293 FALL 1994 PRELIM 2 # 5
2.1.13* Let A and B be $N \times N$ matrices.

a) Complete the following statement: A is singular if and only if $\det(A) = ...$

b) Use the result of part (a) to find the value of λ for which the matrix $\begin{pmatrix} \lambda-1 & 3 \\ 2 & \lambda-2 \end{pmatrix}$ is singular

c) Complete the following statement: $\det(AB) = ...$

d) Use the result of part (c) to show that if A is invertible, $\det\left(A^{-1}\right) = \frac{1}{\det A}$. (Hint: $AA^{-1} = I$)

MATH 293 FALL 1994 PRELIM 2 # 6
2.1.14* Compute

$$\det \begin{pmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{pmatrix}.$$

MATH 293 FALL 1994 PRELIM 3 # 3
2.1.15* Let A be an $nbyn$ matrix. Which of the following is equivalent to the statement: A **is singular?**

a) The $\det(A) = n$.

b) $Ax = 0$ has a nontrivial solution.

c) The rows of A are linearly independent.

d) The rank of A is n.

e) The $\det A = 0$.

f) $Ax = B$ has a unique solution x for each B.

g) A has non-zero nullity.

MATH 293 FALL 1994 FINAL # 3

2.1.16* Evaluate the determinant $\begin{vmatrix} a & b & b & b \\ a & a & b & b \\ a & a & a & b \\ a & a & a & a \end{vmatrix}$ by first using row reduction to convert it to upper triangular form.

MATH 293 FALL 1994 FINAL # 12

2.1.17* If A is a 3 by 3 matrix and $\det(A) = 3$, then $\det\left(\frac{1}{2}A^{-1}\right)$ is:

a) $\frac{1}{24}$
b) $\frac{2}{3}$
c) $\frac{1}{6}$
d) $\frac{8}{3}$
e) $\frac{3}{8}$

MATH 293 FALL 1994 FINAL # 13

2.1.18* If AB is singular, then

a) $\det(A)$ is zero,
b) $\det(B)$ is zero,
c) $\det(A)$ is zero and $\det(B)$ is zero,
d) $\det(A)$ is not zero and $\det(B)$ is not zero,
e) either $\det(A)$ is zero or $\det(B)$ is zero.

MATH 293 FALL 1994 FINAL # 14

2.1.19* Given the system $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. With $p = \det\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$, $q = \det\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix}$, $r = \det\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$, $s = \det\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, by Cramer's Rule, the solution for y is given by

a) $\frac{s}{p}$
b) $\frac{r}{p}$
c) $\frac{p}{r}$
d) $\frac{q}{p}$
e) $\frac{r}{q}$

MATH 293 SPRING 1995 PRELIM 2 # 5

2.1.20* Let A be a 6×6 matrix.

a) Which of the following 3 terms will appear in $\det A$:

$$a_{13}a_{22}a_{36}a_{45}a_{51}a_{64}, a_{15}a_{21}a_{36}a_{45}a_{52}a_{63}, a_{16}a_{25}a_{34}a_{43}a_{52}a_{61}?$$

b) For those which will appear, what will their signs be?
c) How many such terms will there be in all?

MATH 293 SPRING 1995 PRELIM 2 # 6
2.1.21*

a) Calculate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & -1 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -2 & 1 & 0 \end{bmatrix}$$

b) What can you say of the solutions to the equation

$$Ax = 0.$$

MATH 293 SPRING 1996 PRELIM 3 # 9
2.1.22* Let A be an n by n matrix. Which of the following are equivalent to the statement "the determinant of A is not zero"? You do **not** need to show any work. "

a) The columns of A are linearly independent.
b) The rank of A is equal to n.
c) The null space of A is empty.
d) $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b}$ in $\Re^n$.
e) A is not onto.

MATH 293 SPRING 1996 FINAL # 12
2.1.23* The determinant of the matrix below is:

$$\begin{pmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{pmatrix}$$

a) 1
b) -1
c) 2
d) 0
e) none of above

MATH 294 SPRING 1997 FINAL # 2
2.1.24* If the det $A = 2$. Find the det A^{-1}, det A^T.

MATH 294 FALL 1997 FINAL # 6

2.1.25* The equation of a surface S in $\Re^3$ is given as

$$\det\begin{pmatrix} x & y & z & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{pmatrix} = 0$$

where the a_i, b_i, c_i are constants.

a) Does the point $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ lie on S?

b) Do the points $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ lie on S?

c) Find a relationship between the coordinates of a, b, and c such that if this relationship holds, then the origin lies on S.

MATH 294 SPRING 1998 PRELIM 2 # 5

2.1.26* Use cofactor expansion to compute the determinant

$$\begin{bmatrix} 1 & -2 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 0 & 0 & 4 & 4 \end{bmatrix}.$$

At each step choose a row or column that involves the least amount of computation.

MATH 294 SPRING 1998 PRELIM 2 # 5

2.1.27 True or False?

a) The determinant of an $n \times n$ triangular matrix is the product of the entries on the main diagonal.

b) The cofactor expansion of an $n \times n$ matrix down a column is the negative of the cofactor expansion along a row.

MATH 294 FALL 1998 PRELIM 2 # 2

2.1.28* Evaluate the determinant of $B = \begin{bmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix}$.

MATH 293 SPRING ? PRELIM 2 # 3
2.1.29*

a) Compute $\det \begin{bmatrix} 4 & -7 & 2 \\ 5 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

b) If $F(x) = \det \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, then show that $F(737)$ is not zero. (Hint: How many roots can the equation $F(x) = 0$ have?)

MATH 293 UNKNOWN PRACTICE # 2b
2.1.30 A $n \times n$ matrix C is said to be orthogonal if $C^t = C^{-1}$. Show that either $\det C = 1$ or $\det C = -1$. Hint: $CC^T = I$.