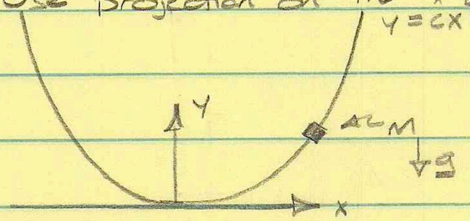
(30 min) 22a. A point-mass sliding along a parabolic wire curve of the form $y = cx^2$. derive equations of motion using Lagrange eq. Use projection on the x-axis as the generalized coordinate.



$\dot{y} = 2c\dot{x}x$

L=T−U where T is total kinetic energy and U is total potential energy

$v = \sqrt{\dot{x}^2 + \dot{y}^2}$

$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right) - \frac{\delta L}{\delta q} = 0 \qquad T = \frac{1}{2}mv^2 = \frac{1}{2}m\sqrt{\dot{x}^2+\dot{y}^2}^{\,2} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$U = mgy = mgcx^2$

$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}mgcx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(2c\dot{x}x)^2 - mgcx^2$

$= \frac{1}{2}m\dot{x}^2 + 2mc^2\dot{x}^2x^2 - mgcx^2 = \dot{x}^2\left(\frac{1}{2}m + 2mc^2x^2\right) - mgcx^2$

$\frac{\delta L}{\delta \dot{x}} = 2\dot{x}\left(\frac{1}{2}m + 2mc^2x^2\right) = \dot{x}m(1 + 4c^2x^2)$

$\frac{\delta L}{\delta x} = 4mc^2\dot{x}^2x - 2mgcx$

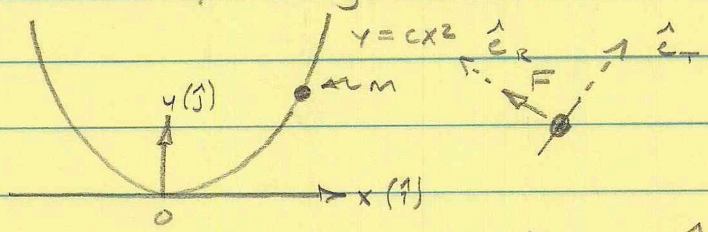
$\frac{d}{dt}\left(\dot{x}m(1 + 4c^2x^2)\right) - 4mc^2\dot{x}^2x + 2mgcx = 0$

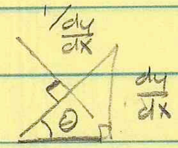
$\ddot{x}m(1 + 4c^2x^2) + \dot{x}m(8c^2\dot{x}x) - 4mc^2\dot{x}^2x + 2mgcx = 0$

$\ddot{x}\cancel{m}(1 + 4c^2x^2) + 8\cancel{m}c^2x\dot{x}^2 - 4\cancel{m}c^2\dot{x}^2x + 2\cancel{m}gcx = 0$

$\boxed{\ddot{x}(1 + 4c^2x^2) + 4c^2x\dot{x}^2 + 2gcx = 0}$ ✓

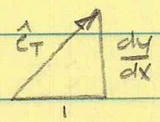
(45 min) 22b. Same setup using Newton's laws.



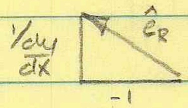


$\tan\theta = \frac{dy}{dx}$

$\theta = \tan^{-1}\left(\frac{dy}{dx}\right)$



$$\hat{e}_T = \frac{\hat{i} + \frac{dy}{dx}\hat{j}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$



$$\hat{e}_R = \frac{-\hat{i} + \frac{1}{dy/dx}\hat{j}}{\sqrt{1+\left(\frac{1}{dy/dx}\right)^2}}$$

Linear momentum balance

$$\sum \vec{F} = m\vec{a} \qquad \frac{dy}{dx} = 2cx$$

$$\hat{e}_T \cdot \left(-mg\hat{j} + F\hat{e}_R = m\vec{a}\right)$$

$$-mg\hat{j}\cdot\hat{e}_T = m\vec{a}\cdot\hat{e}_T = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})\cdot\hat{e}_T$$

$$\frac{-2mgcx}{\cancel{\sqrt{1+(2cx\dot{x})^2}}} = \frac{m\ddot{x}}{\cancel{\sqrt{1+(2cx\dot{x})^2}}} + \frac{m\ddot{y}\cdot 2cx}{\cancel{\sqrt{1+(2cx\dot{x})^2}}}$$

$$-2\cancel{m}gcx = \cancel{m}\ddot{x} + 2\cancel{m}cx\,\ddot{y} \qquad \ddot{y} = \frac{d}{dt}(\dot{y}) = \frac{d}{dt}(2cx\dot{x})$$

$$-2gcx = \ddot{x} + 2cx\left[2c(\dot{x}^2 + \ddot{x}x)\right] \qquad = 2c(\dot{x}^2 + \ddot{x}x)$$

$$-2gcx = \ddot{x} + 2cx\left[2c\dot{x}^2 + 2cx\ddot{x}\right] = \ddot{x} + 4c^2x\dot{x}^2 + 4c^2x^2\ddot{x}$$

$$-2gcx = \ddot{x}(1 + 4c^2x^2) + 4c^2x\dot{x}^2$$

$$\boxed{\ddot{x}(1+4c^2x^2) + 4c^2x\dot{x}^2 + 2gcx = 0}$$

✓

(5 min) 22c. Find frequency for small vibrations in terms of m, g, c.

Assume that all non-linear terms go to zero for small vibrations

$$\ddot{x}(1 + \underbrace{\cancel{4c^2x^2}}_{0}) + \underbrace{\cancel{4cx\dot{x}^2}}_{0} + 2gcx = 0 \longrightarrow \ddot{x} + 2gcx = 0$$

✓

$$\longrightarrow \omega_n = \sqrt{2gc}$$