1.1 Introduction to Linear Systems and Row Reduction

**MATH 294 FALL 1981 PRELIM 1 # 4 204FARP1.P1.Q4.tex**

1.1.1 Solve the following systems of linear equations. If there is no solution, show why. If there are infinitely many solutions, give a general expression.

a) \[
\begin{bmatrix}
2 & 0 & 1 \\
-1 & 2 & 1 \\
1 & 4 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
3 \\
4 \\
-1
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
1 & 3 & -2 \\
-1 & 2 & -3 \\
2 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}
\]

c) \[
\begin{bmatrix}
1 & 4 & 3 \\
-1 & 2 & -3 \\
1 & 3 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
3 \\
2 \\
-1
\end{bmatrix}
\]

**MATH 294 FALL 1982 FINAL # 1 204FARP1.F1.Q1.tex**

1.1.2 a) Find all possible solutions \( \vec{x} \) of \( B\vec{x} = \vec{c} \), where

\[
B = \begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\text{ and } \vec{c} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

b) For the system \( C\vec{x} = \vec{b} \), where

\[
C = \begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 2 \\
1 & 1 & 2
\end{bmatrix},
\]

determine all vectors \( \vec{b} \) for which the system possesses nontrivial solutions \( \vec{x} \).

**MATH 294 SPRING 1983 PRELIM 1 # 2 204SPRP1.P1.Q2.tex**

1.1.3 Consider the system

\[
\begin{align*}
x + y - z + w &= 0 \\
x + 3z + w &= 0 \\
2x + y + 2z + 2w &= 0 \\
3x + 2y + z + 3w &= 0
\end{align*}
\]

a) Find all the solutions to this system.

b) Find a basis for the vector space of solutions to the system above. You need not prove this is a basis.

c) What is the dimension of the vector space of solutions above?

d) Is the the vector \[
\begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix} = \begin{bmatrix}
-2 \\
1 \\
1 \\
2
\end{bmatrix}
\] a solution to the above system?
1.1.4 Find the general solution, or else show that the system has no solutions:

\[
\begin{align*}
7x_1 - 3x_2 + 4x_3 &= -7 \\
2x_1 + x_2 - x_3 + 4x_4 &= 6 \\
x_2 - 3x_4 &= -5
\end{align*}
\]

1.1.5 Find the general solution of the system

\[
\begin{align*}
-x_2 + 3x_3 + 2x_4 &= 1 \\
-2x_1 + 3x_2 + 5x_3 + 4x_4 &= -5 \\
x_1 + x_2 - 2x_3 + x_4 &= 8
\end{align*}
\]

and express your answer in vector form.

1.1.6 a) Solve the linear system \( A\vec{x} = \vec{b} \), where

\[
A = \begin{bmatrix}
1 & 0 & -2 & 4 \\
2 & 1 & -4 & 6 \\
-1 & 2 & 5 & -3 \\
3 & 3 & -5 & 4
\end{bmatrix}
\quad \text{and} \quad \vec{b} = \begin{bmatrix}
4 \\
9 \\
9 \\
15
\end{bmatrix}
\]

b) Solve the linear system \( A\vec{x} = \vec{0} \), where

\[
A = \begin{bmatrix}
-3 & -1 & 0 & 1 & -2 \\
1 & 2 & -1 & 0 & 3 \\
2 & 1 & 1 & -2 & 1 \\
1 & 5 & 2 & -5 & 4
\end{bmatrix}
\]

Express your answer in vector form, and give a basis for the space of solutions.

1.1.7* Find all solutions to:

\[
\begin{align*}
x + z &= 0 \\
y + 4z &= 0 \\
2y - 8z &= 0
\end{align*}
\]
1.1. INTRODUCTION TO LINEAR SYSTEMS AND ROW REDUCTION

MATH 294  FALL 1987  PRELIM 2  # 4

1.1.8  a) Determine the row-reduced form of the matrix:

\[
A = \begin{bmatrix}
0 & 2 & 3 & 5 & 0 \\
0 & 2 & 6 & 8 & -3 \\
0 & 4 & 6 & 10 & 0 \\
0 & 4 & 9 & 13 & -4 \\
\end{bmatrix}
\]

b) Find the general solution of \( A\vec{u} = \vec{0} \), where

\[
\vec{u} = \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
\end{bmatrix}
\text{ and } \vec{0} = \begin{bmatrix}0 \\
0 \\
0 \\
\end{bmatrix}.
\]

MATH 294  FALL 1987  MAKE-UP 2  # 4

1.1.9  Use row reduction to either find the solution or show that no solution exists for the system

\[
\begin{align*}
x_1 & - 2x_2 = -2 \\
3x_1 & + x_2 + 7x_3 = -1 \\
6x_1 & - 5x_2 + 7x_3 = 1
\end{align*}
\]

MATH 294  SPRING 1989  PRELIM 2  # 3

1.1.10  Consider the system of equations,

\[
\begin{align*}
-x_1 & + 2x_2 + 3x_3 = -1 \\
2x_1 & + 5x_2 - 3x_3 = 2 \\
11x_1 & + 14x_2 - 21x_3 = 11
\end{align*}
\]

a) Find all solutions, if any exist, of the system.

b) Is the set of vectors given by,

\[
\begin{bmatrix}
-1 \\
2 \\
11
\end{bmatrix}, \begin{bmatrix}
2 \\
5 \\
14
\end{bmatrix}, \text{ and } \begin{bmatrix}
3 \\
-3 \\
-21
\end{bmatrix}
\]

linearly independent or dependent?

MATH 294  SUMMER 1989  PRELIM 2  # 1

1.1.11  a) Find all solutions to

\[
\begin{align*}
x_1 & + 2x_2 - 4x_3 + 3x_4 = 1 \\
x_1 & + 2x_2 - 2x_3 + 2x_4 = 1 \\
2x_1 & + 4x_2 - 2x_3 + 3x_4 = 2
\end{align*}
\]

using only the row reduction method.
MATH 293  SPRING 1990  PRELIM 1  # 1

1.1.12* Find all the solutions. Write your answers in vector form.
   a) \[2x_1 - 4x_2 - 2x_3 = 0\]
      \[5x_1 - x_2 - 2x_3 = 6\]
      \[-3x_1 + 2x_2 + x_3 = -2\]
   b) \[-x_1 + 3x_2 + 2x_3 = 1\]
      \[3x_1 - 2x_2 - x_3 = 3\]
      \[x_1 + 4x_2 + 3x_3 = 5\]
   c) \[x_1 + 3x_2 - 4x_3 = 0\]
      \[2x_1 - x_2 - x_3 = 0\]
      \[3x_1 - 4x_2 + x_3 = 3\]

MATH 293  SPRING 1990  PRELIM 2  # 2

1.1.13 Consider \(A\vec{x} = \vec{b}\).

Where \(A\) is
\[
\begin{pmatrix}
1 & 3 & 5 & -1 \\
-1 & -2 & -5 & 4 \\
0 & 1 & 1 & -1 \\
1 & 4 & 6 & -2
\end{pmatrix}.
\]

a) Solve for \(\vec{x}\) given \(\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}\).

b) Find a basis for the null space of \(A\).

c) Without carrying out explicit calculation, does a solution exist for any \(\vec{b}\) in \(\mathbb{V}^4\)?

MATH 293  FALL 1991  PRELIM 3  # 2

1.1.14 Solve for the 2 x 2 matrix \(X\) if
\[
\begin{pmatrix}
2 & 3 \\
3 & 5
\end{pmatrix}X = \begin{pmatrix}
-5 & 1 \\
0 & 4
\end{pmatrix}.
\]

MATH 294  SPRING 1992  PRELIM 3  # 2

1.1.15 Here \(A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}\).

a) Find \(A^{-1}\).

b) Find \(X\) if \(AX = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}\).

c) Find \(\vec{v}\) if \(A\vec{v} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}\).
1.1. INTRODUCTION TO LINEAR SYSTEMS AND ROW REDUCTION

MATH 293 FALL 1990 PRELIM 2 # 2

1.1.16 Find all solutions of the system $A\vec{x} = \vec{b}$ if $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 1 & -19 & 12 \end{bmatrix}$ and

a) $\vec{b} = 0.$

b) $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Express all answers in vector form.

MATH 293 FALL 1992 FINAL # 4

1.1.17 a) Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$ 

b) Let $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$. Find a non singular matrix $C$ such that $C^{-1}AC = D$ where $D$ is a diagonal matrix. Find $C^{-1}$ and $D$.

c) For which value of $a$ does the system of equations

$$\begin{align*}
  x + 2y + 3z &= a \\
  3x + 4y + 5z &= 2 \\
  -x + z &= 0
\end{align*}$$

has at least one solution? Explain your answer.

MATH 293 SPRING 1993 FINAL # 1

1.1.18 a) Find the general solution, and write your answer as a particular solution plus the general solution of the associated homogeneous system.

$$\begin{align*}
  x & - 5y + 4z = 3 \\
  2x & - 3y + z = -1 \\
  -3x & + y + 2z = 5
\end{align*}$$

b) Check your answer for part a.

c) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -1 & -2 & -2 \end{bmatrix}.$$ 

d) Check your answer for part c.
MATH 293 SPRING 1993 FINAL # 3

1.1.19 Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$ 

a) Find the vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ such that a solution $\vec{x}$ of the equation $A\vec{x} = \vec{b}$ exists.

b) Find a basis for the column space $\mathbb{R}(A)$ of $A$.

c) It is claimed that $\mathbb{R}(A)$ is a plane in $\mathbb{R}^3$. If you agree, find a vector $\vec{v}$ in $\mathbb{R}^3$ that is normal to this plane. Check your answer.

d) Show that $\vec{v}$ is perpendicular to each of the columns of $A$. Explain carefully why this is true.

MATH 293 SPRING 1994 PRELIM 2 # 2

1.1.20 (True/false) The following properties hold for the matrix

$$A = \begin{pmatrix} 2 & -3 & 7 \\ -1 & 4 & 0 \end{pmatrix}.$$ 

a) If $AM = AN$ then $M = N$, where $M$ and $N$ are $3 \times 2$ matrices.

b) $A$ has an inverse.

c) $A$ is in reduced row echelon form.

d) $A$ is equal to the matrix $B = \begin{pmatrix} -1 & 4 & 0 \\ 2 & -3 & 7 \end{pmatrix}$.

e) $A$ and $B$ are row equivalent.

f) $A$ and $B$ have the same row reduced form.

g) $(A^T)^T = A$.

h) $B^TA = BA^T$. 
1.1. INTRODUCTION TO LINEAR SYSTEMS AND ROW REDUCTION

MATH 293 SPRING 1994 PRELIM 2 # 3

1.1.21* If the reduced row echelon form of

\[
\begin{pmatrix}
1 & -1 & 2 & -2 & 3 & -2 & 6 \\
2 & 0 & 3 & -4 & 1 & -1 & 4 \\
1 & -3 & -1 & -2 & 2 & -5 & -1
\end{pmatrix}
\]

is

\[
\begin{pmatrix}
1 & 0 & 0 & -2 & -7/4 & -1/2 & -20/8 \\
0 & 1 & 0 & 0 & -7/4 & 3/2 & -17/8 \\
0 & 0 & 1 & 0 & 3/2 & 0 & 15/4
\end{pmatrix},
\]

then the general solution of the system

\[
\begin{align*}
x - y + 2z - 2w &= -2 \\
2x + 3y + 3z - 4w &= -1 \\
x - 3y - z - 2w &= -5
\end{align*}
\]

is

a) \( \begin{pmatrix} -1/2 \\ 3/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \), b) \( \frac{1}{2} \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \), c) \( \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7/4 \\ -7/4 \\ 3/2 \end{pmatrix} \),

\[\text{d) } \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} , \text{ e) } \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} .\]

MATH 293 SPRING 1994 PRELIM 2 # 4

1.1.22* The reduced row echelon form of \( A = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 2 & 2 & 0 & 4 \\ 1 & 4 & 3 & -1 \end{pmatrix} \) is

a) \( \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \), b) \( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \), c) \( \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), d) \( \begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), e) \( \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).

(Note! There is one and only one correct answer.)

MATH 293 FALL 1994 PRACTICE 2 # 2

1.1.23 Find the general solution in vector form for the equations

\[
\begin{align*}
x + 2y + 2z - w &= 1 \\
3x + 6y + z + 2w &= 3 \\
-x - 2y + z - 2w &= -1
\end{align*}
\]

\[
\begin{align*}
x + 2y + 2z + 4w &= -1 \\
3x + 6y + 2z + 2w &= 3 \\
x + 2y + z + 2w &= -1
\end{align*}
\]

\[
\begin{pmatrix} -1 \\ -5 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}.
\]

\[
\begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\
3 \\ 6 \\ 1 \\ 2 \\
-1 \\ -2 \\ 1 \\ 2 \\
-1 \\ -2 \\ 1 \\ 2
\end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\
1 \\ 2 \\ 2 \\ 2 \\
1 \\ 2 \\ 1 \\ 2 \\
0 \\ 0 \\ 0 \\ 0
\end{pmatrix}.
\]
1.1.24 Use Gauss-Jordan elimination to find all solutions of

\[
\begin{align*}
x + 2y + 3z &= b_1 \\
x + y + z &= b_2 \\
5x + 7y + 9z &= b_3
\end{align*}
\]

in the cases that

\[
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}
\]

1.1.25 Find the general solution of the system of equations

\[
\begin{align*}
2x_1 + 4x_3 &= 10 \\
2x_1 + x_2 + 3x_3 &= 14 \\
4x_1 + x_2 + 7x_3 + x_4 &= 27 \\
-2x_1 + 2x_2 - 6x_3 + x_4 &= 1
\end{align*}
\]

1.1.26 Find the general solution of the system of equations

\[
\begin{align*}
-x_2 + 3x_3 + 2x_4 &= 0 \\
-2x_1 + 3x_2 + 5x_3 + 4x_4 &= -5 \\
x_1 + x_2 - 2x_3 + x_4 &= 8
\end{align*}
\]

1.1.27* a) Find the general solution of the system of equations

\[
\begin{align*}
2x_1 + 4x_3 &= 10 \\
2x_1 + x_2 + 3x_3 &= 14 \\
4x_1 + x_2 + 7x_3 + x_4 &= 27 \\
-2x_1 + 2x_2 - 6x_3 + x_4 &= 1
\end{align*}
\]

b) Verify your solution.

1.1.28* a) Find the general solution, in vector form, of the equation \(A\vec{x} = \vec{b}\) where

\[
A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 4 & 4 & 3 \\ 0 & -2 & -4 & -2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ -2 \end{bmatrix}
\]

Verify your solution.
1.1. INTRODUCTION TO LINEAR SYSTEMS AND ROW REDUCTION

MATH 293 SPRING 1996 FINAL # 2

1.1.29* Consider the system

\begin{align*}
x_1 + x_2 + x_3 - 2x_4 &= 3 \\
2x_1 + x_2 + 3x_3 + 2x_4 &= 5 \\
-x_2 + x_3 + 6x_4 &= 3
\end{align*}

A solution of these equations is:

a) The trivial solution.
b) \(x_1 = 9, x_2 = 0, x_3 = 0, x_4 = 1\)
c) \(x_1 = 0, x_2 = 3, x_3 = 0, x_4 = 1\)
d) The system has no solution.
e) None of the above.

MATH 293 SPRING 1996 FINAL # 3

1.1.30* Consider the system

\begin{align*}
x + z &= 4 \\
2x + y + 3z &= 5 \\
-3x - 3y + (a^2 - 5a)z &= a - 8
\end{align*}

The value of \(a\) for which the system has infinitely many solutions is:

a) 2
b) 3
c) none
d) 1
e) none of the above.

MATH 293 FALL 1996 PRELIM 2 # 2

1.1.31* Matrix algebra. Let \([A]\) be the matrix

\[A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}\]

and let \(\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\) and \(\vec{c} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\). \(\vec{c}\) is not a solution of \(Ax = \vec{b}\).

a) Find all solutions \(x\) to \(Ax = \vec{b}\) and check your answer by substitution.
b) Find all solutions \(x\) to \(Ax = \vec{c}\) and check your answer by substitution.
c) Give a reason why you believe that \(A^{-1}\) does or does not exist.

MATH 294 SPRING 1997 PRELIM 2 # 1

1.1.32 Find the general solution of the linear system

\begin{align*}
-2 + x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\
x_3 &= 1
\end{align*}

\(x_3 = 1\)
1.1.33* Let \( A = \begin{bmatrix} 9 & 0 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix} \), \( \vec{\delta} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \), and \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \).

a) Find the characteristic polynomial \( \det(A - \lambda I) \) of \( A \), and find all eigenvalues. 

(hint: \( \lambda - 9 \) is one factor of the polynomial.)

b) Find an eigenvector for each eigenvalue.

c) Write the augmented matrix for the system of equations \( A\vec{x} = \vec{\delta} \) and solve the system by row operations.

1.1.34 Solve the following system for \( x_1, x_2, x_3, x_4 \) and express the general solution in parametric form.

\[
\begin{align*}
3x_2 + x_1 - 2x_3 &= 2 \\
-1 + x_2 + x_4 &= 0 \\
x_3 - x_1 &= 1
\end{align*}
\]

1.1.35 a) Consider the problem \( A\vec{x} = \vec{\delta} \), where

\[
A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 2 \\ -1 & 2 & 1 & -3 \end{bmatrix}, \quad \vec{\delta} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.
\]

Determine the general solution to this problem, in vector form.

b) Find a 2 by 2 matrix \( B \), which is not the zero matrix, with \( B^2 = 0 \).

1.1.36 Find the general solution of these equations in vector parametric form.

\[
\begin{align*}
x_3 - x_1 &= 1 \\
x_2 + x_4 &= 2 \\
x_2 - x_1 &= 3
\end{align*}
\]
1.1. **INTRODUCTION TO LINEAR SYSTEMS AND ROW REDUCTION**

MATH 294  **SPRING 1998**  PRELIM 2  # 1  204SPSP2Q1.tex

1.1.37  a) Write the solution set of the system

\[
\begin{align*}
    x_1 - 3x_2 - 2x_3 &= 0 \\
    x_2 - x_3 &= 0 \\
    -2x_1 + 3x_2 + 7x_3 &= 0
\end{align*}
\]

in parametric form.

b) With

\[
A = \begin{bmatrix}
1 & -3 & -2 \\
0 & 1 & -1 \\
-2 & 3 & 7
\end{bmatrix},
\]

find all solutions to the system

\[
A\vec{x} = \begin{bmatrix}
0 \\
1 \\
-3
\end{bmatrix}.
\]

c) True or False?

i) The columns of A are linearly independent.

ii) The solution set of \(A\vec{x} = \vec{b}\) is all vectors of the form \(\vec{w} = \vec{r} + \vec{v}_k\) where \(\vec{v}_k\) is any solution of \(A\vec{v}_k = \vec{0}\) and \(A\vec{r} = \vec{b}\).

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1.1.38* Find all solution to the following matrix equation \(A\vec{x} = \vec{b}\) where

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
2 & 4 & 0 \\
1 & 2 & 2
\end{bmatrix}
\]

for each of the following values of \(\vec{b}\):

a) \(\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\)

b) \(\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\)

c) \(\vec{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}\)
MATH 294 FALL 1998 PRELIM 1 # 3

1.1.39 a) Write the following system of equations as (i) a vector equation (ii) as a matrix equation.

\begin{align*}
-5x_1 + \quad x_3 &= 0 \\
2x_1 - x_2 + 9x_4 &= 1 \\
6x_1 + 2x_2 - 5x_3 + x_4 &= 6
\end{align*}

b) Find all solutions to the linear system

\begin{align*}
2x_1 + 4x_2 + 0x_3 &= 4 \\
x_1 + 2x_2 + x_3 &= 1 \\
x_1 + 2x_2 + x_3 &= 0
\end{align*}

c) Does the above (b) have a solution for any right hand side?

d) Let

\begin{align*}
\vec{u}_1 &= \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -1 \\ -7 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}
\end{align*}

For what value(s) of \( h \) is \( \vec{b} \) in the plane spanned by \( \{ \vec{u}_1, \vec{u}_2 \} \)?

MATH 293 SPRING 1996 PRELIM 2 # 1

1.1.40 Let

\begin{align*}
A &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B &= \begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, C &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}.
\end{align*}

a) Find all \( \vec{x} \) for which \( C\vec{x} = \vec{b} \), where

\begin{align*}
\vec{b} &= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.
\end{align*}
1.1. INTRODUCTION TO LINEAR SYSTEMS AND ROW REDUCTION

MATH 294  FALL 1998  PRELIM 2  # 4

1.1.41* The reduced echelon form of the matrix
\[ A = \begin{bmatrix} 3 & 3 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 1 & 0 & 1 & -2 \\ 0 & -3 & 2 & -1 \end{bmatrix} \]
is
\[ B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

a) What is the rank of \( A \)?
b) What is the dimension of the column space of \( A \)?
c) What is the dimension of the null space of \( A \)?
d) Find a solution to \( A \vec{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \).
e) Find the general solution to \( A \vec{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \).
f) What is the row space of \( A \)?
g) Would any of your answers above change if you changed \( A \) by randomly changing 3 of its entries in the 2nd, third, and fourth columns to different small integers and the corresponding reduced echelon form for \( B \) was presented? (yes?, no?, probably? probably not?, ?)

MATH 294  FALL 1998  FINAL  # 5

1.1.42 Consider \( A \vec{x} = \vec{b} \) with
\[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ -1 & 2 & 5 & 8 \\ 1 & 2 & 3 & 4 \end{bmatrix} \]
and \( \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \). The augmented matrix of this system is
\[ \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
which is row equivalent to
\[ \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ -1 & 2 & 5 & 8 & 1 \end{bmatrix} \]
a) What are the rank of \( A \) and \( \text{dim} \ \text{null} \ A \)? (Justify your answers.)
b) Find bases for \( \text{col} \ A \), \( \text{row} \ A \), and \( \text{null} \ A \).
c) What is the general solution \( \vec{x} \) to \( A \vec{x} = \vec{b} \) with the given \( A \) and \( \vec{b} \)?
d) Select another \( \vec{b} \) for which the above system has a solution. Give the general solution for that \( \vec{b} \).
MATH 293  SPRING ?  FINAL  # 1

1.143* Find the general solution of the following linear system and express it in vector form

\[ \begin{align*}
2x - 3y + 0z - w &= -8 \\
-5x + 2y - 3z + 2w &= -2 \\
2x + 0y + 2z - w &= 4 \\
x - y + z - w &= -2
\end{align*} \]

MATH 293  ???  FINAL  # 3

1.144* a) Give all solutions of the following system in vector form.

\[ \begin{align*}
6x_1 + 4x_3 &= 1 \\
5x_1 - x_2 + 5x_3 &= -1 \\
x_1 + 3x_3 &= 2
\end{align*} \]

b) What is the null space of the matrix of coefficients of the unknowns in a)?

MATH 294  SPRING 1982  PRELIM 1  # 1

1.145 a) Write the system of equations

\[ \begin{align*}
x_1 + 2x_2 + 3x_3 &= 1 \\
2x_1 + 3x_2 + 4x_3 &= -2 \\
3x_1 + 4x_2 + 6x_3 &= 0
\end{align*} \]

in the form \( A\vec{x} = \vec{b} \).

b) Find the \( \det A \) for \( A \) in part (a) above.

c) Does \( A^{-1} \) exist?

d) Solve the above system of equations for \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \).

e) Let \( B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \). Find \( A \cdot B \) (i.e. calculate the product \( AB \)).

MATH 293  FALL 1991  FINAL  # 1

1.146 Write the general solution in vector form:

\[ \begin{align*}
2x - y + z + 3w &= 2 \\
4x + y - 3z + 5w &= 6 \\
-x + 2y - 3z - 2w &= 0 \\
x + 4y - 7z + 0w &= 4
\end{align*} \]

note: This problem is the same as MATH 293 SUMMER 1992 PRELIM 6/20 p.2
1.1. INTRODUCTION TO LINEAR SYSTEMS AND ROW REDUCTION

MATH 293 SPRING 1992 PRELIM 2 #2
1.1.47 Find the general solution solution in vector form for the equations

\[
\begin{align*}
x + 2y + 2z - w &= 1 \\
3x + 6y + z + 2w &= 3 \\
-x - 2y + z - 2w &= -1 \\
\end{align*}
\]

MATH 293 SUMMER 1992 PRELIM 6/30 #2
1.1.48 Find the general solution of the equations

\[
\begin{align*}
2x - y + z + 3w &= 2 \\
4x + y - 3z + 5w &= 6 \\
-x + 2y - 3z - 2w &= 0 \\
x + 4y - 7z + 0w &= 4 \\
\end{align*}
\]

MATH 293 SUMMER 1992 FINAL #1
1.1.49* a) Find the general solution, in vector form, of the equations \(A\vec{x} = \vec{b}\) where

\[
A = \begin{pmatrix} -1 & -3 & 4 & -2 \\
0 & 2 & 5 & 1 \\
0 & 1 & -3 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 5 \\
2 \\
4 \end{pmatrix}
\]

b) Solve \(AX = B\) where

\[
A = \begin{pmatrix} -1 & 2 & -3 \\
2 & 1 & 0 \\
4 & -2 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 3 \\
3 & 2 & 1 \\
1 & 3 & 2 \end{pmatrix}
\]

MATH 293 SPRING 1993 PRELIM 2 #3
1.1.50 a) Solve the linear system:

\[
\begin{align*}
2w - 3x + y - z &= -1 \\
4w + x - 3y - z &= 1 \\
w + 2x - 3y - 2z &= -2 \\
2w - 3x - y - 5z &= -7 \\
\end{align*}
\]

Write the general solution in vector form as the sum of a particular solution plus the general solution of the associated homogeneous equation.

b) Check your answer, and explain what you do to check.

MATH 294 SPRING 1997 PRELIM 2 #10
1.1.51 Two chemicals \(A\) and \(B\), are reacting with each other. After one second has elapsed, 90% of chemical \(A\) stays chemical \(A\), while 10% turns into chemical \(B\); also, 80% of chemical \(B\) stays chemical \(B\), while 20% turns into chemical \(A\). Suppose that the system is in equilibrium, i.e. that there is no change in the amount in grams of chemical \(A\) or \(B\) from one second to the next. If there are 10 grams of chemical \(A\) at equilibrium, how many grams of chemical \(B\) must there be?
1.1.52 A spaceship operator operates daily spaceship service between three planets, A, B, and C. The matrix below shows the traffic during a Monday. The numbers are fractions of the total number of spaceships that start at one location, and go to another destination. For example, the .4 means that 40% of the spaceships that start at C travel to A that day.

\[
\begin{pmatrix}
0 & 1 & .4 \\
.5 & 0 & 6 \\
.5 & 0 & 0
\end{pmatrix}
\]

The distribution of spaceships at planets A, B, and C on Monday is 10,b,c. Find b and c such that the same distribution of spaceships reappears the next day, on Tuesday.

1.1.53 Consider the chemical reaction (unbalanced as written below)

\[C_2H_6O + O_2 \rightarrow CO_2 + H_2O.\]

Let \(x_1, x_2, x_3,\) and \(x_4\) be the number of molecules of each compound (in the order given above). Find integers \(x_1, x_2, x_3, x_4\) that balance this reaction.

**Hint:** If you order your elements and hence equations as

\[
\begin{pmatrix}
\text{Oxygen} \\
\text{Carbon} \\
\text{Hydrogen}
\end{pmatrix}, \text{ or } \begin{pmatrix}
O \\
C \\
H
\end{pmatrix},
\]

you will minimize the number of row operations.

1.1.54 The kingdom of Ferrgrad has three primary industrial sectors: iron, railroad, and coal. Suppose that:

- To produce $1 of steel, the steel sector consumes $2 of steel, $1 of railroad, and $2 of coal.
- To produce $1 of railroad transportation that rail sector consumes $1 of steel, $2 of coal, and $4 of coal.
- To produce $1 of coal, the coal sector consumes $2 of steel, $2 of rail, and $3 of coal.

Ferrgrad does not use all its production in the various sectors to maintain the others. Additionally it exports

\[
\begin{align*}
S_0 &= 1.2 \times 10^6 \text{ of steel}, \\
R_0 &= 0.8 \times 10^6 \text{ of railroad transportation services,} \\
C_0 &= 1.5 \times 10^6 \text{ of coal.}
\end{align*}
\]

Define \(S, R,\) and \(C\) to be the $ values of annual production of steel, rail, and coal. Set and do not solve a matrix equation that will tell you \(S, R,\) and \(C,\) the values of the annual productions of the three sectors. [Hint: first write three simultaneous equations, one for each output, which relate production to output.]