15) what means "rate of change of angular momentum" for a SYSTEM of particles?

For which of these definitions of $\vec{H}_{/c}$ is the following equation of motion true: $\vec{M}_c = \dot{\vec{H}}_{/c}$? $\swarrow \vec{H}_{/c}$

a) $\vec{H}_{/c} = \sum \vec{r}_{i/c'} \times \vec{v}_{i/c'} m_i$

where c' is a point fixed in $\mathcal{F}$ that instantaneously coincides with c

First, we know that $\vec{v}_{i/o} = \vec{v}_{c'/o} + \vec{v}_{i/c'}$

$$\Rightarrow * \quad \vec{v}_{i/c'} = \vec{v}_{i/o} - \vec{v}_{c'/o} \quad *$$

Subbing this into our expression for $\vec{H}_{/c}$:

$$\vec{H}_{/c} = \sum \vec{r}_{i/c'} \times (\vec{v}_{i/o} - \vec{v}_{c'/o}) m_i$$

Since c' is a fixed point in $\mathcal{F}$, $\vec{v}_{c'/o} = \vec{0}$

15) a) (continued)

$$\Rightarrow \quad \vec{H}_{/c} = \Sigma\, \vec{r}_{i/c'} \times (\vec{v}_{i/o} - \vec{0})\, m_i$$

$$\vec{H}_{/c} = \Sigma\, \vec{r}_{i/c'} \times \vec{v}_{i/o}\, m_i$$

$$\Rightarrow \quad \dot{\vec{H}}_{/c} = \Sigma \left[\left(\dot{\vec{r}}_{i/c'} \times \vec{v}_{i/o}\, m_i \right) + \left(\vec{r}_{i/c'} \times \dot{\vec{v}}_{i/o}\, m_i \right) \right]$$

$$\dot{\vec{H}}_{/c} = \Sigma \left[\left(\vec{v}_{i/c'} \times \vec{v}_{i/o}\, m_i \right) + \left(\vec{r}_{i/c'} \times \vec{a}_{i/o}\, m_i \right) \right]$$

Since c' is fixed in $\mathcal{F}$ and the origin o is also fixed, the velocity of particle i with respect to c' and o will be the same

$$\Rightarrow \quad \vec{v}_{i/c'} = \vec{v}_{i/o}$$

0 since a vector crossed w/ itself is 0

$$\Rightarrow \quad \dot{\vec{H}}_{/c} = \Sigma \left[\cancel{\left(\vec{v}_{i/o} \times \vec{v}_{i/o}\, m_i \right)} + \left(\vec{r}_{i/c'} \times \vec{a}_{i/o}\, m_i \right) \right]$$

$$\dot{\vec{H}}_{/c} = \Sigma\, \vec{r}_{i/c'} \times \vec{a}_{i/o}\, m_i$$

15) a) (continued)

Finally, since C' instantaneously coincides with C we have

$$\vec{r}_{i/C'} = \vec{r}_{i/C}$$

$$\Rightarrow \boxed{\dot{\vec{H}}_{/C} = \sum \vec{r}_{i/C} \times \vec{a}_{i/O}\, m_i}$$

✓+

It has been shown that this definition works in general

15) b)

$$\vec{H}_{/c} = \sum \vec{r}_{i/c} \times \vec{v}_{i/o}\, m_i$$

$$\Rightarrow \dot{\vec{H}}_{/c} = \sum \left[(\dot{\vec{r}}_{i/c} \times \vec{v}_{i/o}\, m_i) + (\vec{r}_{i/c} \times \dot{\vec{v}}_{i/o}\, m_i) \right]$$

$$\boxed{\dot{\vec{H}}_{/c} = \sum \left[(\vec{v}_{i/c} \times \vec{v}_{i/o}\, m_i) + (\vec{r}_{i/c} \times \vec{a}_{i/o}\, m_i) \right]}$$

This expressions collapses to $\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o}\, m_i$ for some special cases :

1) <u>If C is stationary</u> ✓

If C is stationary, then we know that $\vec{v}_{i/c} = \vec{v}_{i/o}$ since the origin O is also stationary

$$\Rightarrow \dot{\vec{H}}_{/c} = \sum \left[(\underbrace{\vec{v}_{i/o} \times \vec{v}_{i/o}}_{0}\, m_i) + (\vec{r}_{i/c} \times \vec{a}_{i/o}\, m_i) \right]$$

$$\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o}\, m_i$$ ✓

15) b) (continued)

2) If all particles i are moving away from point C in the same or opposite direction that they are moving away from point O

$\Rightarrow \quad \vec{v}_{i/C} \times \vec{v}_{i/O} = 0$

$\Rightarrow$ The point C is the origin, O

$$\dot{\vec{H}}_{/C} = \Sigma\left[\left(\vec{v}_{i/C} \times \vec{v}_{i/O}\, m_i\right) + \left(\vec{r}_{i/C} \times \vec{a}_{i/O}\, m_i\right)\right]$$

$$\dot{\vec{H}}_{/C} = \Sigma\left[\left(\cancel{\vec{v}_{i/O} \times \vec{v}_{i/O}\, m_i}^{\,0}\right) + \left(\vec{r}_{i/C} \times \vec{a}_{i/O}\, m_i\right)\right]$$

$$\dot{\vec{H}}_{/C} = \Sigma\, \vec{r}_{i/C} \times \vec{a}_{i/O}\, m_i$$

15) b) (continued)

3) All of the systems particles i are "stuck" to the point C

$\Rightarrow \vec{v}_{i/c} = 0$

$$\dot{\vec{H}}_{/c} = \sum\left[\left(\cancel{\vec{v}_{i/c}}^{\,0} \times \vec{v}_{i/o}\, m_i\right) + \left(\vec{r}_{i/c} \times \vec{a}_{i/o}\, m_i\right)\right]$$

$$\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o}\, m_i \quad \checkmark$$

$\Rightarrow$ This definition works for some special cases concerning the motions of the particles i and point C that I have outlined previously

15) c)

$$\vec{H}_{/C} = \sum \vec{r}_{i/C} \times \vec{v}_{i/C}\, m_i$$

$$\Rightarrow \dot{\vec{H}}_{/C} = \sum \left[\left(\dot{\vec{r}}_{i/C} \times \vec{v}_{i/C}\, m_i \right) + \left(\vec{r}_{i/C} \times \dot{\vec{v}}_{i/C}\, m_i \right) \right]$$

$$\dot{\vec{H}}_{/C} = \sum \left[\underbrace{\left(\vec{v}_{i/C} \times \vec{v}_{i/C}\, m_i \right)}_{0} + \left(\vec{r}_{i/C} \times \vec{a}_{i/C}\, m_i \right) \right]$$

$$\boxed{\dot{\vec{H}}_{/C} = \sum \vec{r}_{i/C} \times \vec{a}_{i/C}\, m_i}$$

The expression collapses to $\dot{\vec{H}}_{/C} = \sum \vec{r}_{i/C} \times \vec{a}_{i/O}\, m_i$ for the following cases:

1) <u>C is stationary</u> Can just refer to (a)

If C is fixed, the the velocity of particles i with respect to C will be the same as the velocity of particles with respect to the origin O (since O is also fixed)

$$\Rightarrow \quad \vec{v}_{i/C} = \vec{v}_{i/O}$$

15) c) (continued)

Taking a derivative yields that

* $\vec{a}_{i/c} = \vec{a}_{i/o}$ *

Thus, we have

$$\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/c} m_i$$

$$\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o} m_i \quad \checkmark$$

2) C is the origin, O AND fixed! (not just instantaneously!)

$\Rightarrow \vec{a}_{i/c} = \vec{a}_{i/o}$ and we get

$$\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/c} m_i$$

$$\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o} m_i \quad \checkmark$$

So this is a special case of (2) above. (Not needed)

15) c) (continued)

3) C is the center of mass (COM)

We know that $\sum \vec{F}^{ext} = \sum m_i \vec{a}_{G/O}$

Thus, if the sum of the external forces on this system is zero (a valid assumption) we get that

$* \quad \vec{a}_{G/O} = 0 \quad *$ since $\sum m_i$ will not feasibly be equal to 0

Also, we have that

$$\vec{r}_{i/O} = \vec{r}_{G/O} + \vec{r}_{i/G}$$

Differentiating twice with respect to time we get

$$\vec{a}_{i/O} = \cancelto{0}{\vec{a}_{G/O}} + \vec{a}_{i/G}$$

$$* \quad \vec{a}_{i/O} = \vec{a}_{i/G} \quad *$$

15) c) (continued)

Thus, we had

$$\dot{\vec{H}}_{/C} = \sum \vec{r}_{i/C} \times \vec{a}_{i/C}\, m_i$$

$$\dot{\vec{H}}_{/C} = \sum \vec{r}_{i/C} \times \vec{a}_{i/G}\, m_i \qquad (\text{if } C = G)$$

Using the result that $\vec{a}_{i/O} = \vec{a}_{i/G}$ gives

$$\dot{\vec{H}}_{/C} = \sum \vec{r}_{i/C} \times \vec{a}_{i/O}\, m_i \quad \checkmark$$

$\Rightarrow$ This definition works for some special cases concerning the motions of particles i and point C which have been previously outlined

NE MORE

Also if $\vec{a}_C$ is parallel to $\vec{r}_{G/C}$.

[then $\vec{r}_{G/C} \times \vec{a}_G = \vec{0}$]

is one something lets people get the right answer when they don't deserve to.