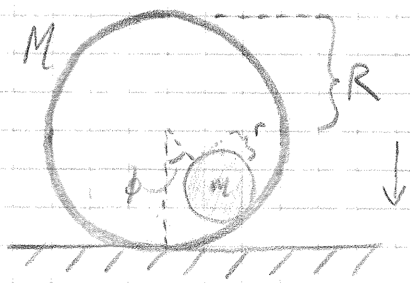
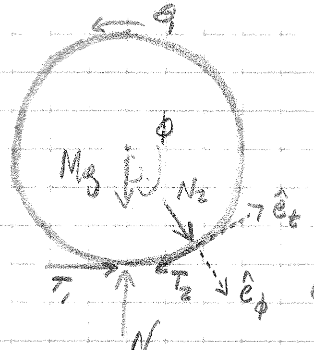
MAE5735 Homeworks

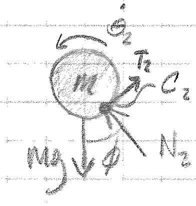
29. A thin-walled cylinder of radius R and mass M rolls without slip on level ground. Inside it rolls (without slip) a disk of radius $r < R$ and mass m. Gravity g acts downward. Find the equations of motion and the modes and frequencies of small amplitude vibrations.

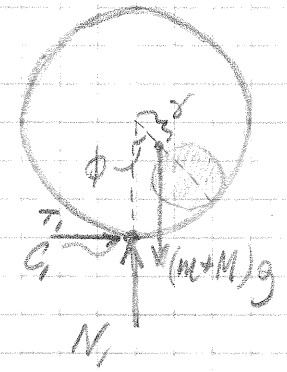




$\hat{e}_t = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{e}_\phi = \sin\phi\,\hat{i} - \cos\phi\,\hat{j}$





Constraint on small mass (no-slip):

$$-r\dot{\theta}_2 = R\dot{\phi} \Rightarrow \text{Let } \theta_2 = \phi \text{ at } t=0\text{, then} \Rightarrow -r\theta_2 = R\phi \;\; \forall t \geq 0$$

Angular momentum balance for m_2 about C_2:

$$\sum \vec{r}_{/C_2} \times \vec{F} = \dot{\vec{H}}_{/C2} \Rightarrow r(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \times (-mg\,\hat{j}) = r(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \times m\ddot{\vec{r}}_2 + I_2\ddot{\theta}_2\hat{k}$$

Find acceleration of large mass:

$$\vec{r}_1 = -R\theta_1\hat{i} + R\hat{j} \Rightarrow \dot{\vec{r}}_1 = -R\dot{\theta}_1\hat{i} \Rightarrow \ddot{\vec{r}}_1 = -R\ddot{\theta}_1\hat{i} \quad \checkmark$$

Find acceleration of small mass:

$$\vec{r}_2 = \vec{r}_1 + (R-r)(\sin\phi\,\hat{i} - \cos\phi\,\hat{j}) \Rightarrow \dot{\vec{r}}_2 = \dot{\vec{r}}_1 + (R-r)(\dot{\phi}\cos\phi\,\hat{i} + \dot{\phi}\sin\phi\,\hat{j}) \Rightarrow \ddot{\vec{r}}_2 = \ddot{\vec{r}}_1 + (R-r)\left((\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)\hat{i} + (\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi)\hat{j}\right)$$

Substitute acceleration for small mass into AMB for small mass:

$$rmg\sin\phi\,\hat{k} = rm\left(R\ddot{\theta}_1\cos\phi + (R-r)(-\ddot{\phi}\cos^2\phi + \dot{\phi}^2\sin\phi\cos\phi - \ddot{\phi}\sin^2\phi - \dot{\phi}^2\cos\phi\sin\phi)\right)\hat{k} + I_2\ddot{\theta}_2\hat{k}$$

29. (continued)

$$\cancel{r}mg\sin\phi = \cancel{r}m\left(R\ddot{\theta}_1\cos\phi - (R-r)\ddot{\phi}\right) + \tfrac{1}{2}mr^2\left(-\tfrac{R}{r}\ddot{\phi}\right)$$

$$\boxed{\left(\tfrac{3}{2}R - r\right)\ddot{\phi} + g\sin\phi = R\ddot{\theta}_1\cos\phi}$$

Angular momentum balance for total system about C_1:

$$\sum \vec{r}_{/C_1}\times\vec{F} = \dot{\vec{H}}_{/C_1} \Rightarrow \gamma(\sin\phi\,\hat{i} - \cos\phi\,\hat{j})\times(-(m+M)g\hat{j}) = \dot{\vec{H}}_{m_1/C_1} + \dot{\vec{H}}_{m_2/C_1}$$

$$-(R-r)mg\sin\phi\hat{k} = (R\hat{j}\times M\ddot{\vec{r}}_1 + I_1\ddot{\theta}_1\hat{k}) + (\vec{r}_{2/C_1}\times m\ddot{\vec{r}}_2 + I_2\ddot{\theta}_2\hat{k})$$

$$= MR^2\ddot{\theta}_1\hat{k} + MR^2\ddot{\theta}_1\hat{k} + ((R-r)\hat{e}_\phi + R\hat{j})\times m((-R\ddot{\theta}_1\hat{i}) + (R-r)(\ddot{\phi}\hat{e}_t - \dot{\phi}^2\hat{e}_\phi)) + \tfrac{1}{2}mr^2\ddot{\theta}_2\hat{k})$$

$$= 2MR^2\ddot{\theta}_1\hat{k} + (-(R-r)mR\ddot{\theta}_1\cos\phi + (R-r)^2m\ddot{\phi} + mR^2\ddot{\theta}_1 + mR(R-r)(-\ddot{\phi}\cos\phi + \dot{\phi}^2\sin\phi)) + \tfrac{1}{2}mr^2\ddot{\theta}_2\hat{k}$$

$$= \ddot{\theta}_1\left(2MR^2 - mR(R-r)\cos\phi + mR^2\right) + \ddot{\phi}\left(m(R-r)^2 - mR(R-r)\cos\phi - \tfrac{1}{2}mrR\right)$$

$$\boxed{\ddot{\theta}_1R^2\left(\frac{2M}{m} - \frac{R-r}{R}\cos\phi + 1\right) + \ddot{\phi}\left((R-r)^2 - R(R-r)\cos\phi - \tfrac{1}{2}rR\right) + (R-r)g\sin\phi = 0}$$

For small-amplitude oscillations: $\cos\phi \approx 1$, $\sin\phi \approx \phi$

$$\left(\tfrac{3}{2}R - r\right)\ddot{\phi} + g\phi = R\ddot{\theta}_1$$

$R^2 - 2rR + r^2 - R^2 + Rr - \frac{1}{2}rR$

$$\ddot{\theta}_1R^2\left(\frac{2M}{m} - \frac{R-r}{R} + 1\right) + \ddot{\phi}\left((R-r)^2 - R(R-r) - \tfrac{1}{2}rR\right) + (R-r)g\phi = 0$$

$$\ddot{\theta}_1R^2\left(\frac{2M}{m} + \frac{r}{R}\right) + \ddot{\phi}r^2\left(1 - \frac{3}{2}\frac{R}{r}\right) + (R-r)g\phi = 0$$

$$R\left(\left(\tfrac{3}{2}R - r\right)\ddot{\phi} + g\phi\right)\left(\frac{2M}{m} + \frac{r}{R}\right) + \ddot{\phi}r^2\left(1 - \frac{3}{2}\frac{R}{r}\right) + (R-r)g\phi = 0$$