EVOLVING DOCUMENT

ME 6700 Advanced Dynamics Homework

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Due at the first class that is 6 or more days of the homework posting. If this is ambiguous, the homework is due later.

1. Learn any non-algabraic "proof" of Euler's theorem. Once you have learned it, write it up neatly without looking at any source.

2. Consider a 90 degree rotation about the y axis followed by a 90° rotation about the x axis.

a) Using geometric reasoning, find the net axis and angle of rotation as best you can.

b) Apply the vector formula, twice in sequence, to get the new positions of unit vectors that were originally on the x, y and z axis, respectively.

c) By taking the previous results as columns of a matrix, or by operating directly on a general vector, find the net rotation of any vector with original components (x, y, z).

3. Race between a wheel and a block. 2D. The ramp is an eighth of a circle (radius R) followed by a level surface. The wheel (radius $r \ll R$) rolls without slip. The block slides with no contact friction. Both have motion resisted by an air friction force proportional to translation speed. Both have the same drag constant so that $F = cv$ for both. Which one rolls farther?

4. Write some Matlab functions (can use this for prob (2) above)

a) `function rp = vecrot(n, theta, r)`

b) function R = rotmat(n, theta)

c) use `VECROT` and the matrix from `ROTMAT` to calculate the rotation for some special cases. Check against the answers you know from geometry.

i) 90° rotation about x axis,

ii) 90° rotation about z axis,

iii) 120° rotation about (1,1,1) axis.

5. Using `PLOT3` (or any other way of making 3D drawings) draw a box with sides b,w, and h and a line through the box in direction n. Then, by repeatedly using PLOT3 rotate the box smoothly rotating about axis n. Use this to illustrate the 120° rotation about the axis $(1, 1, 1)$.

6. In lecture we found the rotation matrix for rotation about the z axis using the general axis, angle formula. Do the same for the x axis and find the rotation matrix associated with a rotation by all three Euler angles. Check that your result agrees with any book. You may use computer algebra.

7. Write a Matlab script that takes rotation matrix as input and calculates the axis and angle of rotation. Check that this works forwards and back with your code from 4 above, both ways, with some odd examples.

8. Back to the wheel and block from problem **??**. Take account of the air friction while on the slope. Can you give a "proof" that one or the other will go farther? What about for quadratic drag? If not a proof, a counter example (you are allowed to alter the shape and height of the slope.

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9. Given the basic postulate of angular momentum balance expressed as

$$\sum \vec{\boldsymbol{M}}_{/C} = \sum \vec{\boldsymbol{r}}_{i/C} \times \vec{\boldsymbol{a}}_{i/\mathcal{F}} m_i$$

show that

$$\sum \vec{\boldsymbol{M}}_{/G} = \frac{d}{dt}\vec{\boldsymbol{H}}_{/G} \qquad \text{where} \qquad \vec{\boldsymbol{H}}_{/G} = \sum \vec{\boldsymbol{r}}_{i/G} \times \vec{\boldsymbol{v}}_{i/G} m_i$$

10. **Block vs wheel, again!**. Two wheel, both with mass m and radius R roll without slip on the same ramp. One has moment of inertia of $I_1 = 0$ the other has a moment of inertia of $I_2 = mR^2/2$ (both about the centers' of mass). The shape of the ramp is described by the function $h(s)$ giving the height h of the wheel center as a function of arc length s along the wheel path. A drag force $F(\dot{s})$ acts on the wheel centers. At some large s the curve becomes flat forever more. The function $F(v)$ is monotonically increasing with v such that at least one of the blocks only goes a finite distance as $t \to \infty$. Is there any curve $h(s)$ and any function $F(v)$ such that the final distance traveled by wheel 1 is greater than that of wheel 2?

11. Rotations. 3D rotations are hard to visualize. They have many representations: axis-angle, rotation vector, vector formula with cross products, dyadic representation of same, a tensor, a matrix, Euler angles (all n versions), Euler parameters and quaternians. The notations for each of these are not universal. And for each there are the rules for composing rotations. Add to this that the same word "Rotation", as for all linear transformations, can sometimes be thought of as a way to change a vector and sometimes as a way to change a given vector's representation. Most people get confused about rotations some times.

We need to describe a vector, say $\vec{\boldsymbol{r}}$, it's components, say r_i and the name of a corresponding column vector of 3 numbers, say r. However, for all but the first of these we also need to know what basis we are using, say the original fixed basis $\mathcal{F}$ or a rotated basis $\mathcal{B}$. And the same issues show for the tensor, say $\underline{\underline{\mathbf{R}}}$, its components, say R_{ij} and the array of numbers used to represent it, say R. Then there is the problem that we are interested in all the same things for the action of $\underline{\underline{\mathbf{R}}}$ on $\vec{\boldsymbol{r}}$, call it, say

$$\vec{\boldsymbol{r}}' = \underline{\underline{\mathbf{R}}} \cdot \vec{\boldsymbol{r}}$$

Consider two frames, the fixed one $\mathcal{F}$ with an orthonormal basis $\hat{\boldsymbol{e}}_i$ and $\mathcal{B}$, which is $\mathcal{F}$ rotated by $\underline{\underline{\mathbf{R}}}$, with basis vectors $\hat{\boldsymbol{e}}_i'$. Here is a notation, including some free choices about how explicit to be about the choice of frame ($\mathcal{F}$ or $\mathcal{B}$).

NOTATION

$\hat{\boldsymbol{e}}_i =$ an orthonormal set of fixed $\mathcal{F}$ basis vectors.

$\hat{\boldsymbol{e}}_i' =$ the rotated $\mathcal{B}$ basis vectors.

$\underline{\underline{\mathbf{R}}} =$ the rotation, a linear transformation, that takes all vectors in $\mathcal{F}$ to $\mathcal{B}$. So

$$\underline{\underline{\mathbf{R}}} \cdot \hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_i'$$
$$\underline{\underline{\mathbf{R}}} \cdot \vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}'$$

$R_{ij} = R_{\mathcal{F}ij} = R_{ij}^{\mathcal{F}} = R_{\mathcal{F}ij}^{\mathcal{F}} =$ various notations, progressively more formal, for the components of $\underline{\underline{\mathbf{R}}}$ in the fixed $\mathcal{F}$ basis. That is, if there is no $\mathcal{F}$ subscript or super script, it is assumed. But if you want to be super clear that the same basis vectors are used for the left and right vectors in the dyads, then you use superscripts and subscripts. If you are never going to use mixed bases, then a single $\mathcal{F}$ will be good enough.

$R_{\mathcal{B}ij} = R_{ij}^{\mathcal{B}} = R_{\mathcal{B}ij}^{\mathcal{B}} =$ various notations, progressively more formal, for the components of $\underline{\underline{\mathbf{R}}}$ in the fixed $\mathcal{B}$ basis. You need at least some $\mathcal{B}$ as a subscript or super script. But if you want to be super clear that the same basis vectors are used for the left and right vectors in the dyads, then you use superscripts and subscripts. If you are never going to use mixed bases, then a single $\mathcal{B}$ will be good enough.

$R_{\mathcal{F}ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = \underline{\underline{\mathbf{R}}} =$ dyadic representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{F}$ basis.

$R_{\mathcal{B}ij}\hat{\boldsymbol{e}}_i'\hat{\boldsymbol{e}}_j' = \underline{\underline{\mathbf{R}}} =$ dyadic representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{B}$ basis.

$\vec{\boldsymbol{r}} = r_i\hat{\boldsymbol{e}}_i = r_{\mathcal{F}i}\hat{\boldsymbol{e}}_i = r_{\mathcal{B}i}\hat{\boldsymbol{e}}_i' =$ some vector of interest.

$\vec{\boldsymbol{r}}' = \underline{\underline{\mathbf{R}}} \cdot \vec{\boldsymbol{r}} = r_{\mathcal{F}i}\hat{\boldsymbol{e}}_i' = r_{\mathcal{F}i}'\hat{\boldsymbol{e}}_i = r_{\mathcal{B}i}'\hat{\boldsymbol{e}}_i' =$ the rotation by $\underline{\underline{\mathbf{R}}}$ of $\vec{\boldsymbol{r}}$.

$R = R_{\mathcal{F}} = R^{\mathcal{F}} = R_{\mathcal{F}}^{\mathcal{F}} = [\underline{\underline{\mathbf{R}}}]_{\mathcal{F}} = [\underline{\underline{\mathbf{R}}}]^{\mathcal{F}} = [\underline{\underline{\mathbf{R}}}]_{\mathcal{F}}^{\mathcal{F}} =$ matrix representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{F}$ basis.

$R_{\mathcal{B}} = R^{\mathcal{B}} = R^{\mathcal{B}}_{\mathcal{B}} = [\underline{\underline{\mathbf{R}}}]_{\mathcal{B}} = [\underline{\underline{\mathbf{R}}}]^{\mathcal{B}} = [\underline{\underline{\mathbf{R}}}]^{\mathcal{B}}_{\mathcal{B}}$ = matrix representation of $\underline{\underline{\mathbf{R}}}$ using $\mathcal{B}$ basis.

$r = r_{\mathcal{F}} = r^{\mathcal{F}} = [\vec{\boldsymbol{r}}]_{\mathcal{F}} = [\vec{\boldsymbol{r}}]^{\mathcal{F}}$ = matrix representation of $\vec{\boldsymbol{r}}$ using $\mathcal{F}$ basis.

$r' = r'_{\mathcal{F}} == [\vec{\boldsymbol{r}}']_{\mathcal{F}} = [\vec{\boldsymbol{r}}']^{\mathcal{F}}$ = matrix representation of $\vec{\boldsymbol{r}}'$ using $\mathcal{F}$ basis.

$r = r_{\mathcal{B}} = r^{\mathcal{B}} = [\vec{\boldsymbol{r}}]_{\mathcal{B}} = [\vec{\boldsymbol{r}}]^{\mathcal{B}}$ = matrix representation of $\vec{\boldsymbol{r}}$ using $\mathcal{B}$ basis.

$r' = r'_{\mathcal{B}} == [\vec{\boldsymbol{r}}']_{\mathcal{B}} = [\vec{\boldsymbol{r}}']^{\mathcal{B}}$ = matrix representation of $\vec{\boldsymbol{r}}'$ using $\mathcal{B}$ basis.

Questions. So that you learn the spirit, the first few questions are answered for you. In all cases, try to derive the results from expressions that you solidly trust, like $\underline{\underline{\mathbf{R}}} \cdot \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}'_1$.

a) Given $\vec{\boldsymbol{r}}$ and the $\hat{\boldsymbol{e}}_i$ what is $[\vec{\boldsymbol{r}}]_{\mathcal{F}}$ = the column vector of the components of $\vec{\boldsymbol{r}}$ using the base vectors from $\mathcal{F}$? Answer: $[\vec{\boldsymbol{r}}]_{\mathcal{F}} = \begin{bmatrix} \vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}_1 \\ \vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}_2 \\ \vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}_3 \end{bmatrix}$.

b) Given $\underline{\underline{\mathbf{R}}}$ and the $\hat{\boldsymbol{e}}_i$ find R_{ij}? Answer: $R_{ij} = \hat{\boldsymbol{e}}_i \cdot \underline{\underline{\mathbf{R}}} \cdot \hat{\boldsymbol{e}}_j$.

c) Given $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}'_i$ find R_{ij}?

d) Given R_{ij} and r_i find r'_i?

e) Given R_{ij} and r_i find $r'_{\mathcal{B}i}$?

f) Given R_{ij} and r'_i find r_i?

g) Given R_{ij} and r'_i find $r_{\mathcal{B}i}$?

h) Given R_{ij} find $R_{\mathcal{B}ij}$?

i) Write $[\vec{\boldsymbol{r}}']_{\mathcal{F}}$ as many different ways as you can.

j) Given $\underline{\underline{\mathbf{A}}} = R_{ij}\hat{\boldsymbol{e}}'_i\hat{\boldsymbol{e}}'_j$, find A_{ij}.

k) Pose and answer a few questions, of the general type above, that most test your confusion.

12. Escape velocity. Assume a non-rotating earth. Assume the central force of gravity on a mass at radius r is

$$F = mg\frac{R^2}{r^2}$$

where R is the radius of the earth.

a) How fast need be a vertical launch so that the projectile goes infinitely far?

b) How does the answer change if the launch is at angle θ from vertical? (be absolutely convincing about your reasoning here).

c) Answer the previous two questions for these two attraction laws:

i) $F = A/r^{1.1}$.

ii) $F = A/r$.

13. See this video for about a minute at the 10:05 mark. http://www.youtube.com/watch?v=dQai9QikTBI That is your HW problem. Does Feynman have it right? Hint: assume a steady precessing solution of a flat plate. Keep clear in your head the relation between these three angular velocities: precession, spin and total.

14. Euler equations. Use the object-fixed coordinate system $\mathcal{B}$ based on $\hat{\boldsymbol{e}}'_i$ in which the moment of inertia matrix is

$$[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

a) Set up numerical solutions of the general non-linear free motion of such an object. That is, given initial values for the angular velocity components, find the angular velocities as a function of time.

b) Using the full non-linear simulation, consider a small deviation to rotation about the x' axis

$$[\vec{\boldsymbol{\omega}}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{\omega}'_1 \\ \hat{\omega}'_2 \\ \hat{\omega}'_2 \end{bmatrix}$$

where the $\hat{\omega}'_i \ll 1$ and are otherwise whatever you like. With a plot show the time histories of the $\hat{\omega}'_i$.

c) Do likewise with rotations about the y' and z' axis.

d) Conclude that axis about one or more of the axes is unstable and that the others are 'stable'.

15. Euler Equations and then some. We have the great Euler equations for a rigid object:

$$\vec{\boldsymbol{M}}_{/G} = \underline{\underline{\mathbf{I}}} \cdot \dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}} \times \left(\underline{\underline{\mathbf{I}}} \cdot \vec{\boldsymbol{\omega}}\right).$$

a) Write the Euler equations entirely in terms of components and or column vectors that are entirely in the $\mathcal{F}$ basis. For simplicity assume that $[\underline{\underline{\mathbf{I}}}]^{\mathcal{B}}$ is diagonal.

b) Set these up to solve in Matlab.

c) Using these equations for the evolution of $\vec{\boldsymbol{\omega}}$, or the ones from the previous body-fixed representation, set up to solve for $\underline{\underline{\mathbf{R}}}(t)$ by setting up the evolution equations for $\underline{\underline{\mathbf{R}}}$, namely

$$\dot{\underline{\underline{\mathbf{R}}}} = \underline{\underline{\mathcal{S}}}(\vec{\boldsymbol{\omega}}) \cdot \underline{\underline{\mathbf{R}}}$$

with matrix form in the $\mathcal{F}$ frame. Alternatively, you can assemble the rotation matrix as

$$R^{\mathcal{F}} = \begin{bmatrix} [\hat{\boldsymbol{e}}'_1]_{\mathcal{F}} & [\hat{\boldsymbol{e}}]'_2]_{\mathcal{F}} & [\hat{\boldsymbol{e}}'_3]_{\mathcal{F}} \end{bmatrix} \qquad \text{and update with} \qquad \dot{\hat{\boldsymbol{e}}}'_i = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{e}}'_i.$$

d) Solve these in Matlab. Animate the motion using some pictorial representation you like (e.g. a box, ellipsoid or Jack) for these problems.

i) Pick some object and initial conditions that interest you.

ii) Consider an initial condition near to rotation nearly about the intermediate axis.

iii) Wobbling of a plate.

In each case you may want to make other plots, or show the trajectories of some special points, in order to clarify some statement you want to make (like about the relative rotation rates of the plate normal and of the whole plate).

e) Instead of representing the rotation with a rotation matrix use Euler angles. That is write the equations of motion using Euler angles and show at least one example where your solution agrees with the rotation matrix solution.

16. Disk on plane. Only steady motions with constant lean angle ϕ and constant pitch rate ω_s. Given disk radius r, mass m, gravity g and inertia

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 2I \end{bmatrix}.$$

Find, restrictions on the radius of the circular motions R, ω_s, the precession rate ω_p, and the lean angle ϕ.

(a) All solutions with no slip

(b) All solutions with no friction.

(c) The intersection of the two (all solutions with no slip and no friction), these are the most famous ones.

17. Steady precession of a top. Find steady precession motions of a top with a fixed point (the contact with the floor). Assume such precession is very close to upright. What is the slowest spin, for given mass and inertia parameters, for which such steady precession solutions exist. This is a non-rigorous guess for the critical spin speed for upright stability.

18. Disk on table cloth. Q-type question. A rolling disk is on a flat table with a table cloth underneath. Someone yanks out the table cloth. After that, the disk slides/skids on the table and then, later, eventually rolls. How fast is it rolling? While on the table cloth you can assume constant force/acceleration pulling or variable pulling and you can assume the disk slides on the table cloth, rolls on the table cloth, or a mixture of both. The answer is the same in every case.

19. Rolling disk in 3D. Find, solve and animate solutions for a rolling disk in 3D. Use any method you like.

20. Slidding disk in 3D. Find, solve and animate solutions for a disk sliding frictionlessly on a plane.

21. Steady precession Using analytic (non-numerical) methods find the steady precession motions for the two problems above. Find the set of solutions that the two problems have in common. For a few cases check that your full solutions to the full equations of motion agree with your steady state solutions.

22. Spinning top in 3D. Find, solve and animate solutions for a spinning top in 3D. Use Euler angles. Find with pencil and paper (or computer algebra) the steady precessing solutions and check that your full equations solved numerically agree with that solution. Is the critical speed for the steady precessing solution indeed the critical speed for the full 3D solutions?

23. Approximate beam modes. Find approximate solutions for first few modes of a vibrating beam. Length is L, mass m, stiffness EI. Half way down is point mass m and hanging from that with spring k another point mass m. Use a value of 1 for all parameters so you can compare solutions with other students.