# VIBRATIONS

## Cornell's MAE 4770
## Spring 2012 Lecture Notes

Richard H. Rand

Compiled on June 14, 2012

## Contents

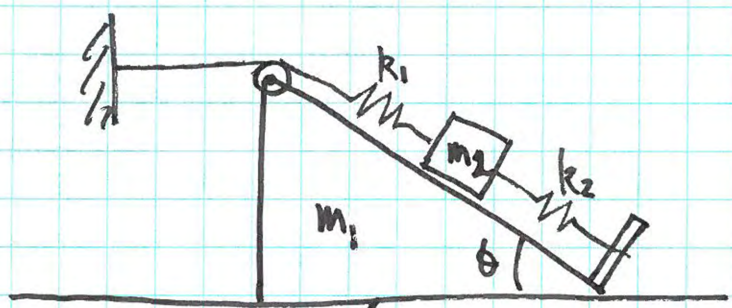
Page numbers here refer to typed numbers on the bottom right of the pages.

## Two DOF Systems

Advantage of using Lagrange's Eqs to derive eqs of motion:

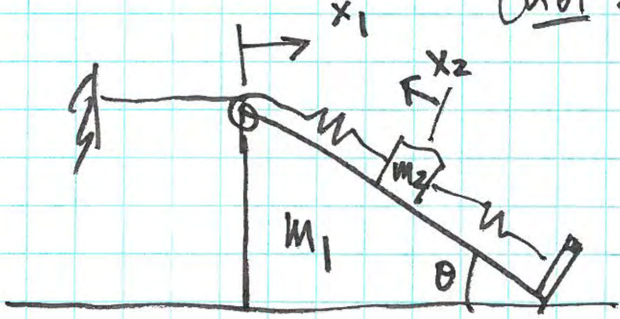
Constraint forces do not appear in the d.e.'s

**Example**



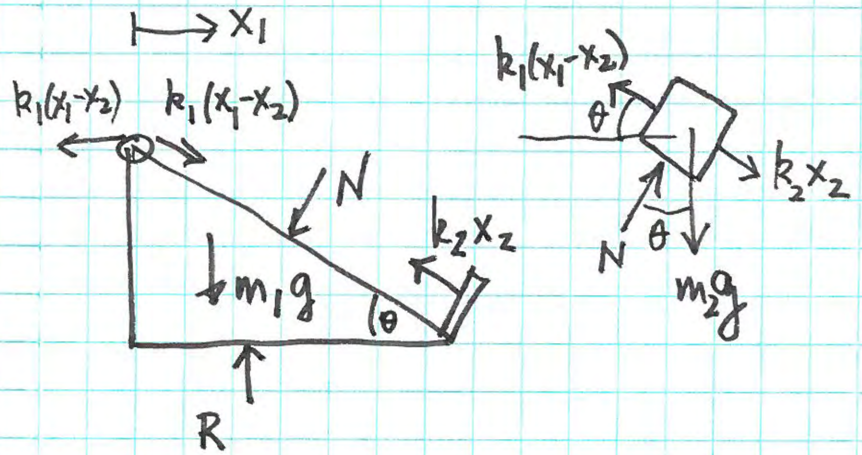

$m_1$ slides on a smooth table, $m_2$ slides on $m_1$ } no friction

Let $x_1, x_2$ be measured from unstretched spring position (not from equilibrium)



Derive eqs of motion in two ways: NE vs. LE

NE free bodies

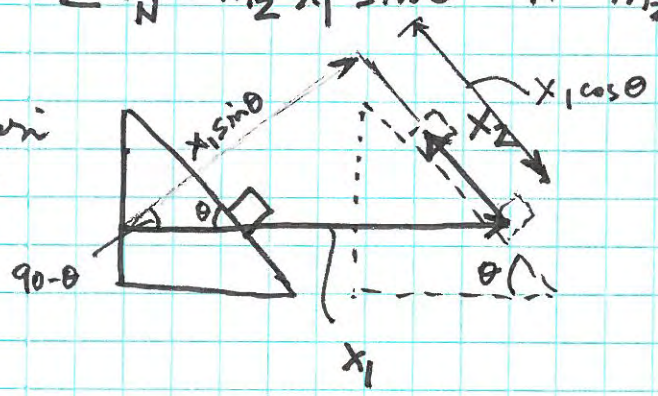



For $m_1$, $\sum F_{x_1} = m_1\ddot{x}_1 = \begin{cases} -k_1(x_1-x_2)+k_1(x_1-x_2)\cos\theta - k_2x_2\cos\theta \\ -N\sin\theta \end{cases}$

For $m_2$, $\sum F_{x_2} = m_2\,(\overline{\ddot{x}_2-\ddot{x}_1\cos\theta}) = +k_1(x_1-x_2) - k_2x_2 - m_2 g\sin\theta$

$$\sum F_N = m_2\ddot{x}_1\sin\theta = N - m_2 g\cos\theta$$

Acceleration of $m_2$:




$$\bar{r}_2 = x_1\sin\theta\,\hat{e}_N + (x_2 - x_1\cos\theta)\,\hat{e}_{x_2}$$

The constraint force N is unknown and must be eliminated from the eqs. manually

$$N = m_2\ddot{x}_1\sin\theta + m_2 g\cos\theta$$

Subst. $N$ into $\Sigma F_{x_1}$ 3

$$m_1 \ddot{x}_1 = -k_1 (x_1 - x_2)(1-\cos\theta) - k_2 x_2 \cos\theta$$
$$- m_2 \ddot{x}_1 \sin^2\theta - m_2 g \sin\theta\cos\theta$$

Mult. $\Sigma F_{x_2}$ by $\cos\theta$

$$m_2 (\ddot{x}_2 - \ddot{x}_1 \cos\theta)\cos\theta = k_1 (x_1 - x_2)\cos\theta - k_2 x_2 \cos\theta$$
$$- m_2 g \sin\theta\cos\theta$$

Now subtract to eliminate $m_2 g$:

$$m_1 \ddot{x}_1 - m_2 \ddot{x}_2 \cos\theta + m_2 \ddot{x}_1 \cancel{(\cos^2\theta + \sin^2\theta)}$$
$$= -k_1 (x_1 - x_2)$$

## LE

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \Big( \underbrace{(\dot{x}_2 - \dot{x}_1 \cos\theta)^2 + \dot{x}_1^2 \sin^2\theta}_{\dot{x}_2^2 - 2\dot{x}_1 \dot{x}_2 \cos\theta + \dot{x}_1^2} \Big)$$

$$V = \frac{1}{2} k_1 (x_1 - x_2)^2 + \frac{1}{2} k_2 x_2^2 + m_2 g \sin\theta \, x_2$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0 , \qquad \mathcal{L} = T - V$$

$\underline{\underline{i=1}}$ $\quad \frac{\partial \mathcal{L}}{\partial x_1} = k_1 (x_1 - x_2) , \quad \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1 - m_2 \dot{x}_2 \cos\theta + m_2 \dot{x}_1$

$$m_1 \ddot{x}_1 + m_2 (\ddot{x}_1 - \ddot{x}_2 \cos\theta) + k_1 (x_1 - x_2) = 0$$

same as derived eq. p.3

$\underline{\underline{i=2}}$ $\quad \frac{\partial \mathcal{L}}{\partial x_2} = -k_1 (x_1 - x_2) + k_2 x_2 + m_2 g \sin\theta$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2 - m_2 \dot{x}_1 \cos\theta$$

same as $\Sigma F_{x_2}$

$$m_2 (\ddot{x}_2 - \ddot{x}_1 \cos\theta) + k_1 x_2 + k_2 x_2 - k_1 x_1 + m_2 g \sin\theta = 0$$

Equilibrium configuration: $\ddot{x}_1 = \ddot{x}_2 = 0 \Rightarrow$

$\underline{\underline{i=1}}$ $\quad x_1 = x_2 , \qquad \underline{\underline{i=2}}$ $\quad x_2 = -\frac{m_2 g}{k_2} \sin\theta$

Can set $\quad \tilde{x}_2 = x_2 + \frac{m_2 g}{k_2} \sin\theta$

which measures $\tilde{x}_2$ from equilibrium, and gives same result as if $g$ was taken $= 0$.

taking $g=0$ we have

$$\begin{bmatrix} m_1+m_2 & -m_2\cos\theta \\ -m_2\cos\theta & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\underset{\text{mass matrix}}{M}\ddot{X} + \underset{\text{stiffness matrix}}{K}X = 0$$

Note that M and K are symmetric matrices

$M = M^t$, $K = K^t$

# Free Vibs of 2 DOF

$$M\ddot{X} + K X = 0 \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$M = M^t, \quad K = K^t$$

Set $\quad X = \underline{X} \cos wt$

i.e. $\left.\begin{aligned} x_1 &= \underline{X}_1 \cos wt \\ x_2 &= \underline{X}_2 \cos wt \end{aligned}\right\}$ vibrations in unison

$$-(m_{11}\underline{X}_1 + m_{12}\underline{X}_2)w^2 + K_{11}\underline{X}_1 + K_{12}\underline{X}_2 = 0$$

$$-(m_{\substack{\cancel{21}\\12}}\underline{X}_1 + m_{22}\underline{X}_2)w^2 + K_{\substack{\cancel{21}\\12}}\underline{X}_1 + K_{22}\underline{X}_2 = 0$$

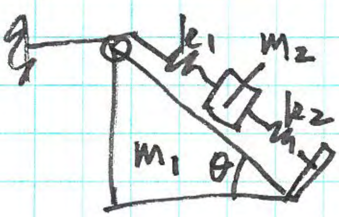
$$\begin{bmatrix} -m_{11}w^2 + K_{11} & -m_{12}w^2 + K_{12} \\ -m_{\substack{\cancel{21}\\12}}w^2 + K_{\substack{\cancel{21}\\12}} & -m_{22}w^2 + k_{22} \end{bmatrix} \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} = \bar{0}$$

For nontrivial solution, det $= 0$

$$(m_{11}m_{22} - m_{12}^2)w^4 - (m_{22}K_{11} + m_{11}K_{22} - 2m_{12}K_{12})w^2 + K_{11}K_{22} - K_{12}^2 = 0$$

# Example

Previous example




take $m_1 = 1$
$m_2 = 1$
$\theta = 60°, \cos\theta = \frac{1}{2}$
$k_1 = 1$
$k_2 = 1$

$$M = \begin{bmatrix} m_1+m_2 & -m_2\cos\theta \\ -m_2\cos\theta & m_2 \end{bmatrix} = \begin{bmatrix} 2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Eq. on $\omega^2$:

$$(m_{11}m_{22} - m_{12}^2)\omega^4 - (m_{22}k_{11} + m_{11}k_{22} - 2m_{12}k_{12})\omega^2 + (k_{11}k_{22} - k_{12}^2) = 0$$

$$\frac{7}{4}\omega^4 - (\underbrace{1 + 4 - 2(\tfrac{1}{2})}_{4})\omega^2 + 1 = 0$$

$$\frac{7}{4}\omega^4 - 4\omega^2 + 1 = 0$$

or $7\omega^4 - 16\omega^2 + 4 = 0$

$$14\omega^2 = +16 \pm \sqrt{256 - 112} = +16 \pm \sqrt{144}$$

$$14\omega^2 = +16 \pm 12 = 28, 4$$

$$\omega^2 = 2, \frac{2}{7}$$

$$\omega = \sqrt{2}, \sqrt{\frac{2}{7}}$$

$\omega^2 = 2$:

$$\begin{bmatrix} -4+1 & 1-1 \\ 1-1 & -4+2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \bar{0} \Rightarrow X_1 = 0$$

In this mode, the $m_1$ block doesn't move.

$\omega^2 = \frac{2}{7}$:

$$\begin{bmatrix} -\frac{4}{7}+1 & \frac{1}{7}-1 \\ \frac{1}{7}-1 & -\frac{2}{7}+2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \bar{0}$$

$$\begin{bmatrix} \frac{3}{7} & -\frac{6}{7} \\ -\frac{6}{7} & \frac{12}{7} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \bar{0} \Rightarrow X_1 = 2X_2$$

Here the $m_1$ block moves with twice the amplitude of the $m_2$ mass.

## General Solution

$$x_1(t) = 2R_1 \cos\left(\sqrt{\tfrac{2}{7}}\,t - \phi_2\right)$$

$$x_2(t) = R_1 \cos\left(\sqrt{\tfrac{2}{7}}\,t - \phi_2\right) + R_2 \cos\left(\sqrt{2}\,t - \phi_1\right)$$

where $R_1, R_2, \phi_1, \phi_2$ are found from IC

# Principal Coordinates

Idea: The original generalized coordinates $x_1, x_2$ were chosen as convenient physical quantities. Principal (or eigencoordinates) coordinates reflect the dynamical structure of the system.

[See text p. 187]

In the previous example, we saw that there were two modes of vibration:

$$\omega_2^2 = 2, \quad {}_2\bar{X} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{modal vector}$$

$$\omega_1^2 = \frac{2}{7}, \quad {}_1\bar{X} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Note that these modal vectors satisfy the eqs

$$\omega_i^2 M({}_i\bar{X}) = K({}_i\bar{X})$$

(from $M\ddot{x} + Kx = 0, \quad x_i = X_i \cos\omega t$

$$\Rightarrow -\omega^2 M X_i + K X_i = 0$$

Define ${}_1\bar{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ corresponding to $\omega_1$, etc.)

<u>Definition</u> Two vectors $\bar{u}$ and $\bar{v}$ are said to be <u>orthogonal</u> with respect to a <u>symmetric</u> matrix $A$ if

$$\bar{u}^t A \bar{v} = 0$$

(Here $\bar{u}, \bar{v}$ are $n \times 1$ and $A$ is $n \times n$.)

Note that $\bar{u}^t A \bar{v}$ is a scalar, and hence equal to its own transpose $\Rightarrow$ $(\bar{u}^t A \bar{v})^t = 0$

$$\bar{v}^t A^t \bar{u} = 0 \quad \text{but } A = A^t$$

$$\therefore \quad \bar{v}^t A \bar{u} = 0$$

Using this definition we may say that the modal vectors ${}_1\bar{X}$ and ${}_2\bar{X}$ are orthogonal with respect to both $M$ and $K$:

$${}_1\bar{X}^t M \, {}_2\bar{X} = 0, \quad {}_1\bar{X}^t K \, {}_2\bar{X} = 0$$

Check it on the previous example where

$$M = \begin{pmatrix} 2 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$${}_1\bar{X}^t M \, {}_2\bar{X} = (2 \ 1) \begin{pmatrix} 2 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (2 \ 1) \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = 0$$

$${}_1\bar{X}^t K \, {}_2\bar{X} = (2 \ 1) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (2 \ 1) \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 0$$

But why is it true?

We have $$\omega_1^2 M ({}_1\bar{X}) = K ({}_1\bar{X}) \qquad (1)$$

and $$\omega_2^2 M ({}_2\bar{X}) = K ({}_2\bar{X}) \qquad (2)$$

Multiply (1) by ${}_2\bar{X}^t$, and multiply (2) by ${}_1\bar{X}^t$

$$\omega_1^2 \; {}_2\bar{X}^t M \, {}_1\bar{X} = {}_2\bar{X}^t K \, {}_1\bar{X} \qquad (3)$$

$$\omega_2^2 \; {}_1\bar{X}^t M \, {}_2\bar{X} = {}_1\bar{X}^t K \, {}_2\bar{X} \qquad (4)$$

Since both $M$ and $K$ are symmetric,

and since a scalar is equal to its own transpose,

(4) may be written in the form:

$$\omega_2^2 \; {}_2\bar{X}^t M \, {}_1\bar{X} = {}_2\bar{X}^t K \, {}_1\bar{X} \qquad (5)$$

Subtracting (5) from (3) gives

$$(\omega_1^2 - \omega_2^2) \; {}_2\bar{X}^t M \, {}_1\bar{X} = 0$$

and since $\omega_1 \neq \omega_2$, we have ${}_2\bar{X}^t M \, {}_1\bar{X} = 0$

which, from (3), also gives that ${}_2\bar{X}^t K \, {}_1\bar{X} = 0$

So that's why the modal vectors are orthogonal with respect to $M$ and $K$.

We are now ready to define principal coordinates.

Notation: principal coords will be $p_1(t)$, $p_2(t)$

These will be linear combinations of $x_1(t)$, $x_2(t)$

$$\underline{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

<u>Definition</u> $x = R\,\underline{P}$

where $R$ is a $2\times 2$ matrix whose columns are the modal vectors:

$$R = \left( {}_1\bar{X}, {}_2\bar{X} \right)$$

<u>Example</u> (as before)

$$R = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

So $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

or $$x_1 = 2p_1$$
$$x_2 = p_1 + p_2$$

What's the point of all these definitions?

First, see what happens when we compute

$$R^t M R$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3\frac{1}{2} & -1/2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Note it's diagonal.}$$

Now try $\quad R^t K R$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{also diagonal.}$$

This works because the columns of R are modal vectors and therefore the rows of $R^t$ are modal vectors.

And the modal vectors are orthog. wrt M, K.

Now watch this:

$$M\ddot{x} + Kx = 0$$

Set $x = R\underline{p}$ where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underline{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

$$MR\ddot{\underline{p}} + KR\underline{p} = 0$$

Now multiply by $R^t$

$$R^t M R \ddot{\underline{p}} + R^t K R \underline{p} = 0$$

But $R^t M R = D_1$, $R^t K R = D_2$ are diagonal

$$D_1 \ddot{\underline{p}} + D_2 \underline{p} = 0$$

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{pmatrix} + \begin{pmatrix} d_3 & 0 \\ 0 & d_4 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$d_1 \ddot{p}_1 + d_3 p_1 = 0$$

$$d_2 \ddot{p}_2 + d_4 p_2 = 0$$

These equations are <u>uncoupled</u>, and therefore are trivial to solve.

Principal coordinates simplify the math and make the dynamics transparent!

# Example of the utility of principal coordinates

Forced vibrations of a 2 DOF system

$$M\ddot{x} + Kx = f(t), \quad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Transform to principal coordinates

$$x = R\,\underline{p}$$

$$MR\ddot{\underline{p}} + KR\underline{p} = f$$

Multiply by $R^t$

$$\underbrace{R^t M R}_{D_1}\ddot{\underline{p}} + \underbrace{R^t K R}_{D_2}\underline{p} = R^t f$$

Note that since $D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $D_1^{-1}$ is easy to compute:

$$D_1^{-1} = \begin{pmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{pmatrix}$$

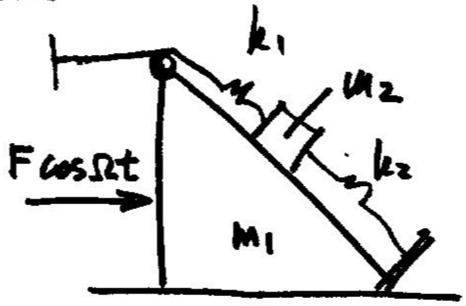
Multiply by $D_1^{-1}$

$$\ddot{\underline{p}} + \underbrace{D_1^{-1} D_2}_{\begin{pmatrix} w_1^2 & 0 \\ 0 & w_2^2 \end{pmatrix}} \underline{p} = D_1^{-1} R^t f \equiv g(t) \text{ forcing function}$$

We have

$$\left.\begin{aligned} \ddot{p}_1 + w_1^2 p_1 &= g_1(t) \\ \ddot{p}_2 + w_2^2 p_2 &= g_2(t) \end{aligned}\right\} \text{ The 2 DOF system is treated as two 1 DOF systems}$$

Example




In NE we just add a force to the free body for $m_1$ giving

$$m_1\ddot{x}_1 = F\cos\Omega t + \text{previous forces}$$

which finally gives

$$m_1\ddot{x}_1 - m_2\ddot{x}_2\cos\theta + m_2\ddot{x}_1 = -k_1(x_1 - x_2) + F\cos\Omega t$$

plus another eq. which is unchanged.

How do we handle applied forces in LE?

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \quad , \quad L = T - V$$

Useful if there are nonconservative or dissipative forces

$q_i$ = generalized coordinate, $Q_i$ = generalized force

# Procedure for finding $Q_i$

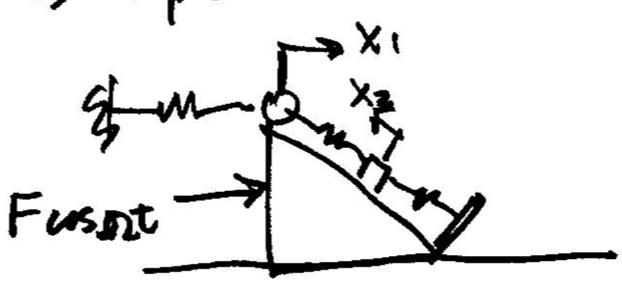
Fix all the gen. coord. but one, and vary that one <u>differentially</u>, write it as $\delta q_i$.

As $\delta q_i$ is varied, <u>freeze time t</u> ($\delta t = 0$), and maintain all constraints.

Compute the work done $\delta W$. It will be of the form $\delta W = (\cdots)\,\delta q_i$

this is $Q_i$ (by definition)

Example




$\delta q_i$ = virtual displacement

$\delta W$ = virtual work

$\delta W$ due to $\delta x_1$: $\quad \delta W = F\cos\Omega t\, \delta x_1 \Rightarrow Q_1 = F\cos\Omega t$

$\delta W$ " " $\delta x_2$: $\quad \delta W = 0 \Rightarrow Q_2 = 0$

So LE become
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = Q_i \quad , \quad i = 1,2$$

$$m_1 \ddot{x}_1 + m_2(\ddot{x}_1 - \ddot{x}_2 \cos\theta) + k_1(x_1 - x_2) = F\cos\Omega t$$

$$m_2(\ddot{x}_2 - \ddot{x}_1 \cos\theta) + k_1 x_2 + k_2 x_2 - k_1 x_1 = 0$$

(see these notes, p.4)

Linearizing about $x_1 = x_2 = 0$ gives

$$M\ddot{x} + Kx = f(t)$$

where $f(t) = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} F\cos\Omega t \\ 0 \end{bmatrix}$

and where $M, K$ are as before, see p. 5, 7 of these notes.

From p. 15 of these notes, we have

$$\ddot{\underline{p}} + D_1^{-1} D_2\, \underline{p} = D_1^{-1} R^t f \quad , \quad \underline{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

where $D_1 = R^t M R = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$ (see p. 13)

$$D_2 = R^t K R = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{(see p. 13)}$$

$$R = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{(see p. 12)}$$

$$\begin{pmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{pmatrix} + \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \overset{D_1^{-1}}{\begin{pmatrix} \frac{1}{7} & 0 \\ 0 & 1 \end{pmatrix}} \overset{R^t}{\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}} \overset{f}{\begin{pmatrix} F\cos\Omega t \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F\cos\Omega t \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} F\cos\Omega t \\ 0 \end{pmatrix}$$

So we have

$$\ddot{p}_1 + \frac{2}{7} p_1 = \frac{2}{7} F \cos \Omega t$$

$$\ddot{p}_2 + 2 p_2 = 0$$

$$p_{1\,part} = C \cos \Omega t$$

$$-\Omega^2 A + \frac{2}{7} C = \frac{2}{7} F$$

$$C = \frac{2/7\,F}{2/7 - \Omega^2}$$

$$p_{2\,part} = 0$$

If we imagine that a very small quantity of damping is present, the complementary solutions will decay and the steady state will consist of

$$p_{\substack{\text{steady}\\\text{state}}} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} C \cos \Omega t \\ 0 \end{pmatrix}$$

This gives $\quad x = R\,p = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \cos \Omega t \\ 0 \end{pmatrix}$

$$\therefore\ x_{ss} = \begin{pmatrix} 2C \cos \Omega t \\ C \cos \Omega t \end{pmatrix} \quad \text{or} \quad \left.\begin{array}{l} x_1 = 2C \cos \Omega t \\ x_2 = C \cos \Omega t \end{array}\right\} \text{steady state}$$

where $C = \dfrac{\frac{2}{7} F}{\frac{2}{7} - \Omega^2} = \dfrac{2F}{2 - 7\Omega^2}$

This result may be checked by direct substitution

$$M\ddot{x} + Kx = f$$

$$2\ddot{x}_1 - \frac{1}{2}\ddot{x}_2 + x_1 - x_2 = F\cos\Omega t$$

$$-\frac{1}{2}\ddot{x}_1 + \ddot{x}_2 - x_1 + 2x_2 = 0$$

Set $x_1 = A\cos\Omega t$
$x_2 = B\cos\Omega t$

$$-2A\Omega^2 + \frac{1}{2}B\Omega^2 + A - B = F \qquad (1)$$

$$\frac{1}{2}A\Omega^2 - B\Omega^2 - A + 2B = 0 \qquad (2)$$

$$(2) \Rightarrow \left(-\frac{\Omega^2}{2}+1\right)A = (2-\Omega^2)B$$

$$\text{So} \quad B = \left(\frac{-\frac{\Omega^2}{2}+1}{2-\Omega^2}\right)A$$

Substitute into (1)

$$(-2\Omega^2+1)A + \underbrace{\left(-1+\frac{\Omega^2}{2}\right)B}_{} = F$$

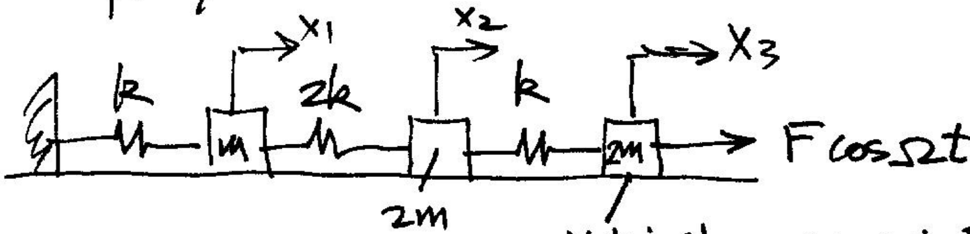
$$+\left(-1+\frac{\Omega^2}{2}\right)\left(\frac{-\frac{\Omega^2}{2}+1}{2-\Omega^2}\right)A$$

Some algebra reveals $\left(\frac{2-7\Omega^2}{4}\right)A = F$, $A = \frac{4F}{2-7\Omega^2}$ ✓ok

$$B = \left(\frac{1-\frac{\Omega^2}{2}}{2-\Omega^2}\right)A = \frac{A}{2} \quad \checkmark\text{ok}$$

# A 3DOF example

Text p.139/5.1
p.140/5.3
p.160/5.30
p.160/5.31




Note: shown as m in Fig. 5-1 but treated as 2m in text

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}(2m)\dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2}(2k)(x_1 - x_2)^2 + \frac{1}{2} k (x_2 - x_3)^2$$

$$\delta W = F\cos\Omega t\, \delta x_3 \Rightarrow Q_3 = F\cos\Omega t$$

Similarly $Q_1 = Q_2 = 0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = Q_i, \quad L = T - V$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} + \frac{\partial V}{\partial x_i} = Q_i$$ (since $T$ does not depend on $x_i$ & $V$ " " " " $\dot{x}_i$)

$$M\ddot{x} + Kx = f, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix}, \quad K = \begin{pmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 0 \\ F\cos\Omega t \end{pmatrix}$$

Begin with the Free Vibrations problem:

$$M\ddot{x} + Kx = 0$$

Set $x = X \cos \omega t$, $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

$$\left( m \begin{bmatrix} -\omega^2 & & \\ & -2\omega^2 & \\ & & -2\omega^2 \end{bmatrix} + k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right) \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Set $\beta = \omega^2 \frac{m}{k}$

$$\begin{vmatrix} 3-\beta & -2 & 0 \\ -2 & 3-2\beta & -1 \\ 0 & -1 & 1-2\beta \end{vmatrix} = 0$$

$$(3-\beta)(3-2\beta)(1-2\beta) - (3-\beta) - 4(1-2\beta) = 0$$

$$\Longrightarrow \quad (\beta-1)(2\beta^2 - 8\beta + 1) = 0$$

$$\beta = 1,\ 2 \pm \sqrt{\frac{7}{2}} = \omega^2 \frac{m}{k}$$

$$\omega = \sqrt{\frac{k}{m}} \left( 1,\ \sqrt{2 \pm \sqrt{\frac{7}{2}}} \right) = \sqrt{\frac{k}{m}},\ .359\sqrt{\frac{k}{m}},\ 1.967\sqrt{\frac{k}{m}}$$

$\omega_2 = \sqrt{\frac{k}{m}}\ ,\ \beta = 1 \Rightarrow$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2X_1 - 2X_2 = 0 \Rightarrow X_2 = X_1$

$-X_2 - X_3 = 0 \Rightarrow X_2 = -X_3$

$${}_2X = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \text{(amplitude arbitrary)}$$

---

$\omega_1 = .359\sqrt{\frac{k}{m}}\ ,\ \beta = .1291$

$(3-\beta)X_1 - 2X_2 = 0 \Rightarrow X_2 = \frac{3-\beta}{2} X_1 = 1.435\, X_1$

$-X_2 + (1-2\beta)X_3 = 0 \Rightarrow X_3 = \frac{1}{1-2\beta} X_2 = 1.348\, X_2$

$= 1.935\, X_1$

$${}_1X = \begin{pmatrix} 1 \\ 1.435 \\ 1.935 \end{pmatrix}$$

---

$\omega_3 = 1.967\sqrt{\frac{k}{m}}\ ,\ \beta = 3.870$

$X_2 = \frac{3-\beta}{2} X_1 = -.435\, X_1$

$X_2 = (1-2\beta)\, X_3 = -6.7416\, X_3$

$X_3 = -.1483\, X_2 = +.06458\, X_1$

$${}_3X = \begin{pmatrix} 1 \\ -.4354 \\ .06458 \end{pmatrix}$$

Next we form the modal matrix R

$$R = \left( {}_1X, {}_2X, {}_3X \right) = \begin{pmatrix} 1 & 1 & 1 \\ 1.435 & 1 & -.435 \\ 1.935 & -1 & 0.06458 \end{pmatrix}$$

Check $R^t M R \stackrel{?}{=} D_1 =$ diagonal

$$\begin{pmatrix} 1 & 1.435 & 1.935 \\ 1 & 1 & -1 \\ 1 & -.435 & .0645 \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} 12.607 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1.3868 \end{pmatrix} m \quad \checkmark \text{ok}$$

$$R^t K R \stackrel{?}{=} D_2$$

$$= \begin{pmatrix} 1.6284 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5.368 \end{pmatrix} k \quad \checkmark \text{ok}$$

Use R to define principal coordinates via $x = R\,p$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1.435 & 1 & -.435 \\ 1.935 & -1 & .0645 \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}}_{p}$$

$$M\ddot{x} + Kx = f$$

$$x = Rp$$

$$MR\ddot{p} + KRp = f$$

Mult by $R^t$

$$\underbrace{R^t M R}_{D_1} \ddot{p} + \underbrace{R^t K R}_{D_2} p = R^t f$$

$$\ddot{p} + D_1^{-1} D_2 p = D_1^{-1} R^t f \equiv g$$

We saw on p. 24 that

$$D_1 = \begin{pmatrix} 12.667m & & \\ & 5m & \\ & & 1.3868m \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1.6284k & & \\ & 5k & \\ & & 5.368k \end{pmatrix}$$

$$\therefore D_1^{-1} = \begin{pmatrix} \frac{1}{m}/12.607 & & \\ & 1/5m & \\ & & 1/1.3868m \end{pmatrix} = \begin{pmatrix} .0793/m & & \\ & .2/m & \\ & & .7210/m \end{pmatrix}$$

$$D_1^{-1} D_2 = \begin{pmatrix} .129\frac{k}{m} & & \\ & k/m & \\ & & 3.87\frac{k}{m} \end{pmatrix} = \begin{pmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \omega_3^2 \end{pmatrix}$$

$$D_1^{-1} R^t = \begin{pmatrix} \frac{.0793}{m} & & \\ & .2/m & \\ & & \frac{.7210}{m} \end{pmatrix} \begin{pmatrix} 1 & 1.435 & 1.935 \\ 1 & 1 & -1 \\ 1 & -.435 & .0645 \end{pmatrix}$$

$$= \begin{pmatrix} .0793 & .1137 & .1534 \\ .2 & .2 & -.2 \\ .7210 & .3136 & .0465 \end{pmatrix} \frac{1}{m}$$

Now with $f = \begin{pmatrix} 0 \\ 0 \\ F\cos\Omega t \end{pmatrix}$

$$D_1^{-1} R^t f = F\cos\Omega t \begin{bmatrix} .1534 \\ -.2 \\ .0465 \end{bmatrix}$$

So we have

$$\ddot{p}_1 + .129\frac{k}{m}p_1 = .153\frac{F}{m}\cos\Omega t$$

$$\ddot{p}_2 + \frac{k}{m}p_2 = -.2\frac{F}{m}\cos\Omega t$$

$$\ddot{p}_3 + 3.87\,k/m\,p_3 = .0465\frac{F}{m}\cos\Omega t$$

Steady state soln (assuming small damping)

$$\ddot{p} + \omega^2 p = E\cos\Omega t$$

Set $p_{part} = C\cos\Omega t$

$$-C\Omega^2 + C\omega^2 = E$$

$$C = \frac{E}{\omega^2 - \Omega^2}$$

$$p_1(t) = \frac{F(.153)/m}{\frac{k}{m}(.129) - \Omega^2}\cos\Omega t$$

$$p_2(t) = \frac{-.2F/m}{\frac{k}{m}(1) - \Omega^2}\cos\Omega t$$

$$p_3(t) = \frac{.0465\ F/m}{\frac{k}{m}(3.87) - \Omega^2}\cos\Omega t$$

Now to find $X_1, X_2, X_3$, use $x = R\,p$

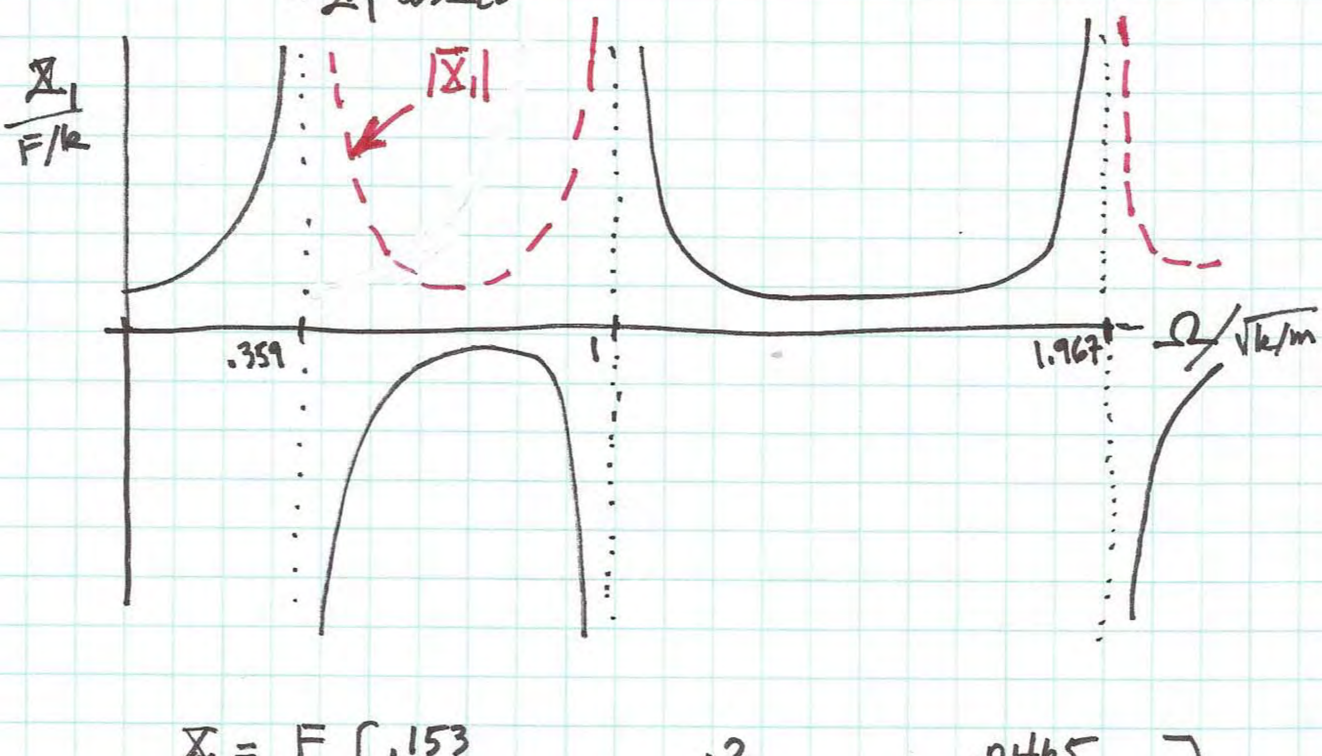
$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1.435 & 1 & -.435 \\ 1.935 & -1 & .06458 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

for example,

$$x_1 = p_1 + p_2 + p_3$$

$$= \frac{F}{m}\cos\Omega t \left[ \frac{.153}{\frac{k}{m}(.129) - \Omega^2} - \frac{.2}{\frac{k}{m} - \Omega^2} + \frac{.0465}{\frac{k}{m}(3.87) - \Omega^2} \right]$$

$$= X_1 \cos\Omega t$$




$$X_1 = \frac{F}{k}\left[ \frac{.153}{.129 - \frac{\Omega^2}{k/m}} - \frac{.2}{1 - \frac{\Omega^2}{k/m}} + \frac{.0465}{3.87 - \frac{\Omega^2}{k/m}} \right]$$

$$= \frac{F}{k}\left[ \frac{.5}{\left(.129 - \frac{\Omega^2}{k/m}\right)\left(1 - \frac{\Omega^2}{k/m}\right)\left(3.87 - \frac{\Omega^2}{k/m}\right)} \right]$$

$$x_2 = 1.435\,p_1 + p_2 - .435\,p_3$$

$$= \frac{F}{k}\cos\Omega t\left[\frac{(1.435)(.153)}{.129 - \frac{\Omega^2}{k/m}} - \frac{.2}{1 - \frac{\Omega^2}{k/m}} - \frac{(.435)(.0465)}{3.87 - \frac{\Omega^2}{k/m}}\right]$$

$$= X_2 \cos\Omega t$$

plot of $X_2$ looks similar to that of $X_1$

$$x_3 = 1.935\,p_1 - p_2 + .06458\,p_3$$

$$= \frac{F}{k}\cos\Omega t\left[\frac{(1.935)(.153)}{.129 - \frac{\Omega^2}{k/m}} + \frac{.2}{1 - \frac{\Omega^2}{k/m}} + \frac{(.06458)(.0465)}{3.87 - \frac{\Omega^2}{k/m}}\right]$$

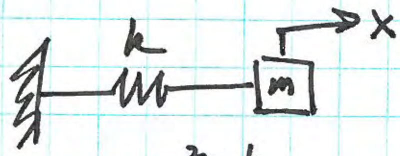

Note that for $\Omega$ chosen here, $X_3 = 0$

This phenomenon could be used when it is desired for a mass to have zero response (vibration absorber)

# The Geometry of 2 DOF Systems

Start with 1 DOF system

$$\ddot{x} + \omega^2 x = 0$$



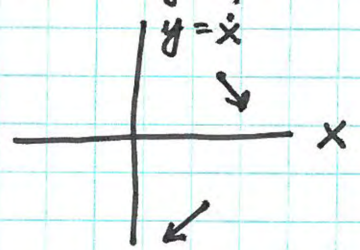

$$\omega^2 = \frac{k}{m}$$

Define $y = \dot{x} \Rightarrow \dot{y} = \ddot{x} = -\omega^2 x$

So we have a system of two first order ODEs:

$$\dot{x} = y$$
$$\dot{y} = -\omega^2 x$$

View this in the x-y plane ("the phase plane")




At each point $(x,y)$ there is a vector $(\dot{x},\dot{y}) = (y, -\omega^2 x)$ "a vector field"

The motion flows along curves in the x-y plane which must satisfy the CONSERVATION OF ENERGY:

$$T + V = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 = h = \text{total energy}$$

Note: 1) $h$ is determined by the initial conditions

2) $T+V=h$ follows from multiplying by $\dot{x}$:

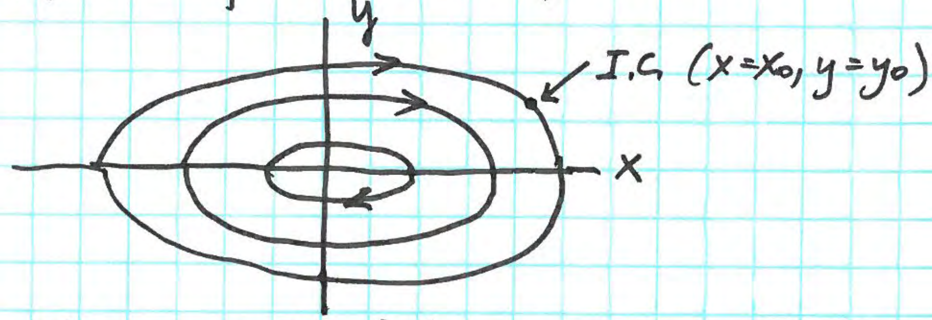
$$\dot{x}(\ddot{x} + \omega^2 x) = 0$$

$$\frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) + \frac{d}{dt}\frac{\omega^2 x^2}{2} = 0$$

$$\frac{d}{dt}(T+V) = 0 \Rightarrow T+V = h$$

$$T+V=h \Rightarrow \underbrace{\frac{1}{2}y^2+\frac{1}{2}\omega^2x^2=h}_{\text{an ellipse}}$$

The x-y plane is filled with ellipses:




Each ellipse is a trajectory corresponding to a specific value of h.
There is particular ellipse and a specific value of h corresponding to each initial condition.

We may also think of $y(x)$ instead of $x(t)$ and $y(t)$. These are related by the chain rule:

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

Here $\frac{dy}{dt}=-\omega^2x$ and $\frac{dx}{dt}=y$, so we have

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{-\omega^2x}{y}$$

Separating variables,

$$y\,dy=-\omega^2x\,dx$$

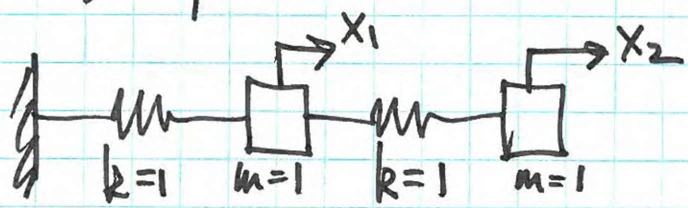
Integrating both sides,

$$\frac{y^2}{2}=-\omega^2\frac{x^2}{2}+\overset{h}{\overset{\shortparallel}{\text{constant}}}$$

$$\Rightarrow \frac{y^2}{2}+\frac{\omega^2x^2}{2}=h \quad \text{(once again)}$$

Now let us go on to think about the geometry of phase space for a 2 DOF system

To make things easier to see, let's think about a specific example:




To begin with, let's find the eqs. of motion and the principal coordinates:

$$T = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2, \quad V = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 - x_2)^2$$

Lagrange's equations:

$$\ddot{x}_1 + 2x_1 - x_2 = 0$$

$$\ddot{x}_2 - x_1 + x_2 = 0$$

Ansatz: $x_1 = X_1 \cos\omega t$, $x_2 = X_2 \cos\omega t$

$$-\omega^2 X_1 + 2X_1 - X_2 = 0$$

$$-\omega^2 X_2 - X_1 + X_2 = 0$$

$$\begin{bmatrix} -\omega^2+2 & -1 \\ -1 & -\omega^2+1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution, det $= 0 \Rightarrow$

$$(-\omega^2+2)(-\omega^2+1) - 1 = 0$$

$$\omega^4 - 3\omega^2 + 1 = 0 \Rightarrow \omega^2 = \frac{3 \pm \sqrt{5}}{2}$$

$$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}} = .618, \quad \omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}} = 1.618$$

Modal vectors:

$$(-\omega^2+2)\,X_1 - X_2 = 0 \Rightarrow X_2 = (-\omega^2+2)\,X_1$$

$$\omega_1 = .618 \Rightarrow {}_1X = \begin{pmatrix} 1 \\ 1.618 \end{pmatrix} \quad \text{in-phase mode}$$

$$\omega_2 = 1.618 \Rightarrow {}_2X = \begin{pmatrix} 1 \\ -.618 \end{pmatrix} \quad \text{out-of-phase mode}$$

Transform to principal coordinates $p_1, p_2$:

$$\text{Define } R = ({}_1X, {}_2X) = \begin{pmatrix} 1 & 1 \\ 1.618 & -.618 \end{pmatrix}$$

Set

$$x = Rp, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

i.e.

$$x_1 = p_1 + p_2$$

$$x_2 = 1.618\,p_1 - .618\,p_2$$

We have

$$M\ddot{x} + Kx = 0$$

$$MR\ddot{p} + KRp = 0$$

$$\underbrace{R^t M R}_{D_1}\ddot{p} + \underbrace{R^t K R}_{D_2} p = 0$$

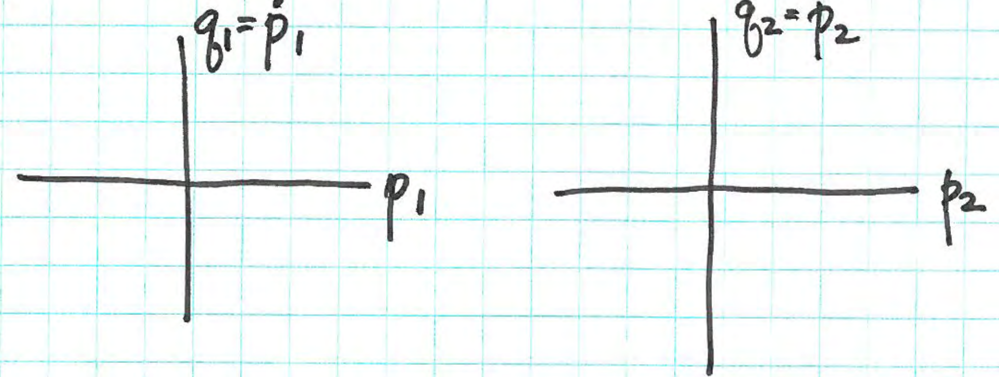
gives, as usual,

$$\ddot{p}_1 + \omega_1^2 p_1 = 0$$

$$\ddot{p}_2 + \omega_2^2 p_2 = 0$$

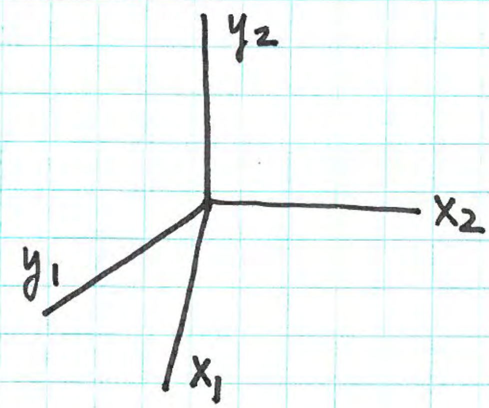
Let $q_1 = \dot{p}_1$, $q_2 = \dot{p}_2$

Then each of the principal modes lives in a 1 DOF $p_i$-$q_i$ phase space:



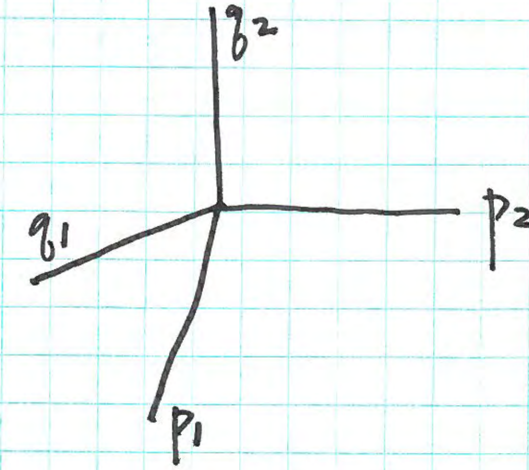

The phase space for the original 2 DOF system can be thought of in two ways: In physical coords,

$$\dot{x}_1 = y_1$$
$$\dot{y}_1 = -2x_1 + x_2$$
$$\dot{x}_2 = y_2$$
$$\dot{y}_2 = x_1 - x_2$$

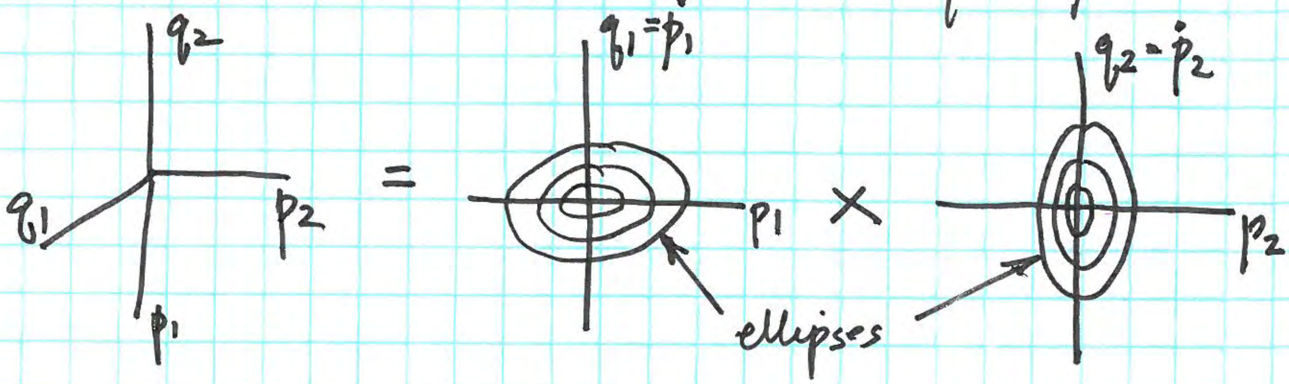



or in modal coordinates,

$$\dot{p}_1 = q_1$$
$$\dot{q}_1 = -\omega_1^2 p_1$$
$$\dot{p}_2 = q_2$$
$$\dot{q}_2 = -\omega_2^2 p_2$$

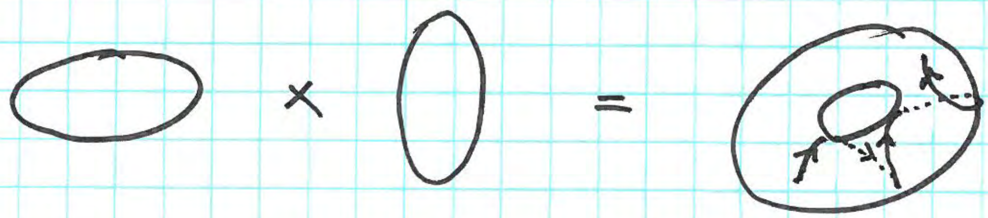


The system in modal coordinates is really two 1 DOF phase planes

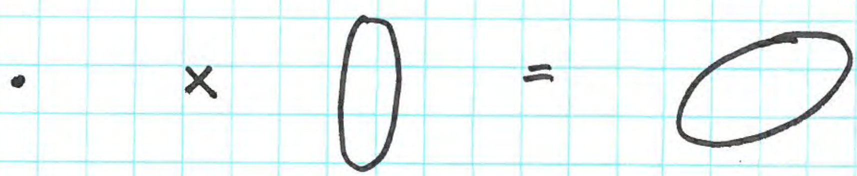



Initial conditions will single out a specific ellipse in the $p_1$-$q_1$ plane, and another in the $p_2$-$q_2$ plane.

Taken together, motion on the two ellipses can be thought of as occurring on a single <u>torus</u>




A special case occurs if the energy in one of the principal modes is zero: then the torus reduces to a circle:



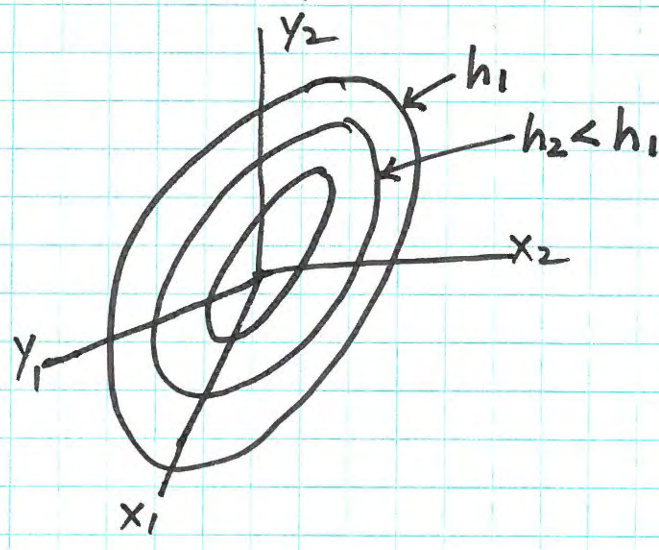

The 4 dimensional phase space is filled with non-intersecting tori. This is the case whether we use principal coordinates, as above, or physical coordinates $x_i$-$y_i$, in which case the tori are moved around by the transformation $x = Rp$.

In order to simplify the picture, we restrict attention to motions corresponding to the same total energy $h$.

$$T + V = h$$

$$\frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + \frac{1}{2} x_1^2 + \frac{1}{2}(x_1 - x_2)^2 = h$$

This object is a kind of 4 dimensional ellipsoid sitting in $x_1$-$x_2$-$y_1$-$y_2$ phase space. The whole space is composed of a continuum of these objects, all nested one within the next:




We will refer to one of these as an "energy manifold". The word "manifold" stands for a surface in $n$ dimensions.

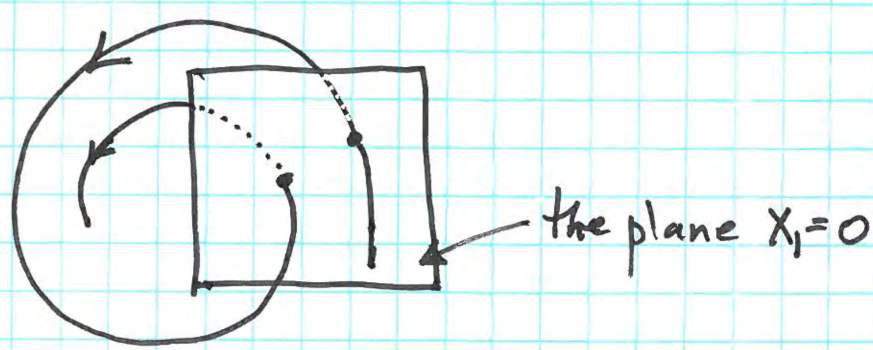
Each energy manifold is composed of a continuum of tori, and each torus represents a particular motion of the original system.

In order to understand how the tori are packed inside the energy manifold, we proceed as follows.

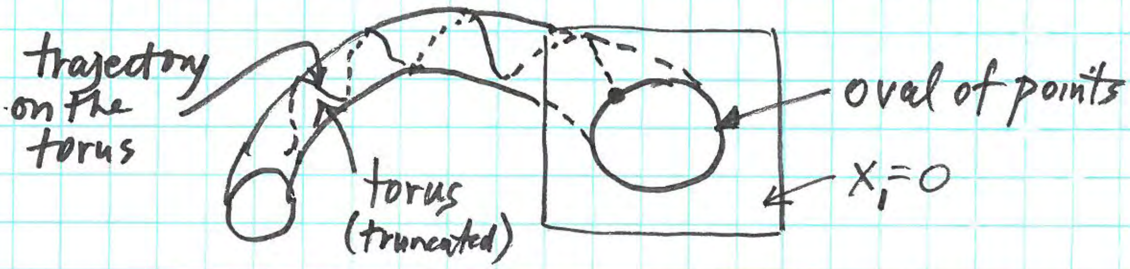
First note that the energy manifold is 3 dimensional. That is, if you choose values for $x_1$, $x_2$ and $y_1$, you can solve $T+V=h$ for $y_2$ (at least locally).

Since it is harder to picture 3 dimensional objects that 2 dimensional objects, we use a trick invented by Poincare to get a 2 dimensional look at the 3 dimensional energy manifold.

The trick is called a <u>Poincare map</u> and involves replacing an entire trajectory by a set of points, these being the places where the trajectory pierces the plane $x_1=0$ (going in the positive $\dot{x}_1$ direction)




So for example, a torus would look like a collection of points in an oval, in the Poincare map:



Now suppose we choose an initial condition which starts out on the plane $x_1=0$:

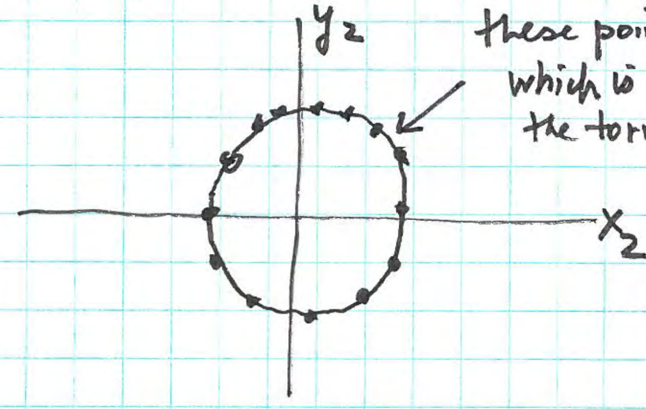
$$x_1=0,\ x_2=x_{20},\ y_2=y_{20}$$

where $(x_{20}, y_{20})$ is a point in the $x_2$-$y_2$ plane and then we COMPUTE $y_1 (=\dot{x}_1)$ at $t=0$ from $T+V=h$:

$$\frac{1}{2}y_1^2+\frac{1}{2}y_2^2+\frac{1}{2}x_1^2+\frac{1}{2}(x_1-x_2)^2=h$$

$$y_{10}=\sqrt{2h-y_{20}^2-x_{20}^2}$$

Now the trajectory with this initial condition takes off & travels through the energy manifold until it hits the plane $x_1=0$ with $\dot{x}_1>0$ again. In this way we obtain a collection of points in the $x_2$-$y_2$ plane:




Had we chosen a different initial condition, we'd have obtained a different curve.
By looking at all such curves we can see how the tori are arranged in the energy manifold.

We proceed now to find analytical expressions for the curves in the $x_2$–$y_2$ plane which are the intersections of the tori with the energy manifold.

From $\ddot{p}_1 + w_1^2 p_1 = 0$ we have

$$\frac{1}{2}\dot{p}_1^2 + \frac{1}{2} w_1^2 p_1^2 = C \quad \text{(conservation of energy)}$$

Next we want to write this in terms of the physical coords.

$$x = Rp \Rightarrow p = R^{-1} x$$

For $R = \begin{pmatrix} 1 & 1 \\ 1.618 & -.618 \end{pmatrix}$, find $R^{-1} = \begin{pmatrix} .2764 & .4472 \\ .7236 & -.4472 \end{pmatrix}$

$$\Rightarrow \quad p_1 = .2764\, x_1 + .4472\, x_2$$
$$p_2 = .7236\, x_1 - .4472\, x_2$$

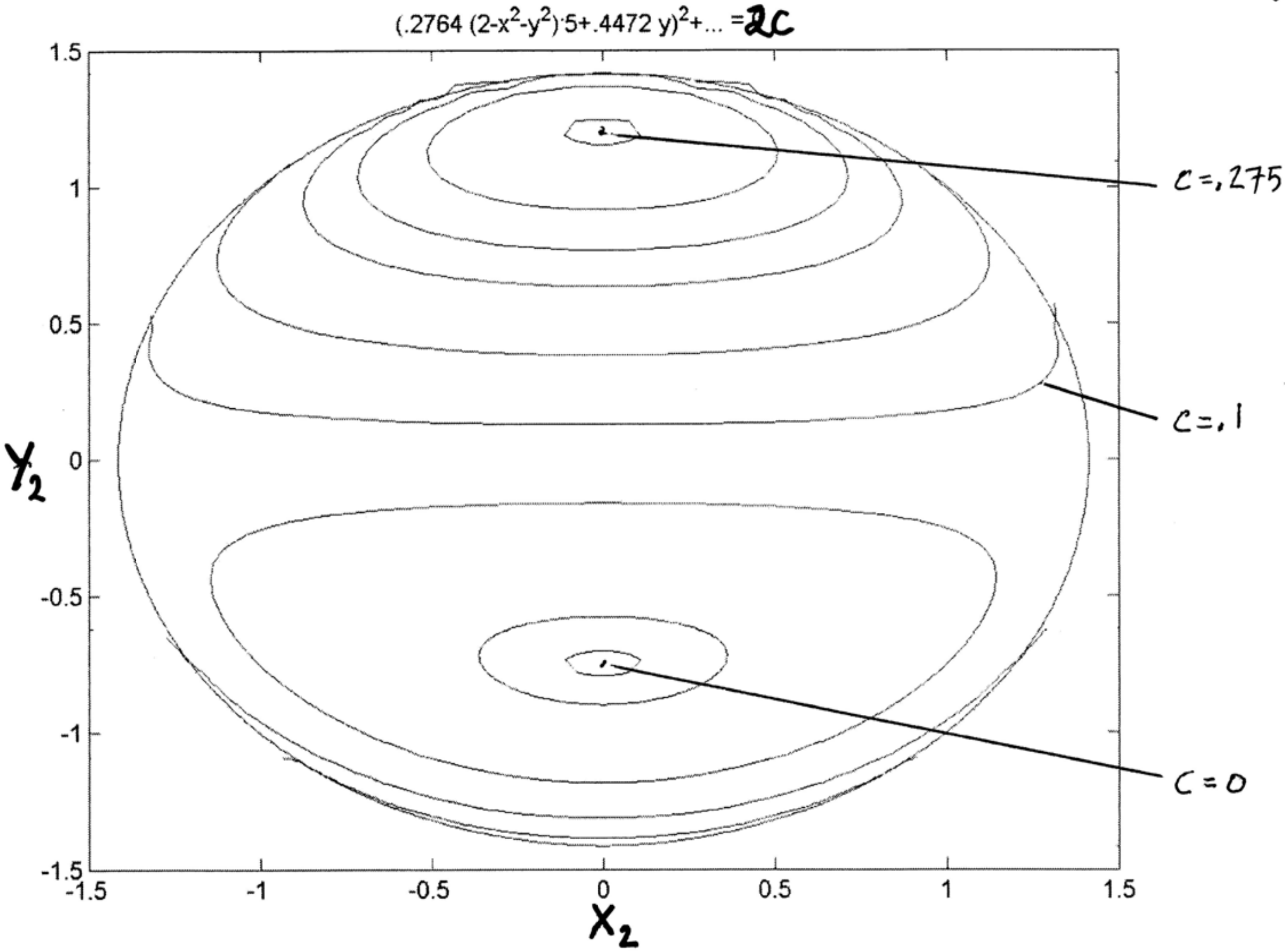
$$\frac{1}{2}\dot{p}_1^2 + \frac{1}{2} w_1^2 p_1^2 = C \Rightarrow \frac{1}{2}(.2764\, \underbrace{\dot{x}_1}_{=y_1} + .4472\, \underbrace{\dot{x}_2}_{=y_2})^2 + \frac{1}{2}(.618)^2(.7236\, x_1 - .4472\, x_2)^2 = C$$

Now set $x_1 = 0$ for Poincare map and

$$y_1 = \sqrt{2h - y_2^2 - x_2^2} \quad \text{from p. G9}$$

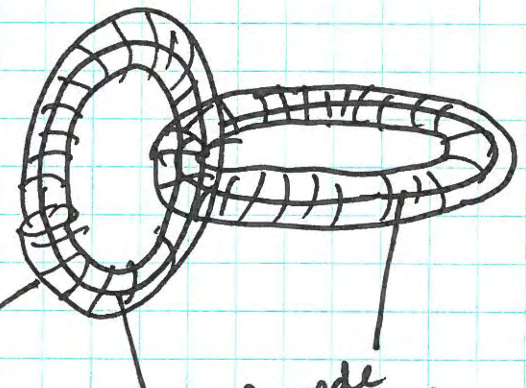
$$\left(.2764\sqrt{2h - y_2^2 - x_2^2} + .4472\, y_2\right)^2 + (.618)^2(.4472\, x_2)^2 = 2C$$

G11




h=1

So we see this arrangement:



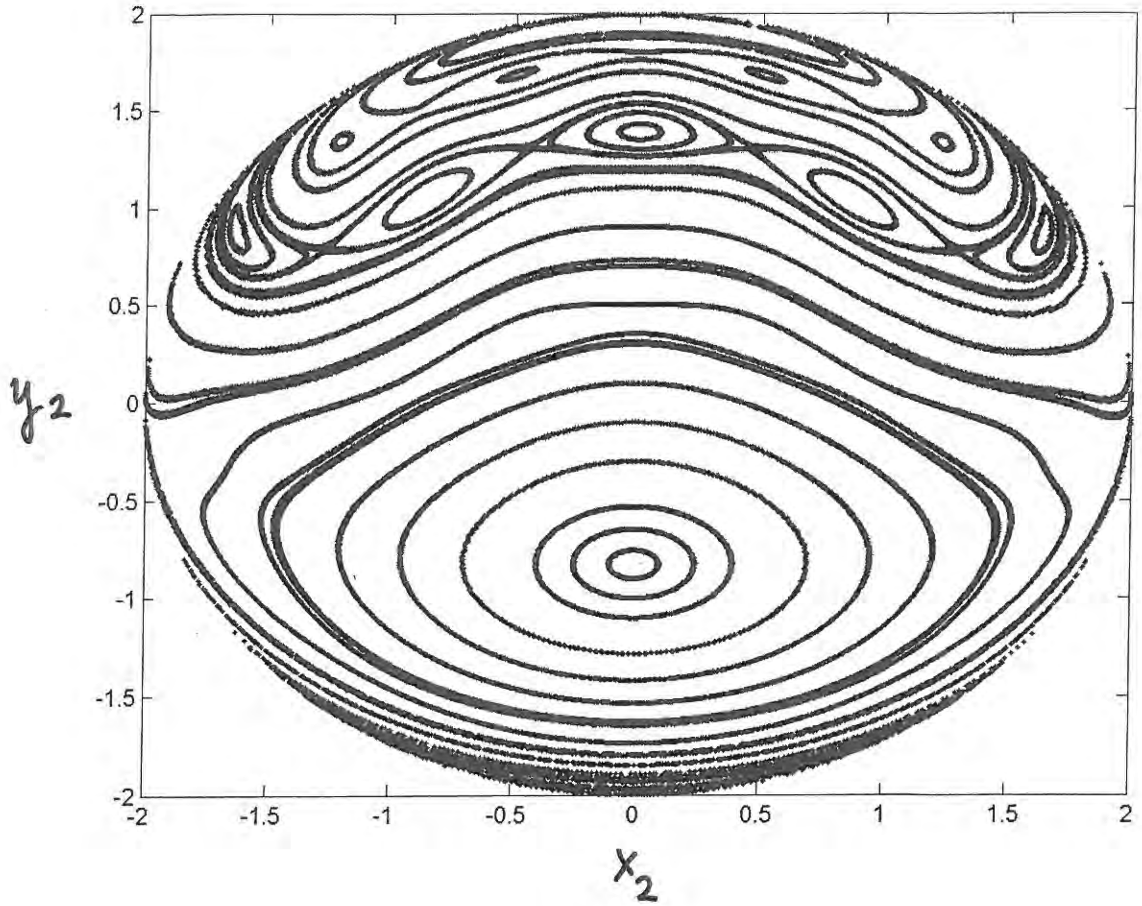

A single parameter family of torus flows go from the nbhd of one principal mode to another.

Diagrams like the foregoing Poincare map are important when nonlinear terms are included in the system.

See next page where the comparable diagram is given for the same system, except a <u>nonlinear spring</u> is included.

The dynamics have become more complicated.

G13

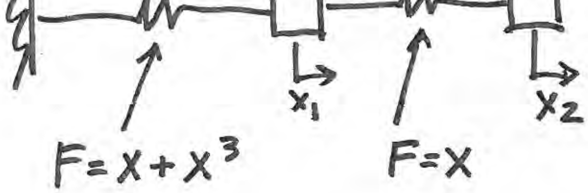



$h=2$

NONLINEAR SPRING

$$V = \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}(x_1 - x_2)^2$$

$$T = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2$$



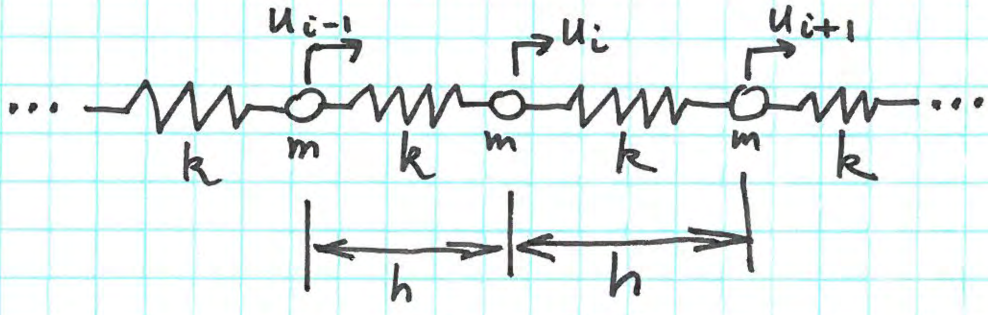

$$\ddot{x}_1 = -x_1 - x_1^3 - (x_1 - x_2)$$

$$\ddot{x}_2 = -(x_2 - x_1)$$

# Longitudinal Vibrations of a Rod

## 3 methods for deriving the EOM

**Method 1** The continuum as a limit of $N$ particles, as $N \to \infty$




The system consists of a line of masses, each one coupled to its nearest neighbors and restrained by linear springs. The displacements $u_i(t)$ are measured from unstretched equilibrium, where $h$ is the distance between the masses.

The EOM:

$$m\frac{d^2u_i}{dt^2} = k(u_{i+1}-u_i) - k(u_i - u_{i-1})$$

Now we wish to pass from this system of ODE's to a PDE via the continuum limit.

Displacement field: $u = u(x,t)$

where $x$ plays the role that $i$ plays in $u_i$

$$u_i(t) = u(x_i,t) \qquad \text{where } x_{i+1} - x_i = h$$

the displacement at time $t$ of a particle which was located at $x = x_i$ at equilibrium

C2

We expand $u_{i+1} = u(x_{i+1}, t)$ in a Taylor Series about $x_i$

$x_{i+1} = x_i + h \Rightarrow$

$$u_{i+1} = u(x_{i+1}, t) = \underset{u_i}{\underset{\shortparallel}{u(x_i, t)}} + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \cdots$$

where the partial derivatives are to be evaluated at $x = x_i$.

Similarly $x_{i-1} = x_i - h \Rightarrow$

$$u_{i-1} = u(x_{i-1}, t) = u_i - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \cdots$$

Substituting these expressions into the EOM $\Rightarrow$

$$m\frac{d^2 u_i}{dt^2} = k(u_{i+1} - 2u_i + u_{i-1})$$

$$\underset{\frac{\partial^2 u}{\partial t^2}}{\shortparallel} = k\left(\left(u_i + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}\right) - 2u_i + \left(u_i - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}\right)\right)$$

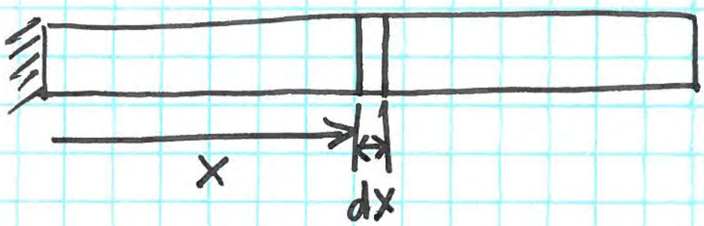
$$m\frac{\partial^2 u}{\partial t^2} = h^2 k\frac{\partial^2 u}{\partial x^2}$$

Absorb the $h^2$ into $x$ with $\tilde{x} = \frac{x}{h}$

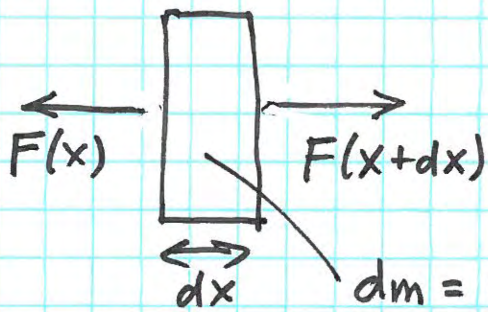
$$\frac{\partial^2 u}{\partial t^2} = \frac{k}{m}\frac{\partial^2 u}{\partial \tilde{x}^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \tilde{x}^2} \quad \text{where } c^2 = \frac{k}{m}$$

## Method 2   Newton's eqs using Hooke's law




Free body of volume element:




$dm = \rho\, dx \quad \left(\rho = \frac{\text{mass}}{\text{length}}\right)$

$$(\rho\, dx)\, u_{tt} = F(x+dx) - F(x)$$

$$\rho\, u_{tt} = \frac{F(x+dx) - F(x)}{dx}$$

$$\rightarrow \frac{\partial F}{\partial x} \quad \text{as} \quad dx \rightarrow 0$$

$F = \sigma A$ where $\sigma =$ normal stress in $x$ direction $= \sigma_{xx}$

$A = A(x) =$ area of cross section

Hooke's Law $\quad \sigma = E\varepsilon$

$E =$ Young's modulus

$\varepsilon = \varepsilon_{xx} =$ strain $= \dfrac{\partial u}{\partial x}$

$$F = EA \frac{\partial u}{\partial x}$$

C4

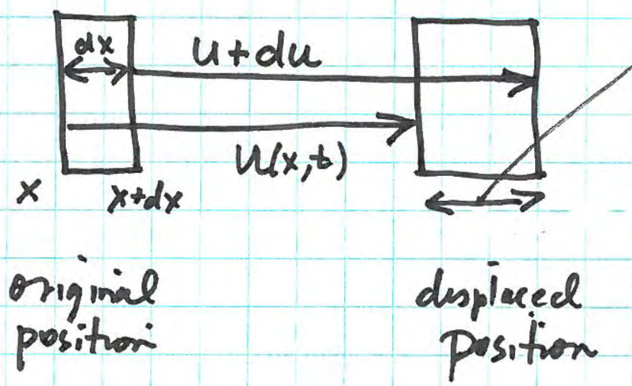
$$\rho u_{tt} = \frac{\partial}{\partial x}\left(E A(x) \frac{\partial u}{\partial x}\right)$$

In the special case of constant cross-section,

$$u_{tt} = \frac{EA}{\rho} u_{xx} = c^2 u_{xx}, \quad c^2 = \frac{EA}{\rho}$$

Detail: Why is strain $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ ?

$$\text{Strain} = \frac{\text{Change in length}}{\text{length}}$$




$(dx+u+du)-u = dx+du$

where

$du = \frac{\partial u}{\partial x} dx$

so $\Delta L = du = u_x \, dx$

original length $= dx$

$$\therefore \frac{\Delta L}{\text{orig. length}} = \text{strain} = \frac{\partial u}{\partial x}$$

Method 3: Lagrange's Eq.

LE for a system of N ODE's is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad \mathcal{L} = T - V$$

LE for a system with 1 PDE on $u(x,t)$

$$\frac{d}{dx}\frac{\partial \mathcal{L}^*}{\partial u_x} + \frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial u_t} - \frac{\partial \mathcal{L}^*}{\partial u} = 0$$

where $\mathcal{L}^*$ = Lagrangian density

$$\mathcal{L} = \int_0^\ell \mathcal{L}^* \, dx$$

$$\mathcal{L}^* = T^* - V^*$$

$$T = \int_0^\ell T^* dx, \quad V = \int_0^\ell V^* dx$$

<u>Example</u> longitudinal vibs of a rod

$$T^* = \frac{1}{2}\rho u_t^2 \qquad (\rho = \text{mass/length})$$

$$T = \int_0^\ell \frac{1}{2}\rho u_t^2 \, dx \qquad (\rho \, dx = dm)$$

$V^*$ = strain energy/length

From theory of elasticity, $\frac{1}{2}\sigma_{xx}\varepsilon_{xx}$ = strain energy/volume

$$V = \int_0^\ell \frac{1}{2}\sigma\varepsilon\,(A\,dx) \Rightarrow V^* = \frac{1}{2}\sigma\varepsilon A$$

$A = A(x)$ = cross-sectional area

C6

For an elastic solid, $\sigma = E\varepsilon$

$$V^* = \frac{1}{2} E \varepsilon^2 A = \frac{1}{2} E A u_x^2$$

$$\mathcal{L}^* = T^* - V^* = \frac{1}{2}\rho u_t^2 - \frac{1}{2} E A(x) u_x^2$$

LE:

$$\frac{d}{dx}\underbrace{\frac{\partial \mathcal{L}^*}{\partial u_x}}_{-EAu_x} + \frac{d}{dt}\underbrace{\frac{\partial \mathcal{L}^*}{\partial u_t}}_{\rho u_t} - \frac{\partial \mathcal{L}^*}{\partial u} = 0$$

$$-E\frac{\partial}{\partial x}(A u_x) + \frac{\partial}{\partial t}(\rho u_t) = 0$$

$$E\frac{\partial}{\partial x}(A(x)\, u_x) = \rho u_{tt} \quad \text{as before}$$

So now we know the PDE for longitudinal vibr of a rod:

Let $\frac{EA}{\rho} = c^2$ then $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ "wave eq."

Look for a solution in the form:

$$u(x,t) = U \cos \omega t \quad \text{where } U = U(x)$$

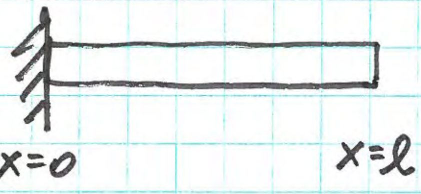
$$\Rightarrow \; -\omega^2 U = c^2 \frac{d^2 U}{dx^2}$$

$$\frac{d^2 U}{dx^2} + \left(\frac{\omega}{c}\right)^2 U = 0$$

$$U(x) = c_1 \sin \frac{\omega}{c} x + c_2 \cos \frac{\omega}{c} x$$

In order to find $c_1, c_2$, we must use the B.C. (boundary conditions)

Example A rod is fixed at $x=0$ and free at $x=l$:




B.C. $u=0$ at $x=0$

$\sigma_{xx}=0$ at $x=l$ $\left(\Rightarrow \frac{\partial u}{\partial x}=0 \text{ at } x=l\right.$ since $\left.\sigma = E\varepsilon = E u_x\right)$

$$U(0) = 0 = C_2$$

$$\frac{dU(\ell)}{dx} = 0 = C_1 \frac{\omega}{c} \cos \frac{\omega}{c} \ell \Rightarrow \left.\begin{array}{l} \frac{\omega \ell}{c} = M \frac{\pi}{2}, \\ M \text{ odd} \end{array}\right\}$$

to remind ourselves that $M$ must be odd, we write $M = 2n-1$, $n = 1, 2, 3, \ldots$

$$\omega_n = (2n-1)\left(\frac{\pi c}{2\ell}\right) \quad \text{frequency of the } n\text{th mode}$$

$$U_n(x) = a_n \sin \frac{\omega_n}{c} x = a_n \sin \frac{2n-1}{2} \frac{\pi}{\ell} x$$

$= n^{th}$ mode shape.

## General solution

$$u(x,t) = \sum_{n=1,2,3,\ldots}^{\infty} U_n(x) \left( a_n \cos \omega_n t + b_n \sin \omega_n t \right)$$

where $a_n, b_n$ are found from IC (initial conditions)

## Example

Suppose in the previous example of a rod fixed at $x=0$ and free at $x=\ell$, we are given that at $t=0$, $\frac{\partial u}{\partial t} = 0$ and $u(x,0) = x/\ell$

C9

IC $\frac{\partial u}{\partial t}(x,0)=0 \Rightarrow b_n=0$

$$u(x,0)=\frac{x}{l} \Rightarrow \frac{x}{l}=\sum a_n U_n(x)$$

$$=\sum_{n=1}^{\infty} a_n \sin\frac{\omega_n}{c}x$$

where $\omega_n=(2n-1)\frac{\pi c}{2l}$

To find the $a_n$ coefficients, we use Fourier series

The mode shapes $U_n(x)$ form an orthogonal set:

$$\int_0^l U_n(x)\,U_m(x)\,dx=0 \quad \text{if } n\neq m$$

Why?

Because each satisfies the d.e. + b.c.:

(1) $$U_n''+\left(\frac{\omega_n}{c}\right)^2 U_n=0, \quad U_n(0)=0,\ U_n'(l)=0$$

Multiply eq. (1) by $U_m$ and integrate from 0 to $l$

(2) $$\int_0^l U_m U_n''\,dx+\left(\frac{\omega_n}{c}\right)^2\int_0^l U_m U_n\,dx=0$$

Integrate the first term by parts: $\int u\,dv=uv-\int v\,du$

$$\int_0^l U_m U_n''\,dx=\int_0^l U_m\,dU_n'=U_m U_n'\Big|_0^l-\int_0^l U_n'\,dU_m$$

$$=U_m(l)\underbrace{U_n'(l)}_{=0}-\underbrace{U_m(0)}_{=0}U_n'(0)$$

$$-\int_0^l U_n' U_m'\,dx$$

C10

$$\therefore \int_0^\ell U_m U_n'' dx = -\int_0^\ell U_n' U_m' dx$$

So that eq. (2) becomes

$$(3) \qquad -\int_0^\ell U_n' U_m' dx + \left(\frac{\omega_n}{c}\right)^2 \int_0^\ell U_m U_n \, dx = 0$$

Now redo everything with $n$ & $m$ interchanged:

$$(4) \qquad -\int_0^\ell U_m' U_n' dx + \left(\frac{\omega_m}{c}\right)^2 \int_0^\ell U_n U_m \, dx = 0$$

And now subtract eq. (4) from eq. (3):

$$\left[\left(\frac{\omega_n}{c}\right)^2 - \left(\frac{\omega_m}{c}\right)^2\right] \int_0^\ell U_n(x) U_m(x) \, dx = 0$$

Q.E.D.

Now back to the original problem of finding the coefficients $a_n$ from the equation

$$\frac{x}{\ell} = \sum_{n=1}^{\infty} a_n \sin \frac{\omega_n}{c} x$$

Multiply by $\sin \frac{\omega_m}{c} x$ and integrate from 0 to $\ell$

$$\int_0^\ell \frac{x}{\ell} \sin \frac{\omega_m}{c} x \, dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^\ell \sin \frac{\omega_m x}{c} \sin \frac{\omega_n x}{c} \, dx}_{\int_0^\ell U_m(x) U_n(x) \, dx = 0}$$

$$= a_m \int_0^\ell \sin^2 \frac{\omega_m x}{c} \, dx$$

C11

We need the following two integrals:

$$\int_0^\ell \frac{x}{\ell} \sin \frac{\omega_m x}{c}\, dx \quad \text{and} \quad \int_0^\ell \sin^2 \frac{\omega_m x}{c}\, dx$$

Do the first one by parts, $\int u\,dv = uv - \int v\,du$

$$\int_0^\ell \frac{x}{\ell} \sin \frac{\omega_m x}{c}\, dx = \int_0^\ell \frac{x}{\ell}\, d\left(\cos \frac{\omega_m x}{c}\right) \cdot \left(-\frac{c}{\omega_m}\right)$$

$$= -\frac{c}{\omega_m} \frac{x}{\ell} \cos \frac{\omega_m x}{c} \Big|_0^\ell$$

$$+ \frac{c}{\omega_m} \int_0^\ell \cos \frac{\omega_m x}{c}\, d\left(\frac{x}{\ell}\right)$$

$$= -\frac{c}{\omega_m} \underbrace{\cos \frac{\omega_m}{c} \ell}_{=0} + \frac{c}{\omega_m \ell} \underbrace{\int_0^\ell \cos \frac{\omega_m x}{c}\, dx}_{\int_0^\ell d\left(\sin \frac{\omega_m x}{c}\right)\left(\frac{c}{\omega_m}\right)}$$

because

$$\omega_m = (2m-1)\frac{\pi c}{2\ell}$$

$$= \frac{c^2}{\omega_m^2 \ell} \sin \frac{\omega_m x}{c} \Big|_0^\ell$$

$$= \frac{c^2}{\omega_m^2 \ell} \sin \frac{\omega_m \ell}{c}$$

$$\sin \frac{\omega_m \ell}{c} = \sin (2m-1)\frac{\pi}{2} = (-1)^{m+1}$$

$$= \frac{c^2}{\omega_m^2 \ell} (-1)^{m+1}$$

We still need $\displaystyle\int_0^l \sin^2 \frac{\omega_m x}{c}\, dx$

$$\sin^2 \frac{\omega_m x}{c} = \frac{1}{2} - \frac{1}{2}\cos\frac{2\omega_m x}{c}$$

gives $\displaystyle\int_0^l \frac{1}{2}\,dx - \frac{1}{2}\int_0^l \underbrace{\cos\frac{2\omega_m x}{c}\,dx}_{d \sin\frac{2\omega_m x}{c}\left(\frac{c}{2\omega_m}\right)}$

$$= \frac{l}{2} - \frac{c}{\omega_m} \underbrace{\sin\frac{2\omega_m x}{c}\Big|_0^l}_{\sin\frac{2\omega_m l}{c} = \sin(2m-1)\pi = 0}$$

$$\therefore \int_0^l \sin^2 \frac{\omega_m x}{c}\, dx = \frac{l}{2}$$

Now we return to p. C10:

$$a_m = \frac{\int_0^l \frac{x}{l} \sin\frac{\omega_m}{c} x\, dx}{\int_0^l \sin^2 \frac{\omega_m x}{c}\, dx}$$

$$= \frac{\frac{c^2}{\omega_m^2 l}(-1)^{m+1}}{l/2}$$

$$= \frac{(-1)^{m+1}\, 2c^2}{\omega_m^2 l^2}$$

$$u(x,t) = \sum_{n=1,2,3,\ldots}^{\infty} a_n U_n(x) \cos \omega_n t \quad , \quad \omega_n = (2n-1)\frac{\pi c}{2\ell}$$

$$= \sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1} 2c^2}{\omega_n^2 \ell^2} \sin \frac{\omega_n x}{c} \cos \omega_n t \right)$$

Example: What is the motion of the end $x=\ell$ ?

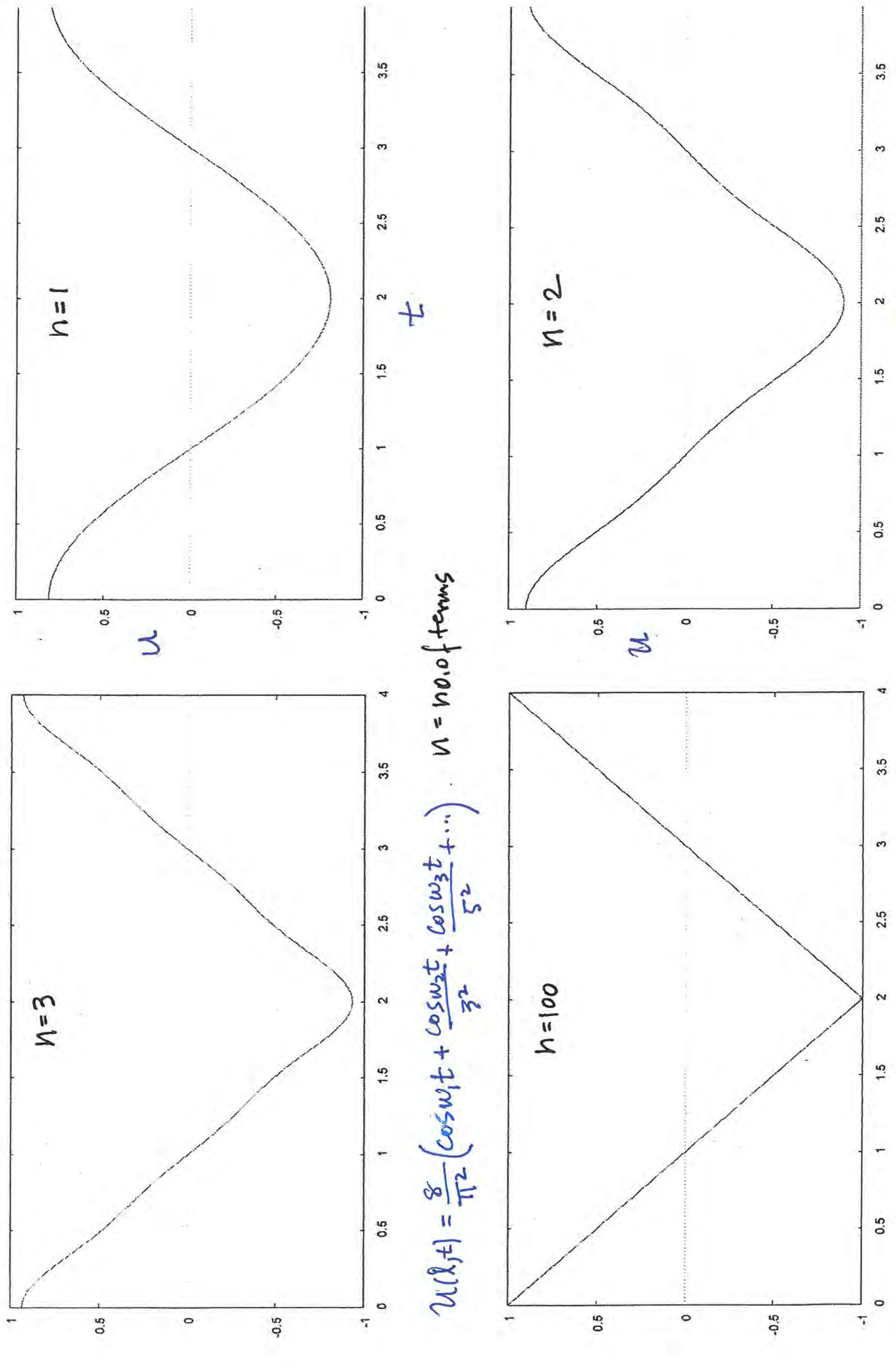
$$u(\ell,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2c^2}{\omega_n^2 \ell^2} \underbrace{\sin \frac{\omega_n \ell}{c}}_{= \sin (2n-1)\frac{\pi}{2} = (-1)^{n+1}} \cos \omega_n t$$

$$= \sum_{n=1}^{\infty} \frac{2c^2}{\omega_n^2 \ell^2} \cos \omega_n t$$

$$= \sum_{n=1}^{\infty} \frac{8}{[(2n-1)\pi]^2} \cos \omega_n t$$

$$= \frac{8}{\pi^2} \left( \cos \omega_1 t + \frac{\cos \omega_2 t}{3^2} + \frac{\cos \omega_3 t}{5^2} + \cdots \right)$$

$$= \frac{8}{\pi^2} \left( \cos \frac{\pi c t}{2\ell} + \frac{1}{9} \cos \frac{3\pi c t}{2\ell} + \frac{1}{25} \cos \frac{5\pi c t}{2\ell} + \cdots \right)$$


n=1
n=2
n=3
n=100
u
t
$u(l,t) = \frac{8}{\pi^2}\left(\cos\omega_1 t + \frac{\cos\omega_2 t}{3^2} + \frac{\cos\omega_3 t}{5^2} + \cdots\right)$
n = no. of terms

F1

## Forced Vibrations

$$u_{tt} = c^2 u_{xx} + f(x,t)$$

Harmonic forcing: $f(x,t) = F(x) \cos \Omega t$

Recall that when we investigated $n$ DOF systems, there were two approaches to solving the forced system: principal coords, and the direct approach. (See Homework No.4)

### Principal Coordinates

Recall in the $n$ DOF system, we obtained principal coordinates as follows:

$$M\ddot{x} + Kx = 0$$

Set $x = Rp$ where $R = [{}_1X, {}_2X, \ldots, {}_nX]$

where ${}_iX$ is the modal vector for $\omega_i$

and where $p = \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{bmatrix}$ are the principal coords.

Note that $x = Rp$, when written out, looks like

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \left[ {}_1X \,\vdots\, {}_2X \,\vdots\, \cdots \,\vdots\, {}_nX \right] \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{bmatrix}$$

$$= p_1({}_1X) + p_2({}_2X) + \cdots + p_n({}_nX)$$

F2

Now in the continuous case, the role played by the modal vectors ${}_i\underline{X}$ is played by $U_i(x)$

This suggests setting

$$u(x,t) = \sum_{i=1}^{\infty} p_i(t)\, U_i(x)$$

Substitute this into $u_{tt} = c^2 u_{xx} + f(x,t)$

$$\sum \ddot{p}_i U_i = c^2 \sum p_i U_i'' + f$$

where $U_i(x)$ satisfies the d.e.

$$U_i'' + \left(\frac{\omega_i}{c}\right)^2 U_i = 0 \Rightarrow U_i'' = -\frac{\omega_i^2}{c^2} U_i$$

$$\therefore \quad \sum \left(\ddot{p}_i + \omega_i^2 p_i\right) U_i = f$$

Now multiply by $U_j$ and $\int_0^l$ and use orthog.

$$\int_0^l U_i U_j\, dx = 0, \; i \neq j$$

Eq. (1) $$\left(\ddot{p}_j + \omega_j^2 p_j\right) \int_0^l U_j^2\, dx = \int_0^l f(x,t)\, U_j(x)\, dx$$

Now this may be written in a convenient way by expanding the forcing function in a series of the principal modes $U_i(x)$.

$$f(x,t) = \sum_{i=1}^{\infty} f_i(t)\, U_i(x)$$

$$f(x,t) = \sum_{i=1}^{\infty} f_i(t)\, U_i(x)$$

Mult. by $U_j(x)$ & $\int_0^l$, use orthogonality:

$$\int_0^l U_j(x) f(x,t)\,dx = \sum f_i(t) \int_0^l U_j U_i\,dx$$

$$= f_j(t) \int_0^l U_j^2\,dx$$

Substituting this into eq. (1), we get

$$\ddot{p}_j + \omega_j^2 p_j = f_j(t) \qquad j = 1, 2, 3, \ldots$$

For example, in the case of harmonic forcing,

$$f(x,t) = F(x) \cos \Omega t$$

we find

$$f_j(t) = \frac{\int_0^l U_j(x) f(x,t)\,dx}{\int_0^l U_j^2\,dx} = \frac{\int_0^l U_j F(x) \cos\Omega t\,dx}{\int_0^l U_j^2\,dx}$$

$$= \left[ \frac{\int_0^l U_j F(x)\,dx}{\int_0^l U_j^2\,dx} \right] \cos \Omega t$$

$$= F_j \cos \Omega t$$

where $F_j = \int_0^l U_j F\,dx \Big/ \int_0^l U_j^2\,dx$

Note that $F_j$ is a coefficient, not a function of $x$

F4

So then the procedure would be to solve

Eq.(2) $$\ddot{p}_j + \omega_j^2 p_j = F_j \cos \Omega t$$

whereupon the solution is

$$u(x,t) = \sum_{i=1}^{\infty} p_i(t)\, U_i(x)$$

Now we may solve eq. (2) easily:

$$p_j(t) = A_j \cos \omega_j t + B_j \sin \omega_j t + \frac{F_j \cos \Omega t}{\omega_j^2 - \Omega^2}$$

But there is still the question of how to find $A_j, B_j$

Naturally these will be found from the I.C.:

$$\left.\begin{aligned} t=0, \quad u(x,0) &= G(x) \\ u_t(x,0) &= H(x) \end{aligned}\right\}$$ these functions are part of the original problem which includes both B.C. and I.C.

How do we use the given function $G(x), H(x)$ to find the constants of integration, $A_j, B_j$?

Answer: Expand $G(x), H(x)$ in series of $U_i(x)$:

$$G(x) = \sum_{i=1}^{\infty} G_i\, U_i(x)$$

Multiply by $U_j(x)$, $\int_0^l$, use orthogonality $\Rightarrow$

$$G_j = \int_0^l G(x)\, U_j(x)\, dx \Big/ \int_0^l U_j^2\, dx$$

F5

and similarly for $H(x) = \sum_{i=1}^{\infty} H_i U_i(x)$

Then we obtain $\quad p_j(0) = G_j = A_j + \dfrac{F_j}{\omega_j^2 - \Omega^2}$

$$\Rightarrow \quad A_j = G_j - \frac{F_j}{\omega_j^2 - \Omega^2}$$

and $\quad \dot{p}_j(0) = H_j = B_j \omega_j$

$$\Rightarrow \quad B_j = \frac{H_j}{\omega_j}$$

In short, everything is expanded in a series of modal functions (eigenfunctions):

1) the solution, $\quad u(x,t) = \sum p_i(t) U_i(x)$

2) the forcing fn, $\quad f(x,t) = \sum f_i(t) U_i(x)$

3) the I.C. $\quad u(x,0) = G(x) = \sum G_i U_i(x)$

$\quad u_t(x,0) = H(x) = \sum H_i U_i(x)$

then $p_i(t)$ is solved for, in terms of $f_i(t), G_i, H_i$.

And the final answer is a series,

$$u(x,t) = \sum p_i(t) U_i(x)$$

F6

Now if there was very light damping, we could ignore the complementary solution, and we would get at steady state:

$$p_j(t) = \frac{F_j \cos \Omega t}{\omega_j^2 - \Omega^2}$$

$$u(x,t)_{\text{steadystate}} = \left( \sum_{j=1}^{\infty} \frac{F_j U_j(x)}{\omega_j^2 - \Omega^2} \right) \cos \Omega t$$

Here we have assumed the forcing is of the form

$$f(x,t) = F(x) \cos \Omega t$$

and then $F_j = \dfrac{\int_0^\ell F(x) U_j(x)\,dx}{\int_0^\ell U_j(x)^2\,dx}$

Direct Approach

$$u_{tt} = c^2 u_{xx} + F(x) \cos \Omega t$$

We look for a particular solution in the form:

$$u(x,t) = S(x) \cos \Omega t$$

We get

$$-\Omega^2 S = c^2 S'' + F$$

We must obtain a particular soln to

$$S'' + \left(\frac{\Omega}{c}\right)^2 S = \frac{F(x)}{c^2}$$

Although this looks simpler than the principal modes method, it will be difficult to solve this DE for a general function $F(x)$. By contrast, solving for steady state soln is easy in the principal modes approach, because everything has been expanded in series.

## Example

$$u_{tt} = u_{xx} + x \cos t$$

$$u = 0 \text{ at } x = 0, 1$$

$$t = 0,\ u = x(1-x),\ u_t = 0$$

First find the frequencies and modes

$$u_{tt} = u_{xx}$$

Ansatz: $u(x,t) = U(x) \cos \omega t$

$$-\omega^2 U = U'', \quad U(0) = U(1) = 0$$

$$U = c_1 \sin \omega x + c_2 \cos \omega x$$

$$U(0) = 0 = c_2$$

$$U(1) = 0 = c_1 \sin \omega \Rightarrow \omega_n = n\pi, \quad n = 1, 2, 3, \ldots$$

$$U_n(x) = \sin n\pi x$$

Now set $u(x,t) = \sum_{n=1}^{\infty} U_n(x)\, p_n(t)$

$$f(x,t) = x \cos t = \sum_{n=1}^{\infty} U_n(x)\, f_n(t)$$

I.C. $u(x,0) = G(x) = x(1-x) = \sum_{n=1}^{\infty} U_n(x)\, G_n$

Compute $f_n(t)$ and $G_n$:

$$\int_0^1 U_m(x)\, x \cos t\, dx = \int_0^1 U_m^2\, dx\ f_m(t)$$

$$f_m(t) = \frac{\int_0^1 x \sin m\pi x\, dx}{\int_0^1 \sin^2 m\pi x\, dx} \cos t$$

$$f_m(t) = \frac{\int_0^1 x \sin m\pi x \, dx}{\int_0^1 \sin^2 m\pi x \, dx} \cos t$$

$$\int_0^1 x \sin m\pi x \, dx = \int_0^1 x \frac{d(\cos m\pi x)}{-m\pi}$$

$$= \frac{x \cos m\pi x}{-m\pi}\Big|_0^1 + \frac{1}{m\pi}\int_0^1 \cos m\pi x \, dx$$

$$= \frac{\cos m\pi}{-m\pi} + \frac{1}{m\pi}\underbrace{\left(\frac{\sin m\pi x}{m\pi}\Big|_0^1\right)}_{=0}$$

$$= \frac{(-1)^{m+1}}{m\pi}$$

$$\int_0^1 \sin^2 m\pi x \, dx = \int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos 2m\pi x\right) dx = \frac{1}{2}$$

$$\therefore f_m(t) = \left[\frac{(-1)^{m+1}}{m\pi} \div \frac{1}{2}\right]\cos t = \frac{2}{m\pi}(-1)^{m+1} \cos t$$

Next compute $G_n$:

$$x(1-x) = \sum_{n=1}^{\infty} U_n(x)\, G_n$$

$$\int_0^1 U_m(x)\, x(1-x)\, dx = \left(\int_0^1 U_m^2 \, dx\right) G_m = \frac{1}{2} G_m$$

$$\parallel$$

$$\int_0^1 (\sin m\pi x)\, x(1-x)\, dx$$

$$\parallel$$

$$\frac{2}{\pi^3 m^3}(1 - \cos \pi m) = \begin{cases} 0, & m = 2,4,6,\ldots \\ \dfrac{4}{\pi^3 m^3}, & m = 1,3,5,\ldots \end{cases}$$

from Wolfram Alpha

F9

So $G_m = \begin{cases} \dfrac{8}{\pi^3 m^3}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}$

Now solve for $p_n(t)$:

$$\ddot{p}_n + w_n^2 p_n = f_n(t) = \underbrace{\frac{2}{n\pi}(-1)^{n+1}}_{F_n}\cos t$$

$$p_n(0) = G_n$$

$$\dot{p}_n(0) = H_n = 0$$

$$p_n(t) = A_n \cos w_n t + \underset{\overset{\shortparallel}{0}}{B_n} \sin w_n t + \frac{F_n \cos t}{w_n^2 - 1}$$

where $w_n = n\pi$

at $t=0$, $G_n = A_n + \dfrac{F_n}{w_n^2-1} \Rightarrow A_n = G_n - \dfrac{F_n}{w_n^2-1}$

$$p_n(t) = G_n \cos w_n t + \frac{F_n}{w_n^2-1}(\cos t - \cos w_n t)$$

where $G_n$, $F_n$, $w_n$ are given above.

Finally,

$$u(x,t) = \sum_{n=1}^{\infty} p_n(t)\, \underset{\overset{\shortparallel}{\sin n\pi x}}{U_n(x)}$$

F10

Write out 1st two terms:

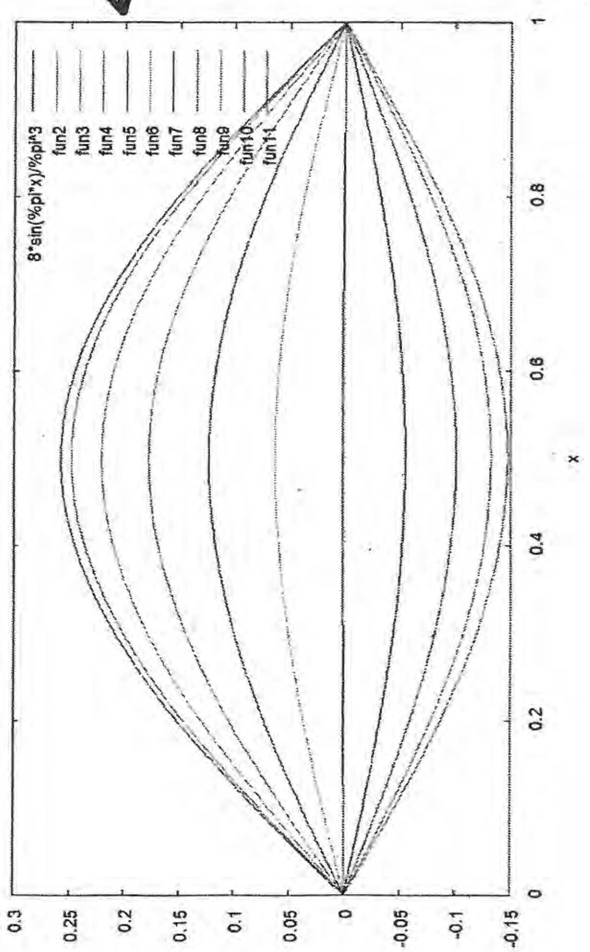
$$u(x,t) = \sin \pi x \left( \frac{8}{\pi^3} \cos \pi t + \frac{2}{\pi} \left( \frac{1}{\pi^2 - 1} \right) (\cos t - \cos \pi t) \right)$$

$$+ \sin 2\pi x \left( -\frac{1}{\pi} \left( \frac{1}{4\pi^2 - 1} \right) (\cos t - \cos 2\pi t) \right)$$

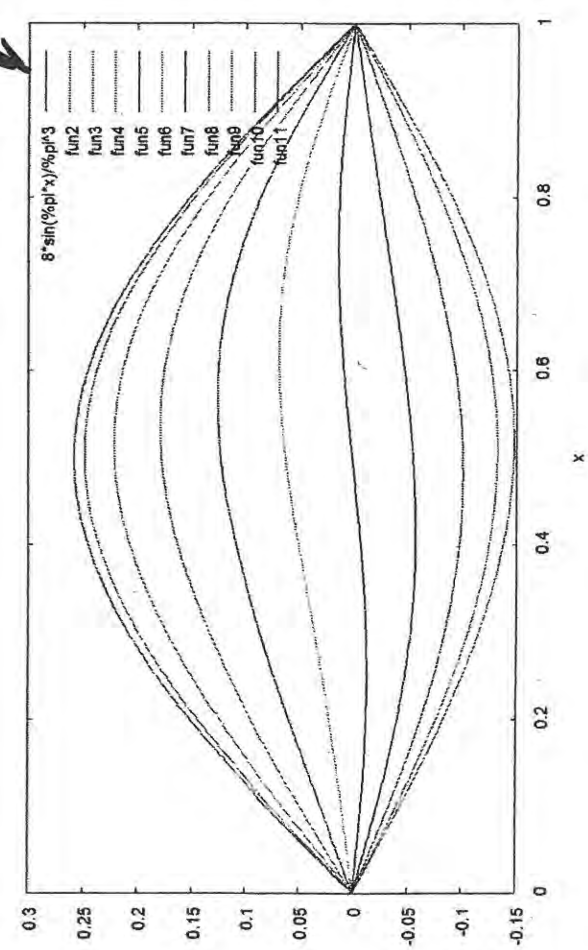
$$+ \cdots$$

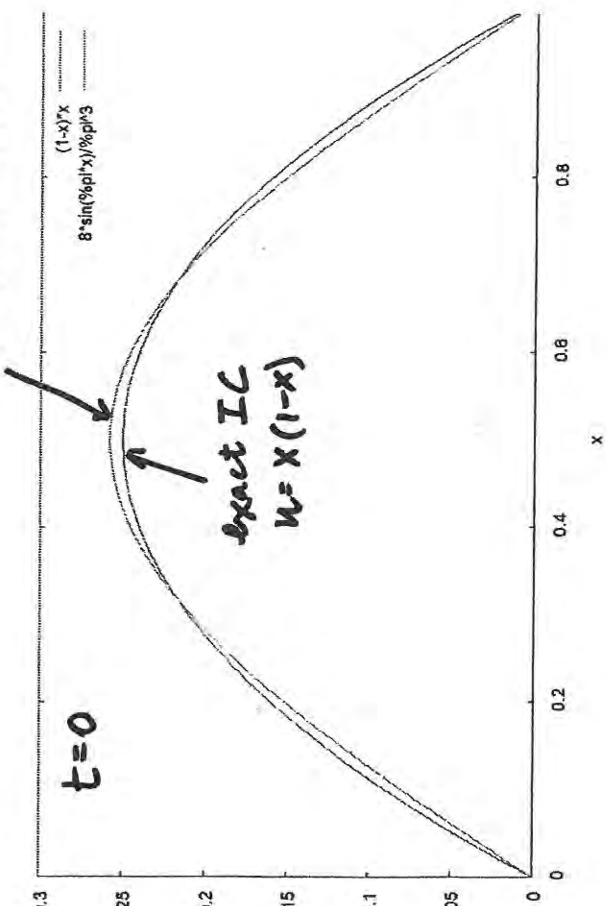
F11

1st term only ($\sin \pi x$ term)
$t = \{0, .1, .2, \ldots, .9, 1.0\}$

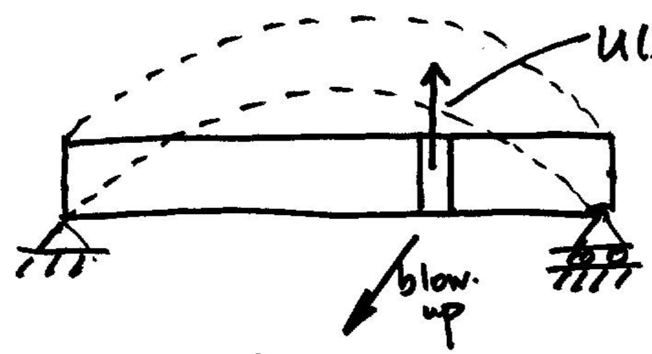



1st two terms
$t = \{0, .1, .2, \ldots, .9, 1.0\}$

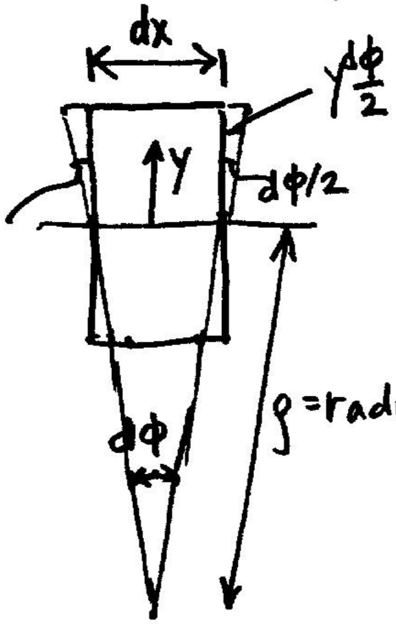


# Theory of Beams (= transverse motion of a rod)



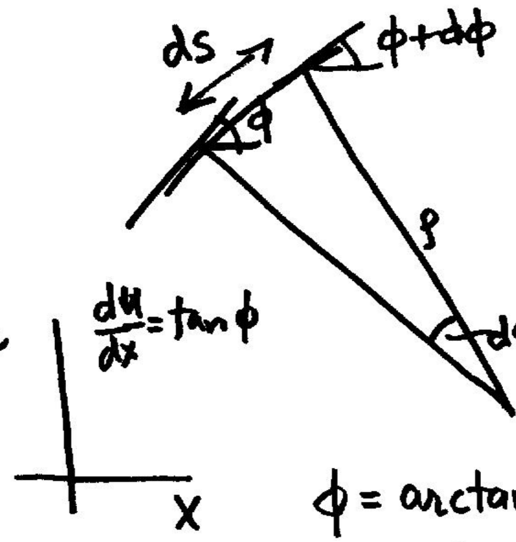

plane sections remain plane




$$\varepsilon = \frac{\Delta L}{L} = \frac{y\,d\phi}{dx} = \frac{y}{\rho}$$

$\rho$ = radius of curvature

$$dx = \rho\, d\phi$$




$$\frac{du}{dx} = \tan\phi$$

$$ds = \sqrt{dx^2 + du^2}$$

$$\rho\, d\phi = ds, \quad \frac{1}{\rho} = \frac{d\phi}{ds} = \frac{u_{xx}}{1+(u_x)^2}\frac{dx}{ds}$$

$$\frac{1}{\rho} = \frac{u_{xx}}{[1+u_x^2]^{3/2}} \approx u_{xx}$$

assume small slope $u_x$

$$\phi = \arctan u_x$$

$$d\phi = \frac{du_x}{1+(u_x)^2} = \frac{u_{xx}\, dx}{1+(u_x)^2}$$

B2

$$\therefore \ \varepsilon = \frac{y}{\rho} \approx y\, u_{xx}$$

$$V = \int_0^{\ell} V^* \, dx$$

$$V^* = \iint_A \frac{1}{2} \sigma \varepsilon \, dA \ , \quad \sigma = E\varepsilon \ \text{Hooke's Law}$$

$$= \iint_A \frac{1}{2} E \varepsilon^2 dA = \iint_A \frac{1}{2} E y^2 u_{xx}^2 \, dA$$

$$= \frac{1}{2} E u_{xx}^2 \underbrace{\iint_A y^2 dA}_{I \,=\, \text{moment of inertia}}$$

$$\left.\begin{aligned} V^* &= \frac{1}{2} E I u_{xx}^2 \\ T^* &= \frac{1}{2} \rho u_t^2 \end{aligned}\right\} \ \mathcal{L}^* = T^* - V^*$$

Lagrange's eq

$$-\frac{d^2}{dx^2} \frac{\partial \mathcal{L}^*}{\partial u_{xx}} + \frac{d}{dx} \frac{\partial \mathcal{L}^*}{\partial u_x} + \frac{d}{dt} \frac{\partial \mathcal{L}^*}{\partial u_t} - \frac{\partial \mathcal{L}^*}{\partial u} = 0$$

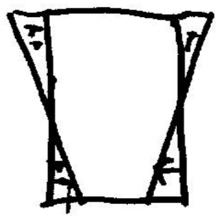
Note this term, usually absent, but included here because $\mathcal{L}^*$ contains $2^{nd}$ derivatives!

LE $\Rightarrow$ $$\frac{d^2}{dx^2} EI u_{xx} + \frac{d}{dt} \rho u_t = 0$$

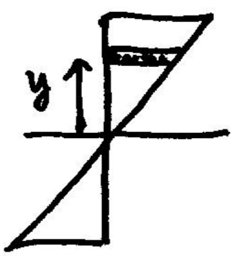
$$\boxed{EI u_{xxxx} + \rho u_{tt} = 0}$$

The beam equation

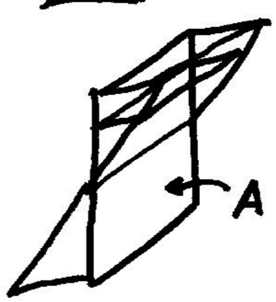
<u>Related facts</u> about beams we will need:



$$\varepsilon = \frac{y}{\rho}, \quad \sigma = E\varepsilon = E\frac{y}{\rho}$$




$$M = \int dM = \int \sigma y \, dA$$

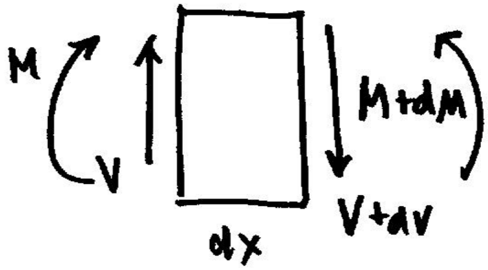



$$M = \int E \frac{y}{\rho} y \, dA$$

$$= \frac{E}{\rho} \int y^2 dA$$

$$= \frac{EI}{\rho}$$

$$\boxed{M \cong EI u_{xx}}$$




$\Sigma M = 0 \Rightarrow$

$V \, dx = dM$

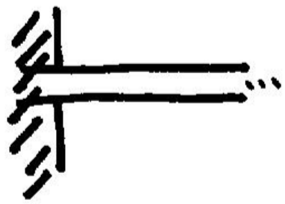
$$\boxed{V = dM/dx = EI u_{xxx}}$$

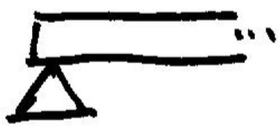
B4

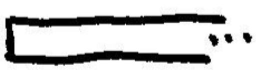
# Forced Vibration of Beams

$$u_{tt} + \frac{EI}{\rho} u_{xxxx} = f(x,t)$$

## Boundary Conditions:

fixed  $u=0,\ u_x=0$

pinned  $u=0,\ u_{xx}=0$
(from $M=0$)

free  $u_{xx}=0,\ u_{xxx}=0$
(from $V=0,\ M=0$)

## Example

$x=0$, $u=0$
$u_x=0$

$x=\ell$, $u_{xx}=0$
$u_{xxx}=0$

B5

Begin with free vibrations, $f(x,t)=0$:

Set $u(x,t) = U(x)\cos\omega t$

$$-\omega^2 U + \frac{EI}{\rho} U^{IV} = 0$$

$$\frac{d^4U}{dx^4} - \beta^4 U = 0 \ , \quad \beta^4 = \frac{\rho\omega^2}{EI}$$

Set $U = e^{\lambda x}$

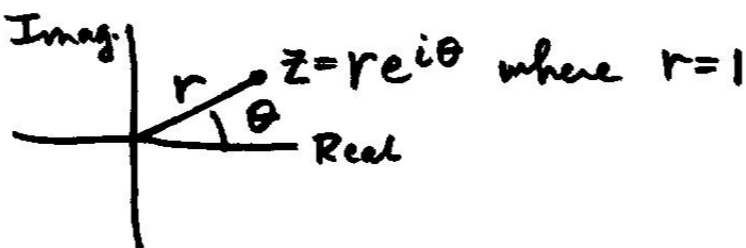
$$\lambda^4 - \beta^4 = 0$$

$$\lambda = (\beta^4)^{1/4} = \beta, -\beta, i\beta, -i\beta$$

(from deMoivre's formula,

that is $1$ may be thought of as $e^{i\theta}$

where $\theta = 0, 2\pi, 4\pi, 6\pi$, etc.




so $(e^{i\theta})^{1/4} = e^{i\frac{\theta}{4}}$

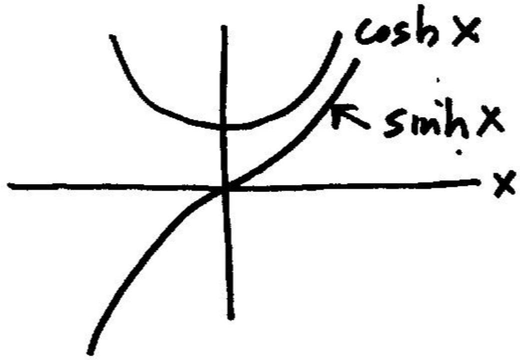
$$\therefore \ 1^{1/4} = e^{i0}, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}, \ldots$$

$$= 1, i, -1, -i \quad )$$

B6

Recall $\dfrac{e^x+e^{-x}}{2} = \cosh x$, $(\cosh x)' = \sinh x$

$$\frac{e^x-e^{-x}}{2} = \sinh x, \quad (\sinh x)' = \cosh x$$




Note:
$\cosh^2 x - \sinh^2 x = 1$

$$\therefore \quad U(x) = c_1 \sin\beta x + c_2\cos\beta x + c_3 \sinh\beta x + c_4\cosh\beta x$$

<u>B.C.</u> $x=0,\ u=0,\ u_x=0$

$$\Rightarrow \quad c_2 + c_4 = 0 \quad \text{and} \quad c_1 + c_3 = 0$$

$x=\ell,\ u_{xx}=0,\ u_{xxx}=0 \Rightarrow$

$$-c_1\sin\beta\ell - c_2\cos\beta\ell + c_3\sinh\beta\ell + c_4\cosh\beta\ell = 0$$

and

$$-c_1\cos\beta\ell + c_2\sin\beta\ell + c_3\cosh\beta\ell + c_4\sinh\beta\ell = 0$$

Use $c_1 = -c_3$ and $c_2 = -c_4$ to get

$$c_3[\sin\beta\ell + \sinh\beta\ell] + c_4[\cos\beta\ell + \cosh\beta\ell] = 0$$

and

$$c_3[\cos\beta\ell + \cosh\beta\ell] + c_4[-\sin\beta\ell + \sinh\beta\ell] = 0$$

For a nontrivial soln for $c_3, c_4$, the determinant $=0$

$$\begin{vmatrix} \sin\beta\ell + \sinh\beta\ell & \cos\beta\ell + \cosh\beta\ell \\ \cos\beta\ell + \cosh\beta\ell & -\sin\beta\ell + \sinh\beta\ell \end{vmatrix} = 0$$

$$(\sin\beta l+\sinh\beta l)(-\sin\beta l+\sinh\beta l)$$
$$-(\cos\beta l+\cosh\beta l)^2=0$$

$$\sinh^2\beta l-\sin^2\beta l-\cos^2\beta l-\cosh^2\beta l-2\cos\beta l\cosh\beta l=0$$

Note: $\left.\begin{array}{l}\cosh^2\beta l-\sinh^2\beta l=1\\ \cos^2\beta l+\sin^2\beta l=1\end{array}\right\}$ gives $\boxed{\cos\beta l\cosh\beta l=-1}$

From "Frequencies of Beams Table" find

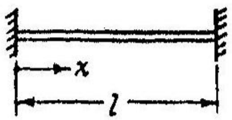
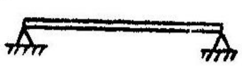
$$\beta l = \underset{=\lambda_1}{1.8751},\ \underset{=\lambda_2}{4.6941},\ \underset{=\lambda_3}{7.8548},\ \underset{=\lambda_4}{10.9955},\ldots$$

then $$U_n(x)=J\left(\frac{\lambda_n x}{l}\right)-\frac{G(\lambda_n)}{F(\lambda_n)}H\left(\frac{\lambda_n x}{l}\right)$$

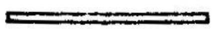
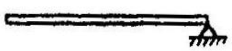
where
$$J(u)=\cosh u-\cos u$$
$$H(u)=\sinh u-\sin u$$
$$F(u)=\sinh u+\sin u$$
$$G(u)=\cosh u+\cos u$$

From "Handbook of Eng'g Mechanics", W. Flugge (editor), McGraw-Hill, 1962

**Table 61.1. Frequencies and Eigenfunctions for Uniform Beams**

| Type | Boundary conditions | Frequency equation | Eigenfunction $\phi_n(x)$ | Roots of frequency equation $\lambda_n$ |
|---|---|---|---|---|
| Clamped-clamped  | $\phi(0) = \phi'(0) = 0$<br>$\phi(l) = \phi'(l) = 0$ | $\cos\lambda \operatorname{Cosh}\lambda = 1$ | $J\left(\frac{\lambda_n x}{l}\right) - \frac{J(\lambda_n)}{H(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$ | $\lambda_1 = 4.7300$<br>$\lambda_2 = 7.8532$<br>$\lambda_3 = 10.9956$<br>$\lambda_4 = 14.1372$<br>For $n$ large,<br>$\lambda_n \approx (2n+1)\pi/2$ |
| Clamped-hinged  | $\phi(0) = \phi'(0) = 0$<br>$\phi(l) = \phi''(l) = 0$ | $\tan\lambda = \operatorname{Tanh}\lambda$ | $J\left(\frac{\lambda_n x}{l}\right) - \frac{J(\lambda_n)}{H(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$ | $\lambda_1 = 3.9266$<br>$\lambda_2 = 7.0686$<br>$\lambda_3 = 10.2102$<br>$\lambda_4 = 13.3518$<br>For $n$ large,<br>$\lambda_n \approx (4n+1)\pi/4$ |
| Clamped-free  | $\phi(0) = \phi'(0) = 0$<br>$\phi''(l) = \phi'''(l) = 0$ | $\cos\lambda \operatorname{Cosh}\lambda = -1$ | $J\left(\frac{\lambda_n x}{l}\right) - \frac{G(\lambda_n)}{F(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$ | $\lambda_1 = 1.8751$<br>$\lambda_2 = 4.6941$<br>$\lambda_3 = 7.8548$<br>$\lambda_4 = 10.9955$<br>For $n$ large,<br>$\lambda_n \approx (2n-1)\pi/2$ |
| Clamped-guided  | $\phi(0) = \phi'(0) = 0$<br>$\phi'(l) = \phi'''(l) = 0$ | $\tan\lambda = -\operatorname{Tanh}\lambda$ | $J\left(\frac{\lambda_n x}{l}\right) - \frac{F(\lambda_n)}{J(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$ | $\lambda_1 = 2.3650$<br>$\lambda_2 = 5.4978$<br>$\lambda_3 = 8.6394$<br>$\lambda_4 = 11.7810$<br>For $n$ large,<br>$\lambda_n \approx (4n-1)\pi/4$ |
| Hinged-hinged  | $\phi(0) = \phi''(0) = 0$<br>$\phi(l) = \phi''(l) = 0$ | $\sin\lambda = 0$ | $\sin\frac{n\pi x}{l}$ | $\lambda_n = n\pi$ |

## Table 61.1. Frequencies and Eigenfunctions for Uniform Beams (*Continued*)

| Type | Boundary conditions | Frequency equation | Eigenfunction $\phi_n(x)$ | Roots of frequency equation $\lambda_n$ |
|---|---|---|---|---|
| Hinged-guided  | $\phi(0) = \phi''(0) = 0$<br>$\phi'(l) = \phi'''(l) = 0$ | $\cos\lambda = 0$ | $\sin\dfrac{(2n-1)\pi x}{2l}$ | $\lambda_n = (2n-1)\pi/2$ |
| Guided-guided  | $\phi'(0) = \phi'''(0) = 0$<br>$\phi'(l) = \phi'''(l) = 0$ | $\sin\lambda = 0$ | $\cos\dfrac{n\pi x}{l}$ | $\lambda_n = n\pi$ |
| Free-free  | $\phi''(0) = \phi'''(0) = 0$<br>$\phi''(l) = \phi'''(l) = 0$ | $\cos\lambda\,\mathrm{Cosh}\,\lambda = 1$ | $G\left(\dfrac{\lambda_n x}{l}\right) - \dfrac{J(\lambda_n)}{H(\lambda_n)} F\left(\dfrac{\lambda_n x}{l}\right)$ | Same as for clamped-clamped beam |
| Free-hinged  | $\phi''(0) = \phi'''(0) = 0$<br>$\phi(l) = \phi''(l) = 0$ | $\tan\lambda = \mathrm{Tanh}\,\lambda$ | $G\left(\dfrac{\lambda_n x}{l}\right) - \dfrac{G(\lambda_n)}{F(\lambda_n)} F\left(\dfrac{\lambda_n x}{l}\right)$ | Same as for clamped-hinged beam |
| Free-guided  | $\phi''(0) = \phi'''(0) = 0$<br>$\phi'(l) = \phi'''(l) = 0$ | $\tan\lambda = -\mathrm{Tanh}\,\lambda$ | $G\left(\dfrac{\lambda_n x}{l}\right) - \dfrac{H(\lambda_n)}{F(\lambda_n)} F\left(\dfrac{\lambda_n x}{l}\right)$ | Same as for clamped-guided beam |

1. The circular frequency is

$$\omega_n = \frac{\lambda_n^2}{l^2}\sqrt{\frac{EI}{\mu}}$$

where

$EI$ = bending stiffness
$\mu$ = mass per unit length
$l$ = length of the beam

2. Notation used in expressions for the eigenfunctions:

$$F(u) = \mathrm{Sinh}\,u + \sin u$$
$$G(u) = \mathrm{Cosh}\,u + \cos u$$
$$H(u) = \mathrm{Sinh}\,u - \sin u$$
$$J(u) = \mathrm{Cosh}\,u - \cos u$$

principal modes

Set $u(x,t) = \sum_{n=1}^{\infty} p_n(t)\, U_n(x)$

in

$$u_{tt} + \frac{EI}{\rho} u_{xxxx} = f(x,t)$$

$$\sum \ddot{p}_n U_n + \frac{EI}{\rho} U_n^{IV} p_n = f(x,t)$$

But $\frac{EI}{\rho} U_n^{IV} = \omega_n^2 U_n$

$$\therefore \sum \left(\ddot{p}_n + \omega_n^2 p_n\right) U_n = f(x,t)$$

Now suppose we could expand

$$f(x,t) = \sum f_n(t) U_n(x)$$

And the I.C.

$$t=0, \quad u(x,0) = G(x) = \sum G_n U_n(x)$$

$$u_t(x,0) = H(x) = \sum H_n U_n(x)$$

use orthogonality of $\{U_n\}$

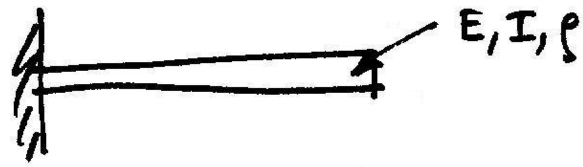
then we would have

$$\left.\begin{aligned} \ddot{p}_n + \omega_n^2 p_n &= f_n(t) \\ p_n(0) &= G_n \\ \dot{p}_n(0) &= H_n \end{aligned}\right\} \text{solved as usual}$$

# Rayleigh's Method

Being a scheme for obtaining an approximation for the lowest frequency of a vibrating system

**Example**




$u = u_x = 0$    $x = 0$

$u_{xx} = u_{xxx} = 0$ (Moment & shear $=0$)    $x = \ell$

The exact solution is obtained by solving the transcendental eq. [from beam freq. table]:

$$\cos\lambda \cosh\lambda = -1$$

$$\Rightarrow \lambda = 1.8751,\ 4.6941,\ 7.8548, \ldots$$

where $\omega = \dfrac{\lambda^2}{\ell^2}\sqrt{\dfrac{EI}{\rho}} \Rightarrow \omega_1 = 3.516\sqrt{\dfrac{EI}{\rho \ell^4}}$

In order to obtain this value, we would need to deal with the exact general soln to the modal equation, $\dfrac{EI}{\rho} U^{IV} = \omega^2 U$, namely

$$U(x) = c_1 \sin\beta x + c_2 \cos\beta x + c_3 \sinh\beta x + c_4 \cosh\beta x$$

where $\beta^4 = \rho\omega^2/EI$, then fit the $c_i$ using B.C. and finally solve $\cos\lambda\cosh\lambda = -1$ numerically.

R2

Rayleigh's method offers a bold alternative (although approximate) to the exact solution. The method involves choosing <u>any</u> function $V(x)$ which satisfies the 4 B.C.

[By contrast, the modal function $U(x)$ also satisfies the B.C., but also satisfies the pde, whereas <u>V(x) does not satisfy the pde</u>!]

Then the method involves computing "Rayleigh's Quotient" Q:

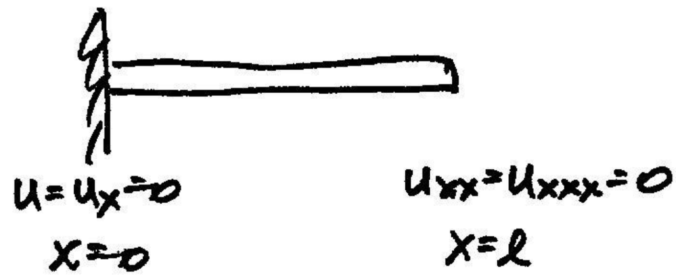
$$Q = \frac{EI}{\rho} \frac{\int_0^\ell \left(V(x)''\right)^2 dx}{\int_0^\ell \left(V(x)\right)^2 dx}$$

Then Q will be approximately equal to $\omega_1^2$. In fact we are guaranteed that

$$\boxed{\omega_1^2 \le Q}$$

R3

Example




For simplicity we choose

$$V(x) = a x^2 + b x^3 + c x^4$$

which satisfies $u = u_x = 0$ at $x = 0$ (i.e. $V = V' = 0$ at $x = 0$)

and we pick $a, b, c$ so that $V'' = V''' = 0$ at $x = \ell$

$$V'' = 2a + 6bx + 12cx^2$$

$$V''' = 6b + 24cx$$

We obtain
$$2a + 6b\ell + 12c\ell^2 = 0$$
$$6b + 24c\ell = 0$$

Take $c = 1$ without loss of generality $\Rightarrow$

$$b = -4\ell$$

$$a = -3b\ell - 6c\ell^2 = 6\ell^2$$

So we have
$$V(x) = 6\ell^2 x^2 - 4\ell x^3 + x^4$$

$$V''(x) = 12\ell^2 - 24\ell x + 12x^2$$

R4

so we have
$$V(x) = 6\ell^2x^2 - 4\ell x^3 + x^4$$
$$V''(x) = 12\ell^2 - 24\ell x + 12x^2$$

Wolfram $\alpha \Rightarrow$ $\displaystyle\int_0^\ell (V'')^2 dx = \frac{144}{5}\ell^5, \quad \int_0^\ell V^2 dx = \frac{104}{45}\ell^9$

giving $\displaystyle Q = \frac{EI}{\rho \ell^4}\frac{144}{5} \div \frac{104}{45} = \frac{162}{13}\frac{EI}{\rho\ell^4}$

$$Q = \frac{162}{13}\frac{EI}{\rho\ell^4} \Rightarrow \sqrt{Q} = 3.530\sqrt{\frac{EI}{\rho\ell^4}}$$

versus exact $\omega_1 = 3.516\sqrt{\dfrac{EI}{\rho\ell^4}}$

FANTASTIC!!

Why does it work?

R5

In order to understand why Rayleigh's method works, we set

$$V(x) = \underset{\substack{\uparrow \\ \text{exact} \\ \text{lowest} \\ \text{mode}}}{U_1(x)} + \underset{\substack{\uparrow \\ \text{the error}}}{\varepsilon\eta(x)}$$

Now Rayleigh's quotient may be thought of as a fn of $\varepsilon$:

$$Q(\varepsilon) = \frac{EI}{\rho} \frac{\int_0^l (U_1'' + \varepsilon\eta'')^2 \, dx}{\int_0^l (U_1 + \varepsilon\eta)^2 \, dx}$$

Note that when $\varepsilon = 0$, $Q = \omega_1^2$. Why?

$$\int_0^l (U_1'')^2 dx = \int_0^l U_1'' \, dU_1' = \underset{\overset{\shortparallel}{0}}{U_1''U_1'}\Big|_0^l - \int_0^l U_1' U_1''' dx$$

$$= -\int_0^l U_1''' \, dU_1 = \underset{\overset{\shortparallel}{0}}{-U_1''' U_1}\Big|_0^l + \int_0^l U_1^{IV} U_1 \, dx$$

So

$$Q(0) = \frac{EI}{\rho} \frac{\int_0^l U_1^{IV} U_1 \, dx}{\int_0^l U_1^2 \, dx}$$

R6

But $U_1(x)$ satisfies the ODE:

$$\frac{EI}{\rho} U_1^{IV} = \omega_1^2 U_1$$

So we get

$$Q(0) = \frac{EI}{\rho} \frac{\int_0^l U_1^{IV} U_1 \, dx}{\int_0^l U_1^2 \, dx} = \frac{\int_0^l \omega_1^2 U_1^2 \, dx}{\int_0^l U_1^2 \, dx} = \omega_1^2$$

Now suppose we are close to $U_1(x)$ when we guess $V(x)$, i.e. suppose $\varepsilon$ is small. Let's expand $Q(\varepsilon)$ for small $\varepsilon$:

$$Q(\varepsilon) = \underbrace{Q(0)}_{= \omega_1^2} + \left.\frac{dQ}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + \frac{1}{2} \left.\frac{d^2Q}{d\varepsilon^2}\right|_{\varepsilon=0} \varepsilon^2 + \cdots$$

Compute $\left.\frac{dQ}{d\varepsilon}\right|_{\varepsilon=0}$ as follows: Start with

$$Q(\varepsilon) \int_0^l (U_1 + \varepsilon\eta)^2 \, dx = \frac{EI}{\rho} \int_0^l (U_1'' + \varepsilon\eta'')^2 \, dx$$

Eq. (*)
$$\frac{dQ}{d\varepsilon} \int_0^l (U_1 + \varepsilon\eta)^2 \, dx + Q \int_0^l 2(U_1 + \varepsilon\eta)\eta \, dx$$
$$= \frac{EI}{\rho} \int_0^l 2(U_1'' + \varepsilon\eta'')\eta'' \, dx$$

Now set $\varepsilon = 0$ to get

$$\left.\frac{dQ}{d\varepsilon}\right|_{\varepsilon=0} = ?$$

R7

Setting $\varepsilon = 0$ in Eq.(*) we get

Eq.(**) $$\left.\frac{dQ}{d\varepsilon}\right|_{\varepsilon=0} \int_0^l U_1^2\,dx + \underset{\omega_1^2}{\overset{\shortparallel}{Q(0)}} \int_0^l 2 U_1 \eta\,dx$$

$$= \frac{EI}{\rho}\int_0^l 2 U_1'' \eta''\,dx$$

$$\int_0^l U_1''\eta''\,dx = \int_0^l U_1''\,d\eta' = \underset{0}{\overset{\shortparallel}{U_1''\eta'\Big|_0^l}} - \int_0^l U_1''' \underbrace{\eta'\,dx}_{d\eta}$$

either from B.C. on $U_1$ or require $\eta'(0) = \eta'(l) = 0$

$$\int_0^l U_1''\eta''\,dx = -\underset{0}{\overset{\shortparallel}{U_1'''\eta\Big|_0^l}} + \int_0^l U_1^{IV}\eta\,dx$$

same idea as above

Eq.(**) becomes

$$\left.\frac{dQ}{d\varepsilon}\right|_{\varepsilon=0} \int_0^l U_1^2\,dx = \frac{EI}{\rho}\int_0^l 2 U_1^{IV}\eta\,dx - \omega_1^2\int_0^l 2 U_1\eta\,dx$$

$$= 0 \quad \text{since } \frac{EI}{\rho} U_1^{IV} = \omega_1^2 U_1$$

$$\therefore \left.\frac{dQ}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

Thus we have that

$$Q(\varepsilon) = Q(0) + \underbrace{\frac{dQ}{d\varepsilon}\bigg|_{\varepsilon=0}}_{=0} \varepsilon + \frac{1}{2}\frac{d^2Q}{d\varepsilon^2}\bigg|_{\varepsilon=0} \varepsilon^2 + \cdots$$

$$= \omega_1^2 + \frac{1}{2}\frac{d^2Q}{d\varepsilon^2}\bigg|_{\varepsilon=0} \varepsilon^2 + \cdots$$

Now we compute $\frac{d^2Q}{d\varepsilon^2}\bigg|_{\varepsilon=0}$ to see if $Q(\varepsilon) \geq \omega_1^2$ or $\leq \omega_1^2$

Differentiate Eq.(*) with respect to $\varepsilon$:

$$\frac{d^2Q}{d\varepsilon^2}\int_0^\ell (U_1+\varepsilon\eta)^2\,dx + 2\frac{dQ}{d\varepsilon}\int_0^\ell 2(U_1+\varepsilon\eta)\eta\,dx$$

$$+ Q\int_0^\ell 2\eta^2\,dx = \frac{EI}{\rho}\int_0^\ell 2\,\eta''^2\,dx$$

Now set $\varepsilon=0$ and use $\frac{dQ}{d\varepsilon}\bigg|_{\varepsilon=0} = 0$ and $Q(0) = \omega_1^2$

Eq.(***) $$\frac{d^2Q}{d\varepsilon^2}\bigg|_{\varepsilon=0}\int_0^\ell U_1^2\,dx = 2\frac{EI}{\rho}\int_0^\ell \eta''^2\,dx - 2\omega_1^2\int_0^\ell \eta^2\,dx$$

Now recall that $V(x) = U_1(x) + \varepsilon\eta(x)$

Thus we may write that

$$\eta(x) = \sum_{i=2}^{\infty} a_i U_i(x) \quad (\text{omit } U_1 \text{ term})$$

R9

$$\eta(x) = \sum_{i=2}^{\infty} a_i U_i(x)$$

$$\Rightarrow \quad \eta''(x) = \sum_{i=2}^{\infty} a_i U_i''(x)$$

We need $\displaystyle\int_0^l \eta''^2 dx = \int_0^l \left(\sum_{i=2}^{\infty} a_i U_i''(x)\right)^2 dx$

$$= \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} a_i a_j \int_0^l U_i'' U_j'' dx$$

Integrate by parts: $\displaystyle\int_0^l U_i'' U_j'' dx = \int_0^l U_i'' \, dU_j'$

$$= \underbrace{U_i'' U_j' \Big|_0^l}_{\substack{= 0 \\ \text{from BC}}} - \int_0^l U_i''' \, dU_j$$

$$= \underbrace{-U_i''' U_j \Big|_0^l}_{=0} + \int_0^l U_i^{IV} U_j \, dx$$

Thus $\displaystyle\frac{EI}{\rho}\int_0^l \eta''^2 dx = \frac{EI}{\rho} \sum_2^{\infty}\sum_2^{\infty} a_i a_j \int_0^l U_i^{IV} U_j \, dx$

$$= \sum_2^{\infty}\sum_2^{\infty} a_i a_j \int_0^l \omega_i^2 U_i U_j \, dx$$

where we have used $\displaystyle\frac{EI}{\rho} U_i^{IV} = \omega_i^2 U_i$

R10

$$\frac{EI}{\rho}\int_0^l \eta''^2 dx = \sum_2 \sum_2 a_i a_j \omega_i^2 \underbrace{\int_0^l U_i U_j \, dx}_{\substack{=0 \text{ unless } i=j \\ \text{by orthogonality}}}$$

$$= \sum_{i=2}^{\infty} a_i^2 \omega_i^2 \int_0^l U_i^2 dx$$

$$> \omega_2^2 \sum_{i=2}^{\infty} a_i^2 \int_0^l U_i^2 dx$$

since $\omega_3 \geq \omega_2$, $\omega_4 \geq \omega_2$, etc.

From Eq.(***):

$$\left.\frac{d^2Q}{d\varepsilon^2}\right|_{\varepsilon=0} \int_0^l U_1^2 dx = 2\left[\frac{EI}{\rho}\int_0^l \eta''^2 dx - \omega_1^2 \int_0^l \eta^2 dx\right]$$

$$= 2\left[\sum_{i=2}^{\infty} a_i^2 \omega_i^2 \int_0^l U_i^2 dx - \omega_1^2 \sum_{i=2}^{\infty} a_i^2 \int_0^l U_i^2 dx\right]$$

$$> 2\left[(\omega_2^2 - \omega_1^2)\sum_{i=2}^{\infty} a_i^2 \int_0^l U_i^2 dx\right]$$

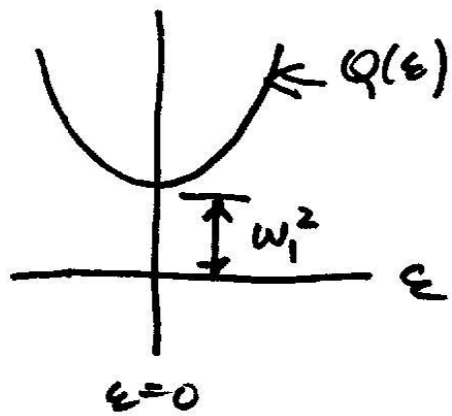
$$> 0$$

$$\Rightarrow \left.\frac{d^2Q}{d\varepsilon^2}\right|_{\varepsilon=0} > 0$$

$$Q(\varepsilon) = \omega_1^2 + \frac{1}{2}\left.\frac{d^2Q}{d\varepsilon^2}\right|_{\varepsilon=0} \varepsilon^2 + \cdots$$

R11

$$Q(\varepsilon) = \omega_1^2 + \frac{1}{2} \underbrace{\left.\frac{d^2Q}{d\varepsilon^2}\right|_{\varepsilon=0}}_{>0} \varepsilon^2 + \cdots$$




$$\therefore \quad Q(\varepsilon) \geq \omega_1^2$$

Rayleigh's method always gives a value $> \omega_1$
(upper bound on $\omega_1$)

# Additional Example of Rayleigh's Method

Example [text, p.233, #7.51]

A simply supported beam has a concentrated mass m at its midspan:


Use Rayleigh's method to approximate the lowest natural frequency.

We take Rayleigh's quotient in the form:

$$Q = \frac{EI \int_0^{\ell} (V'')^2 dx}{\int_0^{\ell} \rho(x) V^2 dx}$$

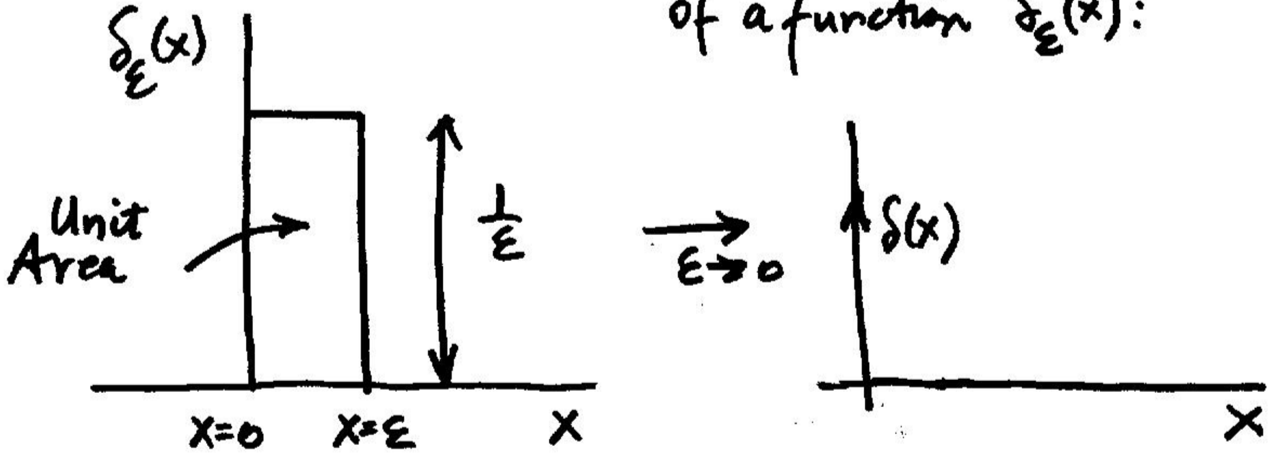
where $\rho$ = mass per unit length $= \rho_0 + m\,\delta(x - \frac{\ell}{2})$

where $\rho_0$ = mass/length of the beam w/out the mass

and $\delta(x-\frac{\ell}{2})$ = Dirac delta function at $x = \frac{\ell}{2}$

## Review of Dirac delta function $\delta(x)$

This may be thought of as the limit as $\varepsilon \to 0$ of a function $\delta_\varepsilon(x)$:




It has the property that it "evaporates" integrals:

$$\int_0^\ell \delta(x-a)\, f(x)\, dx = f(a) \qquad (0<a<\ell)$$

So, for example,

$$\int_0^\ell m\,\delta\left(x-\frac{\ell}{2}\right) V(x)^2\, dx = m\, V\left(\frac{\ell}{2}\right)^2$$

Rayleigh's quotient becomes:

$$Q = \frac{EI\int_0^\ell (V'')^2 dx}{\int_0^\ell (\rho_0 + m\,\delta(x-\frac{\ell}{2}))\,V^2 dx}$$

$$Q = \frac{EI\int_0^\ell (V'')^2 dx}{\rho_0\int_0^\ell V^2 dx + m\,V(\frac{\ell}{2})^2}$$

We choose the function $V(x) = \sin\frac{\pi x}{\ell}$ (= exact solution if $m=0$)

Note that this satisfies the B.C. $V(0)=V(\ell)=V''(0)=V''(\ell)=0$

$$V''(x) = -\frac{\pi^2}{\ell^2}\sin\frac{\pi x}{\ell}$$

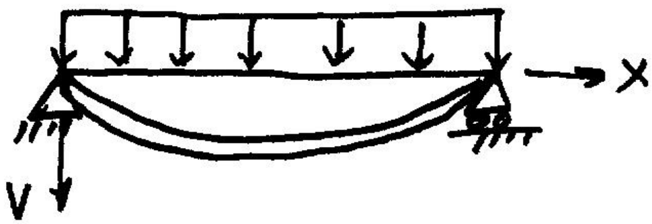
$$\int_0^\ell (V'')^2 dx = \frac{\pi^4}{\ell^4}\cdot\frac{\ell}{2} = \frac{\pi^4}{2\ell^3}$$

$$\int_0^\ell V^2 dx = \frac{\ell}{2}$$

$$\therefore\ Q = \frac{EI\pi^4}{2\ell^3} \div \left(\rho_0\frac{\ell}{2} + m\right)$$

$$\Rightarrow\quad \omega_1 \le \sqrt{Q} = \sqrt{\frac{EI\pi^4}{\rho_0\ell^4 + 2m\ell^3}} \qquad (\pi^4 = 97.41)$$

Compare with $V(x)$ = static deflection due to a uniform load




$$V(x) = x(\ell^3 - 2\ell x^2 + x^3)$$ from Table of Beam Deflections

Note that $V(x)$ satisfies BC, $V(0) = V(\ell) = V''(0) = V''(\ell) = 0$

$$V'' = 12\,x(x-\ell)$$

$$\int_0^\ell (V'')^2 dx = \frac{24}{5}\ell^5 \quad (\text{Wolfram } \alpha)$$

$$\int_0^\ell V^2 dx = \frac{31}{630}\ell^9 \;, \quad V\left(\frac{\ell}{2}\right) = \frac{5}{16}\ell^4$$

$$Q = \frac{EI\,\frac{24}{5}\ell^5}{\frac{31}{630}\rho_0 \ell^9 + m\left(\frac{5}{16}\ell^4\right)^2} = \frac{97.54\,EI}{\rho_0\ell^4 + 1.98\,m\ell^3}$$

Very close to previous choice for $V(x)$ $\left(= \sin\frac{\pi x}{\ell}\right)$

# The Fredholm Alternative

Consider the following system of two linear algebraic equations in two unknowns $x$,$y$:

$$\begin{bmatrix} 1 & 2 \\ 3 & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \quad \text{that is} \quad \begin{array}{c} x + 2y = 1 \\ 3x + \alpha y = \beta \end{array} \tag{1}$$

Depending on the values of $\alpha$ and $\beta$, there are three cases:
Case A: There is exactly one solution for $x, y$. This case corresponds to $\alpha$ taking on any value except 6. When $\alpha$=6 the determinant vanishes.
Case B: There is no solution for $x, y$. This case corresponds to $\alpha = 6$ and $\beta$ taking on any value except 3.
Case C: There are an infinite number of solutions for $x, y$. This case corresponds to $\alpha = 6$ and $\beta = 3$. In this case the second equation is a multiple of the first, so we have only 1 independent equation in 2 unknowns $x, y$.

These three cases are referred to as the Fredholm alternatives. They apply to a linear system of the form $Mz = b$ where $M$ is an nxn matrix and $z$ and $b$ are nx1 column vectors. In case A for such a system, the unique solution is given by $z = M^{-1}b$, where $M^{-1}$ is the inverse of M.

In case C for such a system, how can we characterize the vectors $b$ for which a solution exists? Let $u$ be an nx1 column vector to be determined. Multiply $Mz = b$ on the left by $u^t$, which is a 1xn row vector:

$$u^t Mz = u^t b$$

Note that $u^t Mz$ is a scalar, and as such is equal to its own transpose:

$$u^t Mz = (u^t Mz)^t = z^t M^t u$$

So now we have

$$z^t M^t u = u^t b$$

So far $u$ has been an arbitrary column vector. Now choose $u$ to be in the null space of $M^t$:

$$M^t u = 0$$

Since $z^t M^t u = 0$, we have

$$u^t b = 0$$

That is $b$ **is orthogonal to the null space of** $M^t$.

Let us see how this works for the eq.(1) (in case C, where $\alpha = 6$):

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \Rightarrow M^t = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

$$M^t u = 0 \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u = \begin{bmatrix} c \\ -c/3 \end{bmatrix}$$

where $c$ is arbitrary. Then

$$u^t b = 0 \Rightarrow [c \ \ -c/3] \begin{bmatrix} 1 \\ \beta \end{bmatrix} = 0 \Rightarrow c - c\beta/3 = 0 \Rightarrow \beta = 3$$

In this course we shall be interested in the Fredholm Alternative because it will help us to understand the behavior of linear ODEs. Consider the two-point boundary value problem:

$$\frac{d^2x}{dt^2} + x = \sin \beta t \tag{2}$$

with the boundary conditions

$$x(0) = 0 \quad \text{and} \quad x(\alpha) = 0 \tag{3}$$

Note that this system contains two parameters $\alpha$ and $\beta$. As $\alpha$ and $\beta$ are varied, we will encounter the cases A,B,C of the Fredholm Alternative, in a manner similar to what we saw happen in the algebraic equation (1).

To begin with, we look for the general solution to eq.(2) in the form:

$$x(t) = x_h(t) + x_p(t) \tag{4}$$

where $x_h(t)$ is the general solution to the homogeneous version of eq.(2), and where $x_p(t)$ is a particular solution. We find:

$$x_h(t) = c_1 \sin t + c_2 \cos t \tag{5}$$

and we look for $x_p(t)$ in the form $x_p(t) = A \sin \beta t$ and obtain (assuming $\beta \neq 1$):

$$x_p(t) = \frac{1}{1-\beta^2} \sin \beta t \tag{6}$$

So eq.(4) becomes:

$$x(t) = c_1 \sin t + c_2 \cos t + \frac{1}{1-\beta^2} \sin \beta t \tag{7}$$

Now we apply the boundary conditions (3):

$$x(0) = 0 = c_1 \sin 0 + c_2 \cos 0 + \frac{1}{1-\beta^2} \sin \beta 0 \Rightarrow c_2 = 0 \tag{8}$$

$$x(\alpha) = c_1 \sin \alpha + \frac{1}{1-\beta^2} \sin \beta\alpha = 0 \Rightarrow c_1 = \frac{-1}{1-\beta^2} \frac{\sin \beta\alpha}{\sin \alpha} \tag{9}$$

So we obtain the solution to the boundary value problem as

$$x(t) = \frac{-1}{1-\beta^2} \frac{\sin \beta\alpha}{\sin \alpha} \sin t + \frac{1}{1-\beta^2} \sin \beta t \tag{10}$$

Note that the derived solution (10) is singular when $\sin \alpha = 0$. If $\sin \alpha \neq 0$ there is a unique solution (10) to the boundary value problem (2),(3), and we are in case A.

For the algebraic eqs.(1), case A was characterized by the vanishing of the determinant of the matrix. This is obviously an inappropriate way to characterize case A for eqs.(2),(3). The correct way that works for both eqs.(1) and eqs.(2),(3) is:

**Case A: The homogeneous equation has only the trivial solution.**

That is, the homogeneous equation $\frac{d^2x}{dt^2} + x = 0$ with the boundary condition $x(0) = 0$ has the solution $x(t) = c_1 \sin t$. Then the boundary condition $x(\alpha) = 0$ requires $c_1 \sin \alpha = 0$. This gives $c_1 = 0$ and the trivial solution $x(t) = 0$ unless $\sin \alpha = 0$.

Now let's suppose that we are in the case that $\sin \alpha = 0$, and in particular let's take $\alpha = \pi$. When the boundary condition $x(0) = 0$ is applied to the general solution (7), we get $c_2 = 0$, just as in eq.(8). Then setting $x(\pi) = 0$ gives:

$$x(\pi) = c_1 \sin \pi + \frac{1}{1-\beta^2} \sin \beta\pi = 0 \tag{11}$$

which will not be satisfied unless $\sin \beta\pi = 0$. So for general values of $\beta$ there will be no solution and we are in Case B. However if $\sin \beta\pi = 0$, for example if $\beta = 2$, then we have a solution:

$$x(t) = c_1 \sin t + \frac{1}{1-\beta^2} \sin 2t \tag{12}$$

That is, the ODE (2) and the boundary conditions $x(0) = 0$ and $x(\pi) = 0$ are all satisfied by (12), for any value of $c_1$. Thus the solution is not unique, and we are in Case C.

How can we characterize the values of $\beta$ which give a solution when $\alpha = \pi$? That is, how can we distinguish Case C (infinite number of solutions) from Case B (no solutions)? In the algebraic equations (1), we saw that the answer was that the column vector $b$ had to be orthogonal to the null space of the transpose matrix $M^t$. But such an answer doesn't seem to apply to the ODE problem because:
a) for orthogonality we need an inner product (or dot product), and what could we mean by the dot product of two functions?
and b) what, in the ODE example, could possibly correspond to the transpose of a matrix?

We define the inner product of two functions $f(t), g(t)$ as:

$$(f, g) = \int_0^\pi f(t)g(t)dt \tag{13}$$

Next we write the ODE (2) as $Lx = f$ where $L$ is a linear differential operator:

$$L = \frac{d^2}{dt^2} + 1 \tag{14}$$

and $f = \sin \beta t$. Following the derivation for the matrix problem, we multiplied $Mz = b$ by a vector $u^t$ giving $u^tMz = u^tb$. Of course $u^tb$ is the same as a dot product between $u$ and $b$. So we take an inner product of $Lx = f$ with an as yet undetermined function $u(t)$:

$$(u, Lx) = (u, f) \tag{15}$$

Now in the matrix problem, the next step was to note that $u^tMz$ was a scalar, and thus equal to its own transpose, giving $u^tMz = z^tM^tu$. That is, $u$ dotted into $Mz$ equals $z$ dotted into $M^tu$. The corresponding equation in the ODE problem is:

$$(u, Lx) = (x, L^*u) \tag{16}$$

where $L^*$ plays the role of $M^t$. $L^*$ is called the adjoint of $L$. If we write eq.(16) out, it looks like:

$$\int_0^\pi u(t)Lx(t)dt = \int_0^\pi x(t)L^*u(t)dt \tag{17}$$

The idea is to convert the left hand side of (17) to look like the right hand side through the use of integration by parts. Then we can find $L^*$ by inspection.

Once we know $L^*$, we choose $u$ to be in the null space of $L^*$ and we have

$$(u, Lx) = (x, L^*u) = (u, f) = 0, \qquad L^*u = 0 \tag{18}$$

That is, for a solution to $Lx = f$ in **Case C, $f$ must be orthogonal to the null space of $L^*$.**

For the ODE (2), $L$ is given by (14), and the left hand side of (17) becomes

$$\int_0^\pi u(t)\left(\frac{d^2}{dt^2}+1\right)x(t)dt = \int_0^\pi u(t)\frac{d^2x}{dt^2}dt + \int_0^\pi u(t)x(t)dt \tag{19}$$

Now we integrate the first term on the right hand side of (19) by parts:

$$\int_0^\pi u(t)\frac{d^2x}{dt^2}dt = \int_0^\pi u(t)d\frac{dx}{dt} = u(t)\frac{dx}{dt}\bigg|_0^\pi - \int_0^\pi \frac{dx}{dt}\frac{du}{dt}dt \tag{20}$$

We eliminate the integrated term by requiring that

$$u(0) = u(\pi) = 0 \tag{21}$$

Then we integrate by parts the remaining term one more time:

$$-\int_0^\pi \frac{dx}{dt}\frac{du}{dt}dt = -\int_0^\pi \frac{du}{dt}dx = -\,x(t)\frac{du}{dt}\bigg|_0^\pi + \int_0^\pi x(t)\frac{d^2u}{dt^2}dt \tag{22}$$

We again eliminate the integrated term by using the boundary conditions $x(0) = x(\pi) = 0$, giving finally:

$$\int_0^\pi u(t)\left(\frac{d^2}{dt^2}+1\right)x(t)dt = \int_0^\pi x(t)\frac{d^2u}{dt^2}dt + \int_0^\pi x(t)u(t)dt = \int_0^\pi x(t)\left(\frac{d^2}{dt^2}+1\right)u(t)dt \tag{23}$$

which may be written in the abbreviated form:

$$(u, Lx) = (x, Lu) \tag{24}$$

Comparison with eq.(16) shows that here $L = L^*$, that is, $L$ is a self-adjoint linear differential operator. The comparable item in matrices would be a symmetric matrix $M = M^t$.

So now that we know $L^* = L$ for the boundary value problem (2),(3), we require $f = \sin\beta t$ to be orthogonal to the null space of $L$. That is,

$$\int_0^\pi u(t)\sin\beta t dt = 0 \quad \text{where} \quad \frac{d^2u}{dt^2} + u = 0, \quad u(0) = 0, \quad u(\pi) = 0. \tag{25}$$

The solution to this last differential equation and boundary conditions is $u = c_1 \sin t$ and so the condition on $\beta$ becomes

$$\int_0^\pi \sin t \sin\beta t dt = 0 \tag{26}$$

This integral can be evaluated and gives $\sin\beta\pi = 0$, in agreement with results obtained in eq.(11) above.

**Summary**

**In solving the system $Lx = f$, there are three possibilities:**

**Case A: If the homogeneous system $Lx = 0$ has only the trivial solution then the nonhomogeneous system $Lx = f$ has a unique solution.**

**Case B: If the homogeneous system has a nontrivial solution, then the nonhomogeneous system will have no solution unless**

**Case C: the right hand side $f$ is orthogonal to the null space of the adjoint operator $L^*$, in which case the solution will not be unique since any solution of the homogeneous system can be added to a given solution.**

P1

# The Ritz Method (also known as Rayleigh-Ritz)

Being an extension of Rayleigh's method, providing an approximation for higher freqs, for example $\omega_2$, as well as $\omega_1$.

1. Choose $V(x) = c_1 V_1(x) + c_2 V_2(x)$
   where $V_1(x)$ & $V_2(x)$ satisfy at least the geometric B.C.

2. Compute Rayleigh's quotient

$$Q = \frac{EI}{\rho} \frac{\int_0^\ell (V'')^2 dx}{\int_0^\ell V^2 dx}$$

where $(V'')^2 = (c_1 V_1'' + c_2 V_2'')^2 = c_1^2 (V_1'')^2 + 2 c_1 c_2 V_1'' V_2'' + c_2^2 (V_2'')^2$ etc.

3. Set $\frac{\partial Q}{\partial c_1} = 0, \ \frac{\partial Q}{\partial c_2} = 0$ (to minimize $Q$)

4. This leads to
$$A c_1 + B c_2 = 0$$
$$D c_1 + E c_2 = 0$$
   So set $\det \begin{pmatrix} A & B \\ D & E \end{pmatrix} = 0$ for a nontrivial solution

5. This will give a quadratic on $Q \Rightarrow Q_1, Q_2$, these being bounds on $\omega_1 \le \sqrt{Q_1}$, $\omega_2 \le \sqrt{Q_2}$

clamped

$u = u_x = 0$

$x = 0$

free

$u_{xx} = u_{xxx} = 0$

$x = \ell$

[We know, from the Tables of Beam Frequencies, that $\lambda_1 = 1.8751 \Rightarrow \omega_1 = 3.516\sqrt{\dfrac{EI}{\rho \ell^4}}$

$\lambda_2 = 4.6941 \Rightarrow \omega_2 = 22.034\sqrt{\dfrac{EI}{\rho \ell^4}}$ ]

Choose $V(x) = c_1 x^2 + c_2 x^3$

Note that this choice of $V_1(x) = x^2$, $V_2(x) = x^3$ involves fns which satisfy only the geometric BC $\begin{pmatrix} u = u_x = 0 \\ \text{at } x = 0 \end{pmatrix}$ but not $u_{xx} = u_{xxx} = 0$ at $x = \ell$.

$$V'' = 2c_1 + 6c_2 x$$

$$Q = \frac{EI \int_0^\ell (V'')^2 dx}{\rho \int_0^\ell V^2 dx} = \frac{EI\ell\left(4c_1^2 + 12c_1c_2\ell + 12c_2^2\ell^2\right)}{\rho \ell^5\left(\frac{1}{5}c_1^2 + \frac{1}{3}c_1c_2\ell + \frac{1}{7}c_2^2\ell^2\right)}$$

P3

Define $\bar{c}_2 = c_2 \ell$, $\bar{Q} = Q\left(\frac{\rho \ell^4}{EI}\right)$

then $$\bar{Q} = \frac{4c_1^2 + 12 c_1 \bar{c}_2 + 12 \bar{c}_2^2}{\frac{1}{5}c_1^2 + \frac{1}{3}c_1\bar{c}_2 + \frac{1}{7}\bar{c}_2^2} = \frac{F}{G}$$

$$G\bar{Q} = F$$

$\frac{\partial}{\partial c_1}$ of $\Rightarrow$ $G_1 \bar{Q} + G \cancel{\frac{\partial \bar{Q}}{\partial c_1}} = F_1$ $\quad \left(G_1 = \frac{\partial G}{\partial c_1} \text{ etc.}\right)$

$\frac{\partial}{\partial c_2} \Rightarrow$ $G_2 \bar{Q} + G \cancel{\frac{\partial \bar{Q}}{\partial c_2}} = F_2$

$$\begin{bmatrix} \frac{2}{5}\bar{Q} - 8 & \frac{1}{3}\bar{Q} - 12 \\ \frac{1}{3}\bar{Q} - 12 & \frac{2}{7}\bar{Q} - 24 \end{bmatrix} \begin{bmatrix} c_1 \\ \bar{c}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Set det = 0 for nontrivial $c_1, \bar{c}_2$ $\Rightarrow$

$$\bar{Q}^2 - 1224\bar{Q} + 15120 = 0$$

$\Rightarrow \sqrt{\bar{Q}} = 3.532\ ,\ 34.807$

$\therefore \omega_1 \leq 3.532$ (correct is 3.516)

$\omega_2 \leq 34.807$ (correct is 22.034)

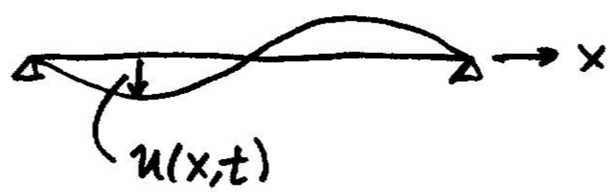
# Vibrations of circular membranes and plates

We have seen that the wave equation,

$$c^2 u_{xx} = u_{tt}$$

describes the longitudinal motion of waves in a rod, where $c^2 = \frac{EA}{\rho}$

It turns out that this same wave equation governs the motion of various other systems (see Table in text on p. 202), including the transverse vibrations of a string

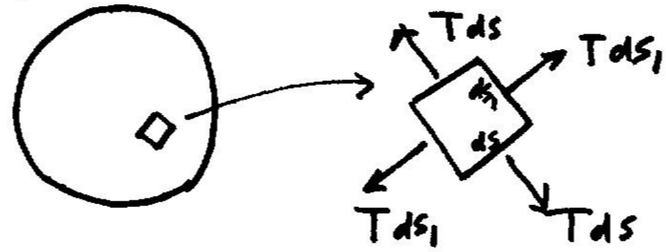



Here $c^2 = \frac{T}{\rho}$, $T$ = tension in the string, $\rho$ = mass/length

A membrane is defined to be a 2D version of a string, namely a perfectly flexible, thin sheet, which is uniformly stretched in all directions by a tension which has a constant value $T$ per unit length along any section or boundary.

Example: a drumhead (a skin stretched over the top of a drum)

So in the case of a circular membrane,




an arbitrary differential element is loaded as shown.

The governing pde is a 2D version of the pde for a string, namely

$$c^2(u_{xx}+u_{yy}) = u_{tt} \quad , \quad c^2=\frac{T}{\rho}$$

Here $u_{xx}+u_{yy}$ is written $\nabla^2 u$

Under a change in variables to polar coordinates, $r, \theta$

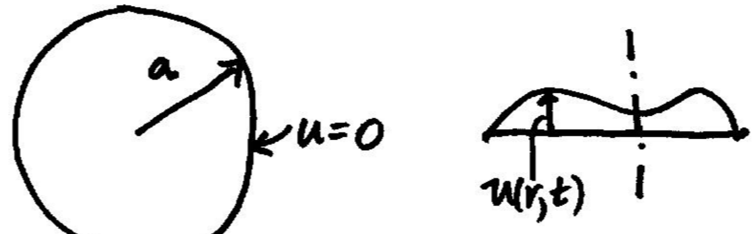
$$x = r\cos\theta, \quad y = r\sin\theta$$

the pde becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2}u_{tt}$$

We shall be interested in axisymmetric motions in which $u=u(r,t)$ i.e. no $\theta$ dependence.

The B.C. is that $u=0$ on the boundary $r=a$




top view

side view

We seek the natural frequencies.

Set $u(r,t) = U(r)\cos\omega t \Rightarrow$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) = -\frac{\omega^2}{c^2}U$$

This is a <u>variable coefficient</u> ODE.

Write it in the form

$$rU'' + U' + \frac{\omega^2}{c^2}rU = 0$$

Define $z = \frac{\omega r}{c}$ giving

$$z\frac{d^2U}{dz^2} + \frac{dU}{dz} + zU = 0$$

plus the B.C. $U=0$ at $r=a$, $z=\frac{\omega a}{c}$

Seek a solution in the form of an infinite series:

$$U = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

$$\frac{dU}{dz} = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots + n a_n z^{n-1}$$

$$\frac{d^2U}{dz^2} = 2a_2 + 6a_3 z + 12a_4 z^2 + \cdots + n(n-1)a_n z^{n-2}$$

$$\begin{aligned} &2a_2 z + 6a_3 z^2 + 12a_4 z^3 + \cdots + n(n-1)a_n z^{n-1} + \cdots \\ &+ a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots + n a_n z^{n-1} + \cdots \\ &+ a_0 z + a_1 z^2 + a_2 z^3 + \cdots + a_n z^{n+1} = 0 \end{aligned}$$

Collect terms

$z^0$: $a_1 = 0$

$z^1$: $a_0 + \cancel{2}4 a_2 = 0$

$z^2$: $9a_3 + a_1 = 0$

$z^3$: $12a_4 + 4a_4 + a_2 = 0$

...

$z^{n-1}$: $n(n-1)a_n + n a_n + a_{n-2} = 0$

$\Rightarrow \quad n^2 a_n + a_{n-2} = 0$

$a_{n-2} = -n^2 a_n$ "recursion relation"

$$a_n = -\frac{a_{n-2}}{n^2}$$

$\therefore \quad a_{odd} = 0$

$$a_2 = -\frac{a_0}{\cancel{2}4}$$

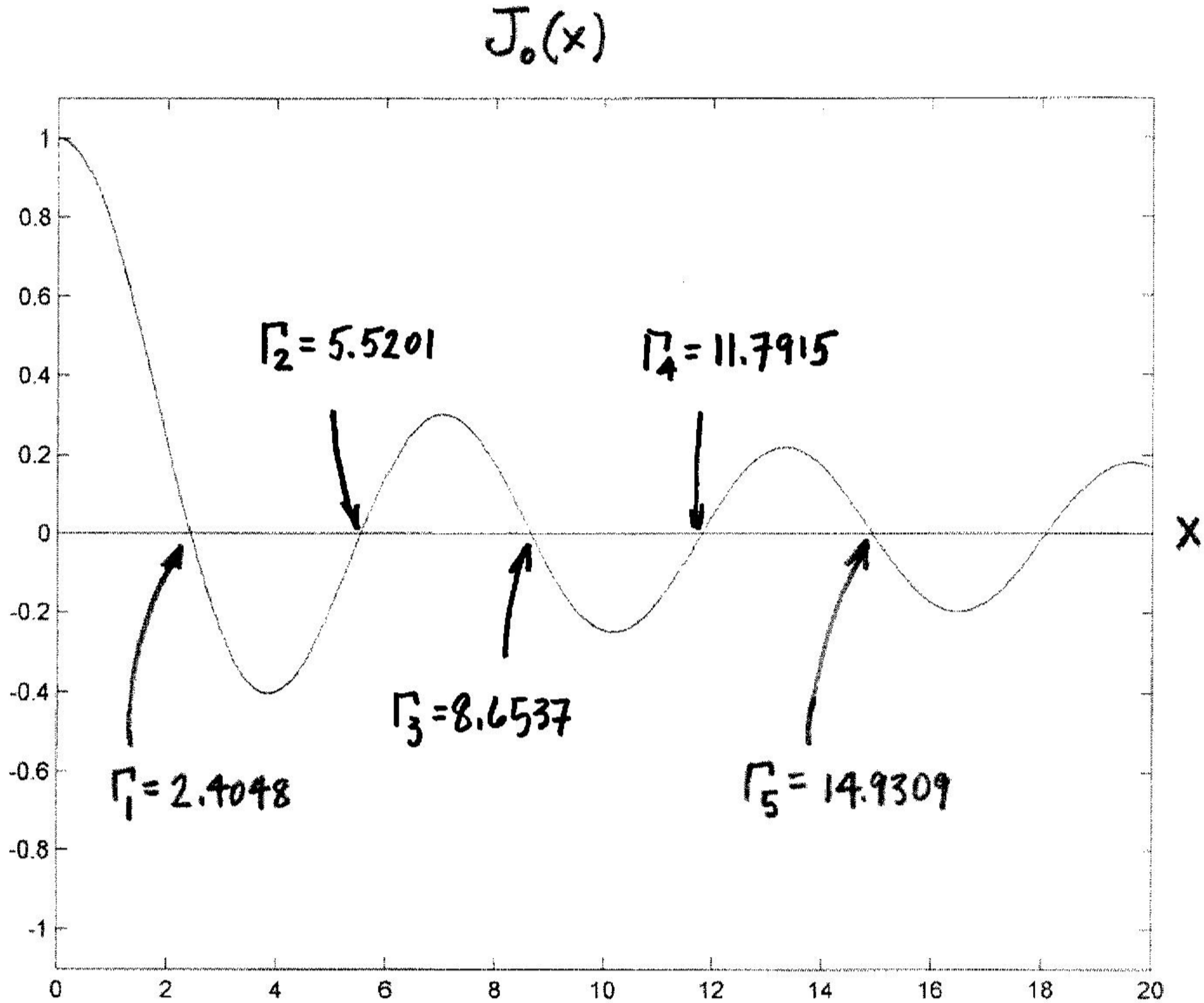
$$a_4 = \frac{-a_2}{16} = \frac{a_0}{\cancel{32}64}$$

$$a_6 = -\frac{a_4}{36} = -\frac{a_0}{36 \cdot \cancel{32}64}$$

$$U(z) = a_0 \underbrace{\left(1 - \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{\left(\frac{1}{4}z^2\right)^2}{(2!)^2} - \frac{\left(\frac{1}{4}z^2\right)^3}{(3!)^2} + \cdots\right)}_{J_0(z)}$$

"Bessel function of order zero of the first kind"

5


$J_0(x)$
$\Gamma_1 = 2.4048$
$\Gamma_2 = 5.5201$
$\Gamma_3 = 8.6537$
$\Gamma_4 = 11.7915$
$\Gamma_5 = 14.9309$
x

$J_0(z)$ plots as a kind of damped cosine

For large $z$ it turns out that

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right)$$

We have $\quad U(z) = J_0(z) = J_0\left(\frac{\omega r}{c}\right)$

B.C. $\quad U(r=a) = 0$

$$\Rightarrow \quad J_0\left(\frac{\omega a}{c}\right) = 0$$

To find the natural frequencies we need the zeros of $J_0$. Let's call these $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$

$$J_0(\Gamma_n) = 0$$

It turns out that $\Gamma_1 = 2.4048$, $\Gamma_2 = 5.5201$, etc. See figure. Thus

$$\frac{\omega_1 a}{c} = \Gamma_1 = 2.4048 \Rightarrow \omega_1 = 2.4048 \frac{c}{a}$$

where $c = \sqrt{\frac{T}{\rho}}$

$$\omega_2 = 5.5201 \sqrt{\frac{T}{\rho a^2}}, \text{ etc.}$$

Functions related to $J_0(x)$

which we will need in this course

$$J_1(x) = -\frac{dJ_0(x)}{dx}$$

$$\left[\text{analogy: } \sin x = -\frac{d}{dx}\cos x\right]$$

$$I_0(x) = J_0(ix)$$

$$\left[\text{analogy: } \cosh x = \cos ix\right]$$

$$I_1(x) = \frac{d}{dx} I_0(x)$$

$$\left[\text{analogy: } \sinh x = \frac{d}{dx}\cosh x\right]$$

In Wolfram alpha, $J_0(x), J_1(x), I_0(x), I_1(x)$

are respectively J∅(x), J1(x), I∅(x), I1(x)

Examples "plot J∅(x) from 0 to 10"

"plot besselJ(∅,x) from 0 to 10"

"plot [besselJ(∅,x), besselI(∅,x)] from 0 to 3"

$I_0(x)$ is called the "modified Bessel fn of order zero of the first kind".

# Circular Plates

Just as a membrane is a 2D version of a string:

| string | membrane |
|---|---|
| $c^2 u_{xx} = u_{tt}$ | $c^2 \nabla^2 u = u_{tt}$ |
| | $c^2 (u_{xx} + u_{yy}) = u_{tt}$ |

So is a plate a 2D version of a beam:

| beam | plate |
|---|---|
| $-EI\, u_{xxxx} = \rho\, u_{tt}$ ($\rho$: mass/length) | $\dfrac{-Eh^3}{12(1-\nu^2)} \nabla^2 \nabla^2 u = \rho\, u_{tt}$ ($\rho$: mass/area) |

where $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)$ (axisymmetric motions)

and so
$$\nabla^2 \nabla^2 u = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right.}_{\nabla^2}\left.\underbrace{\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right\}}_{\nabla^2 u}\right)$$

To find the natural frequencies of a circular plate, we set $u(r,t) = U(r) \cos \omega t$

$$u(r,t) = U(r)\cos\omega t \Longrightarrow$$

$$-\frac{Eh^3}{12(1-\nu^2)}\nabla^2\nabla^2 U(r) = -\rho\omega^2 U$$

$E$ = Young's modulus,
$h$ = plate thickness,
$\nu$ = Poisson's ratio
$\rho$ = density (mass per unit area)

$$\nabla^2\nabla^2 U - \beta^4 U = 0$$

where $\beta^4 = \omega^2\left(\frac{12\rho(1-\nu^2)}{Eh^3}\right)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right)\right)\right) - \beta^4 U = 0$$

Set $z = \beta r$

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{d}{dz}\left(\frac{1}{z}\frac{d}{dz}\left(z\frac{dU}{dz}\right)\right)\right) - U = 0$$

$$U' + zU''$$

$$\frac{1}{z}U' + U''$$

$$-\frac{1}{z^2}U' + \frac{1}{z}U'' + U'''$$

$$-\frac{1}{z}U' + U'' + zU'''$$

$$+\frac{1}{z^2}U' - \frac{1}{z}U'' + 2U''' + zU^{IV}$$

$$\frac{1}{z^3}U' - \frac{1}{z^2}U'' + \frac{2}{z}U''' + U^{IV}$$

$$U^{IV} + \frac{2}{z} U''' - \frac{1}{z^2} U'' + \frac{1}{z^3} U' - U = 0$$

The bounded general solution to this ODE is

$$U(z) = C_1 J_0(z) + C_2 I_0(z)$$

But $z = \beta r \Rightarrow U(\beta r) = C_1 J_0(\beta r) + C_2 I_0(\beta r)$

B.C. for a <u>clamped</u> plate:

$$u = 0, \quad \frac{\partial u}{\partial r} = 0 \text{ at } r = a$$

Plug in the ansatz $u(r,t) = U(r) \cos \omega t \Rightarrow$

$$U(r=a) = 0, \quad U'(r=a) = 0$$

$$U(r=a) = 0 = C_1 J_0(\beta a) + C_2 I_0(\beta a)$$

$$U'(r=a) = 0 = C_1 J_0'(\beta a) + C_2 I_0'(\beta a)$$

$$\begin{bmatrix} J_0(\beta a) & I_0(\beta a) \\ J_0'(\beta a) & I_0'(\beta a) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution $C_1, C_2$ require $\det = 0$:

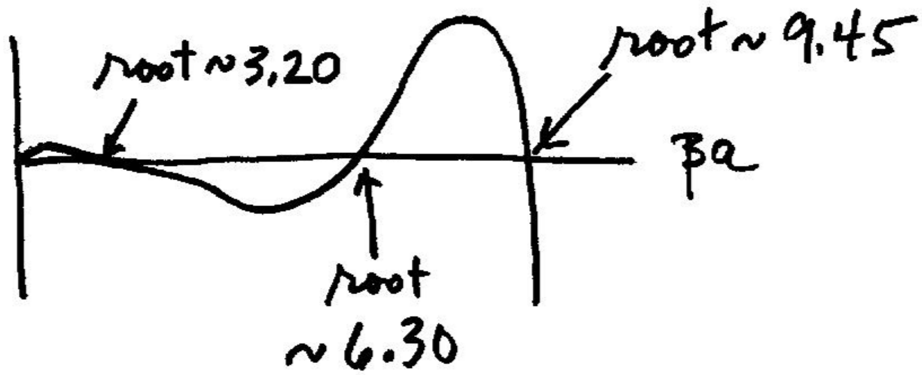
$$J_0(\beta a) \underbrace{I_0'(\beta a)}_{= I_1(\beta a)} - I_0(\beta a) \underbrace{J_0'(\beta a)}_{= -J_1(\beta a)} = 0$$

The frequencies of free vibration are given by the roots of the equation:

$$J_0(\beta a)\, I_1(\beta a) + I_0(\beta a)\, J_1(\beta a) = 0$$

where $\beta^4 = \omega^2 \left( \dfrac{12\rho(1-\nu^2)}{Eh^3} \right)$

plot $\Rightarrow$




$$(\beta a)^4 = \omega^2 \left( \frac{12\rho(1-\nu^2)a^4}{Eh^3} \right) = (3.2)^4$$

$$\omega_1 \approx \underbrace{(3.2)^2}_{=\,10.2} \sqrt{\frac{Eh^3}{12\rho(1-\nu^2)a^4}}$$

Similarly, $\omega_2 \approx 39.8 \sqrt{\;''\;}$

$$\omega_3 \approx 88.9 \sqrt{\;''\;}$$

# Introduction to Nonlinear Vibrations

We have seen that an undamped linear oscillator has the governing ODE:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

In this discussion of nonlinear oscillators, we will modify the linear model as follows:

$$\frac{d^2x}{dt^2} + \omega^2 x = f\left(x, \frac{dx}{dt}\right)$$

f contains nonlinear terms and linear damping terms

To begin with let's redefine time to be nondimensional so that $\bar{t} = \omega t$

which turns the last equation into:

$$\frac{d^2x}{d\bar{t}^2} + x = \bar{f}\left(x, \frac{dx}{d\bar{t}}\right)$$

For convenience we will drop the bars.

$$\ddot{x} + x = f(x, \dot{x})$$

Depending on the form of $f(x, \dot{x})$,

the oscillator will either be

CONSERVATIVE or NONCONSERVATIVE

As an example of the CONSERVATIVE oscillator, we consider "Duffing's Equation"

$$\ddot{x} + x = \alpha x^3$$

This equation has no damping terms and <u>conserves energy</u>

This can be seen in two ways:

(1) Mult. by $\dot{x}$ $\Rightarrow$ $\dot{x}\ddot{x} + \dot{x}x = \alpha \dot{x} x^3$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) + \frac{d}{dt}\left(\frac{x^2}{2} - \alpha\frac{x^4}{4}\right) = 0$$

$$\therefore \underbrace{\frac{1}{2}\dot{x}^2}_{\text{kinetic energy}} + \underbrace{\frac{x^2}{2} - \alpha\frac{x^4}{4}}_{\text{potential energy}} = \text{constant}$$

(2) Write as a first order system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x + \alpha x^3$$

Think of $y$ as a fn of $x$:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \quad \text{(chain rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-x+\alpha x^3}{y}$$

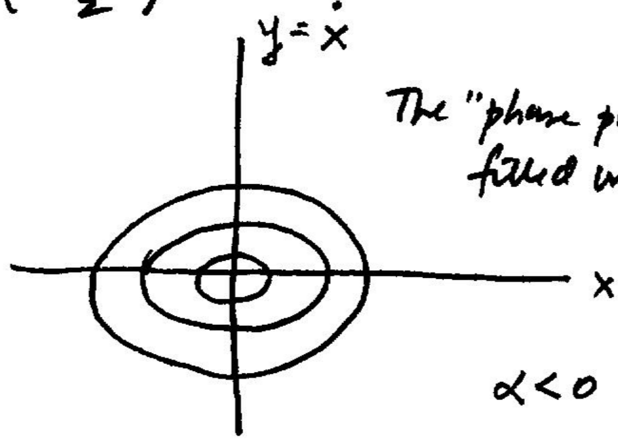
Then

$$y\,dy = (-x+\alpha x^3)\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \alpha\frac{x^4}{4} + \text{const.}$$

kinetic energy $\left(= \frac{\dot{x}^2}{2}\right)$

$-$ potential energy




The "phase plane" is filled with closed curves

(For $\alpha > 0$ it looks different.)

4

Basic question about conservative oscillators:

$$\boxed{\text{What is the relation between period (or frequency) of oscillation and amplitude?}}$$

[For linear oscillator there is no such relation.]

To find out, we use an approximate method called HARMONIC BALANCE

$$\ddot{x} + x = \alpha x^3$$

We look for a solution in the form

$$x = A \cos \omega t$$

Substituting it in the ODE, we get

$$-\omega^2 A \cos \omega t + A \cos \omega t = \alpha A^3 (\cos \omega t)^3$$

Note that $\cos^3 \omega t = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t$

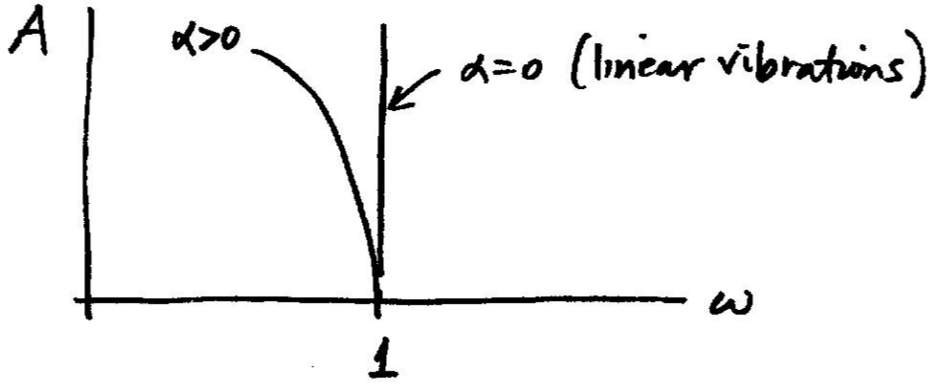
Equating coefficients of $\cos \omega t$ on both sides of the eqn:

$$-\omega^2 A + A = \frac{3}{4} \alpha A^3$$

Solving for $\omega^2$ we get

$$\omega^2 = 1 - \frac{3\alpha}{4} A^2$$

Note that this solution is not exact since the $\cos 3\omega t$ term is not balanced.




$$\omega^2 = 1 - \frac{3}{4}\alpha A^2 \qquad \left(\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1-\frac{3}{4}\alpha A^2}}\right)$$

Example: The pendulum

$$\ddot{x} + \sin x = 0 \qquad \left(\ddot{x} + \frac{g}{l}\sin x = 0 \text{ but } \frac{g}{l} \text{ is absorbed into } \bar{t} = \sqrt{\frac{g}{l}}\,t\right)$$

Now $\sin x \approx x - \frac{x^3}{6}$

So we have

$$\ddot{x} + x = \frac{x^3}{6} \qquad \left(\alpha = \frac{1}{6}\right)$$

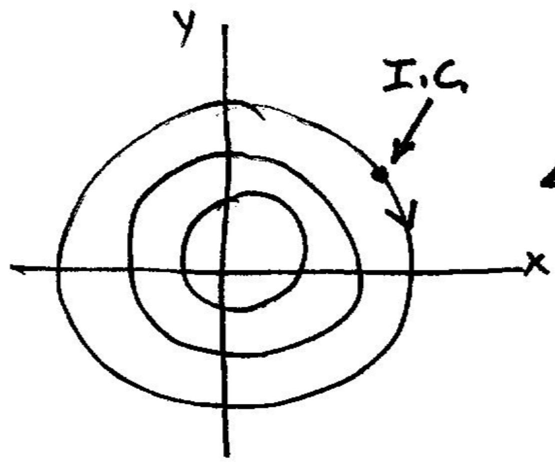
$$\omega = \sqrt{1 - \frac{3}{4}\alpha A^2} = \sqrt{1 - \frac{A^2}{8}}$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1 - A^2/8}}$$

Thus as amplitude increases, frequency decreases and period increases (as observed).

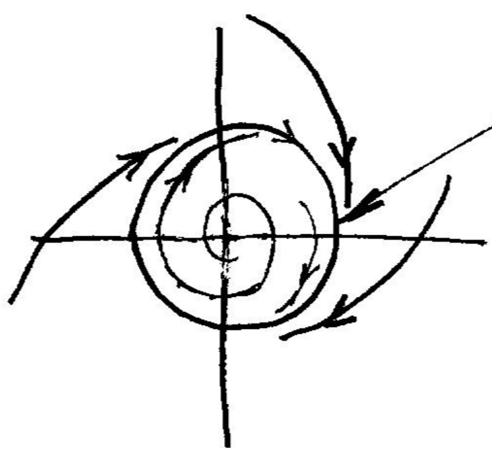
# NONCONSERVATIVE OSCILLATORS

In the case of a conservative oscillator, the amplitude of vibration is dependent on the initial conditions, i.e. which of the continuum of closed curves you are on depends on which one you started on:




CONSERVATIVE CASE
plane is filled with closed curves (periodic motions)

In the NONCONSERVATIVE CASE it is possible to have a unique closed curve to which all I.C.'s are attracted:



The attractive closed curve in this example is called a "LIMIT CYCLE"

Examples of limit cycles:

- * the beating of the heart
- * galloping of high tension wires
- * bowing of a violin

The paradigm example of a limit cycle oscillator is this one, called a VAN DER POL OSCILLATOR:

$$\ddot{x} + x = \underbrace{(1-x^2)\dot{x}}$$

we may think of this as a damping term which changes sign depending on $x$

To obtain an approximation for the LC (limit cycle) we use harmonic balance:

Set $x = A\cos\omega t$

Substituting in the ODE gives

$$-\omega^2 A\cos\omega t + A\cos\omega t = \underbrace{(1-A^2\cos^2\omega t)(-A\omega\sin\omega t)}$$

trigonometrically simplify this (Wolfram Alpha gives)

$$-A\omega\sin\omega t + \frac{A^3\omega}{4}\left(\sin\omega t + \sin 3\omega t\right)$$

Balancing the harmonics: 8

$\cos\omega t$: $(-\omega^2+1)A = 0 \Rightarrow \omega^2 = 1$

$\sin\omega t$: $0 = -A\omega + \frac{A^3\omega}{4} \Rightarrow A = 2$

Conclusion: $x = 2\cos t$ is the approximate solution

<u>The limit cycle has amplitude 2</u>

The onset of vibrations:

## Birth of a Limit Cycle: Hopf bifurcation

We consider a sample system:

$$\ddot{x} + x = \mu \dot{x} - \dot{x}^3$$

Apply harmonic balance:

$$x = A \cos \omega t$$

$$\dot{x} = -A\omega \sin \omega t$$

$$(-\omega^2+1) A \cos \omega t = \mu(-A\omega \sin \omega t) + A^3 \omega^3 \underbrace{(\sin \omega t)^3}_{= \frac{3}{4}\sin \omega t - \frac{1}{4}\sin 3\omega t}$$

$\cos \omega t$: $\quad (-\omega^2+1) A = 0 \quad \Rightarrow \quad \omega = 1$

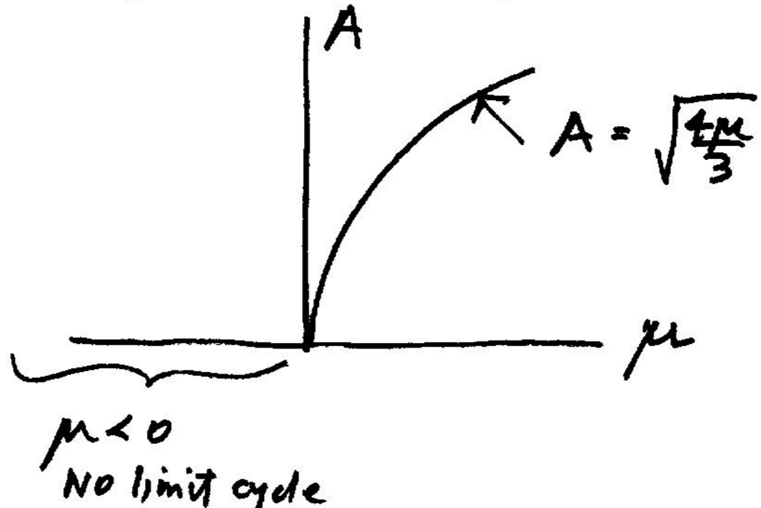
$\sin \omega t$: $\quad 0 = -A\omega\mu + \frac{3}{4}A^3\omega^3$

$$\Rightarrow \quad A^2 = \frac{4\mu}{3}\frac{1}{\omega^2} = \frac{4}{3}\mu \text{ since } \omega = 1$$

For a real solution, $A^2 > 0 \Rightarrow \mu > 0$

The limit cycle is predicted to only exist for $\mu > 0$, and then with amplitude $A = \sqrt{\frac{4\mu}{3}}$

This may be illustrated by plotting $A$ vs. $\mu$:




As $\mu$ increases from a negative value to a positive value, the limit cycle is formed and increases in amplitude.

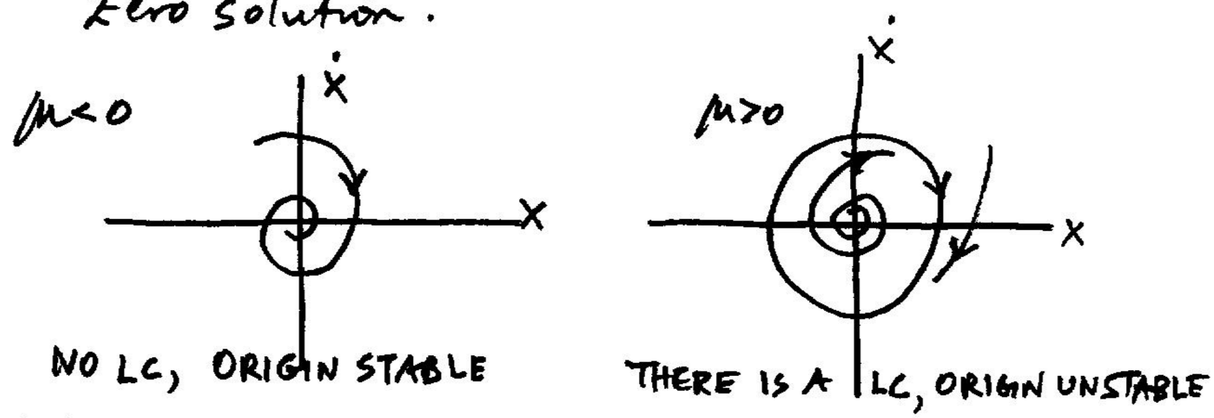
Note that if we keep only linear terms, we get

$$\ddot{x} - \mu \dot{x} + x = 0$$

$\mu$ = coefficient of damping

So for $\mu > 0$, damping is negative and the zero solution $x \equiv 0$ is UNSTABLE

Thus the birth of the LC (limit cycle) is accompanied by a change in stability of the zero solution.



#3

This however is not the whole story.

Suppose we change the sign on the nonlinear term:

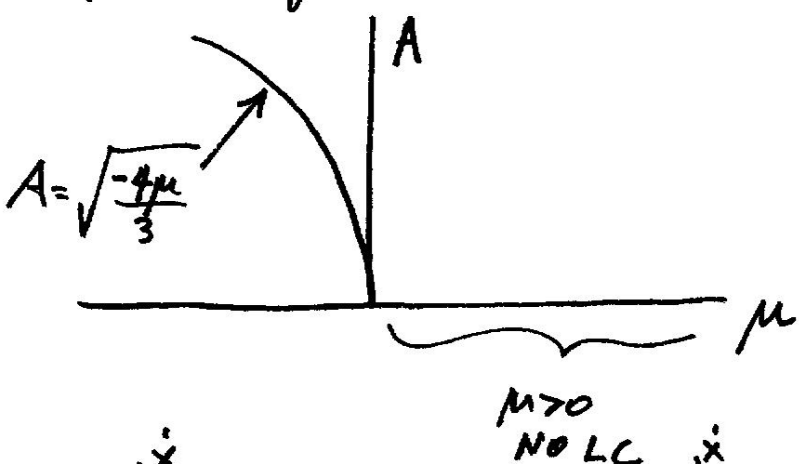
$$\ddot{x} + x = \mu \dot{x} + \dot{x}^3$$

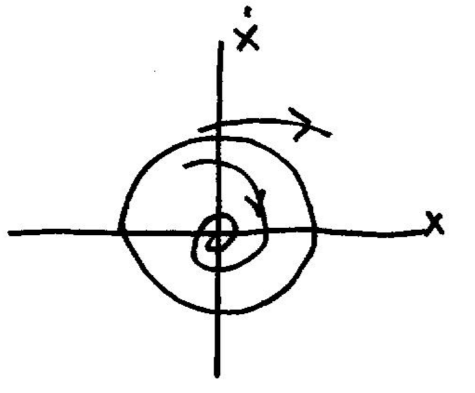
Harmonic balance: $x = A \cos \omega t$

Proceed as before, we obtain:

$$A^2 = -\frac{4}{3}\mu, \quad A = \sqrt{\frac{-4\mu}{3}}$$

So for this equation, the LC occurs for $\mu < 0$.



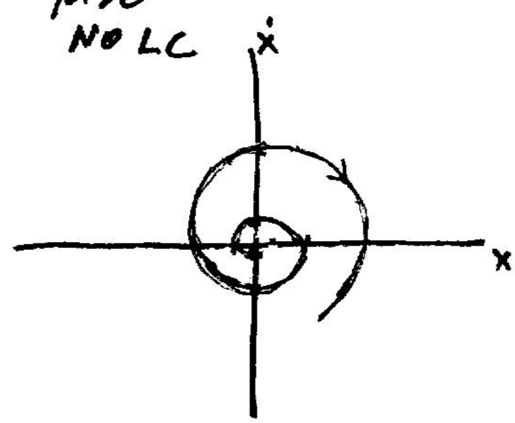




$\mu < 0$

The origin is stable

There is a LC

But it is not attractive




$\mu > 0$

The origin is <u>un</u>stable

NO LC

# JARGON

If the LC is attracting, the Hopf bifurcation is called <u>supercritical</u>

$\ddot{x} + x = \mu \dot{x} - \dot{x}^3$ has a supercritical Hopf

If the LC is repelling, the Hopf is <u>subcritical</u>.

$\ddot{x} + x = \mu \dot{x} + \dot{x}^3$ has a subcritical Hopf.

## The Hopf bifurcation formula

$$\ddot{x} + x = \mu \dot{x} + \beta_1 x^3 + \beta_2 x^2 \dot{x} + \beta_3 x \dot{x}^2 + \beta_4 \dot{x}^3$$

Applying harmonic balance to ✱ gives

$$A = 2\sqrt{\frac{-\mu}{\beta_2 + 3\beta_4}}$$

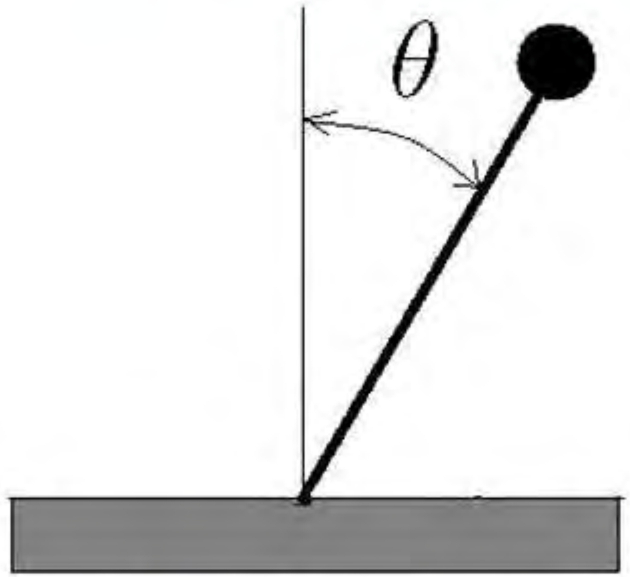
If $\beta_2 + 3\beta_4 < 0$, then $A$ is real for $\mu > 0$
and LC occurs when origin is unstable
$\therefore$ LC is attracting and Hopf is supercritical

If $\beta_2 + 3\beta_4 > 0$, then Hopf is subcritical.

# PARAMETRIC EXCITATION

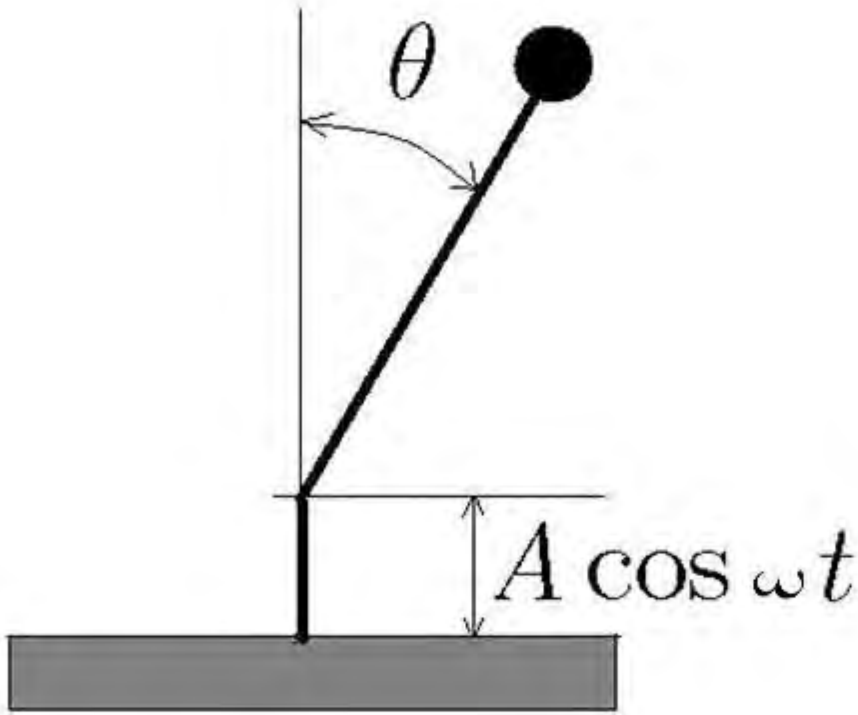
MAE 4770/5770

# A Problem in Control Theory



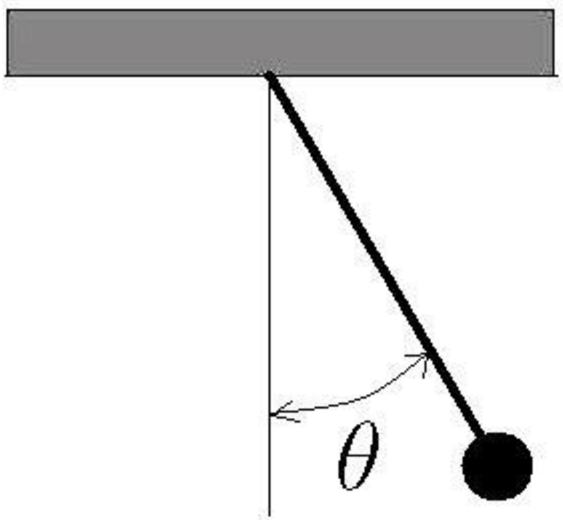

THE UPSIDE-DOWN PENDULUM

**NORMALLY UNSTABLE**



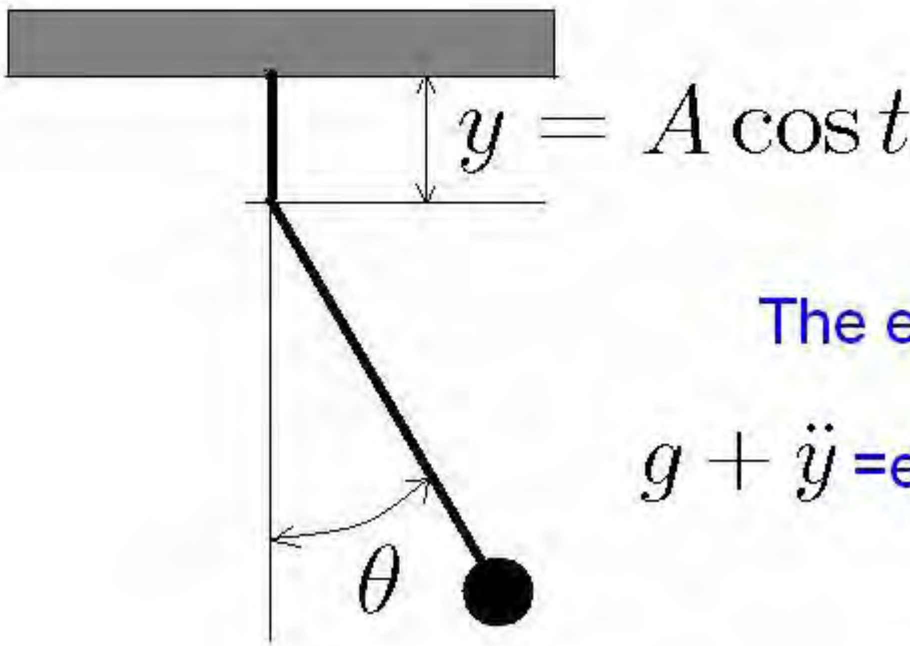

**Can it be made STABLE**
**by driving the base vertically?**

# FREE VIBRATIONS OF A PLANE PENDULUM




$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

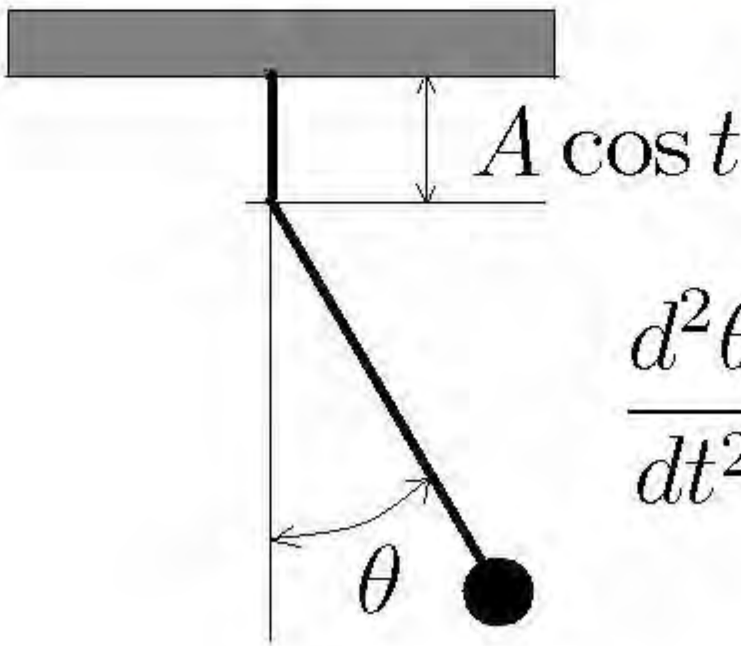
# VERTICALLY-DRIVEN PENDULUM




The effect of the vertical driver $y$
is to replace $g$ by
$g + \ddot{y}$ =effective vertical acceleration

$$\frac{d^2\theta}{dt^2} + \left(\frac{g - A\cos t}{L}\right)\sin\theta = 0$$

# STABILITY OF THE VERTICALLY-DRIVEN PENDULUM




$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{l} - \frac{A}{L}\cos t\right)\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{l} - \frac{A}{L}\cos t\right)\theta = 0$$

## Mathieu's Equation

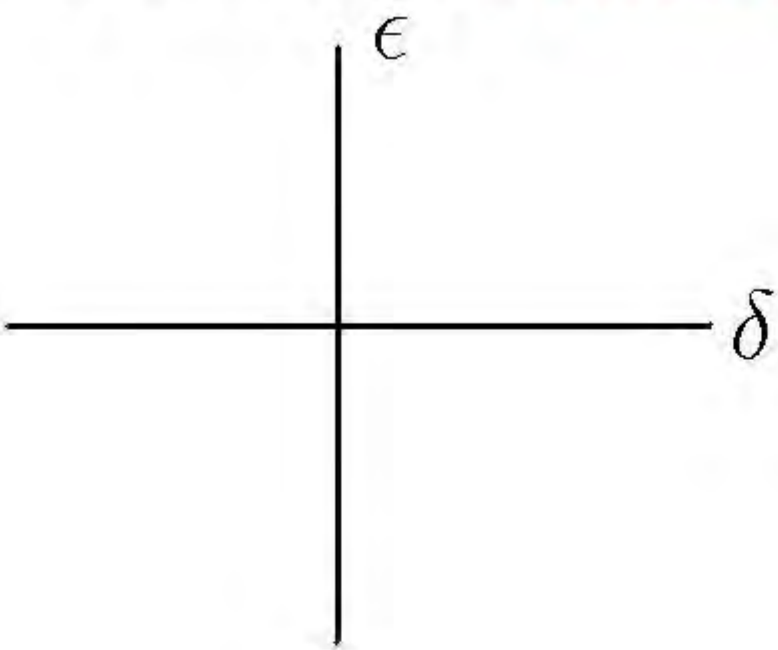
$$\frac{d^2x}{dt^2} + (\delta + \epsilon \cos t)\, x = 0$$

$$\delta = \frac{g}{L}, \qquad \epsilon = -\frac{A}{L}$$

# Mathieu's Equation

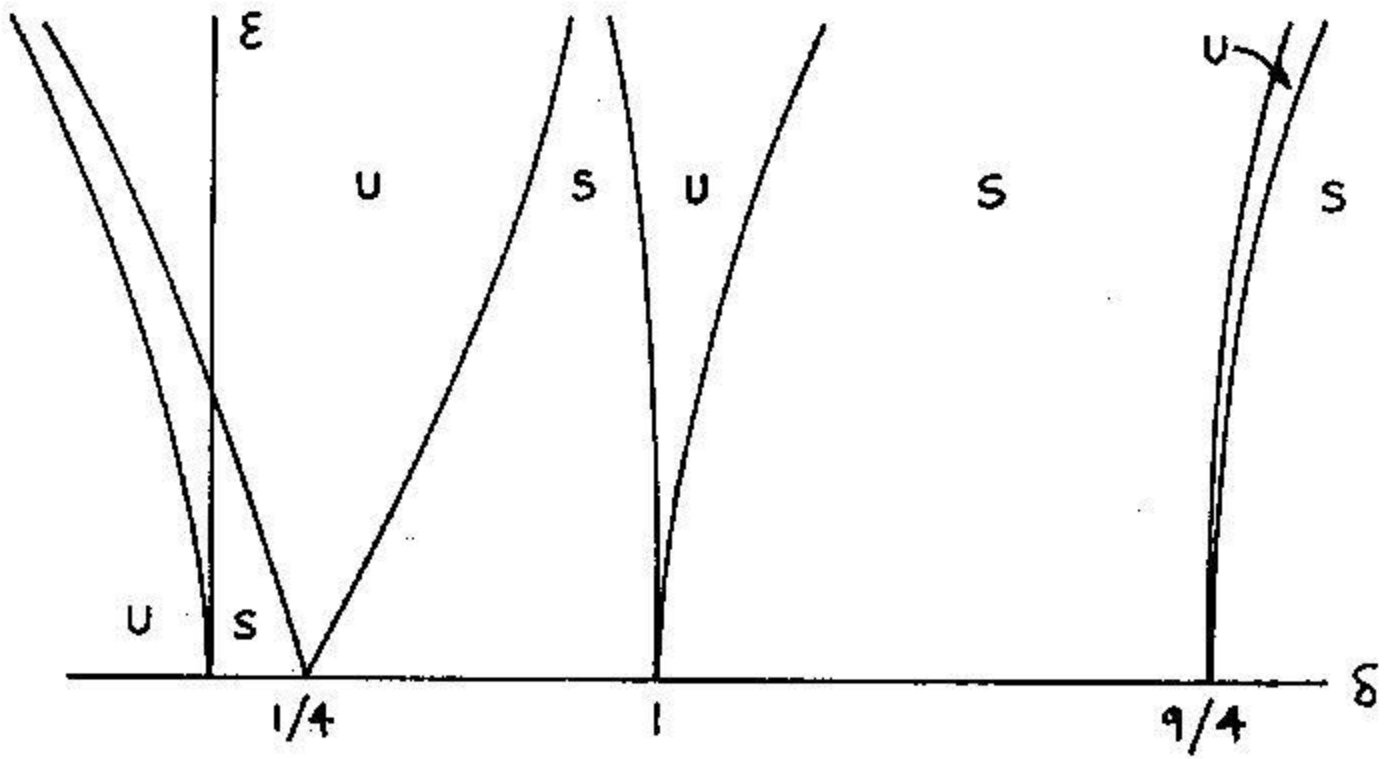
$$\frac{d^2x}{dt^2} + (\delta + \epsilon\cos t)\,x = 0$$

For a given pair of parameters $(\delta, \epsilon)$
either all solutions are bounded (STABLE),
or there exists an unbounded solution (UNSTABLE)



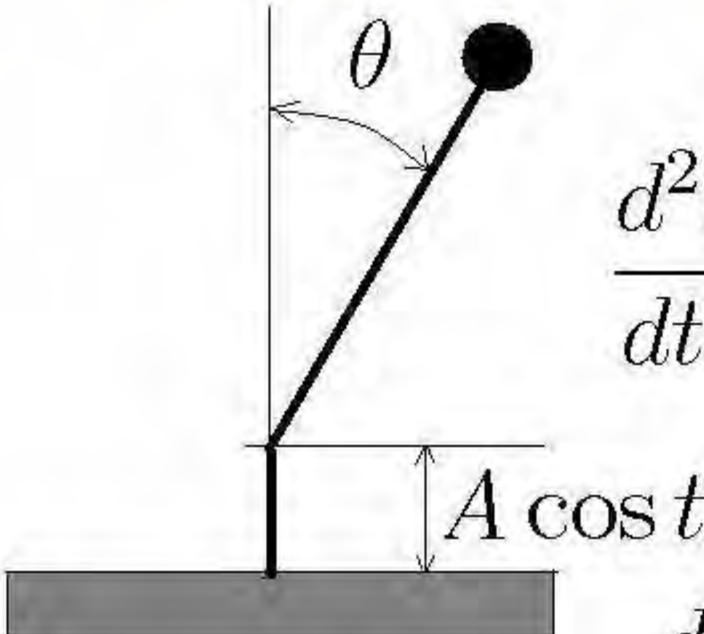

Which points in the $(\delta, \epsilon)$ plane are STABLE vs. UNSTABLE ?

When $\epsilon \neq 0$, transition curves emerge from points on the $\delta$-axis at $\delta = \frac{n^2}{4}$, $n = 0, 1, 2, \cdots$




Transition curves in Mathieu's equation. S=stable, U=unstable.

# STABILITY OF THE UPSIDE-DOWN PENDULUM




$$\frac{d^2\theta}{dt^2} - \left(\frac{g}{l} - \frac{A}{L}\cos t\right)\theta = 0$$

$$\frac{d^2x}{dt^2} + (\delta + \epsilon\cos t)\,x = 0$$

$$\delta = -\frac{g}{L}, \qquad \epsilon = \frac{A}{L}$$

# H. Kauderer, "Nichtlineare Mechanik", 1958

$$\frac{d^2 y}{d x^2} + (\lambda + \mu \cos x)\, y = 0$$

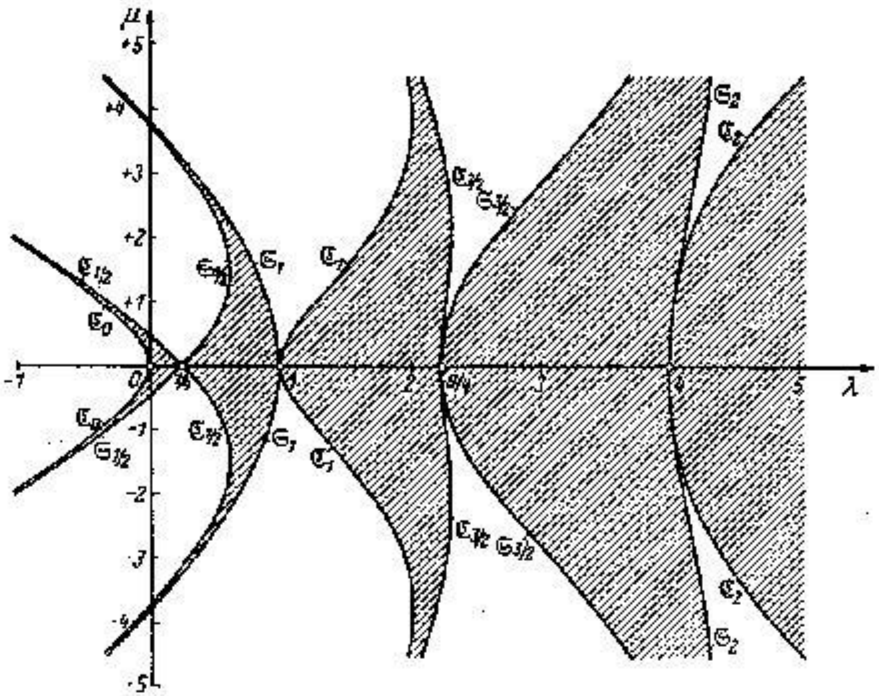



Abb. 184. Stabilitätskarte der MATHIEUschen Gleichung

$$\lambda = -\frac{1}{2}\mu^2 + \frac{7}{32}\mu^4 - \frac{29}{144}\mu^6 + \cdots,$$

$$\lambda = \frac{1}{4} - \frac{1}{2}\mu - \frac{1}{8}\mu^2 + \frac{1}{32}\mu^3 - \frac{1}{384}\mu^4 -$$

$$\lambda = \frac{1}{4} + \frac{1}{2}\mu - \frac{1}{8}\mu^2 - \frac{1}{32}\mu^3 - \frac{1}{384}\mu^4 +$$

$$\lambda = 1 + \frac{5}{12}\mu^2 - \frac{763}{3456}\mu^4 + \cdots,$$

$$\lambda = 1 - \frac{1}{12}\mu^2 + \frac{5}{3456}\mu^4 - \cdots,$$

$$\lambda = \frac{9}{4} + \frac{1}{16}\mu^2 - \frac{1}{32}\mu^3 + \frac{13}{5120}\mu^4 - \cdots,$$

$$\lambda = \frac{9}{4} + \frac{1}{16}\mu^2 + \frac{1}{32}\mu^3 + \frac{13}{5120}\mu^4 + \cdots,$$

$$\lambda = 4 + \frac{1}{30}\mu^2 + \frac{433}{216000}\mu^4 - \cdots,$$

$$\lambda = 4 + \frac{1}{30}\mu^2 \underset{-}{+} \frac{317}{216000}\mu^4 + \cdots.$$

# Upside-down Pendulum

$$\frac{d^2\theta}{dt^2} - \left(\frac{g}{L} + \frac{d^2}{dt^2}(A\cos\omega t)\right)\theta = 0$$

Set $\tau = \omega t$, take $\frac{g}{L} = 1$ WLOG

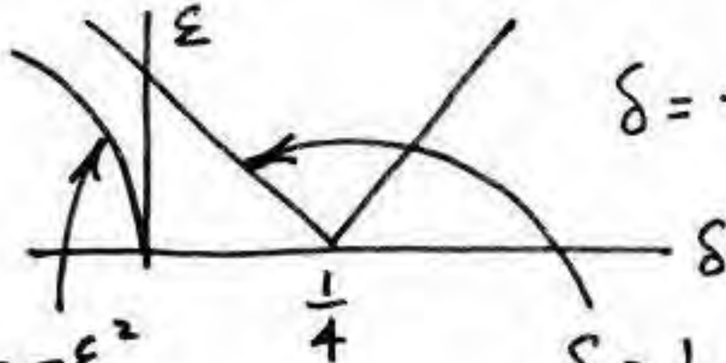
$$\omega^2 \frac{d^2\theta}{d\tau^2} - \left(1 - A\omega^2\cos\tau\right)\theta = 0$$

$$\frac{d^2\theta}{d\tau^2} - \left(\frac{1}{\omega^2} - A\cos\tau\right)\theta = 0$$

$$\frac{d^2\theta}{d\tau^2} - \left(\frac{1}{\omega^2} - A\cos\tau\right)\theta = 0$$

$$\frac{d^2x}{d\tau^2} + (\delta + \varepsilon\cos\tau)\,x = 0$$

$$\delta = -\frac{1}{\omega^2}, \quad \varepsilon = A$$




$$\delta = -\frac{1}{\omega^2}, \quad \varepsilon = A$$

$$\delta = -\frac{\varepsilon^2}{2}$$

$$-\frac{1}{\omega^2} = -\frac{A^2}{2}$$

$$A = \frac{\sqrt{2}}{\omega}$$

$$\omega = 5 \Rightarrow A = \frac{\sqrt{2}}{5} = .28$$

$$\delta = \frac{1}{4} - \frac{\varepsilon}{2}$$

$$-\frac{1}{\omega^2} = \frac{1}{4} - \frac{A}{2}$$

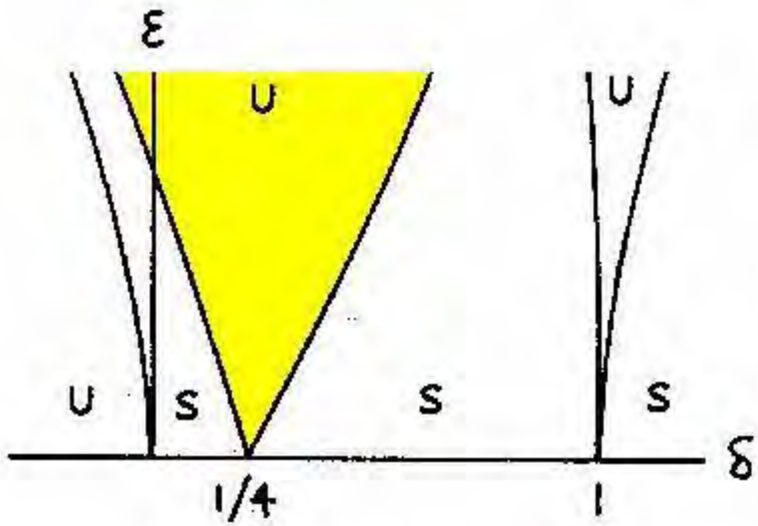
$$A = \frac{1}{2} + \frac{2}{\omega^2}$$

$$\omega = 5 \Rightarrow A = \frac{1}{2} + \frac{2}{25} = .58$$

$$\frac{d^2x}{dt^2} + (\delta + \epsilon \cos t)\, x = 0$$

Natural frequency = $\sqrt{\delta}$

Forcing frequency = $1$




$$\delta = 1/4 \Rightarrow 1 = 2\sqrt{\delta}$$

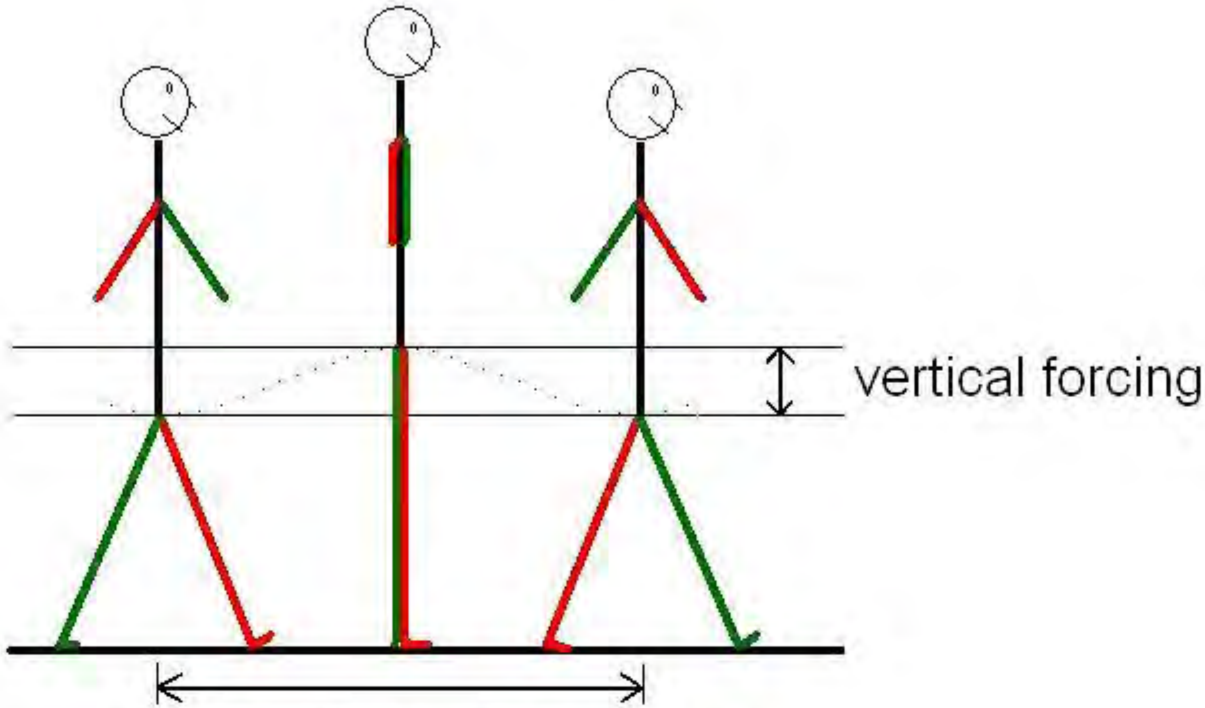
The big instability region corresponds to forcing frequency = 2 X natural frequency

"PARAMETRIC EXCITATION"

2:1 resonance occurs as
we walk and swing our arms.

Arm (as a pendulum)
is being forced vertically
(by leg motion)
at twice its response frequency.

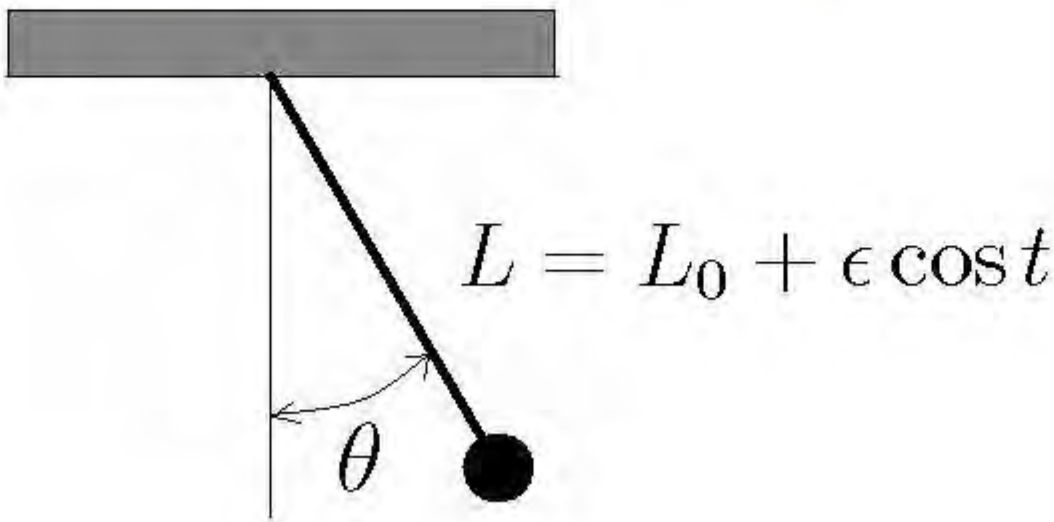
# 2:1 resonance




1 cycle of vertical forcing =
½ cycle of arm-swinging motion

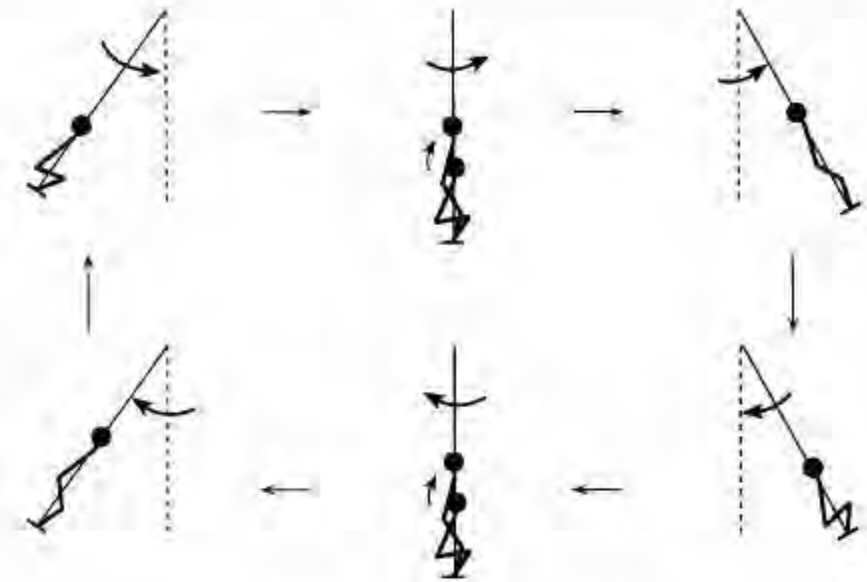
Mathieu's equation governs the pumping of a swing.

Model:
the swing is a pendulum
whose length is periodically changed.



Vol. 29, No. 4, September 1998

# THE COLLEGE MATHEMATICS JOURNAL



Strategy for Pumping a Swing while Standing

***IN THIS ISSUE:***

- *How to Pump a Swing*
- *How Much Money Do You Need for Retirement?*
- *Making Squares from Pythagorean Triples*

$$\frac{d^2x}{dt^2} + (\delta + \epsilon \cos t)\, x = 0$$

$$\frac{d^2x}{dt^2} + (\delta + \epsilon \cos t)\, x = \epsilon \alpha x^3$$

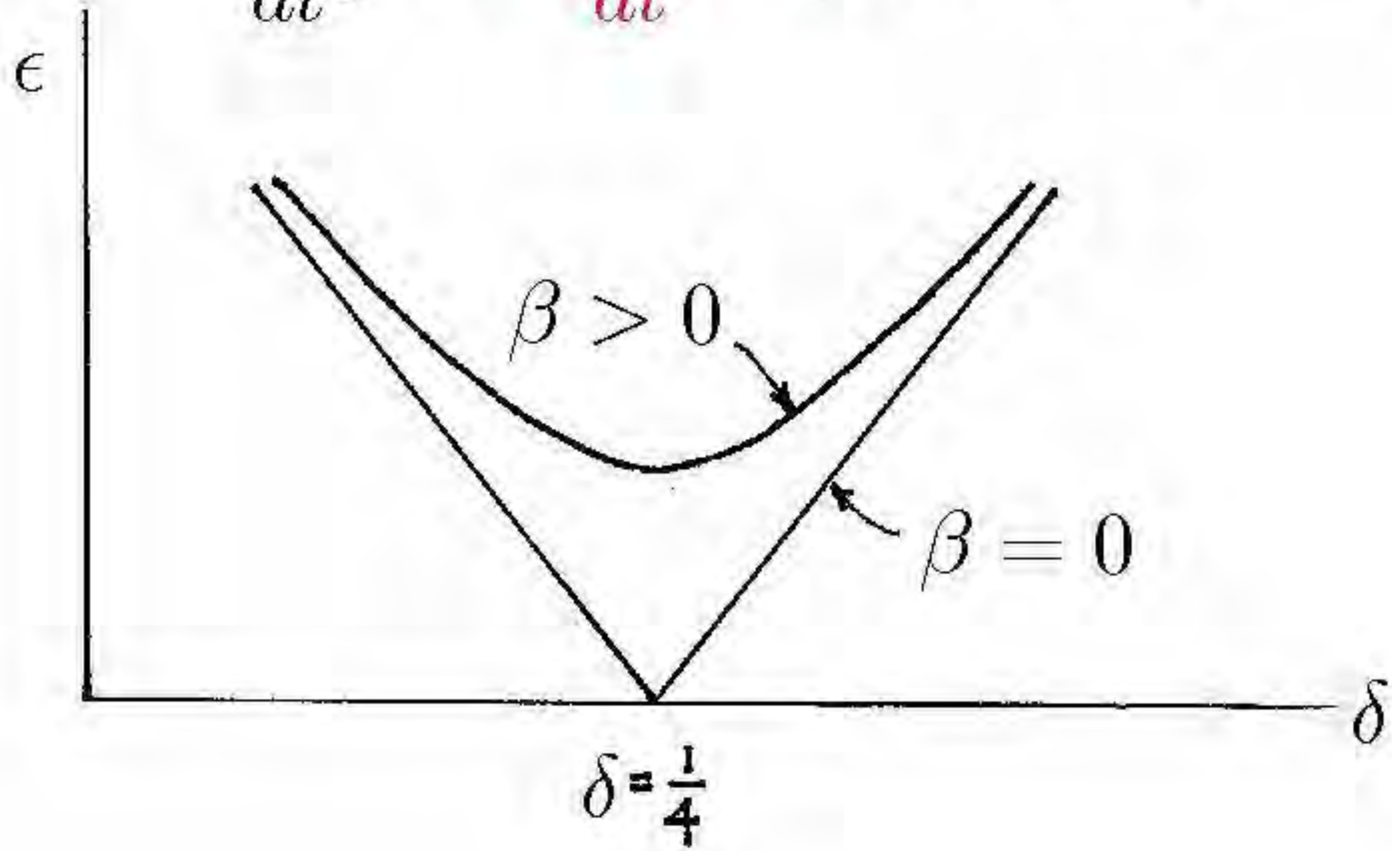
# Why does the nonlinearity prevent unbounded motion?

The nonlinearity produces a relationship between amplitude and frequency.

As the amplitude grows
(due to the resonance),
the frequency changes
and detunes the resonance.

# Effect of Damping

$$\frac{d^2x}{dt^2} + \beta\,\frac{dx}{dt} + (\delta + \epsilon\cos t)\,x = 0$$



# MAE 4770/5770

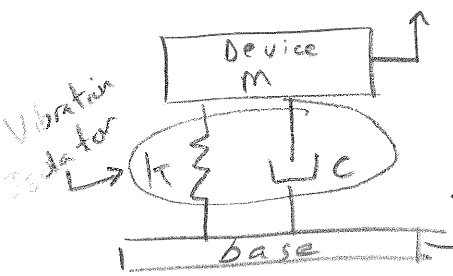
# Passive Vibration Control

- Vibration isolation (See Schaum's 8.1-8.4 or Inman 5.1-5.2)
- Vibration absorbers (Schaum's 8.5-8.6, Inman 5.3-5.4)

## Vibration Isolation

Isolate the source of vibration from the system, or isolate a vibrating system.

## Displacement Transmissibility - Vibrating base/surroundings



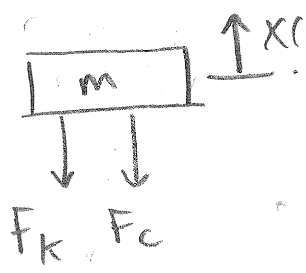

$x(t) = X\sin(\omega_n t - \phi)$

$y(t) = Y\sin(\omega t)$

Goal is to reduce amplitude of displacement of device, $X$

## Base Excitation

Driving system $x(t)$ with a harmonic oscillation $y(t)$




$F_k = k(x-y)$

$F_c = c(\dot{x}-\dot{y})$

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

$$m\ddot{x} + c\dot{x} + kx = cY\omega\cos\omega t + kY\sin\omega t$$

1

Taking $\frac{c}{m} = 2\xi\omega_n$, $\omega_n^2 = \frac{k}{m}$, divide diff Eq by m

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 2\xi\omega_n\omega Y\cos\omega t + \omega_n^2 Y\sin\omega t$$

Two particular solutions, $x_p^{(1)}(t) + x_p^{(2)}(t)$ for mass location

For underdamped case $(0 < \xi < 1)$

$$x_p^{(1)} = \frac{2\xi\omega_n\omega Y}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}}\cos(\omega t - \theta_1) \qquad \theta_1 = \tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

$$x_p^{(2)} = \frac{\omega_n^2 Y}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}}\sin(\omega t - \theta_1)$$

$$x_p(t) = \omega_n Y\left[\frac{\omega_n^2 + (2\xi\omega)^2}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}\right]^{1/2}\cos(\omega t - \theta_1 - \theta_2)$$

where $\theta_2 = \tan^{-1}\left(\frac{\omega_n}{2\xi\omega}\right)$

Defining $r = \omega/\omega_n$ and $x_p = X\cos(\ldots)$

$$X = Y\left[\frac{1 + (2\xi r)^2}{(1-r^2)^2 + (2\xi r)^2}\right]^{1/2} \Rightarrow \boxed{\frac{X}{Y} = \left[\frac{1 + (2\xi r)^2}{(1-r^2)^2 + (2\xi r)^2}\right]^{1/2}}$$

Knowing $F_T$ force transmitted is $= m\ddot{x}(t)$

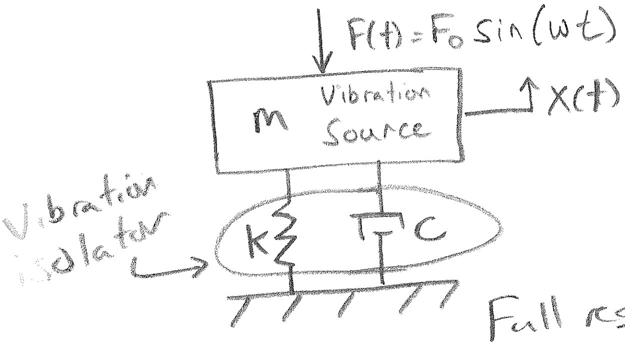
$$F_T = \omega^2 m X$$

$$F_T = \frac{\omega^2}{\omega_n^2}kX = r^2 kX$$

$$\boxed{\frac{F_T}{kY} = r^2\left[\frac{1 + (2\xi r)^2}{(1-r^2)^2 + (2\xi r)^2}\right]^{1/2}}$$

Now we have relationships for both amplitude X and force transmitted as a fn of $\omega/\omega_n$, $\xi$, k, + Y. 2

Force Transmissibility - Isolating a vibrating device from transmitting its vibrations to the surroundings



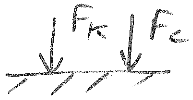

Transmissibility ratio for force

$$TR = \frac{F_T}{F_0} \quad \begin{array}{l} \leftarrow \text{Force transmitted to base} \\ \leftarrow \text{Force input from vibrating device} \end{array}$$

Full response: $x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + X\cos(\omega t - \phi)$

After a period of time of operation of the system, transient response decays to zero and response $x(t)$ modeled by $x(t) = X\cos(\omega t - \phi)$

same as forcing frequency




$\Rightarrow F_k + F_c = F_T(t)$

$$kx + c\dot{x} = F_T(t)$$

combining & rewriting $\dot{x}(t) = -\omega X\sin(\omega t - \phi) = +\omega X\cos(\omega t - \phi + \frac{\pi}{2})$

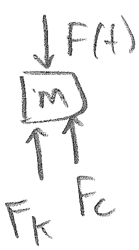
$$F_T(t) = kX\cos(\omega t - \phi) + c\omega X\cos(\omega t - \phi + \frac{\pi}{2})$$

Noting that the two components are $\frac{\pi}{2}$ ($90°$) out of phase, magnitude of $F_T$ calculated as

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = X\sqrt{k^2 + c^2\omega^2}$$

Need to solve for amplitude $X$ knowing $F_0$, $\omega/\omega_n$, $\zeta$, $k$

Force balance




$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0\cos\omega t$$

Solving with $f_0 = \frac{F_0}{m}$ $\quad x = X\cos(\omega t - \phi)$

$$X = \frac{F_0/k}{[(1-r^2)^2 + (2\zeta r)^2]^{1/2}}$$

3

Now, can calculate $\frac{F_T}{F_0}$

$$F_T = \frac{F_0/k}{\left[(1-r^2)^2+(2\xi r)^2\right]^{1/2}}\sqrt{k^2+c^2\omega^2} = F_0 \frac{\sqrt{1+\frac{c^2\omega^2}{k^2}}}{\left[(1-r^2)^2+(2\xi r)^2\right]^{1/2}}$$

$$= F_0\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$$

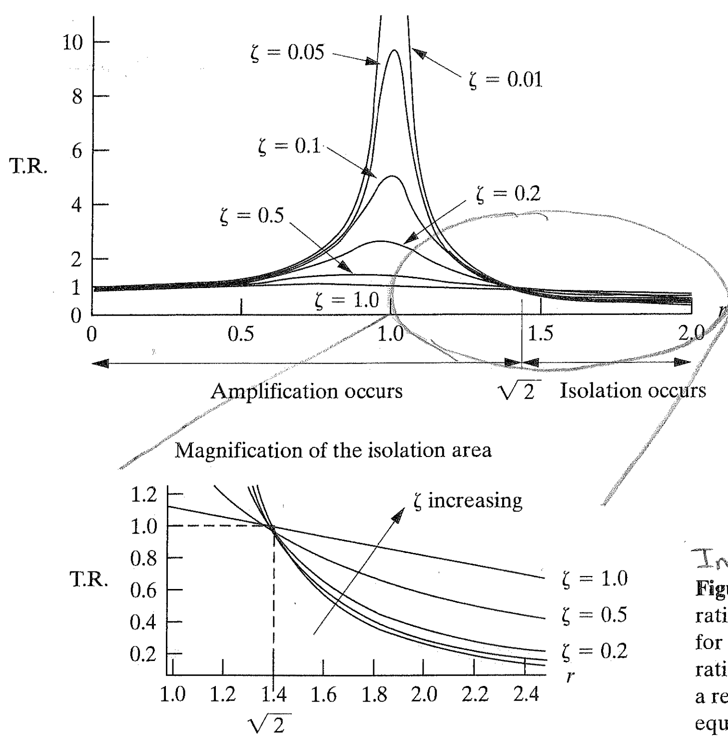
$$\boxed{T.R. = \frac{F_T}{F_0} = \sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}}$$

This is the same as $\frac{X}{Y}$ for base excitation / displacement transmissibility

For Displacement and Force transmissibility,

Amplification occurs for $r < \sqrt{2}$

Isolation occurs for $r > \sqrt{2}$




Inman

**Figure 5.5** Plot of the transmissibility ratio, T.R., indicating the value of T.R. for a variety of choices of the damping ratio $\zeta$ and the frequency ratio $r$. This is a repeat of Figure 2.13 and is a plot of equation (5.7).

Plot of T.R. for various $\xi$, $r$ values

4

As can be seen in Fig 5.5, as $\zeta$ is increased (as $c$ is increased), the TR actually increases in the isolation region ($r > \sqrt{2}$) for a fixed $r$.

However for a system starting up, it must pass through $r=1$ before it reaches its nominal operating $\omega$. At $r=1$, if low damping is present (low $\zeta$), the TR ratio can become quite large, so some damping is required in order to mitigate this.

At a high enough frequency ratio ($r>3$) and low enough damping ($\zeta<0.2$), the TR becomes independent of damping. In this case,

$$TR = \frac{1}{r^2-1} \quad (r>3,\ \zeta<0.2)$$

Reduction in transmissibility $R$ defined as

$$R = 1 - TR$$

Static deflection of spring in vibration isolation system

$$\Delta = \frac{W}{k} = \frac{mg}{k}$$

When designing a system, should choose a clearance between the device and the base/surroundings of at least twice the static deflection $\Delta$ so spring can extend & compress to provide isolation.

5

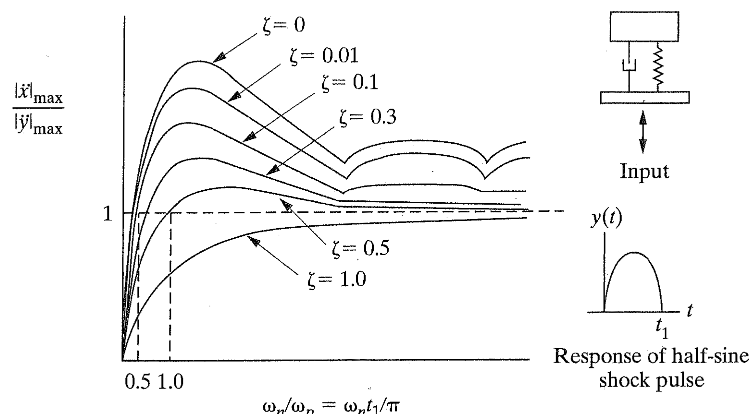
Also of concern is shock isolation, which requires large damping. With shock isolation we are concerned with the transmissibility of acceleration $\frac{|\ddot{x}|_{max}}{|\ddot{y}|_{max}}$ for an input pulse of time length $t_1$.

The input pulse can be modeled as a half-sinusoid input with $\omega_p = \frac{\pi}{t_1}$.

In order to have shock isolation, would need to place system under the $TR = 1.0$ line (seen below)

For $\zeta = 0.5$, need $\frac{\omega_n t_1}{\pi} < 1$, or $k < \frac{m\pi^2}{t_1^2}$

this imposes a bound on the stiffness of the isolator.

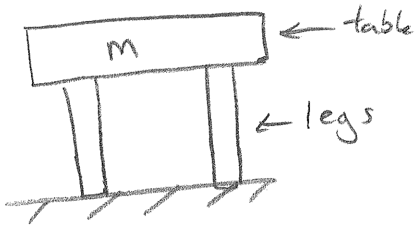



Inman

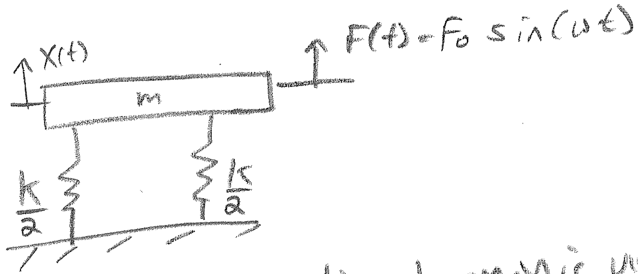
**Figure 5.8** Plot of the ratio of output acceleration magnitude to input acceleration magnitude versus a frequency ratio $(\omega_n/\omega_p)$ for a single-degree-of-freedom system and a base excitation consisting of a shock pulse for different values of damping ratios. Note that a large $\omega_p$ value corresponds to a short pulse width.

6

# Vibration Absorbers

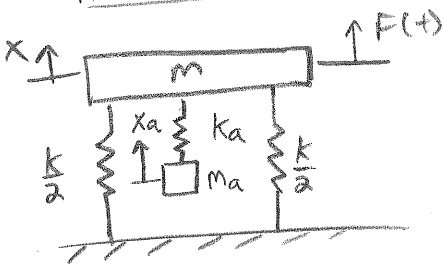



model →




Device generating harmonic vibration
$F(t) = F_0 \sin(\omega t)$

Add a vibration absorber




Design absorber such that the displacement of the primary system (m) is as small as possible.

Write EOM as

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} k+k_a & -k_a \\ -k_a & k_a \end{bmatrix}\begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix}$$

Solve with steady state solution

$$x(t) = \underline{X} \sin \omega t$$
$$x_a(t) = \underline{X}_a \sin \omega t$$

yielding

$$\begin{bmatrix} k+k_a-m\omega^2 & -k_a \\ -k_a & k_a - m_a\omega^2 \end{bmatrix}\begin{bmatrix} \underline{X} \\ \underline{X}_a \end{bmatrix} \sin \omega t = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t$$

Solving for $\begin{bmatrix} \underline{X} \\ \underline{X}_a \end{bmatrix}$ by taking inverse of $A \begin{bmatrix} \underline{X} \\ \underline{X}_a \end{bmatrix} \to A^{-1}$

$$\begin{bmatrix} \underline{X} \\ \underline{X}_a \end{bmatrix} = \frac{1}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2}\begin{bmatrix} (k_a - m_a\omega^2) F_0 \\ k_a F_0 \end{bmatrix}$$

6

We can obtain $X$ of zero by setting

$$\frac{k_a}{m_a} = \omega^2$$

In this case, motion of absorber $x_a(t) = \frac{-F_0}{k_a} \sin \omega t$

So magnitude of absorber motion $|X_a| = \frac{F_0}{k_a}$

This only works for $\frac{k_a}{m_a} = \omega^2$. How about if $\frac{k_a}{m_a}$ near to $\omega^2$?

Define a few ratios & quantities

Mass Ratio $\mu = \frac{m_a}{m}$

$\omega_p = \sqrt{\frac{k}{m}}$ ← Natural freq of system w/o absorber

$\omega_a = \sqrt{\frac{k_a}{m_a}}$ ← Natural freq of absorber w/o system

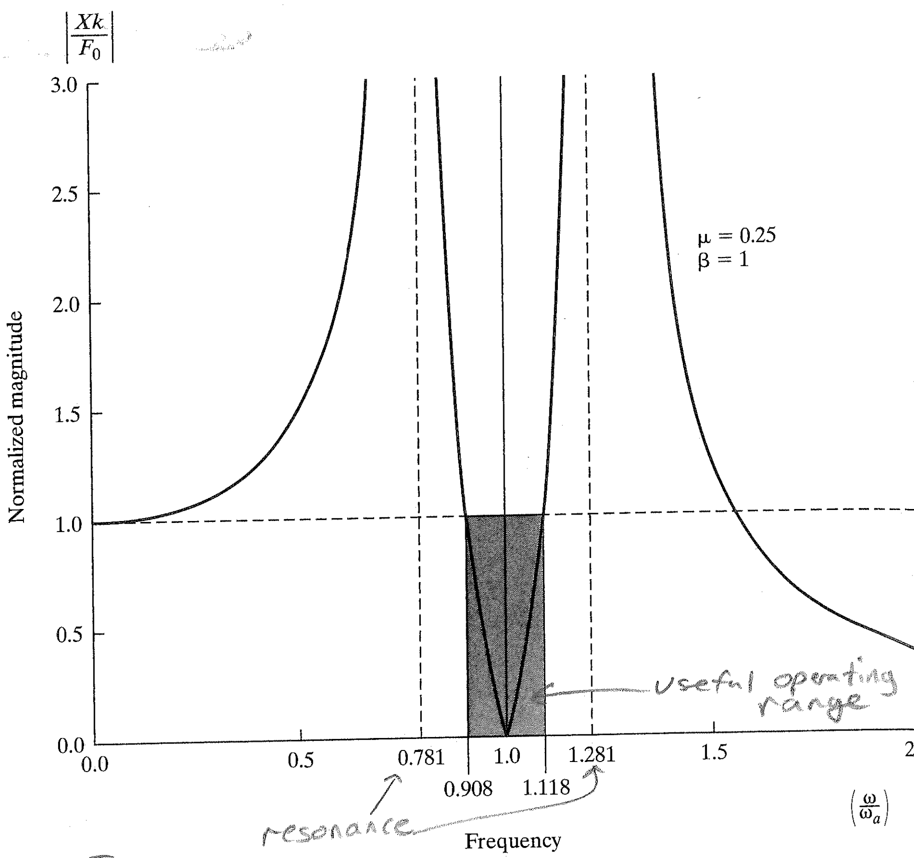
$$\frac{k_a}{k} = \mu \frac{\omega_a^2}{\omega_p^2} = \mu \beta \rightarrow \beta = \frac{\omega_a}{\omega_p} \quad \text{Frequency Ratio}$$

Substitution yields

$$\frac{Xk}{F_0} = \frac{1 - \omega^2/\omega_a^2}{\left[1 + \mu\left(\frac{\omega_a}{\omega_p}\right)^2 - \left(\frac{\omega}{\omega_p}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] - \mu\left(\frac{\omega_a}{\omega_p}\right)^2}$$

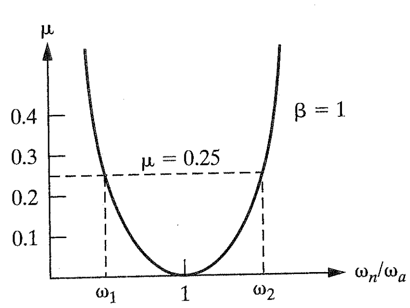
This can be plotted to ~~see~~ how far from $\omega$ that $\omega_a$ can be to still achieve vibration absorption, without amplifying vibrations.

7




Inman

**Figure 5.15** Plot of normalized magnitude of the primary mass versus the normalized driving frequency for the case $\mu = 0.25$. The two natural frequencies of the system occur at 0.781 and 1.281.




Inman

**Figure 5.16** Plot of mass ratio versus system natural frequency (normalized to the frequency of the absorber system), illustrating that increasing the mass ratio increases the useful frequency range of a vibration absorber. Here $\omega_1$ and $\omega_2$ indicate the normalized value of the system's natural frequencies.

As $\mu$ is increased, the range $\omega_1 \to \omega_2$ increases

Too small $\mu \to$ System won't tolerate much deviation in driving freq from $\omega$.

Generally take $\mu = 0.05$ to $0.25$; larger indicates poor design.

8