Contents

Chapter 1

Beginning Linear Algebra

Chapter 2

More Linear Algebra

2.1 Determinants

MATH 293 FALL 1981 PRELIM 1 # 2
2.1.1* Consider the matrices

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 6 \\ 1 & 3 & 4 & 6 \\ 1 & 4 & 5 & 8 \end{bmatrix}.$$

a) Find det A and det B.
b) Find A^{-1} and B^{-1} If they exist. If you think that either of the inverses does not exist, give a reason.

MATH 294 SPRING 1982 PRELIM 1 # 1

2.1.2* **a**) Write the system of equations $\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & -2 \\ 3x_1 & + & 4x_2 & + & 6x_3 & = & 0 \end{array}$ in the form $A\vec{x} = B$.
b) Find det A for A in part (a) above.
c) Does A^{-1} exist?
d) Solve the above system of equations for $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
e) Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$. Find $A \cdot B$ (i.e. calculate the product AB).

MATH 293 FALL 1991 PRELIM 3 # 4

2.1.3* Compute the determinants of the following matrices:

a) $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & -11 & -3 \\ 0 & 0 & 4 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & -5 & 17 & 31 \\ 0 & 3 & 9 & 14 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix}^{-1}$

MATH 293 FALL 1991 PRELIM 3 # 6

2.1.4* True/False

a) All three row operations preserve the absolute value of the determinant of a square matrix.

b) A singular $n \times n$ matrix has a zero determinant.

c) If A is a $n \times n$ matrix $\det(A^t) = \det(A)$.

d) If each entry in a square matrix A is replaced by its reciprocal (inverse), producing a new matrix B, then $\det(B) = (\det(\mathrm{A}))^{-1}$.

e) If a matrix A is nonsingular, then $\det(\mathrm{A}^{-1}) = (\det(\mathrm{A}))^{-1}$

f) For a square matrix A and a scalar k, $\det(kA) = k\det(A)$

g) Let A and B by $n \times n$ matrices

$$\det \begin{pmatrix} A & O_n \\ O_n & B \end{pmatrix} = \det(A)\det(B)$$

where O_n is a $n \times n$ matrix with all elements equal to zero.

MATH 293 SPRING 1992 PRELIM 3 # 1

2.1.5* Compute

a) $\det \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 5 & 0 & 8 \end{bmatrix}$

b) $\det \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

c) $\det \begin{bmatrix} 1 & 1 & 7 & 0 & 0 \\ 1 & 1 & 0 & 3 & 3 \\ 5 & 5 & 1 & 8 & 9 \\ 6 & 6 & 1 & 0 & 1 \\ 6 & 6 & 1 & 0 & 1 \end{bmatrix}$

MATH 293 SUMMER 1992 PRELIM 7_21 # 2

2.1.6* Compute the following determinants:

a) $\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 2 & 3 & 2 & 3 \\ -3 & -5 & 0 & -1 \end{vmatrix}$

b) $\begin{vmatrix} 3-\lambda & 0 & 1 \\ 2 & 1-\lambda & -4 \\ 1 & 0 & -1-\lambda \end{vmatrix}$

MATH 293 FALL 1992 PRELIM 3 # 1

2.1.7* Compute the determinant of the matrix

$$A = \begin{pmatrix} b & a & a & a & a \\ b & b & a & a & a \\ b & b & b & a & a \\ b & b & b & b & a \\ b & b & b & b & b \end{pmatrix}$$

MATH 293 FALL 1992 PRELIM 3 # 4

2.1.8* Let A be an $n \times n$ matrix. Assume that it is known that the equation $Ax = 0$ has nontrivial solutions if and only if $\det(A) = 0$

Let

$$A = \begin{pmatrix} 3-s & 0 & 1 \\ 2 & 1-s & -4 \\ 1 & 0 & -(1+s) \end{pmatrix}$$

where s is an arbitrary scalar.

a) Compute $\det(A)$

b) Find those values of s for which the equation $Ax = 0$ has nontrivial solutions.

MATH 293 FALL 1992 FINAL # 3

2.1.9*

a) Let A be an $n \times n$ nonsingular matrix. Prove that $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$. Hint: You may use the fact that if A and B are $n \times n$ matrices, $\det(AB) = \det(A)\det(B)$.

b) An $n \times n$ matrix A has a nontrivial null space. Find $\det(A)$ and explain your answer.

MATH 293 SPRING 1993 PRELIM 3 # 1

2.1.10* Given the matrix

$$A = \begin{pmatrix} -2 & 1 & 2 \\ -2 & 2 & 2 \\ -9 & 3 & 7 \end{pmatrix}$$

Find $\det A$.

MATH 293 SPRING 1994 PRELIM 2 # 6

2.1.11*

a) Compute the determinant of the matrix $A(\lambda) = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 2 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$, writing your result as a function of λ.

b) Partially check your result by computing the determinant of $A(0)$, and compare this value with the value of the function you found in a) when $\lambda = 0$

MATH 293 FALL 1994 PRELIM 2 # 3

2.1.12* Compute

$$\det \begin{bmatrix} 1 & -2 & 3 \\ -3 & 5 & -8 \\ 2 & 2 & 5 \end{bmatrix}$$

by the following two methods:

a) Use row ops to change the matrix into an upperright triangular matrix with the same det

b) Use cofactors of entries in the first row

MATH 293 FALL 1994 PRELIM 2 # 5

2.1.13* Let A and B be $N \times N$ matrices.

a) Complete the following statement: A is singular if and only if $\det(A) = ...$

b) Use the result of part (a) to find the value of λ for which the matrix $\begin{pmatrix} \lambda-1 & 3 \\ 2 & \lambda-2 \end{pmatrix}$ is singular

c) Complete the following statement: $\det(AB) = ...$

d) Use the result of part (c) to show that if A is invertible, $\det\left(A^{-1}\right) = \frac{1}{\det A}$. (Hint: $AA^{-1} = I$)

MATH 293 FALL 1994 PRELIM 2 # 6

2.1.14* Compute

$$\det \begin{pmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{pmatrix}.$$

MATH 293 FALL 1994 PRELIM 3 # 3

2.1.15* Let A be an *nbyn* matrix. Which of the following is equivalent to the statement: A **is singular?**

a) The $\det(A) = n$.

b) $Ax = 0$ has a nontrivial solution.

c) The rows of A are linearly independent.

d) The rank of A is n.

e) The $\det A = 0$.

f) $Ax = B$ has a unique solution x for each B.

g) A has non-zero nullity.

MATH 293 FALL 1994 FINAL # 3

2.1.16* Evaluate the determinant $\begin{vmatrix} a & b & b & b \\ a & a & b & b \\ a & a & a & b \\ a & a & a & a \end{vmatrix}$ by first using row reduction to convert it to upper triangular form.

MATH 293 FALL 1994 FINAL # 12

2.1.17* If A is a 3 by 3 matrix and $\det(A) = 3$, then $\det\left(\frac{1}{2}A^{-1}\right)$ is:

a) $\frac{1}{24}$
b) $\frac{2}{3}$
c) $\frac{1}{6}$
d) $\frac{8}{3}$
e) $\frac{3}{8}$

MATH 293 FALL 1994 FINAL # 13

2.1.18* If AB is singular, then

a) $\det(A)$ is zero,
b) $\det(B)$ is zero,
c) $\det(A)$ is zero and $\det(B)$ is zero,
d) $\det(A)$ is not zero and $\det(B)$ is not zero,
e) either $\det(A)$ is zero or $\det(B)$ is zero.

MATH 293 FALL 1994 FINAL # 14

2.1.19* Given the system $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. With $p = \det\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$, $q = \det\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix}$, $r = \det\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$, $s = \det\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, by Cramer's Rule, the solution for y is given by

a) $\frac{s}{p}$
b) $\frac{r}{p}$
c) $\frac{p}{r}$
d) $\frac{q}{p}$
e) $\frac{r}{q}$

MATH 293 SPRING 1995 PRELIM 2 # 5

2.1.20* Let A be a 6×6 matrix.

a) Which of the following 3 terms will appear in $\det A$:

$$a_{13}a_{22}a_{36}a_{45}a_{51}a_{64}, a_{15}a_{21}a_{36}a_{45}a_{52}a_{63}, a_{16}a_{25}a_{34}a_{43}a_{52}a_{61}?$$

b) For those which will appear, what will their signs be?
c) How many such terms will there be in all?

MATH 293 SPRING 1995 PRELIM 2 # 6
2.1.21*

a) Calculate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & -1 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -2 & 1 & 0 \end{bmatrix}$$

b) What can you say of the solutions to the equation

$$Ax = 0.$$

MATH 293 SPRING 1996 PRELIM 3 # 9
2.1.22* Let A be an n by n matrix. Which of the following are equivalent to the statement "the determinant of A is not zero"? You do **not** need to show any work. "

a) The columns of A are linearly independent.
b) The rank of A is equal to n.
c) The null space of A is empty.
d) $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b}$ in $\Re^n$.
e) A is not onto.

MATH 293 SPRING 1996 FINAL # 12
2.1.23* The determinant of the matrix below is:

$$\begin{pmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{pmatrix}$$

a) 1
b) -1
c) 2
d) 0
e) none of above

MATH 294 SPRING 1997 FINAL # 2
2.1.24* If the det $A = 2$. Find the det A^{-1}, det A^T.

MATH 294 FALL 1997 FINAL # 6
2.1.25* The equation of a surface S in $\Re^3$ is given as

$$\det \begin{pmatrix} x & y & z & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{pmatrix} = 0$$

where the a_i, b_i, c_i are constants.

a) Does the point $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ lie on S?

b) Do the points $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ lie on S?

c) Find a relationship between the coordinates of a, b, and c such that if this relationship holds, then the origin lies on S.

MATH 294 SPRING 1998 PRELIM 2 # 5
2.1.26* Use cofactor expansion to compute the determinant

$$\begin{bmatrix} 1 & -2 & 5 & -2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 0 & 0 & 4 & 4 \end{bmatrix}.$$

At each step choose a row or column that involves the least amount of computation.

MATH 294 SPRING 1998 PRELIM 2 # 5
2.1.27 True or False?

a) The determinant of an $n \times n$ triangular matrix is the product of the entries on the main diagonal.

b) The cofactor expansion of an $n \times n$ matrix down a column is the negative of the cofactor expansion along a row.

MATH 294 FALL 1998 PRELIM 2 # 2
2.1.28* Evaluate the determinant of $B = \begin{bmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix}$.

MATH 293 SPRING ? PRELIM 2 # 3

2.1.29*

a) Compute $\det \begin{bmatrix} 4 & -7 & 2 \\ 5 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

b) If $F(x) = \det \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, then show that $F(737)$ is not zero. (Hint: How many roots can the equation $F(x) = 0$ have?)

MATH 293 UNKNOWN PRACTICE # 2b

2.1.30 A $n \times n$ matrix C is said to be orthogonal if $C^t = C^{-1}$. Show that either $\det C = 1$ or $\det C = -1$. Hint: $CC^T = I$.

2.2 Intro to Bases

MATH 294 FALL 1981 PRELIM 1 # 3

2.2.1 **a)** Show that the set of vectors

$$\{1+t, 1-t, 1-t^2\}$$

is a basis for the vector space of all polynomials

$$\vec{p} = a_0 + a_1 t + a_2 t^2$$

of degree less than three.

b) Express the vector

$$2 + 3t + 4t^2$$

in terms of the above basis.

MATH 294 SPRING 1982 PRELIM 1 # 2

2.2.2 Let V be the space of all solutions of

$$\vec{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}.$$

Consider the vectors

$$\vec{x}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}, \vec{x}_2(t) = \begin{pmatrix} e^{t} \\ 0 \\ e^{t} \end{pmatrix}.$$

a) Do $\vec{x}_1(t)$, $\vec{x}_2(t)$ belong to V?

b) Are $\vec{x}_1(t)$, $\vec{x}_2(t)$ linearly independent? Give reasons for your answer.

c) Do the vectors $\vec{x}_1(t)$, $\vec{x}_2(t)$ form a basis for V? Give reasons for your answer.

MATH 294 SPRING 1983 FINAL # 10

2.2.3 **a**) Find a basis for the vector space of all 2×2 matrices.

b) A is the matrix given below, $\vec{v}$ is an eigenvector of A. Find any eigenvalue of A.

$$A = \begin{bmatrix} 3 & 0 & 4 & 2 \\ 8 & 5 & 1 & 3 \\ 4 & 0 & 9 & 8 \\ 2 & 0 & 1 & 6 \end{bmatrix} \text{ with } \vec{v} = [\text{an eigenvector of A}] = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

c) Find one solution to each system of equations below, if possible. If not possible, explain why not. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix} \cdot \vec{x} = \vec{b}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

d) Read carefully. Solve for $\vec{x}$ in the equation $A \cdot \vec{b} = \vec{x}$ with: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

e) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

MATH 294 SPRING 1984 FINAL # 2

2.2.4 Determine whether the given vectors form a basis for S, and find the dimension of the subspace. S is the set of all vectors of the form $(a, b, 2a, 2b)$ in $\Re^4$. The given set is $\{(1, 0, 2, 0), (0, 1, 0, 3), (1, -1, 2, -3)\}$.

MATH 294 FALL 1986 FINAL # 1

2.2.5 The vectors (1,0,2,-1,3),(0,1,-1,2,4),(-1,1,-2,1,-3),(0,1,1,-2,-4), and (1,4,2,-1,3) span a subspace S of $\Re^5$.

a) What is the dimension of S?

b) Find a basis for S.

MATH 294 FALL 1986 FINAL # 2

2.2.6 **a**) Solve the linear system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 2 & 1 & -4 & 6 \\ -1 & 2 & 5 & -3 \\ 3 & 3 & -5 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 9 \\ 9 \\ 15 \end{bmatrix}$.

b) Solve the linear system $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix} -3 & -1 & 0 & 1 & -2 \\ 1 & 2 & -1 & 0 & 3 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 5 & 2 & -5 & 4 \end{bmatrix}$ Express your answer in vector form, and give a basis for the space of solutions.

MATH 294 FALL 1987 PRELIM 3 # 6

2.2.7 Find an <u>orthonormal</u> basis for the subspace of $\Re^3$ consisting of all 3-vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x+y+z=0$.

MATH 294 FALL 1989 PRELIM 3 # 3

2.2.8 Let W be the following subspace of $\Re^3$,

$$W = Comb\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}\right)$$

a) Show that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, is a basis for W

For b) and c) below, let T be the following linear transformation $T : W \to \Re^3$.

$$T\left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

for those $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ in $\Re^3$ which belong to W. [You are allowed to use a) even if you did not solve it.]

b) What is the dimension of Range(T)? (Complete reasoning, please.)

c) What is the dimension of Ker(T)? (Complete reasoning, please.)

MATH 293 SPRING 1990 PRELIM 1 # 3

2.2.9 Find the dimension and a basis for the following spaces

a) The space spanned by $\{(1,0,-2,1),(0,3,1,-1),(2,3,-3,1),(3,0,-6,-1)\}$

b) The set of all polynomials $p(t)$ in P^3 satisfying the two conditions

- **i)** $\frac{d^3}{dt^3}p(t) = 0$ for all t
- **ii)** $p(t) + \frac{d}{dt}p(t) = 0$ at $t = 0$

c) The subspace of the space of functions of t spanned by $\{e^{at}, e^{bt}\}$ if $a \neq b$.

d) The space spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ in W, given that $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for W.

MATH 293 SPRING 1990 PRELIM 1 # 4

2.2.10 **a)** Show that $B = \{t^2-1, t^2+1, t\}$ is a basis for P^2

b) Express the vectors in $\{1, t, t^2\}$ in terms of those in B and find the components of $p(t) = (1+t)^2$ with respect to B.

c) Find the components of the vector $\vec{x} = (1,2,3)$ with respect to the basis $\{(1,0,0),(1,1,0),(1,1,1)\}$.

MATH 293 FALL 1990 PRELIM 2 # 1

2.2.11 **a)** Express the vectors $\vec{u}, \vec{v}$ in terms of $\vec{a}, \vec{b}$, given that

$$3\vec{u} + 2\vec{v} = \vec{a}, \vec{u} - \vec{v} = \vec{b}$$

b) If $\vec{a}, \vec{b}$ are linearly independent, find a basis for the span of $\{\ \vec{u}, \vec{v}, \vec{a}, \vec{b}\ \}$

c) Find $\vec{u}, \vec{v}$, if $\vec{a} = (-1, 2, 8), \vec{b} = (-2, -1, 1)$

MATH 293 FALL 1991 PRELIM 3 # 1

2.2.12 Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & 2 & -2 & -2 \\ 2 & 5 & -4 & 1 \\ 1 & 4 & -4 & 0 \end{pmatrix}$$

a) Find a basis for the row space of A.

b) Find a basis for the column space of A.

MATH 293 SPRING 1992 PRELIM 3 # 6

2.2.13 Given $A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 5 \\ 2 & -1 & 3 & 4 \end{pmatrix}$.

a) Find a basis for the null space of A.

b) Find the rank of A.

MATH 293 SUMMER 1992 PRELIM 7/21 # 3

2.2.14 Given a matrix $A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & -2 & 0 & -3 \end{pmatrix}$.

a) Find a basis for the row space W_1 of A.

b) Find a basis for the range W_2 of A.

c) Find the rank of A.

d) Are the two space W_1 and W_2 the same subspace of V_4? Explain your answer carefully in order to get credit for this part.

MATH 293 SPRING 1992 FINAL # 2

2.2.15 **a**) Find a basis for V_4 that contains at least two of the following vectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

b) A is a 3×3 matrix. If $A \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}$ and $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the nullspace of A, then find the general solution $\vec{x}$ of the equation $A\vec{x} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}$.

Find, also, the determinant of A.

MATH 293 SUMMER 1992 PRELIM 7/21 # 4

2.2.16 Given four vectors in V_4

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ -4 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 4 \\ 4 \\ 0 \\ -6 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

a) Find the space W spanned by the vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$

b) Find a basis for W.

c) Find a basis for V_4 that contains as many of the vectors $\vec{v}_1$,$\vec{v}_2$,$\vec{v}_3$ and $\vec{v}_4$ as possible.

MATH 293 FALL 1992 PRELIM 3 # 2

2.2.17 Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 2 & 0 & -6 & -2 \\ -1 & 1 & 5 & 3 \end{pmatrix}$$

a) Find a basis for the column space of A from among the set of column vectors.

b) Find a basis for the row space of A.

c) Find a basis for the null space of A.

d) What is the rank of A and the dimension of the null space (the nullity)?

MATH 293 FALL 1992 PRELIM 3 # 3

2.2.18 Let $C(-\pi,\pi)$ be the vector space of continuous functions on the interval $-\pi \leq x \leq \pi$. Which of the following subsets S of $C(-\pi,\pi)$ are subspaces? If it is not a subspace say why. If it is, then say why and find a basis.
Note: You must show that the basis you choose consists of linearly independent vectors. In what follows a_0, a_1 and a_2 are arbitrary scalars unless otherwise stated.

a) S is the set of functions of the form $f(x) = 1 + a_1 \sin(x) + a_2 \cos(x)$

b) S is the set of functions of the form $f(x) = 1 + a_1 \sin(x) + a_2 \cos(x)$, subject to the condition $\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 2\pi$

c) S is the set of functions of the form $f(x) = 1 + a_1 \sin(x) + a_2 \cos(x)$, subject to the condition $\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0$

MATH 293 FALL 1992 FINAL # 3

2.2.19 **a**) Let A be an $n \times n$ nonsingular matrix. Prove that $\det(A^{-1}) = \frac{1}{\det(A)}$. Hint: You may use the fact that if A and B are $n \times n$ matrices $\det(AB) = \det(A)\det(B)$.

b) An $n \times n$ matrix A has a nontrivial null space. Find $\det(A)$ and explain your answer.

c) Given two vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ in V_3. Find a vector (or vectors) $\vec{w}_1$,$\vec{w}_2$,... in V_3 such that the set $\{\ \vec{v}_1,\vec{v}_2,\vec{w}_1,...\ \}$ is a basis for V_3.

d) Let S be the set of all vectors of the form $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the usual mutually perpendicular unit vectors. Let W be the set of all vectors that are perpendicular to the vector $\vec{v} = \vec{i} + \vec{j} + \vec{k}$. Is W a vector subspace of V_3? Explain your answer.

MATH 293 SPRING 1993 PRELIM 3 # 2

2.2.20 Given the matrix $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 3 & 3 & 4 & 5 \end{pmatrix}$

a) Find a basis for the row space of B

b) Find a basis for the null space of B

MATH 293 SPRING 1993 PRELIM 3 # 14

2.2.21 Consider the following vectors in $\Re^4$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} 2 \\ -3 \\ -8 \\ 2 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}$$

Let W be the subspace of $\Re^4$ spanned by the vectors $\vec{v}_1$,$\vec{v}_2$,$\vec{v}_3$ and $\vec{v}_4$.
Find a basis for W which is contained in (is a subset of) the set $\{\ \vec{v}_1,\vec{v}_2,\vec{v}_3,\ \vec{v}_4\ .\}$

MATH 293 SPRING 1993 PRELIM 3 # 5

2.2.22 **a**) Consider the vector space V whose elements are 3×3 matrices.

i) Find a basis for the subspace W_1 of V which consists of all upper-triangular 3×3 matrices.

ii) Find a basis for the subspace W_1 of V which consists of all upper-triangular 3×3 matrices with zero trace.
The trace of a matrix is the sum of its diagonal elements.

b) Consider the polynomial space P^3 of polynomials with degree ≤ 3 on $0 \leq t \leq 1$. Find a basis for the subspace W of P^3 which consists of polynomials of degree ≤ 3 with the constraint

$$\left[\frac{d^2p}{dt^2} + \frac{dp}{dt}\right]_{t=0} = 0.$$

MATH 293 FALL 1994 PRELIM 3 # 1

2.2.23 Let A be the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & -1 & 2 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

a) Find a basis for the Null Space of A. What is the nullity of A?

b) Find a basis for the Row Space of A. What is its dimension?

c) Find a basis for the Column Space of A. What is its dimension?

d) What is the rank of A?

MATH 293 FALL 1994 FINAL # 4

2.2.24 **a**) Find a basis for the space spanned by: {(1,0,1),(1,1,0),(-1,-4,-3)}.

b) Show that the functions $e^{2x}\cos(x)$ and $e^{2x}\sin(x)$ are linearly independent.

MATH 293 SPRING 1995 PRELIM 3 # 3

2.2.25 Let P_3 be the space of polynomials $p(t)$ of degree ≤ 3. Consider the subspace $S \subset P_3$ of polynomials that satisfy

$$p(0) + \left.\frac{dp}{dt}\right|_{t=0} = 0$$

a) Show that S is a subspace of P_3.

b) Find a basis for S.

c) What is the dimension of S?

MATH 293 SPRING 1995 PRELIM 3 # 5

2.2.26 **a**) Find a basis for the plane $P \subset \Re^3$ of equation

$$x + 2y + 3z = 0$$

b) Find an orthonormal basis for P.

MATH 293 FALL 1995 PRELIM 3 # 5

2.2.27 Let P_3 be the space of polynomials $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ of degree ≤ 3. Consider the subset S of polynomials that satisfy

$$p''(0) = 4p(0) = 0$$

Here $p''(0)$ means, as usual, $\left.\frac{d^2p}{dt^2}\right|_{t=0}$.

a) Show that S is a subspace of P_3. Give reasons.
b) Find a basis for S.
c) What is the dimension of S? Give reasons for your answer.
Hint: What constraint, if any, does the given formula impose on the constants a_0, a_1, a_2, and a_3 of a general $p(t)$?

MATH 293 FALL 1995 FINAL # 2

2.2.28 Consider the subspace W of $\Re^4$ which is defined as

$$W = span \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a) Find a basis for W.
b) What is the dimension of W?
c) It is claimed that W is a "plane" in $\Re^4$. Do you agree? Give reasons for your answer.
d) It is claimed that the "plane" W can be described as the intersection of two 3-D regions $S-1$ and S_2 in $\Re^4$. The equations of $S-1$ and S_2 are:

$$\begin{aligned} S_1 : &\quad x - u = 0 \\ S_2 : &\quad ax + by + cz + du = 0 \end{aligned}$$

where $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ is a generic point in $\Re^4$ and a, b, c, d are real constants.
Find one possible set of values for the constants a, b, c, and d.

MATH 293 SPRING 1996 PRELIM 3 # 1

2.2.29 The set W of vectors in $\Re^3$ of the form (a, b, c), where $a+b+c=0$, is a subspace of $\Re^3$.

a) Verify that the sum of any two vectors in W is again in W.

b) The set of vectors

$$S = (1,-1,0),(1,1,-2),(-1,1,0),(1,2,-3)$$

is in W. Show that S is linearly dependent.

c) Find a subset of S which is a basis for W.

d) If the condition $a+b+c=0$ above is replaced with $a+b+c=1$, is W still a subspace? Why/ why not?

MATH 293 SPRING 1996 PRELIM 3 # 2

2.2.30 Which of the following subsets are bases for $\Re^2$? Show any algebra involved or state a theorem to justify your answer.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}, S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right\}.$$

MATH 293 SPRING 1996 FINAL # 22

2.2.31 Let

$$W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}.$$

Then an orthonormal basis for W is

a) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\}$

d) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix} \right\}$

e) none of the above

MATH 294 FALL 1997 PRELIM 2 # 2

2.2.32 Consider the vector space P_2 of all polynomials of degree ≤ 2. Consider two bases of P_2:

$S : \{1, t, t^2\}$, the standard basis, and

$H : \{1, 2t, -2 + 4t^2\}$, the Hermite basis.

a) Find the matrices $P_{S \leftarrow H}$ and $P_{H \leftarrow S}$.

b) Consider $p_1(t) = 1 + 2t + 3t^2$ in P_2, and $p_2(t) = \frac{d}{dt} p_1(t)$. Find

$$[p_1(t)]_S, [p_2(t)]_S, [p_1(t)]_H, [p_2(t)]_H,$$

i.e. the coordinates of p_1 and p_2 in the bases S and H.

MATH 294 FALL 1997 PRELIM 2 # 3

2.2.33 Let W be the subspace of $\Re^4$ defined as

$$W = span\left(\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -6 \\ 4 \end{pmatrix}\right).$$

a) Find a basis for W. What is the dimension of W?

b) It is claimed that W can be described as the intersection of two linear spaces S_1 and S_2 in $\Re^4$. The equations of S_1 and S_2 are

$$S_1 : x - y = 0,$$

and

$$S_2 : ax + by + cz + dw = 0,$$

where a, b, c, d are real constants that must be determined. Find one possible set of values of a, b, c and d.

MATH 294 FALL 1997 PRELIM 2 # 6

2.2.34 Let V be the vector space of 2×2 matrices.

a) Find a basis for V.

b) Determine whether the following subsets of V are subspaces. If so, find a basis. If not, explain why not.

i) $\{ A \text{ in } V \big| \det A = 0\}$

ii) $\{ A \text{ in } V | A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$.

c) Determine whether the following are linear transformations. Give a short justification for your answers.

i) $T : V \to V$, where $T(A) = A^T$,

ii) $T : V \to \Re^1$, where $T(A) = \det A$,

MATH 294 FALL 1998 FINAL # 4

2.2.35 Here we consider the vector spaces P_1, P_2, and P_3 (the spaces of polynomials of degree 1,2 and 3).

a) Which of the following transformations are linear? (Justify your answer.)

i) $T : P_1 \to P_3, T(p) \equiv t^2 p(t) + p(0)$

ii) $T : P_1 \to P_1, T(p) \equiv p(t) + t$

b) Consider the linear transformation $T : P_2 \to P_2$ defined by $T(a_0 + a_1 t + a_2 t^2) \equiv (-a_1 + a_2) + (-a_0 + a_1)t + (a_2)t^2$. With respect to the standard basis of P_2, $B = \{1, t, t^2\}$, is $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that an eigenvalue/eigenvector pair of A is $\lambda = 1, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Find an eigenvalue/eigenvector (or eigenfunction) pair of T. That is, find λ and $g(t)$ in P_2 such that $T(g(t)) = \lambda g(t)$.

c) Is the set of vectors in $P_2\{3+t, -2+t, 1+t^2\}$ a basis of P_2? (Justify your answer.)

MATH 293 SPRING ? FINAL # C

2.2.36 Give a definition for addition and for scalar multiplication which will turn the set of all pairs $(\vec{u}, \vec{v})$ of vectors, for $\vec{u}, \vec{v}$ in V_2, into a vector space V.

a) What is the zero vector of V?

b) What is the dimension of V?

c) What is a basis for V?

MATH 294 FALL 1987 PRELIM 3 # 2 MAKE-UP

2.2.37 On parts (a) - (g), answer true or false.

a) $span\,(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \Re^3$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

b) The four vectors in (a) are independent.

c) Referring to a again, all vectors $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $span\,(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ satisfy a linear equation $ax_1 + bx_2 + cx_3 = 0$ for scalars a,b,c not all 0.

d) The rank of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ is 3.

e) In Re^n n distinct vectors are independent.

f) $n + 1$ distinct vectors always span $\Re^n$, for $n > 1$.

g) If the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span $\Re^n$, then they are a basis for $\Re^n$.

MATH 293 UNKNOWN PRACTICE # 4a

2.2.38 **a**) Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 6 & 1 & 12 \\ 9 & 18 & 1 & 36 \end{bmatrix}$$

UNKNOWN UNKNOWN UNKNOWN # ?

2.2.39 If A is an $m \times n$ matrix show that $B = A^T A$ and $C = AA^T$ are both square. What are their sizes? Show that $B = B^T, C = C^T$

MATH 294 FALL ? FINAL # 1 MAKE-UP

2.2.40 Consider the homogeneous system of equations $B\vec{x} = \vec{0}$, where

$$B = \begin{bmatrix} 0 & 1 & 0 & -3 & 1 \\ 2 & -1 & 0 & 3 & 0 \\ 2 & -3 & 0 & 0 & 4 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

a) Find a basis for the subspace $W \subset \Re^5$, where $W =$ set of all solutions of $B\vec{x} = \vec{0}$

b) Is B 1-1 (as a transformation of $\Re^5 \to \Re^3$)? Why?

c) Is B:$\Re^5 \to \Re^3$ onto why?

d) Is the set of all solutions of $B\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ a subspace of $\Re^5$? Why?

2.3 Vector Spaces

MATH 294 FALL 1982 PRELIM 1 # 3a

2.3.1 Let $C[0,1]$ denote the space of continuous functions defined on the interval [0,1] (i.e. $f(x)$ is a member of $C[0,1]$ if $f(x)$ is continuous for $0 \leq x \leq 1$). Which one of the following subsets of $C[0,1]$ does **not** form a vector space? Find it and explain why it does not.

MATH 294 SPRING 1982 PRELIM 1 # 3

2.3.2 **a**)

i) The subset of functions f which belongs to $C[0,1]$ for which $\int_0^1 f(s)ds = 0$.

ii) The set of functions f in $C[0,1]$ which vanish at exactly one point (i.e. $f(x) = 0$ for only one x with $0 \leq x \leq 1$). Note different functions may vanish at different points within the interval.

iii) The subset of functions f in $C[0,1]$ for which $f(0) = f(1)$.

b) Let $f(x) = x^3 + 2x + 5$. Consider the four vectors $\vec{v}_1 = f(x)$, $\vec{v}_2 = f'(x)$,$\vec{v}_3 = f''(x)$, $\vec{v}_4 = f'''(x)$, f' means $\frac{df}{dx}$.

i) What is the dimension of the space spanned by the vectors? Justify your answer.

ii) Express $x^2 + 1$ as a linear combination of the $\vec{v}_i$'s.

MATH 294 FALL 1984 FINAL # 1

2.3.3 **a**) Determine which of the following subsets are subspaces of the indicated vector spaces, and for each subspace determine the dimension of th- space. Explain your answer, giving proofs or counterexamples.

i) The set of all vectors in $\Re^2$ with first component equal to 2.

ii) The set of all vectors $\vec{x} = (x_1, x_2, x_3)$ in $\Re^3$ for which $x_1 + x_2 + x_3 = 0$.

iii) The set of all vectors in $\Re^3$ satisfying $x_1^2 + x_2^2 - x_3^2 = 0$.

iv) The set of all functions $f(x)$ in $C[0,1]$ such that $\int_0^1 f(x)dx = 0$. Recall that $C[0,1]$ denotes the space of all real valued continuous functions defined on the closed interval [0,1].

b) Find the equation of the plane passing through the points (0,1,0), (1,1,0) and (1,0,1), and find a unit vector normal to this plane.

MATH 294 SPRING 1985 FINAL # 5

2.3.4 Vectors $\vec{f}$ and $\vec{g}$ both lie in $\Re^n$. The vector $\vec{h} = \vec{f} + \vec{g}$

a) Also lies in $\Re^n$.

b) May or may not lie in $\Re^n$.

c) Lies in $\Re^{\frac{n}{2}}$.

d) Does not lie in $\Re^n$.

MATH 294 SPRING 1985 FINAL # 6

2.3.5 The vector space $\Re^n$

a) Contains the zero vector.

b) May or may not contain the zero vector.

c) Never contains the zero vector.

d) Is a complex vector space.

MATH 294 SPRING 1985 FINAL # 7

2.3.6 Any set of vectors which span a vector space

a) Always contains a subset of vectors which form a basis for that space.
b) May or may not contain a subset of vectors which form a basis for that space.
c) Is a linearly independent set.
d) Form an orthonormal basis for the space.

MATH 294 SPRING 1987 PRELIM 3 # 7

2.3.7 For what values of the constant a are the functions $\{\sin t \text{ and } \sin(t+a)\}$ in C^1_∞ linearly independent?

MATH 294 SPRING 1987 PRELIM 3 # 9

2.3.8 For problems (a) - (c) use the bases B and B' below:

$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

a) Given that $[\vec{v}]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ what is $[\vec{v}]_{B'}$?
b) Using the standard relation between $\Re^2$ and points on the plane make a sketch with the point $\vec{v}$ clearly marked. Also mark the point $\vec{w}$, where $[\vec{w}]_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.
c) Draw the line defined by the points $\vec{v}$ and $\vec{w}$. Do the points on this line represent a subspace of $\Re^2$?

MATH 294 FALL 1987 PRELIM 3 # 2

2.3.9 In parts (a) - (g) answer "true" if V is a vector space and "false" if it is not (no partial credit):

a) $V =$ set of all $x(t)$ in C_∞ such that $x(0) = 0$.
b) $V =$ set of all $x(t)$ in C_∞ such that $x(0) = 1$.
c) $V =$ set of all $x(t)$ in C_∞ such that $(D+1)x(t) = 0$.
d) $V =$ set of all $x(t)$ in C_∞ such that $(D+1)x(t) = e^t$.
e) $V =$ set of all polynomials of degree less than or equal to one with real coefficients.
f) $V =$ set of all rational numbers (a rational number can be written as the ratio of two integers, e.g., $\frac{4}{17}$ is a rational number while $\pi = 3.14\ldots$ is not)
g) $V =$ set of all rational numbers with the added restriction that scalars must also be rational numbers.

MATH 294 FALL 1987 FINAL # 7

2.3.10 Consider the boundary-value problem

$X'' + \lambda X = 0 \; 0 < x < \pi$, $X(0) = X(\pi) = 0$, where λ is a given real number.

a) Is the set of all solutions of this problem a subspace of $C_\infty[0,\pi]$? Why?
b) Let $W =$ set of all functions $X(x)$ in $C_\infty[0,\pi]$ such that $X(0) = X(\pi) = 0$. Is $T \equiv D^2 - \lambda$ linear as a transformation of W into $C_\infty[0,\pi]$? Why?
c) For what values of λ is $Ker(T)$ nontrivial?
d) Choose one of those values of λ and determine $Ker(T)$

MATH 293 SPRING 1990? PRELIM 2 # 3

2.3.11 Is the set of vectors $\{\vec{v}_1 = e^{-t}, \vec{v}_2 = e^t\}$ in C^∞ linearly independent or dependent? (Justify your answer.)

MATH 293 SPRING 1992 PRELIM 2 # 3

2.3.12 W is the subspace of V_4 spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Find the dimension of W and give a basis.

MATH 293 SPRING 1992 PRELIM 2 # 4

2.3.13 V is the vector space consisting of vector-valued functions $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ where $x_1(t)$ and $x_2(t)$ are continuous functions of t in $0 \leq t \leq 1$. W is the subset of V where the functions satisfy the differential equations
$\frac{dx_1}{dt} = x_1 + x_2$ and $\frac{dx_2}{dt} = x_1 - x_2$
Is W a subspace of V?

MATH 293 SPRING 1992 PRELIM 2 # 6

2.3.14 V is the vector space consisting of all 2×2 matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Here the a_{ij} are arbitrary real numbers and the addition and scalar multiplication are defined by
$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$ and $cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$

a) Is $W_1 = \left\{ all \begin{bmatrix} a_{11} & a_{12} \\ 0 & 1 \end{bmatrix} \right\}$ a subspace? If so give a basis for W_1.

b) Same as part (a) for $W_2 = \left\{ \begin{bmatrix} a_{11} & -a_{12} \\ a_{12} & a_{11} \end{bmatrix} \right\}$.

c) Show that $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ are linearly independent.

d) What is the largest possible number of linearly independent vectors in V?

MATH 293 SPRING 1992 FINAL # 7

2.3.15 A "plane" in V_4 means, by definition, the set of all points of the form $\vec{u} + \vec{x}$ where $\vec{u}$ is a constant (fixed) vector and $\vec{x}$ varies over a fixed two-dimensional subspace of V_4. Two planes are "parallel" if their subspaces are the same. It is claimed that the two planes:

1^{st} plane:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ x_3 \\ 0 \end{pmatrix}$$

2^{nd} plane:

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{pmatrix}$$

(where x_2, x_3 and x_4 can assume any scalar values) do not intersect and are not parallel. Do you agree or disagree with this claim? You have to give very clear reasons for your answer in order to get credit for this problem.

MATH 293 SUMMER 1992 FINAL # 3

2.3.16 **a**) Let V be the vector space of all 2 matrices of the form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where a_{ij} , $i, j = 1, 2$, are real scalars.

Consider the set S of all $2 times 2$ matrices of the form $\begin{pmatrix} a+b & a-b \\ b & a \end{pmatrix}$ where a and b are real scalars.

i) Show that S is a subspace. Call it W.

ii) Find a basis for W and the dimension of W.

b) Consider the vector space V $\{f(t) = a + b\sin t + c\cos t\}$, for all real scalars a, b and c and $0 \leq t \leq 1$

Now consider a subspace W of V in which $\frac{df(t)}{dt} + f(t) = 0$ at $t = 0$

Find a basis for the subspace W.

MATH 293 FALL 1992 PRELIM 3 # 3

2.3.17 Let $C(-\pi, \pi)$ be the vector space of continuous functions on the interval $-\pi \leq x \leq \pi$. Which of the following subsets S of $C(-\pi, \pi)$ are subspaces? If it is not a subspace say why. If it is, then say why and find a basis.

Note: You must show that the basis you choose consists of linearly independent vectors. In what follows a_0, a_1 and a_2 are arbitrary scalars unless otherwise stated.

a) S is the set of functions of the form $f(x) = 1 + a_1 \sin x + a_2 \cos x$

b) S is the set of functions of the form $f(x) = 1 + a_1 \sin x + a_2 \cos x$, subject to the condition $\int_{-\pi}^{\pi} f(x)dx = 2\pi$

c) S is the set of functions of the form $f(x) = 1 + a_1 \sin x + a_2 \cos x$, subject to the condition $\int_{-\pi}^{\pi} f(x)dx = 0$

MATH 293 FALL 1992 PRELIM 2 # 5

2.3.18 Consider all polynomials of degree ≤ 3

$$P_3 = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3\}, -\infty < t < \infty$$

They Form a vector space. Now consider the subset S of P_3 consisting of polynomials of degree ≤ 3 with the conditions

$$p(0) = 0, \frac{dp}{dt}(0) = 0$$

Is S a subspace W of P_3? Carefully explain your answer.

MATH 293 FALL 1992 PRELIM 2 # 6

2.3.19 Given a vector space V_4 which is the space of all vectors of the form $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ for all real x_1, x_2, x_3, x_4,
consider the set S of vectors in V_4 of the form

$$S = \left\{ a \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 3 \\ -2 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

for all values of scalars a, b and c.
Is the set S a subspace W of V_4? Explain your answer carefully.

MATH 293 FALL 1992 FINAL # 3d

2.3.20 Let S be the set of all vectors of the form $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the usual mutually perpendicular unit vectors. Let W be the set of all vectors that are perpendicular to the vector $\vec{v}_1 = \vec{i} + \vec{j} + \vec{k}$. Is W a vector subspace of V_3? Explain your answer.

MATH 293 FALL 1994 PRELIM 2 # 5

2.3.21 In each of the following, you are given a vector space V and a subset W. Decide whether W is a subspace of V, and prove that your answer is correct.

a) V is the space $M_{2,2}$ of all 2×2 matrices, and W is the set of 2×2 matrices A such that $A^2 = A$

b) V is the space of differentiable functions, and W is the set of those differentiable functions that satisfy $f'(3) = 0$.

MATH 293 FALL 1994 PRELIM 2 # 4

2.3.22 **a**) Let M denote the set of ordered triples (x, y, z) of real numbers with the operations of addition and multiplication v=by scalars c defined by

$$(x, y, z) \oplus (x', y', z') = (x + z', y + y', z + z')$$

$$c \odot (x, y, z) = (2c, cy, cz).$$

Is M a vector space? Why?

b) Consider the vector space $\Re^4$. Is the subset S of vectors of the form (x_1, x_2, x_3, x_4) where x_1, x_2, and x_3 are arbitrary and $x_4 \leq 0$ a subspace? Why?

c) Consider the vector space P_2 of polynomials of degree ≤ 2. Is the subset S of polynomials of the form $p(t) = a_0 + a_1 t + (a_0 + a_1)t^2$ a subspace? Why?

MATH 293 FALL 1994 PRELIM 3 # 5

2.3.23 Answer each of the following as True or False. If false, explain, by an example.

a) Every spanning set of $\Re^3$ contains at least three vectors.

b) Every orthonormal set of vectors in $\Re^5$ is a basis for $\Re^5$.

c) Let A be a 3 by 5 matrix. Nullity A is at most 3.

d) Let W be a subspace of $\Re^4$. Every basis of W contains at least 4 vectors.

e) In $\Re^n, ||cX|| = |c|\, ||X||$

f) If A is an $n \times n$ symmetric matrix, then rank $A = n$.

MATH 293 FALL 1994 FINAL # 4

2.3.24 **a**) Find a basis for the space spanned by: {(1,0,1),(1,1,0),(-1,-4,3)}.

b) Show that the functions $e^{2x}\cos(x)$ and $e^{2x}\sin(x)$ are linearly independent.

MATH 293 FALL 1994 PRELIM 3 # 2

2.3.25 Which of the following sets of vectors is linearly independent? Show all work.

a) In P_2: $S = \{1, t, t^2\}$

b) In $\Re^3$: $S = \{(1, 2, -1), (6, 3, 0), (4, -1, 2)\}$

MATH 293 SPRING 1995 PRELIM 3 # 3

2.3.26 Let P_3 be the space of polynomials $p(t)$ of degree ≤ 3. Consider the subspace $S \subset P_3$ of polynomials that satisfy

$$p(0) + \left.\frac{dp}{dt}\right|_{t=0} = 0$$

a) Show that S is a subspace of P_3.

b) Find a basis for S.

c) What is the dimension of S?

MATH 293 FALL 1995 PRELIM 3 # 3

2.3.27 Let P_3 be the space of polynomials $p(t) = a_0 + a_1 t + a_2 t + a_3 t^3$ of degree ≤ 3. Consider the subset S of polynomials that satisfy

$$p''(0) + 4p(0) = 0$$

Here $p''(0)$ means, as usual, $\left.\frac{d^2p}{dt^2}\right|_{t=0}$.

a) Show that S is a subspace of P_3. Give reasons.

b) Find a basis for S.

c) What is the dimension of S? Give reasons for your answer.

Hint: What constraint, if any, does the given formula impose on the constants a_0, a_1, a_2, and a_3 of a general $p(t)$?

MATH 293 FALL 1995 PRELIM 3 # 4

2.3.28 We define a new way of "adding" vectors by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 y_2 \end{pmatrix}$$

and use ordinary scalar multiplication.

a) Is the commutative axiom "$x + y = y + x$" satisfied?

b) Is the associative axiom "$x + (y + z) = (x + y) + z$" satisfied?

c) How about the distributive law "$a(x + y) = ax + ay$"?

d) Is this a vector space?

Give reasons.

MATH 293 SPRING 1996 PRELIM 3 # 1

2.3.29 The set W of vectors in $\Re^3$ of the form (a, b, c), where $a + b + c = 0$, is a subspace of $\Re^3$.

a) Verify that the sum of any two vectors in W is again in W.

b) The set of vectors

$$S = \{(1, -1, 0), (1, 1, -2), (-1, 1, 0), (1, 2, -3)\}$$

is in W. Show that S is linearly dependent.

c) Find a subset of S which is a basis for W.

d) If the condition $a + b + c = 0$ above is replaced with $a + b + c = 1$, is W still a subspace? Why/ why not

MATH 293 SPRING 1996 PRELIM 3 # 3

2.3.30 Which of the following subsets are bases for P_2, the vector space of polynomials of degree less than or equal to two? You do *not* need to show your work.

$S_1 = \{1, t, 1 - t, 1 + t\}$, $S_2 = \{t^2, t^2 + 2, t^2 + 2t\}$, $S_3 = \{1 + t + t^2, t, t^2\}$

MATH 293 SPRING 1996 FINAL # 4

2.3.31 Suppose $\vec{v}_1, \ldots, \vec{v}_p$ are vectors in $\Re^n$. Then $Span\{\vec{v}_1, \ldots, \vec{v}_p\}$ is always:

a) a linearly independent set of vectors
b) a linearly dependent set of vectors
c) a basis for a subspace of $\Re^n$
d) the set of all possible linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$,
e) none of the above.

MATH 294 SPRING 1997 FINAL # 7.2

2.3.32 Which of the following subsets are subspaces of the vector space P_2 of polynomials of degree ≤ 2? (No Justification is necessary.) Express your answer as e.g.: SUBSPACE: a,b,c,d; NOT: e

a) $\{\ p(t) \big| p'(t) = 0, \text{ all t}\}$
b) $\{\ p(t) \big| p'(t) - 1 = 0, \text{ all t}\}$
c) $\{\ p(t) \big| p(0) + p(1) = 0\}$
d) $\{\ p(t) \big| p(0) = 0 \text{ and } p(1) = 0\ \}$
e) $\{\ p(t) \big| p(0) = 0 \text{ and } p(1) = 1\ \}$

MATH 294 FALL 1997 PRELIM 2 # 3

2.3.33 Let W be the subspace of $\Re^4$ defined as

$$W = span\left(\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -6 \\ 4 \end{pmatrix}\right)$$

a) Find a basis for W. What is the dimension of W?

b) It is claimed that W can be described as the intersection of two linear spaces S_1 and S_2 in $\Re^4$. The equation of S_1 and S_2 are

$$S_1 : x - y = 0$$

and

$$S_2 : ax + by + cz + dw = 0,$$

where a, b, c, d are real constants that must be determined. Find one possible set of values of a, b, c and d.

MATH 294 FALL 1997 PRELIM 2 # 6

2.3.34 Let V be the vector space of 2×2 matrices.

a) Find a basis for V.

b) Determine whether the following subsets of V are subspaces. If so, find a basis. If not, explain why not.

i) $\{A \text{ in } V \bigm| \det A = 0 \}$

ii) $\{ A \text{ in } V \mid A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$.

c) Determine whether the following are linear transformations. Give a short justification for your answers.

i) $T : V \to V$, where $T(A) = A^T$,

ii) $T : V \to \Re^1$, where $T(A) = \det(A)$.

MATH 294 SPRING 1998 PRELIM 3 # 5

2.3.35 True or False? Justify each answer.

a) In general, if a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis of V.

b) A linearly independent set in a subspace H is a basis for H.

c) An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

d) If an $n \times n$ matrix A is diagonalizable, it is invertible.

MATH 293 SPRING ? FINAL # 6

2.3.36 Give a definition for addition and for scalar multiplication which will turn the set of all pairs $(\vec{u}, \vec{v})$ of vectors, for $\vec{u}, \vec{v}$ in V_2, into a vector space V.

a) What is the zero vector of V?

b) What is the dimension of V?

c) What is a basis for V?

MATH 293 FALL 1998 PRELIM 2 # 2

2.3.37 Given the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix},$$

a) Show by a calculation that its determinant is nonzero.

b) Calculate its inverse by any means.

2.4 Coordinates

MATH 294 SPRING 1987 PRELIM 3 # 9

2.4.1 For problems (a) - (c) use the bases B and B' below:

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \text{ and } B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

a) Given that $[\vec{v}]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ what is $[\vec{v}]_{B'}$?

b) Using the standard relation between $\Re^2$ and points on the plane make a sketch with the point $\vec{v}$ clearly marked. Also mark the point $\vec{w}$, where $[\vec{w}]_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

c) Draw the line defined by the points $\vec{v}$ and $\vec{w}$. Do the points on this line represent a subspace of $\Re^2$?

MATH 294 SPRING 1987 FINAL # 9

2.4.2 A general vector $\vec{v}$ in $\Re^2$ is $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 = b'_1\vec{v}'_1 + b'_2\vec{v}'_2$. where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v}'_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}'_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find a matrix $_{B'}[I]_B$ so that $\begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} =_{B'} [I]_B \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ for all vectors $\vec{v}$ in $\Re^2$.

MATH 293 SPRING 1993 FINAL # 5

2.4.3 **a**) Determine the matrix $H_{E,E}$ which represents reflection of vectors in $\Re^2$ about the y-axis in the standard basis $E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Verify your answer by evaluating the expression

$$H_{E,E} \begin{pmatrix} x \\ y \end{pmatrix}.$$

b) Now consider a basis B which is obtained by rotating each vector of the standard basis by 90 degrees in a counterclockwise direction. Find the change-of-basis matrices $(B : E)$ and $(E : B)$.

c) Find $H_{B,B}$ from the formula $H_{B,B} = (E : B)H_{E,E}(B : E)$.

d) It is claimed that $H_{B,B}$ is equal to the matrix $H_{E,E}$ which represents a reflection about the x-axis in the standard basis. Do you agree? Give geometrical reasons for your answer by drawing a suitable picture.

MATH 293 FALL 1995 PRELIM 3 # 2

2.4.4 Consider the vector space $\Re^3$ and the three bases:

$$\text{the standard basis } E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$\text{the basis } B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}, \text{ and}$$

$$\text{the basis } C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

a) Given the E coordinates of a vector $\vec{x}$, $[\vec{x}]_E = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, find $[x]_C$.

b) Given the B coordinates of a vector $\vec{y}$, $[\vec{y}]_B = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, find the coefficients y_j in $\vec{y} = \vec{y}_1\vec{e}_1 + \vec{y}_2\vec{e}_2 + \vec{y}_3\vec{e}_3$.

c) Find the change-of-coordinates matrix ${}_CP_B$ whose columns consist of the C coordinate vectors of the basis vectors of B.

MATH 293 SPRING 1996 PRELIM 3 # 8

2.4.5 Let

$$B = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}, C = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

a) Find the change of coordinate matrix from B to C.

b) Find the change of coordinate matrix from C to B.

MATH 293 SPRING 1996 FINAL # 8

2.4.6 Let

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}, C = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}.$$

Then the change of coordinates matrix from coordinates with respect to the basis C to coordinates with respect to the basis B is

a) $\begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} -4 & 4 \\ 0 & -4 \end{pmatrix}$

c) $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

d) $\begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}$

e) none of the above

MATH 294 FALL 1995 PRELIM 3 # 8

2.4.7 You are given a vector space V with an inner product $<,>$ and an orthogonal basis $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5$ for V for which $\left|\left|\vec{b}_i\right|\right| = 2, i = 1, \ldots, 5$. Suppose that $\vec{v}$ is in V and

$$\left\langle \vec{v}, \vec{b}_1 \right\rangle = \left\langle \vec{v}, \vec{b}_2 \right\rangle = 0$$

$$\left\langle \vec{v}, \vec{b}_4 \right\rangle = 3, \left\langle \vec{v}, \vec{b}_4 \right\rangle = 4, \left\langle \vec{v}, \vec{b}_5 \right\rangle = 5$$

Find the coordinates of $\vec{v}$ with respect to the basis B i.e. find c_1, c_2, c_3, c_4, c_5 such that

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 + c_5\vec{b}_5$$

MATH 294 SPRING 1998 PRELIM 3 # 3

2.4.8 Let $T : \wp_1 \to \wp_3$ be defined by

$$T\left[p(t)\right] = t^2 p(t)$$

and take

$$B = \{1, 1+t\}$$

to be the basis of $\wp_3$.

a) Find the matrix of T relative to the bases B and C.

b) Use this matrix to find $T[2+t]$.

c) Let $E = \{1, t\}$ be the standard basis for $\wp_1$. Let $[\vec{x}]_B$ be the coordinate vector of $\vec{x}$ in $\wp_1$ relative to the basis B, and let $[\vec{x}]_E$ be the coordinate vector of $\vec{x}$ relative to the basis E. What is the change of coordinate matrix P. such that

$$P\left[\vec{x}\right]_B = \left[\vec{x}\right]_E.$$

[Note: The result of part c) does not depend on the results of parts a) or b)]

MATH 294 SPRING 1998 Final # 4

2.4.9 In P^2, Find the change-of-coordinate matrix from the basis

$$B = \left\{1 - 2t + t^2, 3 - 5t, 2t + 3t^2\right\}$$

to the standard basis

$$E = \{1, t, t^2\}.$$

Then write t^2 as a linear combination of the polynomials in B, i.e. give the coordinates of t^2 with respect to the basis B.

MATH 294 Fall 1998 PRELIM 2 # 3

2.4.10 Besides the standard basis ε here are two bases for $\Re^2$:

$$B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b_2} \right\}, C = \left\{ \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{c_1}, \underbrace{\begin{bmatrix} -4 \\ 4 \end{bmatrix}}_{c_2} \right\}$$

a) What vectors $\vec{x}$ are represented by $[\vec{x}]_B = \begin{bmatrix} 2 \\ 14 \end{bmatrix}$ and $[\vec{x}]_C = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$?

b) Find a single tidy formula to find the components $\begin{bmatrix} d \\ e \end{bmatrix}$ of a vector $\vec{x}$ in the basis B if you are given the components $\begin{bmatrix} f \\ g \end{bmatrix}$ of $\vec{x}$ in the basis C.

c) A student claims that the desired formula is $\begin{bmatrix} d \\ e \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$. Does this formula make the right prediction for the component vector $[\vec{x}]_C = \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$?

MATH 294 FALL 1998 Final # 6

2.4.11 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

a) Find orthogonal eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ of A. [Hint: do not go on to parts d-e below until you have double checked that you have found two orthogonal unit vectors that are eigenvectors of A.]

b) Use the eigenvectors above to diagonalize A.

c) Make a clear sketch that shows the standard basis vectors $\{\vec{e}_1, \vec{e}_2\}$ of $\Re^2$ and the eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ of A

d) Give a geometric interpretation of the change of coordinates matrix, P, that maps coordinates of a vector with respect to the eigen basis to coordinates with respect to the standard basis.

e) Let $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Using orthogonal projection express $\vec{b}$ in terms of $\{\vec{v}_1, \vec{v}_2\}$ the eigenvectors of A.

2.5 Spaces of a Matrix and Dimension

MATH 294 SPRING 1982 PRELIM 1 # 3

2.5.1 **a)** Let $C[0,1]$ denote the space of continuous function defined on the interval $[0,1]$ (i.e. $f(x)$ is a member of $C[0,1]$ if $f(x)$ is continuous for $0 \leq x \leq 1$). Which one of the following subsets of $C[0,1]$ does **not** form a vector space? Find it and explain why it does not.

MATH 294 SPRING 1982 PRELIM 1 # 3

2.5.2 **a)**

i) The subset of functions f which belongs to $C[0,1]$ for which $\int_0^1 f(s)ds = 0$.

ii) The set of functions f in $C[0,1]$ which vanish at exactly one point (i.e. $f(x) = 0$ for only one x with $0 \leq x \leq 1$).
Note different functions may vanish at different points within the interval.

iii) The subset of functions f in $C[0,1]$ for which $f(0) = f(1)$

b) Let $f(x) = x^3 + 2x + 5$. Consider the four vector $v_1 = f(x), v_2 = f'(x), v_3 = f'', v_4 = f'''(x), (f'(x)$ means $\frac{df}{dx})$

i) What is the dimension of the space spanned by the vectors? Justify your answer.

ii) Express $x^2 + 1$ as a linear combination of the v_i's

MATH 294 SPRING 1983 PRELIM 1 # 2

2.5.3 Consider the system

$$\left.\begin{array}{ccccccccc} x & + & y & - & z & + & w & = & 0 \\ x & & & + & 3z & + & w & = & 0 \\ 2x & + & y & + & 2z & + & 2w & = & 0 \\ 3x & + & 2y & + & z & + & 3w & = & 0 \end{array}\right\}$$

a) Find a basis for the vector space of solutions to the system above. You need not prove this is a basis

b) What is the dimension of the vector space of solutions above? Give a reason.

c) Is the vector

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

a solution to the above system?

MATH 294 SPRING 1987 FINAL # 2

2.5.4 Determine whether the given vectors form a basis for S, and find the dimension of the subspace. S is the set of all vectors of the form $(a, b, 2a, 3b$ in R^4. The given set is $\{(1,0,2,0), (0,1,0,3), (1,-1,2,-3)\}$

MATH 294 FALL 1986 FINAL # 1

2.5.5 The vectors $(1,0,2,-1,3),(0,1,-1,2,4),(-1,1,-2,1,-3),(0,1,1,-2,-4)$, and $(1,4,2,-1,3)$ span a subspace S of R^5.

a) What is the dimension of S?

b) Find a basis for$S$

MATH 294 FALL 1986 FINAL # 1

2.5.6 Compute the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

MATH 294 FALL 1987 PRELIM 3 # 3

2.5.7 Find the dimension of the subspace of R^6 consisting of all linear combinations of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

MATH 293 SPRING 1990 PRELIM 1 # 3

2.5.8 Find the dimension and a basis for the following spaces

a) The space spanned by $\{(1,0,-2,1),(0,3,1,-1),(2,3,-3,1),(3,0,-6,-1)\}$

b) The set of all polynomials $p(t)$ in P^3 satisfying the two conditions

i) $fracd^3dt^3p(t) = 0$ for all t

ii) $p(t) + fracdp(t)dt = 0$ at $t = 0$

c) The subspace of the space of functions of t spanned by $\left\{e^{at}, e^{bt}\right\}$ if $a \neq b$

d) The space spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ in W, given that $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for W.

MATH 293 SPRING 1990 PRELIM 2 # 1

2.5.9 Let $A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ -1 & -2 & -5 & 4 \\ 0 & 1 & 1 & -1 \\ 1 & 4 & 6 & -2 \end{bmatrix}$ Find a basis for the **column space** of A

MATH 293 SPRING 1990 PRELIM 2 # 2

2.5.10 Consider the equation

$$Ax = b \; ; \; A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ -1 & -2 & -5 & 4 \\ 0 & 1 & 1 & -1 \\ 1 & 4 & 6 & -2 \end{bmatrix}$$

a) Solve for x given $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}$

b) Find a basis for the null space of A

c) **Without carrying out explicit calculation**, does a solution exist for any b in V^4? (No credit will be given for explicit calculation for).

MATH 293 SPRING 1990 PRELIM 2 # 3

2.5.11 **a)** Find a basis and the dimension of the <u>column</u> space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 2 & 6 & 18 \\ -1 & 1 & 1 \\ 4 & 12 & 36 \end{bmatrix}$$

b) Find a basis for the <u>null</u> space of the above matrix.

MATH 293 SPRING 1990 PRELIM 2 # 4

2.5.12 Which of the following sets form a vector subspace of V_4? Explain.

a) the set of vectors of the form $(x, y, x+y, 0)$

b) the set of vectors of the form $(x, 2x, 3x, 4x)$

c) the set of vectors (x, y, z, w) such that $x + y + w = 1$

d) If the set in (b) is a subspace, find a basis for it and its dimension. In the above, x, y, z, w are any real numbers.

MATH 293 FALL 1991 PRELIM 3 # 6

2.5.13 True-False

True means always true. False means not always true.

a) The column space of a matrix is preserved under row operations

b) The column rank of a matrix is preserved under row operations.

c) For an $n \times n$ matrix, with $m \neq n$, rank plus nullity equals n.

d) The row space of a matrix A is the same vector space as the row space of the row reduced form of A.

e) If two matrices A and B have the same row space, the $A = B$.

MATH 293 FALL 1991 FINAL # 7

2.5.14 Show that the matrices A and B have the same row space:

$$A = \begin{pmatrix} 3 & 1 & 9 \\ 2 & 1 & 7 \\ 1 & 1 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & -1 & 3 \\ 1 & -1 & -1 \\ 2 & -3 & -5 \end{pmatrix}$$

MATH 293 FALL 1991 FINAL # 7

2.5.15 Find the vector in the subspace spanned by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

which is closest to the vector

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

MATH 293 FALL 1991 FINAL # 8

2.5.16 True-False. True means always true, false means not always true. Warning! Matrices are not necessarily square.

a) The rank of A equals the rank of A^T.

b) The nullity of A equals the nullity of A^T.

MATH 293 SUMMER 1992 FINAL # 3

2.5.17 Let V be the vector space of all 2×2 matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where a_{ij},$i, j = 1, 2$, are real scalars.

Consider the set S of all 2×2 matrices of the form

$$\begin{pmatrix} a+b & a-b \\ b & a \end{pmatrix}$$

where a and b are real scalars.

a) Show that S is a subspace. Call it W.

b) Find a basis for W and the dimension of W.

MATH 293 SUMMER 1992 FINAL # 3

2.5.18 Consider the vector space V

$\{f(t) = a + b\sin t + c\cos t\}$, for all real scalars a, b and c and $0 \le t \le 1$

Now consider a subspace

W of V in which $\frac{df(t)}{dt} + f(t) = 0$ at $t = 0$

Find a basis for the subspace W.

MATH 293 FALL 1992 PRELIM 3 # 5

2.5.19 Fill in the blanks of the following statements.
In what follows A is an $m \times n$ matrix

a) The dimension of the row space is 2.
The dimension of the null space is 3.
The number of columns of A is ________ .

b) $Ax = b$ has a solution x if and only if b is in the ________ space of A.

c) If $Ax = 0$ and $Ay = 0$ and if C_1 and C_2 are arbitrary constants then $A(C_1x+C_2y)$ $=$ ________ .

MATH 293 FALL 1992 FINAL # 3

2.5.20 **a)** Let A be an $n \times n$ nonsingular matrix. Prove that $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$. Hint: You may use the fact that if A and B are $n \times n$ matrices $\det(AB) = \det(A)\det(B)$.

b) An $n \times n$ matrix A has a nontrivial null space. Find $\det(A)$ and explain your answer.

c) Given two vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in V_3. Find a vector (or vectors) $w_1, w_2, ...$ in V_3 such that the set $\{v_1, v_2, w_1, ...\}$ is a basis for V_3.

d) Let S be the set set of all vectors of the form $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ where $\vec{i}, \vec{j}$ and $\vec{k}$ are the usual mutually perpendicular unit vectors. Let W be the set of all vectors that are perpendicular to the vector $\vec{v}_1 = \vec{i} + \vec{j} + \vec{k}$. Is W a vector subspace of V_3? Explain your answer.

MATH 293 SPRING 1993 PRELIM 3 # 6

2.5.21 Let A be an $n \times n$ matrix. Suppose the rank of A is r, and that $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r$ are vectors in R^n such that $A\mathbf{u}_1, A\mathbf{u}_2, ..., A\mathbf{u}_r$ is a basis for $R(A)$ (col. space of A). Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-r}$ be a basis for $N(A)$ (null space of A). Then show that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-r}\}$ is a basis for R^n

MATH 293 SPRING 1993 FINAL # 2

2.5.22 **a)** Solve for y, for x near $\frac{\pi}{2}$, if $y' + y \cot x = \cos x$ and $y\left(\frac{\pi}{2}\right) = 0$

b) Find a basis for the null space of the differential operator

$$L = \frac{d^2}{dx^2} - 7\frac{d}{dx} + 12, -\infty < x < -\infty.$$

(Hint: Find as many linearly independent solutions as needed for the equation $L[y(x)] = 0$.)

MATH 293 FALL 1994 PRELIM 3 # 5

2.5.23 Answer each of the following as True or False. **If False, explain, by an example.**

a) Every spanning set of R^3 contains at least three vectors.

b) Every orthogonal set of vectors in R^5 is a basis for R^5.

c) Let A be a 3 by 5 matrix. Nullity A is at most 3.

d) Let W be a subspace of R^4. Every basis of W contains at least 4 vectors.

e) In R^n, $||cX|| = |c|||X||$

f) If A is an $n \times n$ symmetric matrix, then rank $A = n$.

MATH 293 FALL 1994 FINAL # 4

2.5.24 A basis for the null space of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is:

a. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *b.* $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ *c.* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ *d.* $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ *e.* $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

MATH 293 FALL 1994 FINAL # 8

2.5.25 If A is an n by n matrix and rank$(A) < n$. Then

a) A is non singular,
b) The columns of A are linearly independent
c) Some eigenvalue of A is zero
d) $AX = 0$ has only the trivial solution
e) $AX = B$ has a solution for every B

MATH 293 SPRING 1995 PRELIM 3 # 1

2.5.26 Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 0 & 4 & -2 \\ -1 & 3 & 1 & 7 \end{bmatrix}.$$

a) Find a basis for the range of A (i.e., the column space of A).
b) Find a basis for the null-space of A (i.e., the kernel of A).
c) Find a basis for the column space of A^T.

MATH 293 SPRING 1995 PRELIM 3 # 3

2.5.27 Let P_3 be the space of polynomials $p(t)$ of degree ≤ 3. Consider the subspace $S \subset P_3$ of polynomials that satisfy

$$p(0) + \left.\frac{dp}{dt}\right|_{t=0} = 0.$$

a) Show that S is a subspace of P_3.
b) Find a basis for S
c) What is the dimension of S.

MATH 293 FALL 1995 PRELIM 3 # 1

2.5.28 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & -1 & -2 \end{bmatrix}.$$

a) Find a basis for the row space of A.
b) Find a basis for the column space of A
c) What is the rank of A?
d) What is the dimension of the null space?

MATH 293 FALL 1995 PRELIM 3 # 3

2.5.29 Let P_3 be the space of polynomials $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$ of degree ≤ 3. Consider the subset S of polynomials that satisfy

$$p''(0) + 4p(0) = 0$$

Here $p''(0)$ means, as usual, $\left.\frac{d^2p}{dt^2}\right|_{t=0}$.

a) Show that S is a subspace of P_3. Give reasons.
b) Find a basis for S.
c) What is the dimension of S? Give reasons for your answer.
Hint: What constrain, if any, does the given formula impose on the constants a_0, a_1, a_2, and a_3 of a general $p(t)$?

MATH 293 FALL 1995 FINAL # 2

2.5.30 Consider the subspace W of R^4 which is defined as

$$W = span\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

a) Find a basis for W.
b) What is the dimension of W?
c) It is claimed that W is a "plane" in R^4. Do you agree? Give reasons for your answer.
d) It is claimed that the "plane" W can be described as the intersection of two 3-D regions S_1 and S_2 in R^4. The equations of S_1 and S_2 are:
$S_1 : x - u = 0$
$S_2 : ax + by + cz + du = 0$
where $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ is a generic point in R^4 and a, b, c, d are real constants.
Find one possible set of values for the constants $a, b, candd$.

MATH 293 SPRING 1996 PRELIM 3 # 4

2.5.31 Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & -1 & 2 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

a) Find a basis for the null space of A. What is the dimension of the null space of A?

b) Let$\mathbf{x} = (0, \frac{1}{2}, 1, 0)$. We know that $A\mathbf{x} = \mathbf{0}$. True or false:

i) $\mathbf{x}$ is a trivial solution to $A\mathbf{x} = \mathbf{0}$.

ii) $\mathbf{x}$ is in the solution space of $A\mathbf{x} = \mathbf{0}$.

iii) $\mathbf{x}$ is in the null space of A.

iv) $\{\mathbf{x}\}$ is a basis for the null space of A.

c) The vector

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is one vector in a basis for the column subspace of A. Find another vector $\vec{v}$ in a basis for the column subspace of A such that $\{\vec{v}, \vec{w}\}$ is linearly independent.

d) What is the rank of A? How do you know?

MATH 293 SPRING 1996 FINAL # 16

2.5.32 The vector space of all polynomials of degree six or less has dimension:

a) 5

b) 6

c) 7

d) 8

e) none of the above

MATH 293 SPRING 1996 FINAL # 21

2.5.33 A basis for the null space of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is

a) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

e) none of the above

MATH 293 SPRING 1996 FINAL # 23

2.5.34 Suppose A is a matrix with 6 columns and 4 rows. Which of the following must be true?

a) The null space of A has dimension ≥ 2 and the rank of A is 4.
b) The null space of A has dimension ≤ 4 and the rank of A is 2.
c) The null space of A has dimension ≤ 2 and the rank of A is ≤ 4.
d) The null space of A has dimension ≥ 2 and the rank of A is ≤ 4.
e) None of the above

MATH 294 SPRING 1997 FINAL # 2

2.5.35 (All parts are independent problems)

a) If the $\det A = 2$. Find the $\det A^{-1}, \det A^T$

b) From $PA = LU$ find a formula for A^{-1} in terms of P, L and U. Assume P, L, U, A are invertible $n \times n$ matrices.

c) Find the rank of matrix A.

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$$

d) Find a 2×2 matrix E such that for *every* 2×2 matrix A, the second row of EA is equal to the sum of the first two rows of A, e.g. if
$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $EA = \begin{bmatrix} 1 & 2 \\ 3+1 & 4+2 \end{bmatrix}$

e) Write down a 2×2 matrix P which projects every vector onto the x_2 axis. Verify that $P^2 = P$.

MATH 294 SPRING 1997 FINAL # 7

2.5.36 Suppose A is a 6 row by 7 column matrix for which $nulA = Span\{\vec{x}_0\}$ for some $\vec{x}_0 \neq \vec{0}$ in $\Re^7$. Which of the following are always TRUE of A? (NO Justification is necessary.) Express your answer as e.g: TRUE: a,b,c,d; FALSE: e

a) The columns of A are linearly dependent.
b) The linear transformation $\vec{x} \to A\vec{x}$ is onto.
c) $A\vec{x} = \vec{0}$ has only the trivial solution.
d) The columns of A form a basis for $\Re^6$.
e) The columns of A span all of $\Re^6$.

MATH 294 FALL 1997 PRELIM 2 # 1

2.5.37 Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 2 & 0 & 2 \\ 4 & 3 & 1 & 2 \\ -2 & 0 & -2 & 2 \end{pmatrix}.$$

a) Find a basis for the null space N of A. What is the dimension of N?
b) Find a basis for the column space C of A. What is the dimension of C?
c) Find a basis for the row space R of A. What is the dimension of R?

MATH 294 FALL 1997 FINAL # 2

2.5.38 Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ -2 & -1 & 3 & 2 \end{pmatrix}$$

Find bases for the null space of A and the column space of A. What are the dimensions of these two vector spaces?

MATH 294 SPRING 1998 PRELIM 3 # 1

2.5.39 The matrix A is row equivalent to the matrix B:

$$A \equiv \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \equiv B$$

a) Find a basis for $RowA$, $ColA$, and $NulA$.
b) To what vector spaces do the vectors in $RowA$, $ColA$, and $NulA$ belong?
c) What is the rank of A?

MATH 294 SPRING 1998 FINAL # 3
2.5.40 Given that the matrix B is row equivalent to the matrix A where

$$A \equiv \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix} \text{ and } B \equiv \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find rank A and dim Null A.
b) Determine bases for Col A and Null A.
c) Determine a value of c so that the vector $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ c \end{bmatrix}$ is in Col A
d) For this value of c, write the general solution of $A\vec{x} = \vec{b}$.

MATH 294 FALL 1997 PRELIM 2 # 3
2.5.41 Let W be the subspace of $\Re^4$ defined as

$$W = span\left(\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -6 \\ 4 \end{pmatrix}\right).$$

a) Find a basis for W. What is the dimension of W?
b) It is claimed that W can be described as the intersection of two linear spaces S_1 and S_2 in $\Re^4$. The equations of S_1 and S_2 are

$$S_1 : x - y = 0$$

and

$$S_2 : ax + by + cz + dw = 0,$$

where a,b,c,d are real constants that must be determined. Find one possible set of values of a,b,c and d.

MATH 294 FALL 1997 PRELIM 3 # 1
2.5.42 Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}.$$

a) Find an orthofonal basis for the null space of A.
b) Find a basis for the orthogonal complement of $NulA$, i.e. find $(NulA)^{\perp}$.

MATH 294 FALL 1997 PRELIM 3 # 2

2.5.43 Let $A = [\vec{v}_1 \vec{v}_2]$ be a 1000×2 matrix, where $\vec{v}_1, \vec{v}_2$ are the columns of A. You aren't given A. Instead you are given only that

$$A^T A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Find an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ of the column space of A. Your formulas for $\vec{u}_1$ and $\vec{u}_2$ should be written as linear combinations of $\vec{v}_1, \vec{v}_2$. (Hint: what do the entries of the matrix $A^T A$ have to do with dot products?

MATH 294 FALL 1998 PRELIM 2 # 4

2.5.44 The reduced echelon form of the matrix $A = \begin{bmatrix} 3 & 3 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 1 & 0 & 1 & -2 \\ 0 & -3 & 2 & -1 \end{bmatrix}$ is $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

a) What is the rank of A

b) What is the dimension of the column space of A?

c) What is the dimension of the null space of A?

d) Find a solution to $A\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

e) What is the row space of A?

f) Would any of your answers above change if you changed A by randomly changing 3 of its entries in the 2nd, third, and fourth columns to different small integers and the corresponding reduced echelon form for B was presented? (yes?, no?, probably?, probably not?, ?)

MATH 294 FALLL 1998 FINAL # 5

2.5.45 Consider $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ -1 & 2 & 5 & 8 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The augmented matrix of this system is $\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ -1 & 2 & 5 & 8 & 1 \end{bmatrix}$ which is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

a) What are the rank of A and dim nul A? (Justify your answer.)
b) Find bases for col A, row A, and nul A.
c) What is the general solution $\vec{x}$ to $A\vec{x} = \vec{b}$ with the given A and $\vec{b}$?
d) Select another $\vec{b}$ for which the above system has a solution. Give the general solution for that $\vec{b}$.

MATH 294 SPRING 1999 PRELIM 2 # 4

2.5.46 Let A be a matrix where all you know is that it is 5×7 and has rank 3.

a) Define new matrices from A as follows:

- C has as columns a basis for Col A,
- M has as columns a basis for Nul A^T, and
- $T = [CM]$.

Is this enough information to find the size (number of rows and columns) of T?
i) if yes, find the number of rows and columns and justify your answer, or
ii) if no, explain what extra information is needed to find the size of T?

b) Are there any two non-zero vectors $\vec{u}$ and $\vec{v}$ for which:

- $\vec{u}$ is in Col A,
- $\vec{v}$ is in Nul A^T, ***and***
- $\vec{v}$ is a multiple of $\vec{u}$?

i) if yes, why?
ii) if no, why?, or

c) if it depends on information not given, what information? How would that information help?

MATH 294 SPRING 1999 PRELIM 2 # 1

2.5.47 **a**) What is the null space of $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$?

b) What is the column space of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & \pi \\ 3 & 4 & \sqrt{2} \end{bmatrix}$?

d) Are the column of $A = \begin{bmatrix} 1 & 2 & \pi \\ 3 & 4 & \sqrt{2} \\ 3 & 4 & \sqrt{2} \end{bmatrix}$ linearly independent (hint: no long row reductions are needed)?

e) What is the row space of $A = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$?

MATH 293 SUMMER 1992 PRELIM 7/21 # 5

2.5.48 Consider the space P of all polynomials of degree ≤ 3 of the type $\{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3\}$ for all scalars a_0, a_1, a_2, a_3 and $0 \leq t \leq 1$. Now consider a subspace W of P where, for any $p(t) \in W$, we also have

$$\int_0^1 p(t)dt = 0$$

$$\left.\frac{dp}{dt}\right|_{t=0} = 0$$

a) Find a basis for W.
b) What is the dimension of W?

UNKNOWN UNKNOWN UNKNOWN # ?

2.5.49 Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

a) Find the vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that a solution $\vec{x}$ of the equation $A\vec{x} = \vec{b}$ exists.
b) Find a basis for the column space $\mathcal{R}(A)$ of A.
c) It is claimed that $\mathcal{R}(A)$ is a plane on $\Re^3$. If you agree, find a vector $\vec{n}$ in $\Re^3$ that is normal to this plane. Check your answer.
d) Show that $\vec{n}$ is perpendicular to each of the columns of A. Explain carefully why this is true.

MATH 294 FALL 1997 PRELIM 3 # 1 PRACTICE
2.5.50 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & -1 & -2 \end{bmatrix}$$

a) Find a basis for the column space C of A. What is the dimension of C?
b) Find a basis for the column space N of A. What is the dimension of N?
c) Let $W = span\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right)$. Is W orthogonal to N? Please justify your answer by showing your work.

MATH 293 SPRING ? PRELIM 2 # 1
2.5.51 a) Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 2 & 7 & 8 \\ 2 & 0 & 2 & 4 \end{bmatrix}$$

b) Find a basis for the column space of A in (a).

MATH 293 SPRING ? PRELIM 2 # 2
2.5.52 a) If A and B are 4×4 matrices such that

$$AB = \begin{pmatrix} 2 & 1 & 1 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

show that the column space of A is at least three dimensional.

b) Find A^{-1} if $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

MATH 293 SPRING ? FINAL # 2
2.5.53 a) Find a basis for the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

b) Find the rank of A and a basis for its column space, noting that $A = A^T$.
c) Construct an orthonormal basis for the row space of A.

MATH 293 SPRING ? FINAL # 6

2.5.54 Give a definition for addition and for scalar multiplication which will turn the set of all pairs $(\vec{u}, \vec{v})$ of vectors, for $\vec{u}, \vec{v}$ in V_2, into a vector space V.

a) What is the zero vector of V?

b) What is the dimension of V?

c) What is a basis for V?

MATH 293 SPRING ? FINAL # 3

2.5.55 **a**) Give all solutions of the following system in vector form.

$$\begin{aligned} 6x_1 + 4x_3 &= 1 \\ 5x_1 - x_2 + 5x_3 &= -1 \\ x_1 + 3x_3 &= 2 \end{aligned}$$

b) What is the null space of the matrix of coefficients of the unknowns in a)?

MATH 293 SPRING ? FINAL # 4

2.5.56 Let W be the subspace of V_4 spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -8 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

a) Find the dimension and a basis for W.

b) Find an orthogonal basis for W.

MATH 293 UNKNOWN FINAL # 5

2.5.57 **a**) Let A be an $n \times n$ matrix. Show that if $A\vec{x} = \vec{b}$ has a solution then $\vec{b}$ is a linear combination of the column vectors of A.

b) Let A be a 4×4 matrix whose column space is the span of vectors $\vec{v} = (v_1, v_2, v_3, v_4)^T$, satisfying $v_1 - 2v_2 + v_3 - v_4 = 0$. Let $\vec{b} = (1, b_2, b_3, 0)^T$. Find all values of b_2, b_3 for which the matrix equation $A\vec{x} = \vec{b}$ has a solution.

2.6 Finite Difference Equations and Markov Chains

MATH 294 FALL 1997 PRELIM II # ?

2.6.1 Consider the linear difference equation

$$y_{k+3} - 2y_{k+2} - y_{k+1} + 2y_k = 0.$$

a) What is the dimension of the solution set of this equation?
b) Find a basis for this subspace of S.
c) Suppose $u = \{u_k\}$ is a solution to this difference equation where $u_0 = 1, u_1 = 0$, and $u_4 = 4$. Find a formula for u_k. (Hint: Use a linear combination of the basis vectors that you found in part (b) above).

MATH 294 SPRING 1998 PRELIM 3 # 2

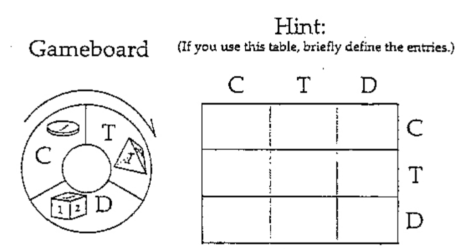
2.6.2 Consider the difference equation

$$y_{k+2} + 4y_{k+1} + y_k = 0$$

for $k = 1, 2, ..., N-2$
a) Find its general solution.
b) Find the particular solution that satisfies the boundary conditions $y_1 = 5000$ and $y_N = 0$.
(The answer involves N.)

MATH 294 FALL 1998 PRELIM 3 # 4

2.6.3 The three "spaces" on the simple board game shown are labeled "C", "I", and "D" for coin, tetrahedron, and dice. On one turn a player advances clockwise a random number of spaces as determined by shaking and dropping the object on their present space (From the C position a player moves 1 or 2 spaces with equal probabilities, from the T space a player moves 1-4 spaces with equal probabilities, and from the D space a player moves 1-6 spaces with equal probabilities.).
In very long game what function of the moves end up on the D space on average? [Hint: Use exact arithmetic rather than truncated decimal representations.]



2.7 Eigen-stuff

MATH 294 FALL 1985 FINAL # 3

2.7.1 Find an angle θ, expressed as a function of a,b, and c so that the matrix product

$$\begin{pmatrix} \cos\theta & sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is a diagonal matrix. In particular, what is θ if $a = c$, and what is the resulting digital matrix? (Hint: $\cos^2\theta - \sin^2\theta = \cos 2\theta$; $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$)

MATH 294 FALL 1985 FINAL # 5

2.7.2 Find all of the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & 3 \\ -1 & -2 & 0 & 4 \\ -2 & -3 & -4 & 0 \end{pmatrix}$$

Show why your answers are correct.

MATH 294 SPRING 1985 FINAL # 9

2.7.3 In general, the eigenvalues of A are (A is a real 2×2 matrix)
a) Always real.
b) Always imaginary.
c) Complex conjugates.
d) Either purely real or purely imaginary.

MATH 294 SPRING 1985 FINAL # 10

2.7.4 If A has purely real eigenvalues, then (A is a real 2×2 matrix)
a) The eigenvalues must be distinct.
b) The eigenvalues must be repeated.
c) The eigenvalues may be distinct or repeated.
d) The eigenvalues must both be zero.

MATH 294 SPRING 1985 FINAL # 11

2.7.5 If A has purely imaginary eigenvalues, then (A is a real 2×2 matrix)
a) The eigenvalues must have the same magnitude but opposite sign.
b) The eigenvalues must be repeated.
c) The eigenvalues may or may not be repeated.
d) The eigenvalues must both be zero.

MATH 294 FALL 1986 FINAL # 3

2.7.6 **a**) Find all eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix}.$$

b) For any square matrix A, show that if $\det(A) \neq 0$, then zero cannot be an eigenvalue of A.

MATH 294 SPRING 1983 FINAL # 10

2.7.7 A is the matrix given below, $\vec{v}$ is an eigenvector of A. Find any eigenvalue of A.

$$A = \begin{bmatrix} 3 & 0 & 4 & 2 \\ 8 & 5 & 1 & 3 \\ 4 & 0 & 9 & 8 \\ 2 & 0 & 1 & 6 \end{bmatrix} \text{ with } \vec{v} = [\text{an eigenvector of } A] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

MATH 294 SPRING 1984 FINAL # 5

2.7.8 Let λ_1 and λ_2 be distinct eigenvalues of a matrix A and let x_1 and x_2 be the associated eigenvectors. Show that x_1 and x_2 are linearly independent.

MATH 294 FALL 1984 FINAL # 2

2.7.9 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

.

MATH 294 FALL 1984 FINAL # 5

2.7.10 Does the matrix with the zero row vector deleted have $\lambda = 0$ as an eigenvalue?

MATH 294 FALL 1986 FINAL # 4

2.7.11 **a**) Find an orthogonal matrix R such that $R^T AR$ is diagonal,where

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

.

b) Write the matrix $D = R^T AR$.

c) What are the eigenvectors and associated eigenvalues of A.

MATH 294 SPRING 1987 PRELIM 2 # 4

2.7.12 Problems (a) and (b) below concern the matrix A:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 4 \\ 0 & 2 & -8 \end{bmatrix}$$

a) One of the eigenvalues of A is 1, what are the other(s)?
b) Find an eigenvector of A.

MATH 294 SPRING 1987 PRELIM 3 # 4

2.7.13 Find *one* eigenvalue of the matrix A below. Three eigenvectors of the matrix are given.

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 1 \\ 1 & 5 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ -1 & 3 & -1 & 5 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

The following three vectors are eigenvectors of A:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

MATH 294 SPRING 1987 FINAL # 10

2.7.14 Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ find R so that $RAR^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

MATH 294 FALL 1987 PRELIM 2 # 1

2.7.15 Find the eigenvalue and eigenvectors of the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

MATH 294 FALL 1987 PRELIM 2 # 2

2.7.16 Find the eigenvalues, eigenvectors and/or generalized eigenvectors of the matrix $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$.

MATH 294 FALL 1989 PRELIM 2 # 1
2.7.17 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

a) Show that $\lambda = 0$ is an eigenvalue of A.
b) Find a corresponding eigenvector.
c) Determine whether the system of equations

$$A\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

has a solution or not.

MATH 294 FALL 1989 PRELIM 3 # 1
2.7.18 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Show that $\lambda = 0$ is a double eigenvalue, and that $\lambda = 1$ is a simple eigenvalue. For b) and c) below, you may use the result of a) even if you did not show it.]
b) Find all linearly independent eigenvectors corresponding to the eigenvalues $\lambda = 0$ and $\lambda = 1$ respectively.
c) Find two linearly independent generalized eigenvectors corresponding to the double eigenvalue $\lambda = 0$.

MATH 293 SPRING 1990 PRELIM 3 # 1
2.7.19 **a)** Find the eigenvalues, eigenvectors and dimension of the subspace of eigenvectors corresponding to each eigenvalue of

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$$

b) For what value of c (if any) is $\lambda = 2$ an eigenvalue of

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & c & 1 \\ -1 & -1 & 1 \end{bmatrix}?$$

In that case find a basis for the subspace of eigenvectors corresponding to $\lambda = 2$.

MATH 294 SPRING 1990 PRELIM 3 # 2

2.7.20 **a**) There is a 2×2 matrix R such that R^tAR is a diagonal matrix, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Find R^tAR. (Hint: You needn't find R; there are two correct answers)

b) Describe the conic $v^tAv = 1$ for v in V_2 and $A = \begin{bmatrix} 4 & -1 \\ -1 & -2 \end{bmatrix}$. Explain why your answer is correct.

MATH 293 FALL 1991 FINAL # 3

2.7.21 Diagonalize the one of the following matrices which can be diagonalized:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, C = \begin{pmatrix} 6 & -10 & 6 \\ 2 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

MATH 293 FALL 1991 PRELIM 3 # 5

2.7.22 Find the eigenvalues and eigenvectors of the matrix A where

$$A = \begin{pmatrix} 3 & 4 & 2 \\ -2 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Hint: $\lambda = 1$ is one eigenvalue of A.

MATH 293 FALL 1991 PRELIM 3 # 6

2.7.23 An $n \times n$ matrix always has n eigenvalues (some possibly complex), but these are not always distinct. (T/F)

MATH 293 FALL 1991 FINAL # 4

2.7.24 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

MATH 293 SPRING 1992 PRELIM 3 # 3

2.7.25 Find the eigenvalues and three linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

MATH 294 SPRING 1992 FINAL # 3

2.7.26 Consider the eigenvalue problem: Find all real numbers λ (eigenvalues) such that the differential equation $-\frac{\partial^2 w}{\partial x^2} = \lambda w, 0 < x < L$ with the boundary conditions $\frac{\partial w}{\partial x}(0) = \frac{\partial w}{\partial x}(L) = 0$ has nontrivial solutions (eigenfunctions). Given that there are no eigenvalues $\lambda < 0$, find all possible eigenvalues $\lambda \geq 0$ and corresponding eigenfunctions. You must derive your result. No credit will be given for simply writing down the answer.

MATH 294 SPRING 1992 FINAL # 3

2.7.27 A vector space V has two bases

$$B_1 : \{e^t, e^{2t}, e^{3t}\} \text{and} B_2 : e^t + e^{2t}, e^{3t}, e^{2t}$$

A linear operator $T : V \to V$ is $T = \frac{\partial}{\partial t}$

a) Find the matrix T_{B_1} which represents T in the basis B_1.

b) For the vectors $v = e^{2t}, w = \frac{\partial v}{\partial t}$, find $\beta_1(v)$ and $\beta_1(w)$ which represent these vectors in the basis B_1.

c) Noting that $T(v) = 2v$, i.e. v is an eigenvector of T with eigenvalue equal to 2, interpret the equation

$$\beta_1(w) = T_{B_1}\beta_1(v)$$

as an eigenvalue-eigenvector equation for T_{B_1}. What are the eigenvalue and eigenvector in this equation?

d) Now consider the basis B_2. Find the matrices $(B_2 : B_1)$ and $(B_1 : B_2)$.

e) Find $\beta_2 v, T_{B_2}$ and $\beta_2(w)$.

f) Is the equation $\beta_2(w) = T_{B_2}\beta_2(v)$ also an eigenvalue-eigenvector equation? If so, what are the eigenvalue and eigenvector in this case?

MATH 293 FALL 1992 FINAL # 4

2.7.28 **a**) Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

b) Let $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$. Find a nonsingular matrix C such that $C^{-1}AC = D$ where D is a diagonal matrix. Find C^{-1} and D.

MATH 294 FALL 1992 FINAL # 6

2.7.29 Find the eigenvalues and eigenfunctions (nontrivial solutions) of the two-point boundary-value problem

$$y'' + \lambda y = 0, 0 < x < 1, (\text{assume} \lambda \geq 0)$$

$$y'(0) = y(1) = 0.$$

MATH 293 SPRING 1993 FINAL # 4

2.7.30 It is known that a 3×3 matrix A has: 1) A twice-repeated eigenvalue $\lambda_1 = 1$ with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and 2) another eigenvalue $\lambda_2 = 0$ with corresponding eigenvectors $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

a) Find a basis for the null space $Nul(B)$ of B, where B is the matrix $B = (A - I_3$, and I_3 is the 3×3 identity matrix.

b) Find a basis for the null space $Nul(A)$ of A.

c) For part a) above, $Null(B)$ is a plane in 3-dimensional space. The equation of this plane can be written in the form $Ax + By + Cz = 0$. Find A, B, and C.

d) For part b above, is $Null(A)$ a line, a plane, or something else? Please explain your answer carefully.

MATH 294 FALL 1994 FINAL # 5

2.7.31 Let $A = \begin{bmatrix} -3 & 0 & -4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A.

b) Find a basis for the eigenspace associated with each eigenvalue. The eigenspace corresponding to an eigenvalue is the set of all eigenvectors associated with the eigenvalue, plus the zero vector.

c) Find an orthogonal matrix P and a diagonal matrix D so that $P^{-1}AP = D$. What is D?

MATH 293 FALL 1994 FINAL # 7

2.7.32 If an $n \times n$ A has n distinct eigenvalues, then

a) $\det(A)$ not zero,

b) $\det(A)$ is zero,

c) A is similar to I_n,

d) A has n linearly independent eigenvectors,

e) $A = A^T$

MATH 294 FALL 1994 FINAL # 10

2.7.33 Which of the following is an eigenvector of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$?

a. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, b. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, c. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, d. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, e. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

MATH 294 SPRING 1995 FINAL # 6

2.7.34 Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

a) Find all eigenvalues of A, and for each an eigenvector.
b) Find a matrix P such that $D = P^{-1}AP$ is diagonal.
c) Find D.

MATH 293 SPRING 1995 FINAL # 7

2.7.35 **a**) Find all eigenvalues of the matrix

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

b) For each eigenvalue find a corresponding eigenvector.
c) Are the eigenvectors orthogonal?

MATH 293 FALL 1995 PRELIM 3 # 5

2.7.36 **a**) One eigenvalue of $A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \end{pmatrix}$ is 2. Find a corresponding eigenvector.

b) Find the characteristic polynomial $\det(A - \lambda I)$ if $A = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix}$. Also find all the eigenvalues of A.

MATH 293 FALL 1995 FINAL # 6

2.7.37 For $A = \begin{bmatrix} 0 & -2 & 1 \\ -2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

a) Show that the characteristic polynomial of A is

$$-\lambda(\lambda^2 - \lambda - 6)$$

b) Find all eigenvalues of A of 3 linearly independent eigenvectors.
c) Check your solution of part (b).

MATH 293 FALL 1995 FINAL # 8

2.7.38 For $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$

a) Find the eigenvalues and 2 linearly independent eigenvectors. Show that these 2 eigenvectors are orthogonal.
b) Find a matrix P and a diagonal matrix D so that $P^{-1}AP = D$
c) If your matrix P in part (b) is not orthogonal, how can it be modified to make it orthogonal? (so that $P^{-1}AP = D$ still holds).

MATH 293 SPRING 1996 PRELIM 3 # 5
2.7.39 Let

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}.$$

a) Find the characteristic polynomial of B.
b) Find the eigenvalues of B. Hint: one eigenvalue is 2.
c) Find eigenvectors corresponding to the eigenvalues other than 2.

MATH 293 SPRING 1996 PRELIM 3 # 6
2.7.40 Let C be a 2-by-2 matrix. Suppose

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are eigenvectors for C with eigenvalues 1 and 0, respectively. Let

$$\vec{x} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

Find $C^{100}\vec{x}$.

MATH 293 SPRING 1996 FINAL # 14
2.7.41 Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \Re$. A complex eigenvector of A is:

a) $\begin{bmatrix} -2 \\ i \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ -i \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ i \end{bmatrix}$ d) $\begin{bmatrix} -i \\ 2 \end{bmatrix}$ e) none of the above

MATH 294 # 5
2.7.42

MATH 293 SPRING 1996 FINAL # 15
2.7.43 Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then
a) A has three linearly independent eigenvectors with eigenvalue 2.
b) The eigenspace corresponding to the eigenvalue 2 has a basis consisting of exactly one eigenvector.
c) The eigenspace corresponding to the eigenvalue 2 has dimension 2.
d) A is not diagonalizable.
e) None of the above.

MATH 293 SPRING 1996 FINAL # 38
2.7.44 The only matrix with 1 as an eigenvalue is the identity matrix. (T/F)

MATH 293 SPRING 1996 FINAL # 39

2.7.45 If A is an $n \times n$ matrix for which $A = PDP^{-1}$, D diagonal, then A cannot have n linearly independent eigenvectors. (T/F)

MATH 293 SPRING 1996 FINAL # 40

2.7.46 If x is an eigenvector of a matrix A corresponding to the eigenvalue λ, then $A^3\vec{x} = \lambda^3\vec{x}$. (T/F)

MATH 293 SPRING 1997 PRELIM 2 # 2

2.7.47 Let $A = \begin{bmatrix} 9 & 0 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

a) Find the characteristic polynomial $\det(A - \lambda I)$ of A, and find all the eigenvalues. (*hint*: $\lambda - 9$ is one factor of the polynomial.

b) Find an eigenvector for each eigenvalue.

MATH 294 SPRING 1997 FINAL # 5

2.7.48 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

a) Find the characteristic polynomial of A. Verify that the eigenvalues of A are: 0, 1, 2

b) For each eigenvalue, find a basis for the corresponding eigenspace.

c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

d) Show that the columns of P form an orthogonal basis for $\Re^3$.

e) Find $A^{10}\vec{x}$ where $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

MATH 294 FALL 1997 PRELIM 2 # 4

2.7.49 The following information is known about a 3×3 matrix A. (Here e_1, e_2, e_3 is the standard basis for $\Re^3$).

i) $Ae_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$,

ii) $e_1 + e_2$ is an eigenvector of A with corresponding eigenvalue 1.

iii) $(e_2 + e_3)$ is an eigenvector of A with corresponding eigenvalue 2.

Find the matrix A.

MATH 294 FALL 1997 PRELIM 3 # 3

2.7.50 Given an $n \times n$ matrix A with n linearly independent eigenvectors, it is possible to find a square root of A (that is, an $n \times n$ matrix $\sqrt{A}$) with $\left(\sqrt{A}^2 = A\right)$) by using the following method:
(1) Find $D = P^{-1}AP$, where D is a diagonal matrix (and P is some suitable matrix).
(2) Find $\sqrt{D}$ by taking the square roots of the entries on the diagonal. (This might involve complex numbers).
(3) The square root of A is then $\sqrt{A} = P\sqrt{D}P^{-1}$.
Use this method to find $sqrtA$ for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

MATH 294 FALL 1997 FINAL # 3

2.7.51 Let

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of A. Is A diagonalizable?

MATH 294 FALL 1997 PRELIM 3 # 4

2.7.52 **a**) Let

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For which values of θ is this matrix diagonalizable?

b) Let A be 2×2 matrix with characteristic polynomial $(\lambda - 1)^2$. Suppose that A is diagonalizable. Find a matrix A with these properties. Now, find *all* possible matrices A with these properties. Justify your answer!

MATH 294 FALL 1997 PRELIM 3 # 5

2.7.53 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

a) Find the characteristic polynomial of A. Verify that the eigenvalues of A are: 0, 1, 2

b) For each eigenvalue, find a basis for the corresponding eigenspace.

c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

d) Show that the columns of P form an orthogonal basis for $\Re^3$.

e) Find $A^{10}\vec{x}$ where $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

MATH 294 SPRING 1998 PRELIM 1 # 2

2.7.54 Find the values of λ (eigenvalues) for which the problem below has a non-trivial solution. Also determine the corresponding non-trivial solutions (eigenfunctions.)

$$y'' + \lambda y = 0 \text{ for } 0 < x < 1$$

$$y(0) = 0, y'(1) = 0.$$

(Hint: λ must be positive for non-trivial solutions to exists. You may assume this.)

MATH 293 SPRING 1998 PRELIM 2 # 5

2.7.55 Consider the matrix A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Find *all* the eigenvalues of A and find a corresponding eigenvector for *each* eigenvalue. (Hint: 1 is an eigenvalue.)

MATH 294 SPRING 1998 PRELIM 3 # 4

2.7.56 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

a) Find the eigenvalues and eigenvectors of A.

b) Diagonalize A. That is, give P and D such $A = PDP^{-1}$.

c) Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be the standard basis vectors of $\Re^2$. Map $\vec{e}_1 \to P\vec{e}_1$ and $\vec{e}_2 \to P\vec{e}_2$ and sketch $P\vec{e}_1$ and $P\vec{e}_2$.

d) Give a geometric interpretation of $\vec{x} \to P\vec{x}$.

MATH 294 SPRING 1998 PRELIM 3 # 5

2.7.57 True or false? Justify each answer.

a) In general, if a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis of V.

b) A linearly independent set in a subspace H is a basis for H.

c) An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

d) If an $n \times n$ matrix A is diagonalizable, it is invertible.

MATH 294 FALL 1998 PRELIM 3 # 3

2.7.58 a)

b) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 10 & -4 & 6 & 5 \\ -4 & 8 & 4 & -6 \\ 6 & 4 & 10 & -1 \\ 5 & -6 & -1 & 5 \end{bmatrix}$. Find an eigenvalue of A.

c) Consider the 20×20 matrix that is all zeros but for the main diagonal. The main diagonal has the numbers 1 to 20 in order. Precisely describe as many eigenvalues and eigenvectors of this matrix as you can.

d) If possible, diagonalize the matrix $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. Explicitly evaluate any relevant matrices (if any inverses are needed the can be left in the form $[\ ]^{-1}$).

MATH 294 FALL 1998 FINAL # 6

2.7.59 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

a) Find orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ of A. [Hint: do not go on to parts d-e below until you have double checked that you have found two orthogonal unit vectors that are eigenvectors of A.]

b) Use the eigenvectors above to diagonalize A.

c) Make a clear sketch that shows the standard basis vectors $\{\vec{e}_1, \vec{e}_2\}$ of $\Re^2$ and the eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ of A.

d) Give a geometric interpretation of the change of coordinates matrix, P, that maps coordinates of a vector with respect to the eigen basis to coordinates with respect to the standard basis.

e) Let $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Using orthogonal projection express $\vec{b}$ in terms of $\{\vec{v}_1, \vec{v}_2\}$ the eigenvectors of A.

MATH 293 SPRING 1993 FINAL # 6

2.7.60 a) Write a matrix $A = S^{-1}AS$ such that A is diagonal, if

$$A = \begin{pmatrix} 6 & -10 & 6 \\ 2 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

b) What is matrix S?

MATH 293 SPRING 1993 FINAL # 6

2.7.61 **a**) Write a matrix $\Lambda = S^{-1}AS$ such that Λ is diagonal, if

$$A = \begin{pmatrix} 6 & -10 & 6 \\ 2 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

b) What is the matrix S?

MATH 293 SPRING 1998 PRELIM 2 # 5

2.7.62 Consider the matrix A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Find all the eigenvalues of A and find a corresponding eigenvector for each eigenvalue. (Hint: 1 is an eigenvalue.)

MATH 294 SPRING 1999 PRELIM 2 # 2b

2.7.63 The matrix $A = \begin{bmatrix} 1 & 3 & 4 & 2.718 \\ 3 & 5 & \pi & 8 \\ \sqrt{5} & 3 & 4 & 1 \\ 3 & 4 & 6 & 7 \end{bmatrix}$ has 4 distinct eigenvalues (no eigenvalue is equal to any of the others. The eigenvectors of A make up the 4 columns of a matrix P. Does the equation $P\vec{x} = \begin{bmatrix} 2.718 \\ \pi \\ \sqrt{5} \\ 4 \end{bmatrix}$ have

a) no solution (why?), or
b) exactly one solution (why?), or
c) exactly two solutions (why?), or
d) an infinite number of solutions (why?), or
e) it depends on information not given (what information?, how would that information tell you the answer?)?

MATH 294 SPRING 1999 PRELIM 2 # 2c

2.7.64 The matrix $A = \begin{bmatrix} 66 & -52 & 8 & -4 \\ -52 & 83 & -26 & -24 \\ 8 & -26 & 54 & -52 \\ -4 & -24 & -52 & 22 \end{bmatrix}$ has four eigenvalues $\lambda_1 = -30, \lambda_2 = 30, \lambda_3 = 90, \lambda_4 = 135$. Some four vectors $\vec{v}_i$ satisfy $A\vec{v}_i = \lambda_i \vec{v}_i$ (for i = 1,2,3,4). This is all you are told about the vectors $\vec{v}_i$. The vectors $\vec{v}_i$ make up, in the order given, the columns of a matrix P. If this is sufficient to answer each of the questions below, then answer the questions, if not explain why you need more information. No credit for unjustified correct answers. [Hint: massive amounts of arithmetic are not needed for any of the three parts].

a) What is the element in the third row and second column of $P^T P$?

b) What is the element in the third row and third column of $P^{-1}AP$?

c) What is the element in the second row and second column of $P^T P$?

MATH 294 SPRING 1999 PRELIM 2 # 3

2.7.65 A couch potato spends *all* of his/her time either smoking a cigarette ('C') *or* eating a bag of fries ('F') *or* watching a TV show ('T'). Since there is no smoking inside and no food allowed in the living room, he/she only does one thing at a time.

- After a cigarette there is a 50% chance he/she will light up again, but right after smoking he/she never eats, (If you think this is ambiguous please reread the initial statement,)
- After eating a bag of fries he/she has a 50% chance of going out for a smoke, a 25% chance of eating another bag of fries, and a 25% chance of turning the TV on.
- After watching TV show he/she only wants to eat.

On average he/she watches 300 TV shows a month. **On average, how many cigarettes does he/she smoke in a month?**

MATH 293 UNKNOWN FINAL # 8

2.7.66 **a**) Let A be a nonsingular $n \times n$ matrix, $X, B n \times n$ matrices. Solve the equation $\left[A \times A^T\right]^T - B^T$ for X and show that X is symmetric.

b) Let $\vec{u} = [u_1, \ldots, u_n]^T$ and $C = I - \vec{u}\vec{u}^T$. Express the entries c_{ij} of the matrix C in terms of the u_i. Show that C has zero as an eigenvalue provided that $||\vec{u}|| = 1$. Determine the corresponding <u>unit</u> eigenvector. (Hint: Do not attempt to evaluate the characteristic polynomial of C. Use instead the definition of eigenvalue and eigenvector.)

MATH 293 SPRING ? FINAL # 4

2.7.67 **a**) Determine the real numbers a,b,c,d,e,f, given that $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ d & e & f \end{pmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

b) What are the corresponding eigenvalues?

MATH 294 FALL 1987 PRELIM 3 # 1 PRELIM

2.7.68 **a**) Find the general solution of the system $\vec{x}' = A\vec{x}$ if

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

b) How many independent eigenvectors can we find for the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}?$$

MATH 293 SUMMER 1992 FINAL # 2

2.7.69 Given

$$A = \frac{1}{2}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 4 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

a) Find all the eigenvalues of A.
b) Find all linearly independent eigenvectors of A.
c) Can A be diagonalized by a change of basis? If so, let $D = (B:S)^{-1}A(B:S)$ where D is diagonal.
Find $(B:S)$ and D.

2.8 Linear Transformation II

MATH 294 SPRING 1987 PRELIM 3 # 3

2.8.1 Consider the subspace of C^2_∞ given by all things of the form

$$\vec{x}(t) = \begin{bmatrix} a\sin t + b\cos t \\ c\sin t + d\cos t \end{bmatrix},$$

where a,b,c & d are arbitrary constants. Find a matrix representation of the linear transformation

$$T(\vec{x}) = D\vec{x}, \text{ where } D\vec{x} \equiv \dot{\vec{x}}.$$

carefully define any terms you need in order to make this representation. Hint: A good basis for this vector space starts something like this

$$\left\{ \begin{pmatrix} \sin t \\ 0 \end{pmatrix}, \ldots \right\}.$$

MATH 294 SPRING 1987 PRELIM 3 # 5

2.8.2 The idea of eigenvalue λ and eigenvector $\mathbf{v}$ can be generalized from matrices and $\Re^n$ to linear transformations and their related vector spaces. If $T(\mathbf{v}) = \lambda\mathbf{v}$ (and $\mathbf{v} \neq 0$) then λ is an eigenvalue of T, and $\mathbf{v}$ is its associated eigenvector.
For the subspace of $\mathbf{x}(t)$ in C^1_∞ with $\mathbf{x}(0) = \mathbf{x}(1) = 0$ find *an* eigenvalue and eigenvector of $T(\mathbf{x}) = D^2\mathbf{x}$, where $D^2\mathbf{x} \equiv \ddot{\mathbf{x}} - \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{x}$. What is the kernel of T?

MATH 294 spr97 FINAL # 2

2.8.3 T is linear transformation from C^2_∞ to C^2_∞ which is given by $T(\mathbf{x}) = \dot{\mathbf{x}}$

MATH 294 FALL 1987 PRELIM 3 # 14

2.8.4 Find the kernel of the linear transformation

$$T(\mathbf{x}(t)) \equiv \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where T transforms C^2_∞ into C^2_∞

MATH 294 FALL 1997 PRELIM 3 # 5

2.8.5 Define $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \equiv \begin{bmatrix} x+y \\ x-z \\ y+z \end{bmatrix}$, which is a linear transformation of $\Re^3$ into itself.

a) Is T 1-1?
b) Is T onto?
c) Is T an isomorphism?
Substantiate your answers.

MATH 294 FALL 1987 FINAL # 1
2.8.6 T is a linear transformation of $\Re^3$ into $\Re^2$ such that

$$T\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix},\ T\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\ T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}.$$

a) Is T 1-1?
b) Determine the matrix of T relative to the standard bases in $\Re^3$ and $\Re^2$.

MATH 294 FALL 1987 FINAL # 7a
2.8.7 Consider the boundary-value problem
$X'' + \lambda X = 0,\ \ 0 < x < \pi,\ X(0) = X(\pi) = 0$, where λ is a given real number.
a) Is the set of all solutions of this problem a subspace of $C_\infty[0, \pi]$? why?
b) Let W = set of all functions $X(x)$ in $C_infty[0, \pi]$ such that $X(0) = X(\pi) = 0$. Is $T \equiv D^2 - \lambda$ linear as a transformation of W into $C_\infty[0, \pi]$? Why?
c) For what values of λ is $Ker(T)$ nontrivial?
d) Choose one of those values of λ and determine $Ker(T)$

MATH 294 FALL 1989 PRELIM 3 # 3
2.8.8 Let W be the following subspace of $\Re^3$,

$$W = Comb\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix},\begin{bmatrix}3\\3\\-3\end{bmatrix}\right).$$

a) Show that $\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix}$ is a basis for W.

For b) and c) below, let T be the following linear transformation $T : W \to \Re^3$,,

$$T\left(\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right)=\begin{bmatrix}1&0&-1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$$

for those $w = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ in $\Re^3$ which belong to W.

[You are allowed to use a) even if you did not solve it.]
b) What is the dimension of $Range(T)$? (Complete reasoning, please.
c) What is the dimension of $Ker(T)$? (Complete reasoning, please.

MATH 294 FALL 1989 FINAL # 7

2.8.9 Let $T : \Re^2 \to \Re^2$ be the linear transformation given in the standard basis for $\Re^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 0 \end{bmatrix}.$$

a) Find the matrix of T in the standard basis for $\Re^2$

b) Show that $\beta = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ is also a basis for $\Re^2$.

In c) below, you may use the result of b) even if you did not show it.

c) Find the matrix of T in the basis β given in b). (I.e., in $T : \Re^2 \to \Re^2$ both copies of $\Re^2$ have the basis β.

MATH 294 SPRING 1990? PRELIM 2 # 4

2.8.10 Let A be a linear transformation from a vector space V to another vector space U. Let $(\vec{v}_1, \dots, \vec{v}_n)$ be a basis for V and let $(\vec{u}_1, \dots, \vec{u}_n)$ be a basis for U.
Suppose it is known that

$$\begin{aligned}
&A(\vec{v}_1) = 2u_2 \\
&A(\vec{v}_2) = 3u_3 \\
&\vdots \\
&A(\vec{v}_i) = (i+1)\vec{u}_{i+1} \\
&\vdots \\
&A(\vec{v}_{n-1}) = n\vec{u}_n
\end{aligned}$$

and $A(\vec{v}_n) = 0 \leftarrow$ zero vector in U.
Can you find $A(\vec{v})$ in terms of the $\vec{u}_i$'s where

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_n = \sum_{i=1}^{n} \vec{v}_i$$

MATH 294 FALL 1991 FINAL # 8

2.8.11 T/F

c) If $T : V \to W$ is a linear transformation, then the range of T is a subspace of V.

d) If the range of $T : V \to W$ is W, then T is 1-1.

e) If the null space of $T : V \to W$ is $\{0\}$, then T is 1-1.

f) Every change of basis matrix is a product of elementary matrices.

g) If $T : U \to V$ and $S : V \to W$ are linear transformations, and S is not 1-1, then $ST : U \to W$ is not 1-1.

h) If V is a vector space with an inner product, (,), if $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ is an orthonormal basis for V, and if $\vec{v}$ is a vector in V, then $\vec{v} = \sum_{i=1}^{n} (\vec{v}, \vec{w}_i)\vec{w}_i$.

i) $T : V_n \to V_n$ is an isomorphism if and only if the matrix which represents T in any basis is non-singular.

j) If S and T are linear transformations of V_n into V_n, and in a given basis, S is represented by a matrix A, and T is represented by a matrix B, then ST is represented by the matrix AB

-note- Matrices are not necessarily square.

MATH 294 SPRING 1992 PRELIM 3 # 5

2.8.12 The vector space V_3 has the standard basis $S = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and the basis $B = \left(2\vec{e}_2, -\frac{1}{2}\vec{e}_1, \vec{e}_3\right)$.

a) Find the change of basis matrices $(B : S)$ and $(S : B)$. If a vector $\vec{v}$ has the representation $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the standard basis, find its representation $\beta(\vec{v})$ in the B basis.

b) A transformation T is defined as follows: $T\vec{v} =$ the reflection of $\vec{v}$ across the $x-z$ plane in the standard basis. (For reflection, in V_2 the reflection of $a\vec{i} + b\vec{j}$ across the x axis would be $a\vec{i} - b\vec{j}$. Find a formula for T in the standard basis. Why is T a linear transformation?

c) Find T_B, the matrix of T in the B basis.

d) Interpret T geometrically in the B basis, i.e., describe T_B in terms of rotations, reflections, etc.

MATH 294 FALL 1992 FINAL # 6

2.8.13 Let $C^2(-\infty, \infty)$ be the vector space of twice continuously differentiable functions on $-\infty < x < \infty$ and $C^0(infty, \infty)$ be the vector space of continuous functions on $-\infty < x < \infty$.

a) Show that the transformation $L : C^2(\infty, \infty) \to C^0(-\infty, \infty)$ defined by $Ly = \frac{\partial^2 y}{\partial x^2} - 4y$ is linear.

b) Find a basis for the null space of L. Note: You must show that the vectors you choose are linearly independent.

MATH 293 SPRING 1995 FINAL # 2

2.8.14 Let P^3 be the vector space of polynomials of degree ≤ 3, and let $L : P^3 \to P^3$ be given by

$$L(p)(t) = t\frac{\partial^2 p}{\partial t^2}(t) + 2p(t).$$

a) Show that L is a linear transformation.
b) Find the matrix of L in the basis $\left(1, t, t^2, t^3\right)$.
c) Find a solution of the differential equation

$$t\frac{\partial^2 p}{\partial t^2} + 2p(t) = t^3.$$

Do you think that you have found the general solution?

MATH 293 SPRING 1995 FINAL # 3

2.8.15 Let V be the vector space of real 3×3 matrices.
a) Find a basis of V. What is the dimension of V?
Now consider the transformation $L : V \to V$ given by $L(A) = A + A^T$.
b) Show that L is a linear transformation.
c) Find a basis for the null space (kernel) of L.

MATH 294 SPRING 1997 FINAL # 10

2.8.16 Let P_2 be the vector space of polynomials of degree ≤ 2, equipped with the inner product

$$< p(t), q(t) >= \int_{-1}^{1} p(t)q(t)dt$$

Let $T : P_2 \to P_2$ be the transformation which sends the polynomial $p(t)$ to the polynomial

$$(1 - t^2)p''(t) - 2tp'(t) + 6p(t)$$

a) Show that T is linear.
b) Verify that $T(1) = 6$ and $T(t) = 4t$. Find $T(t^2)$.
c) Find the matrix A of T with respect to the standard basis $\epsilon = \left(1, t, t^2\right)$ for P_2.
d) Find the basis for $Nul(A)$ and $Col(A)$.
e) Use the Gram-Schmidt process to find an orthogonal basis B for P_2 starting form ϵ.

MATH 294 FALL 1997 PRELIM 3 # 5

2.8.17 Let $T : \Re^2 \to \Re^2$ be the linear transformation that rotates every vector (starting at the origin) by θ degrees in the counterclockwise direction. Consider the following two bases for $\Re^2$:

$$B = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

and

$$C = \left(\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \right).$$

a) Find the matrix $[T]_B$ of T in the standard basis B.
b) Find the matrix $[T]_C$ of T in the basis C. Does $[T]_C$ depend on the angle α?

MATH 294 FALL 1997 FINAL # 9

2.8.18 Consider the vector space V of 2 matrices. Define a transformation $T : V \to V$ by $T(A) = A^T$, where A is an element of V (that is, it is a 2×2 matrix), and A^T is the transpose of A.
a) Show that T is linear transformation.
The value λ is an *eigenvalue* for T, and $\vec{v} \neq 0$ is the corresponding eigenvector, if $T(\vec{v}) = \lambda\vec{v}$. (*Note*: here $\vec{v}$ is a 2×2 matrix).
b) Find an eigenvalue of T (You need only find one, not all of them). (*Hint*: Search for matrices A such that $T(A)$ is a scalar multiple of A.)
c) Find an eigenvector for the particular eigenvalue that yl=ou found in part (b).
d) Let W be the complete eigenspace of T with the eigenvalue from part (b) above. Find a basis for W. What is the dimension of W?

MATH 294 SPRING 1998 FINAL # 6

2.8.19 Let $T : P^2 \to P^3$ be the transformation that maps the second order polynomial $p(t)$ into $(1 + 2t)p(t)$,
a) Calculate $T(1), T(t)$, and $T(t^2)$.
b) Show that T is a linear transformation.
c) Write the components of $T(1), T(t), T(t^2)$ with respect to the basis $C = \{1, t, t^2, 1 + t^3\}$.
d) Find the matrix of T relative to the bases $B = \{1, t, t^2\}$ and $C = \{1, t, t^2, 1 + t^3\}$.

MATH 294 FALL 1998 PRELIM 3 # 1

2.8.20 Consider the following three vectors in $\Re^3$

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{and} \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

[Note: $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.].

a) Find the orthogonal projection of $\vec{y}$ onto the subspace of $\Re^3$ spanned by $\vec{u}_1$ and $\vec{u}_2$.

b) What is the distance between $\vec{y}$ and $span\{\vec{u}_1, \vec{u}_2\}$?

c) In terms of the standard basis for $\Re^3$, find the matrix of the linear transformation that orthogonally projects vectors onto $span\{\vec{u}_1, \vec{u}_2\}$.

MATH 294 FALL 1998 FINAL # 4

2.8.21 Here we consider the vector spaces P_1, P_2, and P_3 (the spaces of polynomials of degree 1,2 and 3).

a) Which of the following transformations are linear? (Justify your answer.)

i) $T : P_1 \to P_3,\ T(p) \equiv t^2 p(t) + p(0)$

ii) $T : P_1 \to P_1,\ T(p) \equiv p(t) + t$

b) Consider the linear transformation $T : P_2 \to P_2$ defined by $T(a_0 + a_1 t + a_2 t^2) \equiv (-a_1 + a_2) + (-a_0 + a_1)t + (a_2)t^2$. with respect to the standard basis of $P_2, \beta = \{1, t, t^2\}$, is $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that an eigenvalue/eigenvector pair of A is $\lambda = 1, v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Find an eigenvaue/eigenvector (or eigenfunction) pair of T. That is, find λ and $g(t)$ in P_2 such that $T(g(t)) = \lambda g(t)$.

c) Is the set of vectors in $P_2\{3+t, -2+t, 1+t^2\}$ a basis of P_2? (Justify your answer.)

MATH 293 SPRING 19? PRELIM 2 # 4

2.8.22 Let M be the transformation from P^n to P^n such that

$$Mp(t) = \frac{1}{2}[p(t) + p(-t)](\text{t real})$$

a) If $n = 3$ find the matrix of this transformation with respect to the basis $\{1, t, t^2, t^3\}$.

b) Let $N = I - M$. What is $Np(t)$ in terms of $p(t)$? Show that $M^2 = MM = M$, $MN = MN = 0$

MATH 294 FALL 1987 PRELIM 2 # 3 MAKE-UP

2.8.23 a) If A is an $n \times n$ matrix with $rank(A) = r$, then what is the dimension of the vector space of all solutions of the system of linear equations $A\vec{x} = \vec{0}$

b) What is the dimension of the kernel of the linear transformation from $\Re^n$ to $\Re^n$ which has A for its matrix in the standard basis.

MATH 294 FALL 1987 PRELIM 2 # 14 MAKE-UP

2.8.24 Show that if $T: V \to W$ is a linear transformation from V to W, and kernel$(T) = \vec{0}$, then T is 1-1. (Recall: kernel$(T) = \left\{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \right\}$.)

MATH 294 FALL 1987 FINAL # 6 MAKE-UP

2.8.25 Let $T: \Re^2 \to \Re^4$ be a linear transformation.

a) If $T \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ and $T \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, what is $T \begin{bmatrix} -9 \\ 26 \end{bmatrix}$?

b) What are $T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

c) What is the matrix of T in the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\Re^2$, and the standard basis for $\Re^4$?

MATH 294 SUMMER 1989 PRELIM 2 # 1

2.8.26 a)

b) Find a basis for $ker(L)$, where L is linear transformation from $\Re^4$ to $\Re^3$ defined by

$$L(\vec{x}) \equiv \begin{bmatrix} 1 & 2 & -4 & 3 \\ 1 & 2 & -2 & 2 \\ 2 & 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

c) What is the dimension of $ker(L)$?

d) Is the vector $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in $range(L)$? (Justify your answer.) If so, find all vectors $\vec{x}$ in $\Re^4$ which satisfy $L(\vec{x}) = \vec{y}$

MATH 294 SUMMER 1989 PRELIM 2 # 4

2.8.27 Let P be the linear transformation from $\Re^3$ to $\Re^3$ defined by

$$P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

a) Find a basis for $ker(P)$.

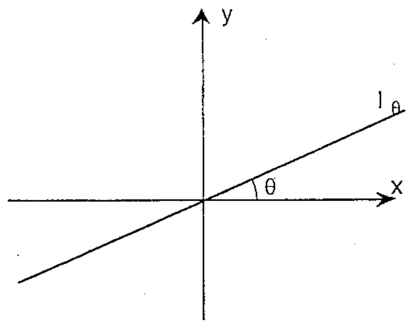
b) Find a basis for $range(P)$.

c) Find all vectors $\vec{x}$ in $\Re^3$ such that $P\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

d) Find all vectors $\vec{x}$ in $\Re^3$ such that $P\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

MATH 293 SPRING 1995 PRELIM 3 # 4

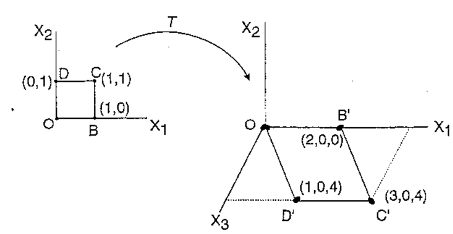
2.8.28 Let $L_\theta : \Re^2 \to \Re^2$ be the linear transformation which represent orthogonal projection onto the line ℓ_θ forming angle θ with the x-axis.



a) Find the matrix T of L_θ (with respect to the standard basis of $\Re^2$).
b) Is L_θ invertible. Explain your answer geometrically.
c) Find all the eigenvalues of T.

MATH 294 FALL 1998 PRELIM 2 # 1

2.8.29 The unit square $OBCD$ below gets mapped to the parallelogram $OB'C'D'$ (on the $x_1 - x_3$ plane) by the linear transformation $T : \Re^2 \to \Re^3$ shown.



Problems (b) - (e) below can be answered with or without use of the matrix A from part (a).
a) Is this transformation one-to-one? For this and all other short answer questions on this test, some explanation is needed.)
b) What is the null space of A?
c) What is the column space of A?
d) Is A invertible? (No need to find the inverse if it exists.)

MATH 294 FALL ? FINAL # 1 MAKE-UP

2.8.30 Consider the homogeneous system of equations $B\vec{x} = \vec{0}$, where

$$B = \begin{bmatrix} 0 & 1 & 0 & -3 & 1 \\ 2 & -1 & 0 & 3 & 0 \\ 2 & -3 & 0 & 0 & 4 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

a) Find a basis for the subspace $W \subset \Re^5$, where $W =$ set of all solutions of $B\vec{x} = \vec{0}$.

b) Is B 1-1 (as a transformation of $\Re^5 \to \Re^3$)? Why?

c) Is $B : \Re^5 \to \Re^3$ onto? Why?

d) Is the set of all solutions of $B\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ a subspace of $\Re^5$? Why?

2.9 Orthogonality

MATH 294 SPRING 1987 PRELIM 3 # 8

2.9.1 Find c_3 so that:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ -4 \\ -4 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}$

Note that the four vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and $\vec{v}_4$ are mutually orthogonal.

MATH 294 SPRING 1992 FINAL # 6

2.9.2 Given $A = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

a) Find an orthogonal matrix C such that $C^{-1}AC$ is diagonal. (The columns of an orthogonal matrix are orthonormal vectors.)

b) If $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2 + d\vec{v}_3$
where $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are the columns of C, find the scalars a, b and d.

MATH 293 FINAL SPRING 1993 # 3

2.9.3 Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

a) Find the vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that a solution $\vec{x}$ of the equation $A\vec{x} = \vec{b}$ exists.

b) Find a basis for the column space $\mathcal{R}(A)$ of A

c) It is claim that $\mathcal{R}(A)$ is a plane in $\Re^3$. If you agree, find a vector n in $\Re^3$ that is normal to this plane. Check your answer.

d) Show that n is perpendicular to each of the columns of A. Explain carefully why this is true.

MATH 293 FALL 1994 PRELIM 3 # 5

2.9.4 True/False

Answer each of the following as True or False. If False, explain, by an example.

a) Every spanning set of $\Re^3$ contain at least three vectors.

b) Every orthonormal set of vectors in $\Re^5$ is a basis for $\Re^5$.

c) Let A be a 3 by 5 matrix. Nullity A is at most 3.

d) Let W be a subspace of $\Re^4$. Every basis of W contain at least 4 vectors.

e) In $\Re^n, ||cX|| = |c|||X||$

f) If A is an $n \times n$ symmetric matrix, then rank $A = n$.

MATH 294 FALL 1997 PRELIM 3 # 4

2.9.5 Consider $\mathcal{W}$, a subspace of $\Re^4$, defined as $\sqsupseteq\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$\mathcal{W}$ is a "plane" in $\Re^4$.

a) Find a basis for a subspace $\mathcal{U}$ of $\Re^4$ which is orthogonal to $\mathcal{W}$.

Hint: Find *all* vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that are perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.

b) What is the geometrical nature of $\mathcal{U}$?

c) Find the vector in $\mathcal{W}$ that is closest to the vector $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

MATH 294 unknown unknown # ?

2.9.6 Let W be the subspace of $\Re^3$ spanned by the orthonormal set $\left\{\frac{(1,2,-1)}{\sqrt{6}}, \frac{(1,0,1)}{\sqrt{2}}\right\}$. Let $X = (1,1,1)$. Find a vector Z, in W, and a vector Y, perpendicular to every vector in W, such that $X = Z + Y$. What is the distance from X to W?

MATH 294 SPRING 1999 PRELIM 3 # 1

2.9.7 Let the functions $f_1 = 1, f_2 = tf_3 = t^2$ be three "vectors" which span a subspace, S, in the vector space of continuous functions on the interval $-1 \leq t \leq 1 (C[-1,1])$, with inner product

$$< f, g > \equiv \int_{-1}^{1} f(t)g(t)dt.$$

Find three orthogonal vectors, $u_1 = 1, u_2 = ?, u_3 = ?$ that span S.

MATH 294 SPRING ? FINAL # 10

2.9.8 Consider the vector space $C_0(-\pi,\pi)$ of continuous functions in the interval $-\pi \leq x \leq \pi$, with inner product conjugation. Consider the following set of functions $b = \left\{\dots e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, \dots\right\}$.

a) Are they linearly independent? (Hint: Show that they are orthogonal, that is $\left(e^{inx}, e^{imx}\right) = 0$ for $n \neq m$.
$\left(e^{inx}, e^{imx}\right) \neq 0$ for $n = m$.

b) Ignoring the issue of convergence for the moment, let $f(x)$ be in $C_0(-\pi,\pi)$. Express $f(x)$ as a linear combination of the basis B. That is,

$$f = \dots a_{-2}e^{-2ix} + a_{-1}e^{-ix} + a_0 + a_1e^{ix} + a_2e^{2ix} + \dots$$

find the coefficients $\{a_n\}$ of each of the basis vectors. Use the results from (a).

c) How does this relate to the Fourier series? Are the coefficients $\{a_n\}$ real or complex? What if B is a set of arbitrary orthogonal functions?

MATH 294 SPRING 1999 PRELIM 2 # 2a

2.9.9 **a**) Three matrices A, B, and P have:

i) $A = P^{-1}BP$,

ii) B is symmetric ($B^T = B$), and

iii) P is orthogonal ($P^T = P^{-1}$).

Is it necessary true that A is symmetric? If so, prove it. If not, find a counter example (say three 2×2 matrices A, B and P where (i) - (iii) above are true and A is not symmetric).

MATH 294 SPRING 1999 PRELIM 3 # 4

2.9.10 The temperature, $u(x,y)$, in a rectangular plate was measured at six locations. The (x,y) coordinates and measured temperatures, u, are given in the table below.

x	y	u
0	0	11
$\frac{\pi}{2}$	0	19
0	1	1
$\frac{\pi}{2}$	1	14

Assume that $u(x,y)$ is supposed to obey the equation (this is *not* a PDE question)

$$u(x,y) = \beta_0 + \beta_1 e^{-y} \sin x.$$

Set up, but do not solve, a system of equations for the parameters, β_0, β_1, that provide the least-squares best fit of the measured data to the equation above.
Extra credit Neatly write out a sequence of Matlab commands that will give you the parameters β_0, β_1.

2.10 Orthogonal Projection / Gram Schmidt

MATH 293 FALL 1995 PRELIM 1 # 3

2.10.1 **a**) Find the orthogonal (scalar) projection of the vector $\vec{v} = \vec{i} + \vec{j} + \vec{k}$ in the direction of the vector $\vec{w} = 5\vec{i} + 12\vec{j}$

b) Consider the two vectors

$$\vec{a} = 3\vec{i} - 4\vec{j}$$

$$\vec{b} = 3\vec{i} + 4\vec{j}$$

The vector $\vec{u}$ has orthogonal projections $-\frac{1}{5}$ and $\frac{7}{5}$ along the vectors $\vec{a}$ and $\vec{b}$, respectively. Find $\vec{u}$.
Hint: Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$

MATH 293 SPRING 1995 FINAL # 8

2.10.2 **a**) What is the formula for the scalar orthogonal projection of a vector $\vec{v} \in \Re^n$ onto the line spanned by a vector $\vec{w}$.
Let

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Suppose $\vec{v}_1$ has orthogonal projection 3 and 7 onto the lines spanned by $\vec{b}_1$ and $\vec{b}_2$ respectively.

b) Find $\vec{v}_1$.

c) Suppose $\vec{v}_2$ has orthogonal projections -6 and -14 onto the lines spanned by $\vec{b}_1$ and $\vec{b}_2$ respectively. Find $\vec{v}_2$.

d) Are $\vec{v}_1$ and $\vec{v}_2$ linearly independent.

MATH 294 FALL 1997 PRELIM 3 # 6

2.10.3 As part of their plan to take over the world, lab assistant Pinky has collected 100 points of data

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{100}, y_{100},$$

(which represent some devious no-good data) which his partner, Brain, will analyze. A computer program boils down this data into the following set of numbers:

$$\sum_1^{100} x_i = 10, \sum_1^{100} x_i^2 = 20, \sum_1^{100} x_i^3 = 100, \sum_1^{100} x_i^6 = 200,$$

and

$$\sum_1^{100} y_i = 200, \sum_1^{100} x_i y_i = 230, \sum_1^{100} x_i^2 y_i = 250, \sum_1^{100} x_i^3 y_i = 300.$$

Brain has determined that the data is probably of the form $y = a + bx^3$. Your job is to find the least-squares solution to this problem (i.e. find the a and b that gives the least-squares solution).

MATH 294 FALL 1997 PRELIM 3 # 4

2.10.4 Consider $\mathcal{W}$, a subspace of $\Re^4$, defined as $\sqsupset\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$\mathcal{W}$ is a "plane" in $\Re^4$.

a) Find a basis for a subspace $\mathcal{U}$ of $\Re^4$ which is orthogonal to $\mathcal{W}$.

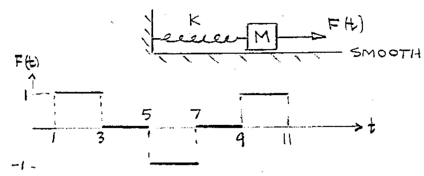
Hint: Find *all* vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that are perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.

b) What is the geometrical nature of $\mathcal{U}$?

c) Find the vector in $\mathcal{W}$ that is closest to the vector $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

MATH 294 FALL 1997 FINAL # 8

2.10.5 The following figure shows numerical results y_i, for $i = 1, 2, \ldots, n$. It is known that the exact solution of the problem is a formula of the form $y = c$, for some constant c. Find the least squares solution for the constant c in terms of $y_1, y_2, \ldots, y_n$, and n.



MATH 294 FALL 1998 FINAL # 6

2.10.6 **a)** Find orthonormal eigenvectors $\{\vec{v}_1, vecv_2\}$ of A. [Hint: do not go on to parts d-e below until you have double checked that you have found two orthogonal unit vectors that are eigenvectors of A.

b) Use the eigenvectors above to diagonalize A.

c) Make a clear sketch that shows the standard basis vectors $\{\vec{e}_1, \vec{e}_2\}$ of $\Re^2$ and the eigenvectors $\vec{v}_1, \vec{v}_2$ of A.

d) Give a geometric interpretation of the change of coordinates matrix, P, that maps coordinates of a vector with respect to the eigen basis to coordinates with respect to the standard basis.

e) Let $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Using orthogonal projection express $\vec{b}$ in terms of $\{\vec{v}_1, \vec{v}_2\}$ the eigenvectors of A.

MATH 294 FALL 1998 PRELIM 3 # 1

2.10.7 Consider the following three vectors in $\Re^3$:

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

[Note: $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.].

a) Find the orthogonal projection of $\vec{y}$ onto the subspace of $\Re^3$ spanned by $\vec{u}_1$ and $\vec{u}_2$.

b) What is the distance between $\vec{y}$ and $span\,\{\vec{u}_1, \vec{u}_2\}$?

c) In terms of the standard basis for $\Re^3$, find the matrix of the linear transformation that orthogonally projects vectors onto $span\,\{\vec{u}_1, \vec{u}_2\}$.

MATH 293 FALL 1994 PRELIM 3 # 14

2.10.8 The vectors $\{(1, 0, 0, -1), (1, -1, 0, 0), (0, 1, 0, 1)\}$ are linearly independent and span a subspace S of $\Re^4$. Use the Gram-Schmidt process to find an orthogonal basis for the subspace of S that is orthogonal to the first vector of the given set, $(1, 0, 0, -1)$.

MATH 293 FALL 1995 FINAL # 7

2.10.9 a) Find an orthonormal basis for the space of vectors in $\Re^3$ having the form $\begin{bmatrix} c_1 - c_2 \\ c_2 \\ 2c_2 \end{bmatrix}$. You may use Gram-Schmidt or any other method.

b) If $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\right\}$ is an orthonormal basis for $\Re^4$,

$$\begin{bmatrix} 1 \\ -9 \\ 0 \\ \sqrt{5} \end{bmatrix} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4, \text{ and } \vec{b}_2 = \begin{bmatrix} 0.5 \\ 0 \\ \alpha \\ 0 \end{bmatrix},$$

(where c_i are real constants), find the possible values of c_2 and α

MATH 294 FALL 1997 PRELIM 3 # 1

2.10.10 Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}.$$

a) Find an orthogonal basis for the null space of A.

b) Find a basis for the orthogonal complement of $Nul(A)$, i.e. find $(Nul(A))^T$.

MATH 294 FALL 1997 PRELIM 3 # 12

2.10.11 Let $A = [\vec{v}_1\vec{v}_2]$ be a 1000×2 matrix, where $\vec{v}_1, \vec{v}_2$ are the columns of A. You aren't given A. Instead you are given only that

$$A^TA = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Find an **orthonormal**basis $\{\vec{u}_1, \vec{u}_2\}$ of the column space of A. Your formulas for $\vec{u}_1$ and $\vec{u}_2$ should be written as linear combinations of $\vec{v}_1, \vec{v}_2$. (Hint: what do the entries of the matrix A^TA have to so with dot products?)

MATH 293 SPRING ? FINAL # 2

2.10.12 a) Find a basis for the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

b) Find the rank of A and a basis for its column space, noting that $A = A^T$.

c) Construct an orthonormal basis for the row space of A.

2.11 Inner Product Spaces

MATH 294 SPRING 1983 FINAL # 3

2.11.1 Consider the vector space of functions over the interval $0 \leq x \leq 1$ and the inner product

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Find an orthogonal basis for the subspace spanned by $1, e^x, 2e^x, e^{-x}$.

MATH 294 SPRING 1987 PRELIM 3 # 6

2.11.2 Find an element of the vector space V which is functions of the form $ae^{-t} + be^{-2t}$ (where a and b are arbitrary constants) which is orthogonal to E^{-t}. Use the following inner product: $< f(t), g(t) > \equiv \int_0^\infty f(t)g(t)dt$.

MATH 294 FALL 1987 PRELIM 3 # 5 MAKEUP

2.11.3 Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an orthonormal basis for $\Re^n$, and $\vec{v}$ is a vector in $\Re^n$ such that $< \vec{v}, \vec{v}_i >= 0$ for all $i = 1, 2, \ldots, n$. (Here $< v, w >$ denotes the standard inner product in $\Re^n$.) Then what can you conclude about $\vec{v}$? Why? (Hint: write $\vec{v}$ as a linear combination of the basis elements $\vec{v}_1, \ldots, \vec{v}_n$, then apply the condition $< \vec{v}, \vec{v}_i >= 0$.)

MATH 294 FALL 1987 PRELIM 3 # 6 MAKE-UP

2.11.4 With the inner product $< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt$ in the vector space of continuous functions defined on $[-\pi, \pi]$ what is $||a \sin x + b \cos x||$?

MATH 294 FALL 1987 FINAL # 8

2.11.5 Let $< f, g >= \int_0^1 f(t)g(t)dt$ be an inner product on $C_\infty[0, 1]$. Determine:

a) the component of $f(x) = 1 + x$ along $g(x) = x^2$.

b) the component of $f(x)$ perpendicular to $g(x)$.

MATH 294 SUMMER 1989 PRELIM 2 # 3

2.11.6 Consider the vector space $C[0, 1]$ of continuous functions over the interval $0 \leq t \leq 1$ and inner product

$$< f, g >= \int_0^1 f(t)g(t)dt.$$

a) Show that $\left\{1, t, t^2\right\}$ is a set of linearly independent vectors in $C[0, 1]$.

b) Find an orthogonal basis for the subspace of $C[, 1]$ spanned by $1, t, 4t, t^2$.

MATH 294 FALL 1989 PRELIM 3 # 4

2.11.7 Let $C[-1,1]$ denote the continuous real-valued functions on $[-1,1]$, and let W be the following subspace thereof:

$$W = \left\{ c_1 + c_2 t + c_3 t^4 : c_1, c_2, c_3 \text{ real numbers } \right\}.$$

a) Show that W is three dimensional.

b) For functions $p(t), q(t)$ i W, introduce the following inner product:

$$< p, q >= \int_{-1}^{1} p(t)q(t)dt.$$

Find an orthogonal basis for W which contains the function $p(t) \equiv 1$ as one element.

MATH 293 SPRING 1992 FINAL # 5

2.11.8 In V_4, $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

a) Using the standard inner product

$$(\vec{u}, \vec{v}) = \sum_{i=1}^{4} u_i v_i$$

find the length of $\vec{u}$ and determine whether the angle between $\vec{u}$and $\vec{v}$ is greater or less than 90 degrees.

b) Using the nonstandard inner product

$$(\vec{x}, \vec{y}) = \sum_{k=1}^{4} k x_k y_k = x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 + 4x_4 y_4$$

find the length of $\vec{u}$and determine whether the angle between $\vec{u}$ and $\vec{v}$ is more or less than 90 degrees.

MATH 293 SUMMER 1992 FINAL # 5

2.11.9 Given the set of functions

$$\left\{1, x, x^2, x^3\right\} \text{ with } -1 \le x \le 1$$

and the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$

a) Find an orthogonal basis $(w_1(x), w_2(x), w_3(x), w_4(x))$ of the space spanned by the functions $1, x, x^2$ and x^3. Use the Schmidt orthogonalization procedure.

b) Given $\phi(x) = 1 + 2x + 3x^2$ and $\phi(x) = c_1 w_1(x) + c_2 w_2(x) + c_3 w_3(x) + c_4 w_4(x)$, find c_2.

MATH 294 SPRING 1997 FINAL # 8

2.11.10 You are given a vector space V with an inner product $<,>$ and an orthogonal basis $B = \left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\right\}$ for which $||\vec{b}_|| = 2, i = 1, \ldots, 5$. Suppose that $\vec{v}$ is in V and

$$< \vec{v}, \vec{b}_1 >=< \vec{v}, \vec{b}_2 >= 0$$

$$\text{and } < \vec{v}, \vec{b}_3 >= 3, < \vec{v}, \vec{b}_4 >= 4, < \vec{v}, \vec{b}_5 >= 5$$

Find the coordinates of $\vec{v}$ with respect to the basis B i.e. find c_1, c_2, c_3, c_4, c_5 such that

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 + c_5\vec{b}_5$$

MATH 294 FALL 1997 FINAL # 7

2.11.11 Consider the subspace

$$W = span\{1, t\}, \text{ for } 0 \le t \le 1,$$

equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Find the best approximation to the function $f(t) = e^t$ in W.

MATH 294 SPRING 1998 FINAL # 7

2.11.12 Regard P^2 as a subspace of $C[-1, 1]$ and construct an orthogonal basis from the standard basis $E = \left\{1, t, t^2\right\}$ using the inner product

$$\langle p, q\rangle = \int_{-1}^1 p(t)q(t)dt.$$

MATH 294 SPRING 1998 FINAL # 8

2.11.13 Consider P^2 to be a subspace of $C[-1, 1]$

a) Check that $\left\{1, t, t^2 - \frac{2}{3}\right\}$ is an orthogonal basis for this subspace with respect to the inner product

$$< f, g >\equiv f(-1)g(-1) + f(0)g(0) + f(1)g(1).$$

b) Determine the beat second order polynomial approximation, $a_0, a_1t + a_2t$, to the function e^t with respect to this inner product.

MATH 294 FALL 1998 PRELIM 3 # 2

2.11.14 For all the questions below use the vector space $C[-\pi,\pi]$, the set of all continuous functions between $-\pi$ and π, and the inner product:

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$$

[Hint $\int_{-\pi}^{\pi}\sin^2(t)dt = \int_{-\pi}^{\pi}\cos^2 dt = \pi$.]

a) What is the "distance" between the function t and the function 1?

b) Do the three functions $\{1,\sin(t),\cos(t)\}$ form an orthogonal basis for a subspace of $C[-\pi,\pi]$?

c) What value should A have if $A\sin(t)$ is to be the best possible fit to the function t?

MATH 293 SPRING ? PRELIM 2 # 5

2.11.15 Consider $C^0([1,3])$. Is the constant function $g(x) = 1(1 \le x \le 3)$ a unit vector? Find the orthogonal projection $w(x)$ of $f(x) = \frac{1}{x}(1 \le x \le 3)$ onto the span of $g(x)$.

$$[\text{Here } f\cdot g = \int_1^3 f(x)h(x)dx]$$

MATH 293 SPRING ? FINAL # 5

2.11.16 In the space of continuous functions of x in the interval $1 \le x \le 2$ one may define an inner product (other than the usual one) as follows: $f\cdot g = \int_1^2 \frac{1}{x}f(x)g(x)dx$.

a) Using this inner product of f and g find

$$||f|| \text{ (norm of } f \text{) if } f(x) = \sqrt{x} \text{ for } 1 \le x \le 2,$$

b) Determine the real constants a,b,c which make the set $\{a, b+cx\}$ orthonormal (leave $\ln 2$ as is in answers).

MATH 293 # 3 PRACTICE

2.11.17 Let $V = C^0[-1,1]$ with the inner product

$$f\cdot g = \int_{-1}^{1} f(x)g(x)dx.$$

a) Find an orthogonal basis for the space spanned by the functions $f_1(x) = 1, f_2(x) = x, f_3(x) = x^2$.

b) Find the orthogonal projection of x^3 onto the subspace spanned by the functions $1, x, x^2$.

Chapter 3

Ordinary Differential Equations

Chapter 4

Multi-Variate Integrals

Chapter 5

Fourier and Partial Differential Equations

5.1 Fourier

MATH 294 SPRING 1982 FINAL # 5

5.1.1 Consider the function $f(x) = 2x, 0 \leq x \leq 1$.

a) Sketch the odd extension of this function on $-1 \leq x \leq 1$.

b) Expand the function $f(x)$ in a Fourier sine series on $0 \leq x \leq 1$.

MATH 294 SPRING 1983 PRELIM 3 # 2

5.1.2 Find the Fourier sine series for the function $f(x) = x, 0 \leq x \leq \pi$.

MATH 294 SPRING 1983 PRELIM 3 # 4

5.1.3 **a**) Consider $f(x) = x + 1, 0 \leq x \leq 1$. Make an accurate sketch of the function $g(x)$ which is the odd extension of $f(x)$ over the interval $-1 < x \leq 1$.

b) What is the value of the Fourier series for $g(x)$ in part (a) when $x = 0$?

MATH 294 SPRING 1983 PRELIM 3 # 5

5.1.4 **a**) Name one function $f(x)$ that is both even **and** odd over the interval $-1 < x \leq 1$

b) What is the Fourier sine series of the function from part (a) above?

c) What is the Fourier cosine series of the function $f(x) = 1$ for $0 \leq x \leq 7$

MATH 294 SPRING 1983 FINAL # 2

5.1.5 **a**) Find the Fourier series for the function $f(x) = |x|, -2 \leq x \leq 2$

b) What is the value of the series from part (a) at $x = -\frac{1}{2}$?

MATH 294 FALL 1984 FINAL # 4

5.1.6 **a**) Compute the Fourier Cosine series of the function $f(x)$ given for $0 \leq x \leq L$ by

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \frac{L}{2} \\ 0 & \frac{L}{2} \leq x \leq L \end{cases}$$

.

MATH 294 FALL 1984 FINAL # 5

5.1.7 **a)** Compute the Fourier Series solution of the problem

$$\frac{d^2y}{dx^2} - 4y = g(x), 0 < x < L$$

if

$$y(0) = y(L) = 0$$

and

$$g(x) = \begin{cases} 1, & 0 \leq x < \frac{L}{2} \\ -1, & -\frac{L}{2} \leq x \leq L \end{cases}$$

.

MATH 294 SPRING 1985 FINAL # 4

5.1.8 **a)** Compute the Fourier Series of the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \leq x < 0 \\ 2 & \text{for} \quad 0 \leq x \leq \pi \end{cases}$$

on the interval $[-\pi, \pi]$.

b) State, for each x in $[-\pi, \pi]$, the what the Fourier series for f converges

MATH 294 SPRING 1985 FINAL # 13

5.1.9 What is the Fourier series for the function $f(x) = \sin x$ on the interval $[-\pi, \pi]$?

a) $cosx$

b) $\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nx$

c) $\sin x$

d) none of these.

MATH 294 FALL 1987 PRELIM 2 # 1

5.1.10 Consider the function $f(x) \equiv 1, 0 \leq x \leq 1$.

a) Extend the function $on - 1 \leq x \leq 1$ in such a way that the Fourier series (of the extended function) converges to $\frac{1}{2}$ at $x = 0$ and at $x = 1$.

b) Compute the Fourier series for **your** extension. (Remark: (a) does not have a unique answer, but (b) forces you to make the simplest choice.)

MATH 294 FALL 1987 PRELIM 2 # 2

5.1.11 Consider the function $f(x) = x$ on $0 \leq x \leq 1$. Compute the Fourier series of the **odd extension** of f on $-1 \leq x \leq 1$. To what value does this series converge when $x = 0; x = 1; x = 39.75$ (3 answers are required)?

MATH 294 SPRING 1985 FINAL # 14

5.1.12 What is the Fourier series for the function $f(x) = \sin x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

a) $cosx$

b) $\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nx$

c) $\sin x$

d) none of these.

MATH 294 SPRING 1985 FINAL # 15

5.1.13 Compute $\int_{-\pi}^{\pi} \cos 2x \cos 3x dx$.

a) 1
b) $\frac{1}{5}$
c) $\frac{1}{6}$
d) 0
e) none of these.

MATH 294 SPRING 1985 FINAL # 16

5.1.14 To what does the Fourier series, of the function $f(x) = x$ on the interval $[-1, 1]$, converge at $x = 1$?

a) -1
b) 0
c) 1
d) none of these.

MATH 294 SPRING 1985 FINAL # 16

5.1.15 To what does the Fourier series, of the function $f(x) = x$ on the interval $[-1, 1]$, converge at $x = 10$?

a) -1
b) 0
c) 1
d) 10
e) none of these.

MATH 294 SPRING 1987 PRELIM 1 # 12

5.1.16 A MuMath (primitive version of MAPLE) command can be used for full credit on one of these (your choice).

a) What is the **Fourier** series for $\sin 6\pi x$ on the interval $-3 \leq x < 3$.
b) What is the Fourier **sine** series for the function $\sin 6\pi x$ on the interval $-3 \leq x < 3$.
c) What is the Fourier **cosine** series for $\sin 6\pi x$ on the interval $-3 \leq x < 3$.
d) Write out the first four non-zero terms of the **Fourier** series for the function below in the interval $(-3 \leq x < 3)$

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0; \\ 2 & \text{if} \quad x > 0 \end{cases}$$

.

MATH 294 SPRING 1989 PRELIM 2 # 1

5.1.17 Consider the function $f(x) = 1 - x$ defined on $0 \leq x \leq 1$.

a) Sketch the odd extension of $f(x)$ over the interval $-1 \leq x \leq 1$.
b) Find the Fourier sine series for $f(x)$.
c) What does the series converge to on the interval $0 \leq x \leq 1$?

MATH 294 SPRING 1989 PRELIM 2 # 2

5.1.18 Let $f(x)$ is defined for all x,

a) Show that

$$g(x) = \frac{f(x) - f(-x)}{2}$$

is even.

b) Compute $\int_{-\pi}^{\pi} g(x)\sin(x)dx$

MATH 294 FALL 1989 FINAL # 4

5.1.19 Find the Fourier series for the function $f(x) = x^2, -1 \leq x \leq 1$.
Hint: $a_k = \int_{-1}^{1} f(x)\cos(\pi kx)dx, b_k = \int_{-1}^{1} f(x)\sin(\pi kx)dx$.

MATH 294 SPRING 1990 PRELIM 2 # 4

5.1.20 **a)** Find the Fourier series of

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \text{ and } \frac{\pi}{2} < x \leq \pi \\ 1 & 0 \leq x \leq \frac{\pi}{2} \end{cases}$$

b) Find the Fourier series of

$$g(x) = \begin{cases} 0 & \frac{\pi}{2} \leq x < \pi \\ 1 & 0 \leq x \leq \frac{\pi}{2} \end{cases}$$

c) Find the Fourier sine series of $g(x)$ (defined in (b)).

MATH 294 SPRING 1990 PRELIM 3 # 6

5.1.21 Let $f(x) = 1 - x, 0 \leq x \leq 1$.

a) Use the *even* extension of $f(x)$ onto the interval $[-1, 1]$ to get a Fourier cosine series that represents $f(x)$.

b) Sketch the graph of $f(x)$ and its even extension, and on the same graph sketch the 2^{nd} partial sum of the cosine series.

MATH 294 FALL 1990 FINAL # 16

5.1.22 Given the function $f(x) = 1 - x$ on $0 \leq x \leq 1$.

a) Determine its Fourier *sine* series. What value does this series have at $x = 0$?

b) Write down the integral forms for the coefficients a_n and b_n of the *full* Fourier series.

MATH 294 SPRING 1991 PRELIM 1 # 1

5.1.23 Given the function $f(x) = 1 + x$ on $-1 \leq x \leq 1$, determine its Fourier series. To what values does the series converge to at $x = -1, x = 0$, and $x = 1$?

MATH 294 SPRING 1991 PRELIM 1 # 2

5.1.24 Given the function $f(x) = 1$ on $0 \leq x \leq 1$.

a) Determine its Fourier *sine* series. To what values does the series converge to at $x = 0, x = \frac{1}{2}$, and $x = 1$?

b) Determine its Fourier *cosine* series. To what values does the series converge to at $x = 0, x = \frac{1}{2}$, and $x = 1$?

MATH 294 SPRING 1991 FINAL # 7

5.1.25 Given the function $f(x) = 1 - x$ on $0 \leq x \leq 1$.

a) Determine its Fourier *sine* series. What value does this series have at $x = 0$?

b) Write down the integral forms for the coefficients a_n and b_n of the *full* Fourier series on $0 \leq x \leq 1$:

$$1 - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi\left(x - \frac{1}{2}\right)}{1/2} + b_n \sin \frac{n\pi\left(x - \frac{1}{2}\right)}{1/2}.$$

MATH 294 FALL 1991 PRELIM 1 # 1

5.1.26 Given the function $f(x)$, defined on the interval $(-\pi, \pi)$:

$$f(x) = \begin{cases} 1 + \sin x, & 0 \leq x < \pi \\ \sin x, & -\pi \leq x < 0 \end{cases}$$

Determine the Fourier Series of its periodic extension.

MATH 294 FALL 1991 PRELIM 1 # 1

5.1.27 Given the function $f(x) = x - \pi$ on $(0, 2\pi)$.
Determine the Fourier Series of its periodic extension. What value does the Fourier Series converge to at $x = 2\pi$?

MATH 294 FALL 1991 PRELIM 1 # 3

5.1.28 Let $f(x)$ be given by

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$$

Determine the Fourier Series for the odd, periodic extension of $f(x)$ (i.e. the Fourier Sine Series).

MATH 294 FALL 1992 FINAL # 7

5.1.29 Find the Fourier cosine series of the function $f(x) = x^2$ on the interval $0 \leq x \leq 1$.

MATH 294 FALL 1992 FINAL # 7

5.1.30 For each of the following Fourier series representations,
$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin[(2n-1)x], 0 < x < \pi,$
$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, 0 < x < \pi,$
$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos[(2n-1)x], 0 \leq x < \pi,$

a) Find the numerical value of the series at $x = -\frac{\pi}{3}, \pi$, and $12\pi + 0.2$ (9 answers required).

b) Find the Fourier series for $|x|, -\pi < x < \pi$. (Think - this is easy !).

MATH 294 SPRING 1985 FINAL # 4
5.1.31 Find the Fourier series of period 2 for

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 0 \\ 0 & 0 < x \leq 1 \end{cases}$$

MATH 294 FALL 1993 FINAL # 1
5.1.32 Each problem has equal weight. Show all work.
Let $f(x) = 1(0 < x < 2)$. Consider f_e and f_o to be the *even* and *odd* periodic extensions of f having period 4.
a) Find the Fourier Series of f_0.
b) List the values $f_e(x), f_0(x)$ for $x = 1$, and 3. You should have a total of 4 answers to this part.
c) One main idea underlying Fourier series is the "orthogonality" of functions. Give an example of a function g which is orthogonal (over $-2 \leq x \leq 2$) to x^2 in other words. $\int_{-2}^{2} g(x)x^2 dx = 0$ with g not identically 0.

MATH 294 FALL 1993 PRELIM 1 # 6
5.1.33 Given the function $f(x) = 1 - x$ on $0 \leq x \leq 1$.
a) Determine its Fourier *sine* series. What value does this series have at $x = 0$?
b) Write down the integral forms for the coefficients a_n and b_n of the *full* Fourier series.

MATH 294 FALL 1994 PRELIM 3 # 3
5.1.34 Find the Fourier series for the period 4 function $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 \leq x < 2 \end{cases}$ and state for which values of x the function is equal to its Fourier series.

MATH 294 FALL 1994 PRELIM 3 # 4
5.1.35 **a)** A certain Fourier series is given by

$$f(x) = \cos 2x + \frac{\cos 4x}{4} + \frac{\cos 6x}{9} + ...$$

i) sketch 1^{st} and 2^{nd} terms of the series.
ii) sketch the sum of the 1^{st} and 2^{nd} terms
iii) sketch $f(x)$ over several periods, noting the period length.

MATH 294 SPRING 1995 PRELIM 3 # 1
5.1.36 Let

$$f(x) = \left\{ \begin{array}{ll} -1, & \text{if } -1 < x < 0 \\ 1 - x, & \text{if } 0 \leq x \leq 1 \end{array} \right\}$$

a) Graph on the interval $[-5, 5]$ the function $g(x)$ such that
i) $g(x) = f(x)$if $-1 < x \leq 1$
ii) $g(2 - x) = g(x)$ if $1 < x \leq 3$
iii) $g(x) = g(x + 4)$ for all x.
b) Write an algebraic expression for $g(x)$ like the one for $f(x)$.

MATH 294 FALL 1995 PRELIM 3 # 2

5.1.37 For the function f defined by $f(x)\begin{cases} 0, & \text{if } 0 \le x < \pi \\ 3, & \text{if } \pi \le x < 2\pi \\ f(-x), & \text{for all } x \\ f(x+4\pi), & \text{for all } x \end{cases}$

a) Calculate the Fourier series of f. Write out the first few terms of the series very explicitly.

b) Make a sketch showing the graph of the function to which the series converges on the interval $-8\pi < x < 12\pi$.

MATH 294 FALL 1995 FINAL # 4

5.1.38 For $f(x) = \begin{cases} 1 & \text{if } |x| \le c \\ 0 & c < |x| < \pi \\ f(x+2\pi) & \text{for all } x \end{cases}$

you are given the Fourier series $f(x) = \frac{c}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin nc}{n\pi} \cos nx$. Here $0 < c < \pi$.

a) Verify that the given Fourier coefficients are correct by deriving them.

b) Evaluate f and its series when $c = \pi$ and $x = \frac{\pi}{2}$, and use the result to derive the formula

$$\frac{\pi}{8} = \frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \frac{1}{18} - ....$$

c) Sketch the graph of the function to which the series converges on the interval $[-3\pi, 6\pi]$.

d) Use the series to help you solve

$$\begin{cases} u_t = u_{xx} \\ u_z(0,t) = u_x(\pi,t) = 0 \\ u(x,o) = \begin{cases} 1 \text{ if } 0 < x < c \\ 0 \text{ if } 0 < x < \pi \end{cases} \end{cases}$$

MATH 294 FALL 1996 PRELIM 3 # 2

5.1.39 For each of the following Fourier series expansion:

i) $f_i(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, -\pi < x < \pi$

ii) $f_{ii}(x) = 1 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left[(2n-1)\frac{\pi}{2}x\right], 0 < x < 2$

iii) $f_{iii}(x) = x = \pi - \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left[(2n-1)\frac{x}{2}\right], 0 \le x \le 2\pi$

a) Give the numerical value of the series at $x = -\pi/3, \pi$ and $12.5\pi = 39.3$. (9 answers required)

b) Find the Fourier series for $|x|, -2\pi < x < 2\pi$.

c) Does $\int x dx = \frac{x^2}{2} = 2\sum_{n=1}^{\infty}(-1)^n n^2(\cos nx - 1); -\pi < x < \pi$?
Does $\frac{dx}{dx} = 1 = 2\sum_{n=1}^{\infty} \cos nx; -\pi < x < \pi$?

Give one reason why you answered "yes" or "no" to these questions.

MATH 294 SPRING 1996 FINAL # 5

5.1.40 Let $u(x,y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} e^{-ny} \sin(nx)$

You are also given the Fourier Series $x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} \sin(nx)$ for $0 < x < \pi$

True or False (reason not required)

i) $u_{xx} + u_{yy} = 0$

ii) $u(0,y) = 0$

iii) $lim_{y \to \infty} u(x,y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} \sin(nx)$

iv) $u_x(0,y) = 0$

v) $u_x(\pi,y) = 0$

vi) $u(\pi,y) = 0$

vii) $\nabla^2 u = 0$

viii) $u(x,0) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} \sin(nx)$

ix) $u(x,0) = x(\pi - x)$ if $0 < x < \pi$

x) $div(\vec{\nabla} u) = 0$

MATH 294 SPRING 1997 FINAL # 4

5.1.41 Let $f(x) = 1, 0 < x < \pi$

a) Find a Fourier series for the odd (period 2π extension of $f(x)$.

b) Let $U = Span\{\sin x, \sin 2x, \sin 3x, \sin 4x, \sin 5x\}$. Find $\hat{f}$, the best approximation in U for $f(x)$ with respect to the inner product $< f, g >= \int_{-\pi}^{\pi} f(t)g(t)dt$.

c) Find a Fourier series for the even (period 2π) extension for $f(x)$.

d) Let $V = Span\{\cos x, \cos 2x, \cos 3x, \cos 4x, \cos 5x\}$. Find $\hat{f}$, the best approximation in V for $f(x)$ with respect to the same inner product as above.

MATH 294 FALL 1997 FINAL # 5

5.1.42 **a)** Find the Fourier cosine series for $f(x) = 1 + x$, on $0 \le x \le 1$.

b) Solve the equation $u_{xx} = 2u_t$, subject to the constrains $u_x(0,t) = u_x(1,t) = 0$, and $u(x,0) = 1 + x$, for $0 \le x \le 1$.

MATH 294 SPRING 1998 PRELIM 1 # 1

5.1.43 Let $f(x) = \begin{cases} 1 & -\frac{\pi}{2} < x < 0 \\ 0 & 0 < x < \frac{\pi}{2} \end{cases}$

a) Extend $f(x)$ as a periodic function, with period π. Sketch this function over several periods.

b) Compute the Fourier series for $f(x)$.

c) Write out the first three non-zero terms of the Fourier series.

d) To what values does the Fourier series converge at $x = -\frac{pi}{4}, x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$?

MATH 294 SPRING 1984 FINAL # 10

5.1.44 Consider the vector space $C-0(-\pi,\pi)$ of continuous functions in the interval $-\pi \leq x \leq \pi$, with inner product $(f,g) = \int_{-\pi}^{\pi} f(x)(g(x))^*$ where * denotes complex conjugation. Consider the following set of functions $b = \{...e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, ...\}$.

a) Are they linearly independent? (Hint: Show that they are orthogonal, that is
$(e^{inx}, e^{imx}) = 0$ for $n \neq m$
$(e^{inx}, e^{imx}) \neq 0$ for $n = m$

b) Ignoring the issue of convergence for the moment, let $f(x)$ be in $C_0(-\pi,\pi)$. Express $f(x)$ as a linear combination of the basis B. That is,

$$f = ...a_{-2}e^{-2ix} + a_{-1}e^{-ix} + a_0 + a_1e^{ix} + a_2e^{2ix} + ...$$

find the coefficients $\{a_n\}$ of each of the basis vectors. Use the results from (a).

c) How does this relate to the Fourier series? Are there coefficients $\{a_n\}$ real or complex? What if B is a set of arbitrary orthogonal functions?

MATH 294 SPRING 1996 PRELIM 3 # 2

5.1.45 **a**) Consider the function f defined by

$$f(x) = \begin{cases} 0, & \text{if } -\pi \leq x < 0, \\ 3 & \text{if } 0 \leq x < \pi, \\ f(x+2\pi), & \text{for all } x. \end{cases}$$

Calculate the Fourier series of f. Write out the first few terms of the series explicitly. Make a sketch showing the graph of the function to which the series converges on the interval $-4\pi < x < 4\pi$. To what value does the series converge at $x = 0$?

b) Consider the partial differential equation $u_t + u = 3u_x$ (which is not the heat equation). Assuming the product form $u(x,t) = X(x)T(t)$, find ordinary differential equations satisfied by X and T. (You are not asked to solve them.)

MATH 294 FALL 1992 FINAL # 7

5.1.46 For each of the following Fourier series representations,
$1 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin[(2n-1)x], 0 < x < \pi,$
$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, -\pi < x < \pi,$
$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos[(2n-1)x], 0 \leq x < \pi,$

a) Find the numerical value of the series at $x = -\frac{\pi}{3}, \pi$ and $12\pi + 0.2$ (9 answers required).

b) Find the Fourier series for $|x|, -\pi < x < \pi$. (Think - this is easy!).

MATH 294 FALL 1998 FINAL # 1

5.1.47 Consider the functions $f(x), S(x)$, and $C(x)$ defined below. [Note that $S(x)$ and $C(x)$ can be evaluated for any x even though $f(x)$ is only defined over a finite interval.]
$f(x) = 1$
$S(x) =$ the function to which the Fourier sin series for $f(x)$ converges (using $L = \pi$), and
$C(x) =$ the function to which the Fourier cos series for $f(x)$ converges (using $L = \pi$).

a) Sketch $S(x)$ over the interval $-3\pi \leq x < 3\pi$. (This can be done without finding any terms in the sin series.)
b) Sketch $C(x)$ over the interval $-3\pi \leq x < 3\pi$. (This can be done without finding any terms in the cos series.)
c) Find $S(x)$ explicitly. (This requires some simple integration.)
d) Compute $C(x)$ explicitly. (This can be done with no integration. If done with integration all integrals are trivial.)

MATH 294 SPRING 1999 PRELIM 3 # 3

5.1.48 Consider the function $f(x) = 1, 0 < x < 3$.

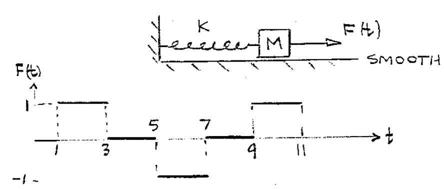
a) Calculate the Fourier sine series for $f(x)$ on $0 < x < 3$.
b) Although the function $f(x)$ is defined only over $0 < x < 3$, the Fourier sine series exists for all x. Sketch over $-6 \leq x \leq 6$ the function to which the Fourier sine series for $f(x)$ converges. (Note that you should be able to do this part even if you don't have the correct solution to part a).

MATH 294 SPRING 1983 PRELIM 3 # 1 MAKE-UP

5.1.49 Find the Fourier Cosine Series for the function $f(x) = x, 0 \leq x \leq 1$.

MATH 294 SPRING 1984 FINAL # 11

5.1.50 A simple harmonic oscillator of mass M and stiffness K is acted on by the pulsed periodic force $F(t)$ shown in the figure.



a) Determine the forced response of the oscillator (particular solution) to this excitation – in the form of an infinite series.

First note that the excitation function can be written in the form:

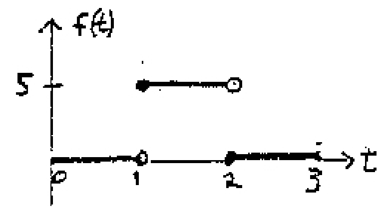
$$F(t) = \frac{4}{\pi} \sum_{i=1}^{n} \frac{\sin \frac{n\pi}{2} \sin \frac{n\pi}{4}}{n} \sin \frac{n\pi t}{4}.$$

Write a brief explanation of this representation.

MATH 294 FALL 1987 FINAL # 4 MAKEUP
5.1.51 Consider the function $f(x) = 3 \; 0 \leq x \leq \pi$

a) Compute the Fourier series of the odd extension of f on $[-\pi, \pi]$.
b) To what value does the series (obtained in (a)) converge when $x = 0, x = 1$, and $x = 54$? (3 answers required).
c) Compute the Fourier series of the even extension of f on $[-\pi, \pi]$.
d) To what value does the series (obtained in (c)) converge when $x = 0, x = 1$, and $x = 105, 326$?

MATH 294 SPRING 1987 FINAL # 8 *
5.1.52 It is claimed that the function $f(t)$ graphed below



is equal to the series

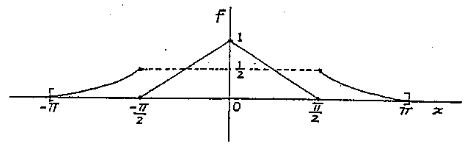
$$S(t) = \frac{a_0}{2} + \sum_{i=1}^{n} a_n \cos\left(\frac{n\pi t}{3}\right) + b_n \sin\left(\frac{n\pi t}{3}\right)$$

at all points $0 < t < 3$ except perhaps $t = 1$ and $t = 2$.

a) Extend $f(t)$ any way that you like over the whole interval $-3 < t < 3$ and graph your extension. (There are many answers to this question, and 3 particularly nice ones.)
b) For the extension you have drawn, find b_{17}.
c) What is $S(1)$?
d) What is $S(7.75)$?

MATH 294 SPRING 1988 PRELIM 1 # 1

5.1.53 A function $f(x)$ in the interval $(-\pi, \pi)$ is graphed below.



The Fourier series for this function is:

$$\frac{a_0}{2} + \frac{1}{3}\cos(x) + \frac{1}{7}\cos(2x) + a_3\cos(3x) + \ldots + b_1\sin(x) + b_2\sin(2x) + \ldots$$

a) What is the value of $\int_{-\pi}^{\pi} f(x)\cos(2x)dx$? (A number is wanted.)
b) What is the value of the Fourier Series at $x = 0$?
c) What is the value of Fourier Series at $x = \frac{\pi}{2}$?
d) What is your estimate for the value of b_1? (Any well justified answer within .2 of the instructors' best estimate will get full credit.)
e) What is your estimate for the value of $\frac{a_0}{2}$? (Any well justified answer within .2 of the instructors' best estimate will get full credit.)

MATH 294 SUMMER 1990 PRELIM 2 # 6

5.1.54 Let $f(x) = 1 - x, 0 \leq x \leq 1$.
a) Use the *even* extension of $f(x)$ onto the interval $[-1, 1]$ to get a Fourier cosine series that represents $f(x)$.
b) Sketch the graph of $f(x)$ and its even extension, and on the same graph sketch the 2^{nd} partial sum of the cosine series.

MATH 294 FALL 1990 FINAL # 5 MAKEUP

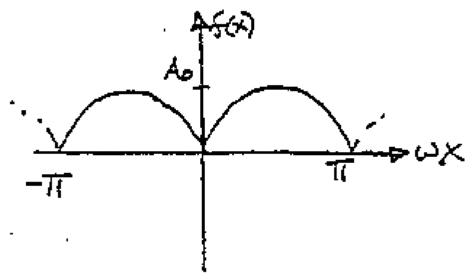
5.1.55 Given the function $f(x) = x - 1$ on $0 \leq x \leq 1$
a) Determine its Fourier cosine series. What value does this series have at $x = 0$?
b) Write down the integral forms for the coefficients a_n and b_n of the full Fourier series.

MATH 294 FALL 1993 PRELIM 3 # 1 *

5.1.56 **a**) Develop a Fourier Series for a rectified sine wave

$$f(x) = \begin{cases} A_0 \sin \omega x & 0 < \omega x < \pi \\ -A_0 \sin \omega x & -\pi < \omega x < 0 \end{cases}$$

$$\text{and } f\left(x + \frac{2\pi}{\omega}\right) = f(x)$$



b) What is the Fourier series for the (unrectified) sine wave: $f(x) = A_0 \sin \omega x$?

c) What is the value of the Fourier series in parts a.) and b.) when evaluated at $x = \frac{3\pi}{2\omega}$?

d) Comment on the derivative of $f(x)$ at $x = 0$. Assuming that the Fourier Series can be differentiated term by term, what is its derivative at $x = 0$?

MATH 294 FALL 1996 PRELIM 3 # 2 MAKE-UP

5.1.57 Let $f(x) = \pi; 0 < x < \pi$.

NOTE: (In parts a and b it is unnecessary to evaluate the integrals for any coefficients a_n or b_n, but the integrals do need to be written explicitly.

a) Express $f(x)$ as a Fourier series of period 2π that involves an infinite series of $\sin\left(\frac{n\pi x}{L}\right)$ terms alone ; $n = 1, 2, 3, \ldots$ Sketch the function to which the Fourier series converges for $-3\pi \leq x \leq 3\pi$.

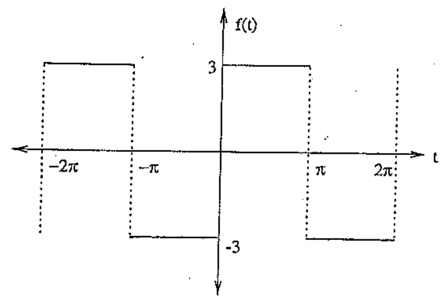
b) Express $f(x)$ as a Fourier series of period 4π that involves an infinite series of $\cos\left(\frac{n\pi x}{L}\right)$ terms alone ; $n = 1, 2, 3, \ldots$ Sketch the function to which the Fourier series converges for $-3\pi \leq x \leq 3\pi$.

c) Sketch an extension of $f(x)$ of period 6π such that the Fourier series of this $f(x)$ contains both sine and cosine terms.

d) Write the simplest possible Fourier series for $f(x)$ (i.e., one containing the fewest terms).

MATH 294 SPRING 1997 PRELIM 1 # 3

5.1.58 Consider the periodic function $f(t)$ shown in the figure below.



a) Find a general explicit expression for the Fourier sine coefficients b_n of $f(t)$

b) Find, explicitly, the first three nonzero terms in the Fourier series for $f(t)$.

5.2 General PDEs

MATH 294 SPRING 1996 PRELIM 2 # 8

5.2.1 Consider the PDE $u_t = -6u_x$

a) What is the most general solution to this equation you can find?

b) Consider the initial condition $u(x, 0) = \sin(x)$. What does $u(x, t)$ look like for a very small but not zero t?

MATH 294 SPRING 1983 PRELIM 3 # 5

5.2.2 Consider $u_t = u_x$. Which of the functions below are solutions to this equation? (Show your reasoning.)

a) $3e^{-\lambda t} \sin \sqrt{\lambda t}$

b) $3\ e^{-3t}e^{-3x} + 5e^{-5t}e^{-5x}$

c) $ae^{-3t}e^{-5x}$

d) $\sin(x)\cos(t) + \cos(x)\sin(t)$

e) $sinh^{-1}\left[(x+t)^3\right]$.

MATH 294 SPRING 1984 FINAL # 12

5.2.3 Determine if the following equation is of the form of a linear partial differential equation. If not, explain why.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0.$$

MATH 294 SPRING 1984 FINAL # 13

5.2.4 Verify that the given function is a solution of the given partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0, u(x, y) = f\left(\frac{y}{x}\right), x = 0$$

$f(\cdot)$ is a differentiable function of **one** variable.

MATH 294 FALL 1991 PRELIM 2 # 1

5.2.5 **a)** Find the general solution of
$(y + 2x \sin y dx + (x + x^2 \cos y)dy = 0$.

b) Determine the solution of the initial-value problem
$(x^2 + 4)\frac{dy}{dx} + 2xy = x$, with $y(0) = 0$.

MATH 294 SPring 1992 FINAL # 9

5.2.6 Consider the initial boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < L, t > 0$$

with the boundary conditions

$$u(0,t) = u(L,t) = 0, t \geq 0$$

and the initial condition

$$u(x,0) = f(x), 0 \leq x \leq L.$$

a) Use the method of separation of variables to derive the solution of this problem. You may use the fact that the equation $\dot{X} + \lambda X = 0, 0 < x < L$ with the boundary condition $X(0) = X(L) = 0$ has nontrivial solutions only for an interface number of constants $lambda_n = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, \ldots$. This corresponding solutions are of the form $X_n = A_n \sin \frac{n\pi x}{L}$.

b) Find the solution when $L = 1$ and $f(x) = -6 \sin 4\pi x + \sin 7\pi x$.

MATH 294 FALL 1992 FINAL # 4

5.2.7 For the PDE $u_x + 4u_y = 0$:

a) Solve it by separation of variables.

b) Show that any function of the form $u(x,y) = f(ax + y)$ is a solution if f is differentiable and the constant a is chosen correctly.

c) Solve the PDE with boundary condition $u(x,0) = \cos x$. (You may use (b) rather than (a).)

MATH 294 SPRING 1994 FINAL # 3

5.2.8 Let D be a region in the (x,y) plane and C be its boundary curve with counter-clockwise orientation. If the function $u(x,y)$ satisfies $u_{xx} + u_{yy} = 0$ in D, show that

$$\oint_C u u_x dy - u u_y dx = \int\int_D \left(u_x^2 + u_y^2\right) dx dy.$$

MATH 294 SPRING 1984 FINAL # 15

5.2.9 Show that the partial differential equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right)$$

can be reduced to

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$

by setting $u(t,x) = e^{\alpha x + \beta t} v(t,x)$ and choosing the constants α and β appropriately.

MATH 294 SPRING 1988 PRELIM 2 # 3

5.2.10 Find **one** non-zero solution to the equation below. Do not leave any free constants in your solution (that is, assign some specific numerical values to any constants in your solution). Note that you do not have to satisfy any specific initial conditions or boundary conditions.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u$$

)

MATH 294 FALL 1993 PRELIM 3 # 3

5.2.11 **a**) Find the two **ordinary** differential equations that arise from the **partial** differential equation
$\alpha^2 u_{xx} = utt$ for $0 < x < \ell, t \geq 0$ when the equation is solved by separation of variables using a separation constant $\sigma = -\lambda^2 < 0$.

b) Solve the ordinary differential equation which gives the x dependence. Use the boundary conditions $u(0,t) = u(\ell,t) = 0$ for $t \geq 0$

c) Solve the ordinary differential equation which gives the time dependence.

MATH 294 FALL 1994 PRELIM 3 # 2

5.2.12 Given the partial differential equation (P.D.E)

$$u_x + u_{yy} + u = 0,$$

a) Use separation of variables to replace the equation with two ordinary differential equations.

b) Find a non-zero solution to the P.D.E.

MATH 294 SPRING 1995 FINAL # 4

5.2.13 Consider the *first order* partial differential equation

$$u_t + cu_x = 0, -\infty < x < \infty, o < t < \infty, (*)$$

where c is a constant. We wish to solve this in two different ways.

a) Find a general solution to (*) by first writing the equation with the change of variables, $\xi = x - ct, \eta = t$.

b) Now solve (*) using a separation of variables technique. What are the units of c if x is in meters and t is in second?

c) Find $u(x,t)$ if $u(x,0) = ke^{-x^2}$.

d) Discuss the nature of your solution.

MATH 294 FALL 1995 PRELIM 3 # 3

5.2.14 While separating variables for a PDE, Professor X was faced with the problem of finding positive numbers λ and functions X which are not identically zero and

$$X''(x) = -\lambda X(x)$$

$$X(0) = 0, X'(1) = 0$$

Find the values of λ and corresponding functions X which will solve the professor's problem.

MATH 294 FALL 1995 PRELIM 3 # 4

5.2.15 For the PDE $yu_{xx} + u_y = 0$, (which is not the heat equation)

a) Assuming the product form $u(x, y) = X(x)Y(y)$, find ODE's satisfied by X and Y.

b) Find solutions to the ODE's.

c) Write down at least one non-constant solution to the PDE.

MATH 294 SPRING 1996 PRELIM 3 # 2

5.2.16 a) Consider the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 3 & \text{if } 0 \le x < \pi \\ f(x + 2\pi) & \text{for all } x. \end{cases}$$

Calculate the Fourier series of f. Write out the first few terms of the series explicitly. Make a sketch showing the graph of the function to which the series converges on the interval $-4\pi < x < 4\pi$. To what values does the series converges at $x = 0$?

b) Consider the partial differential equation $u_t + u = 3u_x$ which is not the heat equation). Assuming the product form $u(x, t) = X(x)T(t)$, find ordinary differential equations satisfied by X and T, (You are not asked to solve them.)

MATH 294 SPRING 1996 FINAL # 6

5.2.17 Consider the equation for a vibrating string moving in an elastic medium

$$a^2 u_{xx} - b^2 u = u_{tt}$$

where a and b are constants. (a would be the wave speed if not for the elastic constant b.) Assume the ends are fixed at $x = 0, L$ and initially the string is displaced by $u(x, 0) = f(x)$, but not moving $u_t(x, o) = 0$.

a) Find a general solution for these conditions. (If you need help, you may wish to work part (b) first.)

b) If the first term in the general solution to part (a) is

$$u_1(x, t) = c_1 \cos(\lambda_1 t) \sin\left(\frac{\pi x}{L}\right)$$

where $(\lambda_1)^2 = \left(\frac{\pi a}{L}\right)^2 + b^2$, find the solution when the string starts from $u(x, 0) = 2\sin\left(\frac{\pi x}{L}\right)$

MATH 294 SPRING 1997 FINAL # 9

5.2.18 Consider the PDE

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t);\ 0 \leq x < \pi;\ t \geq 0 \tag{7}$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0 \tag{8}$$

$$u(x,0) = \sin(x) \tag{9}$$

a) Define

$$v(x,t) = e^t u)x,t) \tag{10}$$

If $u(x,t)$ satisfies equations (7, 8 ,9), show that $v(x,t)$ satisfies the standard heat equation

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} \frac{\partial v(x,t)}{\partial t} \tag{11}$$

with boundary conditions and initial conditions

$$v(0,t) = v(\pi,t) = 0 \tag{12}$$

$$v(x,0) = \sin x \tag{13}$$

b) The general solution of equations (11,12) is

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin nx \tag{14}$$

Find the unique solution $v)x,t)$ of equations (11,12,13).

c) Now find the unique solution of equations (7, 8, 9).

MATH 294 FALL 1992 FINAL # 7

5.2.19 For each of the following Fourier series representations,

$1 = \frac{4}{\pi}\sum^i nfty_{n=1} \frac{1}{2n-1} \sin[(2n-1)x],\ 0 < x < \pi,$

$x = 2\sum^i nfty_{n=1} \frac{(-1)^{n+1}}{n} \sin nx,\ -\pi < x < \pi,$

$x = \frac{\pi}{2} - \frac{4}{\pi}\sum^i nfty_{n=1} \frac{1}{(2n-1)^2} \cos[(2n-1)x],\ 0 \leq x < \pi.$

a) Find the numerical value of the series at $x = -\frac{\pi}{3}, \pi$ and $12\pi + 0.2$ (9 answers required).

b) Find the Fourier series for $|x|,\ -\pi < x < \pi$. (Think - this is easy!).

MATH 294 FALL 1992 FINAL # 8

5.2.20 Solve the initial-boundary-value problem

$$T_t = T_{xx},\ 0 < x < \pi,\ t > 0,$$

$$T_x(0,t) = T_x(\pi,t) = 0,$$

$$T(x,0) = 3x.$$

(You may use information from problem 7 if this helps.)

MATH 294 SPRING 1998 PRELIM 1 # 5

5.2.21 Consider the partial differential equation for $u(x,t)$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with the initial conditions

$$u(x,0) = 1 - x,\ 0 \le x \le 1,\ u(x,0) = 0 \text{ elsewhere.}$$

(No boundary conditions are necessary).
Use a centered difference approximation for the space derivative:

$$\frac{\partial u}{\partial x}(x,t) \approx \frac{1}{2h}[u(x+h,t) - u(x-h,t)],$$

and a forward difference approximation for the time derivative:

$$u(x,t+k) \approx u(x,t) + k\frac{\partial u}{\partial t}(x,t).$$

Then introduce a grid with $N+1$ spatial points $x_i = ih$, with $i = 0,1,2,\dots N$ (where h is the grid spacing) and times $t_j = jk$ (where k is the time step). Let $u(x_i,t_j) \equiv u[i,j]$.

a) With $h = 0.25$, and $k = 0.25$, write down the values of the initial condition at each grid point for $0 \le x \le 2$, i.e. $u[i,0],\ i = 0,\dots,8$.

b) Obtain the expression relating $u[i,j+1]$ to $u[i-1,j]$ and $u[i+1,j]$.

c) Use the initial data (with $h = 0.25$ and $k = 0.25$) to determine the approximate the value of u at $x_i = 1.0,\ t_k = 0.5$.

MATH 294 FALL 1998 FINAL # 3

5.2.22 Consider the partial differential equation $\frac{\partial u}{\partial x} - \beta \frac{\partial u}{\partial t} = 0$ (Note that this is not the heat equation.) with the initial condition $u(x, 0) = x^2$. In an approximate solution u is to be evaluated on a grid of points spaces by h on the x axis and δt on the t axis: $x_i = (i - 1)h$ and $t_j = (j - 1)\delta t$. The values of $u(x_i, t_j)$ are contained in the array $\hat{u}_{ij} \equiv \hat{u}(i, j) \equiv u(x_i, t_j)$. Here are the forward difference approximations: $\frac{\partial u(x,t)}{\partial x} = \frac{1}{h}[u(x + h, t) - u(x, t)]$ and $\frac{\partial u(x,t)}{\partial t} = \frac{1}{\delta t}[u(x, t + \delta t) - u(x, t)]$

a) Derive a finite difference algorithm for this equation. That is, find an expression for $\hat{u}(i, j + 1)$ in terms of $\hat{u}(i, j)$ and $\hat{u}(i + 1, j)$.

b) Let $h = 1, \delta t = \frac{1}{2}$, and $\beta = 1$. Use your approximate scheme above and the given initial condition to find approximate values for $u\left(2, \frac{1}{2}\right)$, $u\left(3, \frac{1}{2}\right)$, and $u(2, 1)$.

c) Find the exact solution to the partial differential equation and given boundary condition. (Do not waste time with Fourier series formulae).

MATH 294 SPRING 1999 PRELIM 3 # 2

5.2.23 Parts a), b), and c) are related, but each part can be done independently of the other parts. Consider the following problem consisting of a PDE for $u = u(x, t)$, two B.C.'s and an I.C.:

$$\frac{\partial^2 u}{\partial x^2} = u + \frac{\partial u}{\partial t}$$

$$\text{B.C.'s: } u(0, t) = 0, \frac{\partial u(2, t)}{\partial x} = 0, t > 0$$

$$I.C.: u(x, 0) = \sin \frac{\pi x}{4},\ \ 0 \leq x \leq 2.$$

a) Use separation of variables on the PDE to obtain two ODE's for $X(x)$ and $T(t)$, and the B.C. for $X(x)$.

b) In some separation of variables problem, a student obtained the following ODE plus B.C.'s on $X(x)$:

$$\frac{d^2 X}{dx^2} + \lambda X = 0\ X(0) = 0, \frac{dX(2)}{dx} = 0.$$

Find all nontrivial solutions to the ODE with these B.C.'s.

c) Which, if any, of the equations given below is a solution to the PDE's, B.C.'s and I.C. at the top of the page? (Justification of your answer is required to get credit. Note that you may have to check several boundary conditions as well as the PDE.)

i) $u(x, t) = e^{\left(-1-\frac{\pi^2}{16}\right)t} \sin \frac{\pi x}{4}$

ii) $u(x, t) = e^{-\frac{\pi^2 t}{16}} \sin \frac{\pi x}{4}$

iii) $u(x, t) = e^{-\frac{\pi^2 t}{16}} \cos \pi x$

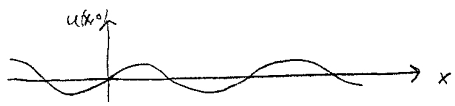
iv) $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 \pi^2 t}{16}} \sin \frac{n\pi x}{4},\ b_n = \int_0^2 sin\left(\frac{\pi x}{4}\right) sin\left(\frac{n\pi x}{2}\right)$

MATH 294 SPRING 1983 PRELIM 3 # 2

5.2.24 Consider the PDE $u_t = -6u_x$.

a) What is the most general solution to this equation you can find?

b) Consider the initial condition $U(x, 0) = \sin(x)$. What does $u(x, t)$ look like for a very small but not zero t?



MATH 294 FALL 1987 PRELIM 2 # 5 MAKE-UP

5.2.25 Find the solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \begin{array}{l} 0 < x < 1 \\ 0 < y < 1 \end{array} ,$$

$$u(0, y) = u(x, 0) \equiv 0,$$

$$u(1, y) = u(x, 1) \equiv 1.$$

MATH 294 SPRING 1996 FINAL # 7 MAKE-UP

5.2.26 Solve the problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$u(1, \theta) = 4\sin(\theta) - 3\cos(2\theta)$$

$$\lim_{r\to 0} u(r, \theta) = 0$$

You may use the fact that $r^{\pm n}\cos(n\theta)$ and $r^{\pm n}\sin(n\theta)$ are some of the solutions to this partial differential equation.

MATH 294 FALL 1996 PRELIM 3 # 3 MAKE-UP

5.2.27 **a**) Find all solutions of the form $u(x, t) = X(x)T(t)$ for

$$xu_x = 2u_t \ ;$$

subscripts indicate differentiation with respect to that variable.

b) If $u(x,0) = 2x - 3x^2$, find $u(x,t)$.

MATH 294 SPRING 1996 FINAL # 7

5.2.28 In the problem below, choose the solution that corresponds to the given physical problem. Justify your choice. (Note that a sketch is required with (a), and that to make the right choices you will probably have to check several of the boundary/initial conditions as well as the appropriate partial differential equation.)

a) A taut string stretching to infinity in both directions has a wave speed a and an initial displacement $y(x,0) = \frac{1}{(1+8x^2)}$ but no initial velocity.

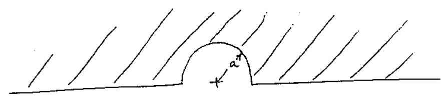
i) $y(x,t) = \frac{\frac{1}{2}}{1+8(x-at)^2} + \frac{\frac{1}{2}}{1+8(x+at)^2}$

ii) $y(x,t) = \frac{1}{1+8x^2} + \frac{\frac{1}{2}}{1+8(x-at)^2} + \frac{\frac{1}{2}}{1+8(x+at)^2}$

iii) $y(x,t) = \sum\limits_{i=1}^{n} c_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi at}{L}\right)$ where $c_n = \frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

iv) (no choice, do this) Plot the solution initially and when $at = 1$

b) The steady state temperature exterior to a semicircular hole $(r > a, 0 < \theta < \pi)$ with boundary conditions $u(r,0) = 0\ andu(r,\pi) = 0$ for $a < r < \infty$ and $u(a,\theta) = f(\theta)$ and $\lim_{r\to\infty} u(r,\theta) = 0$ for $0 \leq \theta \leq \pi$.



In all of the choices, $c_n = \frac{2a^n}{\pi}\int_0^\pi f(\theta)\sin(n\theta)d\theta$.

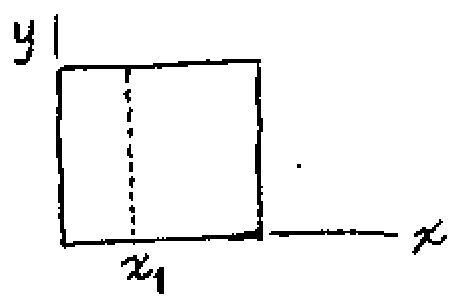
i) $u(r,\theta) = \sum\limits_{i=1}^{n} c_n r^{-n}\sin(n\theta)$

ii) $u(r,\theta) = \sum\limits_{i=1}^{n} c_n r^{-n}\cos(n\theta)$

iii) $u(r,\theta) = \sum\limits_{i=1}^{n} c_n r^{n}\sin(n\theta)$

iv) $u(r,\theta) = \sum\limits_{i=1}^{n} c_n r^{n}\cos(n\theta)$

c) A square copper plate with sides L has all four edges maintained at $0°$



A line across the plate at $x = x_1, 0 < y < L$ is heated to T_1 by an external heat source until a steady state results. The temperature in the plate is:

i) $T = \begin{cases} T_1 \frac{x}{x_1} & \text{if} \quad x \leq x_1 \\ T_1 \frac{L-x}{L-x_1} & \text{if} \quad x > x_1 \end{cases}$

ii) $T = \sum_{i=1}^{n} b_n e^{-\frac{n^2\pi^2 kt}{L}}$ where $b_n = \frac{1}{L}\int_0^L \frac{T_1 x}{x_1} \sin\left(\frac{n\pi x}{L}\right) dx$

iii) $T = \frac{4T_1}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi(L-y)}{L}\right)}{n\sinh(n\pi)}$

iv) None of the above.

5.3 Laplace Equation

MATH 294 FALL 1982 FINAL # 6

5.3.1 **a)** Determine the solution to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on $0 < x < 1,\ 0 < y < 1$, subject to the boundary conditions $u(0, y) = u(x, 0) = u(1, y) = 0$, $u(x, 1) = 2x$

b) Use this solution and the linearity of Laplace's equation to obtain the solution to the boundary value problem $u(0, y) = u(x, 0) = 0$, $u(1, y) = 2y$, $u(x, 1) = 2x$.

MATH 294 SPRING 1983 FINAL # 6

5.3.2 Solve Laplace's equation on the rectangle $0 \le x \le 4, 0 \le y \le 3$ with the given boundary conditions.

$u_{xx} + u_{yy} = 0,$
$u(x, 0) = u(0, y) = 0,$
$u(4, y) = 2\sin\left(\frac{\pi y}{3}\right),$
$u(x, 3) = 5\sin\left(\frac{\pi x}{4}\right)$

MATH 294 SPRING 1990 PRELIM 3 # 6

5.3.3 **a)** Solve Laplace's equation $\delta^2 u = 0$ on the square $0 < x < \pi, 0 < y < \pi$, subject to $u = 0$ on the three sides $x = 0, y = 0$ and $x = \pi$, and $u(x, \pi) = g(x)$, where g is defined in 4(b). (Hint: $u_n(x, y) = \sin(nx)\sinh(ny), n = 1, 2, \dots$)

b) Repeat (a) if the b.c. $u(\pi, y) = 0$ is replaced by $u(\pi, y) = 2\sin 3y - 14\sin 9y$

MATH 294 FALL 1993 FINAL # 5

5.3.4 The solution to any partial differential equation depends on the domain in which the solution is to be valid as well as the boundary conditions and/or initial conditions that the solution must satisfy. We wish to consider physically plausible (i.e. u does not approach ∞) solutions to Laplace's equation in circular regions, $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$, for various boundary conditions. Below you are given some functions that satisfy the Laplace equation and certain conditions. You are asked to choose (and to give arguments that led to that choice) what problem is being discussed. You do not necessarily have to solve any equation.

a) $u(r,\theta) = fracC_n 2 + \sum_{i=1}^{n} r^{-n}(C_n \cos n\theta + K_n \sin n\theta)$

$C_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$; $K_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$

i) a disk $(r \le a)$ with $u(a,\theta) = f(\theta)$

ii) an annulus of inner radius a $(a \le r < infty)$ with $\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$

iii) an annulus of inner radius a $(a \le r < infty)$ with $u(a,\theta) = f(\theta)$

iv) none of above

b) $u(r,\theta) = \sum_{i=1}^{n} C - nr^n \sin n\theta$; $C_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta d\theta$

i) An annulus $(a \le r < \infty)$, satisfying $u(a,\theta) = f(\theta),\ 0 \le \theta < \pi$

ii) A half-disk $(r \le a;\ 0 \le \theta \le \pi)$, with $u(r,\pi) = 0, u(a,\theta) = f(\theta)$

iii) A disk $(r \le a)$ with $u(a,\theta) = f(\theta)$ for $0 \le \theta < \pi$

iv) none of the above

c) $u(r,\theta) = \frac{u_b \ln\left(\frac{r}{a}\right) + u_b \ln\left(\frac{b}{r}\right)}{\ln\left(\frac{b}{a}\right)}$

i) a pie-shaped wedge $(0 \le r \le a)$ of angle $\tan\theta_0 = \frac{b}{a}$ with $u(a,\theta) = u_a\theta + u_b\left(\frac{b}{r}\right)$

ii) An annulus $(a \le r \le b)$ with constant specified values on the inner and outer bounding circles

iii) a disk $(r \le a)$ satisfying $u(a,\theta) = u_a + \theta u_b$

iv) none of the above

d) Describe a situation where Laplace's equation arises in cartesian coordinates. Discuss the meaning of the appropriate boundary and initial conditions for the problem that you have chosen.

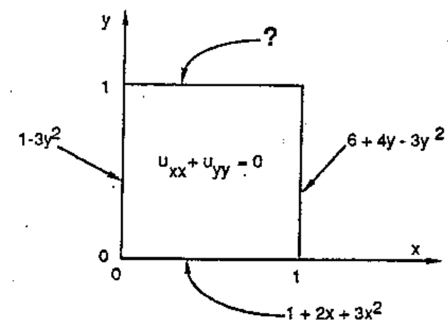
MATH 294 SPRING 1988 PRELIM 2 # 4

5.3.5 **a**) Find constants $A, B, C, D, E,$ and F (real numbers) so that the function

$$u = A + Bx + Cx^2 + Dx^3 + Ey^2 + Fxy$$

is
i) a solution to Laplace's equation in the rectangle shown, and
ii) satisfies the three boundary conditions shown.

b) What boundary condition is satisfied on the fourth boundary?



UNKNOWN UNKNOWN UNKNOWN # UNKNOWN

5.3.6 Solve: $U_{xx} + u_{yy} = 0, 0 < x < \pi, 0 < y < 1$ subject to the boundary conditions $u_x(0, y) = u_x(\pi, y) = 0, 0 \leq y \leq 1, u(x, 0) = 4\cos(6x) + \cos(7x), 0 \leq x \leq \pi, u(x, 1) = 0, 0 \leq x \leq \pi$.

MATH 294 SPRING 1996 FINAL # 6 MAKE-UP

5.3.7 Consider the Laplace equation in the semi-infinite strip $0 < x < L, y > 0$, with boundary conditions $u(0, y) = 0, u(L, y) = 0, u(x, 0) = f(x)$, and u must not approach ∞ as y approaches ∞.

a) Find a general solution for these conditions.

b) Write out the solution for this problem in the case that $f(x) = x$. You are given the Fourier sine and cosine series for $x, (0 < x < L)$

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$x = \frac{2L}{\pi} \sum_{i=1}^{n} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

MATH 294 FALL 1987 FINAL # 3 MAKE-UP
5.3.8 Find the solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\left\{\begin{array}{l} 0<x<\pi \\ 0<y<\pi \end{array}\right.,$$

$$\frac{\partial u}{\partial x}(0,y)=\frac{\partial u}{\partial x}(\pi,y)=0,\ 0<y<\pi,$$

$$u(x,0)=0, u(x,\pi)=9,\ 0<x<\pi.$$

5.4 Heat Equation

MATH 294 SPRING 1985 FINAL # 1

5.4.1 **a**) Find the solution to the partial differential equation given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \text{where} u = u(x,t)$$

with boundary conditions
$u(0,t) = 0$
$u(L,t) = 0$
and initial condition
$u(x,0) = 1;\ 0 < x < L$

b) What does the solution $u(x,t)$ approach as $t \to \infty$. Briefly explain this answer. (No credit will be given for guessing.)

MATH 294 FALL 1986 FINAL # 10

5.4.2 Solve for $u = u(x,t)$ where

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, (0 < x < 2, t > 0),$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(2,t) = 0 \end{array}\right\} t > 0$$

$$u(x,0) = \cos(2\pi x), 0 \leq x \leq 2.$$

MATH 294 FALL 1986 FINAL # 13

5.4.3 Find the solution to the initial/boundary value problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < L, t > 0$$

$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, t > 0$$

$$u(x,0) \equiv 1, 0 < x < L.$$

You may use symmetry to solve a more familiar problem on $0 < x < 2L$.

MATH 294 FALL 1987 PRELIM 1 # 8

5.4.4 Find **any** non-zero solution to the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

that satisfies the boundary conditions $u(0,t) = u(5,t) = 0$. (You need not go into great detail explaining how you find this solution.)

MATH 294 SPRING 1983 PRELIM 3 # 3

5.4.5 Let $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < 10$ have boundary conditions

$$u_x(0,t) = u_x(10,t) = 0$$

a) What is the most general solution to this boundary value problem you can find.
b) Given $u(x,0) = x^2$, what is $u(5, t \to \infty)$ in the above problem?

MATH 294 SPRING 1983 PRELIM 3 # 4

5.4.6 Let $u_t = u_{xx}, 0 < x < \pi$ have boundary conditions $u_x(0,t) = 0, u(\pi,t) = 0$.
a) What is the most general solution you can find to this equation and the given boundary conditions.
b) For the initial condition $u(x,0) = x^2 - e^x \sin x$ what is $u(1, t \to \infty)$?

MATH 294 SPRING 1983 FINAL # 4

5.4.7 A bat of length 1 is assumed to satisfy the heat equation with $\alpha^2 = 1$. The ends of the bar are in ice water at temperature $u = 0$. At time $t = 0$ the temperature of the bar is $u(x,0) = 100 \sin \pi x$. What is the temperature in the middle of the bar at $t = 2$?

MATH 294 SPRING 1984 FINAL # 16

5.4.8 Heat conduction in a closed-loop wire of radius 1 can be described by $u(t,x)$ where

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\ 0 \leq x < 2\pi,\ t > 0$$

and $u(t, x = 0) = u(t, x = 2\pi)\ \ t \geq 0$. The initial distribution of temperature is

$$u(t = 0, x) = \begin{cases} 0 & 0 < x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x \leq \frac{3\pi}{2} \\ 0 & \frac{3\pi}{2} < x \leq 2\pi \end{cases}$$

find $u(t,x)$ by separation of variables, what is the temperature distribution as $t \to \infty$?

MATH 294 FALL 1984 FINAL # 14

b) Solve the following initial-boundary value problem for the heat equation

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \; 0 \leq x \leq L, \; t \geq 0$

$\frac{\partial u}{\partial x}(0,t) = 0, \; \frac{\partial (}{\partial L}, t) = 0$

$u(x,0) = f(x)$ where $f(x)$ is the function given in IV A.

c) Determine the value of $H(t)$ given by $H(t) = \int_0^L u(x,t)dx$. If you have been unable to find the solution in IV B, then form $\frac{\partial H}{\partial t}$ and use the differential equation for u to find an ordinary differential equation for H.

MATH 294 FALL 1987 PRELIM 3 # 3

5.4.9 Find the solution of the initial-boundary-value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; 0 < x < \frac{\pi}{2}, \; t > 0,$

$u(0,t) = \frac{\partial u}{\partial x}\left(\frac{\pi}{2}, t\right) = 0 \; t > 0,$

$u(x,0) = 7\sin x - 14\sin 9x$

(Hint: symmetry and superposition)

MATH 294 FALL 1987 PRELIM 2 # 5

5.4.10 Find the solution of the initial-boundary-value problem

5.4.11 Find the solution of the initial-boundary-value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; 0 < x < \pi, \; t > 0,$

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \; t > 0,$

$u(x,0) = 10\sin{}^2 4x$

MATH 294 FALL 1987 PRELIM 3 # 4

5.4.12 a) Find the solution of the initial-boundary-value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; 0 < x < \pi, \; t > 0,$

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \; t > 0,$

$u(x,0) = 1 + x \;\; 0 < x < \pi$

b) What is $\lim_{t\to\infty} u(x,t)$, where $u(x,t)$ is the solution of part (a).

c) Show that your result from part (b) is a time-independent (equilibrium) solution of the first two equations in part (a).

MATH 294 FALL 1988 PRELIM 3 # 5

5.4.13 Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $\frac{\partial 0,t}{\partial =}0$ and $\frac{\partial u}{\partial x}(\pi, t) = 0$ and initial condition $u(x,0) = 4\cos(x) - 5\cos(4x)$.

MATH 294 FALL 1989 FINAL # 5

5.4.14 Let $\alpha > 0$. Find the solution $u(x,t)$ of

$u_{xx} = \frac{1}{\alpha}u_t, \; t > 0 \;, 0 < x < 1,$

$u_x(0,t) = u_x(1,t) = 0, \; t > 0,$

$u(x,0) = 1 + 3\cos(2\pi x) - 2\cos(5\pi x), \; 0 < x < 1.$

MATH 294 SPRING 1991 FINAL # 2

5.4.15 Consider the conduction of heat through a wire of unit length that is insulated on its lateral surface and at its ends.

a) Use the method of separation of variables to show that the solution of the initial value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $0 \le 1 \le 0, 0 \le t \le \infty$;

with $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$; and $u(x,0) = f(x)$

is given in the form

$u(x,t) = \frac{a_0}{2} + \sum_{i=1}^{n} a_n e^{-n^2\pi^2 t} \cos n\pi x.$

Hint: The equation $X'' + \lambda X = 0, 0 \le x \le 1$, with $X'(0) = X'(1) = 0$ has nonzero solutions only for an infinite number of constants $\lambda = n^2\pi^2$, for $n = 0, 1, 2, 3, \dots$. The corresponding solutions are $X_n(x) = A_n \cos n\pi x$.

MATH 294 FALL 1994 FINAL # 5

5.4.16 Solve

$$\begin{cases} u_t = u_{xx} \text{on} 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \begin{cases} 1 & \frac{\ell}{4} \le x \le \frac{3\ell}{4} \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

You may leave Fourier coefficients unsimplified after doing the integrals.

MATH 294 SPRING 1993 FINAL # 5

5.4.17 a) The solution to

$$\begin{aligned} U_{tt} &= u_{xx}; \;\; -\infty < x < \infty \\ u(x,0) &= e^{-x^2} \\ u_t(x,0) &= 0 \end{aligned}$$

is the form $u(x,t) = \varphi(x+t) + \varphi(x-t)$. Find the solution without using Fourier series.

b) Find the solution of

$$\begin{aligned} u_{xx} &= u_t \quad 0 \le x \le 1 \\ u(0,t) &= 1 \\ u(1,t) &= 2 \\ u(x,0) &= 1 + x \end{aligned}$$

Hint: The solution may be time-independent.

MATH 294 SPRING 1994 FINAL # 7

5.4.18 Consider

$$\begin{aligned} u_{xx} &= u_t, \;\; 0 < x < 1, \;\; t > 0 \\ u(0,t) &= 0, \\ u(1,t) &= -1, \\ u(x,0) &= 0 \end{aligned}$$

a) Find $v(x)$ if $u(x,t) \to v(x)$ as $t \to \infty$

b) Find a Fourier series solution for $u(x,t)$.

MATH 294 SPRING 1995 PRELIM 3 # 2

5.4.19 Consider the condition of heat through a wire of unit length that is insulated on its lateral surface and at its ends. This implies boundary conditions $u_x(0,t) = 0 = u_x(1,t), t \geq 0$.

a) Verify that solutions $u(x,t)$ to the heat equation with the initial condition $u(x,0) = f(x)$ piecewise continuous first derivatives may be given in the form

$$u(x,t) = \frac{a_0}{2} + \sum_{i=1}^{n} a_n e^{-n^2\pi^2 t} \cos(n\pi x)$$

b) Find $u(x,t)$ when $f(x) = 2 + 5\cos(3\pi x)$.?

MATH 294 SPRING 1996 PRELIM 3 # 3

5.4.20 Consider the heat equation

$$\begin{aligned} u_t &= 0.04 u_{xx} \\ u(x,0) &= \sin\left(\frac{\pi x}{2}\right) - \frac{1}{2}\sin(\pi x) \\ u(0,t) &= 0 \\ u(2,t) &= 0 \end{aligned}$$

a) Find the solution to this problem.

b) Verify by substitution that your answer to part (a) does in fact satisfy all four of these equations. (You can get full credit for this part by checking everything, even if your answer to (a) is *wrong*.

MATH 294 FALL 1996 PRELIM 3 # 3

5.4.21 **a**) Find the full solution to

$$\begin{aligned} u_{xx} &= ut \;\; 0 < x < \pi, \;\; t > 0 \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= x \end{aligned}$$

You may find problem 2 helpful in solving this.

b) Using only the first two terms in your solution, write out $u(x,0)$ and $u(x,1)$. Sketch these terms and their sum. Comment on your plot.

$$\begin{aligned} e^{0} &= 1 \\ e^{-1} &= .368 \\ e^{-2} &= .135 \\ e^{-3} &= .050 \\ e^{-4} &= 0.018 \\ e^{-5} &= .007 \end{aligned}$$

MATH 294 SPRING 1998 PRELIM 1 # 3

5.4.22 Consider the one dimensional heat transfer problem

$$u_{xx} = u_t, \; 0 \leq x \leq 1$$

with boundary conditions $u(0,t) = 0, u(1,t) = 1, t > 0$,
and initial conditions $u(x,0) = 0, \; 0 \leq x \leq 1$.

a) Find the long time, i.e. time independent, solution reached as $t \to \infty$.

b) Find the time-dependent solution $u(x,t)$ that satisfies the given boundary and initial conditions.

MATH 294 SPRING 1998 FINAL # 2

5.4.23 Find the solution of the initial value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \text{on} 0 < x < 1 \text{for} t > 0,$$

with

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$

and

$$u(x,0) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

MATH 294 FALL 1998 FINAL # 2

5.4.24 Solve, for $t > 0$ and $0 < x < \pi$, the partial differential equation $2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with the boundary conditions that $u(0,t) = 0,\ u(\pi,t) = \pi$,
and the initial condition that $u(x,0) = x + \sin(x)$.
(This problem can be solved completely without any integrations.)

MATH 294 UNKNOWN 1990 UNKNOWN # 5

5.4.25 Solve the initial boundary value problem

$$u_t = u_{xx},\ 0 < x < \pi, T > 0,$$

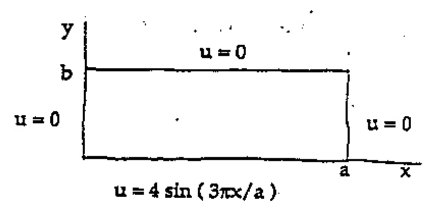
$$u(0,t) = u_x(\pi,t) = 0,$$

$$u(x,0) = 18\sin\left(\frac{9x}{2}\right).$$

(Hint: $u(x,y) = X(x)T(t)$; you may use the fact that the only nontrivial solution here occurs when X is a linear combination of a cosine and a sine.)

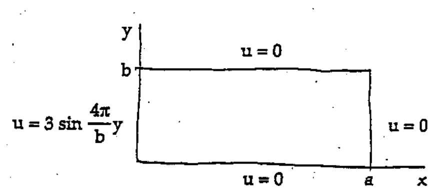
MATH 294 FALL 1990 FINAL # 5

5.4.26 Find the steady state temperature distribution in the plate shown.



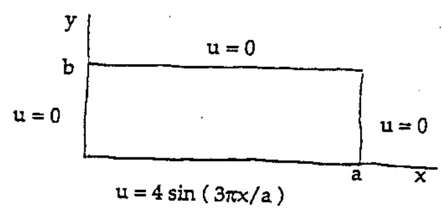
MATH 294 FALL 1990 FINAL # 4 MAKE-UP

5.4.27 Find the steady state temperature distribution in the plate shown.



MATH 294 FALL 1994 PRELIM 1 # 5

5.4.28 Find the steady state temperature distribution in the plate shown.



5.5 Wave

MATH 294 SPRING 1983 FINAL # 5

5.5.1 **a)** Solve the wave equation with wave speed $c = 1$, boundary conditions: $u(0,t) = u(6,t) = 0$ and initial conditions $u(x,0) = 0, u_t(x,0) = 5\sin\left(\frac{\pi x}{3}\right)$.

b) Make a clearly labeled graph of $u(3,t)$ vs. t for your solution in part (a) above.

MATH 294 SPRING 1994 FINAL # 14

5.5.2 Verify that $u(t,x) = \frac{1}{2}[f(x+t) + f(x-t)]$ solves the initial value problem:

$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad t > 0, \quad -\infty < x < \infty,$
$u(t=0,x) = f(x)$
$u_t(t=0,x) = 0.$

MATH 294 FALL 1986 FINAL # 9

5.5.3 **a)** Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $(0 < x < 1, 0 < y < 1)$ where $u = u(x,y)$ and $u(0,y) = 0, u(1,y) = 0, u(x,0) = 0$, and $u(x,1) = 2\sin(2\pi x)$.

b) Use your result from part (a) to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, (0 < x < 1, 0 < y < 1)$$

where $u = u(x,y)$ and $u(0,y) = 0, u(1,y) = 2\sin(2\pi y), u(x,0) = 0, u(x,1) = 0$.

c) Use your result from part (a) and (b) to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, (0 < x < 1, 0 < y < 1)$$

where $u = u(x,y)$ and $u(0,y) = 0, u(x,0) = 0, u(1,y) = 2\sin(2\pi y), u(x,1) = 2\sin(2\pi x)$.

MATH 294 FALL 1986 FINAL # 12

5.5.4 Find the solution to the initial/boundary value problem

$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < L, t > 0$
$u(0,t) = u(L,t) = 0, t > 0$
$u(x,0) = 0, 0 < x < L$
$\frac{\partial u}{\partial t}(x,0) = \sin\left(36\pi \frac{x}{L}\right), 0 < x < L.$

MATH 294 SPRING 1987 PRELIM 2 # 2

5.5.5 Find the value of u at $x = t = 1$ if $u(x,t)$ satisfies:

$$\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial^2 u}{\partial x^2}$$

$$0 = u(0,t) = u(3\pi,t)$$

with

$$u(x,0) = \sin(5x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sin x$$

MATH 294 SPRING 1987 FINAL # 7

5.5.6 Find *any* non-zero solution $u(x,t)$ to

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \text{with} 0 = \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t)$$

and the extra restriction that $u(0,0) \neq u(1,0)$.

MATH 294 FALL 1987 PRELIM 2 # 3

5.5.7 Find the solution of the initial-boundary-value problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 \quad t > 0$$
$$u(0,t) = u(1,t) = 0 \quad t > 0$$
$$\left.\begin{aligned} u(x,0) &= 0 \\ \frac{\partial u}{\partial t}(x,0) &= x. \end{aligned}\right. \quad 0 < x < 1$$

MATH 294 SPRING 1988 PRELIM 2 # 5

5.5.8 Once released, the deflection u of a taught string satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

where x is position along the string and t is time. It is held fixed (no deflection) at its ends at $x = 0$ and $x = 2$. At time $t = 0$ it is released from rest with the deflected shape $u = 3\sin\left(\frac{\pi x}{2}\right)$. Make a plot of $u(1,t)$ versus t for $0 \leq t \leq 2$. Label the axes at points of intersection with the curve. (You may quote any results that you remember.)

MATH 294 FALL 1991 FINAL # 3

5.5.9 The displacement $u(x, y)$ of a vibrating string satisfies

$$\frac{\partial^2 x}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

in $0 \leq x \leq 4,\ t \geq 0$ and the boundary and initial conditions

$$u(0,t) = 0, u(4,t) = 0, u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = f(x),$$

where

$$f(x) = \begin{cases} 1, & \text{when } 0 \leq x \leq 2 \\ 0, & \text{when } 2 \leq x \leq 4 \end{cases}$$

a) Find a series representation for the solution.
b) Write down the equation for the displacement of the string at $t = 4$.

MATH 294 FALL 1993 FINAL # 14

5.5.10 a) Solve the wave equation $(a^2 u_{xx} = u_{tt}$ to find the displacement $u(x,t)$ of an elastic string of length ℓ. Both ends of the string are always free $[u_x(0,t) = 0;\ u_x(\ell,t) = 0]$ and the string is set in motion from its equilibrium position, $u(x,0) = 0$, with an initial velocity, $u_t(x,0) = V_0 \cos\frac{3\pi x}{\ell}$. Assume for this problem that it is legitimate to differentiate any Fourier series term-by-term. If you use separation of variables, consider only the case with a negative separation constant.
b) Write the solution to the wave equation $\left(a^2 u_{xx} = u_{tt}\right)$ for the boundary conditions $u(0,t) = h_L$ and $u(\ell,t) = h_R$ with initial conditions $u(x,0) = h_L + (h_R - h_L)\frac{x}{\ell}$ and $u_t(x,0) = 0$.

MATH 294 SPRING 1994 FINAL # 8

5.5.11 Consider

$$u_{xx} = u_{tt} \quad -\infty < x < \infty,\ \ t > 0$$

$$u(x,0) = 0, u_t(x,0) = g(x),$$

where $g(x)$ is a given function.
a) Show that $u(x,t) = G(x+t) - G(x-t)$ satisfies the above wave equation and initial conditions for a suitable function $G(x)$. How are $G(x)$ and $g(x)$ related?
b) Find $u(x,t)$ if $u_t(x,0) = g(x) = \frac{x}{1+x^2}$.

MATH 294 SPRING 1993 FINAL # 15

5.5.12 **a**) The solution to

$u_{tt} = u_{xx} \quad -\infty < x < \infty$
$u(x, 0) = e^{-x^2}$
$u_t(x, 0) = 0$ is of the form $u(x, t) = \varphi(x + t) + \varphi(x - t)$. Find the solution without using Fourier series.

b) Find the solution of

$u_{xx} = u_t \;\; 0 \le x \le 1$
$u(0, t) = 1$
$u(1, t) = 2$
$u(x, 0) = 1 + x$

Hint: The solution may be time-independent.

MATH 294 FALL 1995 FINAL # 15

5.5.13 If $u(x, t) = F(x + t) + G(x - t)$ for some functions F and G,

a) Find expressions for $u(x, 0)$ and $u_t(x, 0)$ in terms of F and G.

b) If also $\begin{cases} u_{tt} = u_{xx} \quad -\infty < x < \infty \\ u(x, 0) = e^{-x^2} \\ u_t x, 0 = 0 \end{cases}$ find expressions for F and G, and sketch the graph of $u(x, t)$ when $t = 0, 1$, and 2.

MATH 294 SPRING 1998 PRELIM 1 # 4

5.5.14 Consider the following partial differential equation for $u(x, t)$,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \;\; 0 \le x \le 1,$$

with boundary conditions $u(0, t) = u(1, t) = 0, \; t > 0$,
and initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = 0, \;\; 0 \le x \le 1$.
which, if any, of the functions below is a solution to the initial/boundary-value problem? Justify your answer.

a) $u(x, t) = \sum_{i=1}^{n} b_n e^{-\pi^2 n^2 t} \sin n\pi x, \; b_n = 2\int_0^1 f(x) \sin n\pi x dx$

b) $u(x, t) = \sum_{i=1}^{n} b_n cos n\pi t \sin n\pi x, \; b_n = 2\int_0^1 f(x) \sin n\pi x dx$

MATH 294 SUMMER 1990 PRELIM 2 # 5
5.5.15 Consider the partial differential equation

$$(*)\ \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},\ \text{for } 0 \le x \le L,$$

with conditions
i) $u(0,t) = 0,$
ii) $u(\ell,t) = 0$
iii) $\frac{\partial u}{\partial t}(x,0) = 0,$
and
iv) $u(x,0) = f(x).$

a) Verify that $u(x,t) = \sum_{i=1}^{n} C_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$ is a solution to (*) and the conditions (i), (ii), and (iii).
b) Suppose $f(x) = \sin\left(\frac{\pi x}{L}\right)$. What values for the C_n's will satisfy condition (iv)?
c) For a general piecewise smooth function $f(x)$; Determine the formula for the C_n so that condition (iv) is satisfied.

MATH 294 FALL 1992 UNKNOWN # 4
5.5.16 Solve the initial-boundary-value problem

$$u_{tt} = u_{xx}, 0 < x < 1, t > 0,$$

$$u(0,t) = u(1,t) = 0, t > 0,$$

$$u(x,0) = 8\sin 13\pi x - 2\sin 31\pi x,$$

$$u_t(x,0) = -\sin 8\pi x + 12\sin 88\pi x.$$

MATH 294 SPRING 1996 FINAL # 5 MAKE-UP
5.5.17 Consider $u(x,y,z,t) = w(ax+by+cz+dt)$ where w is some differentiable function of one variable, and the expression $ax+by+cz+dt$ has been substituted for that variable.
a) Find restrictions on the constants $a, b, c,$ and d so that u will be a solution to the three dimensional wave equation $u_{xx} + u_{yy} + u_{zz} = u_{tt}$
b) Find a solution to the wave equation if (a) having $u(x,y,z,0) = 5\cos x$ and $u_t(x,y,z,0) = 0.$

Chapter 6

Outliers

6.1 Some Geometry and Kinematics

MATH 294 UNKNOWN FINAL # 2

6.1.1 Let $R(t) = e^t \cos t\vec{i} + t\vec{j} + te^t\vec{k}$ be position of a particle moving in space at time t.

a) Set up, but do not evaluate, a definite integral equal to the distance traveled by the particle from $t = 0$ to $t = \pi$.

b) Find all points on the curve where the velocity vector is orthogonal to the acceleration vector.

MATH 293 UNKNOWN PRELIM 2 # 2

6.1.2 If $\vec{r}(t) = \cos t\vec{i} - 3\sin t\vec{j}$ gives the position of a particle

a) find the velocity and acceleration

b) sketch the curve, and sketch the acceleration and velocity vectors at one point of the curve (you choose the point)

c) what is the torsion (if you can do this without computation, that is acceptable - but please give reasons for your answer).

MATH 293 UNKNOWN FINAL # 1

6.1.3 **a**) Find an equation for the plane containing the points $(1, 0, 1), (-1, 2, 0), (1, 1, 1)$.

b) Find the cosine of the angle between the plane in a) and the plane $x-2y+z-5 = 0$.

MATH 294 SPRING 1984 FINAL # 1

6.1.4 Prove that for any vector $\vec{F}$:

$$\vec{F} = \left(\vec{F} \cdot \vec{i}\right)\vec{i} + \left(\vec{F} \cdot \vec{j}\right)\vec{j} + \left(\vec{F} \cdot \vec{k}\right)\vec{k}.$$

MATH 294 FALL 1985 FINAL # 1

6.1.5 Find a unit vector in $\Re^3$ which is perpendicular to both $\vec{i} + \vec{j}$ and $\vec{k}$.

MATH 294 SPRING 1985 FINAL # 5

6.1.6 Find a solution defined in the right half-plane $\{(x, y)|x > 0\}$ whose gradient is the vector field $\frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$.

MATH 294 FALL 1985 FINAL # 7

6.1.7 Let S be the surface with equation $x^2 + xy = z^2 + 2y$.

a) Find the equation of the plane tangent to S at the point $(1, 0, 1)$

b) Find all points on S at which the tangent plane is parallel to the xy plane.

MATH 294 FALL 1986 FINAL # 8

6.1.8 Consider the curve $C : x = t, y = \frac{1}{t}, z = \ln t$, and the line $L : x = 1 + \tau, y = 1 + 2\tau, z = -\tau$. The curve and the line intersect at the point $P = (1, 1, 0)$. Let $\vec{v}$ be a unit vector tangent to C at $P, \vec{w}$ a unit vector tangent to L at P. Compute the cosine of the angle θ between $\vec{v}$ and $\vec{w}$.

MATH 294 SPRING 1987 PRELIM 1 # 4

6.1.9 Consider the function $f(x, y, z) = 1 - 2x^2 - 3y^4$.

a) Find a unit vector that points in the direction of maximum increase of f at the point $R = (1, 1, 1)$.

b) Find the outward unit normal to the surface $f = -4$ at any point of your choice (clearly indicate your choice near your answer).

MATH 294 SPRING 1987 FINAL # 6

6.1.10 A particle moves with velocity $\vec{v}$ that depends on position $(x, y).\vec{v} = (a + y)\vec{i} + (-x + y)\vec{j}$. At $t = 0$ the particle is at $x = 1, y = 0$. Where is the particle at $t = 1$?

MATH 294 SPRING 1988 PRELIM 2 # 1

6.1.11 Given $f = xy \sin z$ and $\vec{F} = (xy)\vec{i} + (e^{yz}\vec{j} + (z^2)\vec{k}$, evaluate:

a) $\vec{\nabla} f = grad(f)$ at $(x, y, z) = (1, 2, 3)$

b) $\vec{\nabla} \cdot \vec{F}$ at $(x, y, z) = (1, 2, 3)$

MATH 294 FALL 1988 PRELIM 3 # 1

6.1.12 Curve C is the line of intersection of the paraboloid$z = x^2 + y^2$ and the plane $z = x + \frac{3}{4}$. The positive direction on C is the counterclockwise direction direction viewed from above, i.e. from a point (x, y, z) with $z > 0$. Calculate the length of the curve C.

MATH 293 SUMMER 1990 PRELIM 1 # 1

6.1.13 A parallelogram $ABCD$ has vertices at $A(2, -1, 4), B(1, 0, -1), C(1, 2, 3)$ and D.

a) Find the coordinates of D.

b) Find the cosine of the interior angle at B.

c) Find the vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$

d) Find the area of $ABCD$.

e) Find an equation for the plane in which $ABCD$ lies.

MATH 294 SUMMER 1990 PRELIM 1 # 1

6.1.14 Given the function $f(x, y) = e^{-x^2} + y - e^y$:

a) Compute the directional derivative at $(1, -1)$ in the direction of the origin;

b) Find all relative extreme points and classify them as maximum, minimum, or saddle points;

c) Give the linearization of f about $(1, -1)$

MATH 293 SUMMER 1990 PRELIM 1 # 5 3

6.1.15 The position vector of a particle

$$\vec{R}(t) \text{ is given by } \vec{R}(t) = t\cos t\vec{i} + t\sin t\vec{j} + \left(\frac{2\sqrt{2}}{3}\right) t^{\frac{3}{2}}\vec{k}$$

a) Find the velocity and acceleration of the particle at $t = \pi$

b) Find the total distance travelled by the particle in space from $t = 0$ to $t = \pi$.

MATH 293 SUMMER 1990 PRELIM 1 # 4

6.1.16 Find $\vec{T}, \vec{N}, \vec{B}$ and κ at $t = 0$ for the space curve defined by

$$\vec{R}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + t\vec{k}$$

MATH 294 SPRING 1990 FINAL # 10

6.1.17 Find t e shortest distance from the plane $3x + y - z = 5$ to the point $(1, 1, 1)$.

MATH 293 FALL 1990 PRELIM 1 # 1

6.1.18 **a**) Find the equation of the plane P which contains the point $R = (2, 1, -1)$ and is perpendicular to the straight line $L : x = -1 + 2t, y = 5 - 4t, z = t$.

b) Find the point of intersection of the lint L and the plane P.

c) Use b) to find the distance of the point R from the line L.

MATH 294 UNKNOWN 1990 UNKNOWN # ?

6.1.19 **a**) Determine the rate of change of the function $f(x, y, z) = e^x \cos yz$ in the direction of the vector $A = 2\vec{i} + \vec{j} - 2\vec{k}$ at the point $(0, 1, 0)$.

b) Determine the equation of the plane tangent to the surface $e^x \cos yz = 1$ at the point $(0, 1, 0)$.

MATH 293 FALL 1990 PRELIM 1 # 2

6.1.20 **a**) Find a <u>unit</u> vector which lies in the plane of $\vec{a}$ and $\vec{b}$ and is orthogonal to $\vec{c}$ if

$$\vec{a} = 2\vec{i} - \vec{j} + \vec{k}, \vec{b} = \vec{i} + 2\vec{j} - \vec{k}, \vec{c} = \vec{i} + \vec{j} - 2\vec{k}$$

b) Find the vector projection of $\vec{b}$ onto $\vec{a}$.

MATH 293 FALL 1990 PRELIM 1 # 3

6.1.21 Show that the following are true

a) $\left(\vec{a} \cdot \vec{i}\right)^2 + \left(\vec{a} \cdot \vec{j}\right)^2 + \left(\vec{a} \cdot \vec{k}\right)^2 = |\vec{a}|^2$

b) $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$
(hint: use the angle between them)

c) $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ is orthogonal to $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$

MATH 293 FALL 1990 PRELIM 1 # 4

6.1.22 **a**) Find $\vec{v}$ and $\vec{a}$ for the motion

$$\vec{R}(t) = t\vec{i} + t^3\vec{j}$$

b) Sketch the curve including $\vec{v}, \vec{a}$.

c) Find the speed at $t = 2$.

MATH 293 FALL 1990 PRELIM 1 # 5

6.1.23 Let $\vec{R}(t) = (\cos 2t)\vec{i} + (\sin 2t)\vec{j} + t\vec{k}$.

a) Find the length of the curve from $t = 0$ to $t = 1$.

b) Find the unit tangent $\vec{T}$, the principal unit normal $\vec{N}$ and the curvature κ at $t = 1$.

MATH 294 FALL 1990 FINAL # 1

6.1.24 Given the surface $z = x^2 + 2y^2$. At the point $(1, 1)$ in the x-y plane:

a) determine the direction of greatest increase of z

b) determine a unit normal to the surface.

Given the vector field $\vec{F} = 6xy^z\vec{i} - 2y^3z\vec{j} + 4z\vec{k}$,

c) calculate its divergence

d) use the divergence theorem to calculate the outward flux of the vector field over the surface of a sphere of unit radius centered at the origin.

MATH 293 SPRING 1992 PRELIM 1 # 1

6.1.25 Given the vectors

$$\vec{A} = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{B} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\vec{C} = \vec{i} - 2\vec{j} + \vec{k}$$

where $\vec{i}, \vec{j}$ and $\vec{k}$ are mutually perpendicular unit vectors.

Evaluate

a) $\vec{A} \cdot \vec{B}$

b) $\vec{A} \times \vec{B}$

c) $\left(\vec{A} \times \vec{B}\right) \cdot \vec{C}$

d) $\left(\vec{A} \times \vec{B}\right) \times \vec{C}$

MATH 293 SPRING 1992 PRELIM 1 # 4

6.1.26 Consider the plane $x + 2y + 3z = 17$ and the line through the points $P : (0, 3, 4)$ and $Q : (0, 6, 2)$.

a) Is the line parallel to the plane? Five clear reasons for your answer.

b) Find the point of intersection, if any, of the line and the plane.

MATH 293 SPRING 1992 PRELIM 1 # 2

6.1.27 The acceleration of a point moving on a curve in space is given by $\vec{a} = -\vec{i}b\cos t - \vec{j}c\sin t + 2d\vec{k}$ where $\vec{i}, \vec{j},$ and $\vec{k}$ are mutually perpendicular unit vectors and b,c and d are scalars. Also, the position vector $\vec{R}(t)$ and velocity vector $\vec{v}(t)$ have the initial values

$$\vec{R}(0) = \vec{i}(b+1), \vec{v}(0) = \vec{j}c$$

Find $\vec{R}(t)$ and $\vec{v}(t)$

MATH 293 SPRING 1992 PRELIM 1 # 5

6.1.28 Consider the curve

$$\vec{R}(t) = 3\vec{i} + \vec{j}\cos t + \vec{k}\sin t, 0 \leq t \leq 2\pi$$

where $\vec{i}, \vec{j}$ and $\vec{k}$ are mutually perpendicular unit vectors.

a) Sketch and describe the curve in words.

b) Determine the unit tangent, principal normal and binormal vectors ($\vec{T}, \vec{N}$ and $\vec{B}$) to the curve at the point $t = \frac{\pi}{2}$

c) Sketch the vectors $\vec{T}, \vec{N}$ and $\vec{B}$ at $t = \frac{\pi}{2}$.

MATH 293 SPRING 1992 PRELIM 1 # 6

6.1.29 The position vector of a point moving along a curve is

$$\vec{R}(t) = t\vec{i} + e^{2t}\vec{j}$$

where $\vec{i}$ and $\vec{j}$ are mutually perpendicular unit vectors and t is time. The acceleration vector $\vec{a}$ at the time $t = 0$ can be written as

$$\vec{a}(0) = c\vec{T} + d\vec{N}$$

where $\vec{T}$ and $\vec{N}$ are the unit tangent and principal normal vectors to the curve at the time $t = 0$.
Find the scalars c and d

MATH 293 SPRING 1992 FINAL # 1

6.1.30 A point is moving on a spiral given by the equation

$$\vec{R}(t) = e^t \cos t\vec{i} + e^t \sin t\vec{j}$$

where $\vec{i}$ and $\vec{j}$ are the usual mutually perpendicular unit vectors. Find

a) The speed (the magnitude of the velocity) of the point at $t = 0$.

b) The curvature of the spiral at $t = 0$.

MATH 293 SUMMER 1992 PRELIM 6/30 # 1

6.1.31 Find the equation of the plane which passes through the points

$$A(0,0,0), B(-1,1,0) \text{ and } C(-1,1,1).$$

MATH 293 SUMMER 1992 PRELIM 6/30 # 5

6.1.32 A point P, starting at the origin $(0,0,0)$ is moving along a smooth curve. At any time, the distance s travelled by the point from the origin, is observed to be

$$s = 2t$$

Also, the unit tangent vector to the curve, as this point, is

$$\vec{T} = -\frac{\sin t}{2}\vec{i} + \frac{\cos t}{2}\vec{j} + \frac{\sqrt{3}}{2}\vec{k}$$

a) Find the acceleration $\vec{a}$ of P as a function of time.

b) Find the position vector $\vec{R}(t)$ of P.

MATH 293 SUMMER 1992 FINAL # 6

6.1.33 A point P is moving on a curve defined as

$$x(t) = \cos \alpha t$$
$$y(t) = 2t$$
$$z(t) = 3\cos t + 6t + 3(\alpha - 1)t^2$$

Find value(s) of α such that the curve defined above lies in a plane for all $0 \le t \le \infty$. Hint: The idea of torsion of a curve should be useful here!

MATH 293 FALL 1992 PRELIM 1 # 3

6.1.34 Let $P_1(-1,0,-1), P_2(1,1,-1)$ and $P_3(1,-1,1)$ be three points and let $\vec{A} = \overrightarrow{P_1P_2} = 2\vec{i} + \vec{j}$ and $\vec{B} = \overrightarrow{P_1P_3} = 2\vec{i} - \vec{j} + 2\vec{k}$.

a) Find a vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

b) Find the area of the parallelogram whose edges are $\vec{A}$ and $\vec{B}$.

c) Find the equation of the plane passing through the points P_1, P_2 and P_3.

MATH 293 FALL 1992 PRELIM 1 # 4

6.1.35 Let $P_1(-1,0,-1), P_2(1,1,-1)$ and $P_3(1,-1,1)$ be three points and let $\vec{A} = \overrightarrow{P_1P_2} = 2\vec{i} + \vec{j}$ and $\vec{B} = \overrightarrow{P_1P_3} = 2\vec{i} - \vec{j} + 2\vec{k}$.

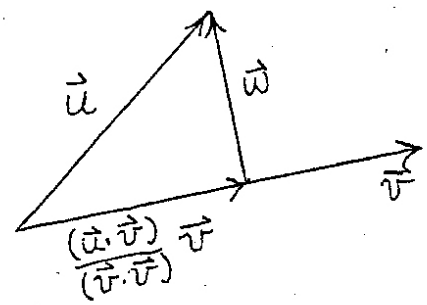
a) Find the distance from the point $(1,1,1)$ to the plane passing through the points P_1, P_2 and P_3.

b) Find the equation of the line passing through the point P_3 and parallel to the line passing through P_1 and P_2.

c) Find the vector projection of $\vec{A}$ in the direction of $\vec{B}$ and the scalar component of $\vec{A}$ in the direction of $\vec{B}$.

MATH 293 FALL 1992 PRELIM 1 # 5

6.1.36 Let $\vec{u}$ and $\vec{v}$ be two given vectors. The vector projection of $\vec{u}$ in the direction of $\vec{v}$ is $\frac{(\vec{u}\cdot\vec{v})}{(\vec{v}\cdot\vec{v})}\vec{v}$. Consider the vector $\vec{w} = \vec{u} - \frac{(\vec{u}\cdot\vec{v})}{(\vec{v}\cdot\vec{v})}\vec{v}$. By taking the scalar product of $\vec{w}$ with $\vec{v}$ show that $\vec{w}$ is perpendicular (orthogonal) to $\vec{v}$.



MATH 293 FALL 1992 PRELIM 2 # 2

6.1.37 Find all points (x, y, z) which lie on the intersection of the planes

$$x + y + z = 6, -x + 2z = 1, y + 3z = 7$$

Is this set of points a single point, a line or a plane?

MATH 293 FALL 1992 PRELIM 2 # 3

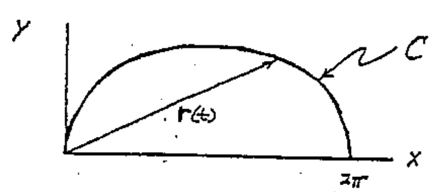
6.1.38 A point move on a space curve with the position vector

$$\vec{v}(t) = e^t \cos t\vec{i} + e^t \sin t\vec{j} + 2\vec{k}$$

Find the velocity $\vec{v}$, speed, unit tangent vector $\vec{T}$, unit principal normal $\vec{N}$, acceleration $\vec{a}$ and curvature κ as functions of time. Also check that $\vec{N}$ is perpendicular to $\vec{T}$.

MATH 294 FALL 1992 PRELIM 2 # 3?

6.1.39 Determine the arc-length, $\int_C ds$ of the curve C (a cycloid) given by: $r(t) = (t - \sin t)\vec{i} + (1 - \cos t)\vec{j}, 0 \leq t \leq 2\pi$ (see figure below).



MATH 293 FALL 1993 PRELIM 1 # 4

6.1.40 The four corners of a parallelopiped are given as $(1, 1, 1), (1, 4, 2), (4, 2, 3)$ and $(1, 1, 4)$ in xyz-space. Using $(1, 1, 4)$ as the common point of three vectors lying along the parallelopiped's edges, calculate the volume of the parallelopiped.

MATH 293 FALL 1992 FINAL # 2

6.1.41 A particle is moving along the positive branch of the curve $y = 1 + x^2$ and its x coordinates is controlled as a function of time according to $x(t) = 2t$. Find

a) The tangential component of the particle's acceleration, a_T, at time $t = 0$.

b) The normal component of the particle's acceleration, A_N, at time $t = 0$.

c) The radius of curvature ρ of the curve, along which the particle is moving, at the point $(0, 1)$. Hint: $|\tilde{a}|^2 = a_N^2 + a_T^2, a_T = \frac{d\tilde{v}}{dt}, a_N = \frac{|\tilde{v}|^2}{\rho}$.

MATH 293 FALL 1993 PRELIM 1 # 6

6.1.42 Find the equation of the plane that contains the intersecting lines L_1 and L_2 given by:

$$L_1 : \left| \begin{array}{l} x = 1 + t \\ y = 2 + t \\ z = 1 + t \end{array} \right. \quad L_2 : \left| \begin{array}{l} x = 1 - t \\ y = 2 - t \\ z = 1 \end{array} \right.$$

Sketch the plane.

MATH 293 FALL 1993 PRELIM 1 # 5

6.1.43 A line contains the two points $(1, 2, 3)$ and $(-2, 1, 4)$. Find parametric equations of the line and calculate the distance from the line to the point $(5, 5, 5)$.

MATH 293 FALL 1993 PRELIM 1 # 3

6.1.44 Calculate the volume of the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

by imaging it to be comprised of a set of thin elliptical disks, of thickness dz, oriented parallel to the x-y plane.

MATH 293 SPRING 1993 PRELIM 1 # 2

6.1.45 **a**) Solve the initial value problem

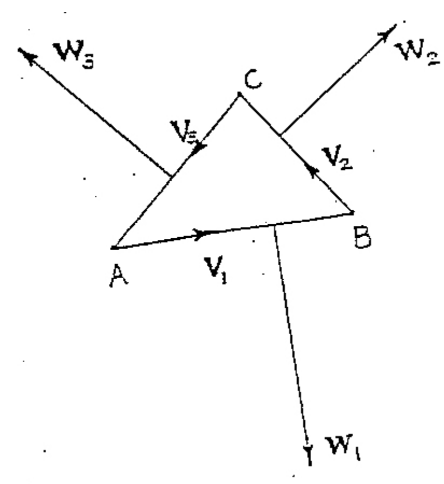
$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

if $y = 0$ when $x = 1$

b) Consider a triangle ABC with three vectors defined as

$$\vec{v}_1 = \overrightarrow{AB}, \vec{v}_2 = \overrightarrow{BC}, \vec{v}_3 = \overrightarrow{CA}$$

From three points, one on each side of the triangle, draw vectors $\vec{w}_1, \vec{w}_2,$ and $\vec{w}_3$ in plane of the triangle. Each of these vectors is perpendicular to its side (i.e. $\vec{w}_1$ is perpendicular to $\overrightarrow{AB}$ and so on) with length equal to the length of the side and pointing out of the triangle.



i) Find $\vec{w}_1, \vec{w}_2,$ and $\vec{w}_3$ in terms of the components of $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$.

c) Show that $\vec{w}_1 + \vec{w}_2 + \vec{w}_3 = 0$

MATH 294 FALL 1993 PRELIM 1 # 2

6.1.46 C is the curve given by

$$\vec{r}(t) = e^{-t}\cos t\vec{i} + e^{-t}\sin t\vec{j} + \sqrt{1 - e^{-2t}}\vec{k},\ (0 \le t < \infty).$$

Show that C lies on the sphere $x^2 + y^2 + z^2 = 1$ and describe the curve with words and a sketch.
You may use the fact that $\cos t\vec{i} + \sin t\vec{j}$ $(0 \le t \le 2\pi)$ is a parametrization of the unit circle.

MATH 293 SPRING 1993 PRELIM 1 # 3

6.1.47 Find the equation of the plane that contains the intersecting lines L_1 and L_2 where:

$$L_1 : \left| \begin{array}{l} x = 1 + t \\ y = 1 + t \\ z = 1 + t \end{array} \right. \quad L_2 : \left| \begin{array}{l} x = 1 - t \\ y = 1 - t \\ z = 1 \end{array} \right.$$

MATH 293 SPRING 1993 PRELIM 1 # 4

6.1.48 Find the equation of the plane through the points $(2, 2, 1)$ and $(-1, 1, -1)$ that is perpendicular to the plane $2x - 3y + z = 3$.

MATH 293 SPRING 1993 PRELIM 1 # 5

6.1.49 Consider a point (x, y). Let d_1 be the distance from (x, y) to the line $x + y = 0$ and d_2 be the distance from (x, y) to the line $x - y = 0$.
Given $d_1 d_2 = 1$, find the locus of all such points, i.e., say what the curve is and find its equation.

MATH 293 SPRING 1993 PRELIM 2 # 1

6.1.50 A point P is moving along a plane curve. The unit tangent and principal normal vectors of this curve are, (for $t \geq 0$),

$$\vec{T}(t) = -\vec{i}\sin(t) + \vec{j}\cos(t)$$

$$\vec{N}(t) = -\vec{i}\cos(t) - \vec{j}\sin(t)$$

(where $\vec{i}$ and $\vec{j}$ are the usual mutually perpendicular unit vectors), and the tangential component of the velocity vector of P, (the speed), is

$$\vec{v}_T = t.$$

a) Find the velocity vector $\vec{v}(t)$ of P.
b) Find the acceleration vector $\vec{a}(t)$ of P.
c) Find the tangential ($\vec{a}_T$) and normal ($\vec{a}_n$) components of the acceleration vector.
d) Find the radius of curvature $\rho(t)$ of the curve.

MATH 293 SPRING 1994 PRELIM 1 # 1

6.1.51 Find the distance from the point $(2, 1, 3)$ to the plane which contains the points $(2, 1, 0), (0, 1, 1), (0, 0, 2)$.

MATH 293 SPRING 1994 PRELIM 1 # 2

6.1.52 Find the point on the segment from $P_1 = (1, 0, -1)$ to $P_2 = (4, 3, 2)$ which is twice as far from P_2 as it is from P_1.

MATH 293 SPRING 1994 PRELIM 1 # 3

6.1.53 A particle moves on the sphere of radius a centered at the origin. Its position vector $\vec{r}(t)$ is a differentiable function of the time, t. Show that the velocity vector $\vec{v}(t)$ of the particle is always perpendicular to its position vector, $\vec{r}(t)$.

MATH 293 SPRING 1994 PRELIM 1 # 4

6.1.54 A parallelogram, P, is determine by the two vectors $\vec{i}+\vec{j}+\vec{k}$ and $2\vec{i}-\vec{j}-\vec{k}$.

a) What is the area of P?
b) What is the area of the orthogonal projection of P in the xy-plane?
c) What is the area of the orthogonal projection of P in the xz-plane?
d) What is the area of the orthogonal projection of P in the yz-plane?
e) What is the area of the orthogonal projection of P in the plane $x+y-z=0$?

MATH 293 SPRING 1994 PRELIM 2 # 1

6.1.55 A point P is moving along the spiral

$$x = e^t \cos(t)$$

$$y = e^t \sin(t).$$

a) Find the curvature of the given spiral at $t=0$.
b) The acceleration of P is written as

$$\vec{a} = a_T\vec{T} + a_N\vec{N}.$$

Find a_T and a_N at $t=0$.

MATH 293 FALL 1994 PRELIM 1 # 1

6.1.56 Let

$$\vec{A} = 2\vec{i}-\vec{j}+\vec{k}$$

$$\vec{B} = \vec{i}+\vec{j}+\vec{k}$$

$$\vec{C} = \vec{i}+2\vec{j}+\vec{k}$$

a) Find the vector projection of A onto the direction of $\vec{B}$.
b) Show that $\vec{A} - proj_{\vec{B}}\vec{A}$ is perpendicular to $\vec{B}$.
c) Find the area of the parallelogram with edges $\vec{A}$ and $\vec{B}$.
d) Find the volume of the box with edges $\vec{A}, \vec{B}$ and $\vec{C}$.
e) Find the parametric equation of the line through $(0,0,0)$ and parallel to the intersection of the planes with normals $\vec{A}$ and $\vec{B}$.

MATH 293 FALL 1994 PRELIM 1 # 2

6.1.57 Let $\vec{a}$ and $\vec{b}$ be vectors. Show that

a) $|\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$, and
b) that $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$

MATH 293 FALL 1994 PRELIM 1 # 3

6.1.58 Graph $z = x^2 + y^2 + 1$ and label any intersection the surface may have with any axis. Describe the curves that are the intersections of the surface with the planes $z =$ constant $(z > 1)$.

MATH 293 FALL 1994 PRELIM 2 # 1

6.1.59 Consider the path traversed by a particle given parametrically by $\vec{r}(t) = (e^t \cos t)\vec{i} + (e^t \sin t)\vec{j} + e^t\vec{k}$. Find the

a) velocity vector
b) speed
c) acceleration vector
d) length of the path from $t = 0$ to $t = \ln 4$

MATH 293 FALL 1994 FINAL # 1

6.1.60 The level curves of the function $f(x, y, z) = z + x^2 + y^2 + 1$ are:

a) Hyperboloids
b) Planes
c) Cones
d) Paraboloids
e) Spheres

MATH 293 FALL 1994 FINAL # 3

6.1.61 The vector projection of $(1, 0, 1, 0)$ in the direction of $(1, 1, 1, 1)$ is:

a) $\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$,
b) $(0, 1, 0, 1)$,
c) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$,
d) $(0, -1, 0, -1)$
e) $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

MATH 293 FALL 1994 FINAL # 5

6.1.62 Any non-zero vector perpendicular to the vectors $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{k}$ is

a) Perpendicular to $2\vec{i} + \vec{j} + \vec{k}$,
b) Parallel to $\vec{i} + \vec{j} + \vec{k}$,
c) Perpendicular to $2\vec{i} - \vec{j} - \vec{k}$,
d) Parallel to $2\vec{i} + \vec{j} + \vec{k}$,
e) Parallel to $2\vec{i} - \vec{j} - \vec{k}$

MATH 293 SPRING 1995 PRELIM 1 # 2

6.1.63 Consider the planar curve

$$y^2 = 4x.$$

Find parametric equations of the following lines.

a) Tangent to the above curve at $P(1, 2)$.
b) Normal to the above curve at $O(0, 0)$.

MATH 293 SPRING 1995 PRELIM 1 # 3

6.1.64 **a**) Show that the points $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ are the vertices of a parallelogram.

b) What is the area of this parallelogram?

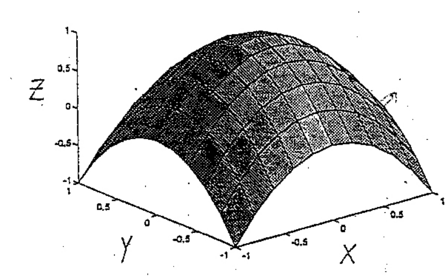
MATH 294 SPRING 1995 PRELIM 1 # 5

6.1.65 The surface S drawn below can be described in two ways, i.e.

as $z = f(x, y) = 1 - x^2 - y^2, -1 \leq x \leq 1, -1 \leq y \leq 1$

or $g(x, y, z) = z + x^2 + y^2 = 1, -1 \leq x \leq 1, -1 \leq y \leq 1$

Evaluate and sketch the gradient fields $\vec{\nabla} f$ and $\vec{\nabla} g$. Explain the relationship between these two vector fields.



MATH 293 SPRING 1995 PRELIM 2 # 2

6.1.66 Let

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

be a space curve; let $\vec{v}(t)$ be the velocity vector and $\vec{a}(t)$ the acceleration vector.

a) Give the formula which gives the curvature of the curve in terms of $\vec{v}$ and $\vec{a}$.

b) By differenting $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$, find a formula for $\frac{d\vec{v}}{dt}$ in terms of $\vec{v}$ and $\vec{a}$

c) If at some instant we have

$$|\vec{v}| = 3\ m/s, \frac{d|\vec{v}|}{dt} = 4\ m/s^2, |\vec{a}| = 5\ m/s^2$$

what is the radius of curvature in meters.

MATH 293 FALL 1995 PRELIM 1 # 1 *

6.1.67 This is a two-dimentional problem. Consider the parabola

$$y^2 = 4x \text{ and the point } P(1,2) \text{ on it.}$$

a) Find an unit vector $\vec{t}$ that is tangential to the parabola at P.

b) Find the equation of the tangent line to the parabola at P. Any correct form of the equation is acceptable.

c) Find an unit vector $\vec{n}$ that is normal to the parabola at P.

d) Find the equation of the normal line to the parabola at P. Any correct form of the equation is acceptable.

MATH 294 FALL 1995 PRELIM 1 # 1 *

6.1.68 **a**) For $f = x^2 + 8y^2$, show that $(4,2)$ lies on the level curve $f(x,y) = 48$. Sketch this level curve.

b) Find the vector field $\vec{\nabla} f$

c) Evaluate $\vec{\nabla} f$ at $(x,y) = (\sqrt{48}, 0), (4,2), (4,-2), (0, \sqrt{6})$ and sketch these vectors, showing very clearly their relation to the level curve.

MATH 293 FALL 1995 PRELIM 1 # 2

6.1.69 Consider two straight lines in space given by the equations:

$$L_1 : \left| \begin{array}{l} x = 2 + t \\ y = 2 + t \\ z = -t \end{array} \right. \quad -\infty \le t \le \infty$$

$$L_2 : \left| \begin{array}{l} x = 3 + u \\ y = -2u \\ z = 1 + u \end{array} \right. \quad -\infty \le u \le \infty$$

a) Do these lines intersect? If so, find the coordinates of the point of intersection.

b) Find a vector $\vec{u}$ along L_1 and a vector $\vec{v}$ along L_2.

c) Find, if possible, the equation of the plane that contains the lines L_1 and L_2.

MATH 293 FALL 1995 PRELIM 1 # 4a *

6.1.70 Describe the set of points defined by the equations

$$\begin{array}{rcl} x^2 + y^2 + z^2 & \le & 4 \\ z & \le & 1 \end{array}$$

Also, draw a sketch showing this set of points.

MATH 293 FALL 1995 FINAL # 5 *

6.1.71 A point P is moving on a plane curve with the position vector

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, t \geq 0$$

where t is time and $\vec{i}$ and $\vec{j}$ are the usual orthogonal Cartesian unit vectors. The position components $x(t)$ and $y(t)$ satisfy the equations

$$t\frac{dx}{dt} + x = t^2, x(0) = 0$$

$$\text{and } \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0, y(0) = 0, \frac{dy}{dt}(0) = 1$$

a) Find $x(t)$ as an explicit function of time.
b) Find $y(t)$ as an explicit function of time.
c) Find $\vec{v}(t) = \frac{d\vec{r}}{dt}$, the velocity of P as a function of time.

MATH 293 SPRING 1996 FINAL # 17 *

6.1.72 A bug flies around the room so that at time t, the position of the bug is given by $x = t^2, y = t^{\frac{3}{2}}, z = t^2$. The velocity at time $t = 1$ is
a) 10.25
b) $2\vec{i} + \frac{3}{2}\vec{j} + 2\vec{k}$
c) $\vec{i} + \vec{j} + \vec{k}$
d) 3
e) none of the above

MATH 293 SPRING 1996 FINAL # 18 *

6.1.73 The speed of the bug above at time $t = 1$ is
a) 40
b) 19
c) 3
d) $\vec{i} + 3\vec{j} + 8\vec{k}$
e) none of the above

MATH 293 SPRING 1996 FINAL # 19 *

6.1.74 The position of the bug above at time $t = 1$ is
a) $\sqrt{3}$
b) 40
c) 19
d) 3
e) none of the above

MATH 293 SPRING 1996 FINAL # 25 *

6.1.75 A cannon fires a cannonball at an angle of 45 degrees from horizontal. The cannonball lands 1000 meters away. Taking Newton's gravitational constant g to be 10 meters per second squared, the speed of the cannonball when leaving the cannon in meters per second is

a) 10
b) $10\sqrt{10}$
c) 100
d) $\frac{2000}{\sqrt{2}}$
e) none of the above

MATH 293 SPRING 1996 FINAL # 20 *

6.1.76 The projection of the vector (1,0,1,0) in the direction of (1,1,1,1) is

a) $\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$
b) (0,1,0,1)
c) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
d) (0,-1,0,1)
e) none of the above

MATH 293 SPRING 1996 FINAL # 24 *

6.1.77 Let $P = (1,1,1), Q = (1,0,0), R = (0,1,0)$. Then the equation of the plane in $\Re^3$ containing the triangle PQR is

a) $x + y - z = -1$
b) $x - y + z = 1$
c) $-x + y - z = -1$
d) $x + y - z = 1$
e) none of the above

MATH 293 SPRING 1996 FINAL # 30 *

6.1.78 If $\vec{u}, \vec{v}, \vec{w}$ are vectors in $\Re^3$, then $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{w} \times \vec{v}) \cdot \vec{u}$ (T/F)

MATH 294 SPRING 1996 FINAL # 1 MAKE-UP

6.1.79 In this problem $f(x, y) = x - y^2$.

a) Sketch the level curve $f(x, y) = -7$
b) Evaluate $\vec{\nabla} f(2, 3)$ and sketch it on the graph, showing the relation to the level curve.
c) Find the to the right flux of $\vec{\nabla} f$ across the segment $0 \leq y \leq 5, x = 0$

MATH 293 FALL 1997 PRELIM 3 # 4

6.1.80 Evaluate the line integral

$$\int_C \frac{zydx + zxdy + (z - xy)dz}{z^2}$$

where C is the curve given by parametric equations $x(t) = \cos(\pi t), y(t) = \sin(\pi t), z(t) = t, (1 \leq t \leq 2)$.

MATH 293 SUMMER 1992 PRELIM 6/30 # 3

6.1.81 A point is moving along a curve given by the parametric equations

$$x(t) = t$$

$$y(t) = 2t^2$$

Find, as functions of time t

a) The velocity of the point, $\vec{v}$

b) The acceleration of the point, $\vec{a}$

c) The curvature κ of the curve

d) If $\vec{a} = a_N\vec{N} + a_T\vec{T}$, where $\vec{N}$ is the principal unit normal and $\vec{T}$ is the unit tangent vector to the curve at some point on it, find a_N and a_T.

MATH 293 SPRING 1995 PRELIM 2 # 1

6.1.82 Consider the spiral parametrized by

$$t \mapsto \begin{bmatrix} e^{-t}\cos t \\ e^{-t}\sin t \end{bmatrix} \quad 0 \le t < \infty.$$

a) Sketch the curve.

b) Find its length or show that it has infinite length.

MATH 293 FALL 1994 PRELIM 2 # 1

6.1.83 Find the arc length parametrization of the space curve:

$$\vec{r}(t) = \cos(2t)\vec{i} + \sin(2t)\vec{j} + \frac{2}{3}t^{\frac{3}{2}}\vec{k}, \text{ with } 0 \le t \le 5.$$

MATH 294 SUMMER 1995 PRELIM (1) # 1

6.1.84 The surface

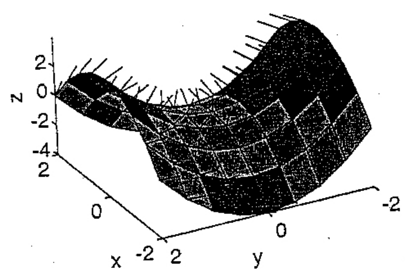
$$z = f(x, y) = y^2 - x^2; -2 \leq x \leq 2, -2 \leq y \leq 2$$

is shown below along with its normal vectors.

a) Sketch the contour lines of the surface in the (x, y) plane, i.e. draw the curves such that $z =$ constant, for example $z = -2, -1, 0, +1, +2$.

b) On your sketch for part (a) sketch the vector field $\vec{\nabla} f$.

c) Find an expression for the unit normal vectors $\vec{n}$ of the surface.

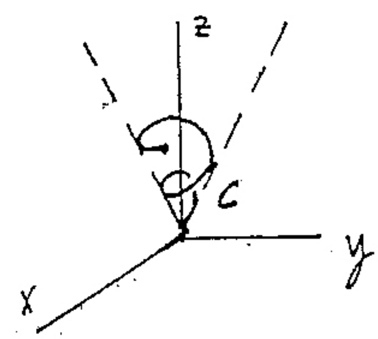


MATH 294 FALL 1992 FINAL # 2

6.1.85 Consider the curve $C : \vec{r}(t) = t\cos t\vec{i} + t\sin t\vec{j} + t\vec{k}, 0 \leq t \leq 4\pi$, which corresponds to the conical spiral shown below.

a) Set up, but so not evaluate, the integral yielding the arc-length of C.

b) Compute $\int_C (y + z)dx + (z + x)dy + (x + y)dz$.



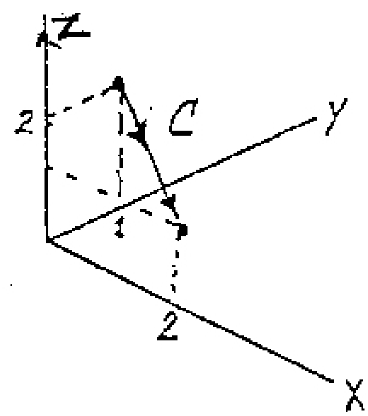
MATH 293 FALL 1994 FINAL # 2

6.1.86 A bug flies around the room along a path parametrized by $x = t^2, y = t^{\frac{3}{2}}, z = t^2$. If the temperature at any point (x,y,z) is given by $T(x, y, z) = x^2y + z^2$, the rate at which the bug feels the temperature change when $t = 1$ is

a) 3
b) -3
c) $\frac{19}{2}$
d) 0
e) $\frac{15}{2}$

MATH 294 FALL 1994 PRELIM 1 # 1 *

6.1.87 C is the line segment from (0,1,2) to (2,0,1).



a) which of the following is a parametrization of C?
i) $x = 2t, y = 1 - t, z = 2 - t, 0 \leq t \leq 1$
ii) $x = 2 - 2t, y = -2t, z = 1 - 2t, 0 \leq t \leq \frac{1}{2}$
iii) $x = 2\cos t, y = \sin t, z = 1 + \sin t, 0 \leq t \leq \frac{\pi}{2}$
b) evaluate $\int_C 3z\vec{j} \cdot d\vec{r}$

MATH 294 SPRING 1996 PRELIM 1 # 1 *

6.1.88 **a**) Evaluate $\int_{(0,0,0)}^{(4,0,2)} 2xz^3dx + 3x^2z^2dz$ on any path.
b) Write parametric equations for the line segment from (1,0,3) to (2,5,0).

MATH 293 SPRING 1992 PRELIM 1 # 3

6.1.89 Given a plane $x - 5y + z = 21$ and a point R with coordinates (1,2,3), find
a) The parametric equations of a line perpendicular to the plane and passing through R.
b) The point of intersection of the line and the plane.
c) The distance from R to the plane.

MATH 293 FALL 1997 PRELIM 3 # 3

6.1.90 Consider the sphere $x^2 + y^2 + z^2 = 25$.

a) Express the equation of the sphere in cylindrical coordinates (r, θ, z) and find volume inside it by evaluating a triple integral in cylindrical coordinates.

b) Now consider the region that you get by starting with the solid interior of the sphere as before, and removing the points which are contained inside the cone $z = \sqrt{x^2 + y^2}$. This means that our new region consists of points having $x^2 + y^2 + z^2 \leq 25, z \leq \sqrt{x^2 + y^2}$. Find the volume of this region by evaluating a triple integral spherical coordinates (ρ, ϕ, θ).

MATH 293 SUMMER 1992 PRELIM 6/30 # 6

6.1.91 A consider is described by the parametric equations

$$x = 2\cos t$$

$$y = 2\sin t$$

A point P inside the circle has coordinates (1,1). The line, normal to the circle, through P, intersects the circle at two points Q_1 and $Q_2 . Q_1$ is the point nearer to P.

a) Find the vector $\vec{N}$ along the line PQ_1

b) Find the parametric equations of the line segment PQ_1.

c) Find the distance PQ_1.

MATH 293 SUMMER 1990 PRELIM 1 # 2

6.1.92 **a**) Find the distance between the point $P = (0, 0, 0)$ and the line L defined parametrically by

$$X = t + 1$$

$$Y = t + 1$$

$$Z = t$$

b) Find an equation of the line through P that is perpendicular to the line L.