EVOLVING DOCUMENT
TAM 4735/5735 Intermediate Dynamics and Vibrations
Syllabus and HW

Date of this version: November 27, 2012

Note: Things in blue are hyperlinks.

Contents

Syllabus

41 Lectures, 50 minutes each

1. Wed. Aug 22, 2012, Fundamentals of Mechanics

Reading: Ruina and Pratap Chapter 1

1. Space, time and mass
2. Force, Free Body Diagrams (FBDs), action and re-action
3. The three Pillars:
   (a) Constitutive Laws *e.g.,* $F = kx$
   (b) Geometry *e.g.,* $a = d^2x/dt^2, \vec{\boldsymbol{a}} = d^2\vec{\boldsymbol{r}}/dt^2$
   (c) Laws of mechanics: Linear and angular momentum, Energy

*The single homework problem associated with this lecture (however loosely) is Homework problem **1**. Due Wednesday August 29

2. Fri. Aug. 24, Particle Mechanics

1. Linear Momentum Balance: $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$
2. $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ with $\vec{\boldsymbol{F}} = \vec{f}(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t)$
3. Set up eqns. from FBD
4. Separate into 1st order eqns
5. Numerical Soln. *e.g.,* Euler's method
6. Example: Cannon ball with gravity and quadratic drag.

1. Special cases
   (a) $F = -kr$
   (b) $F = -c/r^2$ (*e.g.,* $c = GmM$)

*Homework problem **2** due Wed Sept 5.

3. Monday Aug 27. Theorems for motion of a single particle

1. Balance and Conservation of linear and angular momentum
2. Work and energy (Power and rate of change of energy)
3. Conservative forces. All of these things are equivalent for a force that depends on position in some region of interest, $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ (and not if $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(t)$, $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(\vec{\boldsymbol{v}})$, $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}, t)$ *etc.*)
   - (a) $\vec{\boldsymbol{F}}$ is conservative
   - (b) In the whole region ($\vec{\nabla}$ is sometimes written ∇)
     $\vec{\nabla} \times \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$.
   - (c) For all closed paths
     $\oint dW \equiv \oint \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = 0.$
   - (d) Work is independent of path for all paths between A and B, for all points A and B:
     $\int_A^B dW \equiv \int_A^B \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ is path independent.
   - (e) A potential V exists so that
     $\vec{\boldsymbol{F}} = -\vec{\nabla} V$.
     Notation: in this class we use $V = E_{\mathrm{P}}$. V is only determined up to an additive constant.
   - (f) A unique potential $V(\vec{\boldsymbol{r}})$ exists so that
     $V(\vec{\boldsymbol{r}}) = \int_C^{\vec{\boldsymbol{r}}} dW = \int_C^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}'$
     for any reference point A, fixed once and for all, and any path(s) from A. Choosing a different reference point only changes V by an additive constant.

   Any of the things in the list above implies all of the others. The Wikipedia page on Conservative Forces is decent.
4. Examples
   - (a) $\vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{k}} \quad \Rightarrow \quad E_{\mathrm{P}} = mgz$ (near earth gravity)
   - (b) $\vec{\boldsymbol{F}} = -k\vec{\boldsymbol{r}} \quad \Rightarrow \quad E_{\mathrm{P}} = kr^2/2$ (important for vibrations, *the* canonical example)
   - (c) $\vec{\boldsymbol{F}} = -GmM\vec{\boldsymbol{r}}/r^3 \quad \Rightarrow \quad E_{\mathrm{P}} = -GmM/r$ (inverse square gravity)
   - (d) $\vec{\boldsymbol{F}} = -c\vec{\boldsymbol{v}}$ is not conservative
   - (e) $\vec{\boldsymbol{F}} = -cv\vec{\boldsymbol{v}}$ is not conservative
   - (f) $\vec{\boldsymbol{F}} = F_0 \sin(\omega t)\hat{\boldsymbol{\imath}}$ is not conservative

*Homework problems **3** and **4**. Due Wednesday Sept 5.

4. Wed. Aug 29. Integration methods. Intro to central force motion

1. Euler's method vs midpoint method: truncation (method) errors and roundoff errors. This www page has a decent write up, although it doesn't acknowledge the random cancellation, thus $\sqrt{n}$, aspect of roundoff error.
2. Ease of setting up and integrating eqs of motion for any force on one particle.
3. Central Force motion: Assume $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{e}}_r$. Numerical examples for $F = -GmM/r^2$ and $F = -kr$.

4. Equal areas in equal times (Newton's proof). See lecture 2 of Feynman's Messenger Lectures at Cornell where he reproduces Newton's derivation Youtube (starting 14 minutes in).

*Homework problem **5**. Due Wed. Sept 5.

5. Fri. Aug. 31. Central Force Motion (cont'd). Mechanics with many particles.

1. Theorems for central force motion
   (a) Angular momentum
   (b) Power, work and energy
   (c) apogee, perigee and circular orbits
2. Systems of many particles
   (a) Internal vs external forces
   (b) Pairwise equal and opposite forces, and why that is not always reasonable

*Homework problems **6** and **7**. Due Wed. Sept 12.

6. Wed. Sept 5. Mechanics with many particles (cont'd).

Reading: Ruina and Pratap chapters 9, 11 and 12. Greenwood, Intermediate Dynamics, Sections 4.1-4.3.

1. Center of Mass: $\vec{\boldsymbol{r}}_{\mathrm{G}} m_{\mathrm{tot}} = \sum m_i \vec{\boldsymbol{r}}_i$
2. Momentum $\vec{\boldsymbol{L}} = m_{\mathrm{tot}} \vec{\boldsymbol{r}}_{\mathrm{G}}$
3. Angular momentum: $\vec{\boldsymbol{H}}_{/\mathrm{C}} = \vec{\boldsymbol{H}}_{\mathrm{G}/\mathrm{C}} + \vec{\boldsymbol{H}}_{/\mathrm{G}}$
4. Power, work and energy (partially in Homework).

*Homework problem **8**. Due Wed. Sept 12.

7. Fri. Sept 7. Mechanics with many particles (cont'd).

1. LMB using center of mass
2. AMB using center of mass
3. Numerical example using ODE23: Two masses and a spring.

*Homework problem **9**. Due Wed. Sept 19.

8. Mon. Sept 10. Astronauts and constraints.

1. Skylab shows that with no external forces an astronaut can rotate with zero angular momentum, but cannot translate.
2. The simplest constraint example: 1D, 2-masses connected by a rod with a force on one of them.
3. Approaches to constraints:
   (a) I. Don't use them: write equations with the higher DOF system and approximately enforce the constraint with springs and/or dampers as connections.

(b) II.1 Write DAEs (Differential Algebraic Equations) and solve for constraint forces with accelerations
(c) II.2 Manipulate the DAEs to eliminate constraint forces and reduce the order of the equations.
(d) II.3 Finesse constraint forces by special equations of motion that don't contain them.

Here is one good **cat video**. Also see more analytic discussion, with some other rotary locomotion puzzles in this **discussion** starting at minute 20:32.

*Homework problem **10**. Due Wed. Sept 19.

9. Wed Sept 12. Constraints (2 masses, cont'd).

1. Using methods I and II.1-3 with two masses connected by a rod. Including various methods for II.3, including:
   (a) Find appropriate momentum (or angular momentum) equations that do not contain the constraint forces and do not involve degrees of freedom that are constrained out (e.g., system LMB using COM position as the configuration variable).
   (b) Energy conservation ($E_{\text{K}} + E_{\text{P}} = Constant$), works for 1 DOF systems
   (c) Power balance ($P_{tot} = \dot{E_{\text{K}}}$), works for 1 DOF systems
   (d) Lagrange equations:
   $$\frac{\partial \mathcal{L}}{\partial x} - \frac{dt}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0,$$
   where
   $$\mathcal{L} = E_{\text{K}} - E_{\text{P}} = \mathcal{L}(x, \dot{x}) \qquad (= \text{“}T - V\text{”}).$$
2. Simple pendulum.

*No Homework

10. Fri. Sept 14. Constraints & Simple pendulum (Cont'd)

A point mass is connect to a string/rod. Some of the various methods to get to equations of motion are enumerated

I. Hang the mass from a spring with rest length ℓ_0 and high spring constant k.
II.1 Write out all DAEs and

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \quad \text{and} \quad \frac{d^2}{dt^t}\left\{x^2 + y^2 = \ell_0^2\right\}$$
$$\Rightarrow \underbrace{\begin{bmatrix} m & 0 & x/\ell_0 \\ 0 & m & y/\ell_0 \\ x/\ell_0 & y/\ell_0 & 0 \end{bmatrix}}_{\overbrace{\begin{bmatrix} M & J \\ J' & 0 \end{bmatrix}}} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ mg \\ -(\dot{x}^2 + \dot{y}^2)/\ell_0 \end{bmatrix}$$

The first two equations are $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and the third can be interpreted as 'the radial component of acceleration is the centripetal acceleration'.

II.2: The first two equations (in the array above) can be manipulated to eliminate T and $x^2 + y^2 = \ell_0^2$ (with it's first and second derivatives) can be used to eliminate $y, \dot{y}$ and $\ddot{y}$.

II.3 Finessing use of the constraint by measuring all motions with minimal coordinates (in this case, for example, θ or x) and using equations of motion that do not involve the constraints. For example:

1. $\text{AMB}_{/0}$,
2. $\{\text{LMB}\}\cdot\hat{\boldsymbol{e}}_\theta$,
3. $\vec{\boldsymbol{r}}\times\{\text{LMB}\}$,
4. $\dot{E}_{tot} = 0$,
5. $P = \dot{E}_{\text{K}}$.
6. Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{dt}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \quad \text{where} \qquad \mathcal{L} = m\left(\ell_0 \dot{\theta}\right)^2/2 + mg\cos\theta$$

Note that within general methods one has to choose coordinates (*e.g.,* x or θ) and make other choices as well (*e.g.,* whether or not to substitute in ℓ_0 for $x^2 + y^2$ where those expressions appear).

*Homework problems **11**, **12** and **13**. Due Wed. Sept 26.

11. Mon Sept 17. The 'rigid-object' constraint (2D)

1. Naive approach: For n particles
   (a) Write $\vec{\boldsymbol{F}}_i = m_i \vec{\boldsymbol{a}}_i$, for the $i = 1 \ldots n$ particles $\Rightarrow$ $2n$ 2nd order ODEs (2 for each particle.
   (b) Imagine some pairs of the particles, i and j, are connected by massless rigid rods with unknown tensions T_{ij}. These unknown tensions appear in the ODEs above.
   (c) For each rod also write $2n - 3$ algebraic kinematic constraint equations, setting the lengths ℓ_{ij} of the rods to be fixed.
   (d) For n particles in 2D you need $2n - 3$ rods. The constraints thus lower the system from $2n$ degrees of freedom to the 3 degrees of freedom of a 2D rigid object (*e.g.,* the position of one point and the angle of one bar, this is one parameterization of the 3 DOFs of the ***pose*** of a 2D rigid object.

   For the 3DOFs of a rigid object this approach uses $4n - 3$ DAEs. It is thus a lot of setup and a lot of numerical solution. On top of which the constraints may drift apart in time. Almost no-one (exactly no-one?) does the mechanics of rigid objects this way.
2. Use the rigid object constraint and finesse the constraint equations: rotations of a rigid object. Thus we need to think about rotations (θ, $\omega = \dot{\theta}$, $\ddot{\theta} = \dot{\omega}$, $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$) and coordinate systems, including rotating coordinate systems ($\mathcal{B}, x', y', \hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}'$).

   Reading: Ruina/Pratap 14.1,14.2, 15.1, 15.2

*Homework problem **14**. Due Wed. Sept 26.

12. Wed Sept 19. The 'rigid-object' constraint (2D), cont'd.

Derivation of

$$\dot{\vec{H}}_{/\mathrm{G}} = I^G \omega \hat{\boldsymbol{k}}$$

Read Ruina/Pratap sections 15.3, 15.4, 16.1, 16.2, 17.1, 17.2. Homework problem **15** due Wed Oct 3.

13. Fri Sept 21. Movie.

Angular momentum and non-holonomic constraints
(Andy Ruina at **Dynamic Walking 2010)**

14. Mon Sept 24. Rigid body example: The compound or physical pendulum.

Deriving

$$\ddot{\theta} = -\frac{mgd}{I + md^2} \sin\theta$$

In class quiz questions:

1. Is the reaction force on the rod from the hinge along the rod? more vertical? more horizontal? totally vertical? totally horizontal?
2. If a point mass was added at the center of mass would the pendulum speed up? slow down? oscillate at the same frequency?

15. Wed Sept 26. Rigid body example: Chaplygen Sleigh

Consideration of modeling wheels and casters. Derivation of governing equations

$$\begin{aligned} \{\mathrm{LMB}\} \cdot \hat{\boldsymbol{e}}_t \quad &\Rightarrow \quad \dot{v} = d\dot{\theta}^2 \\ \{\mathrm{AMB}_{/C}\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad \ddot{\theta} = -\frac{md}{I + md^2} v\dot{\theta} \end{aligned}$$

(Derivation on page 7 of **Ruina**, *Non-Holonomic Stability Aspects of blah blah blah*)

Homework problems **16** and **17** due Wed Oct 3.

16. Fri Sept 27. Chaplygen Sleigh (cont'd)

The complete governing equations were found. Also the equations linearized about the zero-rotation constant velocity solution. Features of the equations were discussed.

Quiz question: How many degrees of freedom? Answer 3 configuration degrees of freedom and 2 velocity degrees of freedom $3 > 2 \quad \Rightarrow \quad$ non-holonomic.

No new homework for this lecture (related problem assigned with previous lecture).

17. Mon Oct 1. Matlab, Friction and Double Pendulum intro.

1. Matlab, 3 examples shown in lecture (also **posted** on course www page):
   (a) How to use ODESET to set 'abstol' and 'reltol'
   (b) How to use 'events' to stop integration when something happens before the end of span.
   (c) How to do animations of simple line drawings
2. Friction: Coulombs law: a curve on a plane and the friction cone (see Ruina/Pratap section 4.3 and Appendix B.
3. Setup of the double pendulum equations of motion using angular momentum balance.

18. Wed Oct 3. Double pendulum continued.

1. review of setup using angular momentum balance
2. getting the equations onto a computer for simulation.

HW problem **18** and **19** due Wed Oct 10.

19. Fri Oct 5. Double Pend (dont'd), Mathie Equation

1. Consider still the double pendulum with shoulder at 0 and elbow at E. As before, the upper arm $\mathrm{OG_1E}$ is object 1 and the forearm $\mathrm{EG_2}$ is object 2.
   (a) First option. Last class we used $\mathrm{AMB}_{/0}$ for the system and $\mathrm{AMB}_{/E}$ for the fore-arm. We then set out to calculate all positions, velocities and accelerations in the expression for $\dot{\vec{\boldsymbol{H}}}_{/0}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{E}}$ in terms of θ_1 and θ_2 and their derivatives $(\dot\theta_1, \ddot\theta_1, \dot\theta_2, \ddot\theta_2)$. To do this we used $\vec{\boldsymbol{\omega}}_1 = \dot\theta_1 \hat{\boldsymbol{k}}$ and expressions based on $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ and $\dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}$. The alternative approach is to write the positions of all points in terms of the θs and to merely differentiate. For example:

$$\vec{\boldsymbol{v}}_{\mathrm{G}_1} = \frac{d}{dt}\vec{\boldsymbol{r}}_{\mathrm{G}_1} = \frac{d}{dt}\{d_1 \cos\theta_1 \hat{\boldsymbol{\imath}} + d_1 \sin\theta_1 \hat{\boldsymbol{\jmath}}\} = -(d_1 \sin\theta_1)\dot\theta_1 \hat{\boldsymbol{\imath}} + d_1(\cos\theta_1)\dot\theta_1 \hat{\boldsymbol{\jmath}}.$$

   Accelerations can be found likewise. The net result, when all is reduced, is the same set of equations as before.
   (b) **Eqns for each object, manipulated to eliminate constraint forces**. If we break the arm into two links we can write linear momentum balance for each and also angular momentum balance for each. This is 6 equations total. They include the 4 constraint force components from the force at the shoulder $\vec{\boldsymbol{R}}_0$ (2) and elbow $\vec{\boldsymbol{F}}_{\mathrm{E}}$ (2). By adding and subtracting these equations appropriately we can try to eliminate the constraint forces. However this is just what we did already using angular momentum balance about appropriate points. That is, calling, for example linear momentum balance for system 1, LMB_1 we have, taking cross products of both sides of the equations:

$$AMB_{12/0} = \vec{\boldsymbol{r}}_{\mathrm{G}_1/0} \times \mathrm{LMB}_1 + \mathrm{AMB}_{1/G_1} + \vec{\boldsymbol{r}}_{\mathrm{G}_2/0} \times \mathrm{LMB}_2 + \mathrm{AMB}_{2/G_2}$$

   and

$$AMB_{2/E} = \vec{\boldsymbol{r}}_{\mathrm{G}_2/\mathrm{E}} \times \mathrm{LMB}_2 + \mathrm{AMB}_{2/G_2}$$

That is, the two minimal equations from last lecture are just linear combinations of the 6 equation of linear and angular momentum balance of the two independent objects. The objects are not really independent, however, because we use the constraints (that they are jointed together) when calculating the velocities and accelerations in terms of minimal coordinates.

2. **Mathieu Equation.** Imagine an inverted pendulum, tipped an angle θ from the vertical. The base at 0 has some given vertical upwards acceleration $a_0(t)$. By taking $\text{AMB}_{/0}$ we find (you should all be able to get here on your own in a couple of minutes):

$$\ddot{\theta} = \frac{m\,(g + a(t))\,d}{I + md^2}\sin\theta$$

That is, the acceleration is the same as if it was a still pendulum and gravity had been altered. It is a general theorem in mechanics that the equations are the same if

   (a) You put the whole system in an accelerating frame (which adds, say $\vec{\boldsymbol{a}}$ to all accelerations, or

   (b) You add a body force at the center of mass of every object in a fixed direction and with size proportional to mass which adds, say, a body force $-\vec{\boldsymbol{a}}m$ to each mass.

It's easy to prove. Try it. Einstein elevated this to a fundamental postulate in his development of general relativity, so I've heard.

On the computer we find that for an oscillating $a(t)$ a pendulum can be stabilized when it is upright. The physical demo in the lecture also showed this. A hard challenge is to make intuitive sense of this. It was a puzzle for me for about 25 years then I heard an explanation from Mark Levi (mathematician at Penn State). Then, last year, I thought of two more. Can you make sense of it?

20. Wed Oct 10. Five term acceleration formula review.

The only explicit review request was for derivation of, and explanation of, the "five term acceleration formula" and it's relation to "the Q-dot" formula (the so-called *transport theorem*. Read about these in Ruina/Pratap sections 17.2-4.

HW problem **20** due Wed Oct 24.

21. Fri Oct 12 . Comments on inverted pendulum with inverted base + intro to vibrations.

1. **Intuitions about stability of Mathieu equation.** Try to understand why vertical oscillations of the base cause a net alignment force. In all cases we have

$$\ddot{\theta} = \frac{g + a_b}{\ell}\sin\theta$$

where a_b is the vertical acceleration of the base. The first thing to notice is that is that if $g + a_b > 0$ for all time then if the pendulum starts with $\theta > 0$ and $\dot{\theta} > 0$ that θ is monotonically increasing. So stability of the inverted pendulum depends on $a_b < -g$ for at least parts of the motion; the downwards acceleration of the base needs to be bigger than g at least some times.
Given that, for simplicity lets look at the alignment force when there is base shaking but no gravity. Here are three intuitive explanations, in brief:

(a) Replace the oscillating base with an oscillating gravity. This would cause slight oscillations of the mass. But when the mass is lower in that oscillation cycle the force is upwards and more tangent to the circle of motion than when the mass is higher. Thus there is a righting torque.

(b) Look at the motions as the bar, a 'two-force-member' pushes and pulls on the mass. When it pushes is at the low point in the base motion, and the bar is more vertical. Thus the average of the pushing and pulling has a component upwards along the circle.

(c) Imagine the mass is pushed and pulled up and down a track shaped like a tractrix (pursuit) curve. The bar force is always tangent to that track. A centrifugal force from the track fights the centripetal acceleration of the mass moving on a curved track. But the track isn't there and the missing force is an alignment force.

2. **You should review 1DOF oscillations**

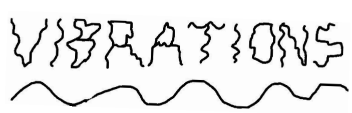
Homework: Read/review/learn Inman chapters 1-2 and Ruina/Pratap sections 10.1-2. More or less this means: know and/or be able to derive the formulas on the inside cover of Inman. The key bits of freshman math that you need to use are:

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \qquad \text{(the quadratic formula)}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{(the Euler formula)}$$

$$\cos(b-a) = \cos a\cos b + \sin a\sin b \qquad \text{(addition formula for cosines)}$$

$$(b-a) = \cos a\sin b - \sin a\cos b \qquad \text{(addition formula for sines)}$$

21.Mon Oct 15 . Introduction to .

The main thing is solution of this 'forced, damped oscillator' equation

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

especially for

$$f(t) = F_0 \sin\omega_0 t, \qquad \text{(sinusoidal forcing)}$$

but also (chapter 3 of Inman) for a delta function, a Heaviside function, and an arbitrary function (using convolution).

First off is numerical solution, which is straightforward by writing the 2nd order governing equation as a pair of 1st order ODEs:

$$\dot{x} = v$$
$$\dot{v} = \frac{-cv - kx + F_0\sin\omega_0 t}{m}.$$

With these you can find the solution with any initial conditions x_0, v_0. As shown in class you can see underdamped motion, over damped motion, resonance, etc.

For analytic solution of the motion the key things are the homogeneous x_h and particular x_p solutions:

$$x_h = Ae^{-\omega_n \zeta t} \sin(\omega_d t + \phi) \qquad \text{(transient response)}$$
$$x_p = X \sin(\omega_0 t + \phi) \qquad \text{(steady state response)}$$

In terms of m, c, k, F_0 and ω_0 the constants above are:

$$\omega_n = \sqrt{k/m} \qquad \text{(undamped natural frequency)}$$
$$r = \omega_0/\omega_n \qquad \text{(ratio of forcing to undamped natural frequencies)}$$
$$c_{cr} = 2\sqrt{km} \qquad \text{(critical damping, if } c = c_{cr} \text{ the } \surd \text{ is zero)}$$
$$\zeta = c/c_{cr} = c/(2\sqrt{km}) \qquad \text{(damping ratio)}$$
$$\omega_d = \omega_n \sqrt{1-\zeta^2} \qquad \text{(damped natural frequency)}$$
$$\phi \qquad \text{(Phase. For } x_h \text{ this depends on ICs.)}$$
$$\phi = \tan^{-1}\left(2\zeta r/(1-r^2)\right) \qquad \text{(phase for } x_p\text{)}$$
$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \text{(Amplitude of steady state response.)}$$
$$\omega_{max} = \omega_n \sqrt{1-2\zeta^2} \qquad \text{(max response is when } \omega_0 = \omega_{max}\text{)}$$

Mostly we are interested in underdamped motions, possibly with $\zeta \ll 1$. Then:

1. The damped natural frequency ω_d and the peak in the response curve ω_{max} are both close to the undamped natural frequency ω_n.
2. For low frequency forcing the motion is nearly in phase with the forcing: $r \ll 1 \quad \Rightarrow \quad \phi \approx 0$. For high frequency forcing the motion is nearly out of phase with the forcing: $r \gg 1 \quad \Rightarrow \quad \phi \approx \pi$.

Homework **21** and **22** due Wednesday Oct 24.

22.Wed Oct 17 . Introduction to normal modes.

For now consider linear systems with 2 or more degrees of freedom and no damping. The governing equation is:

$$M\ddot{\vec{x}} + K\vec{x} = \vec{0}$$

where
$\vec{x}$ is a list of n minimal coordinates, say the positions of masses;
M is an $n \times n$ mass matrix. It is symmetric and positive definite;
K is an $n \times n$ stiffness matrix. It is symmetric and positive semidefinite.

Solving this numerically is easy enough by using this set of $2n$ first order ODEs:

$$\dot{\vec{x}} = \vec{v}$$
$$\dot{\vec{v}} = -M^{-1}K\vec{x} \qquad \text{(or equivalent using backslash in Matlab)}$$

Thus, given parameters and initial conditions it is easy enough to find the motions.

We solve this analytically by making the guess that

$$\vec{x} = e^{i\omega t}\vec{\mathbf{u}}$$

where $\mathbf{u}$ is a constant vector and ω is a constant number. Plugging this guess into the governing ODEs we get:

$$\begin{aligned}
& -\omega^2 M\vec{\mathbf{u}}e^{i\omega t} + K\vec{\mathbf{u}}e^{i\omega t} = \vec{\mathbf{0}} \\
\Rightarrow\quad & -\omega^2 M\vec{\mathbf{u}} + K\vec{\mathbf{u}} = \vec{\mathbf{0}} \\
\Rightarrow\quad & -\omega^2 M^{-1}M\vec{\mathbf{u}} + M^{-1}K\vec{\mathbf{u}} = \vec{\mathbf{0}} \\
\Rightarrow\quad & [M^{-1}K - \omega^2 I]\vec{\mathbf{u}} = \vec{\mathbf{0}}.
\end{aligned}$$

This is of the form of a standard eigenvalue problem: $[A - \lambda I]\vec{v} = \vec{\mathbf{0}}$. Even though M and K are symmetric there is no reason to think that $M^{-1}K$ is symmetric. And it usually isn't. Nonetheless, we live in a friendly universe and $M^{-1}K$ will be found to have n linearly independent eigenvectors and n eigenvalues. Not only that, the eigenvalues turn out to be positive, so the ωs, the square roots of the eigenvalues, are real.

Because both real and imaginary parts are real solutions. The most general solution we can construct with our guess is;

$$\vec{x}(t) = \sum_i^n [A_i \cos\omega_i t + B_i \sin\omega_i t]\vec{\mathbf{u}}_i.$$

There are enough free constants here to satisfy any initial conditions $\vec{x}_0$, $\dot{\vec{x}}_0$ so this is the general solution.

Thus, the general solution is a sum of 'normal mode' vibrations. For each mode, all parts (all components of $\vec{x}$) move synchronously.

Reading: Inman chapter 4.

Homework **23**, **24** due Wednesday Oct 24.

22. Fri. Oct 19 . Normal modes cont'd.

The collection of mode shapes (eigenvectors $\vec{\mathbf{u}}_i$ of $M^{-1}K$) can be stored in a matrix P. Each column of P is an eigenvector of $M^{-1}K$. The eigenvalues ω_i^2 can be stored as the diagonal elements of a square matrix D. We can find P and D with the single Matlab command

```
[P D] = eig( M^(-1)*K ) ;  % type >>help eig.
```

If, say, initial conditions are given as $\vec{x}(0) = \vec{x}_0$ and $\vec{v}(0) = \vec{\mathbf{0}}$ then we can find the coefficients

$$\vec{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ etc \end{bmatrix}$$

by solving $\vec{x}_0 = P\vec{A}$ for $\vec{A}$ using the Matlab command:

```
A = P\x0;  % type >>help \  .
```

Thus we can put together the solution as a sum of normal modes. Example **code** was shown in lecture and the solution compared with direct integration of the ODEs.

Some things to note:

1. The columns of P (the eigenvectors $\vec{\mathbf{u}}_i$ of $M^{-1}K$ are generally *not* orthogonal.
2. The normal modes are simple synchronous oscillations, but the superposition of them can be wild and chaotic looking (but they are technically *not* 'chaotic'). They are 'quasi-periodic', meaning that all positions as functions of time are a sum of periodic functions (namely sine waves for linear vibrations).
3. This method, using $M^{-1}K$ is downgraded by most professionals, including Inman, because it doesn't let you use reasoning associated with symmetric matrices (note that $M^{-1}K$ is not symmetric).
4. The 'professional' method is based on a change of variables using the square root $M^{1/2}$ of the mass matrix (coming lectures).

Reading: Inman chapter 4.

Homework **25** and **26** due Wednesday Oct 31.

23. Mon. Oct 22. Personal introspective contemplation of mechanics.

No class due to, say, communication difficulties. ☹

24. Wed. Oct 24. Normal modes cont'd.

Various things

1. M and K don't always come out symmetric from the equations of motion. But they can always be made symmetric by adding and subtracting rows (adding and subtracting equations).
2. M is always positive definite: $2E_{\mathrm{P}} = x'Mx > 0$ for $x \neq 0$.
3. K is always positive semi-definite: $2E_{\mathrm{P}} = x'Kx \geq 0$ for $x \neq 0$. The cases for which non-zero deformations ($x \neq 0$) give no energy are motions where no springs are stretched or compressed.
4. The directions for which $Ku = 0$ are eigenvectors of $M^{-1}K$ giving vibration frequencies of zero. For those mode shapes you need to use $x = (A + Bt)u$ instead of $x = (A \sin \omega_i t + B \cos \omega_i t)u$. This requires the use of an IF statement when summing normal modes to get a general solution.
5. If we make the change of coordinates $q = M^{1/2}x$ then the governing vibration equation turns to
$$\ddot{q} + M^{-1/2}KM^{-1/2}q = 0.$$
The eigenvectors v_i of $M^{-1/2}KM^{-1/2}$ are (or can be taken to be) ortho-normal. Read about this official approach to vibrations in Inman 4.1-4 and Ruina/Pratap 10.3.

Homework **27** and **28** due Wednesday Oct 31.

25. Fri. Oct 26, Normal modes cont'd.

How to get the matrices in K and M in $M\ddot{x} + Kx = 0$? Linearize non-linear ODES, or truncate LE to quadratic order, or ad hoc linearization (e.g. double pendulum example).

26. Mon Oct 29. Normal modes cont'd.

Normal modes and q coordinates: $\ddot{q} + \tilde{K}q = 0 \Rightarrow$ separated equations for modal coordinates: $\ddot{r}_i + \omega_i^2 r = 0$. In a normal mode vibration *each* mass feels itself to be a harmonic oscillator with the same frequency. Thus each mass must have the same $\left(\frac{k}{m}\right)^{eff}$.

27. Wed. Oct 31. Normal modes cont'd.

Example with three masses in a line. Introduction to stiffness matrix for 2D structures: a single spring with a mass at each end.

28. Fri. Nov 2. Normal modes cont'd.

Filling in the structural stiffness matrix (using the k matrix from a single spring).

29. Mon Nov 5. Normal modes cont'd.

Example, a triangle floating in 2D space.

30. Wed. Nov 7. Normal modes cont'd.

Review of HW. Introduction to damping: $\dot{z} = Az$ with A being a constant matrix.

31. Fri. Nov 9. Normal modes cont'd.

e^{At}.

32. Mon. Nov 12. Normal modes cont'd.

Numerical experiments comparing ODE45 versus e^{At} versus superposition of complex e-vector solutions.

33. Wed. Nov 14. Normal modes cont'd.

Numerical experiments comparing ODE45 versus e^{At} versus superposition of complex e-vector solutions.

34. Fri Nov 16. Forcing of multi-DOF systems.

Modal equations and modal forcing.

35. Mon. Nov 19. Wave equation.

Derivation of wave equation for strings and rods.

36. Wed. Nov 21. String (cont'd)

Wave equation: separation of variables, model solutions, harmonics, music theory.

37. Mon. Nov 26. Approximate normal mode for wave equation

Assume a modal shape $f(x)$. Assume $u(x,t) = Bf(x)g(t)$. Then calculate E_K and E_P. Then use either Lagrange Equations or Conservation of energy to get ODE for $g(t)$: $\ddot{g} + \omega_n^2 g = 0$. Thus ω_n (which depends on the guessed shape) an estimate of a natural frequency has been found.

Empirical fact: In 1D, 2D and 3D, for discrete and for continuous systems, if you guess a modal shape that satisfies all BCs and looks reasonably smooth, the prediction of modal frequency ω_n is often quite accurate.

Example: Vibrations of a string of length L $\ddot{u} = c^2 u''$ with $u(0,t) = u(L,t) = 0$. Using separation of variables we found the lowest mode (the fundamental, the zeroth harmonic) was, exactly.

$$u(x,t) = \left(A\cos(\pi ct/L) + Bsin(\underbrace{\pi ct/L}_{\omega_n t}) \right) \sin(\pi x/L).$$

Now, with the guess that $f(x) = x(L-x)$, instead of the exact $\sin(\pi x/L)$, we turn the crank and find $\omega_n = \sqrt{10}\ c/L$ instead of the exact, from separation of variables, $\omega_n = \pi c/L$. Note that $sqrt 10/\pi \approx 1.007$ (less than 1% error).

38. Wed. Nov 28. Vibration isolation.

A quick introduction.

39. Fri. Nov 30. Guest lecture: Richard Rand

Coupled oscillators.

Homework problems

1. **Euler's method.** If needed, review Euler's method of numerical solution of ordinary differential equations (ODEs) in your ODE text or from the WWW. Consider the differential equation $\dot{x} = x$ with initial conditions $x_0 = 1$ solved over the interval $0 \leq t \leq 1$. In all cases write your own code and do not use any Matlab ODE solvers.

   (a) Use any time step you like to calculate $x(1)$ and check that the result is reasonably close to $e \approx 2.718281828...$.

   (b) Solve the equation many times using time steps of $h = 10^{-n}$ there n = 0, 1, 2, ... as high as your computer and patience will allow. [hint: for large values of n storing intermediate values in the calculation is time consuming. And if you do store intermediate values, initialize variables with code like:

   ```
   x = zeros(1,10^n).
   ```

   Note that large n will take much time in any case.

   (c) Plot the error vs n on a log-log plot.

   (d) For what value of n is the error smallest?

   (e) Can you rationalize that result?

   (f) If you did not know the analytic result, how could you determine the optimal value of n for the most accurate solution?

2. **Canon ball.** A cannon ball m is launched at angle θ and speed v_0. It is acted on by gravity g and a viscous drag with magnitude $|cv|$.

   (a) Find position vs time analytically.

   (b) Find a numerical solution using $\theta = \pi/4$, $v_0 = 1\,\text{m/s}$, $g = 1\,\text{m/s}^2$, $m = 1\,\text{kg}$. Use Euler's method programmed by you.

   (c) Compare the numeric and analytic solutions. At $t = 2$ how big is the error? How does the error depend on step size?

   (d) Use larger and larger values of v_0 and for each trajectory choose a time interval so the canon at least gets back to the ground. Plot the trajectories (using equal scale for the x and y axis. As $v \to \infty$ what is the eventual shape? [Hint: the answer is simple and interesting.]

   (e) For any given v_0 there is a best launch angle θ^* for maximizing the range. As $v_0 \to \infty$ to what angle does θ^* tend? Justify your answer as best you can with careful numerics, analytical work, or both.

3. **Mass hanging from spring.** Consider a point mass hanging from a zero-rest-length linear spring in a constant gravitational field.

   (a) Set up equations. Set up for numerical solution. Plot 2D projection of 3D trajectories.

   (b) By playing around with initial conditions, find the most wild motion you can find. Make one or more revealing plots. [Hint: Make sure the features you observe are properties of the system and not due to numerical errors. That is, check that the features do not change when the numerics is refined.]

   (c) Using analytical methods justify your answer to part (b).

4. **What means “rate of change of angular momentum”?** Consider a moving particle P. Consider also a moving point C (moving relative to a Newtonian frame $\mathcal{F}$ that has an origin 0). For which of these definitions of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ Is the following equation of motion true (that is, consistent with $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$)?

$$\vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$$

In each case say whether the definition works i) in general, or ii) for some special cases concerning the motions of P and C that you name.

(a) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C'}} \times \vec{\boldsymbol{v}}_{\mathrm{P/0}} m$,
where C’ is a point fixed in $\mathcal{F}$ that instantaneously coincides with C.

(b) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{v}}_{\mathrm{P/0}} m$.

(c) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{v}}_{\mathrm{P/C}} m$.

That is, for each possible definition of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ you need to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ by differentiation and see if and when you get $\vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{F}}$.

5. **Periodic motions for a central force.** By numerical experiments, and trial and error, try to find a period motion that is *neither* circular nor a straight line for some central force *besides* $F = -kr$ or $F = -GmM/r^2$. In your failed searches, before you find a periodic motion, do the motions always have regular patterns or are they sometimes chaotic looking (include some pretty pictures)?

6. **Mechanics of two or more particles**

   (a) For two particles with mass m_1 and m_2 what is the period of circular motion if the distance between the particles is d and the only force is the force between them, $F = Gm_1m_2/r^2$?

   (b) Pick numbers for G, m_1, m_2 and r and, using appropriate initial conditions, test your analytical result with a numerical simulation. Make any plots needed to make your result convincing.

   (c) For three equal particles, $m_1 = m_2 = m_3 = 1$ and $G = 1$ what is the angular speed for circular motion on a circle with diameter of $d = 1$?

   (d) Check your result with a numerical simulation.

7. **Montgomery’s eight.** (From Ruina/Pratap). Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$\begin{aligned}(x1, y1) &= (-0.97000436, 0.24308753)\\ (x2, y2) &= (-x1, -y1)\\ (x3, y3) &= (0, 0)\end{aligned}$$

and initial velocities

$$\begin{aligned}(vx3, vy3) &= (0.93240737, 0.86473146)\\ (vx1, vy1) &= -(vx3, vy3)/2\\ (vx2, vy2) &= -(vx3, vy3)/2.\end{aligned}$$

For each of the problems below show accurate computer plots and explain any curiosities.

(a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units.
(b) Same as above, but run for 10 time units.
(c) Same as above, but change the initial conditions slightly.
(d) Same as above, but change the initial conditions more and run for a much longer time.

8. **Konig's Theorem** The total kinetic energy of a system of particles is

$$E_{\mathrm{K}} = \frac{1}{2}\sum m_i v_i^2.$$

(a) Derive an expression of this form

$$E_{\mathrm{K}} = \frac{1}{2} m_{\mathrm{tot}} v_G^2 + \text{...you fill in the rest....}$$

(b) Is it always true that

$$\left(\sum \vec{\boldsymbol{F}}^{ext}\right) \cdot \vec{\boldsymbol{v}}_{\mathrm{G}} = \frac{d}{dt}\left(\frac{1}{2} m_{\mathrm{tot}} v_G^2\right)?$$

Defend your answer with unassailable clear reasoning (that is, a proof or a counter-example).

(c) Is it always true that the power of internal forces is equal to the rate of change of the quantity you filled in in part (a) above? Provide a proof or a counter-example. (A good solution is expected from those in 5735).

9. **Two masses** This problem has 3 independent educational goals:

(a) Introduction to ODE23. Towards that end you should study the lecture example until you can write it yourself without looking at any reference.
(b) Introduce the simplest of a class of vibrations problems you should master. At this point it is mastery of derivation of the equations. You should check that you can reproduce the lecture example with *no* sign errors without looking up anything.
(c) Motivate the use of kinematic constraints.

Two masses m_1 and m_2 are constrained to move frictionlessly on the x axis. Initially they are stationary at positions $x_1(0) = 0$ and $x_2(0) = \ell_0$. They are connected with a linear spring with constant k and rest length ℓ_0. A force is applied to the second mass. It is a step, or '**Heaviside**' function

$$F(t) = F_0 H(t) = \begin{cases} 0 & \text{if } t < 0 \\ F_0 & \text{if } t \geq 0 \end{cases}$$

(a) Write code to calculate, plot and (optionally) animate the motions for arbitrary values of the given constants.

(b) Within numerical precision, should your numerical solution always have the property that $F = (m_1 + m_2)a_G$ where $x_G = (x_1 m_1 + x_2 m_2)/(m_1 + m_2)$? (As always in this course, yes or no questions are not multiple choice, but need justification that another student, one who got the opposite answer, would find convincing.)

(c) Use your numerics to demonstrate that if k is large the motion of each mass is, for time scales large compared to the oscillations, close to the center of mass motion.

(d) 5735 only: Using analytic arguments, perhaps inspired by and buttressed with numerical examples, make the following statement as precise as possible:

For high values of k the system nearly behaves like a single mass.

Of course, in detail, the system has 2 degrees of freedom (DOF). So you are looking for a way to measure the extent to which the system is 1 DOF, and in which conditions (for which extreme values of parameters and times) the system is close to 1 DOF by that measure. There is not a simple single unique answer to this question.

10. **Two masses constrained** This is an elaboration of the problem above, replacing the two masses with a rigid rod. As per lecture, set up the DAEs and solve them using Matlab using numbers of your choice. Note the increasing error (as time progresses) in the satisfaction of the constraint. Compare this solution with the the method from the problem above (where you use some very large value of k). Which one is faster? more accurate in predicting COM motion?

11. **Simple pendulum.** Derive the simple pendulum equation $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ as many ways as you can without looking anything up in books. For example, in all cases using polar coordinates,

(a) linear momentum and manipulate the equations to eliminate constraint force

(b) linear momentum, dot with $\hat{\boldsymbol{e}}_\theta$

(c) linear momentum, cross with $\vec{\boldsymbol{r}}$

(d) angular momentum

(e) conservation of energy

(f) power balance

(g) Lagrange equations

12. **Pendulum numerics.** Set up the pendulum in cartesian coordinates. Express the constant length constraint as a set of linear equations restricting the acceleration. Solve these (3 2nd order) DAE equations with numerical integration and initial conditions and parameters of your choosing. No polar coordinates allowed. Quantitatively compare your solution with a solution of the simple pendulum equations (For the comparison you need to either compute x from θ or *vice versa*. Integrate for a long enough time so you can detect drift away from satisfying the kinematic constraint.

13. **Pendulum with an awkward parameterization** By any means you like, for a simple pendulum find the equations of motion using y (horizontal position) as your parameterization of the configuration. That is, find a 2nd order differential

equation determining $\ddot{y}$ in terms of y, $\dot{y}$ and physical parameters (g, m, ℓ). Using numerics, quantitatively compare the solution of this ODE with a solution of the simple pendulum equations (Note, you can assume the pendulum is hanging down, hence $x > 0$. This problem is different from the DAE problem above in that you should obtain a single 2nd order ODE, not a set of 3 equations.

14. **2D Dumbell.** Two equal masses $m = 1$ are constrained by a rod to be a distance $\ell = 1$ apart. At $t = 0$ they have equal and opposite velocities ($v = 1$) perpendicular to the rod. Use a set of 3DAEs ($\vec{F} = m\vec{a}$ & the constraint equation $(x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell^2$ with numerical integration to find the subsequent motion. Use plots and/or animation to help debug your code. Using what you know about systems of particles (*e.g.,* momentum, angular momentum, constraint equation, energy) quantify as many different numerical errors as you can.

15. **What means "rate of change of angular momentum" for a SYSTEM of particles?** Consider a system of moving particles with moving center of mass at G. Consider also a moving point C (moving relative to a Newtonian frame $\mathcal{F}$ that has an origin 0). For which of these definitions of $\vec{H}_{/\mathrm{C}}$ Is the following equation of motion true (that is, consistent with $\vec{F} = m\vec{a}$)?

$$\vec{M}_{\mathrm{C}} = \dot{\vec{H}}_{/\mathrm{C}}$$

In each case say whether the definition works i) in general, or ii) for some special cases concerning the motions of P and C that you name.

(a) $\vec{H}_{/C} = \sum \vec{r}_{\mathrm{i/C'}} \times \vec{v}_{\mathrm{i/C'}} m_i$,
where C' is a point fixed in $\mathcal{F}$ that instantaneously coincides with C.
(Hint: his definition is good one, always!)

(b) $\vec{H}_{/C} = \sum \vec{r}_{\mathrm{i/C}} \times \vec{v}_{\mathrm{i/0}} m_i$.
(This strange definition is used in the classic book by Housner and Hudson)

(c) $\vec{H}_{/C} = \sum \vec{r}_{i/C} \times \vec{v}_{i/C} m_i$.
(Hint: this is the most important candidate definition, but it's only good for special kinds of C, namely: C = COM, C is fixed and ...)

That is, for each possible definition of $\vec{H}_{/\mathrm{C}}$ you need to calculate $\dot{\vec{H}}_{/\mathrm{C}}$ by differentiation and see if and when you get $\sum \vec{r}_{\mathrm{i/C}} \times \vec{a}_{\mathrm{i/0}}$. If you are short for time just consider cases (a) and (c) and note their agreement if C is stationary or if C=G.

16. **Braking stability** 2D, looking down. Consider the steering stability of a car going straight ahead with either the front brakes locked or the rear brakes locked. The steering is locked and straight ahead. For simplicity assume that the center of mass is at ground height between the front and back wheels. Assume that the locked wheels act the same as a single dragging point on the centerline of the car midway between the wheels.

(a) Develop the equations of motion.

(b) Set them up for computer solution.

(c) For some reasonable parameters and initial conditions find the motion and make informative plots that answer the question about steering stability. Note, in this problem where there is no steady state solution you have to make up a reasonable definition of steering stability.

(d) See what analytical results you can get about the steering stability (as dependent on the car geometry, mass distribution, the coefficient of friction and the car speed). As much as you have time and interest, illustrate your results with graphs and animations of numerical integrations.

(e) **Hints.** Check that your governing equations reduce to the lecture equations when the friction is zero. Check special cases of the numerical solutions with solutions you know other ways. A challenge is to think of as many of these as you can (even if you don't check all of them). That is, for some special parameter values and/or initial conditions you know features of the solution (examples: 1) no friction means energy is conserved 2) with friction and no initial rotation rate slowing is with constant acceleration, etc).

17. **Review.** Spend 3 hours doing problems of your choice from Ruina/Pratap. Pick problems at the edge of your confidence level. Make sure to draw clear free body diagrams and to use clear problem setups and clear vector notation. Hand in your work.

18. **Review again.** Again spend 3 hours doing problems of your choice from Ruina/Pratap. Pick problems at the edge of your confidence level. Make sure to draw clear free body diagrams and to use clear problem setups and clear vector notation. Hand in your work.

19. **Double pendulum** Consider the double pendulum made with two bars. Hinges are at the origin 0 and the elbow E. For definiteness (and so we can check solutions against each other) both bars are uniform with the same length $\ell = 1$ (in some consistent unit system). $g = 10$. Neglect all friction and assume there are no joint motors.

    (a) Set up and numerically solve (there is no analytic solution) to the governing equations that you find using AMB. You may refer to lecture notes, but you should be able to do it on your own by the time you hand in the work. Assume that at $t = 0$ the upper arm is horizontal, sticking to the right, and the fore-arm is vertical up (like looking from the front at a driver using hand signals to signal a right turn). Integrate until $t = 10$. Draw the (crazy) trajectory of the end of the forearm.

    (b) Use Lagrange equations to find the governing equations and, either by comparing equations or by comparing numerical solutions, show that the governing equations are the same as those obtained using AMB.

20. **Mass in slot on turntable.** A rigid turntable (m_t, I_t) is free to rotate about a hinge at it's center. It has in it a straight frictionless slot that passes a distance d from it's center. A mass m_s slides in the slot. For minimal coordinates use rotation of the disk θ from the position in which the slot is horizontal and below the disk center, and the distance s the mass is from the point where the slot is closest to the center of the disk.

    (a) find the acceleration of the mass in terms of $d, \theta\dot{\theta}, \ddot{\theta}, s, \dot{s}$ and $\ddot{s}$. Do this three different ways and check that all give the same answer when reduced to x and y coordinates.

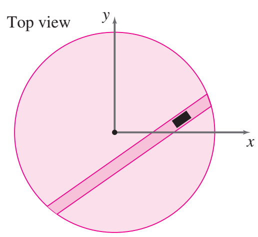
    i. Write the position of the mass in terms of d, θ and s using base vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. Differentiate twice.

    ii. Write the position using $\hat{\boldsymbol{\imath}}'$, which aligns with the slot, and $\hat{\boldsymbol{\jmath}}'$. Differentiate twice using that $\dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}' = \omega \hat{\boldsymbol{\jmath}}'$ and $\dot{\hat{\boldsymbol{\jmath}}}' = -\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}' = -\omega \hat{\boldsymbol{\imath}}'$

    iii. Use the five-term acceleration formula (using $\vec{\boldsymbol{v}}_{\text{rel}} = \dot{s}\hat{\boldsymbol{\imath}}'$ and $\vec{\boldsymbol{a}}_{\text{rel}} = \ddot{s}\hat{\boldsymbol{\imath}}'$).

    (b) Using the most convenient expression above, find the equations of motion (That is, find $\ddot{\theta}$ and $\ddot{s}$ in terms of fixed parameters and position and velocity variables.). One way to do this would be to use AMB for the whole system about the center and to use {LMB for the mass}$\cdot\hat{\boldsymbol{\imath}}'$.

    (c) Assume ICs that $s(0) = 0, \dot{s}(0) = v_0, \theta(0) = 0$ and $\dot{\theta}(0) = \omega_0 > 0$. As $t \to \infty$ does $\theta \to \infty$? (As for all questions, please explain in a way that would convince a non-believer.)



21. **Mass and spring vibration.** The harmonically forced vibration of a damped oscillator is given by this equation:

$$m\ddot{x} + c\dot{x} + kx = A\cos\omega_0 t + B\sin\omega_0 t$$

    (a) Assume that the mass is connected to the spring and to the dashpot, the other ends of which are at C and D, respectively. Define x as the displacement of the mass in inertial space. For each of the cases below, find A and B. The latter three cases are from excitation by a moving base.

    i. C and D are fixed and a force $F = F_0 \sin\omega_0 t$ acts on the mass.

    ii. C is fixed and D oscillates with $\delta = \delta_0 \sin\omega_0 t$.

    iii. D is fixed and C oscillates with $\delta = \delta_0 \sin\omega_0 t$.

    iv. C and D oscillate together with $\delta = \delta_0 \sin\omega_0 t$.

    (b) For the following problems, solve the governing equation above numerically using, say ODE45 using various appropriate forcings and initial conditions. For definiteness use the underdamped case $m = 1, c = 1, k = 1$.

i. Set $A = 0, B = 0$ and $c = 0$. Using numerics find the natural frequency ω_n. Do this using 'events' (for the mass released from rest, find the time until the velocity gets to zero from above). Then calculate the frequency from this measure period. Compare the result with the analytic result $\omega_n = \sqrt{k/m}$.
ii. Now set $A = 0, B = 0$ and $c = 1$ and find the damped natural frequency ω_d. Compare this with the analytic result.
iii. Using 'events' that do not terminate the integration, and using the logarithmic decrement method, find the damping ratio ζ ('zeta'). Compare with the analytic result.
iv. Draw a frequency response curve (each point on this curve requires a full simulation). For example, use A or $B = 0$ and the other equal to 1 and look at the amplitude of steady state response. The hard part here is running the simulation long enough so that the response is "steady state". Compare this curve with an analytically derived curve. Using the numerics, find, as accurately as you can, the frequency at which the amplitude of the steady state response is maximum.
v. Compare the three frequencies: 1) natural frequency, 2) damped natural frequency and 3) 'resonance' frequency (frequency which gives maximum amplitude response). Note their order and note how close, or not, they are to each other.

22. **Bead on parabolic wire** For a frictionless point-mass bead sliding on a rigid wire on the curve $y = cx^2$ with gravity in the $-y$ direction, find the equation of motion.

    (a) Derive the equations of motion using Lagrange equations. Use, say the projection of the position on the x axis as the generalized coordinate.
    (b) Derive the equations of motion using Newton's laws. First write $\vec{F} = m\vec{a}$ with an unknown constraint force orthogonal to the wire. Then dot both sides with a vector tangent to the wire. You should get the same answer as for part (a) with a very similar amount of algebra.
    (c) Find the frequency of small vibrations (find a formula for this in terms of some or all of m,g and c.

23. **Three masses normal modes.** Three equal masses are in a line between two rigid walls. They are separated from each other and the walls by four equal springs.

    (a) Write the equations of motion in matrix form.
    (b) By guessing/intuition find one of the normal modes.
    (c) Using the MATLAB eig function find all three normal modes.
    (d) Using numerical integration, with masses released from rest with a normal mode shape ($\vec{x}_0 = \vec{\mathbf{u}}_i$), show that you get normal mode (synchronous) oscillations.

24. **Double Pendulum normal modes.** Use your double pendulum solutions with the following simplifications:

    (a) both links have the same length ℓ;
    (b) all mass is in two equal point masses (one at the elbow, one at the hand), so $I_1 = I_2 = 0$;

(c) linearize: drop all terms that involve products like $\dot\theta_1^2, \dot\theta_2^2$ or $\dot\theta_1^2$. For both θs replace $\sin\theta$ with θ and $\cos\theta$ with 1.

Thus write the small amplitude double pendulum equations in this form:

$$M\ddot{\vec{\theta}} + K\vec{\theta} = \vec{\mathbf{0}}.$$

Here, $\vec{\theta} = [\theta_1 \quad \theta_2]'$, and M and K are 2×2 symmetric matrices whose entries are expressions involving, m,g and ℓ. Use the normal mode approach (e.g., the problem above) to find the normal modes. Use one of these, with small amplitude, as as initial conditions for your full non-linear simulator and show that you get (nearly) synchronous motion.

25. **Normal modes, solving an IVP.** (IVP = Initial Value Problem). For your three mass system (above), find the motion if you are given initial positions and velocities. In some special cases (including at least one normal mode shape) check your motion against direct integration of the ODEs.

26. **Rolling cylinder** A uniform cylinder with mass m and radius r rolls without slip inside a cylinder with radius R. Gravity g pulls it down.

    (a) Are the full non-linear differential equations the same as those of a pendulum? If so, or not, explain why this is expected.

    (b) In terms of some or all of m, g, r and R find the frequency of small oscillation near the bottom.

27. **Cart and pendulum** A cart m_1 slides frictionlessly on a level surface. A massless stick with length ℓ is hinged to it with a mass m_2 at the end. Take $\theta = 0$ to be the configuration when the pendulum is straight down. Use gravity g.

    (a) Find the full non-linear governing equations at least two different ways and show that they agree.

    (b) Linearize the equations for small deviations from the configuration where the pendulum hangs straight down.

    (c) Write the equations in standard vibration form: $M\ddot{\vec{x}} + K\vec{x} = \vec{0}$.

28. **Normal Mode Numerics** Much of this problem solution can be done by recycling previous solutions. Given M, K, x_0, v_0 one can find $x(t)$ and $v(t)$ three ways.

    (a) Write a matlab function `SOLVENUM`

    ```
    [xmatrix] = solvenum(M,K,x0,v0,tspan);
    ```

    The output is an array, each row of which are the values of x at the corresponding time in span. Use `ODE45` and any other functions you write.

    (b) Write a matlab function `SOLVEMODE`

    ```
    [xmatrix] = solvemode(M,K,x0,v0,tspan);
    ```

    that solves the same problem by using eigenvectors of $M^{-1}K$. Make sure your function works even when K is indefinite (even when there are motions that have no potential energy).

    (c) Write a matlab function `SOLVENORM`

    ```
    [xmatrix] = solvenorm(M,K,x0,v0,tspan);
    ```

    that solves the same problem by using eigenvectors of $M^{-1/2}KM^{-1/2}$. Make sure your function works even when K is indefinite (even when there are motions that have no potential energy).

    (d) Show that all three functions above give the same solution for the linearized cart and pendulum.

29. **Cylinder in a pipe.** . A thin-walled hollow cylinder with radius R and mass M rolls without slip on level ground. Inside it rolls, without slip, a disk with radius $r < R$ and mass m. Gravity g points down. Find the equations of motion (two ways if you have time and energy and would find it educational). Find the modes of small oscillation and their frequencies. One of these should make clear intuitive sense. Can you find at least one special case in which you can check the other? [For those interested in such: can you find a conservation law associated with the translation invariance of the governing equations? (There is one, but I don't know what it is)]

30. **Mass in slot on a turntable.** 2D. A turntable with mass M and moment of inertia I is held in place at its center with a bearing and a torsional spring k_t. Along one diameter of the disk is a slot in which a mass m slides with no friction. A zero-rest-length spring pulls it to the center with spring constant k. Find the equations of motion at least two different ways. Find the normal modes and frequencies.

31. **Normal modes by inspection.** For each of the systems below find as many normal modes, and their frequencies, as you can **without** doing matrix calculations. Then, if you like and can, check your work with matrix calculations.

    (a) 1D. Three equal masses in a line connected by two springs. No springs are connected to ground.

    (b) 3D. Two unequal masses, m_1 and m_2, are at points $\vec{r}_1$ and $\vec{r}_2$ in 3D space and are connected by one spring k. No springs are connected to ground.

    (c) 2D. 4 point masses are arranged in a square. The 4 edges are equal massless springs.

    (d) A regular hexagon has equal point masses at the vertices and equal springs on the edges. No springs are connected to ground. Just find one mode of vibration that does *not* have zero frequency.

    (e) 1D. An infinite line of equal point masses m is connected by an infinite line of equal springs k. One normal mode oscillation is given by

    $$\vec{v} = [\cdots -1 \quad 1 \quad -1 \quad 1 \quad -1 \ldots]'$$

    with $\omega = 2\sqrt{k/m}$. Find another mode and frequency. Challenge: find more. [Hint 1: this problem has impenetrably beautiful simple solutions which you might, with some luck, guess and, with some skill, check. Hint 2: If you assume the solution is periodic with period n then the stiffness matrix can be written as an $n \times n$ matrix. You can use this to numerically find normal modes which, if you look at them (say, plot the vector components vs their indecies) should reveal a pattern. Then you can use that same matrix to check if you detected the pattern correctly.]

32. **Intro to damped modes.** Consider the set of ODEs

$$\dot{z} = Az$$

where z is a list of scalar functions of time and A is a constant real matrix. Here you are to test, in Matlab, the basic theory of the solutions of equations like this.

(a) Generate a fairly random $n \times n$ matrix A using `RAND` or any other way. You can use any positive integer $3 \leq n \leq 100$ that pleases you.

(b) Find any eigenvalue λ and associated eigenvector v of A (these will undoubtedly be complex).

(c) For a sequence of, say, 100 or 1000 times, starting at $t = 0$, plot the real part of $e^{\lambda t} v_1$ versus t, where v_1 is the first component of the eigenvector. Pick a length of time where the curve is variable enough to be interesting, but not so variable that no details can be detected.

(d) Find the vector w which is the real part of v.

(e) Solve

$$\dot{z} = Az$$

using `ODE45` with the initial condition $z_0 = w$. Plot z(1) vs t from this solution and compare it with the plot above. If nothing pops out, someone made a mistake. Explain the interesting relation as best you can.

33. **Old Qualifying exams.** Four documents on the course WWW page have old qualifying exams. Pick a 2D dynamics problem from that set. This should be a problem that you are challenged by. Take the poorly worded question and sloppy figure and make a clear figure and clear question. Write up a clear solution. If this takes less than 4 hours, do another and another until 4 hours are used up.

34. **Damped normal modes.** Consider 3 equal masses in a line held between 2 walls by 4 equal springs. A single small dashpot ($c = .1\sqrt{mk}$) connects the leftmost mass from the wall at its left. Assume the initial velocity is zero. Assume the initial position is a mode shape with mass one having a displacement of 1. Plot position vs time for the first mass for all three mode shapes. Comment on the similarity and differences between the results for the three methods below. Make any other revealing plot(s) you can think of.

    (a) Numerical ODE solution (an arbitrarily exact numerical method).

    (b) Solution using first order odes and the matrix exponential (an exact method, numerically evaluated).

    (c) Solution using modal damping:

        i. Pick α and β so that $C = \alpha M + \beta K$ gives about the right decay rate for the fastest and slowest modes (a numerically evaluated analytic expression for the slightly wrong problem).

        ii. Use the change of coordinates that reduces undamped problem to diagonal form in terms of modal coordinates:

$$\ddot{\vec{r}} + \underbrace{P' M^{-1/2} C M^{-1/2} P}_{\hat{C}} \dot{\vec{r}} + \Lambda \vec{r} = \vec{\mathbf{0}}.$$

        Only if you are lucky is this $\hat{C}$ diagonal (for example if $C = \alpha M + \beta K$). If it isn't, which it isn't for this HW problem, replace the $\hat{C}$ matrix with the diagonal part of $\hat{C}$. Then find the solution for each mode (A numerically evaluated analytic solution to a more nearby problem).

35. **a) Old Qualifying exam, 2nd try.**
    Observation: No honest student could say "Could do, but I won't learn from this."

    i Pick a 2D dynamics Q-exam problem, hopefully one that is not trivial for you.

    ii Don't write on page backs. Staple this problem separate from your other HW. Page 1: problem statement, Page 2: start of solution. No more than 3 pages total.

    iii This should not be a record of your brainstorming, it should be a clear write-up of a solution. (Practice brainstorming. Just don't show that here.)

    iv State assumptions. Describe methods if the calculation is too long for details to show. Name generalizations and how you could/would deal with them.

    v Write in such a way that you will make a competent reader, say a TA or professor in a course like this, think you are a clear thinker with mastery of the topic and good communication skills. Meanwhile convince another imagined skeptical member of this class that your solution is correct.

    vi MAIN INITIAL GOAL: Reduce the problem to one or more precisely defined mechanics problems. State reasonable assumptions. Don't need complete sentences. Clear sentence fragments & shorthand ok.

    vii SECONDARY GOAL: Use precise mechanics reasoning to solve, or set up solutions.

    viii THIRD GOAL: Describe why your solution does or does not make sense.

35. **b,c,d,...) Same as (a).**

36. **Rolling eccentric cylinder** A cylinder with radius R has center of mass G offset from the cylinder center C by a distance $d < R$. It has total mass M, radius R and moment of inertia I^G about it's center of mass. It rolls without slip down a ramp with slope γ, propelled by gravity g.

    (a) Find the equations of motion.
    (b) Find the needed coefficient of friction to enforce the rolling constraint.
    (c) After release from rest how far does it roll before it skips into the air.

    It's ok to use numerical solutions based on any non-trivial parameter choices. No need for parameter sweeps.

    Interesting extension if you have *lots* of time: With appropriate initial conditions, this can roll on a level ramp then skip, then do about a full revolution in the air, then land with no no relative velocity at impact (thus conserving energy) and continue rolling and skipping. This is explained, somewhat, in a paper on Ruina's www page: "A collisional model ..." (figure 4). The calculation details are like those in this paper "Persistent Passive Hopping ..."

37. **Modal forcing** This problem is interesting. Consider our favorite 3 mass system, from problem 34 above: 3 equal masses in a line separated by 4 springs. Assume that all springs are parallel to dampers with $c = .1\sqrt{mk}$. If you are short of time, leave off the damping (use $c = 0$). Now consider this problem.

    (a) The system starts from rest at $t = 0$.
    (b) A force $F = F_0 \sin(\sqrt{\frac{2k}{m}}t)$ is applied to the left mass. You can think of this as a small force if you like (although the words "small" and "large" have no meaning in the solution of linear problems).

    Now solve this problem various ways and notice some interesting features by making relevant plots.

    (a) Solve using your favorite Matlab ODE solver.
    (b) Plot the positions of all three masses as a function of time. Make two plots: 1) Use a long enough time scale so that you get a sense of steady state response; 2) Use a short enough time so you can see the amplitude growing.
    (c) Use modal forcing for the mode $\vec{v} = [1 \quad 0 \quad -1]'$. Note that for this diagonalizeable problem, the modal solution is exact.
    (d) Make similar plots. Note that in this solution the middle mass does not move at all.

    Main question: Given that the modal solution is exact. And given that in that modal solution the middle mass does not move. What makes the right mass move?

38. **Write a final exam question.** You should write a clear candidate final exam question on one page. You should write a clear solution on the following page(s), 2 page max. Single sided (don't write on backs). These 2 or 3 pages should be stapled together separate from your other homework problems. Your question should not be like that of anyone else you know. Hand printing and hand drawing are fine. Some

of these will be scanned and posted before the final exam. At least one from the scanned bunch, possibly slightly modified, will be on the final exam.

A good final exam question has these properties:

(a) The question is clear.
(b) Most of the people who could do it well in a relaxed 6 hours could do it decently in just 30 minutes.
(c) Most people who mastered all the prerequisite course material, all of the course homeworks, all the lecture material and all of the readings should be able to do it.
(d) Most people who had the pre-requisite courses but didn't take this course (or its equivalent) should not be able to do it.
(e) Getting the problem right should be indicative of having (hopefully useful) skills and knowledge related to this course. For example, asking "What did Professor Ruina dress up as on Halloween?" might fulfill all of the requirements above, but not this one.
(f) Not too many jokes in the problem statement or in the solution. If it's too cute people get annoyed.
(g) The solution should be maximally illuminating. With minimum reading effort, someone who doesn't know how to do the problem should learn and understand how.
(h) One good type of problem would be of a type that you couldn't do at the course start, can do well now, and wish was on the final exam.

39. **Final computation project.** Due at the end of the semester. This is an extension of the double pendulum homework. The minimal version is to simulate and animate both a triple pendulum and also a 4-bar linkage. For the triple-pendulum the equations of motion should be found two different ways. In both problems the numerical solutions should be checked as many ways as possible (Energy conservation, limiting cases where simple-pendulum motion is expected, etc). Optional extras are a) to simulate and animate more complicated mechanisms of your choice (e.g., 4,5, n link pendulum or closed kinematic loop) and b) to find periodic motions.

    Deliverables:

    (a) Send one zip file called YourName4735.zip or Yourame5735.zip. By midnight Dec 4.
        i. That should be a compressed version of a single folder.
        ii. In that folder should be a collection of Matlab files
        iii. In that folder should be a *README file explaining how to use the Matlab files. It should be VERY EASY to use the files for simple demonstrations
        iv. In that folder should be a file called REPORT.pdf. It could be made from WORD, LateX or scanned handwork, or any mixture of those. It should explain what you have done, how, and give sample output. This is the main demonstration of your effort. As appendices this should include your documented matlab files.
    (b) On Dec 4, you will have 5 minutes to demonstrate animations of your simulations on your own laptop. Sign up for time on the course www page.

No more problems to come $\ddot{\smile}$.