

15) What means "rate of change of angular momentum" for a SYSTEM of particles?

For which of these definitions of $\vec{\dot{H}}_{ic}$ is the following equation of motion true: $\vec{M}_c = \vec{\dot{H}}_{ic}$?

a) $\vec{\dot{H}}_{ic} = \sum \vec{r}_{i/c'} \times \vec{v}_{i/c'} m_i$

where C' is a point fixed in \mathcal{C} that instantaneously coincides with C

First, we know that $\vec{v}_{i/o} = \vec{v}_{c'/o} + \vec{v}_{i/c'}$

$$\Rightarrow * \quad \vec{v}_{i/c'} = \vec{v}_{i/o} - \vec{v}_{c'/o} *$$

Subbing this into our expression for $\vec{\dot{H}}_{ic}$:

$$\vec{\dot{H}}_{ic} = \sum \vec{r}_{i/c'} \times (\vec{v}_{i/o} - \vec{v}_{c'/o}) m_i$$

Since C' is a fixed point in \mathcal{C} , $\vec{v}_{c'/o} = \vec{0}$

15) a) (continued)

$$\Rightarrow \vec{H}_{ic} = \sum \vec{r}_{i/c'} \times (\vec{v}_{i/o} - \vec{o}) m_i$$

$$\vec{H}_{ic} = \sum \vec{r}_{i/c'} \times \vec{v}_{i/o} m_i$$

$$\Rightarrow \dot{\vec{H}}_{ic} = \sum [(\vec{r}_{i/c'} \times \vec{v}_{i/o} m_i) + (\vec{r}_{i/c'} \times \dot{\vec{v}}_{i/o} m_i)]$$

$$\dot{\vec{H}}_{ic} = \sum [(\vec{v}_{i/c'} \times \vec{v}_{i/o} m_i) + (\vec{r}_{i/c'} \times \vec{a}_{i/o} m_i)]$$

Since C' is fixed in F and the origin o is also fixed, the velocity of particle i with respect to c' and o will be the same

$$\Rightarrow \vec{v}_{i/c'} = \vec{v}_{i/o}$$

$$\Rightarrow \dot{\vec{H}}_{ic} = \sum [(\vec{v}_{i/o} \times \vec{v}_{i/o} m_i) + (\vec{r}_{i/c'} \times \vec{a}_{i/o} m_i)]$$

~~$\vec{v}_{i/o} \times \vec{v}_{i/o}$~~ O since a vector crossed w/ itself is 0

$$\dot{\vec{H}}_{ic} = \sum \vec{r}_{i/c'} \times \vec{a}_{i/o} m_i$$

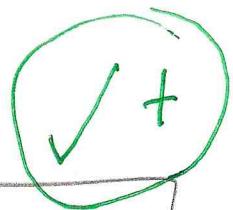
15) a) (continued)

Finally, since C' instantaneously coincides with C we have

$$\vec{r}_{i/C'} = \vec{r}_{i/C}$$

$$\Rightarrow \dot{\vec{r}}_{i/C} = \sum \vec{r}_{i/C} \times \vec{\omega}_{i/o} m_i$$

It has been shown that this definition works in general



15) b)

$$\vec{H}_{ic} = \sum \vec{r}_{i/c} \times \vec{v}_{i/o} m_i$$

$$\Rightarrow \dot{\vec{H}}_{ic} = \sum [(\dot{\vec{r}}_{i/c} \times \vec{v}_{i/o} m_i) + (\vec{r}_{i/c} \times \dot{\vec{v}}_{i/o} m_i)]$$

$$\boxed{\dot{\vec{H}}_{ic} = \sum [(\vec{v}_{i/c} \times \vec{v}_{i/o} m_i) + (\vec{r}_{i/c} \times \vec{a}_{i/o} m_i)]}$$

This expression collapses to $\dot{\vec{H}}_{ic} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o} m_i$ for some special cases :

i) If C is stationary

If C is stationary, then we know that $\vec{v}_{i/c} = \vec{v}_{i/o}$
since the origin O is also stationary

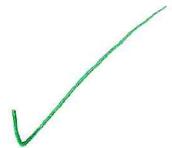
$$\Rightarrow \dot{\vec{H}}_{ic} = \sum [(\vec{v}_{i/o} \times \vec{v}_{i/o} m_i) + (\vec{r}_{i/c} \times \vec{a}_{i/o} m_i)]$$

$$\dot{\vec{H}}_{ic} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o} m_i \quad \checkmark$$

15) b) (continued)

2) If all particles i are moving away from point C in the same or opposite direction that they are moving away from point O

$$\Rightarrow \vec{v}_{ilc} \times \vec{r}_{ilo} = 0$$



\Rightarrow The point C is the origin, O

$$\dot{\vec{r}}_{ilc} = \sum \{ (\vec{v}_{ilc} \times \vec{r}_{ilo} m_i) + (\vec{r}_{ilc} \times \vec{\alpha}_{ilo} m_i) \}$$

$$\dot{\vec{r}}_{ilc} = \sum \{ (\vec{v}_{ilc} \times \overset{O}{\vec{r}_{ilo}} m_i) + (\vec{r}_{ilc} \times \vec{\alpha}_{ilo} m_i) \}$$

$$\dot{\vec{r}}_{ilc} = \sum \vec{r}_{ilc} \times \vec{\alpha}_{ilo} m_i \quad \checkmark$$

15) b) (continued)

3) All of the systems particles i are "stuck" to the point C

$$\Rightarrow \vec{v}_{i/c} = 0$$

$$\dot{\vec{r}}_{i/c} = \sum \left[(\vec{r}_{i/c} \times \vec{\omega}_{i/o} m_i) + (\vec{r}_{i/c} \times \vec{\alpha}_{i/o} m_i) \right]$$

$$\dot{\vec{r}}_{i/c} = \sum \vec{r}_{i/c} \times \vec{\alpha}_{i/o} m_i \quad \checkmark$$

=>

This definition works for some special cases concerning the motions of the particles i and point C that I have outlined previously

15) c)

$$\vec{H}_{ic} = \sum \vec{r}_{ic} \times \vec{v}_{ic} m_i$$

$$\Rightarrow \dot{\vec{H}}_{ic} = \sum [(\dot{\vec{r}}_{ic} \times \vec{v}_{ic} m_i) + (\vec{r}_{ic} \times \dot{\vec{v}}_{ic} m_i)]$$

$$\dot{\vec{H}}_{ic} = \sum [(\vec{v}_{ic} \times \vec{v}_{ic} m_i) + (\vec{r}_{ic} \times \vec{a}_{ic} m_i)]$$

$$\boxed{\dot{\vec{H}}_{ic} = \sum \vec{r}_{ic} \times \vec{a}_{ic} m_i}$$

The expression collapses to $\dot{\vec{H}}_{ic} = \sum \vec{r}_{ic} \times \vec{a}_{io} m_i$ for the following cases:

1) C is stationary (or just refer to (a))

If C is fixed, the the velocity of particles i with respect to C will be the same as the velocity of particles with respect to the origin O (since O is also fixed)

$$\Rightarrow \vec{v}_{ic} = \vec{v}_{io}$$

15) c) (continued)

Taking a derivative yields that

$$*\vec{a}_{i/c} = \vec{a}_{i/o} *$$

Thus, we have

$$\dot{\vec{r}}_{i/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/c} m_i$$

$$\dot{\vec{r}}_{i/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o} m_i \quad \checkmark$$

2) C is the origin, O

AND fixed!
(not just
instantaneously.)

$$\Rightarrow \vec{a}_{i/c} = \vec{a}_{i/o} \text{ and we get}$$

$$\dot{\vec{r}}_{i/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/c} m_i$$

$$\dot{\vec{r}}_{i/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/o} m_i \quad \checkmark$$

So this is
a special case
of (2) above.
(Not needed)

15) c) (continued)

3) C is the center of mass (COM)

$$\text{We know that } \sum \vec{F}^{\text{ext}} = \sum m_i \vec{a}_{G/0}$$

Thus, if the sum of the external forces on this system is zero (a valid assumption) we get that

$$* \quad \vec{a}_{G/0} = 0 \quad * \quad \text{since } \sum m_i \text{ will not feasibly be equal to 0}$$

Also, we have that

$$\vec{r}_{i/0} = \vec{r}_{G/0} + \vec{r}_{i/G}$$

Differentiating twice with respect to time we get

$$\vec{a}_{i/0} = \cancel{\vec{a}_{G/0}^0} + \vec{a}_{i/G}$$

$$* \quad \vec{a}_{i/0} = \vec{a}_{i/G} \quad *$$

15) c) (continued)

Thus, we had

$$\vec{H}_{i/c} = \sum \vec{r}_{i/c} \times \vec{\alpha}_{i/c} m_i$$

$$\vec{H}_{i/c} = \sum \vec{r}_{i/c} \times \vec{\alpha}_{i/G} m_i \quad (\text{if } C = G)$$

Using the result that $\vec{\alpha}_{i/o} = \vec{\alpha}_{i/G}$ gives

$$\vec{H}_{i/c} = \sum \vec{r}_{i/c} \times \vec{\alpha}_{i/o} m_i \quad \checkmark$$

→ This definition works for some special cases concerning the motions of particles i and point C which have been previously outlined

NE MORE

Also if $\vec{\alpha}_c$ is parallel to $\vec{r}_{G/c}$,

[then $\vec{r}_{G/c} \times \vec{\alpha}_c = \vec{0}$] {when they
is one sometimes lets people get the right answer} {don't deserve
to.}