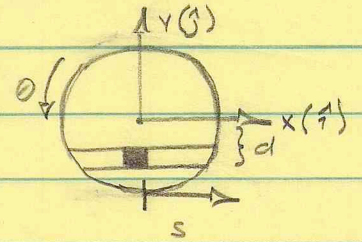
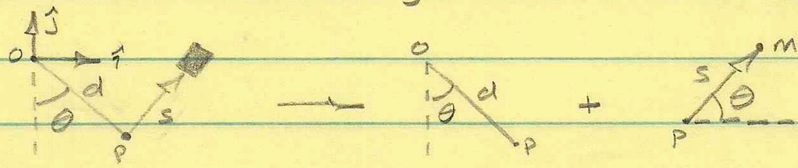
8 hrs

20 Find acceleration of a mass in a slot on a rigid turntable. (m_T, I_T)



30 min 9.1 Write position of mass in terms of d, θ, s using base base vectors $\hat i$ and $\hat j$. Differentiate twice.



$\vec r_{M/O} = \vec r_{P/O} + \vec r_{M/P}$ where $\vec r_{P/O} = d\sin\theta\,\hat i - d\cos\theta\,\hat j$

$\vec r_{M/P} = s\cos\theta\,\hat i + s\sin\theta\,\hat j$

$$\vec r_{M/O} = [d\sin\theta\,\hat i - d\cos\theta\,\hat j] + [s\cos\theta\,\hat i + s\sin\theta\,\hat j]$$
$$= (d\sin\theta + s\cos\theta)\hat i + (s\sin\theta - d\cos\theta)\hat j$$

$$\frac{d}{dt}(\vec r_{M/O}) = (\dot\theta d\cos\theta + \dot s\cos\theta - \dot\theta s\sin\theta)\hat i + (\dot s\sin\theta + \dot\theta s\cos\theta + \dot\theta d\sin\theta)\hat j$$

$$\frac{d}{dt}(\vec v_{M/O}) = [(\ddot\theta d\cos\theta - \dot\theta^2 d\sin\theta) + (\ddot s\cos\theta - \dot\theta\dot s\sin\theta) - (\ddot\theta s\sin\theta + \dot\theta\frac{d}{dt}(s\sin\theta))]\hat i + [(\ddot s\sin\theta + \dot\theta\dot s\cos\theta) + (\ddot\theta s\cos\theta \; \dot\theta\frac{d}{dt}(s\cos\theta)) + (\ddot\theta d\sin\theta + \dot\theta^2 d\cos\theta)]\hat j$$

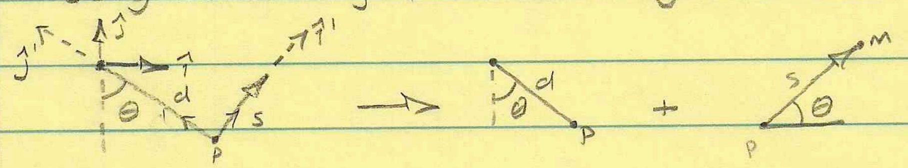
where $\frac{d}{dt}(s\sin\theta) = \dot s\sin\theta + \dot\theta s\cos\theta$

$\frac{d}{dt}(s\cos\theta) = \dot s\cos\theta - \dot\theta s\sin\theta$

$$\frac{d}{dt}(\vec{v}_{M/O}) = [\ddot\theta d\cos\theta - \dot\theta^2 d\sin\theta + \ddot s\cos\theta - \dot\theta\dot s\sin\theta - \ddot\theta s\sin\theta$$
$$- \dot\theta\dot s\sin\theta - \dot\theta^2 s\cos\theta]\hat{\imath}$$
$$+ [\ddot s\sin\theta + \dot\theta\dot s\cos\theta + \ddot\theta s\cos\theta + \dot\theta\dot s\cos\theta - \dot\theta^2 s\sin\theta$$
$$+ \ddot\theta d\sin\theta + \dot\theta^2 d\cos\theta]\hat{\jmath}$$

$$\boxed{\begin{aligned}\vec{a}_{M/O} = {} & [\ddot\theta(d\cos\theta - s\sin\theta) - \dot\theta^2(d\sin\theta + s\cos\theta) - 2\dot\theta\dot s\sin\theta \\ & + \ddot s\cos\theta]\hat{\imath} \\ & + [\ddot\theta(s\cos\theta + d\sin\theta) + \dot\theta^2(d\cos\theta - s\sin\theta) + 2\dot\theta\dot s\cos\theta \\ & + \ddot s\sin\theta]\hat{\jmath}\end{aligned}}$$

(75 min) ii. Using $\hat{\imath}'$ and $\hat{\jmath}'$, which align with the slot. Differentiate twice



$\vec{r}_{M/O} = \vec{r}_{P/O} + \vec{r}_{M/P}$ where $\vec{r}_{P/O} = d\hat{\jmath}'$ and $\vec{r}_{M/P} = s\hat{\imath}$

$$= -d\hat{\jmath}' + s\hat{\imath}'$$

$$\frac{d}{dt}(\vec{r}_{M/O}) = -d\dot{\hat{\jmath}}' + \dot s\hat{\imath}' + s\dot{\hat{\imath}}'$$

$$\frac{d}{dt}(\vec{v}_{M/O}) = -d\ddot{\hat{\jmath}}' + \ddot s\hat{\imath}' + \dot s\dot{\hat{\imath}}' + \dot s\dot{\hat{\imath}}' + s\ddot{\hat{\imath}}' = -d\ddot{\hat{\jmath}}' + \ddot s\hat{\imath}' + 2\dot s\dot{\hat{\imath}}' + s\ddot{\hat{\imath}}'$$

$$\vec{a}_{M/O} = -d\cdot\frac{d}{dt}(-\vec\omega\times\hat{\jmath}') + \ddot s\hat{\imath}' + 2\dot s\omega\hat{\jmath}' + s\frac{d}{dt}(\vec\omega\times\hat{\imath}')$$
$$= -d[-\dot{\vec\omega}\times\hat{\jmath}' - \vec\omega\times\dot{\hat{\jmath}}'] + \ddot s\hat{\imath}' + 2\dot s\dot\theta\hat{\jmath}' + s[\dot{\vec\omega}\times\hat{\imath}' + \vec\omega\times\dot{\hat{\imath}}']$$
$$= -d[-\ddot\theta\times\hat{\jmath}' - \vec\omega\times(-\omega\hat{\imath}')] + \ddot s\hat{\imath}' + 2\dot s\dot\theta\hat{\jmath}' + s[\ddot\theta\times\hat{\imath}' + \vec\omega\times(-\omega\hat{\jmath}')]$$
$$= -d[-\ddot\theta\hat{\imath}' - \dot\theta^2\hat{\jmath}'] + \ddot s\hat{\imath}' + 2\dot s\dot\theta\hat{\jmath}' + s[\ddot\theta\hat{\jmath}' - \dot\theta^2\hat{\imath}']$$

where $\hat{\imath}' = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$, $\hat{\jmath}' = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

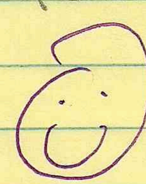
$$\vec{a}_{M/O} = -d\left[-\ddot{\theta}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) - \dot{\theta}^2(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\right]$$
$$+ \ddot{s}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + 2\dot{s}\dot{\theta}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$
$$+ s\left[\ddot{\theta}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) - \dot{\theta}^2(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\right]$$

?

$$= \left[\ddot{\theta}(d\cos\theta - s\sin\theta) - \dot{\theta}^2(s\cos\theta - d\sin\theta) - 2\dot{s}\dot{\theta}(\sin\theta) + \ddot{s}\cos\theta\right]\hat{i}$$
$$+ \left[\ddot{\theta}(d\sin\theta + s\cos\theta) + \dot{\theta}^2(d\cos\theta - s\sin\theta) + 2\dot{s}\dot{\theta}\cos\theta + \ddot{s}\sin\theta\right]\hat{j}$$

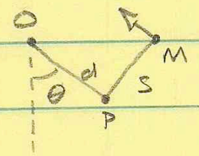
iii. Using the five term acceleration equation.

$$\vec{a}_P = \vec{a}_{O'} + \vec{a}_{rel} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/O'}) + \dot{\vec{\omega}}\times\vec{r}_{P/O} + 2\vec{\omega}\times\vec{v}_{rel}$$
$$\vec{a}_{M/O} = \vec{a}_P + \vec{a}_{rel} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{M/P}) + \dot{\vec{\omega}}\times\vec{r}_{M/O} + 2\vec{\omega}\times\vec{v}_{rel}$$
$$= \vec{a}_P + \ddot{s}\,\hat{i}' + \vec{\omega}\times[\vec{\omega}\times -d\,\hat{j}'] + \dot{\vec{\omega}}\times(-d\,\hat{j}' + s\,\hat{i}') + 2\vec{\omega}\times\dot{s}\,\hat{i}'$$
$$= \vec{a}_P + \ddot{s}\,\hat{i}' + \dot{\theta}\hat{k}\times[-\dot{\theta}d\,\hat{i}'] + \ddot{\theta}(-d\,\hat{i}' + s\,\hat{j}') + 2\dot{\theta}\dot{s}\,\hat{j}'$$
$$= \cancel{\vec{a}_P} + \ddot{s}\,\hat{i}' - \dot{\theta}^2 d\,\hat{j}' + \ddot{\theta}(-d\,\hat{i}' + s\,\hat{j}') + 2\dot{\theta}\dot{s}\,\hat{j}'$$
$$= \ddot{s}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) - \dot{\theta}^2 d(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$
$$+ \ddot{\theta}\left[-d(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + s(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\right] + 2\dot{\theta}\dot{s}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

Trivial algebra conversion to reduced form 

(30min) 20b. Find equations of motion for $\ddot{\theta}, \ddot{s}$ given fixed parameters

$$\sum \vec{M}_o = \dot{\vec{H}}_o = \sum \vec{F}_o \times \vec{a}_o M + I_T \dot{\omega}_S \hat{k} \quad \text{where } \vec{a}_o = 0$$



$$\vec{F}_{M/o} \times \vec{F}_M = I_T \ddot{\theta} \hat{k} \quad \text{where } \vec{F}_M = m\vec{a}_M \cdot \hat{j}'$$

$$\vec{F}_{M/o} \times m(\vec{a}_M \cdot \hat{j}') = I_T \ddot{\theta} \hat{k}$$

$$(d\hat{j}' + s\hat{i}') \times m(\vec{a}_M \cdot \hat{j}') = I_T \ddot{\theta} \hat{k} \quad \text{where } \vec{a}_M = -d\ddot{\hat{j}}' + \ddot{s}\hat{i}' + 2\dot{s}\dot{\hat{i}}' + s\ddot{\hat{i}}'$$

$$(d\hat{j}' + s\hat{i}') \times m[(-d\ddot{\hat{j}}' + \ddot{s}\hat{i}' + 2\dot{s}\dot{\hat{i}}' + s\ddot{\hat{i}}') \cdot \hat{j}'] = I_T \ddot{\theta} \hat{k}$$

where $\ddot{\hat{j}}' = \frac{d}{dt}(-\omega \times \hat{j}') = -\dot{\omega} \times \hat{j}' - \dot{\hat{j}} \times \omega = -\dot{\omega} \times \hat{j}' - \omega \times (-\omega \hat{i}')$

$$(d\hat{j}' + s\hat{i}') \times m[(-d(-\dot{\omega} \times \hat{j}' - \omega \times (-\omega\hat{i}')) + \ddot{s}\hat{i}' + 2\dot{s}\dot{\hat{i}}' + s\ddot{\hat{i}}'$$

Using previous equation for acceleration with $\hat{i}'$ and $\hat{j}'$

$$\vec{F}_{M/o} \times m[-d\dot{\theta}^2\hat{j}' + 2\dot{s}\dot{\theta}\hat{j}' + s\ddot{\theta}\hat{j}'] = I_T \ddot{\theta} \hat{k}$$

$$(d\hat{j}' + s\hat{i}') \times m[d\dot{\theta}^2 + 2\dot{s}\dot{\theta} + s\ddot{\theta}]\hat{j}' = I_T \ddot{\theta} \hat{k}$$

$$ms(d\dot{\theta}^2 + 2\dot{s}\dot{\theta} + s\ddot{\theta}) = I_T \ddot{\theta}$$

$$-\dot{\theta}^2 msd - 2\dot{\theta} ms\dot{s} - \ddot{\theta} ms^2 = I_T \ddot{\theta}$$

$$\ddot{\theta}(I_T + ms^2) = \dot{\theta}^2 msd + 2\dot{\theta} ms\dot{s}$$

$$\boxed{\ddot{\theta} = \frac{-2\dot{\theta} ms\dot{s} - \dot{\theta}^2 msd}{I_T + ms^2}}$$

Linear momentum balance for mass in slot:

$$\dot{\vec{L}} = m\vec{a} = \vec{F} \quad \text{where } \vec{F} \cdot \hat{i}' = 0$$

$$m\vec{a} \cdot \hat{i}' = 0 = [-\dot{\theta}^2 s\hat{i}' + \ddot{\theta} s\hat{j}' + 2\dot{s}\dot{\theta}\hat{j}' + \ddot{s}\hat{i}' + \dot{\theta}^2 d\hat{j}' + \ddot{\theta} d\hat{i}'] \cdot \hat{i}'$$

$$0 = -\dot{\theta}^2 s + \ddot{s} + \ddot{\theta} d$$

$$\ddot{s} = \dot{\theta}^2 s - \ddot{\theta} d \longrightarrow \boxed{\ddot{s} = \dot{\theta}^2 s - d\left[\frac{-2\dot{\theta} ms\dot{s} - \dot{\theta}^2 msd}{I_T - ms^2}\right]}$$

20c. Given I.C.s of $s(0)=\theta(0)=0$, $\dot{s}(0)=v_0$, $\dot{\theta}(0)=\omega_0$

Finding angular acceleration at time t infinitessimally after $t=0$

$$\ddot{\theta} = \frac{-2\omega_0 m v_0(t) - \omega_0^2 m d(t)}{I_T + m(t)^2}$$

This results in a negative angular acceleration term, which means that as soon as the mass passes the radial line, it will begin to spin slower. To rationalize, because angular momentum must stay constant, as the mass goes away from center the rotational speed must decrease to account for mass's increased polar moment.

So $\theta \to \infty$? e.g. $\dot{\theta} \sim 1/t$

or $\theta \to$ const ? e.g. $\dot{\theta} \sim 1/t^2$