GEOMETRY, KINEMATICS, STATICS, AND DYNAMICS

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August 6, 2018

Contents

Chapter One
Introduction

The principles of kinematics and dynamics presented in this book are consistent with the numerous available books on these subjects. However, the presentation differs from other books in crucial ways. In particular, we define concepts and properties of idealized objects with extreme care in order to provide a precise foundation for the key results, which are derived and proved in a rigorous, mathematical style. This approach is intended to add clarity to the basic ideas of kinematics and dynamics, which are often obscure in traditional textbooks. We also find that a precise treatment of the concepts, notation, and terminology underlying kinematics and dynamics is valuable for solving problems.

1.1 Points, Particles, and Bodies

A *point* has zero size and zero mass. A *particle* has zero size and nonnegative mass. Therefore, a point can be viewed as a particle with zero mass. Points and particles have position *relative* to other points and particles. Two points, two particles, or a point and a particle are *colocated* if they are located at the same place. If two particles are in contact, then they are colocated.

A *reference point* (such as the origin of a frame) is a point relative to which the positions of other points are determined. Any point can be used as a reference point.

Points and particles can have translational motion relative to other points and particles. Translational motion includes velocity and acceleration. For example, the point or particle x has a position relative to the point or particle y. Likewise, the point or particle x has a velocity and acceleration *relative* to the point or particle y and with *respect* to frame $\mathrm{F_A}$. Points and particles cannot rotate.

A *body* (not necessarily rigid) is a finite collection of particles and rigid massless links and joints. A *rigid body* is a body whose shape does not change. A continuum body has an infinite number of particles. Each particle in a body may be subject to *internal forces*, which are *reaction forces* due to the interaction between each particle and all other particles in the body. An *external force* is a force on a particle in a body that is not due to interactions with other particles in the body. A collection of forces applied to a body is *balanced* if its sum is zero. A moment that arises from a collection of balanced forces is a *torque*.

The role of points, particles, frames, and bodies in kinematics and dynamics is summarized in Table 1.1.

All bodies are assumed to be *Newtonian*, which means that the internal forces between every pair of particles are equal in magnitude, opposite in direction, and parallel to the line that passes through the particles. Newton's law of gravity, which involves only attractive forces directed along the line passing through the particles, satisfies this assumption, as does Coulomb's law for electric charges, where the forces may be either attractive or repulsive depending on whether the electric

	Translation	Rotation
Geometry and Kinematics	Point	Frame
Dynamics	Particle	Body

Table 1.1: Conceptual roadmap for kinematics and dynamics. For translational geometry and kinematics, mass is irrelevant, and thus a particle is effectively a point. Furthermore, for rotational geometry and kinematics, mass distribution is irrelevant, and thus a body is viewed as a frame. Points and particles cannot rotate, and thus rotational geometry, kinematics, and dynamics apply only to frames and bodies.

charges are the same or opposite. A rigid body that is Newtonian can be viewed as a plane or space truss with particles connected by rigid, massless members that support only compression and tensile forces [8]. A rigid body consisting of interconnected permanent magnets does not fit into this framework since the internal forces that attract or repel the magnets follow curved field lines. Although a proof is outside the scope of this book, the total internal force and the total moment on a rigid body containing permanent magnets are both zero.

A body consisting of at least two noncolocated particles is rigid if the distance between every pair of particles is constant. Thus, a particle is not a rigid body. The particles in a *trivial* rigid body lie along a single line, whereas the particles in a *nontrivial* rigid body do not lie along a single line. A nontrivial rigid body thus contains three particles that form a triangle. Henceforth, unless stated otherwise, the phrase “rigid body” refers to a nontrivial rigid body. A rigid body thus has positive size and positive mass, consists of at least three particles that do not lie along a line, and does not change shape. Only a rigid body can possess a body-fixed frame. A trivial rigid body, that is, a body whose particles lie along a line, can rotate around its transverse axes but has no meaningful dynamics around its longitudinal axis. However, for the sake of kinematics or when dynamics around only the transverse axes are of interest, rigid massless links can be attached orthogonally to the body to define a body-fixed frame. The rotational motion of a rigid body is described by its angular velocity and angular acceleration.

A *massive particle* is a particle with infinite mass. Massive particles are unaffected by all forces, and thus every massive particle is effectively an unforced particle. A *massive body* is a rigid body that has at least three massive particles that are not along a single line. A massive body is unaffected by all forces and moments. Consequently, every particle in a massive body is unaffected by forces, and thus every particle in a massive body is unforced. If not preceded by the word “massive,” the word “body” assumes finite mass.

	Space	Motion	Force
Geometry	Yes	No	No
Kinematics	Yes	Yes	No
Statics	Yes	No	Yes
Dynamics	Yes	Yes	Yes

Table 1.2: Definitions of the various branches of mechanics.

A massive body is *inertially nonrotating* if its angular velocity relative to an inertial frame is zero; otherwise it is *inertially rotating*. Consequently, every body-fixed frame associated with an inertially nonrotating massive body $\mathcal{B}$ is an inertial frame, and every particle in $\mathcal{B}$ is unforced. The Earth is not a massive body since it is affected by central gravity from the Sun and other planets. In addition, the Earth is rotating relative to inertial frames. However, the assumption that the Earth is an inertially nonrotating massive body is useful in many applications.

In order to describe the dynamics of a body it is necessary to specify an unforced particle. If reaction forces are applied to the particle, then it is convenient to assume that the point is fixed in an inertially nonrotating massive body. Wall, ceiling, ground, and floor are conceptual examples of inertially nonrotating massive bodies to which mechanical systems can be attached for the study of dynamics.

If a body interacts with a massive body, then the resulting reaction forces and torques are classified as internal forces and torques. Consequently, the phrases "internal force" and "reaction force" are identical even though, strictly speaking, the reaction forces between a body and a massive body are not internal to the body.

The study of mechanics may include either time or force. The various branches of mechanics are outlined in Table 1.2.

	1DOF	2DOF	3DOF
Revolute	Pin Hinge	Dual Pin Universal Joint	Ball Joint
Prismatic	Sleeve	Dual Sleeve x-y Stage	x-y-z Stage
Combined		Slot Collar	Dual Sleeve-Pin Collar-Pin

Table 1.3: Terminology for revolute and prismatic joints. A slot is a groove within which a pin translates. A collar is a ring that slides along a shaft while rotating.

1.2 Mechanical Interconnection and Newton's Third Law

Rigid bodies that interact through joints are *articulated*. A prismatic joint allows translational motion along a line, curve, or surface. Friction may or may not be present in these mechanisms. The reaction force in a frictionless prismatic joint is zero in the direction of translation. A revolute joint allows rotation around one or more directions. The reaction torque in a frictionless revolute joint is zero along the axes of rotation. Joint terminology is summarized in Table 1.3. Bodies may also interact indirectly through gravity or through interconnections consisting of springs, dashpots, and inerters.

A mechanical system consists of particles and rigid bodies (either discrete or continuum, massive or not massive) interconnected by springs, dashpots, inerters, and joints and with direct contact that is either rolling, slipping, or impulsive. Rolling, slipping, and joints may involve friction or may be frictionless. We consider five types of joints. A *pin* allows rotation around a single axis and no translation. A *sleeve* allows translation along a single direction but no rotation. A *collar* is a combination of a pin and a sleeve that allows rotation around a single axis and translation along a single direction, where the axis of rotation and the direction of translation are parallel. A *slot* is a combination of a pin and a sleeve that allows rotation around a single axis and translation along a single direction, where the axis of rotation is perpendicular to the direction of translation. A *ball* allows rotation around three axes.

The dynamics of a particle depend on the total force acting on the particle. Likewise, the dynamics of a rigid body depend on the total force and torque acting on the body. These observations provide the ability to analyze the dynamics of the particles and rigid bodies within a body in terms of a *free-body diagram* for each rigid body.

A body may consist of a collection of particles and rigid bodies that interact with each other in various ways. Direct contact between rigid bodies may involve one or more fixed contact points in each rigid body. Direct contact may occur as a collision or through a revolute joint, or it may involve time-varying contact points, as in the case of a prismatic joint, sliding (relative translation without relative rotation), or rolling (relative translation and relative rotation with or without slipping). Interaction involving time-varying contact points can occur with or without friction. Indirect contact between particles and rigid bodies can occur through springs, dashpots, and inerters.

Newton's third law concerns the reaction forces between a pair of rigid bodies either through direct contact, indirect contact, or noncontacting forces. Newton's third law applies to all cases of direct contact between rigid bodies as well as all cases of indirect contact between particles and rigid bodies. Noncontacting forces can occur through gravitational, electrostatic, and magnetic forces. Newton's third law applies in these cases as well, except that the reaction forces arising from magnetic forces are not aligned with the line passing through the magnetic dipoles. However, Newton's third law may not hold in the case of electrodynamics; for details, see [4, pp. 349–351].

1.3 Physical Vectors and Frames

The notion of location is *relative*; in other words, the location of a point or particle is meaningful only when given in terms of other points and particles. An analogous statement applies to motion. We do not ascribe meaning to the word "stationary"; in fact, the location of a point or particle can be "fixed" only relative to other points and particles. Consequently, the position, velocity, and acceleration of a point or particle are meaningful only when used in a relative sense. Analogous statements apply to bodies under translation and rotation.

We develop kinematics and dynamics in terms of 14 types of physical vectors. A *physical vector* has a magnitude (which may be dimensional or dimensionless) and direction, but it has no physical location. For example, although the points x and y have locations relative to each other, the physical position vector $\vec{r}_{x/y}$ has no physical location. Likewise, although the force vector $\vec{f}$ can represent a force applied to a particle or body, the physical force vector $\vec{f}$ has no physical location. Consequently, the total force on a particle or the center of mass of a rigid body can be determined by summing individual forces by plotting them tip-to-tail.

Two nonzero physical vectors are *parallel* if one is a scalar multiple of the other. Two nonzero physical vectors are *aligned* if they are parallel and one is a positive multiple of the other.

The purpose of a *frame*, which is a set of three linearly independent (and usually mutually perpendicular) physical vectors, is to define directions in three-dimensional space. For example, the frame $\mathrm{F_A}$ is represented by the row vectrix $\mathrm{F_A} = [\hat{\imath}_\mathrm{A}\ \hat{\jmath}_\mathrm{A}\ \hat{k}_\mathrm{A}]$ and the column vectrix $\mathcal{F}_\mathrm{A} = \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix}$. Since physical vectors have no physical location, a frame has no location. Since a frame has no physical location, it is meaningless to refer to its velocity and acceleration. This conception of a frame, which is a unique feature of this book, stresses its role in defining direction as distinct from location. A body-fixed frame is a frame that is rigidly attached to a rigid body and thus rotates as the rigid body rotates.

It is traditional but not necessary to associate a frame with a reference point that is designated as the *origin* of the frame. Like any other point, the origin of a frame has position, velocity, and acceleration relative to other points, and it can be used to define the position, velocity, and acceleration of other points in a relative sense. In fact, the traditional notion of the "acceleration of a frame" as used in physics refers to the motion of its origin rather than the axes of the frame. A frame need not be assigned an origin; however, we often do this for convenience.

It is not meaningful to say that a point is "fixed in a frame," although it is meaningful to say that a point is fixed in a rigid body. A point may p be fixed relative to a rigid body but not part of the rigid body. For convenience, we view p as rigidly attached to the rigid body by means of a massless rigid link, and we simply say that p is fixed in the body. Unlike a point, it is meaningful to say that a direction is fixed in a rigid body or a frame in the sense that the direction of the vector depends on the orientation of the frame. We almost exclusively consider only *standard frames*, which are orthogonal, right-handed frames with dimensionless, unit-length axes. Frames that are nonstandard are considered only in Section 2.19.

Velocity and acceleration depend on the frame with *respect* to which changes are observed. Hence, derivatives of physical vectors are defined only with respect to frames. We refer to these derivatives as *frame derivatives*.

An *unforced particle* is a particle that has no force (that is, zero total force) applied to it. The motion of an unforced particle is thus determined by its initial position and velocity. An *inertial frame* is a frame that has the property that the relative acceleration with respect to the frame of every pair of unforced particles is zero. This is Newton's first law. Like all frames, an inertial frame has no location, and thus the velocity and acceleration of an inertial frame are meaningless. There are an infinite number of inertial frames, and the relative angular velocity of each pair of inertial frames is zero. We do not recognize the notion of an "absolute" frame.

1.4 Remarks on Notation

The notation in this book differs from other books on dynamics. With modest effort, the reader will find that this notation is extremely helpful for understanding the principles of kinematics and dynamics and for solving problems. Some of the features of this notation are described below.

First, a half arrow over a symbol such as $\vec{r}_{x/y}$, where x and y are points or particles, emphasizes the fact that $\vec{r}_{x/y}$ denotes a physical vector, which is not *resolved* in a frame. Derivations and calculations can be carried out to the greatest possible extent without resolving physical vectors in a specific frame. At a later stage, every vector in the equation can be resolved in any frame of interest to obtain mathematical vectors, which are column vectors with numerical or symbolic components. This approach facilitates the physical interpretation of the components of mathematical vectors.

The notation used in this book strives to be 1) independent of context, 2) explicit, and 3) unambiguous. The meaning of each symbol (with the exception of force and moment vectors) can be determined by its appearance alone, without the need for additional verbiage, commentary, or explanation. This interpretation is facilitated by subscripts. For example, $\vec{r}_{x/y}$ denotes the position of the point or particle x relative to the point or particle y.

If x, y, and z are points, then we have the "slash and split" identity

$$\vec{r}_{z/x} = \vec{r}_{z/y} + \vec{r}_{y/x}. \tag{1.4.1}$$

Likewise,

$$\vec{v}_{z/x/\mathrm{A}} = \vec{v}_{z/y/\mathrm{A}} + \vec{v}_{y/x/\mathrm{A}}, \tag{1.4.2}$$

$$\vec{a}_{z/x/\mathrm{A}} = \vec{a}_{z/y/\mathrm{A}} + \vec{a}_{y/x/\mathrm{A}}. \tag{1.4.3}$$

Note that

$$\vec{r}_{x/y} = -\vec{r}_{y/x}, \tag{1.4.4}$$

$$\vec{v}_{x/y/\mathrm{A}} = -\vec{v}_{y/x/\mathrm{A}}, \tag{1.4.5}$$

$$\vec{a}_{x/y/\mathrm{A}} = -\vec{a}_{y/x/\mathrm{A}}. \tag{1.4.6}$$

A physical vector can be multiplied by a real scalar, as in $3\vec{f}$ or $-6\vec{f}$. The zero physical vector is denoted by $\vec{0}$, and therefore $\vec{r}_{x/x} = \vec{0}$. A physical vector such as $\vec{r}_{x/y}(t)$ can be a function of time. For a nonzero physical vector $\vec{x}$, the notation $\hat{x}$ represents a dimensionless, unit-length physical vector whose direction is the same as the direction of $\vec{x}$.

When rate is involved, an additional subscript is included to denote the frame used for the frame derivative as in, for example, $\vec{v}_{x/y/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{x/y}$, which denotes the velocity of the point or particle x relative to the point or particle y with respect to the frame $\mathrm{F_A}$. Frame derivatives are denoted by $\overset{\mathrm{A}\bullet}{\vec{r}}_{x/y}$, $\overset{\mathrm{B}\bullet}{\vec{r}}_{x/y}$, $\overset{\mathrm{C}\bullet}{\vec{r}}_{x/y}$, and so forth.

A physical matrix, which is denoted by $\vec{M}$, can be viewed as a 3×3 matrix that is not resolved in a frame. A physical matrix is traditionally called a dyad or second-order tensor. Physical rotation matrices and physical inertia matrices are physical matrices that play key roles in kinematics and dynamics.

We use only a single font for all symbols, without need for bold. This style allows easy presentation on a whiteboard or blackboard without the need for underscores and undertildes. We also avoid superscripts, both pre and post, which are pervasive in many texts.

1.5 Resolving Physical Vectors

A physical vector has no components, and thus it is distinct from a mathematical column vector of the form $[1 \;\; -6 \;\; 3]^{\mathrm{T}}$. However, any physical vector can be resolved in any frame. For example, the position vector $\vec{r}_{y/x}$ can be resolved in $\mathrm{F_A}$ by writing

$$\vec{r}_{y/x}\Big|_{\mathrm{A}}. \tag{1.5.1}$$

The resolved vector can also be represented by

$$r_{y/x|\mathrm{A}} = \vec{r}_{y/x}\Big|_{\mathrm{A}}. \tag{1.5.2}$$

1.6 Types of Physical Vectors

A *physical vector* (as distinct from a mathematical vector, which is a column of numbers) is an abstract quantity having a tip and a tail and thus magnitude and direction. A physical vector is not a physical object, and thus it is not located anywhere, although we can envision its tail located at an arbitrary location for convenience. A physical vector is denoted with a half arrow or hat over the symbol denoting the physical quantity. For example, $\vec{f}$ is a physical vector representing a force applied to a particle in a body, while $\vec{r}_{x/y}$ is the physical vector representing the position of the point x *relative* to the point y. We may envision the tip of $\vec{r}_{x/y}$ at x and its tail at y. However, the physical vector $\vec{r}_{x/y}$ has no physical location. The magnitude of $\vec{x}$ is denoted by $|\vec{x}|$.

A physical vector may have dimensions or it may be dimensionless. A frame consists of three unit, dimensionless physical vectors that are mutually orthogonal. The motion of a point, particle, or body relative to another point, particle, or body is determined with *respect* to a frame. Frame differentiation is discussed in Chapter 4.

Statics, kinematics, and dynamics are based on 13 types of physical vectors, namely:

i) Dimensionless. A dimensionless physical vector has no physical units associated with it. A unit dimensionless physical vector is written as $\hat{\imath}$. Three mutually orthogonal unit dimensionless physical vectors comprise a frame. The unit dimensionless physical vector that points in the direction of the nonzero physical vector $\vec{x}$ is denoted by $\hat{x}$. Hence, $\vec{x} = |\vec{x}|\hat{x}$. If $\vec{x} = 0$, then $\hat{x}$ is not defined.

ii) Unit angle vector. The unit angle vector of the physical vector $\vec{y}$ relative to the physical vector $\vec{x}$, where $\vec{y}$ and $\vec{x}$ are nonzero and not parallel, is written as $\hat{\theta}_{\vec{y}/\vec{x}}$. The direction of $\hat{\theta}_{\vec{y}/\vec{x}}$ is given by the right hand rule with the fingers curled from $\vec{x}$ to $\vec{y}$ through the short-way angle $\theta_{\vec{y}/\vec{x}} = \theta_{\vec{x}/\vec{y}}$ between $\vec{x}$ and $\vec{y}$. Hence, $\theta_{\vec{x}/\vec{y}} = -\theta_{\vec{y}/\vec{x}}$

iii) Angle. The angle vector of the physical vector $\vec{y}$ relative to the physical vector $\vec{x}$, where $\vec{y}$ and $\vec{x}$ are nonzero and not parallel, is given by $\vec{\theta}_{\vec{y}/\vec{x}} = \theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}$, where $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$ is the short-way angle between $\vec{x}$ and $\vec{y}$. $\vec{\theta}_{\vec{y}/\vec{x}}$ is aligned with the physical cross-product vector $\vec{x} \times \vec{y}$, and $|\vec{\theta}_{\vec{y}/\vec{x}}| = \theta_{\vec{y}/\vec{x}}$.

iv) Position. The position of the point y relative to the point x is written as $\vec{r}_{y/x}$.

v) Velocity. The velocity of the point y relative to the point x with respect to the frame F_A is written as $\vec{v}_{y/x/A}$.

vi) Acceleration. The acceleration of the point y relative to the point x with respect to the frame F_A is written as $\vec{a}_{y/x/A}$.

vii) Momentum. The momentum of the particle y relative to the point x with respect to the frame F_A is written as $\vec{p}_{y/x/A}$. The momentum of the body $\mathcal{B}$ relative to the point x with respect to the frame F_A is written as $\vec{p}_{\mathcal{B}/x/A}$.

viii) Force. A force $\vec{f}$ can be applied to a point or a particle, where a point is viewed as a particle with zero mass. We allow a force to be applied to a point as long as neither infinite acceleration nor infinite angular acceleration can occur. For example, a force can be applied to a point along a rigid massless link in a body. A force on a particle in a body can be either an internal force or an external force. Reaction forces are forces due to the interaction between particles; these forces may or may not involve contact. Reaction forces due to the interaction with a massive body are viewed as external forces. The force on a particle or body due to gravity can be either uniform, that is, a function of mass, or central, that is, a function of mass and position.

ix) Angular velocity. The angular velocity of the frame F_B relative to the frame F_A is written as $\vec{\omega}_{B/A}$.

x) Angular acceleration. The angular acceleration of the frame F_B relative to the frame F_A with respect to the frame F_C is written as $\vec{\alpha}_{B/A/C}$.

xi) Angular momentum. The angular momentum of the particle x relative to the point w with respect to the frame F_A is written as $\vec{H}_{x/w/A}$. The angular momentum of the body $\mathcal{B}$ relative to the point w with respect to the frame F_A is written as $\vec{H}_{\mathcal{B}/w/A}$.

xii) Moment. A moment can be applied to either a particle or a body. A moment on the particle x relative to the point y is written as $\vec{M}_{x/y}$. A moment on the body $\mathcal{B}$ relative to the point y is written as $\vec{M}_{\mathcal{B}/y}$. A moment can be applied to a trivial rigid body as long as infinite angular acceleration cannot occur.

xiii) Torque. A torque can be applied to a body. A torque on the body $\mathcal{B}$ is written as $\vec{M}_{\mathcal{B}}$. A torque can be applied to a trivial rigid body as long as infinite angular acceleration cannot occur.

Position, velocity, acceleration, momentum, and force can be projected onto a direction $\hat{n}$; the resulting vector is the position, velocity, acceleration, momentum, and force *along* $\hat{n}$. Angular velocity, angular acceleration, angular momentum, moment, and torque can be projected onto a direction $\hat{n}$; the resulting vector is the angular velocity, angular acceleration, angular momentum, moment, and torque *around* $\hat{n}$.

Energy is a scalar quantity that is associated with a particle or a body. Potential energy can be defined in terms of a spring or a uniform or central gravitational vector field. Kinetic energy is defined in terms of velocity with respect to an inertial frame.

1.7 Mechanical Systems

We apply the techniques of kinematics and dynamics to various types of mechanical systems. These systems may be one-dimensional (linear), two-dimensional (planar), or three-dimensional (spatial), depending on whether the motion occurs along a line, in a plane, or in three-dimensional space. The systems may involve various joints, they may involve one or more rigid bodies, they may include the effect of gravity, they may include springs and dashpots (either linear or torsional), they may involve friction, and they may involve rolling, slipping, or collisions. For convenience, we refer to these examples by the following terminology:

Pendulum. A pendulum is a planar or spatial mechanical system connected to a massive body by means of a revolute joint. Gravity is usually present. Springs and dashpots can be included, as well as multiple rigid bodies.

MCK system. Rigid bodies, springs, and dashpots can be connected to form a planar or spatial mechanical system.

Gimbal. A gimbal is a spatial mechanical system with multiple revolute joints. Springs and dashpots can be included, as well as a spinning payload.

Shaft. A shaft is a three-dimensional mechanical system consisting of a rotating rigid body connected to a massive body by means of a revolute joint. Additional rigid bodies may be connected to the shaft by means of revolute or prismatic joints.

Linkage. A linkage is a planar or spatial device involving multiple rigid bodies connected by revolute or prismatic joints. Springs and dashpots can be included, as well as rolling disks.

Rolling body. A disk, sphere, and cone can roll over a surface that is either flat or curved.

Spinning top. A top is a spinning body connected to a massive body by means of a ball joint.

The above classification is not precise and is for convenience only. For example, a pendulum can be viewed as a linkage, a gimbal can be viewed as a type of a shaft, and rolling bodies can be combined with other types of mechanical systems.

1.8 Classification of Forces and Moments

Forces that do not entail a loss of energy are called *conservative forces*, while forces that give rise to a loss of energy are called *dissipative forces*.

Table 1.4 classifies reaction and non-reaction forces and moments in terms of energy conservation and dissipation. Reaction forces due to elastic collisions, rolling, frictionless sliding, slipping, and pivoting, as well as springs and inerters are conservative. Forces due to friction (except rolling without slipping), inelastic collisions, and dashpots are dissipative. When two bodies are in contact, the reaction force may be either tangential or normal. The reaction force between two bodies that are in contact is frictionless if the tangential contact force is zero. Reaction forces may be associated with the Coriolis, angular-acceleration, and centripetal components of acceleration. Angular-acceleration and centripetal reaction forces involve zero relative motion, whereas a Coriolis contact force involves nonzero relative motion (such as a particle sliding with friction within a groove on a rotating platform).

	Conservative	Dissipative
Reaction forces	Elastic impact Joint constraint Frictionless sliding Rolling without slipping Frictionless slipping Frictionless pivoting Spring Inerter Central gravity Electrostatic force magnetic force	Inelastic impact Sliding with friction Slipping with friction Dashpot
Reaction torques	Joint constraint Rotational spring Rotational inerter	Pivoting with friction Rotational dashpot
Non-reaction forces and moments	Uniform gravity	Control Disturbance

Table 1.4: Classification of reaction and non-reaction forces, torques, and moments. Dissipative forces and moments entail a loss of energy, whereas energy is conserved by conservative forces and torques. Control and disturbances forces and moments can increase or decrease energy. Direct contact forces include normal and tangential reaction forces due to collision, rolling, sliding, and pivoting. Indirect contact forces may be due to springs, dashpots, and inerters. Noncontacting forces include gravitational and electromagnetic forces.

Chapter Two
Geometry

2.1 Angle and Dot Product

An angle $\theta \in [0,\pi]$ between two physical vectors represents the "short way" between the vectors. An angle confined to $(-\pi,\pi]$ is a *wrapped angle*; otherwise, $\theta \in \mathbb{R}$ is an *unwrapped angle*.

Since θ and $\theta + 2n\pi$, where n is an integer, represent the same angle, wrapped angles represent all possible angles between a pair of physical vectors. However, sums and differences of angles can violate this constraint, and thus it is sometimes convenient to use unwrapped angles but extend the notion of "equal" angles. Hence, for $a, b \in \mathbb{R}$, the notation $a \equiv b$ means that $a - b$ is an integer multiple of 2π.

Fact 2.1.1. Let $a, b, c, d \in \mathbb{R}$. Then, the following statements hold:

i) $a \equiv b$ if and only if $a - b \equiv 0$.

ii) $a \equiv b$ if and only if $-a \equiv -b$.

iii) If $a \equiv b$ and n is an integer, then $na \equiv nb$.

iv) Let n and m be integers such that $n + m$ is even. Then, $a \equiv n\pi$ if and only if $a \equiv m\pi$.

v) $a \equiv -a$ if and only if there exists an integer n such that $a = n\pi$.

vi) The following statements are equivalent:

a) $a \equiv -a \equiv 0$.

b) There exists an even integer n such that $a = n\pi$.

c) $a \equiv 0$.

vii) The following statements are equivalent:

a) $a \equiv -a \equiv \pi$.

b) There exists an odd integer n such that $a = n\pi$.

c) $a \equiv \pi$.

viii) If $a \equiv b$ and $c \equiv d$, then $a + c \equiv b + d$.

Let $\theta_{\vec{y}/\vec{x}} = \theta_{\vec{x}/\vec{y}} \in [0,\pi]$ denote the *angle* between two physical vectors $\vec{x}$ and $\vec{y}$. If either $\vec{x}$ or $\vec{y}$ is the zero physical vector $\vec{0}$ (also written as just 0), then $\theta_{\vec{y}/\vec{x}} = \theta_{\vec{x}/\vec{y}} = 0$. The *dot product* $\vec{x} \cdot \vec{y}$

of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \cdot \vec{y} \triangleq |\vec{x}||\vec{y}| \cos \theta_{\vec{y}/\vec{x}}. \tag{2.1.1}$$

Hence,

$$|\vec{x} \cdot \vec{y}| \triangleq |\vec{x}||\vec{y}||\cos \theta_{\vec{y}/\vec{x}}|. \tag{2.1.2}$$

If $\vec{x}$ and $\vec{y}$ are nonzero, then

$$\hat{x} \cdot \hat{y} = \cos \theta_{\vec{y}/\vec{x}}, \tag{2.1.3}$$

and thus

$$\theta_{\vec{y}/\vec{x}} = \cos^{-1} \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|} = \cos^{-1} \hat{x} \cdot \hat{y} \in [0, \pi]. \tag{2.1.4}$$

Note that the range of the function $\cos^{-1}$ is $[0, \pi]$.

Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors. Then $\vec{x}$ and $\vec{y}$ are *mutually orthogonal* if $\vec{x} \cdot \vec{y} = 0$, that is, if $\theta_{\vec{y}/\vec{x}} = \pi/2$. Equivalently, we say that $\vec{x}$ is *orthogonal to* $\vec{y}$, and $\vec{y}$ is *orthogonal to* $\vec{x}$. Furthermore, $\vec{x}$ and $\vec{y}$ are *parallel* if either $\theta_{\vec{y}/\vec{x}} = 0$ or $\theta_{\vec{y}/\vec{x}} = \pi$. Note that $\vec{x}$ and $\vec{y}$ are parallel if and only if

$$|\vec{x} \cdot \vec{y}| = |\vec{x}||\vec{y}|. \tag{2.1.5}$$

We define

$$\vec{x}'\vec{y} \triangleq \vec{x} \cdot \vec{y}, \tag{2.1.6}$$

where the *physical covector* $\vec{x}'$ can be viewed as an operator on the physical vector $\vec{y}$ that produces the real scalar $\vec{x} \cdot \vec{y}$. The physical covector $\vec{x}'$ is the *coform* of the physical vector $\vec{x}$. The set of physical covectors corresponding to a set $\mathcal{V}$ of physical vectors is denoted by $\mathcal{V}'$. Each physical covector can be associated with a hyperplane in the space of physical vectors, that is, a plane that is translated away from the origin. Specifically, for the physical covector $\vec{x}'$, define

$$\mathcal{H}(\vec{x}') \triangleq \{\vec{y} \in \mathcal{V} \colon \vec{x}'\vec{y} = 1\}. \tag{2.1.7}$$

To show that $\mathcal{H}(\vec{x}')$ is a hyperplane, let $\vec{y}_0$ satisfy $\vec{x}'\vec{y}_0 = 1$. Then,

$$\mathcal{H}(\vec{x}') = \vec{y}_0 + \{\vec{y} \in \mathcal{V} \colon \vec{x}'\vec{y} = 0\}. \tag{2.1.8}$$

We define $(\vec{x}')' \triangleq \vec{x}$.

Fact 2.1.2. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then, $\vec{x} = \vec{y}$ if and only if $\vec{x}' = \vec{y}'$.

2.2 Angle Vector and Cross Product

Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors that are not parallel so that $\theta_{\vec{y}/\vec{x}} \in (0, \pi)$. The *unit angle vector* $\hat{\theta}_{\vec{y}/\vec{x}}$ of $\vec{y}$ relative to $\vec{x}$ is the unit dimensionless physical vector orthogonal to both $\vec{x}$ and $\vec{y}$ whose direction is determined by the right hand rule with the fingers curled from $\vec{x}$ to $\vec{y}$ through the angle $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$ between $\vec{x}$ and $\vec{y}$. See Figure 2.2.1. The *angle vector* $\vec{\theta}_{\vec{y}/\vec{x}}$ of $\vec{y}$ relative to $\vec{x}$ is defined by

$$\vec{\theta}_{\vec{y}/\vec{x}} \triangleq \theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}. \tag{2.2.1}$$

See Figure 2.2.2. Note that the magnitude of $\vec{\theta}_{\vec{y}/\vec{x}}$ is $\theta_{\vec{y}/\vec{x}}$. Furthermore,

$$\hat{\theta}_{\vec{x}/\vec{y}} = -\hat{\theta}_{\vec{y}/\vec{x}}, \tag{2.2.2}$$

$$\vec{\theta}_{\vec{x}/\vec{y}} = -\vec{\theta}_{\vec{y}/\vec{x}}. \tag{2.2.3}$$

$\hat{\theta}_{\vec{y}/\vec{x}}$ is not defined in the case where $\vec{x}$ and $\vec{y}$ are parallel, that is, $\theta_{\vec{y}/\vec{x}} \in \{0,\pi\}$.

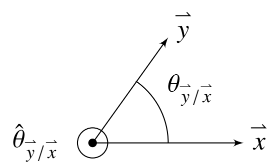


Figure 2.2.1: The unit angle vector $\hat{\theta}_{\vec{y}/\vec{x}}$ of $\vec{y}$ relative to $\vec{x}$ is the dimensionless physical vector whose direction is determined by the direction of the right-hand thumb with the fingers curled from $\vec{x}$ to $\vec{y}$ through the short-way positive angle $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$. For the example shown, $\hat{\theta}_{\vec{y}/\vec{x}}$ points out of the page. Note that $\theta_{\vec{y}/\vec{x}} = \theta_{\vec{x}/\vec{y}}$ and $\hat{\theta}_{\vec{x}/\vec{y}} = -\hat{\theta}_{\vec{y}/\vec{x}}$.

Let $\vec{x}$ and $\vec{y}$ be physical vectors. If $\theta_{\vec{y}/\vec{x}}$ is either 0 or π, then the *cross product* $\vec{x}\times\vec{y}$ is the zero physical vector. On the other hand, if $\vec{x}$ and $\vec{y}$ are nonzero and not parallel, then the *cross product* of $\vec{x}$ and $\vec{y}$ is defined as

$$\vec{x}\times\vec{y} \triangleq |\vec{x}||\vec{y}|(\sin\theta_{\vec{y}/\vec{x}})\hat{\theta}_{\vec{y}/\vec{x}}. \tag{2.2.4}$$

Note that, since $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$, it follows that $\sin\theta_{\vec{y}/\vec{x}} > 0$. Therefore,

$$\hat{x}\times\hat{y} = (\sin\theta_{\vec{y}/\vec{x}})\hat{\theta}_{\vec{y}/\vec{x}}, \tag{2.2.5}$$

$$|\vec{x}\times\vec{y}| = |\vec{x}||\vec{y}|\sin\theta_{\vec{y}/\vec{x}}, \tag{2.2.6}$$

$$\hat{\theta}_{\vec{y}/\vec{x}} = \frac{1}{|\vec{x}\times\vec{y}|}\vec{x}\times\vec{y} = \frac{1}{|\vec{x}||\vec{y}|\sin\theta_{\vec{y}/\vec{x}}}\vec{x}\times\vec{y} = \frac{1}{\sin\theta_{\vec{y}/\vec{x}}}\hat{x}\times\hat{y}, \tag{2.2.7}$$

$$\vec{y}\times\vec{x} = -(\vec{x}\times\vec{y}) = (-\vec{x})\times\vec{y} = \vec{x}\times(-\vec{y}), \tag{2.2.8}$$

$$\vec{x}\times\vec{x} = 0. \tag{2.2.9}$$

Figure 2.2.2 shows that $\vec{\theta}_{\vec{y}/\vec{x}}$ is the vector that is orthogonal to both $\vec{x}$ and $\vec{y}$, whose length is $\theta_{\vec{y}/\vec{x}}$, and whose direction is determined by the right-hand rule with the thumb pointing in the direction of $\hat{k}$.

Fact 2.2.1. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors that are not parallel. Then,

$$\vec{\theta}_{\vec{y}/\vec{x}} = \frac{\theta_{\vec{y}/\vec{x}}}{|\vec{x}\times\vec{y}|}\vec{x}\times\vec{y} = \frac{\theta_{\vec{y}/\vec{x}}}{|\vec{x}||\vec{y}|\sin\theta_{\vec{y}/\vec{x}}}\vec{x}\times\vec{y} = \frac{\theta_{\vec{y}/\vec{x}}}{\sin\theta_{\vec{y}/\vec{x}}}\hat{x}\times\hat{y}. \tag{2.2.10}$$

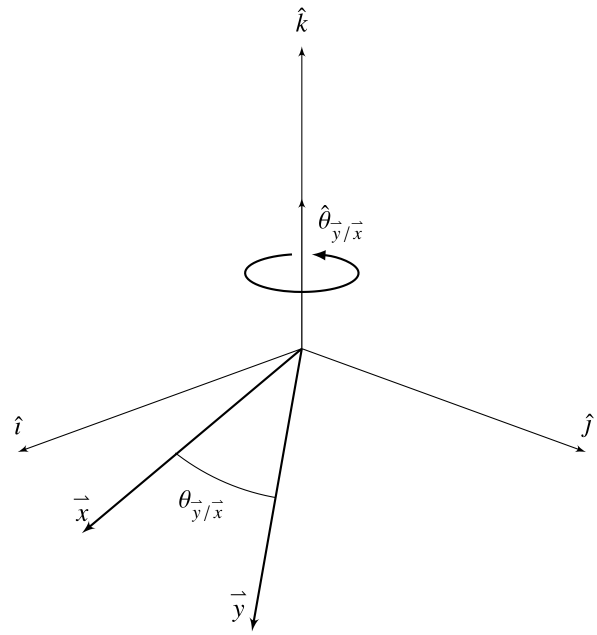


Figure 2.2.2: In this example, the angle vector $\vec{\theta}_{\vec{y}/\vec{x}}$ of $\vec{y}$ relative to $\vec{x}$, both of which lie in the $\hat{\imath}$-$\hat{\jmath}$ plane, points in the direction of $\hat{k}$. The magnitude of $\vec{\theta}_{\vec{y}/\vec{x}}$ is the number $\theta_{\vec{y}/\vec{x}} \in (0, \pi)$ of radians in the "short-way" angle between $\vec{y}$ and $\vec{x}$. The direction of $\vec{\theta}_{\vec{y}/\vec{x}}$, which is the same as the direction of $\hat{\theta}_{\vec{y}/\vec{x}}$, is determined by the direction of the right-hand thumb with the fingers curled from $\vec{x}$ to $\vec{y}$ through the angle $\theta_{\vec{y}/\vec{x}}$.

2.3 Directed Angles

Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors, and let $\hat{n}$ be a unit dimensionless physical vector that is orthogonal to both $\vec{x}$ and $\vec{y}$; that is, either $\hat{n} = \hat{\theta}_{\vec{y}/\vec{x}}$ or $\hat{n} = -\hat{\theta}_{\vec{y}/\vec{x}}$. The *directed angle* $\theta_{\vec{y}/\vec{x}/\hat{n}}$ *from* $\vec{x}$ to $\vec{y}$ *around* $\hat{n}$ is defined by

$$\theta_{\vec{y}/\vec{x}/\hat{n}} \triangleq \begin{cases} 0, & \text{if } \theta_{\vec{y}/\vec{x}} = 0, \\ \theta_{\vec{y}/\vec{x}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in (0, \pi) \text{ and } \hat{n} = \hat{\theta}_{\vec{y}/\vec{x}}, \\ -\theta_{\vec{y}/\vec{x}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in (0, \pi) \text{ and } \hat{n} = -\hat{\theta}_{\vec{y}/\vec{x}}, \\ \pi, & \text{if } \theta_{\vec{y}/\vec{x}} = \pi. \end{cases} \tag{2.3.1}$$

In the first and last cases, $\vec{x}$ and $\vec{y}$ are parallel. In the second case, the directed angle $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is positive; in the third case, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is negative. Note that $\theta_{\vec{y}/\vec{x}/\hat{n}} \in (-\pi, \pi]$, and thus $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is a wrapped angle. Figure 2.3.1 shows that $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is the angle from $\vec{x}$ to $\vec{y}$, as determined by the right-hand rule with the thumb pointing in the direction of $\hat{n}$. Hence, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ increases as $\vec{y}$ rotates relative to $\vec{x}$ in the direction of the curled fingers. Note that

$$\theta_{\vec{x}/\vec{y}/\hat{n}} = \begin{cases} -\theta_{\vec{y}/\vec{x}/\hat{n}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in [0,\pi), \\ \pi, & \text{if } \theta_{\vec{y}/\vec{x}} = \pi, \end{cases} \tag{2.3.2}$$

$$\theta_{\vec{y}/\vec{x}/-\hat{n}} = \begin{cases} -\theta_{\vec{y}/\vec{x}/\hat{n}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in [0,\pi), \\ \pi, & \text{if } \theta_{\vec{y}/\vec{x}} = \pi. \end{cases} \tag{2.3.3}$$

Hence,

$$\theta_{\vec{x}/\vec{y}/-\hat{n}} = \theta_{\vec{y}/\vec{x}/\hat{n}}. \tag{2.3.4}$$

Finally, if $\vec{x}$ and $\vec{y}$ are not parallel, then

$$\theta_{\vec{y}/\vec{x}/\hat{\theta}_{\vec{y}/\vec{x}}} = \theta_{\vec{y}/\vec{x}}, \tag{2.3.5}$$

$$\theta_{\vec{y}/\vec{x}/-\hat{\theta}_{\vec{y}/\vec{x}}} = -\theta_{\vec{y}/\vec{x}}. \tag{2.3.6}$$

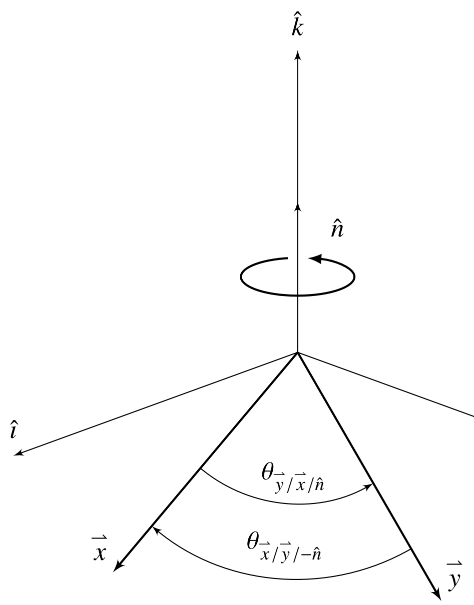


Figure 2.3.1: This example illustrates the directed angle $\theta_{\vec{y}/\vec{x}/\hat{n}}$ from $\vec{x}$ to $\vec{y}$ around $\hat{n}$. The value of $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is determined by the curled fingers of the right hand when the right-hand thumb is pointing in the direction of $\hat{n}$. The arrowhead on the curved arc indicates that the directed angle $\theta_{\vec{y}/\vec{x}/\hat{n}}$ becomes more positive as $\vec{y}$ rotates in the indicated direction. The directed angle $\theta_{\vec{x}/\vec{y}/-\hat{n}}$ from $\vec{y}$ to $\vec{x}$ around $-\hat{n}$ is also shown, and it can be seen that $\theta_{\vec{x}/\vec{y}/-\hat{n}} = \theta_{\vec{y}/\vec{x}/\hat{n}}$, which is positive as shown. Similarly, $\theta_{\vec{y}/\vec{x}/-\hat{n}} = \theta_{\vec{x}/\vec{y}/\hat{n}}$ are negative; these angles are not shown.

The directed angle $\theta_{\vec{y}/\vec{x}/\hat{n}} \in (-\pi,\pi]$ can be understood in the following way. Define a frame $\mathrm{F} = [\hat{\imath}\ \hat{\jmath}\ \hat{k}]$ such that $\hat{\imath} = \hat{x}$, $\vec{y}$ lies in the $\hat{\imath}$-$\hat{\jmath}$ plane, and $\hat{k} = \hat{n}$. Furthermore, write $\vec{y} = y_1\hat{\imath} + y_2\hat{\jmath}$. Next, we view $\hat{\imath}$ and $\hat{\jmath}$ as defining a complex plane, where $\hat{\imath}$ points the direction of the positive real axis, and $\hat{\jmath}$ points in the direction of the positive imaginary axis. Then, the vector $\vec{y}$ can be viewed as the position of the complex number $y_1 + y_2 \jmath$ relative to the origin. With this construction, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is the angle of the complex number $y_1 + y_2 \jmath$ in the complex plane, where the usual convention is that clockwise rotations correspond to more positive angles with zero radians assigned to points on the positive real axis. Hence,

$$\tan \theta_{\vec{y}/\vec{x}/\hat{n}} = \frac{y_2}{y_1}. \tag{2.3.7}$$

Furthermore,

$$\theta_{\vec{y}/\vec{x}/\hat{n}} = \operatorname{atan2}(y_2, y_1), \tag{2.3.8}$$

where atan2 is the four-quadrant inverse of the tangent function, that is,

$$\operatorname{atan2}(y, x) = \begin{cases} 0, & y = x = 0, \\ \tan^{-1} \frac{y}{x}, & x > 0, \\ -\pi/2, & y < 0, x = 0, \\ \pi/2, & y > 0, x = 0, \\ -\pi + \tan^{-1} \frac{y}{x}, & y < 0, x < 0, \\ \pi + \tan^{-1} \frac{y}{x}, & y \geq 0, x < 0. \end{cases} \tag{2.3.9}$$

Note that the range of the function $\tan^{-1}$ is $(-\pi/2, \pi/2)$, whereas the range of the function atan2 is $(-\pi, \pi]$. Equivalently,

$$\operatorname{atan2}(y, x) = \begin{cases} 0, & y = x = 0, \\ 2\tan^{-1} \frac{y}{\sqrt{x^2+y^2}+x}, & \sqrt{x^2 + y^2} + x > 0, \\ \pi, & y = 0, x < 0. \end{cases} \tag{2.3.10}$$

2.4 Frames

A frame is a collection of three unit dimensionless physical vectors, called *axes*, that are mutually orthogonal. Since each frame vector is a physical vector, the notion of the “location” of the frame is meaningless. In addition, since a frame has no location, it cannot translate, and thus a frame has no velocity or acceleration.

Nevertheless, it is often useful to associate a *reference point* with a frame. When we do this, we call the reference point the *origin of the frame*, and we draw the frame as if all of the axes were located at the reference point, which may have nonzero velocity and acceleration. Hence, the notion of a “translating frame” refers to the motion of the origin of the frame but not the axes that comprise the frame. A frame has no position, velocity, or acceleration since a physical vector has no location and thus cannot translate.

Let $\mathrm{F_A}$ be a frame with axes $\hat{\imath}_\mathrm{A}, \hat{\jmath}_\mathrm{A}, \hat{k}_\mathrm{A}$. Therefore,

$$\hat{\imath}_\mathrm{A} \cdot \hat{\imath}_\mathrm{A} = \hat{\jmath}_\mathrm{A} \cdot \hat{\jmath}_\mathrm{A} = \hat{k}_\mathrm{A} \cdot \hat{k}_\mathrm{A} = 1, \tag{2.4.1}$$

$$\hat{\imath}_\mathrm{A} \cdot \hat{\jmath}_\mathrm{A} = \hat{\jmath}_\mathrm{A} \cdot \hat{k}_\mathrm{A} = \hat{k}_\mathrm{A} \cdot \hat{\imath}_\mathrm{A} = 0. \tag{2.4.2}$$

The frame $\mathrm{F_A}$ is *right handed* if the labeling of the axes conforms to

$$\hat{\imath}_\mathrm{A} \times \hat{\jmath}_\mathrm{A} = \hat{k}_\mathrm{A}.$$

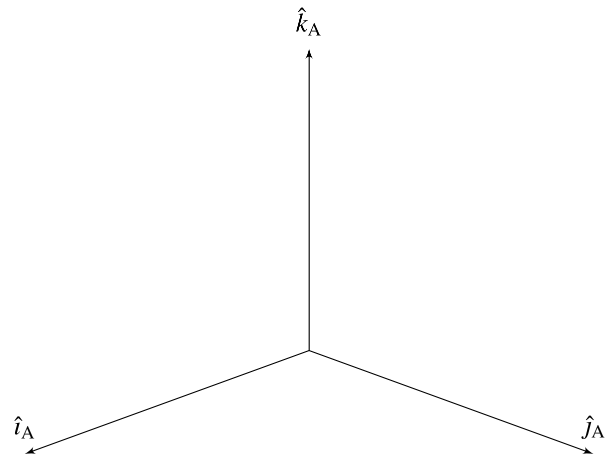


Figure 2.4.1: Right-handed frame $\mathrm{F_A}$ with mutually orthogonal axes $\hat{\imath}_\mathrm{A}, \hat{\jmath}_\mathrm{A}, \hat{k}_\mathrm{A}$.

Consequently,

$$\hat{\jmath}_\mathrm{A} \times \hat{k}_\mathrm{A} = \hat{\imath}_\mathrm{A},$$
$$\hat{k}_\mathrm{A} \times \hat{\imath}_\mathrm{A} = \hat{\jmath}_\mathrm{A}.$$

See Figure 2.4.1. All orthogonal frames in this book are right handed.

For convenience, we write

$$\mathrm{F_A} = \begin{bmatrix} \hat{\imath}_\mathrm{A} & \hat{\jmath}_\mathrm{A} & \hat{k}_\mathrm{A} \end{bmatrix}, \tag{2.4.3}$$

which is a *row vectrix*, as well as

$$\mathcal{F}_\mathrm{A} \triangleq \mathrm{F_A^T} = \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix}, \tag{2.4.4}$$

which is a *column vectrix*. Furthermore, we define the *coframe*

$$\mathrm{F'_A} \triangleq \begin{bmatrix} \hat{\imath}'_\mathrm{A} & \hat{\jmath}'_\mathrm{A} & \hat{k}'_\mathrm{A} \end{bmatrix}. \tag{2.4.5}$$

Therefore,

$$\mathcal{F}'_\mathrm{A} = \mathrm{F_A^{T\prime}} = \begin{bmatrix} \hat{\imath}'_\mathrm{A} \\ \hat{\jmath}'_\mathrm{A} \\ \hat{k}'_\mathrm{A} \end{bmatrix}. \tag{2.4.6}$$

The coframe $\mathrm{F'_A}$ is a *row covectrix*, whereas its transpose $\mathcal{F}'_\mathrm{A}$ is a *column covectrix*. The axes $\hat{\imath}'_\mathrm{A}, \hat{\jmath}'_\mathrm{A}, \hat{k}'_\mathrm{A}$ of the coframe can be viewed as a basis for the space $\mathcal{V}'$ of covectors. Note that the components of a vectrix are physical vectors, whereas the components of a covectrix are physical covectors. More generally, let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors. Then,

$$\begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}, \quad \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}, \tag{2.4.7}$$

$$\begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}' = \begin{bmatrix} \vec{x}' & \vec{y}' & \vec{z}' \end{bmatrix}, \quad \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}' = \begin{bmatrix} \vec{x}' \\ \vec{y}' \\ \vec{z}' \end{bmatrix}, \tag{2.4.8}$$

$$\begin{bmatrix} \vec{x}' & \vec{y}' & \vec{z}' \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \vec{x}' \\ \vec{y}' \\ \vec{z}' \end{bmatrix}, \quad \begin{bmatrix} \vec{x}' \\ \vec{y}' \\ \vec{z}' \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \vec{x}' & \vec{y}' & \vec{z}' \end{bmatrix}. \tag{2.4.9}$$

Vectrices and covectrices are multiplied according to the rules

$$\begin{bmatrix} \vec{x}_1' & \vec{y}_1' & \vec{z}_1' \end{bmatrix} \begin{bmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{bmatrix} = \vec{x}_1'\vec{x}_2 + \vec{y}_1'\vec{y}_2 + \vec{z}_1'\vec{z}_2, \tag{2.4.10}$$

$$\begin{bmatrix} \vec{x}_1' \\ \vec{y}_1' \\ \vec{z}_1' \end{bmatrix} \begin{bmatrix} \vec{x}_2 & \vec{y}_2 & \vec{z}_2 \end{bmatrix} = \begin{bmatrix} \vec{x}_1'\vec{x}_2 & \vec{x}_1'\vec{y}_2 & \vec{x}_1'\vec{z}_2 \\ \vec{y}_1'\vec{x}_2 & \vec{y}_1'\vec{y}_2 & \vec{y}_1'\vec{z}_2 \\ \vec{z}_1'\vec{x}_2 & \vec{z}_1'\vec{y}_2 & \vec{z}_1'\vec{z}_2 \end{bmatrix}, \tag{2.4.11}$$

$$\begin{bmatrix} \vec{x}_1 & \vec{y}_1 & \vec{z}_1 \end{bmatrix} \begin{bmatrix} \vec{x}_2' \\ \vec{y}_2' \\ \vec{z}_2' \end{bmatrix} = \vec{x}_1\vec{x}_2' + \vec{y}_1\vec{y}_2' + \vec{z}_1\vec{z}_2'. \tag{2.4.12}$$

In particular,

$$\mathrm{F}_{\mathrm{A}}'\mathcal{F}_{\mathrm{A}} = 3, \quad \mathcal{F}_{\mathrm{A}}'\mathrm{F}_{\mathrm{A}} = I_3, \quad \mathrm{F}_{\mathrm{A}}\mathcal{F}_{\mathrm{A}}' = \vec{I}, \tag{2.4.13}$$

where I_3 is the 3×3 identity matrix and $\vec{I}$ is defined in Section 2.8

Let F_{A} be a frame, and let $\vec{x}$ be a physical vector. Then, $\vec{x}\big|_{\mathrm{A}}$ is the physical vector $\vec{x}$ *resolved* in F_{A}. In fact, $\vec{x}\big|_{\mathrm{A}}$ is the *mathematical vector* defined by

$$\vec{x}\Big|_{\mathrm{A}} \triangleq \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \cdot \vec{x} \\ \hat{\jmath}_{\mathrm{A}} \cdot \vec{x} \\ \hat{k}_{\mathrm{A}} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{2.4.14}$$

where x_1, x_2, and x_3 are the *components* of the physical vector $\vec{x}$ resolved in F_{A}. Every physical vector is uniquely specified by resolving it in a frame. In particular, $\vec{x}$ can be reconstructed from

$\vec{x}\big|_{\mathrm{A}}$ by means of

$$\vec{x} = \begin{bmatrix} \hat{\imath}_{\mathrm{A}} & \hat{\jmath}_{\mathrm{A}} & \hat{k}_{\mathrm{A}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}. \tag{2.4.15}$$

In other words,

$$\vec{x} = \mathrm{F}_{\mathrm{A}}\left(\vec{x}\big|_{\mathrm{A}}\right) = \vec{x}\big|_{\mathrm{A}}^{\mathrm{T}} \mathcal{F}_{\mathrm{A}}. \tag{2.4.16}$$

A shorthand notation for $\vec{x}\big|_{\mathrm{A}}$ is given by

$$x_{|\mathrm{A}} \triangleq \vec{x}\big|_{\mathrm{A}}. \tag{2.4.17}$$

Fact 2.4.1. Let F_{A} be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{x} = \vec{y} \tag{2.4.18}$$

if and only if

$$\vec{x}\big|_{\mathrm{A}} = \vec{y}\big|_{\mathrm{A}}. \tag{2.4.19}$$

The physical covector $\vec{x}'$ is resolved according to

$$\vec{x}'\big|_{\mathrm{A}} \triangleq \vec{x}\big|_{\mathrm{A}}^{\mathrm{T}}, \tag{2.4.20}$$

and vectrices and covectrices are resolved as

$$\begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}\Big|_{\mathrm{A}} = \begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix}, \tag{2.4.21}$$

$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \\ \vec{z}' \end{bmatrix}\Bigg|_{\mathrm{A}} = \begin{bmatrix} \vec{x}\big|_{\mathrm{A}}^{\mathrm{T}} \\ \vec{y}\big|_{\mathrm{A}}^{\mathrm{T}} \\ \vec{z}\big|_{\mathrm{A}}^{\mathrm{T}} \end{bmatrix}. \tag{2.4.22}$$

In particular,

$$\mathrm{F}_{\mathrm{A}}|_{\mathrm{A}} = \mathcal{F}'_{\mathrm{A}}\big|_{\mathrm{A}} = \mathrm{F}_{\mathrm{A}}^{\mathrm{T}\prime}\big|_{\mathrm{A}} = I_3. \tag{2.4.23}$$

However, $\mathrm{F}_{\mathrm{A}}^{\mathrm{T}}\big|_{\mathrm{A}}$ and $\mathrm{F}'_{\mathrm{A}}\big|_{\mathrm{A}}$ are not defined.

Let F_{A} be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors, where

$$\vec{x}\big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{y}\big|_{\mathrm{A}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{2.4.24}$$

Then,

$$\begin{aligned} \vec{x} \cdot \vec{y} &= (x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}) \cdot (y_1\hat{\imath}_{\mathrm{A}} + y_2\hat{\jmath}_{\mathrm{A}} + y_3\hat{k}_{\mathrm{A}}) \\ &= x_1y_1 + x_2y_2 + x_3y_3 \end{aligned}$$

$$= \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}} \vec{y}\Big|_{\mathrm{A}}. \tag{2.4.25}$$

Note that

$$\vec{x}'\vec{y} = \vec{x}\cdot\vec{y} = \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}} \vec{y}\Big|_{\mathrm{A}} = \vec{x}'\Big|_{\mathrm{A}} \vec{y}\Big|_{\mathrm{A}}. \tag{2.4.26}$$

The following result expresses the length of the physical vector $\vec{x}$ in terms of its components in an arbitrary frame. For $x = [x_1 \ \ x_2 \ \ x_3]^{\mathrm{T}} \in \mathbb{R}^3$, define

$$\|x\| \triangleq \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{2.4.27}$$

Fact 2.4.2. Let F_{A} be a frame, and let $\vec{x}$ be a physical vector. Then,

$$|\vec{x}| = \sqrt{\vec{x}'\vec{x}} = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{\vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}} \vec{x}\Big|_{\mathrm{A}}} = \Big\|\vec{x}\Big|_{\mathrm{A}}\Big\|. \tag{2.4.28}$$

Note that $\vec{x}'\vec{x} = |\vec{x}|^2$.

Let $x, y \in \mathbb{R}^3$, let F_{A} be a frame, and define $\vec{x} \triangleq \mathrm{F}_{\mathrm{A}} x$ and $\vec{y} \triangleq \mathrm{F}_{\mathrm{A}} y$. Then, the cross product of x and y is defined by

$$x \times y = \vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}} \triangleq (\vec{x}\times\vec{y})\Big|_{\mathrm{A}}. \tag{2.4.29}$$

Therefore,

$$\begin{aligned}
\vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}} &= (\vec{x}\times\vec{y})\Big|_{\mathrm{A}} \\
&= [(x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}) \times (y_1\hat{\imath}_{\mathrm{A}} + y_2\hat{\jmath}_{\mathrm{A}} + y_3\hat{k}_{\mathrm{A}})]\Big|_{\mathrm{A}} \\
&= [(x_2y_3 - x_3y_2)\hat{\imath}_{\mathrm{A}} - (x_1y_3 - x_3y_1)\hat{\jmath}_{\mathrm{A}} + (x_1y_2 - x_2y_1)\hat{k}_{\mathrm{A}}]\Big|_{\mathrm{A}} \\
&= \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.
\end{aligned} \tag{2.4.30}$$

Defining the cross-product matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{\times} \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \tag{2.4.31}$$

which is a 3×3 skew-symmetric matrix, it follows that

$$\vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\times} \vec{y}\Big|_{\mathrm{A}}. \tag{2.4.32}$$

Finally, we have the formal identity

$$\vec{x}\times\vec{y} = \det\begin{bmatrix} \hat{\imath}_{\mathrm{A}} & \hat{\jmath}_{\mathrm{A}} & \hat{k}_{\mathrm{A}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}. \tag{2.4.33}$$

Fact 2.4.3. Let $\vec{x}$, $\vec{y}$, $\vec{z}$ be physical vectors. Then,

$$\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{x} \cdot \vec{y})\vec{z}, \tag{2.4.34}$$

$$(\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{y} \cdot \vec{z})\vec{x}, \tag{2.4.35}$$

$$(\vec{x} \times \vec{y}) \cdot \vec{z} = \vec{x} \cdot (\vec{y} \times \vec{z}). \tag{2.4.36}$$

Furthermore, let $\mathrm{F_A}$ be a frame. Then,

$$(\vec{x} \times \vec{y}) \cdot \vec{z} = \left(\vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}}\right)^{\mathrm{T}} \vec{z}\Big|_{\mathrm{A}} = \det\left[\begin{array}{ccc} \vec{x}\Big|_{\mathrm{A}} & \vec{y}\Big|_{\mathrm{A}} & \vec{z}\Big|_{\mathrm{A}} \end{array}\right]. \tag{2.4.37}$$

If a frame rotates according to the rotation of a rigid body, then the frame is a *body-fixed frame*. A body-fixed frame can be painted on a rigid body. The origin of a body-fixed frame is usually taken to be a point in the body. Vice versa, the orientation of a rigid body is usually defined by the orientation of a body-fixed frame.

The physical vectors $\vec{x}, \vec{y}, \vec{z}$ are *linearly independent* if the only the real numbers α, β, γ that satisfy

$$\alpha\vec{x} + \beta\vec{y} + \gamma\vec{z} = \vec{0} \tag{2.4.38}$$

are $\alpha = \beta = \gamma = 0$. Now, let $\mathrm{F_A}$ be a frame. Then, it can be seen that the physical vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ are linearly independent if and only if the only the real numbers α, β, γ that satisfy

$$\alpha\vec{x}\Big|_{\mathrm{A}} + \beta\vec{y}\Big|_{\mathrm{A}} + \gamma\vec{z}\Big|_{\mathrm{A}} = \vec{0} \tag{2.4.39}$$

are $\alpha = \beta = \gamma = 0$.

Fact 2.4.4. The physical vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ are linearly independent if and only if

$$\vec{x} \cdot (\vec{y} \times \vec{z}) \neq 0. \tag{2.4.40}$$

Proof. Let $\mathrm{F_A}$ be a frame. Then, it follows from (2.4.37) that (2.4.40) is equivalent to

$$\det\left[\begin{array}{ccc} \vec{x}\Big|_{\mathrm{A}} & \vec{y}\Big|_{\mathrm{A}} & \vec{z}\Big|_{\mathrm{A}} \end{array}\right] \neq 0,$$

which is equivalent to the fact that the mathematical vectors $\vec{x}\Big|_{\mathrm{A}}$, $\vec{y}\Big|_{\mathrm{A}}$, $\vec{z}\Big|_{\mathrm{A}}$ are linearly independent, and thus the physical vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, are linearly independent. □

Fact 2.4.5. The physical vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ are linearly independent if and only if, for every physical vector $\vec{w}$, there exist unique real numbers α, β, γ such that

$$\vec{w} = \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}. \tag{2.4.41}$$

2.5 Position Vector

Let x and y be points. Then, the *position of y relative to x* is denoted by $\vec{r}_{y/x}$. Note that $\vec{r}_{x/y} = -\vec{r}_{y/x}$. If, in addition, z is a point, then vector addition yields the "slash and split" identity

$$\vec{r}_{y/x} = \vec{r}_{y/z} + \vec{r}_{z/x}. \tag{2.5.1}$$

This identity can be resolved in $\mathrm{F_A}$ by writing

$$\vec{r}_{y/x}\Big|_{\mathrm{A}} = \vec{r}_{y/z}\Big|_{\mathrm{A}} + \vec{r}_{z/x}\Big|_{\mathrm{A}}. \tag{2.5.2}$$

Equivalently, we can write

$$r_{y/x|\mathrm{A}} = r_{y/z|\mathrm{A}} + r_{z/x|\mathrm{A}}. \tag{2.5.3}$$

2.6 Physical Matrices

Let $\vec{x}_1, \ldots, \vec{x}_n$ and $\vec{y}_1, \ldots, \vec{y}_n$ be physical vectors. Then,

$$\vec{M} \triangleq \sum_{i=1}^{n} \vec{x}_i \vec{y}_i' \tag{2.6.1}$$

is a *physical matrix*. A physical matrix is a second-order component-free tensor. The zero physical matrix is denoted by $\vec{0}$ or just 0. Physical matrices operate on physical vectors according to the rules given below.

Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors, and define

$$\vec{M} \triangleq \vec{x}\vec{y}'. \tag{2.6.2}$$

Then, $\vec{M}$ satisfies the multiplication rules

$$\vec{M}\vec{z} = (\vec{x}\vec{y}')\vec{z} \triangleq \vec{x}\vec{y} \cdot \vec{z} = (\vec{y} \cdot \vec{z})\vec{x}, \tag{2.6.3}$$

$$\vec{z}'\vec{M} = \vec{z}'(\vec{x}\vec{y}') = (\vec{z} \cdot \vec{x})\vec{y}'. \tag{2.6.4}$$

Let $\vec{w}$ and $\vec{v}$ be physical vectors, and define

$$\vec{N} \triangleq \vec{w}\vec{v}'. \tag{2.6.5}$$

Then,

$$\vec{M}\vec{N} = \vec{M}\vec{w}\vec{v}' = \vec{x}(\vec{y} \cdot \vec{w})\vec{v}' = (\vec{y} \cdot \vec{w})\vec{x}\vec{v}', \tag{2.6.6}$$

$$\vec{M}\vec{N}\vec{z} = (\vec{x}\vec{y}')(\vec{w}\vec{v}')\vec{z} = \vec{x}(\vec{y} \cdot \vec{w})(\vec{v} \cdot \vec{z}) = (\vec{y} \cdot \vec{w})(\vec{v} \cdot \vec{z})\vec{x}. \tag{2.6.7}$$

Let $\vec{x}$ and $\vec{y}$ be physical vectors, and define

$$\vec{M} \triangleq \vec{x}\vec{y}'. \tag{2.6.8}$$

Then, the *coform* $\vec{M}'$ of $\vec{M}$ is defined by

$$\vec{M}' \triangleq \vec{y}\vec{x}', \tag{2.6.9}$$

which is also a physical matrix. Furthermore, let $\vec{N}$ and $\vec{L}$ be physical matrices. Then,

$$(\vec{N}+\vec{L})' = \vec{N}' + \vec{L}', \tag{2.6.10}$$

$$(\vec{N}\vec{L})' = \vec{L}'\vec{N}'. \tag{2.6.11}$$

Finally, if $\vec{z}$ is a physical vector, then

$$(\vec{M}\vec{z})' = \vec{z}'\vec{M}'. \tag{2.6.12}$$

Let $\vec{M}$ be a physical matrix, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{M}\vec{x}\vec{y}' = \vec{M}(\vec{x}\vec{y}') = (\vec{M}\vec{x})\vec{y}'. \tag{2.6.13}$$

The physical matrix $\vec{M}$ is *symmetric* if $\vec{M}' = \vec{M}$ and *skew symmetric* if $\vec{M}' = -\vec{M}$.

Fact 2.6.1. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and define

$$\vec{M} \triangleq \vec{x}\vec{y}' - \vec{y}\vec{x}'. \tag{2.6.14}$$

Then, $\vec{M}$ is skew symmetric.

Let $\vec{x}$ and $\vec{y}$ be physical vectors, and let $\mathrm{F_A}$ be a frame. Then, we define

$$(\vec{x}\vec{y}')\Big|_{\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}} \vec{y}\Big|_{\mathrm{A}}^{\mathrm{T}}. \tag{2.6.15}$$

Note that $(\vec{x}\vec{y}')\Big|_{\mathrm{A}}$ is a 3×3 matrix whose rank is 1 if and only if $\vec{x}$ and $\vec{y}$ are nonzero, and whose rank is 0 if and only if either $\vec{x}$ or $\vec{y}$ is zero. Furthermore, if $\vec{w}$ and $\vec{z}$ are physical vectors, then

$$(\vec{x}\vec{y}' + \vec{w}\vec{z}')\Big|_{\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}} \vec{y}\Big|_{\mathrm{A}}^{\mathrm{T}} + \vec{w}\Big|_{\mathrm{A}} \vec{z}\Big|_{\mathrm{A}}^{\mathrm{T}}. \tag{2.6.16}$$

Fact 2.6.2. Let $\vec{M}$ be a physical matrix, let $\vec{z}$ be a physical vector, and let $\mathrm{F_A}$ be a frame. Then,

$$(\vec{M}\vec{z})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} \vec{z}\Big|_{\mathrm{A}}. \tag{2.6.17}$$

Proof. Assuming that $\vec{M}$ has the form (2.6.1),

$$(\vec{M}\vec{z})\Big|_{\mathrm{A}} = \sum_{i=1}^{n} (\vec{y}_i \cdot \vec{z})\vec{x}_i\Bigg|_{\mathrm{A}} = \sum_{i=1}^{n} \vec{y}_i\Big|_{\mathrm{A}}^{\mathrm{T}} \vec{z}\Big|_{\mathrm{A}} \vec{x}_i\Big|_{\mathrm{A}} = \sum_{i=1}^{n} \vec{x}_i\Big|_{\mathrm{A}} \vec{y}_i\Big|_{\mathrm{A}}^{\mathrm{T}} \vec{z}\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} \vec{z}\Big|_{\mathrm{A}}. \qquad \square$$

The following result is analogous to Fact 2.4.1.

Fact 2.6.3. Let $\vec{M}$ and $\vec{N}$ be physical matrices. Then,

$$\vec{M} = \vec{N} \tag{2.6.18}$$

if and only if

$$\vec{M}\Big|_{\mathrm{A}} = \vec{N}\Big|_{\mathrm{A}}. \tag{2.6.19}$$

Fact 2.6.4. Let $\vec{M}$ and $\vec{N}$ be physical matrices. Then,

$$\vec{M} = \vec{N} \tag{2.6.20}$$

if and only if, for all physical vectors $\vec{x}$,

$$\vec{M}\vec{x} = \vec{N}\vec{x}. \tag{2.6.21}$$

Fact 2.6.5. Let $\mathrm{F_A}$ be a frame, let $\vec{M}$ and $\vec{N}$ be physical matrices, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{M}'\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}}^{\mathrm{T}}, \tag{2.6.22}$$

$$(\vec{M}+\vec{N})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} + \vec{N}\Big|_{\mathrm{A}}, \tag{2.6.23}$$

$$(\vec{M}\vec{N})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}}\vec{N}\Big|_{\mathrm{A}}, \tag{2.6.24}$$

$$(\vec{M}\vec{x})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}}\vec{x}\Big|_{\mathrm{A}}, \tag{2.6.25}$$

$$(\vec{x}'\vec{M})\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}}\vec{M}\Big|_{\mathrm{A}}, \tag{2.6.26}$$

$$\vec{x}'\vec{M}\vec{y} = \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}}\vec{M}\Big|_{\mathrm{A}}\vec{y}\Big|_{\mathrm{A}}, \tag{2.6.27}$$

$$\vec{M}\Big|_{\mathrm{A}} = \begin{bmatrix} \hat{\imath}'_{\mathrm{A}}\vec{M}\hat{\imath}_{\mathrm{A}} & \hat{\imath}'_{\mathrm{A}}\vec{M}\hat{\jmath}_{\mathrm{A}} & \hat{\imath}'_{\mathrm{A}}\vec{M}\hat{k}_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}}\vec{M}\hat{\imath}_{\mathrm{A}} & \hat{\jmath}'_{\mathrm{A}}\vec{M}\hat{\jmath}_{\mathrm{A}} & \hat{\jmath}'_{\mathrm{A}}\vec{M}\hat{k}_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}}\vec{M}\hat{\imath}_{\mathrm{A}} & \hat{k}'_{\mathrm{A}}\vec{M}\hat{\jmath}_{\mathrm{A}} & \hat{k}'_{\mathrm{A}}\vec{M}\hat{k}_{\mathrm{A}} \end{bmatrix}. \tag{2.6.28}$$

It can be seen that the coform of a physical vector or a physical matrix is analogous to the transpose of a mathematical vector or a mathematical matrix.

The following definition concerns eigenvalues and eigenvectors of physical matrices.

Definition 2.6.6. Let $\vec{M}$ be a physical matrix, let $\vec{x}$ be a nonzero dimensionless physical vector, let λ be a complex number, and assume that

$$\vec{M}\vec{x} = \lambda\vec{x}. \tag{2.6.29}$$

Then, λ is an *eigenvalue* of $\vec{M}$, and $\vec{x}$ is an *eigenvector* of $\vec{M}$ associated with λ.

The following result shows that the eigenvalues and eigenvectors of a physical matrix correspond to the eigenvalues and eigenvectors of 3×3 matrices.

Fact 2.6.7. Let $\vec{M}$ be a physical matrix, let λ be an eigenvalue of $\vec{M}$, let $\vec{x}$ be a eigenvector of $\vec{M}$ associated with λ, and let $\mathrm{F_A}$ be a frame. Then, λ is an eigenvalue of $\vec{M}\Big|_{\mathrm{A}}$, and $\vec{x}\Big|_{\mathrm{A}}$ is an eigenvector of $\vec{M}\Big|_{\mathrm{A}}$ associated with λ.

2.7 Physical Projector Matrices

Let $\vec{y}$ be a nonzero physical vector. Then, the *physical projector matrix* $\vec{P}_{\vec{y}}$ *onto the line spanned by* $\vec{y}$ is defined by

$$\vec{P}_{\vec{y}} \triangleq \frac{1}{|\vec{y}|^2}\vec{y}\,\vec{y}\,', \tag{2.7.1}$$

and the *physical projector matrix* $\vec{P}_{\vec{y}}$ *onto the plane perpendicular to* $\vec{y}$ is defined by

$$\vec{P}_{\vec{y}\perp} \triangleq \vec{I} - \vec{P}_{\vec{y}}. \tag{2.7.2}$$

Note that

$$\vec{P}_{\vec{y}}^{\,2} = \vec{P}_{\vec{y}}, \tag{2.7.3}$$

$$\vec{P}_{\vec{y}\perp}^{\,2} = \vec{P}_{\vec{y}\perp}. \tag{2.7.4}$$

If $\vec{y}$ has unit length, then

$$\vec{P}_{\hat{y}} = \hat{y}\hat{y}', \tag{2.7.5}$$

$$\vec{P}_{\hat{y}\perp} = \vec{I} - \hat{y}\hat{y}'. \tag{2.7.6}$$

Let $\vec{y}$ be a nonzero physical vector, and let $\vec{x}$ be a physical vector. Then, the *projection of* $\vec{x}$ *onto the line spanned by* $\vec{y}$ is given by

$$\vec{P}_{\vec{y}}\vec{x} = \frac{\vec{x}\cdot\vec{y}}{|\vec{y}|^2}\vec{y}, \tag{2.7.7}$$

and the *projection of* $\vec{x}$ *onto the plane that is perpendicular to* $\vec{y}$ is given by

$$\vec{P}_{\vec{y}\perp}\vec{x} = (\vec{I} - \vec{P}_{\vec{y}})\vec{x} = \vec{x} - \frac{\vec{x}\cdot\vec{y}}{|\vec{y}|^2}\vec{y}. \tag{2.7.8}$$

Note that $\vec{P}_{\vec{y}}\vec{y} = \vec{y}$ and $\vec{P}_{\vec{y}\perp}\vec{y} = \vec{0}$.

Fact 2.7.1. Let $\vec{y}$ be a nonzero physical vector, and let $\vec{x}$ be a physical vector. Then,

$$|\vec{P}_{\vec{y}}\vec{x}| = \frac{|\vec{x}\cdot\vec{y}|}{|\vec{y}|} = |\vec{x}||\cos\theta_{\vec{y}/\vec{x}}|. \tag{2.7.9}$$

Figure 2.7.1 illustrates the physical projector matrix.

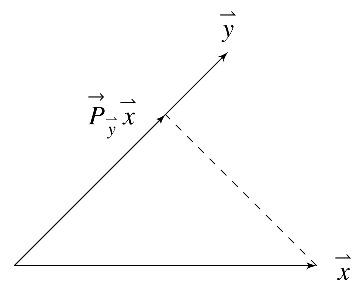


Figure 2.7.1: The projection $\vec{P}_{\vec{y}}\vec{x}$ of $\vec{x}$ onto $\vec{y}$.

Now, let $\vec{y}$ and $\vec{z}$ be nonzero physical vectors that are orthogonal. Then, the *physical projector matrix* $\vec{P}_{\vec{y},\vec{z}}$ *onto the plane spanned by* $\vec{y}$ *and* $\vec{z}$ is defined by

$$\vec{P}_{\vec{y},\vec{z}} \triangleq \vec{P}_{\vec{y}} + \vec{P}_{\vec{z}}. \tag{2.7.10}$$

For each physical vector $\vec{x}$, the *projection of* $\vec{x}$ *onto the plane spanned by* $\vec{y}$ *and* $\vec{z}$ is the physical vector $\vec{P}_{\vec{y},\vec{z}}\vec{x}$ given by

$$\vec{P}_{\vec{y},\vec{z}}\vec{x} = \vec{P}_{\vec{y}}\vec{x} + \vec{P}_{\vec{z}}\vec{x} = \frac{\vec{x}\cdot\vec{y}}{|\vec{y}|^2}\vec{y} + \frac{\vec{x}\cdot\vec{z}}{|\vec{z}|^2}\vec{z}. \tag{2.7.11}$$

If $\vec{y}$ and $\vec{z}$ have unit length, then

$$\vec{P}_{\hat{y},\hat{z}} = \hat{y}\hat{y}' + \hat{z}\hat{z}'. \tag{2.7.12}$$

Finally, if $\vec{y}$ and $\vec{z}$ are not orthogonal, then $\vec{y} - P_{\vec{z}}\vec{y}$ and $\vec{y}$ are orthogonal, and we define

$$\vec{P}_{\vec{y},\vec{z}} \triangleq \vec{P}_{\vec{y}-\vec{P}_{\vec{z}}\vec{y}} + \vec{P}_{\vec{z}}. \tag{2.7.13}$$

Problem 2.26.7 shows that this definition does not depend on the order of $\vec{y}$ and $\vec{z}$.

2.8 Physical Rotation Matrices

Let $\mathrm{F_A}$ be a frame. Then, the *physical identity matrix* $\vec{I}$ is defined by

$$\vec{I} \triangleq \hat{\imath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} = \mathrm{F_A}\mathcal{F}'_{\mathrm{A}}. \tag{2.8.1}$$

The following result shows that $\vec{I}$ is independent of the choice of frame in (2.8.1). Let I_3 denote the 3×3 identity matrix, and let e_i denote the ith column of I_3.

Fact 2.8.1. Let $\mathrm{F_A}$ be a frame, and define $\vec{I}$ by (2.8.1). Then, for all physical vectors $\vec{x}$,

$$\vec{I}\vec{x} = \vec{x}, \tag{2.8.2}$$

and, for all physical covectors $\vec{x}'$,

$$\vec{x}'\vec{I} = \vec{x}'. \tag{2.8.3}$$

Now, let $\mathrm{F_B}$ be a frame. Then,

$$\vec{I}\Big|_{\mathrm{B}} = I_3. \tag{2.8.4}$$

Proof. Writing $\vec{x} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}$, it follows that

$$\vec{I}\vec{x} = (\hat{\imath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\hat{k}'_{\mathrm{A}})(x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}) = \vec{x}.$$

Consequently,

$$\vec{I}\Big|_{\mathrm{B}}\,\vec{x}\Big|_{\mathrm{B}} = \vec{x}\Big|_{\mathrm{B}}.$$

Therefore,

$$\begin{aligned}\vec{I}\Big|_{\mathrm{B}} &= \vec{I}\Big|_{\mathrm{B}} I_3 = \begin{bmatrix} \vec{I}\Big|_{\mathrm{B}} e_1 & \vec{I}\Big|_{\mathrm{B}} e_2 & \vec{I}\Big|_{\mathrm{B}} e_3 \end{bmatrix} = \begin{bmatrix} (\vec{I}\hat{\imath}_{\mathrm{B}})\Big|_{\mathrm{B}} & (\vec{I}\hat{\jmath}_{\mathrm{B}})\Big|_{\mathrm{B}} & (\vec{I}\hat{k}_{\mathrm{B}})\Big|_{\mathrm{B}} \end{bmatrix}\\ &= \begin{bmatrix} \hat{\imath}_{\mathrm{B}}|_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}}|_{\mathrm{B}} & \hat{k}_{\mathrm{B}}|_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = I_3. \qquad \square\end{aligned}$$

Let $\vec{M}$ and $\vec{N}$ be physical matrices. If $\vec{M}\vec{N} = \vec{I}$, then we define

$$\vec{M}^{-1} \triangleq \vec{N}. \tag{2.8.5}$$

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, the *physical rotation matrix* $\vec{R}_{\mathrm{B/A}}$ is defined by

$$\vec{R}_{\mathrm{B/A}} \triangleq \hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{A}}. \tag{2.8.6}$$

Note that

$$\vec{R}_{\mathrm{B/A}} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix}\begin{bmatrix} \hat{\imath}'_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}} \end{bmatrix} = \mathrm{F_B}\mathcal{F}'_{\mathrm{A}}, \tag{2.8.7}$$

$$\vec{R}_{\mathrm{A/A}} = \mathrm{F_A}\mathcal{F}'_{\mathrm{A}} = \vec{I}. \tag{2.8.8}$$

A physical matrix $\vec{R}$ is a *physical rotation matrix* if there exist frames $\mathrm{F_A}$ and $\mathrm{F_B}$ such that $\vec{R} = \vec{R}_{\mathrm{B/A}}$. The following result shows that $\vec{R}_{\mathrm{B/A}}$ rotates $\mathrm{F_A}$ to $\mathrm{F_B}$.

Fact 2.8.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\hat{\imath}_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\hat{\imath}_{\mathrm{A}}, \tag{2.8.9}$$

$$\hat{\jmath}_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\hat{\jmath}_{\mathrm{A}}, \tag{2.8.10}$$

$$\hat{k}_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\hat{k}_{\mathrm{A}}. \tag{2.8.11}$$

Furthermore,

$$\vec{R}_{\mathrm{B/A}} = \vec{R}'_{\mathrm{A/B}} \tag{2.8.12}$$

$$\vec{R}_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}} = \vec{I}, \tag{2.8.13}$$

$$\vec{R}_{\mathrm{B/A}} = \vec{R}^{-1}_{\mathrm{A/B}} = \vec{R}'_{\mathrm{A/B}}. \tag{2.8.14}$$

We thus have

$$\begin{aligned} \mathrm{F_B} &= \begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} \\ &= \begin{bmatrix} \vec{R}_{\mathrm{B/A}}\hat{\imath}_{\mathrm{A}} & \vec{R}_{\mathrm{B/A}}\hat{\jmath}_{\mathrm{A}} & \vec{R}_{\mathrm{B/A}}\hat{k}_{\mathrm{A}} \end{bmatrix} \\ &= \vec{R}_{\mathrm{B/A}}\begin{bmatrix} \hat{\imath}_{\mathrm{A}} & \hat{\jmath}_{\mathrm{A}} & \hat{k}_{\mathrm{A}} \end{bmatrix} \\ &= \vec{R}_{\mathrm{B/A}}\mathrm{F_A}. \end{aligned} \tag{2.8.15}$$

Since $\vec{R}^{-1}_{\mathrm{A/B}} = \vec{R}'_{\mathrm{A/B}}$, it follows that $\vec{R}_{\mathrm{A/B}}$ is an *orthogonal physical matrix*.

Fact 2.8.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, there exists a unique physical rotation matrix $\vec{R}$ such that $\mathrm{F_B} = \vec{R}\mathrm{F_A}$. In particular, $\vec{R} = \vec{R}_{\mathrm{B/A}}$.

2.9 Physical Cross Product Matrix

Let $\vec{x}$ be a physical vector. Then, for all physical vectors $\vec{y}$, the *physical cross product matrix* $\vec{M} \triangleq \vec{x}^{\times}$ is defined by

$$\vec{M}\vec{y} = \vec{x}^{\times}\vec{y} \triangleq \vec{x} \times \vec{y}. \tag{2.9.1}$$

Fact 2.9.1. Let $\vec{x}$ be a physical vector, and let $\mathrm{F_A}$ be a frame. Then,

$$\vec{x}^{\times}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\times} = \begin{bmatrix} 0 & -\hat{k}_{\mathrm{A}}\cdot\vec{x} & \hat{\jmath}_{\mathrm{A}}\cdot\vec{x} \\ \hat{k}_{\mathrm{A}}\cdot\vec{x} & 0 & -\hat{\imath}_{\mathrm{A}}\cdot\vec{x} \\ -\hat{\jmath}_{\mathrm{A}}\cdot\vec{x} & \hat{\imath}_{\mathrm{A}}\cdot\vec{x} & 0 \end{bmatrix}, \tag{2.9.2}$$

$$\vec{x}^{\times} = (\hat{\imath}_{\mathrm{A}}\cdot\vec{x})(\hat{k}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}}) + (\hat{\jmath}_{\mathrm{A}}\cdot\vec{x})(\hat{\imath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}}) + (\hat{k}_{\mathrm{A}}\cdot\vec{x})(\hat{\jmath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}}). \tag{2.9.3}$$

Proof. Let $\vec{y}$ be a physical vector. We thus have

$$\vec{x}^{\times}\Big|_{\mathrm{A}}\vec{y}\Big|_{\mathrm{A}} = (\vec{x}^{\times}\vec{y})\Big|_{\mathrm{A}} = (\vec{x}\times\vec{y})\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\times}\vec{y}\Big|_{\mathrm{A}}.$$

It thus follows from Fact 2.6.4 that $\vec{x}^{\times}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\times}$. The second equality in (2.9.2) follows from (2.4.31). Finally, resolving the right-hand side of (2.9.3) yields the matrix in (2.9.2), and thus the second statement follows from Fact 2.6.3. □

Fact 2.9.2. Let $\vec{x}$ be a physical vector. Then,

$$\vec{x}^{\times\prime} = -\vec{x}^{\times}, \tag{2.9.4}$$

$$\vec{x}^{\times}\vec{x} = 0, \tag{2.9.5}$$

$$\vec{x}^{\prime}\vec{x}^{\times} = 0, \tag{2.9.6}$$

$$\vec{x}^{\times 2} = \vec{x}\vec{x}^{\prime} - |\vec{x}|^2\vec{I}, \tag{2.9.7}$$

$$(\vec{I} + \vec{x}^{\times})^{-1} = \frac{1}{1+|\vec{x}|^2}(\vec{I} + \vec{x}\vec{x}^{\prime} - \vec{x}^{\times}) \tag{2.9.8}$$

$$= \vec{I} + \frac{1}{1+|\vec{x}|^2}(\vec{x}^{\times 2} - \vec{x}^{\times}). \tag{2.9.9}$$

Now, let F_A be a frame, and define $x \triangleq \vec{x}\big|_A$. Then,

$$x^{\times T} = -x^{\times}, \tag{2.9.10}$$

$$x^{\times}x = 0, \tag{2.9.11}$$

$$x^{T}x^{\times} = 0, \tag{2.9.12}$$

$$x^{\times 2} = xx^{T} - x^{T}xI_3, \tag{2.9.13}$$

$$(I_3 + x^{\times})^{-1} = \frac{1}{1+\|x\|^2}(I_3 + xx^{T} - x^{\times}) \tag{2.9.14}$$

$$= I_3 + \frac{1}{1+\|x\|^2}(x^{\times 2} - x^{\times}). \tag{2.9.15}$$

Equation (2.9.4) shows that the physical cross product matrix $\vec{x}^{\times}$ is skew symmetric. The following result provides the converse result, namely, that if the physical matrix $\vec{M}$ is skew symmetric, then it must be a physical cross product matrix.

Fact 2.9.3. Let $\vec{M}$ be a physical matrix, and assume that $\vec{M}$ is skew symmetric. Then, there exists a physical vector $\vec{x}$ such that $\vec{M} = \vec{x}^{\times}$.

Proof. Let F_A be a frame, and define $M \triangleq \vec{M}\big|_A$. Furthermore, define $\vec{x} = -M_{(2,3)}\hat{\imath}_A + M_{(1,3)}\hat{\jmath}_A - M_{(1,2)}\hat{k}_A$. Then, $\vec{M}\big|_A = \vec{x}^{\times}\big|_A$, and thus $\vec{M} = \vec{x}^{\times}$. □

Fact 2.9.4. Let $\vec{x}$ be a physical vector, let α and β be real numbers, and assume that either $\alpha \neq 0$ or $\beta|\vec{x}|^2 \neq 1$. Then,

$$(\vec{I} + \alpha\vec{x}^{\times} + \beta\vec{x}^{\times 2})^{-1} = \vec{I} - \frac{\alpha}{\alpha^2|\vec{x}|^2 + (\beta|\vec{x}|^2 - 1)^2}\vec{x}^{\times} + \frac{\alpha^2 + \beta^2|\vec{x}|^2 - \beta}{\alpha^2|\vec{x}|^2 + (\beta|\vec{x}|^2 - 1)^2}\vec{x}^{\times 2}. \tag{2.9.16}$$

Now, let F_A be a frame, and define $x \triangleq \left.\vec{x}\right|_A$. Then,

$$(I_3 + \alpha x^\times + \beta x^{\times 2})^{-1} = I_3 - \frac{\alpha}{\alpha^2\|x\|^2 + (\beta\|x\|^2 - 1)^2} x^\times + \frac{\alpha^2 + \beta^2\|x\|^2 - \beta}{\alpha^2\|x\|^2 + (\beta\|x\|^2 - 1)^2} x^{\times 2}. \tag{2.9.17}$$

Fact 2.9.5. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$(\vec{x} \times \vec{y})' = -\vec{y}'\vec{x}^\times, \tag{2.9.18}$$

$$(\vec{x} \times \vec{y})^\times = \vec{y}\vec{x}' - \vec{x}\vec{y}', \tag{2.9.19}$$

$$\vec{x}^\times \vec{y}^\times = \vec{y}\vec{x}' - (\vec{y}'\vec{x})\vec{I}. \tag{2.9.20}$$

Now, let F_A be a frame, and define $x \triangleq \left.\vec{x}\right|_A$ and $y \triangleq \left.\vec{y}\right|_A$ Then,

$$(x \times y)^T = -y^T x^\times, \tag{2.9.21}$$

$$(x \times y)^\times = yx^T - xy^T, \tag{2.9.22}$$

$$x^\times y^\times = yx^T - y^T x I_3. \tag{2.9.23}$$

Proof. To prove (2.9.18), note that it follows from (2.9.4) that

$$(\vec{x} \times \vec{y})' = (\vec{x}^\times \vec{y})' = \vec{y}'\vec{x}^{\times\prime} = -\vec{y}'\vec{x}^\times.$$

Next, to prove (2.9.19) let $\vec{z}$ be a physical vector. Then, Fact 2.4.3 implies that

$$(\vec{x} \times \vec{y})^\times \vec{z} = (\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x}'\vec{z})\vec{y} - (\vec{y}'\vec{z})\vec{x} = (\vec{y}\vec{x}' - \vec{x}\vec{y}')\vec{z}.$$

Finally, to prove (2.9.20), let $\vec{z}$ be a physical vector. Then, Fact 2.4.3 implies that

$$\vec{x}^\times \vec{y}^\times \vec{z} = \vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x}'\vec{z})\vec{y} - (\vec{x}'\vec{y})\vec{z} = [\vec{y}\vec{x}' - (\vec{y}'\vec{x})\vec{I}\,]\vec{z}.$$

□

Fact 2.9.6. Let $\mathcal{S}$ be a parallelogram with vertices a, b, c, d so that $\vec{r}_{b/a} = \vec{r}_{d/c}$ and $\vec{r}_{c/a} = \vec{r}_{d/b}$, and let $\theta \in (0, \pi)$ be the angle between $\vec{r}_{b/a}$ and $\vec{r}_{c/a}$. Then,

$$\operatorname{area}(\mathcal{S}) = |\vec{r}_{b/a}||\vec{r}_{c/a}| \sin\theta = |\vec{r}_{b/a} \times \vec{r}_{c/a}| = |(\vec{r}_{b/a}\vec{r}\,'_{c/a} - \vec{r}_{c/a}\vec{r}\,'_{b/a})^{-\times}|. \tag{2.9.24}$$

Now, define $x \triangleq \left.\vec{r}_{b/a}\right|_A$ and $y \triangleq \left.\vec{r}_{c/a}\right|_A$. Then,

$$\operatorname{area}(\mathcal{S}) = \|x\|\|y\| \sin\theta = \|x \times y\| = \|(x \times y)^\times\|_F = \|(xy^T - yx^T)^{-\times}\| = \|xy^T - yx^T\|_F. \tag{2.9.25}$$

Fact 2.9.6 shows that the cross product $\vec{r}_{b/a} \times \vec{r}_{c/a}$ can be viewed as a *directed area*, and likewise for the physical matrix $\vec{r}_{b/a}\vec{r}\,'_{c/a} - \vec{r}_{c/a}\vec{r}\,'_{b/a}$. It will be shown in Chapter 3 that $\vec{r}_{b/a} \wedge \vec{r}\,'_{c/a} = \vec{r}_{b/a} \otimes \vec{r}\,'_{c/a} - \vec{r}_{c/a} \otimes \vec{r}\,'_{b/a} = \vec{r}_{b/a}\vec{r}\,'_{c/a} - \vec{r}_{c/a}\vec{r}\,'_{b/a}$, where $\vec{r}_{b/a} \wedge \vec{r}\,'_{c/a}$ is a bivector

Fact 2.9.7. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors, and let F_A be a frame. Then,

$$(\vec{x} \times \vec{y})'\vec{z} = \vec{x}'(\vec{y} \times \vec{z}) = \det\begin{bmatrix} \left.\vec{x}\right|_A & \left.\vec{y}\right|_A & \left.\vec{z}\right|_A \end{bmatrix}. \tag{2.9.26}$$

Proof. Note that

$$(\vec{x} \times \vec{y})'\vec{z} = -(\vec{y} \times \vec{x})'\vec{z} = -(\vec{y}^{\times}\vec{x})'\vec{z} = -\vec{x}'\vec{y}^{\times\prime}\vec{z} = \vec{x}'\vec{y}^{\times}\vec{z} = \vec{x}'(\vec{y} \times \vec{z}).$$

Finally, note that

$$(\vec{x} \times \vec{y})'\vec{z} = \left(\vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}}\right)^{\mathrm{T}} \vec{z}\Big|_{\mathrm{A}} = \det\begin{bmatrix} \vec{x}\Big|_{\mathrm{A}} & \vec{y}\Big|_{\mathrm{A}} & \vec{z}\Big|_{\mathrm{A}} \end{bmatrix}. \qquad \square$$

Fact 2.9.8. Let $\vec{x}$ be a physical vector, and let $\vec{R}$ be a physical rotation matrix. Then,

$$(\vec{R}\vec{x})^{\times} = \vec{R}\vec{x}^{\times}\vec{R}'. \tag{2.9.27}$$

Proof. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames such that $\vec{R} = \vec{R}_{\mathrm{B/A}}$, and write $\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Using (2.9.3) we have

$$\begin{aligned}
\vec{R}_{\mathrm{B/A}}\vec{x}^{\times}\vec{R}_{\mathrm{A/B}} &= \vec{R}_{\mathrm{B/A}}[x_1(\hat{k}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}}) + x_2(\hat{\imath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}}) + x_3(\hat{\jmath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}})]\vec{R}_{\mathrm{A/B}} \\
&= x_1(\hat{k}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} - \hat{\jmath}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}) + x_2(\hat{\imath}_{\mathrm{B}}\hat{k}'_{\mathrm{B}} - \hat{k}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}}) + x_3(\hat{\jmath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} - \hat{\imath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}}) \\
&= x_1\hat{\imath}^{\times}_{\mathrm{B}} + x_2\hat{\jmath}^{\times}_{\mathrm{B}} + x_3\hat{k}^{\times}_{\mathrm{B}} = (x_1\hat{\imath}_{\mathrm{B}} + x_2\hat{\jmath}_{\mathrm{B}} + x_3\hat{k}_{\mathrm{B}})^{\times} = (\vec{R}_{\mathrm{B/A}}\vec{x})^{\times}. \qquad \square
\end{aligned}$$

Fact 2.9.9. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and let $\vec{R}$ be a physical rotation matrix. Then,

$$\vec{R}(\vec{x} \times \vec{y}) = (\vec{R}\vec{x}) \times (\vec{R}\vec{y}). \tag{2.9.28}$$

Now, let $\mathrm{F_A}$ be a frame and define $x \triangleq \vec{x}\Big|_{\mathrm{A}}$, $y \triangleq \vec{y}\Big|_{\mathrm{A}}$, and $\mathcal{R} \triangleq \vec{R}\Big|_{\mathrm{A}}$. Then,

$$\mathcal{R}(x \times y) = (\mathcal{R}x) \times (\mathcal{R}y). \tag{2.9.29}$$

Proof. Using (2.9.27) it follows that

$$\vec{R}(\vec{x} \times \vec{y}) = \vec{R}\vec{x}^{\times}\vec{y} = \vec{R}\vec{x}^{\times}\vec{R}'\vec{R}\vec{y} = (\vec{R}\vec{x})^{\times}\vec{R}\vec{y} = (\vec{R}\vec{x}) \times (\vec{R}\vec{y}). \qquad \square$$

2.10 Rotation and Orientation Matrices

The following result is needed for the subsequent development.

Fact 2.10.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \cdot \hat{\imath}_{\mathrm{B}} & \hat{\imath}_{\mathrm{A}} \cdot \hat{\jmath}_{\mathrm{B}} & \hat{\imath}_{\mathrm{A}} \cdot \hat{k}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{A}} \cdot \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{A}} \cdot \hat{\jmath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{A}} \cdot \hat{k}_{\mathrm{B}} \\ \hat{k}_{\mathrm{A}} \cdot \hat{\imath}_{\mathrm{B}} & \hat{k}_{\mathrm{A}} \cdot \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{A}} \cdot \hat{k}_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{k}_{\mathrm{B}}|_{\mathrm{A}} \end{bmatrix} = \mathrm{F_B}|_{\mathrm{A}}. \tag{2.10.1}$$

Proof. Note that

$$
\begin{aligned}
\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} &= e_1\hat{\imath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} + e_2\,\hat{\jmath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} + e_3\hat{k}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}\begin{bmatrix} \hat{\imath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \\ \hat{\jmath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \\ \hat{k}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} \hat{\imath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \\ \hat{\jmath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \\ \hat{k}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \\ \hat{\jmath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \\ \hat{k}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \\
&= \begin{bmatrix} \hat{\imath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}\,e_1 & \hat{\imath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}\,e_2 & \hat{\imath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}\,e_3 \\ \hat{\jmath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}\,e_1 & \hat{\jmath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}\,e_2 & \hat{\jmath}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}\,e_3 \\ \hat{k}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}e_1 & \hat{k}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}e_2 & \hat{k}_{\mathrm{A}}|_{\mathrm{B}}^{\mathrm{T}}e_3 \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\mathrm{A}}\cdot\hat{\imath}_{\mathrm{B}} & \hat{\imath}_{\mathrm{A}}\cdot\hat{\jmath}_{\mathrm{B}} & \hat{\imath}_{\mathrm{A}}\cdot\hat{k}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{A}}\cdot\hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{A}}\cdot\hat{\jmath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{A}}\cdot\hat{k}_{\mathrm{B}} \\ \hat{k}_{\mathrm{A}}\cdot\hat{\imath}_{\mathrm{B}} & \hat{k}_{\mathrm{A}}\cdot\hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{A}}\cdot\hat{k}_{\mathrm{B}} \end{bmatrix} \\
&= \begin{bmatrix} \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{k}_{\mathrm{B}}|_{\mathrm{A}} \end{bmatrix} = \mathrm{F_B}|_{\mathrm{A}} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{k}_{\mathrm{B}}|_{\mathrm{A}} \end{bmatrix}\begin{bmatrix} e_1^{\mathrm{T}} \\ e_2^{\mathrm{T}} \\ e_3^{\mathrm{T}} \end{bmatrix} \\
&= \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}}\,e_1^{\mathrm{T}} + \hat{\jmath}_{\mathrm{B}}|_{\mathrm{A}}\,e_2^{\mathrm{T}} + \hat{k}_{\mathrm{B}}|_{\mathrm{A}}e_3^{\mathrm{T}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}. \qquad \square
\end{aligned}
$$

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define the *rotation matrix from* $\mathrm{F_A}$ *to* $\mathrm{F_B}$ to be the 3×3 matrix

$$\mathcal{R}_{\mathrm{B/A}} \triangleq R_{\mathrm{B/A|B}} = R_{\mathrm{B/A|A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \mathrm{F_B}|_{\mathrm{A}}. \tag{2.10.2}$$

Furthermore, define the *orientation matrix of* $\mathrm{F_A}$ *relative* to $\mathrm{F_B}$ to be the 3×3 matrix

$$\mathcal{O}_{\mathrm{A/B}} \triangleq \mathcal{R}_{\mathrm{B/A}}. \tag{2.10.3}$$

Hence,

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = R_{\mathrm{A/B|A}} = R_{\mathrm{A/B|B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} = \mathrm{F_A}|_{\mathrm{B}}. \tag{2.10.4}$$

Fact 2.10.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{\mathrm{T}} = \mathcal{O}_{\mathrm{A/B}}^{\mathrm{T}}, \tag{2.10.5}$$

$$\mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{-1}, \tag{2.10.6}$$

$$\mathcal{O}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{B/A}}^{-1}. \tag{2.10.7}$$

Therefore,

$$\mathcal{R}_{\mathrm{B/A}}^{\mathrm{T}} = \mathcal{R}_{\mathrm{B/A}}^{-1}, \tag{2.10.8}$$

$$\mathcal{O}_{\mathrm{B/A}}^{\mathrm{T}} = \mathcal{O}_{\mathrm{B/A}}^{-1}. \tag{2.10.9}$$

Proof. Note that

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}} = \vec{R}'_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathrm{T}} = \mathcal{R}_{\mathrm{B/A}}^{\mathrm{T}} = \mathcal{O}_{\mathrm{A/B}}^{\mathrm{T}}.$$

Next, since $\vec{I} = \vec{R}_{\mathrm{A/B}}\vec{R}_{\mathrm{B/A}}$, it follows from (2.10.1) that

$$I_3 = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}}\,\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}}\,\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\mathcal{R}_{\mathrm{B/A}}.$$

Hence, $\mathcal{R}_{A/B} = \mathcal{R}_{B/A}^{-1}$. □

It follows from (2.10.8) and (2.10.9) that $\mathcal{R}_{A/B}$ and $\mathcal{O}_{A/B}$ are orthogonal matrices.

Fact 2.10.3. Let F_A and F_B be frames. Then,

$$\mathcal{O}_{A/B} = \begin{bmatrix} \hat{\imath}_A \cdot \hat{\imath}_B & \hat{\imath}_A \cdot \hat{\jmath}_B & \hat{\imath}_A \cdot \hat{k}_B \\ \hat{\jmath}_A \cdot \hat{\imath}_B & \hat{\jmath}_A \cdot \hat{\jmath}_B & \hat{\jmath}_A \cdot \hat{k}_B \\ \hat{k}_A \cdot \hat{\imath}_B & \hat{k}_A \cdot \hat{\jmath}_B & \hat{k}_A \cdot \hat{k}_B \end{bmatrix} = \begin{bmatrix} \hat{\imath}_B|_A & \hat{\jmath}_B|_A & \hat{k}_B|_A \end{bmatrix} = F_B|_A. \tag{2.10.10}$$

We can write $\mathcal{O}_{A/B}$ in terms of row and column vectrices as

$$\mathcal{O}_{A/B} = \begin{bmatrix} \hat{\imath}_A \\ \hat{\jmath}_A \\ \hat{k}_A \end{bmatrix} \cdot \begin{bmatrix} \hat{\imath}_B & \hat{\jmath}_B & \hat{k}_B \end{bmatrix} = \begin{bmatrix} \hat{\imath}'_A \\ \hat{\jmath}'_A \\ \hat{k}'_A \end{bmatrix} \begin{bmatrix} \hat{\imath}_B & \hat{\jmath}_B & \hat{k}_B \end{bmatrix} = \mathcal{F}'_A F_B. \tag{2.10.11}$$

The following result shows that the entries of $\mathcal{O}_{A/B}$ are the cosines of the angles between pairs of vectors in frames F_A and F_B. Consequently, $\mathcal{O}_{A/B}$ is a *direction cosine matrix*.

Fact 2.10.4. Let F_A and F_B be frames. Then,

$$\mathcal{O}_{A/B} = \begin{bmatrix} \cos\theta_{\hat{\imath}_A/\hat{\imath}_B} & \cos\theta_{\hat{\imath}_A/\hat{\jmath}_B} & \cos\theta_{\hat{\imath}_A/\hat{k}_B} \\ \cos\theta_{\hat{\jmath}_A/\hat{\imath}_B} & \cos\theta_{\hat{\jmath}_A/\hat{\jmath}_B} & \cos\theta_{\hat{\jmath}_A/\hat{k}_B} \\ \cos\theta_{\hat{k}_A/\hat{\imath}_B} & \cos\theta_{\hat{k}_A/\hat{\jmath}_B} & \cos\theta_{\hat{k}_A/\hat{k}_B} \end{bmatrix}. \tag{2.10.12}$$

The following result relates column vectrices that represent different frames.

Fact 2.10.5. Let F_A and F_B be frames. Then,

$$\begin{bmatrix} \hat{\imath}_B \\ \hat{\jmath}_B \\ \hat{k}_B \end{bmatrix} = \mathcal{O}_{B/A} \begin{bmatrix} \hat{\imath}_A \\ \hat{\jmath}_A \\ \hat{k}_A \end{bmatrix}, \tag{2.10.13}$$

where

$$\mathcal{O}_{B/A} = \begin{bmatrix} \hat{\imath}_B \cdot \hat{\imath}_A & \hat{\imath}_B \cdot \hat{\jmath}_A & \hat{\imath}_B \cdot \hat{k}_A \\ \hat{\jmath}_B \cdot \hat{\imath}_A & \hat{\jmath}_B \cdot \hat{\jmath}_A & \hat{\jmath}_B \cdot \hat{k}_A \\ \hat{k}_B \cdot \hat{\imath}_A & \hat{k}_B \cdot \hat{\jmath}_A & \hat{k}_B \cdot \hat{k}_A \end{bmatrix}. \tag{2.10.14}$$

Proof. Note that

$$\begin{bmatrix} \hat{\imath}_B \\ \hat{\jmath}_B \\ \hat{k}_B \end{bmatrix} = \begin{bmatrix} \vec{I}\,\hat{\imath}_B \\ \vec{I}\,\hat{\jmath}_B \\ \vec{I}\,\hat{k}_B \end{bmatrix} = \begin{bmatrix} (\hat{\imath}_A\hat{\imath}'_A + \hat{\jmath}_A\hat{\jmath}'_A + \hat{k}_A\hat{k}'_A)\hat{\imath}_B \\ (\hat{\imath}_A\hat{\imath}'_A + \hat{\jmath}_A\hat{\jmath}'_A + \hat{k}_A\hat{k}'_A)\hat{\jmath}_B \\ (\hat{\imath}_A\hat{\imath}'_A + \hat{\jmath}_A\hat{\jmath}'_A + \hat{k}_A\hat{k}'_A)\hat{k}_B \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\imath}'_A\hat{\imath}_B & \hat{\jmath}'_A\hat{\imath}_B & \hat{k}'_A\hat{\imath}_B \\ \hat{\imath}'_A\hat{\jmath}_B & \hat{\jmath}'_A\hat{\jmath}_B & \hat{k}'_A\hat{\jmath}_B \\ \hat{\imath}'_A\hat{k}_B & \hat{\jmath}'_A\hat{k}_B & \hat{k}'_A\hat{k}_B \end{bmatrix} \begin{bmatrix} \hat{\imath}_A \\ \hat{\jmath}_A \\ \hat{k}_A \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\imath}_{\mathrm{B}} \cdot \hat{\imath}_{\mathrm{A}} & \hat{\imath}_{\mathrm{B}} \cdot \hat{\jmath}_{\mathrm{A}} & \hat{\imath}_{\mathrm{B}} \cdot \hat{k}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{B}} \cdot \hat{\imath}_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}} \cdot \hat{\jmath}_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}} \cdot \hat{k}_{\mathrm{A}} \\ \hat{k}_{\mathrm{B}} \cdot \hat{\imath}_{\mathrm{A}} & \hat{k}_{\mathrm{B}} \cdot \hat{\jmath}_{\mathrm{A}} & \hat{k}_{\mathrm{B}} \cdot \hat{k}_{\mathrm{A}} \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix}$$

$$= \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix}. \qquad \square$$

Note that (2.10.13) can be written as

$$\mathcal{F}_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\mathcal{F}_{\mathrm{A}} = \mathcal{R}_{\mathrm{A/B}}\mathcal{F}_{\mathrm{A}}. \tag{2.10.15}$$

Therefore,

$$\mathrm{F}_{\mathrm{B}} = \mathrm{F}_{\mathrm{A}}\mathcal{O}_{\mathrm{A/B}} = \mathrm{F}_{\mathrm{A}}\mathcal{R}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\mathrm{F}_{\mathrm{A}}. \tag{2.10.16}$$

Note the "commuting" property in the last equality in (2.10.16), which implies that

$$\mathrm{F}_{\mathrm{A}} = \vec{R}_{\mathrm{B/A}}\mathrm{F}_{\mathrm{A}}\mathcal{R}_{\mathrm{A/B}} = \vec{R}_{\mathrm{A/B}}\mathrm{F}_{\mathrm{A}}\mathcal{R}_{\mathrm{B/A}}. \tag{2.10.17}$$

Resolving (2.10.16) in F_{A} yields

$$\mathcal{O}_{\mathrm{A/B}} = \mathrm{F}_{\mathrm{B}}\big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}, \tag{2.10.18}$$

whereas resolving (2.10.16) in F_{B} yields

$$I_3 = \mathcal{O}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{B/A}}. \tag{2.10.19}$$

To directly show the equality between the second and fourth terms in (2.10.16), note that

$$\vec{R}_{\mathrm{B/A}}\mathrm{F}_{\mathrm{A}} = \mathrm{F}_{\mathrm{B}}\mathcal{F}'_{\mathrm{A}}\mathrm{F}_{\mathrm{A}} = \mathrm{F}_{\mathrm{B}}I_3 = \mathrm{F}_{\mathrm{B}} = \vec{I}\,\mathrm{F}_{\mathrm{B}} = \mathrm{F}_{\mathrm{A}}\mathcal{F}'_{\mathrm{A}}\mathrm{F}_{\mathrm{B}} = \mathrm{F}_{\mathrm{A}}\mathcal{O}_{\mathrm{A/B}}. \tag{2.10.20}$$

Note that the last three equalities in (2.10.20) show that

$$\mathcal{F}_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\mathcal{F}_{\mathrm{A}}, \tag{2.10.21}$$

which is (2.10.13). Finally, it follows from (2.10.20) and (2.4.13) that

$$\vec{R}_{\mathrm{B/A}} = \mathrm{F}_{\mathrm{A}}\mathcal{O}_{\mathrm{A/B}}\mathcal{F}'_{\mathrm{A}} = \mathrm{F}_{\mathrm{A}}\mathcal{R}_{\mathrm{B/A}}\mathcal{F}'_{\mathrm{A}}. \tag{2.10.22}$$

To relate (2.10.13) to (2.10.10), note that it follows from (2.10.13) that

$$\begin{bmatrix} \hat{\imath}'_{\mathrm{B}} \\ \hat{\jmath}'_{\mathrm{B}} \\ \hat{k}'_{\mathrm{B}} \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}'_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}} \end{bmatrix}, \tag{2.10.23}$$

that is,

$$\mathcal{F}'_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\mathcal{F}'_{\mathrm{A}}. \tag{2.10.24}$$

Now, taking the transpose of (2.10.24) and multiplying on the right by $\mathcal{O}_{\mathrm{B/A}}$ yields

$$\mathrm{F}'_{\mathrm{A}} = \mathrm{F}'_{\mathrm{B}}\mathcal{O}_{\mathrm{B/A}}, \tag{2.10.25}$$

that is,

$$\begin{bmatrix} \hat{\imath}'_{\mathrm{A}} & \hat{\jmath}'_{\mathrm{A}} & \hat{k}'_{\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \hat{\imath}'_{\mathrm{B}} & \hat{\jmath}'_{\mathrm{B}} & \hat{k}'_{\mathrm{B}} \end{bmatrix}\mathcal{O}_{\mathrm{B/A}}. \tag{2.10.26}$$

Furthermore, resolving (2.10.23) in $\mathrm{F_A}$ yields

$$\begin{bmatrix} \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}}^{\mathrm{T}} \\ \hat{\jmath}_{\mathrm{B}}|_{\mathrm{A}}^{\mathrm{T}} \\ \hat{k}_{\mathrm{B}}|_{\mathrm{A}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \hat{\imath}'_{\mathrm{B}}\big|_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{B}}\big|_{\mathrm{A}} \\ \hat{k}'_{\mathrm{B}}\big|_{\mathrm{A}} \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}}, \tag{2.10.27}$$

which implies that

$$\mathcal{O}_{\mathrm{A/B}} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{k}_{\mathrm{B}}|_{\mathrm{A}} \end{bmatrix} = \mathrm{F_B}|_{\mathrm{A}}. \tag{2.10.28}$$

The following identities are useful.

Fact 2.10.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{I} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}'_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}} \end{bmatrix}, \tag{2.10.29}$$

$$\vec{R}_{\mathrm{B/A}} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} \mathcal{O}_{\mathrm{A/B}} \begin{bmatrix} \hat{\imath}'_{\mathrm{B}} \\ \hat{\jmath}'_{\mathrm{B}} \\ \hat{k}'_{\mathrm{B}} \end{bmatrix}. \tag{2.10.30}$$

Proof. It follows from (2.10.23) that

$$\begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}'_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} \begin{bmatrix} \hat{\imath}'_{\mathrm{B}} \\ \hat{\jmath}'_{\mathrm{B}} \\ \hat{k}'_{\mathrm{B}} \end{bmatrix} = \hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + \hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}} = \vec{I}. \qquad \square$$

The following result is especially useful.

Fact 2.10.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_{\mathrm{A}}, \tag{2.10.31}$$

$$\vec{x}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\ \vec{x}\Big|_{\mathrm{A}}. \tag{2.10.32}$$

Proof. Note that

$$\vec{x}\Big|_{\mathrm{B}} = I_3\ \vec{x}\Big|_{\mathrm{B}} = \mathcal{F}'_{\mathrm{B}}\mathrm{F_B}\ \vec{x}\Big|_{\mathrm{B}} = \mathcal{F}'_{\mathrm{B}}\mathrm{F_A}\ \vec{x}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_{\mathrm{A}}. \qquad \square$$

Fact 2.10.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\vec{x}$ be a physical vector, and let $\vec{y} = \vec{R}_{\mathrm{B/A}}\vec{x}$. Then,

$$\vec{y}\Big|_{\mathrm{A}} = \mathcal{R}_{\mathrm{B/A}}\ \vec{x}\Big|_{\mathrm{A}} = \mathcal{R}^2_{\mathrm{B/A}}\ \vec{x}\Big|_{\mathrm{B}}, \tag{2.10.33}$$

$$\vec{y}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{B/A}}\ \vec{x}\Big|_{\mathrm{B}} = \vec{x}\Big|_{\mathrm{A}}. \tag{2.10.34}$$

The following result shows that the orthogonal matrix $\mathcal{O}_{\mathrm{A/B}}$ is proper, that is, $\mathcal{O}_{\mathrm{A/B}}$ is a *rotation matrix*.

Fact 2.10.9. Let F_A and F_B be frames. Then,

$$\det \mathcal{O}_{B/A} = 1. \tag{2.10.35}$$

Proof. Using Fact 2.9.7, it follows that

$$\det \mathcal{O}_{B/A} = \det \left[\begin{array}{ccc} \hat{\imath}_A|_B & \hat{\jmath}_A|_B & \hat{k}_A|_B \end{array}\right] = (\hat{\imath}_A|_B \times \hat{\jmath}_A|_B)^T \hat{k}_A\Big|_B = \hat{k}_A\Big|_B^T \hat{k}_A\Big|_B = 1. \qquad \square$$

Example 2.10.10. Let F_A and F_B be frames such that

$$\hat{\imath}_B = -\hat{k}_A,\ \hat{\jmath}_B = \hat{\jmath}_A,\ \hat{k}_B = \hat{\imath}_A. \tag{2.10.36}$$

Therefore, $\vec{R}_{B/A}$ rotates F_A by $\pi/2$ rad according to the right hand rule around $\hat{\jmath}_A$. Furthermore,

$$\mathcal{O}_{B/A} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{2.10.37}$$

which satisfies (2.10.35). Finally,

$$\left[\begin{array}{ccc} \hat{\imath}_B & \hat{\jmath}_B & \hat{k}_B \end{array}\right] \mathcal{O}_{B/A} \begin{bmatrix} \hat{\imath}'_A \\ \hat{\jmath}'_A \\ \hat{k}'_A \end{bmatrix} = -\hat{\imath}_B\hat{k}'_A + \hat{k}_B\hat{\imath}'_A + \hat{\jmath}_B\hat{\jmath}'_A = \hat{\imath}_A\hat{\imath}'_A + \hat{\jmath}_A\hat{\jmath}'_A + \hat{k}_A\hat{k}'_A = \vec{I},$$

which confirms (2.10.29).

Fact 2.10.11. Let $\vec{M}$ be a physical matrix. Then,

$$\vec{M}\Big|_B = \mathcal{O}_{B/A}\, \vec{M}\Big|_A\, \mathcal{O}_{A/B}. \tag{2.10.38}$$

Proof. Write

$$\vec{M} = \sum_{i=1}^n \vec{x}_i \vec{y}'_i.$$

We thus have

$$\begin{aligned} \vec{M}\Big|_B &= \sum_{i=1}^n \vec{x}_i\Big|_B\, \vec{y}_i\Big|_B^T = \sum_{i=1}^n \mathcal{O}_{B/A}\, \vec{x}_i\Big|_A \left(\mathcal{O}_{B/A}\, \vec{y}_i\Big|_A\right)^T \\ &= \mathcal{O}_{B/A} \sum_{i=1}^n \vec{x}_i\Big|_A\, \vec{y}_i\Big|_A^T \mathcal{O}_{B/A}^T = \mathcal{O}_{B/A}\, \vec{M}\Big|_A\, \mathcal{O}_{A/B}. \qquad \square \end{aligned}$$

Fact 2.10.12. Let $\vec{x}$ be a physical vector, and let F_A and F_B be frames. Then,

$$(\mathcal{R}_{B/A}x)^\times = \mathcal{R}_{B/A}x^\times\mathcal{R}_{A/B}, \tag{2.10.39}$$

$$(\mathcal{O}_{B/A}x)^\times = \mathcal{O}_{B/A}x^\times\mathcal{O}_{A/B}. \tag{2.10.40}$$

Proof. The result follows from Fact 2.9.8. $\square$

Fact 2.10.13. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{O}_{\mathrm{B/A}}\left(\vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}}\right) = \vec{x}\Big|_{\mathrm{B}} \times \vec{y}\Big|_{\mathrm{B}}. \tag{2.10.41}$$

Proof. The result follows from Fact 2.9.9. □

Fact 2.10.14. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Then,

$$\mathcal{O}_{\mathrm{A/B}}\ \vec{x}^{\times}\Big|_{\mathrm{B}} = \vec{x}^{\times}\Big|_{\mathrm{A}}\mathcal{O}_{\mathrm{A/B}}. \tag{2.10.42}$$

That is,

$$\mathcal{O}_{\mathrm{A/B}}\begin{bmatrix} 0 & -\hat{k}_{\mathrm{B}}\cdot\vec{x} & \hat{\jmath}_{\mathrm{B}}\cdot\vec{x} \\ \hat{k}_{\mathrm{B}}\cdot\vec{x} & 0 & -\hat{\imath}_{\mathrm{B}}\cdot\vec{x} \\ -\hat{\jmath}_{\mathrm{B}}\cdot\vec{x} & \hat{\imath}_{\mathrm{B}}\cdot\vec{x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\hat{k}_{\mathrm{A}}\cdot\vec{x} & \hat{\jmath}_{\mathrm{A}}\cdot\vec{x} \\ \hat{k}_{\mathrm{A}}\cdot\vec{x} & 0 & -\hat{\imath}_{\mathrm{A}}\cdot\vec{x} \\ -\hat{\jmath}_{\mathrm{A}}\cdot\vec{x} & \hat{\imath}_{\mathrm{A}}\cdot\vec{x} & 0 \end{bmatrix}\mathcal{O}_{\mathrm{A/B}}. \tag{2.10.43}$$

Fact 2.10.15. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$\vec{R}_{\mathrm{C/A}} = \vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}}. \tag{2.10.44}$$

Furthermore,

$$\mathcal{O}_{\mathrm{C/A}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}, \tag{2.10.45}$$

$$\mathcal{R}_{\mathrm{C/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{C/B}}. \tag{2.10.46}$$

Proof. The first equality follows directly from the definition of the physical rotation matrix. Next, using (2.10.44), (2.6.24), and (2.10.38), we have

$$\mathcal{O}_{\mathrm{C/A}} = \vec{R}_{\mathrm{A/C}}\Big|_{\mathrm{C}} = \left(\vec{R}_{\mathrm{A/B}}\vec{R}_{\mathrm{B/C}}\right)\Big|_{\mathrm{C}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{C}}\vec{R}_{\mathrm{B/C}}\Big|_{\mathrm{C}} = \mathcal{O}_{\mathrm{C/B}}\ \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}}\mathcal{O}_{\mathrm{B/C}}\mathcal{O}_{\mathrm{C/B}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}.$$ □

For four frames we have the following immediate extension.

Fact 2.10.16. Let $\mathrm{F_A}$, $\mathrm{F_B}$, $\mathrm{F_C}$, and $\mathrm{F_D}$ be frames. Then,

$$\vec{R}_{\mathrm{D/A}} = \vec{R}_{\mathrm{D/C}}\vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}}. \tag{2.10.47}$$

Furthermore,

$$\mathcal{O}_{\mathrm{D/A}} = \mathcal{O}_{\mathrm{D/C}}\mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}, \tag{2.10.48}$$

$$\mathcal{R}_{\mathrm{D/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{C/B}}\mathcal{R}_{\mathrm{D/C}}. \tag{2.10.49}$$

For the 3×3 matrix M, the trace of M, which is denoted by $\operatorname{tr} M$, is the sum of the diagonal entries of M. For the physical matrix $\vec{M}$, we define

$$\operatorname{tr}\vec{M} \triangleq \operatorname{tr}\vec{M}\Big|_{\mathrm{A}}, \tag{2.10.50}$$

where $\mathrm{F_A}$ is an arbitrary frame. This definition is independent of the choice of frame since, if $\mathrm{F_B}$ is also a frame, then

$$\operatorname{tr}\vec{M}\Big|_{\mathrm{A}} = \operatorname{tr}\left(\mathcal{O}_{\mathrm{A/B}}\vec{M}\Big|_{\mathrm{B}}\mathcal{O}_{\mathrm{B/A}}\right) = \operatorname{tr}\left(\mathcal{O}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}\vec{M}\Big|_{\mathrm{B}}\right) = \operatorname{tr}\vec{M}\Big|_{\mathrm{B}}. \tag{2.10.51}$$

Note that

$$\operatorname{tr} \vec{M} = \hat{\imath}'_{\mathrm{A}} \vec{M} \hat{\imath}_{\mathrm{A}} + \hat{\jmath}'_{\mathrm{A}} \vec{M} \hat{\jmath}_{\mathrm{A}} + \hat{k}'_{\mathrm{A}} \vec{M} \hat{k}_{\mathrm{A}}. \tag{2.10.52}$$

Likewise, we define

$$\det \vec{M} \triangleq \det \vec{M}\Big|_{\mathrm{A}}, \tag{2.10.53}$$

which is also independent of the choice of frame.

Fact 2.10.17. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\operatorname{tr} \vec{x}\vec{y}' = \vec{y}'\vec{x}. \tag{2.10.54}$$

The following result shows that the trace of a physical rotation matrix lies in the range $[-1, 3]$.

Fact 2.10.18. Let $\vec{R}$ be a physical rotation matrix. Then,

$$-1 \le \operatorname{tr} \vec{R} \le 3. \tag{2.10.55}$$

Furthermore, $\operatorname{tr} \vec{R} = 3$ if and only if $\vec{R} = \vec{I}$.

Proof. Let F_{A} be a frame, and define $\mathcal{R} \triangleq \vec{R}\Big|_{\mathrm{A}}$. Then, it follows from Problem 2.26.15 that the eigenvalues of $\mathcal{R}$ are 1, λ, and $\bar{\lambda}$, where $|\lambda| = |\bar{\lambda}| = 1$. Therefore, $-2 \le \lambda + \bar{\lambda} \le 2$, and thus $-1 \le \lambda + \bar{\lambda} + 1 = \operatorname{tr} \vec{R} \le 3$. Therefore, $\operatorname{tr} \mathcal{R} = \operatorname{tr} \vec{R}\Big|_{\mathrm{A}} = \operatorname{tr} \vec{R} = 3$ if and only if $\mathcal{R} = I_3$, which is the case if and only if $\vec{R} = \vec{I}$. □

The following result is the *Cayley-Hamilton* theorem for physical matrices. This result shows that every physical matrix satisfies a polynomial of degree 3.

Fact 2.10.19. Let $\vec{M}$ be a physical matrix. Then, $\vec{M}$ satisfies

$$\vec{M}^3 - (\operatorname{tr} \vec{M})\vec{M}^2 + \tfrac{1}{2}[(\operatorname{tr} \vec{M})^2 - \operatorname{tr} \vec{M}^2]\vec{M} - (\det \vec{M})\vec{I} = 0. \tag{2.10.56}$$

In addition,

$$\det \vec{M} = \tfrac{1}{3} \operatorname{tr} \vec{M}^3 - \tfrac{1}{2}(\operatorname{tr} \vec{M}) \operatorname{tr} \vec{M}^2 + \tfrac{1}{6}(\operatorname{tr} \vec{M})^3. \tag{2.10.57}$$

Finally, if $\vec{M}$ is nonsingular, then

$$\operatorname{tr} \vec{M}^{-1} = \frac{(\operatorname{tr} \vec{M})^2 - \operatorname{tr} \vec{M}^2}{2 \det \vec{M}}. \tag{2.10.58}$$

Proof. See [1, p. 283] or [5, p. 87]. □

2.11 Eigenaxis Rotations and Rodrigues's Formula

Let $\hat{n}$ be a unit dimensionless physical vector, let $\theta \in (-\pi, \pi]$, and define the physical matrix

$$\vec{R}_{\hat{n}}(\theta) \triangleq (\cos\theta)\vec{I} + (1-\cos\theta)\hat{n}\hat{n}' + (\sin\theta)\hat{n}^{\times}, \tag{2.11.1}$$

which is *Rodrigues's formula*. Equivalently,

$$\vec{R}_{\hat{n}}(\theta) = \hat{n}\hat{n}' + (\cos\theta)(\vec{I} - \hat{n}\hat{n}') + (\sin\theta)\hat{n}^{\times}. \tag{2.11.2}$$

Using (2.9.7), an equivalent form of (2.11.1) is given by

$$\vec{R}_{\hat{n}}(\theta) = \vec{I} + (1-\cos\theta)\hat{n}^{\times 2} + (\sin\theta)\hat{n}^{\times}. \tag{2.11.3}$$

Resolving (2.11.1), (2.11.2), and (2.11.3) in $\mathrm{F_A}$ yields

$$\begin{aligned}\mathcal{R}_n(\theta) &= (\cos\theta)I_3 + (1-\cos\theta)nn^{\mathrm{T}} + (\sin\theta)n^{\times} &(2.11.4)\\ &= nn^{\mathrm{T}} + (\cos\theta)(I_3 - nn^{\mathrm{T}}) + (\sin\theta)n^{\times} &(2.11.5)\\ &= I_3 + (1-\cos\theta)n^{\times 2} + (\sin\theta)n^{\times}, &(2.11.6)\end{aligned}$$

where

$$\mathcal{R}_n(\theta) \triangleq \vec{R}_{\hat{n}}(\theta)\Big|_{\mathrm{A}}, \tag{2.11.7}$$

$$n \triangleq \hat{n}|_{\mathrm{A}}. \tag{2.11.8}$$

Now, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames and assume that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. Then, we write

$$\mathrm{F_A} \xrightarrow[\hat{n}]{\theta} \mathrm{F_B}, \tag{2.11.9}$$

which is equivalent to

$$\mathrm{F_B} \xrightarrow[\hat{n}]{-\theta} \mathrm{F_A}, \tag{2.11.10}$$

$$\mathrm{F_B} \xrightarrow[-\hat{n}]{\theta} \mathrm{F_A}, \tag{2.11.11}$$

$$\mathrm{F_A} \xrightarrow[-\hat{n}]{-\theta} \mathrm{F_B}. \tag{2.11.12}$$

This angle is the *eigenangle* of $\vec{R}_{\hat{n}}(\theta)$, while the unit dimensionless physical $\hat{n}$ is the *eigenaxis* of $\vec{R}_{\hat{n}}(\theta)$. The following result shows that $\vec{R}_{\hat{n}}(\theta)$ is the physical rotation matrix that rotates vectors around the eigenaxis $\hat{n}$ through the eigenangle θ.

Fact 2.11.1. Let $\hat{n}$ be a unit dimensionless physical vector, and let $\theta \in (-\pi, \pi]$. Then, $\vec{R}_{\hat{n}}(\theta)$ is a physical rotation matrix. Furthermore, let $\vec{x}$ be a nonzero physical vector, and let $\vec{x}_{\perp} \triangleq \vec{P}_{\hat{n}\perp}\vec{x}$ be the component of $\vec{x}$ that is perpendicular to $\hat{n}$. Then, the physical vector $\vec{y} = \vec{R}_{\hat{n}}(\theta)\vec{x}$ is obtained by rotating $\vec{x}$ according to the right hand rule around $\hat{n}$ by the angle θ, which is the directed angle

$$\theta = \theta_{\vec{R}_{\hat{n}}(\theta)\vec{x}_{\perp}/\vec{x}_{\perp}/\hat{n}}, \tag{2.11.13}$$

In particular,

$$\vec{R}_{\hat{n}}(\theta)\hat{n} = \hat{n}. \tag{2.11.14}$$

Furthermore,

$$\vec{R}_{\hat{n}}(-\theta) = \vec{R}_{-\hat{n}}(\theta) = \vec{R}{}'_{\hat{n}}(\theta), \tag{2.11.15}$$

$$\vec{R}_{\hat{n}}(\theta) = \vec{R}_{-\hat{n}}(-\theta) = \vec{R}{}'_{\hat{n}}(-\theta). \tag{2.11.16}$$

In addition,

$$\cos\theta = \tfrac{1}{2}(\operatorname{tr}\vec{R}_{\hat{n}}(\theta) - 1), \tag{2.11.17}$$

$$(\sin\theta)\hat{n}^{\times} = \tfrac{1}{2}\left(\vec{R}_{\hat{n}}(\theta) - \vec{R}{}'_{\hat{n}}(\theta)\right). \tag{2.11.18}$$

Furthermore, if $\theta \neq 0$ and $\theta \neq \pi$, then

$$\hat{n}^{\times} = \frac{1}{2\sin\theta}\left(\vec{R}_{\hat{n}}(\theta) - \vec{R}{}'_{\hat{n}}(\theta)\right). \tag{2.11.19}$$

Finally, if $\theta \neq 0$, $\theta \neq \pi$, and $\vec{x}$ and $\hat{n}$ are orthogonal, then

$$\hat{n} = \begin{cases} \hat{\theta}_{\vec{R}_{\hat{n}}(\theta)\vec{x}/\vec{x}}, & \theta > 0, \\ -\hat{\theta}_{\vec{R}_{\hat{n}}(\theta)\vec{x}/\vec{x}}, & \theta < 0. \end{cases} \tag{2.11.20}$$

Proof. Using (2.9.7) we have

$$\begin{aligned}
\vec{R}_{\hat{n}}(\theta)\vec{R}{}'_{\hat{n}}(\theta) &= (\cos\theta)^2\vec{I} + 2(\cos\theta)(1-\cos\theta)\hat{n}\hat{n}' - (\cos\theta)(\sin\theta)\hat{n}^{\times} \\
&\quad + (\cos\theta)(\sin\theta)\hat{n}^{\times} + (1-\cos\theta)^2\hat{n}\hat{n}' - (\sin\theta)^2(\hat{n}^{\times})^2 \\
&= (\cos\theta)^2\vec{I} + (1-\cos^2\theta)\hat{n}\hat{n}' - (\sin\theta)^2(\hat{n}\hat{n}' - \vec{I}) \\
&= (\cos^2\theta + \sin^2\theta)\vec{I} + (1 - \cos^2\theta - \sin^2\theta)\hat{n}\hat{n}' \\
&= \vec{I}.
\end{aligned}$$

To prove that $\vec{R}_{\hat{n}}(\theta)$ is proper, let $\mathrm{F_A}$ be such that $\hat{n} = \hat{\imath}_{\mathrm{A}}$. Then,

$$\det \vec{R}_{\hat{n}}(\theta)\Big|_{\mathrm{A}} = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = 1.$$

Next, to demonstrate the effect of applying $\vec{R}_{\hat{n}}(\theta)$ to $\vec{x}$, we write $\vec{x} = x_{\mathrm{par}}\hat{n} + \vec{x}_{\perp}$. We then have

$$\begin{aligned}
\vec{R}_{\hat{n}}(\theta)\vec{x} &= (\cos\theta)\vec{x} + x_{\mathrm{par}}(1-\cos\theta)\hat{n} + (\sin\theta)\hat{n}\times\vec{x}_{\perp} \\
&= x_{\mathrm{par}}(\cos\theta)\hat{n} + (\cos\theta)\vec{x}_{\perp} + x_{\mathrm{par}}(1-\cos\theta)\hat{n} + (\sin\theta)\hat{n}\times\vec{x}_{\perp} \\
&= x_{\mathrm{par}}\hat{n} + [(\cos\theta)\hat{p} + (\sin\theta)\hat{n}\times\vec{x}_{\perp}],
\end{aligned}$$

which shows that $\vec{R}_{\hat{n}}(\theta)$ rotates $\vec{x}$ according to the right hand rule around $\hat{n}$ by the angle θ.

Finally, (2.11.17) and (2.11.18) follow from (2.11.1). □

Fact 2.11.2. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors, assume that $\vec{x}$ and $\vec{y}$ are not parallel, let $\hat{n}$ be a unit dimensionless physical vector that is orthogonal to both $\vec{x}$ and $\vec{y}$, and define $\theta \triangleq \theta_{\vec{y}/\vec{x}/\hat{n}}$. Then,

$$\vec{y} = \frac{|\vec{y}|}{|\vec{x}|}\vec{R}_{\hat{n}}(\theta)\vec{x} \tag{2.11.21}$$

$$= \frac{|\vec{y}|}{|\vec{x}|}\Big[(\cos\theta)\vec{x} + (\sin\theta)\hat{n}\times\vec{x}\Big]. \tag{2.11.22}$$

Proof. First note that $\hat{n} = (\operatorname{sign}\theta)(\vec{x}\times\vec{y})/|\vec{x}\times\vec{y}|$. It thus follows from (2.11.1), (2.4.35), and (2.1.1) that

$$\begin{aligned}
\vec{R}_{\hat{n}}(\theta)\vec{x} &= [(\cos\theta)\vec{I} + (1-\cos\theta)\hat{n}\hat{n}' + (\sin\theta)\hat{n}^\times]\vec{x} \\
&= (\cos\theta)\vec{x} + (1-\cos\theta)\hat{n}\hat{n}'\vec{x} + (\sin\theta)\hat{n}\times\vec{x} \\
&= (\cos\theta)\vec{x} + (\sin\theta)\hat{n}\times\vec{x} \\
&= (\cos\theta)\vec{x} + \frac{(\operatorname{sign}\theta)\sin\theta}{|\vec{x}\times\vec{y}|}(\vec{x}\times\vec{y})\times\vec{x} \\
&= (\cos\theta)\vec{x} + \frac{(\operatorname{sign}\theta)\sin\theta}{|\vec{x}\times\vec{y}|}[(\vec{x}\cdot\vec{x})\vec{y} - (\vec{x}\cdot\vec{y})\vec{x}] \\
&= (\cos\theta)\vec{x} + \frac{(\operatorname{sign}\theta)\sin\theta}{|\vec{x}||\vec{y}||\sin\theta|}[|\vec{x}|^2\vec{y} - |\vec{x}||\vec{y}|(\cos\theta)\vec{x}] \\
&= (\cos\theta)\vec{x} + \frac{1}{|\vec{x}||\vec{y}|}[|\vec{x}|^2\vec{y} - |\vec{x}||\vec{y}|(\cos\theta)\vec{x}] \\
&= \frac{|\vec{x}|}{|\vec{y}|}\vec{y}.
\end{aligned}$$

□

Note that (2.11.22) can be written as

$$\vec{y} = \frac{|\vec{y}|}{|\vec{x}|}\vec{M}\vec{x}, \tag{2.11.23}$$

where

$$\vec{M} \triangleq (\cos\theta)\vec{I} + (\sin\theta)\hat{\theta}^\times_{\vec{y}/\vec{x}}. \tag{2.11.24}$$

However, $\vec{M}$ does not necessarily satisfy $\vec{M}\vec{M}' = \vec{I}$, and thus is not necessarily a physical rotation matrix.

The following result shows that the eigenaxis vector $\hat{n}$ has the same components when resolved in both frames. Consequently, when the two frames coincide with the same body-fixed frame for a rigid body before and after rotation, the vector $\hat{n}$ can be viewed as body-fixed.

Fact 2.11.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\hat{n}$ be a unit dimensionless physical vector, let $\theta \in (-\pi, \pi]$, and assume that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. Then,

$$\hat{n}|_{\mathrm{B}} = \hat{n}|_{\mathrm{A}}\,. \tag{2.11.25}$$

Proof. Note that

$$\hat{n}|_{\mathrm{B}} = (\vec{R}_{\mathrm{B/A}}\hat{n})\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}}\ \hat{n}|_{\mathrm{B}} = \mathcal{O}_{\mathrm{A/B}}\ \hat{n}|_{\mathrm{B}} = \hat{n}|_{\mathrm{A}}\,. \qquad \square$$

Each pair of frames $\mathrm{F_A}$ and $\mathrm{F_B}$ is related by a unique physical rotation matrix, namely, $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A}$. The following result is *Euler's theorem*, which states that every physical rotation matrix can be expressed in the eigenaxis/eigenangle form (2.11.1). In particular, this result provides explicit expressions for the eigenaxis and eigenangle for a given physical rotation matrix.

We consider three cases separately. In the first case, the rotation is through an eigenangle of 0 rad, and thus the eigenaxis is arbitrary. In the second case, the frames are related by a rotation through an eigenangle of π rad, and the eigenaxis can be chosen in two distinct ways. In the last case, there are two distinct eigenangles in the range $(-\pi, \pi)$ with corresponding the eigenaxes. These cases are distinguished by the trace inequalities given by Fact 2.10.18.

We first consider a rotation through 0 rad. This case corresponds to the condition $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = 3$, which occurs when the upper bound is attained in (2.10.55).

Fact 2.11.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that

$$\operatorname{tr}\vec{R}_{\mathrm{B/A}} = 3. \tag{2.11.26}$$

Then, for every unit dimensionless physical vector $\hat{n}$, $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(0) = \vec{I}$.

We next consider a rotation through an eigenangle of π rad. This case corresponds to the condition $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = -1$, which occurs when the lower bound is attained in (2.10.55). In this case there is one eigenangle and two distinct eigenaxes.

Fact 2.11.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that

$$\operatorname{tr}\vec{R}_{\mathrm{B/A}} = -1. \tag{2.11.27}$$

Then, there exist exactly two representations of $\vec{R}_{\mathrm{B/A}}$ of the form (2.11.1). In particular, let $\hat{n}_{\mathrm{B/A}}$ be a unit dimensionless physical vector satisfying

$$\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} = \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} + \vec{I}\right) \tag{2.11.28}$$

or, equivalently,

$$\hat{n}^{\times 2}_{\mathrm{B/A}} = \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} - \vec{I}\right). \tag{2.11.29}$$

Therefore,

$$n_{\mathrm{B/A}}n^{\mathrm{T}}_{\mathrm{B/A}} = \tfrac{1}{2}(\mathcal{R}_{\mathrm{B/A}} + I_3), \tag{2.11.30}$$

$$n^{\times 2}_{\mathrm{B/A}} = \tfrac{1}{2}(\mathcal{R}_{\mathrm{B/A}} - I_3). \tag{2.11.31}$$

Then, $\vec{R}_{\mathrm{B/A}}$ has the two representations

$$\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}_{\mathrm{B/A}}}(\pi) = \vec{R}_{-\hat{n}_{\mathrm{B/A}}}(\pi). \tag{2.11.32}$$

Furthermore,

$$\mathcal{R}_{\mathrm{B/A}} = -I_3 + 2n_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\mathrm{T}} \tag{2.11.33}$$

$$= I_3 + 2n_{\mathrm{B/A}}^{\times 2}, \tag{2.11.34}$$

where

$$n_{\mathrm{B/A}} \triangleq \hat{n}_{\mathrm{B/A}}\big|_{\mathrm{B}} = \hat{n}_{\mathrm{B/A}}\big|_{\mathrm{A}}. \tag{2.11.35}$$

Finally, the eigenvalues of $\mathcal{R}_{\mathrm{B/A}}$ are 1, -1, and -1,

$$\mathcal{R}_{\mathrm{B/A}}n_{\mathrm{B/A}} = n_{\mathrm{B/A}}, \tag{2.11.36}$$

and, if $m \in \mathbb{R}^3$ satisfies $m^{\mathrm{T}}n_{\mathrm{B/A}} = 0$, then

$$\mathcal{R}_{\mathrm{B/A}}m = -m. \tag{2.11.37}$$

It follows from (2.11.28) that

$$|\hat{n}_{\mathrm{B/A}}|^2 = \hat{n}_{\mathrm{B/A}}'\hat{n}_{\mathrm{B/A}} = \operatorname{tr}\hat{n}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}' = \tfrac{1}{2}\operatorname{tr}\left(\vec{R}_{\mathrm{B/A}} + \vec{I}\right) = \tfrac{1}{2}(-1+3) = 1.$$

Therefore, $\hat{n}_{\mathrm{B/A}}$ satisfying (2.11.28) is a unit vector.

In Fact 2.11.5 the unit vector $n_{\mathrm{B/A}}$ satisfies (2.11.30). Consequently, the matrix $\frac{1}{2}(\mathcal{R}_{\mathrm{B/A}} + I_3)$ is positive semidefinite and has rank 1. Thus, there exist exactly two vectors $x \in \mathbb{R}^3$ that satisfy $xx' = \frac{1}{2}(\mathcal{R}_{\mathrm{B/A}} + I_3)$, namely, $x = n_{\mathrm{B/A}}$ and $x = -n_{\mathrm{B/A}}$.

In the last case, the eigenangle is assumed to be neither 0 rad nor π rad. This condition is equivalent to strict inequality in the lower and upper bounds in (2.11.38). In this case, there are two distinct eigenangles and two distinct eigenaxes.

Fact 2.11.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that

$$-1 < \operatorname{tr}\vec{R}_{\mathrm{B/A}} < 3. \tag{2.11.38}$$

Then, there exist exactly two representations of $\vec{R}_{\mathrm{B/A}}$ of the form (2.11.1). In particular, let $\theta_{\mathrm{B/A}} \in (0,\pi)$ satisfy

$$\cos\theta_{\mathrm{B/A}} = \tfrac{1}{2}(\operatorname{tr}\vec{R}_{\mathrm{B/A}} - 1), \tag{2.11.39}$$

and let $\hat{n}_{\mathrm{B/A}}$ be the unit dimensionless physical vector satisfying

$$\hat{n}_{\mathrm{B/A}}^{\times} = \frac{1}{2\sin\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}_{\mathrm{B/A}}'\right). \tag{2.11.40}$$

Then, the eigenvalues of $\vec{R}_{\mathrm{B/A}}$ are $1, \lambda, \overline{\lambda}$, where $\lambda \triangleq \cos\theta_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\jmath$. Furthermore,

$$\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) = \vec{R}_{-\hat{n}_{\mathrm{B/A}}}(-\theta_{\mathrm{B/A}}), \tag{2.11.41}$$

and thus

$$\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}, \tag{2.11.42}$$

$$n_{\mathrm{B/A}}^{\times} = \frac{1}{2\sin\theta_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}}), \tag{2.11.43}$$

$$\mathcal{R}_{\mathrm{B/A}} = (\cos\theta_{\mathrm{B/A}})I_3 + (1-\cos\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\mathrm{T}} + (\sin\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times} \tag{2.11.44}$$

$$= I_3 + (1-\cos\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times 2} + (\sin\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times}, \tag{2.11.45}$$

where

$$n_{\mathrm{B/A}} \triangleq \hat{n}_{\mathrm{B/A}}\big|_{\mathrm{B}} = \hat{n}_{\mathrm{B/A}}\big|_{\mathrm{A}}. \tag{2.11.46}$$

Furthermore,

$$\mathcal{R}_{\mathrm{B/A}}n_{\mathrm{B/A}} = n_{\mathrm{B/A}}. \tag{2.11.47}$$

Finally, let $m = m_1 + m_2 \jmath \in \mathbb{C}^3$, where $m_1, m_2 \in \mathbb{R}^3$ satisfy $n_{\mathrm{B/A}}^{\mathrm{T}}m_1 = n_{\mathrm{B/A}}^{\mathrm{T}}m_2 = m_1^{\mathrm{T}}m_2 = 0$. Then,

$$\mathcal{R}_{\mathrm{B/A}}m = [\cos\theta_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\jmath]m. \tag{2.11.48}$$

Proof. Since $\vec{R}_{\mathrm{B/A}}$ is a physical rotation matrix, it follows that one of its eigenvalues is 1. It follows from (2.11.38) that the remaining eigenvalues of $\vec{R}_{\mathrm{B/A}}$ are not real. Furthermore, it follows from (2.11.39) that $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = 1 + 2\cos\theta_{\mathrm{B/A}}$, and thus the eigenvalues of $\vec{R}_{\mathrm{B/A}}$ are $1, \lambda, \overline{\lambda}$, where λ satisfies $|\lambda| = 1$ and λ is neither 1 nor -1.

To show that $\hat{n}_{\mathrm{B/A}}$ given by (2.11.40) is a unit vector, note that

$$\hat{n}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}' - |\hat{n}_{\mathrm{B/A}}|^2\vec{I} = \hat{n}_{\mathrm{B/A}}^{\times 2} = \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}_{\mathrm{B/A}}'\right)^2 = \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}}^2 - 2\vec{I} + \vec{R}_{\mathrm{B/A}}'^2\right).$$

Taking the trace yields

$$-2|\hat{n}_{\mathrm{B/A}}|^2 = \frac{\operatorname{tr}\vec{R}_{\mathrm{B/A}}^2 - 3}{2\sin^2\theta_{\mathrm{B/A}}}.$$

Hence

$$|\hat{n}_{\mathrm{B/A}}|^2 = \frac{3 - \operatorname{tr}\vec{R}_{\mathrm{B/A}}^2}{4\sin^2\theta_{\mathrm{B/A}}} = \frac{3 - (1 + \lambda^2 + \overline{\lambda}^2)}{4\sin^2\theta_{\mathrm{B/A}}} = \frac{1 + \sin^2\theta_{\mathrm{B/A}} - \cos^2\theta_{\mathrm{B/A}}}{2\sin^2\theta_{\mathrm{B/A}}} = 1.$$

Next, note that

$$\begin{aligned}
\hat{n}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}' &= \hat{n}_{\mathrm{B/A}}^{\times 2} + \vec{I} \\
&= \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}_{\mathrm{B/A}}'\right)^2 + \vec{I} \\
&= \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}}^2 - 2\vec{I} + \vec{R}_{\mathrm{B/A}}'^2\right) + \vec{I} \\
&= \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}}^2 + \vec{R}_{\mathrm{B/A}}'^2\right) + \vec{I} - \frac{1}{2\sin^2\theta_{\mathrm{B/A}}}\vec{I}.
\end{aligned}$$

Therefore, using (2.11.39), (2.11.40), and Problem 2.26.17 it follows that

$$\begin{aligned}
&\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}})\\
&= (\cos\theta_{\mathrm{B/A}})\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}\\
&= (\cos\theta_{\mathrm{B/A}})\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}^2_{\mathrm{B/A}} + \vec{R}'^2_{\mathrm{B/A}}\right) + (1-\cos\theta_{\mathrm{B/A}})\vec{I} - \frac{1-\cos\theta_{\mathrm{B/A}}}{2\sin^2\theta_{\mathrm{B/A}}}\vec{I} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}\\
&= \vec{I} + \frac{1}{4(1+\cos\theta_{\mathrm{B/A}})}\left(\vec{R}^2_{\mathrm{B/A}} + \vec{R}'^2_{\mathrm{B/A}}\right) - \frac{1}{2(1+\cos\theta_{\mathrm{B/A}})}\vec{I} + \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right)\\
&= \frac{1+2\cos\theta_{\mathrm{B/A}}}{2(1+\cos\theta_{\mathrm{B/A}})}\vec{I} + \frac{1}{4(1+\cos\theta_{\mathrm{B/A}})}\left(\vec{R}^2_{\mathrm{B/A}} + \vec{R}'^2_{\mathrm{B/A}}\right) + \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right) = \vec{R}_{\mathrm{B/A}}. \qquad \square
\end{aligned}$$

Note that, in Fact 2.11.6, $\theta_{\mathrm{B/A}}$ is defined such that $\theta_{\mathrm{B/A}} \in (0,\pi)$. In fact,

$$\theta_{\mathrm{B/A}} = \cos^{-1}[\tfrac{1}{2}(\mathrm{tr}\,\vec{R}_{\mathrm{B/A}} - 1)]. \tag{2.11.49}$$

Furthermore, since Rodrigues's formula (2.11.1) is defined for $\theta \in (-\pi,\pi]$, within the context of Fact 2.11.4, we define $\theta_{\mathrm{B/A}} = 0$, while, within the context of Fact 2.11.5, we define $\theta_{\mathrm{B/A}} = \pi$. Consequently, in all cases, (2.11.49) is valid, and the notation $\theta_{\mathrm{B/A}}$ denotes an element of $[0,\pi]$, despite the fact that θ in Rodrigues's formula (2.11.1) may be an element of $(-\pi,\pi]$. Since, by definition, $\theta_{\mathrm{B/A}} \in [0,\pi]$ in all cases, it follows that

$$\theta_{\mathrm{B/A}} = \theta_{\mathrm{A/B}}. \tag{2.11.50}$$

If $\theta_{\mathrm{B/A}} \in (0,\pi)$, then the corresponding eigenaxis $\hat{n}_{\mathrm{B/A}}$ is unique. Therefore, if $\theta_{\mathrm{B/A}} \in (0,\pi)$, then

$$\hat{n}_{\mathrm{B/A}} = -\hat{n}_{\mathrm{A/B}}. \tag{2.11.51}$$

However, in the case $\theta_{\mathrm{B/A}} = 0$, the eigenaxis $\hat{n}_{\mathrm{B/A}}$ is arbitrary, and thus (2.11.51) is not meaningful. Furthermore, in the case $\theta_{\mathrm{A/B}} = \pi$, there exist exactly two choices of $\hat{n}_{\mathrm{B/A}}$, which are related by the factor -1. Therefore, the notation $\hat{n}_{\mathrm{B/A}}$ is ambiguous, and thus (2.11.51) is not meaningful.

The following result states Euler's theorem for the three cases considered in Fact 2.11.4, Fact 2.11.5, and Fact 2.11.6.

Fact 2.11.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, there exist a unit dimensionless physical vector $\hat{n}$ and $\theta \in (-\pi,\pi]$ such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. In addition,

$$\hat{n}\big|_{\mathrm{B}} = \hat{n}\big|_{\mathrm{A}}. \tag{2.11.52}$$

Proof. Since $\vec{R}_{\mathrm{B/A}}$ is a physical rotation matrix, there exists a unit dimensionless physical vector $\hat{n}$ such that $\hat{n} = \vec{R}_{\mathrm{B/A}}\hat{n}$. If $\vec{R}_{\mathrm{B/A}} = \vec{I}$, then the result holds with $\theta = 0$. Now assume that $\vec{R}_{\mathrm{B/A}} \neq \vec{I}$. Since $\vec{R}_{\mathrm{B/A}}$ has exactly one eigenvalue equal to 1 and since $\hat{n}$ is a unit dimensionless eigenvector of $\vec{R}_{\mathrm{B/A}}$, it follows that $\hat{n}$ must be equal to either $\hat{n}_{\mathrm{B/A}}$ or $-\hat{n}_{\mathrm{B/A}}$. In both cases, there exists $\theta \in (-\pi,\pi]$ such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. In particular, if $\hat{n} = \hat{n}_{\mathrm{B/A}}$, then $\theta = \theta_{\mathrm{B/A}}$, whereas, if $\hat{n} = -\hat{n}_{\mathrm{B/A}}$, then $\theta = \theta_{\mathrm{B/A}} - 2\pi$. $\square$

The following result considers the reverse rotation in all three cases. This is a restatement of (2.11.1), (2.11.2), and (2.11.3) with $\theta = \theta_{\rm B/A}$ and $\hat{n} = \hat{n}_{\rm B/A}$ using (2.11.15), (2.11.50), and (2.11.51).

Fact 2.11.8. Let $\rm F_A$ and $\rm F_B$ be frames, and assume that $\theta_{\rm B/A} \in (0,\pi)$. Then,

$$\vec{R}_{\rm B/A} = (\cos\theta_{\rm B/A})\vec{I} + (1-\cos\theta_{\rm B/A})\hat{n}_{\rm B/A}\hat{n}'_{\rm B/A} + (\sin\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A}, \tag{2.11.53}$$

$$\vec{R}_{\rm A/B} = (\cos\theta_{\rm B/A})\vec{I} + (1-\cos\theta_{\rm B/A})\hat{n}_{\rm B/A}\hat{n}'_{\rm B/A} - (\sin\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A}, \tag{2.11.54}$$

$$\vec{R}_{\rm B/A} = \hat{n}_{\rm B/A}\hat{n}'_{\rm B/A} + (\cos\theta_{\rm B/A})(\vec{I} - \hat{n}_{\rm B/A}\hat{n}'_{\rm B/A}) + (\sin\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A}, \tag{2.11.55}$$

$$\vec{R}_{\rm A/B} = \hat{n}_{\rm B/A}\hat{n}'_{\rm B/A} + (\cos\theta_{\rm B/A})(\vec{I} - \hat{n}_{\rm B/A}\hat{n}'_{\rm B/A}) - (\sin\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A}, \tag{2.11.56}$$

$$\vec{R}_{\rm B/A} = \vec{I} + (1-\cos\theta_{\rm B/A})\hat{n}^{\times 2}_{\rm B/A} + (\sin\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A}, \tag{2.11.57}$$

$$\vec{R}_{\rm A/B} = \vec{I} + (1-\cos\theta_{\rm B/A})\hat{n}^{\times 2}_{\rm B/A} - (\sin\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A}. \tag{2.11.58}$$

The following result assigns a physical rotation matrix to a pair of physical vectors of the same length. This physical rotation matrix is defined in terms of an eigenaxis rotation, which is unique if and only if the eigenangle is neither 0 rad nor π rad.

Fact 2.11.9. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and assume that $|\vec{x}| = |\vec{y}| \neq 0$. Then, there exists a physical rotation matrix $\vec{R}$ such that $\vec{y} = \vec{R}\vec{x}$. Furthermore, suppose that $\vec{R}$ is a physical rotation matrix such that $\vec{y} = \vec{R}\vec{x}$. Then, the following statements hold:

i) If either $\theta_{\hat{y}/\hat{x}} = 0$ or $\theta_{\hat{y}/\hat{x}} = \pi$, then, for every unit dimensionless physical vector $\hat{n}$ such that $\hat{n}'\vec{x} = 0$, it follows that $\vec{R} = \vec{R}_{\hat{n}}(\theta_{\hat{y}/\hat{x}})$.

ii) If $\theta_{\hat{y}/\hat{x}} \in (0,\pi)$, then there exists a unique unit dimensionless physical vector $\hat{n}$ such that $\hat{n}'\vec{x} = \hat{n}'\vec{y} = 0$ and $\vec{R} = \vec{R}_{\hat{n}}(\theta_{\hat{y}/\hat{x}})$. In particular, $\hat{n} = \hat{\theta}_{\hat{y}/\hat{x}}$. In addition, $\vec{R} = \vec{R}_{-\hat{n}}(-\theta_{\hat{y}/\hat{x}})$.

The following result replaces $\hat{n}^{\times}_{\rm B/A}$ in (2.11.1) by a difference of physical matrices

Fact 2.11.10. Let $\rm F_A$ and $\rm F_B$ be frames, and let $\hat{v}_{\rm B/A}$ and $\hat{w}_{\rm B/A}$ satisfy $\hat{n}_{\rm B/A} = \hat{v}_{\rm B/A} \times \hat{w}_{\rm B/A}$. Then,

$$\vec{R}_{\rm B/A} = (\cos\theta_{\rm B/A})\vec{I} + (1-\cos\theta_{\rm B/A})\hat{n}_{\rm B/A}\hat{n}'_{\rm B/A} + (\sin\theta_{\rm B/A})(\hat{w}_{\rm B/A}\hat{v}'_{\rm B/A} - \hat{v}_{\rm B/A}\hat{w}'_{\rm B/A}). \tag{2.11.59}$$

Proof. The result follows from (2.9.19). □

The following result determines the eigenaxis rotation arising from a pair of eigenaxis rotations.

Fact 2.11.11. Let $\rm F_A$, $\rm F_B$, and $\rm F_C$ be frames. Then,

$$\cos\tfrac{1}{2}\theta_{\rm C/A} = (\cos\tfrac{1}{2}\theta_{\rm C/B})(\cos\tfrac{1}{2}\theta_{\rm B/A}) - (\sin\tfrac{1}{2}\theta_{\rm C/B})(\sin\tfrac{1}{2}\theta_{\rm B/A})\hat{n}'_{\rm C/B}\hat{n}_{\rm B/A}, \tag{2.11.60}$$

$$\begin{aligned}\hat{n}_{\rm C/A} = (\csc\tfrac{1}{2}\theta_{\rm C/B})[&(\sin\tfrac{1}{2}\theta_{\rm C/B})(\cos\tfrac{1}{2}\theta_{\rm B/A})\hat{n}_{\rm C/B} + (\cos\tfrac{1}{2}\theta_{\rm C/B})(\sin\tfrac{1}{2}\theta_{\rm B/A})\hat{n}_{\rm B/A}\\ &+ (\sin\tfrac{1}{2}\theta_{\rm C/B})(\sin\tfrac{1}{2}\theta_{\rm B/A})(\hat{n}_{\rm C/B}\times\hat{n}_{\rm B/A})]\end{aligned}$$

$$= \frac{\cot\frac{1}{2}\theta_{\mathrm{C/A}}}{1-\hat{n}'_{\mathrm{C/B}}\hat{n}_{\mathrm{B/A}}(\tan\frac{1}{2}\theta_{\mathrm{C/B}})\tan\frac{1}{2}\theta_{\mathrm{B/A}}}[(\tan\tfrac{1}{2}\theta_{\mathrm{C/B}})\hat{n}_{\mathrm{C/B}}+(\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}$$
$$+(\tan\tfrac{1}{2}\theta_{\mathrm{C/B}})(\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})(\hat{n}'_{\mathrm{C/B}}\hat{n}_{\mathrm{B/A}})]. \tag{2.11.61}$$

2.12 Euler Rotations and Euler Angles

A rotation of one frame to yield another frame can be achieved through a sequence of three eigenaxis rotations, where each eigenaxis is chosen to be an axis of either the initial frame or the frame resulting from the preceding rotation. Consequently, every physical rotation matrix can be expressed as the product of three physical rotation matrices, where each physical rotation matrix is an eigenaxis rotation represented by Rodrigues's formula. The three rotations involve a total of four frames, namely, the initial and final frames as well as two intermediate frames. Each eigenaxis rotation is an *Euler rotation*, and the directed angles that define the transformations are the *Euler angles*. Consequently, the orientation of the final frame relative to the initial frame can be expressed as the product of three orientation matrices.

There are twelve different Euler-angle rotation sequences, for example, 1-2-3, where the first rotation is around the $\hat{\imath}$ axis of the initial frame, the second rotation is about the $\hat{\jmath}$ axis of the second frame, and the third rotation is about the $\hat{k}$ axis of the third frame. The 3-2-1 and 3-1-3 rotation sequences are the most frequently used, where 1, 2, and 3 refer to rotations around the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ axes, respectively, of the original and intermediate frames. By renaming the frame axes, it can be seen that there are only two distinct Euler-angle sequences, which can be represented by the rotation sequences 3-2-1 and 3-1-3. Each of the remaining ten rotation sequences is equivalent to one of these two rotation sequences under a renaming of the frame axes.

As an example of an Euler rotation, suppose that the frame $\mathrm{F_B}$ is obtained by rotating the frame $\mathrm{F_A}$ around the eigenaxis $\hat{k}_\mathrm{A}$. Then,

$$\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}}) = \vec{R}_{\hat{k}_\mathrm{A}}(\theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}).$$

Hence, using the notation of (2.11.9),

$$\mathrm{F_A} \xrightarrow[\hat{k}_\mathrm{A}]{\theta} \mathrm{F_B}, \tag{2.12.1}$$

and, since $\hat{k}_\mathrm{B} = \hat{k}_\mathrm{A}$,

$$\mathrm{F_A} \xrightarrow[\hat{k}_\mathrm{B}]{\theta} \mathrm{F_B}, \tag{2.12.2}$$

where θ is the name of the eigenangle, that is, $\theta = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}} = \theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}$. For convenience, we write

$$\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}. \tag{2.12.3}$$

Note that (2.12.1) is equivalent to

$$\mathrm{F_A} \xrightarrow[-\hat{k}_\mathrm{A}]{-\theta} \mathrm{F_B}, \tag{2.12.4}$$

which we write as

$$\mathrm{F_A} \xrightarrow[-3]{-\theta} \mathrm{F_B}. \tag{2.12.5}$$

Equivalently,

$$\mathrm{F_B} \xrightarrow[3]{-\theta} \mathrm{F_A}, \tag{2.12.6}$$

$$\mathrm{F_B} \xrightarrow[-3]{\theta} \mathrm{F_A}. \tag{2.12.7}$$

If $\theta = \pi$, then $-\theta$ must be replaced by π in the above expressions. See (2.3.2).

The following result considers (2.11.1) in the case where the eigenaxis is a frame axis.

Fact 2.12.1. Let $\mathrm{F_A}$ be a frame, and let $\theta \in (-\pi, \pi]$. If $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{\imath}_\mathrm{A}}(\theta)\mathrm{F_A}$, then

$$\mathcal{O}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathrm{T}} = \vec{R}_{\hat{\imath}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{2.12.8}$$

$$\theta = \theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{\imath}_\mathrm{A}} = \theta_{\hat{k}_\mathrm{B}/\hat{k}_\mathrm{A}/\hat{\imath}_\mathrm{A}}. \tag{2.12.9}$$

If $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{\jmath}_\mathrm{A}}(\theta)\mathrm{F_A}$, then

$$\mathcal{O}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathrm{T}} = \vec{R}_{\hat{\jmath}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.12.10}$$

$$\theta = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{\jmath}_\mathrm{A}} = \theta_{\hat{k}_\mathrm{B}/\hat{k}_\mathrm{A}/\hat{\jmath}_\mathrm{A}}. \tag{2.12.11}$$

If $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\mathrm{F_A}$, then

$$\mathcal{O}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathrm{T}} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.12.12}$$

$$\theta = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}} = \theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}. \tag{2.12.13}$$

For convenience we define the *Euler orientation matrices*

$$\mathcal{O}_1(\theta) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{2.12.14}$$

$$\mathcal{O}_2(\theta) \triangleq \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.12.15}$$

$$\mathcal{O}_3(\theta) \triangleq \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.12.16}$$

and the *Euler rotation matrices*

$$\mathcal{R}_1(\theta) \triangleq \mathcal{O}_1^{\mathrm{T}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \tag{2.12.17}$$

$$\mathcal{R}_2(\theta) \triangleq \mathcal{O}_2^{\mathrm{T}}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.12.18}$$

$$\mathcal{R}_3(\theta) \triangleq \mathcal{O}_3^{\mathrm{T}}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.12.19}$$

Note that, for all angles θ, ϕ,

$$\mathcal{O}_1(\theta)\mathcal{O}_1(\phi) = \mathcal{O}_1(\theta + \phi), \tag{2.12.20}$$

and likewise for $\mathcal{O}_2$ and $\mathcal{O}_3$. Note that, if θ and ϕ are wrapped angles, then $\theta + \phi$ is necessarily a wrapped angle.

For a 3-2-1 rotation sequence, the Euler rotations and Euler angles are denoted by

$$\mathrm{F_A} \xrightarrow[3]{\Psi} \mathrm{F_B} \xrightarrow[2]{\Theta} \mathrm{F_C} \xrightarrow[1]{\Phi} \mathrm{F_D}. \tag{2.12.21}$$

In aircraft kinematics and dynamics, the 3-2-1 Euler angles Ψ, Θ, and Φ refer to *yaw*, *pitch*, and *roll*, respectively. Thus,

$$\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\Psi)\mathrm{F_A}, \tag{2.12.22}$$

$$\mathrm{F_C} = \vec{R}_{\mathrm{C/B}}\mathrm{F_B} = \vec{R}_{\hat{\jmath}_{\mathrm{B}}}(\Theta)\mathrm{F_B}, \tag{2.12.23}$$

$$\mathrm{F_D} = \vec{R}_{\mathrm{D/C}}\mathrm{F_C} = \vec{R}_{\hat{\imath}_{\mathrm{C}}}(\Phi)\mathrm{F_C}, \tag{2.12.24}$$

where

$$\Psi = \theta_{\hat{\imath}_{\mathrm{B}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}} = \theta_{\hat{\jmath}_{\mathrm{B}}/\hat{\jmath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}, \tag{2.12.25}$$

$$\Theta = \theta_{\hat{\imath}_{\mathrm{C}}/\hat{\imath}_{\mathrm{B}}/\hat{\jmath}_{\mathrm{B}}} = \theta_{\hat{k}_{\mathrm{C}}/\hat{k}_{\mathrm{B}}/\hat{\jmath}_{\mathrm{B}}}, \tag{2.12.26}$$

$$\Phi = \theta_{\hat{\jmath}_{\mathrm{D}}/\hat{\jmath}_{\mathrm{C}}/\hat{\imath}_{\mathrm{C}}} = \theta_{\hat{k}_{\mathrm{D}}/\hat{k}_{\mathrm{C}}/\hat{\imath}_{\mathrm{C}}}. \tag{2.12.27}$$

Hence,

$$\vec{R}_{\mathrm{D/A}} = \vec{R}_{\mathrm{D/C}}\vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{\imath}_{\mathrm{C}}}(\Phi)\vec{R}_{\hat{\jmath}_{\mathrm{B}}}(\Theta)\vec{R}_{\hat{k}_{\mathrm{A}}}(\Psi). \tag{2.12.28}$$

Each step can be interpreted equivalently as either a rotation or an orientation. The matrix

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{\mathrm{T}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathrm{T}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\Psi)\Big|_{\mathrm{A}}^{\mathrm{T}} = \mathcal{O}_3(\Psi) \tag{2.12.29}$$

gives the orientation of $\mathrm{F_B}$ with respect to $\mathrm{F_A}$ as a function of the eigenangle Ψ, which is measured from $\hat{\imath}_{\mathrm{A}}$ to $\hat{\imath}_{\mathrm{B}}$ or from $\hat{\jmath}_{\mathrm{A}}$ to $\hat{\jmath}_{\mathrm{B}}$ as shown in Figure 2.12.1. Likewise, as illustrated in Figure 2.12.2 and Figure 2.12.3,

$$\mathcal{O}_{\mathrm{C/B}} = \mathcal{R}_{\mathrm{B/C}} = \mathcal{R}_{\mathrm{C/B}}^{\mathrm{T}} = \vec{R}_{\mathrm{C/B}}\Big|_{\mathrm{B}}^{\mathrm{T}} = \vec{R}_{\hat{\jmath}_{\mathrm{B}}}(\Theta)\Big|_{\mathrm{B}}^{\mathrm{T}} = \mathcal{O}_2(\Theta), \tag{2.12.30}$$

$$\mathcal{O}_{\mathrm{D/C}} = \mathcal{R}_{\mathrm{C/D}} = \mathcal{R}_{\mathrm{D/C}}^{\mathrm{T}} = \vec{R}_{\mathrm{D/C}}\Big|_{\mathrm{C}}^{\mathrm{T}} = \vec{R}_{\hat{\imath}_{\mathrm{C}}}(\Phi)\Big|_{\mathrm{C}}^{\mathrm{T}} = \mathcal{O}_1(\Phi). \tag{2.12.31}$$

Using Fact 2.10.16 to combine the 3-2-1 rotation sequence yields the product of Euler orientation matrices given by

$$\mathcal{O}_{\mathrm{D/A}} = \mathcal{O}_{\mathrm{D/C}}\mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi). \tag{2.12.32}$$

Similarly, in terms of Euler rotation matrices we have

$$\mathcal{R}_{\mathrm{D/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{C/B}}\mathcal{R}_{\mathrm{D/C}} = \mathcal{R}_3(\Psi)\mathcal{R}_2(\Theta)\mathcal{R}_1(\Phi). \tag{2.12.33}$$

Consequently,

$$\mathcal{O}_{\mathrm{D/A}} = \mathcal{O}_{\mathrm{D/C}}\mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi). \tag{2.12.34}$$

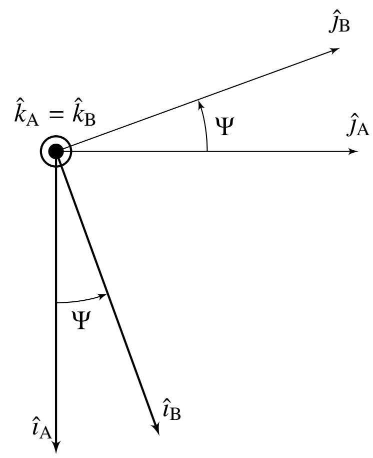


Figure 2.12.1: Rotation from $\mathrm{F_A}$ to $\mathrm{F_B}$.

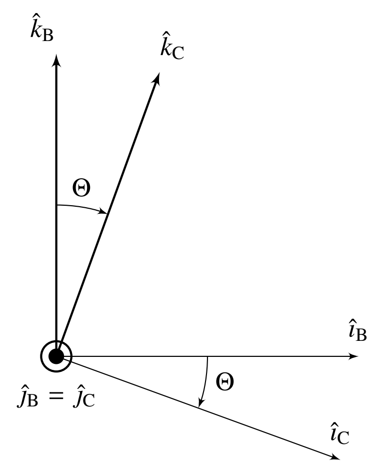


Figure 2.12.2: Rotation from $\mathrm{F_B}$ to $\mathrm{F_C}$.

For a 3-1-3 rotation sequence, the Euler rotations and Euler angles are denoted by

$$\mathrm{F_A} \xrightarrow[3]{\Phi} \mathrm{F_B} \xrightarrow[1]{\Theta} \mathrm{F_C} \xrightarrow[3]{\Psi} \mathrm{F_D}. \tag{2.12.35}$$

The orientation matrices for the 3-1-3 sequence are thus given by

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{\mathrm{T}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathrm{T}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\Phi)\Big|_{\mathrm{A}}^{\mathrm{T}} = \mathcal{O}_3(\Phi), \tag{2.12.36}$$

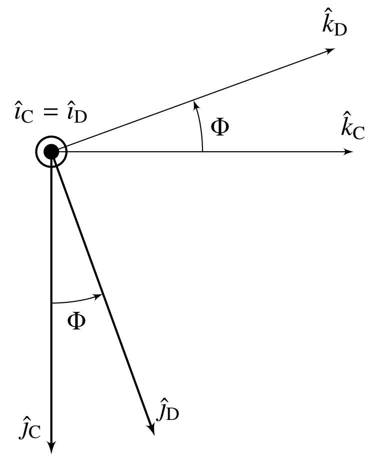


Figure 2.12.3: Rotation from $\rm F_C$ to $\rm F_D$.

$$\mathcal{O}_{\rm C/B} = \mathcal{R}_{\rm B/C} = \mathcal{R}_{\rm C/B}^{\rm T} = \vec{R}_{\rm C/B}\Big|_{\rm B}^{\rm T} = \vec{R}_{\hat{\imath}_{\rm B}}(\Theta)\Big|_{\rm B}^{\rm T} = \mathcal{O}_1(\Theta), \tag{2.12.37}$$

$$\mathcal{O}_{\rm D/C} = \mathcal{R}_{\rm C/D} = \mathcal{R}_{\rm D/C}^{\rm T} = \vec{R}_{\rm D/C}\Big|_{\rm C}^{\rm T} = \vec{R}_{\hat{k}_{\rm C}}(\Psi)\Big|_{\rm C}^{\rm T} = \mathcal{O}_3(\Psi). \tag{2.12.38}$$

Using Fact 2.10.15 to combine the 3-1-3 rotation sequence yields the product of Euler orientation matrices given by

$$\mathcal{O}_{\rm D/A} = \mathcal{O}_{\rm D/C}\mathcal{O}_{\rm C/B}\mathcal{O}_{\rm B/A} = \mathcal{O}_3(\Psi)\mathcal{O}_1(\Theta)\mathcal{O}_3(\Phi). \tag{2.12.39}$$

Similarly, in terms of Euler rotation matrices we have

$$\mathcal{R}_{\rm D/A} = \mathcal{R}_{\rm B/A}\mathcal{R}_{\rm C/B}\mathcal{R}_{\rm D/C} = \mathcal{R}_3(\Phi)\mathcal{R}_1(\Theta)\mathcal{R}_3(\Psi). \tag{2.12.40}$$

For spacecraft attitude dynamics, the 3-1-3 Euler angles Φ, Θ, and Ψ refer to *precession*, *nutation*, and *spin*, respectively. Consequently,

$$\mathcal{O}_{\rm D/A} = \mathcal{O}_{\rm D/C}\mathcal{O}_{\rm C/B}\mathcal{O}_{\rm B/A} = \mathcal{O}_3(\Psi)\mathcal{O}_1(\Theta)\mathcal{O}_3(\Phi). \tag{2.12.41}$$

For satellite orbital dynamics, the 3-1-3 Euler angles Ω, i, and ω refer to *right ascension of the ascending node*, *inclination*, and *argument of periapsis*, respectively. Consequently,

$$\mathcal{O}_{\rm D/A} = \mathcal{O}_{\rm D/C}\mathcal{O}_{\rm C/B}\mathcal{O}_{\rm B/A} = \mathcal{O}_3(\omega)\mathcal{O}_1(i)\mathcal{O}_3(\Omega). \tag{2.12.42}$$

2.13 Products of Euler Orientation Matrices

Fact 2.13.1. The following statements hold:

i) $\mathcal{O}_1(0) = \mathcal{O}_2(0) = \mathcal{O}_3(0) = I_3$.

ii) $\mathcal{O}_1(\pi) = \mathrm{diag}(1,-1,-1)$, $\mathcal{O}_2(\pi) = \mathrm{diag}(-1,1,-1)$, and $\mathcal{O}_3(\pi) = \mathrm{diag}(-1,-1,1)$.

iii) Let $i, j, k \in \{1,2,3\}$ be distinct. Then, $\mathcal{O}_i(\pi) = \mathcal{O}_j(\pi)\mathcal{O}_k(\pi)$.

iv) Let $a \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, the following statements are equivalent:

a) $\mathcal{O}_i(a)$ is symmetric.

b) $\mathcal{O}_i(a)$ is diagonal.

c) Either $a \equiv 0$ or $a \equiv \pi$.

v) Let $a \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, $\mathcal{O}_i(a) = I$ if and only if $a \equiv 0$.

vi) Let $a \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, $\mathcal{O}_i(-a) = \mathcal{O}_i(a)^{-1} = \mathcal{O}_i(a)^{\mathrm{T}}$.

vii) Let $a, b \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, $\mathcal{O}_i(a)\mathcal{O}_i(b) = \mathcal{O}_i(a+b)$.

viii) Let $a, b \in \mathbb{R}$, and let $i, j \in \{1,2,3\}$ be distinct. Then, the following statements are equivalent:

a) $\mathcal{O}_i(a) = \mathcal{O}_j(b)$.

b) $\mathcal{O}_i(a)\mathcal{O}_j(b) = I$.

c) $a \equiv b \equiv 0$.

The entries of $R \in \mathbb{R}^{3\times 3}$ are written as

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{2.13.1}$$

Note that, if R is an orientationi matrix, then, for all $i, j \in \{1,2,3\}$, $|r_{ij}| \le 1$.

3-2-1 factorizations of an orientation matrix are considered in two cases. In the case where $|r_{13}| < 1$, b can assume two distinct values and a and c are uniquely determined by b. In the case where $|r_{13}| = 1$, b is unique and a and c can assume infinitely many values. For both proofs it is useful to note that

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) = \begin{bmatrix} cbcc & cbsc & -sb \\ ccsasb - casc & cacc + sasbsc & cbsa \\ sasc + caccsb & casbsc - ccsa & cacb \end{bmatrix}. \tag{2.13.2}$$

The follow result shows that, in the case where $|r_{13}| < 1$, the entries r_{21}, r_{22}, r_{31}, and r_{32} of an orientation matrix R are uniquely determined by the entries r_{11}, r_{12}, r_{13}, r_{23}, and r_{33}.

Fact 2.13.2. Let $R \in \mathbb{R}^{3\times 3}$, and assume that $r_{11}^2 + r_{12}^2 > 0$ and $r_{11}^2 + r_{12}^2 + r_{13}^2 = r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$. Then, R is an orientation matrix if and only if

$$r_{21} = -\frac{r_{11}r_{13}r_{23} + r_{12}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{22} = \frac{r_{11}r_{33} - r_{12}r_{13}r_{23}}{r_{11}^2 + r_{12}^2}, \tag{2.13.3}$$

$$r_{31} = \frac{r_{12}r_{23} - r_{11}r_{13}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{32} = -\frac{r_{12}r_{13}r_{33} + r_{11}r_{23}}{r_{11}^2 + r_{12}^2}. \tag{2.13.4}$$

Proof. To prove sufficiency, it can be shown that, with (2.13.3) and (2.13.4), R satisfies $RR^{\mathrm{T}} = I$ and $\det R = 1$. To prove necessity, note that $RR^{\mathrm{T}} = I$ yields five equalities involving r_{21}, r_{22}, r_{31}, and r_{32}, namely,

$$r_{11}r_{21} + r_{12}r_{22} + r_{13}r_{23} = 0, \tag{2.13.5}$$

$$r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} = 0, \tag{2.13.6}$$

$$r_{21}^2 + r_{22}^2 + r_{23}^2 = 1, \tag{2.13.7}$$

$$r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} = 0, \tag{2.13.8}$$

$$r_{31}^2 + r_{32}^2 + r_{33}^2 = 1. \tag{2.13.9}$$

Since $r_{11}^2 + r_{12}^2 > 0$, it follows that either $r_{11} \neq 0$ or $r_{12} \neq 0$. The case where $r_{12} \neq 0$ is considered first. In this case, (2.13.5) implies that

$$r_{22} = -(r_{11}r_{21} + r_{13}r_{23})/r_{12} = 0, \tag{2.13.10}$$

which, using (2.13.7) yields

$$(r_{11}^2 + r_{12}^2)r_{21}^2 + 2r_{11}r_{23}r_{13}r_{21} + r_{23}^2(r_{13}^2 + r_{12}^2) - r_{12}^2 = 0. \tag{2.13.11}$$

Solving (2.13.11) for r_{21} yields

$$\begin{aligned} r_{21} &= \frac{-r_{11}r_{23}r_{13} \pm \sqrt{r_{11}^2r_{23}^2r_{13}^2 - (r_{11}^2 + r_{12}^2)(r_{23}^2(r_{13}^2 + r_{12}^2) - r_{12}^2)}}{r_{11}^2 + r_{12}^2} \\ &= \frac{-r_{11}r_{23}r_{13} \pm r_{12}r_{33}}{r_{11}^2 + r_{12}^2}. \end{aligned} \tag{2.13.12}$$

Substituting r_{21} given by (2.13.12) into (2.13.10) yields

$$r_{22} = \frac{\mp r_{11}r_{33} - r_{23}r_{13}r_{12}}{r_{11}^2 + r_{12}^2}. \tag{2.13.13}$$

The same expressions (2.13.12) and (2.13.13) are obtained in the case where $r_{11} \neq 0$. Note that (2.13.12) and (2.13.13) imply that r_{21} and r_{22} are given by either (2.13.3) or

$$r_{21} = -\frac{r_{11}r_{13}r_{23} - r_{12}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{22} = \frac{-r_{11}r_{33} - r_{12}r_{13}r_{23}}{r_{11}^2 + r_{12}^2}. \tag{2.13.14}$$

Following a similar procedure using (2.13.6) and (2.13.9), it follows that r_{31} and r_{32} are given by either (2.13.4) or

$$r_{31} = \frac{-r_{12}r_{23} - r_{11}r_{13}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{32} = -\frac{r_{12}r_{13}r_{33} - r_{11}r_{23}}{r_{11}^2 + r_{12}^2}. \tag{2.13.15}$$

It thus follows from (2.13.3), (2.13.14), (2.13.4), and (2.13.15) that r_{21}, r_{22}, r_{31}, and r_{32} are given by either *i*) (2.13.3) and (2.13.4), *ii*) (2.13.3) and (2.13.15), *iii*) (2.13.14) and (2.13.4), or *iv*) (2.13.14) and (2.13.15). In case *i*), $RR^{\mathrm{T}} = I$ and $\det R = 1$. In case *ii*), $RR^{\mathrm{T}} \neq I$ and $\det R = -1$. In particular, the $(2,3)$ and $(3,2)$ entries of RR^{T} are not zero. In case *iii*), $RR^{\mathrm{T}} \neq I$ and $\det R = 1$. In particular, the $(2,3)$ and $(3,2)$ entries of RR^{T} are not zero. In case *iv*), $RR^{\mathrm{T}} = I$ and $\det R = -1$. Therefore, cases *ii*), *iii*), and *iv*) are spurious, and thus r_{21}, r_{22}, r_{31}, and r_{32} satisfy (2.13.3) and (2.13.4). $\square$

The following result considers the case where a 3-2-1 product of Euler rotation matrices is equal to an arbitrary rotation matrix.

Fact 2.13.3. Let R be an orientation matrix, and assume that $|r_{13}| < 1$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$R = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) \tag{2.13.16}$$

if and only if either

$$b \equiv -r_{13} \tag{2.13.17}$$

or

$$b \equiv r_{13} + \pi \tag{2.13.18}$$

and

$$a \equiv \left(\frac{r_{23}}{\cos b}, \frac{r_{33}}{\cos b}\right), \quad c \equiv \left(\frac{r_{12}}{\cos b}, \frac{r_{11}}{\cos b}\right). \tag{2.13.19}$$

Proof. To prove necessity, note that it follows from the $(1,3)$ entry of (2.13.2) that b satisfies either (2.13.17) or (2.13.18). Furthermore, it follows from the $(1,1)$, $(1,2)$, $(2,3)$, and $(3,3)$ entries of (2.13.2) that a and c are given by (2.13.19).

To prove sufficiency note that it follows from (2.13.17) that the $(1,3)$ entry of (2.13.2) is $-\sin b = \sin r_{13} = r_{13}$. Likewise, it follows from (2.13.18) that the $(1,3)$ entry of (2.13.2) is $-\sin b = -\sin(r_{13}+\pi) = r_{13}$. Furthermore, it follows from (2.13.19) that the $(1,1)$ entry of (2.13.2) is given by $(\cos b)\cos c = (\cos b)r_{11}/(\cos b) = r_{11}$ and likewise for the $(1,2)$, $(2,3)$, and $(3,3)$ entries of (2.13.2). Next, since $\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)$ is a rotation matrix, Lemma 1 implies that its $(2,1)$, $(2,2)$, $(3,1)$, $(3,2)$, entries are determined by its $(1,1)$, $(1,2)$, $(1,3)$, $(2,3)$, $(3,3)$, entries. In particular, the $(2,1)$ entry of $\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)$ is given by

$$\begin{aligned} ccsasb - casc &= \frac{r_{11}}{\cos b}\frac{r_{23}}{\cos b}\sin b - \frac{r_{33}}{\cos b}\frac{r_{12}}{\cos b} \\ &= \frac{r_{11}r_{23}\sin b - r_{33}r_{12}}{\cos^2 b} \\ &= \frac{-r_{11}r_{23}r_{13} - r_{33}r_{12}}{1 - r_{13}^2} \\ &= r_{21}, \end{aligned}$$

where the last equality follows from (2.13.3) of Lemma 3. Similar calculations show that the $(2,2)$, $(3,1)$, and $(3,2)$ of $\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)$ are given by r_{22}, r_{31}, and r_{32}, respectively. Consequently, (2.13.16) is satisfied. $\square$

In the special case where $r_{13} = 0$, it follows from Theorem 1 that $a, b, c \in \mathbb{R}$ satisfy (2.13.16) if and only if either $b \equiv 0$ or $b \equiv \pi$ and

$$a \equiv (r_{23}, r_{33}), \quad c \equiv (r_{12}, r_{11}). \tag{2.13.20}$$

Setting $R = \mathcal{O}_2(d)$ in Fact 2.13.3 yields the following result.

Fact 2.13.4. Let $d \in \mathbb{R}$, and assume that $d \not\equiv \pi/2$ and $d \not\equiv -\pi/2$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c). \tag{2.13.21}$$

if and only if either *i)* $a \equiv c \equiv 0$ and $b \equiv d$ or *ii)* $a \equiv c \equiv \pi$ and $b \equiv \pi - d$.

The following result extends Fact 2.13.5 to the case where $|r_{13}| = 1$, that is, $r_{11} = r_{12} = 0$. The proof is omitted.

Fact 2.13.5. Let R be a rotation matrix, and assume that $|r_{13}| = 1$. Then, $a, b, c \in \mathbb{R}$ satisfy (2.13.16) if and only if either

$$b \equiv \pi/2, \quad a \equiv c + (r_{21}, r_{22}) \tag{2.13.22}$$

or

$$b \equiv -\pi/2, \quad a \equiv -c + (-r_{21}, r_{22}). \tag{2.13.23}$$

In the case where $|r_{13}| = 1$, it follows that $r_{31} = -r_{13}r_{22}$ and $r_{32} = r_{13}r_{21}$. Therefore, in Fact

2.13.5, (2.13.22) and (2.13.23) can be replaced, respectively, by

$$b \equiv \pi/2, \quad a \equiv c + (-r_{32}, r_{31}), \tag{2.13.24}$$

$$b \equiv -\pi/2, \quad a \equiv -c + (-r_{32}, -r_{31}). \tag{2.13.25}$$

Setting $R = \mathcal{O}_2(d)$ in Fact 2.13.5 yields the following result.

Fact 2.13.6. Let $d \in \mathbb{R}$, and assume that either $d \equiv \pi/2$ or $d \equiv -\pi/2$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c). \tag{2.13.26}$$

if and only if either *i)* $a \equiv c$ and $b \equiv d \equiv \pi/2$, *ii)* $a \equiv -c$ and $b \equiv d \equiv -\pi/2$.

1-2-1 factorizations of a rotation matrix R are considered in two cases. In the case where $|r_{11}| < 1$, b can assume two distinct values and a and c are uniquely determined by b. In the case where $|r_{11}| = 1$, b is unique and a and c can assume infinitely many values. For both proofs it is useful to note that

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c) = \begin{bmatrix} cb & sbsc & -sbcc \\ sasb & cacc - cbsasc & casc + cbccsa \\ casb & -ccsa - cacbsc & cacbcc - sasc \end{bmatrix}. \tag{2.13.27}$$

The following result considers the case where a 1-2-1 product of Euler rotation matrices is equal to an arbitrary rotation matrix.

Fact 2.13.7. Let R be a rotation matrix, and assume that $|r_{11}| < 1$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$R = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c). \tag{2.13.28}$$

if and only if either

$$b \equiv \operatorname{acos} r_{11} \tag{2.13.29}$$

or

$$b \equiv -\operatorname{acos} r_{11} \tag{2.13.30}$$

and

$$a \equiv \left(\frac{r_{21}}{\sin b}, \frac{r_{31}}{\sin b}\right), \quad c \equiv \left(\frac{r_{12}}{\sin b}, \frac{-r_{13}}{\sin b}\right). \tag{2.13.31}$$

The following result extends Fact 2.13.7 to the case where $|r_{11}| = 1$.

Fact 2.13.8. Let R be a rotation matrix, and assume that $|r_{11}| = 1$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$R = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c). \tag{2.13.32}$$

if and only if either

$$b \equiv 0, \quad a \equiv -c + (r_{23}, r_{22}) \tag{2.13.33}$$

or

$$b \equiv \pi, \quad a \equiv c + (r_{23}, r_{22}). \tag{2.13.34}$$

In the case where $|r_{11}| = 1$, it follows that $r_{32} = -r_{11}r_{23}$ and $r_{33} = r_{11}r_{22}$. Therefore, in Fact 2.13.8, (2.13.33) and (2.13.34) can be replaced, respectively, by

$$b \equiv 0, \quad a \equiv -c + (-r_{32}, r_{33}), \tag{2.13.35}$$

$$b \equiv \pi, \quad a \equiv c + (r_{32}, -r_{33}). \tag{2.13.36}$$

The following result considers the case where a 1-2-1 product of Euler rotation matrices is equal to the identity matrix.

Fact 2.13.9. Let $a, b, c \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c) = I \tag{2.13.37}$$

if and only if $b \equiv 0$ and $a \equiv -c$.

Proof. Rewriting (2.13.37) as

$$\mathcal{O}_1(a + c) = \mathcal{O}_2(-b), \tag{2.13.38}$$

viii) of Lemma 2 implies that (2.13.38) holds if and only if $a + c \equiv b \equiv 0$. □

The following result considers the case where a 3-2-1 product of Euler rotation matrices is equal to the identity matrix.

Fact 2.13.10. Let $a, b, c \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) = I \tag{2.13.39}$$

if and only if either $a \equiv b \equiv c \equiv 0$ or $a \equiv b \equiv c \equiv \pi$.

Proof. Sufficiency is immediate. To prove necessity, note that, by rewriting (2.13.39) as

$$\begin{bmatrix} cbcc & cbsc & -sb \\ ccsasb - casc & cacc + sasbsc & cbsa \\ sasc + caccsb & casbsc - ccsa & cacb \end{bmatrix} = I,$$

it follows from the (1,1) entry that $cb \neq 0$, and thus from the (1,2), (1,3), and (2,3) entries that $sa = sb = sc = 0$. Hence, it follows from the (2,2) and (3,3) entries that $cacc = cacb = 1$, and thus either $ca = cb = cc = 1$ or $ca = cb = cc = -1$. Hence, either $a \equiv b \equiv c \equiv 0$ or $a \equiv b \equiv c \equiv \pi$. □

The following result considers the case where a 2-3-2-1 product of Euler orientation matrices is equal to the identity matrix.

Fact 2.13.11. Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)\mathcal{O}_2(d) = I \tag{2.13.40}$$

if and only if either *i*) $b \equiv -d \equiv \pi/2$ and $a \equiv c$, *ii*) $b \equiv -d \equiv -\pi/2$ and $a \equiv -c$, *iii*) $a \equiv c \equiv 0$ and $b \equiv -d$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv d + \pi$.

Proof. Sufficiency is immediate. To prove necessity, note that (2.13.40) implies

$$\begin{bmatrix} cbccd - sbsd & cbsc & -cdsb - cbccsd \\ cbsasd - cd(casc - ccsasb) & cacc + sasbsc & sd(casc - ccsasb) + cbcdsa \\ cd(sasc + caccsb) + cacbsd & casbsc - ccsa & cacbcd - sd(sasc + caccsb) \end{bmatrix} = I,$$

Since $cbsc = 0$, it follows that either $cb = 0$ or $sc = 0$. Therefore, either *i*) $b \equiv \pi/2$, *ii*) $b \equiv -\pi/2$, *iii*) $c \equiv 0$, or *iv*) $c \equiv \pi$.

Case *i*)**:** $b \equiv \pi/2$. In this case,

$$\begin{bmatrix} -sd & 0 & -cd \\ -cd(casc - ccsa) & cacc + sasc & sd(casc - ccsa) + cbcdsa \\ cd(sasc + cacc) & casc - ccsa & -sd(sasc + cacc) \end{bmatrix} = I,$$

Since $sd = -1$ and $cd = 0$, it follows that $d \equiv -\pi/2$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cacc + sasc & -(casc - ccsa) \\ 0 & casc - ccsa & sasc + cacc \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a - c) & \sin(a - c) \\ 0 & -\sin(a - c) & \cos(a - c) \end{bmatrix} = I.$$

Hence, $a - c \equiv 0$.

Case *ii*)**:** $b \equiv -\pi/2$. In this case,

$$\begin{bmatrix} sd & 0 & cd \\ -cd(casc + ccsa) & cacc - sasc & sd(casc + ccsa) \\ cd(sasc - cacc) & -casc - ccsa & -sd(sasc - cacc) \end{bmatrix} = I,$$

Since $sd = 1$ and $cd = 0$, it follows that $d \equiv \pi/2$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cacc - sasc & casc + ccsa \\ 0 & -casc - ccsa & -(sasc - cacc) \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a + c) & \sin(a + c) \\ 0 & -\sin(a + c) & \cos(a + c) \end{bmatrix} = I.$$

Hence, $a + c \equiv 0$.

Case *iii*)**:** $c \equiv 0$. In this case,

$$\begin{bmatrix} cbcd - sbsd & 0 & -cdsb - cbsd \\ cbsasd + cdsasb) & ca & -sdsasb + cbcdsa \\ cdcasb + cacbsd & -sa & cacbcd - sdcasb \end{bmatrix} = I,$$

Since $ca = 1$ and $sa = 0$, it follows that $a \equiv 0$. Hence,

$$\begin{bmatrix} cbcd - sbsd & 0 & -cdsb - cbsd \\ 0 & 1 & 0 \\ cdsb + cbsd & 0 & cbcd - sdsb \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix} \cos(b + d) & 0 & -\sin(b + d) \\ 0 & 1 & 0 \\ \sin(b + d) & 0 & \cos(b + d) \end{bmatrix} = I,$$

Hence, $b + d \equiv 0$.

Case *iv*)**:** $c \equiv \pi$. In this case,

$$\begin{bmatrix} -cbcd - sbsd & 0 & -cdsb + cbsd \\ cbsasd - cdsasb & -ca & sdsasb + cbcdsa \\ -cdcasb + cacbsd & sa & cacbcd + sdcasb \end{bmatrix} = I,$$

Since $ca = -1$ and $sa = 0$, it follows that $a \equiv \pi$. Hence,

$$\begin{bmatrix} -cbcd - sbsd & 0 & -cdsb + cbsd \\ 0 & 1 & 0 \\ cdsb - cbsd & 0 & -cbcd - sdsb \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix} -\cos(b-d) & 0 & -\sin(b-d) \\ 0 & 1 & 0 \\ \sin(b-d) & 0 & -\cos(b-d) \end{bmatrix} = I,$$

Hence, $b - d \equiv \pi$. $\square$

The following result considers the case where a 3-2-1 product of Euler orientation matrices is equal to a 2-axis Euler orientation matrix.

Fact 2.13.12. $a, b, c, d \in \mathbb{R}$ satisfy

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c). \tag{2.13.41}$$

if and only if either *i*) $a \equiv c$ and $b \equiv d \equiv \pi/2$, *ii*) $a \equiv -c$ and $b \equiv d \equiv -\pi/2$, *iii*) $a \equiv c \equiv 0$ and $b \equiv d$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv \pi - d$.

The following result considers the case where a 2-1-2-1 product of Euler orientation matrices is equal to the identity matrix.

Fact 2.13.13. Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c)\mathcal{O}_2(d) = I \tag{2.13.42}$$

if and only if either *i*) $b \equiv d \equiv 0$ and $a \equiv -c$, *ii*) $a \equiv c \equiv 0$ and $b \equiv -d$, *iii*) $b \equiv d \equiv \pi$ and $a \equiv c$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv d$.

Proof. Sufficiency is immediate. To prove necessity, note that (2.13.42) implies

$$\begin{bmatrix} cbcd - ccsbsd & sbsc & -cbsd - cccdsb \\ sd(casc + cbccsa) + cdsasb & cacc - cbsasc & cd(casc + cbccsa) - sasbsd \\ cacdsb - sd(sasc - cacbcc) & -ccsa - cacbsc & -cd(sasc - cacbcc) - casbsd \end{bmatrix} = I.$$

Since $sbsc = 0$, it follows that either *i*) $b \equiv 0$, *ii*) $c \equiv 0$, *iii*) $b \equiv \pi$, or *iv*) $c \equiv \pi$.

Case *i*)**:** $b \equiv 0$. In this case,

$$\begin{bmatrix} cd & 0 & -sd \\ sd(casc + ccsa) & cacc - sasc & cd(casc + ccsa) \\ -sd(sasc - cacc) & -ccsa - casc & -cd(sasc - cacc) \end{bmatrix} = I.$$

Since $cd = 1$ and $sd = 0$, it follows that $d \equiv 0$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cacc - sasc & casc + ccsa \\ 0 & -ccsa - casc & -sasc + cacc \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a+c) & \sin(a+c) \\ 0 & -\sin(a+c) & \cos(a+c) \end{bmatrix} = I.$$

Hence, $a + c \equiv 0$.

Case *ii*)**:** $c \equiv 0$. In this case,

$$\begin{bmatrix} cbcd - sbsd & 0 & -cbsd - cdsb \\ sdcbsa + cdsasb & ca & cdcbsa - sasbsd \\ cacdsb + sdcacb & -sa & cdcacb - casbsd \end{bmatrix} = I.$$

Since $ca = 1$ and $sa = 0$, it follows that $a = 0$. Hence,

$$\begin{bmatrix} cbcd - sbsd & 0 & -cbsd - cdsb \\ 0 & 1 & 0 \\ cdsb + sdcb & 0 & cdcb - sbsd \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} \cos(b+d) & 0 & -\sin(b+d) \\ 0 & 1 & 0 \\ \sin(b+d) & 0 & \cos(b+d) \end{bmatrix} = I.$$

Hence, $b + d \equiv 0$.

Case *iii*)**:** $b \equiv \pi$. In this case,

$$\begin{bmatrix} -cd & 0 & sd \\ sd(casc - ccsa) & cacc + sasc & cd(casc - ccsa) \\ -sd(sasc + cacc) & -ccsa + casc & -cd(sasc + cacc) \end{bmatrix} = I.$$

Since $cd = -1$ and $sd = 0$, it follows that $d \equiv \pi$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cacc + sasc & -(casc - ccsa) \\ 0 & -ccsa + casc & sasc + cacc \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a-c) & \sin(a-c) \\ 0 & -\sin(a-c) & \cos(a-c) \end{bmatrix} = I.$$

Hence, $a \equiv c$.

Case *iv*)**:** $c \equiv \pi$. In this case,

$$\begin{bmatrix} cbcd + sbsd & 0 & -cbsd + cdsb \\ -sdcbsa + cdsasb & -ca & -cdcbsa - sasbsd \\ cacdsb - sdcacb & sa & -cdcacb - casbsd \end{bmatrix} = I.$$

Since $ca = -1$ and $sa = 0$, it follows that $a = \pi$. Hence,

$$\begin{bmatrix} cbcd + sbsd & 0 & -cbsd + cdsb \\ 0 & 1 & 0 \\ -cdsb + sdcb & 0 & cdcb + sbsd \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} \cos(d-b) & 0 & -\sin(d-b) \\ 0 & 1 & 0 \\ \sin(d-b) & 0 & \cos(d-b) \end{bmatrix} = I.$$

Hence, $d - b \equiv 0$. □

The following result considers the case where a 1-2-1 product of Euler orientation matrices is equal to a 2-axis Euler orientation matrix.

Corollary 2. $a, b, c, d \in \mathbb{R}$ satisfy

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c). \tag{2.13.43}$$

if and only if either *i*) $b \equiv -d \equiv 0$ and $a \equiv -c$, *ii*) $a \equiv c \equiv 0$ and $b \equiv d$, *iii*) $b \equiv -d \equiv \pi$ and $a \equiv c$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv -d$.

Fact 2.13.13 yields the following result on commuting Euler orientation matrices.

Fact 2.13.14. Let $a, b \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b) = \mathcal{O}_2(b)\mathcal{O}_1(a) \tag{2.13.44}$$

if and only if either $a \equiv 0$, $b \equiv 0$, or $a \equiv b \equiv \pi$.

The following result considers the case where a 3-1-2-1 product of Euler orientation matrices is equal to the identity matrix.

Fact 2.13.15. Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c)\mathcal{O}_3(d) = I \tag{2.13.45}$$

if and only if either *i*) $b \equiv d \equiv 0$ and $a \equiv -c$, *ii*) $b \equiv d \equiv \pi$ and $a \equiv c + \pi$, *iii*) $a \equiv -c \equiv -\pi/2$ and $b \equiv -d$, or *iv*) $a \equiv -c \equiv \pi/2$ and $b \equiv d$.

Proof. Sufficiency is immediate. To prove necessity, note that (2.13.45) implies

$$\begin{bmatrix} cbcd - sbscsd & cbsd + cdsbsc & -ccsb \\ cdsasb - sd(cacc - cbsasc) & cd(cacc - cbsasc) + sasbsd & casc + cbccsa \\ sd(ccsa + cacbsc) + cacdsb & casbsd - cd(ccsa + cacbsc) & cacbcc - sasc \end{bmatrix} = I.$$

Since $ccsb = 0$, it follows that either *i*) $b \equiv 0$, *ii*) $b \equiv \pi$, *iii*) $c \equiv \pi/2$, or *iv*) $c \equiv -\pi/2$.

Case *i*): $b \equiv 0$. In this case,

$$\begin{bmatrix} cd & sd & 0 \\ sd(sasc - cacc) & cd(cacc - sasc) & casc + ccsa \\ sd(ccsa + casc) & -cd(ccsa + casc) & cacc - sasc \end{bmatrix} = I.$$

Since $cd = 1$ and $sd = 0$, it follows that $d \equiv 0$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cacc - sasc & casc + ccsa \\ 0 & -ccsa - casc & -sasc + cacc \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a+c) & \sin(a+c) \\ 0 & -\sin(a+c) & \cos(a+c) \end{bmatrix} = I.$$

Hence, $a + c \equiv 0$.

Case *ii*)**:** $b \equiv \pi$. In this case,

$$\begin{bmatrix} -cd & -sd & 0 \\ -sd(cacc + sasc) & cd(cacc + sasc) & casc - ccsa \\ sd(ccsa - casc) & cd(casc - ccsa) & -cacc - sasc \end{bmatrix} = I.$$

Since $cd = -1$ and $sd = 0$, it follows that $d \equiv \pi$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -cacc - sasc) & casc - ccsa \\ 0 & ccsa - casc & -cacc - sasc \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos(a-c) & -\sin(a-c) \\ 0 & \sin(a-c) & -\cos(a-c) \end{bmatrix} = I.$$

Hence, $a - c \equiv \pi$.

Case *iii*)**:** $c \equiv \pi/2$. In this case,

$$\begin{bmatrix} cbcd - sbsd & cbsd + cdsb & 0 \\ sa(cdsb + sdcb) & sa(sbsd - cdcb) & ca \\ ca(sdcb + cdsb) & ca(sbsd - cdcb) & -sa \end{bmatrix} = I.$$

Since $ca = 0$ and $sa = -1$, it follows that $a \equiv -\pi/2$. Hence,

$$\begin{bmatrix} cbcd - sbsd & cbsd + cdsb & 0 \\ -cdsb - sdcb & cdcb - sbsd & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} \cos(b+d) & \sin(b+d) & 0 \\ -\sin(b+d) & \cos(b+d) & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, $b + d \equiv 0$.

Case *iv*)**:** $c \equiv -\pi/2$. In this case,

$$\begin{bmatrix} cbcd + sbsd & cbsd - cdsb & 0 \\ sa(cdsb + sdcb) & sa(sbsd - cdcb) & -ca \\ ca(cdsb - sdcb) & ca(sbsd + cdcb) & sa \end{bmatrix} = I.$$

Since $sa = 1$ and $ca = 0$, it follows that $a \equiv \pi/2$. Hence,

$$\begin{bmatrix} cbcd + sbsd & cbsd - cdsb & 0 \\ cdsb + sdcb & sbsd + cdcb & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix} \cos(b-d) & -\sin(b-d) & 0 \\ \sin(b-d) & \cos(b-d) & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, $b - d \equiv 0$. □

These results show that there is one permutationally distinct product of two Euler orientation matrices whose product is the identity; two permutationally distinct products of three Euler orientation matrices whose product is the identity; and three permutationally distinct products of four Euler orientation matrices whose product is the identity. Products of more than four Euler orientation matrices can be considered. For example, the number of permutationally distinct products of five, six, seven, and eight Euler orientation matrices whose product is the identity is five, 11, 21, and 33, respectively. Hence, (10.8.5) is one of 21 permutationally distinct products of seven Euler orientation matrices. More generally, letting a_n denote the number of permutationally distinct products of n Euler orientation matrices whose product is the identity, it follows that

$$a_{n+2} = a_{n+1} + 2a_n. \tag{2.13.46}$$

2.14 Exponential Representation of Rotation Matrices and the Eigenaxis Angle Vector

An alternative way to express a physical rotation matrix is in terms of the exponential of a skew-symmetric physical matrix. For the physical matrix $\vec{M}$ define the *exponential* of $\vec{M}$ by

$$\exp(\vec{M}) \triangleq \vec{I} + \vec{M} + \tfrac{1}{2}\vec{M}^2 + \tfrac{1}{3!}\vec{M}^3 + \cdots. \tag{2.14.1}$$

Therefore, for each frame $\mathrm{F_A}$,

$$\exp(\vec{M})\Big|_{\mathrm{A}} = e^M = I_3 + M + \tfrac{1}{2}M^2 + \tfrac{1}{3!}M^3 + \cdots, \tag{2.14.2}$$

where $M \triangleq \vec{M}\Big|_{\mathrm{A}}$.

The following result provides an exponential analogue of Fact 2.11.9.

Fact 2.14.1. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors that have the same magnitude and are not parallel. Then,

$$\vec{R}_{\hat{\theta}_{\vec{y}/\vec{x}}}(\theta_{\vec{y}/\vec{x}}) = \exp\left(\vec{\theta}_{\vec{y}/\vec{x}}^{\times}\right) = \exp\left(\theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}^{\times}\right) = \exp\left(\frac{\theta_{\vec{y}/\vec{x}}}{\sin\theta_{\vec{y}/\vec{x}}}(\hat{x}\times\hat{y})^{\times}\right). \tag{2.14.3}$$

Furthermore,

$$\vec{y} = \exp\left(\vec{\theta}_{\vec{y}/\vec{x}}^{\times}\right)\vec{x}. \tag{2.14.4}$$

Proof. The result follows from Fact 11.11.6 in [1]. □

Recall that, if $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = 3$, then $\theta_{\mathrm{B/A}} = 0$ and $\hat{n}_{\mathrm{B/A}}$ is an arbitrary unit dimensionless physical vector. Furthermore, if $\operatorname{tr}\vec{R}_{\mathrm{B/A}} \in (-1,3)$, then $\theta_{\mathrm{B/A}} = \cos^{-1}[\frac{1}{2}(\operatorname{tr}\vec{R}_{\mathrm{B/A}} - 1)]$ and $\hat{n}^{\times}_{\mathrm{B/A}} = \frac{1}{2\sin\theta_{\mathrm{B/A}}}(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}})$, which is uniquely defined. Finally, if $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = -1$, then $\theta_{\mathrm{B/A}} = \pi$ and $\hat{n}_{\mathrm{B/A}}$ satisfies $\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} = \frac{1}{2}(\vec{R}_{\mathrm{B/A}} + \vec{I})$. In this case, there are two eigenaxes in the sense that, if $\hat{n}_{\mathrm{B/A}}$ is an eigenaxis then so is $-\hat{n}_{\mathrm{B/A}}$.

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. If $\theta_{\mathrm{B/A}} \in [0,\pi)$, then we define the *eigenaxis angle vector* by

$$\vec{\Theta}_{\mathrm{B/A}} \triangleq \begin{cases} \vec{0}, & \theta_{\mathrm{B/A}} = 0, \\ \theta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}, & \theta_{\mathrm{B/A}} \in (0,\pi). \end{cases} \tag{2.14.5}$$

Therefore, for all $\theta_{\mathrm{B/A}} \in [0,\pi)$, it follows that

$$\vec{\Theta}_{\mathrm{A/B}} = -\vec{\Theta}_{\mathrm{B/A}}, \tag{2.14.6}$$

$$\hat{\Theta}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}. \tag{2.14.7}$$

Finally, define

$$\Theta_{\mathrm{B/A}} \triangleq \vec{\Theta}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{\Theta}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \theta_{\mathrm{B/A}}n_{\mathrm{B/A}}. \tag{2.14.8}$$

If $\theta_{\mathrm{B/A}} = \pi$, then $\vec{R}_{\mathrm{B/A}}$ has two eigenaxes related by -1, and, therefore, $\hat{n}_{\mathrm{B/A}}$ is not uniquely defined. Consequently, $\vec{\Theta}_{\mathrm{B/A}}$ is also not uniquely defined. In this case, we follow the convention that $\hat{n}_{\mathrm{B/A}}$ represents one of two possible eigenaxes, and that $\vec{\Theta}_{\mathrm{B/A}}$ represents one of two possible eigenaxis angle vectors.

In terms of $\theta_{\mathrm{B/A}}$ and $\hat{n}_{\mathrm{B/A}}$, (2.11.1), (2.11.2), and (2.11.3) can be written as

$$\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) = (\cos\theta_{\mathrm{B/A}})\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \tag{2.14.9}$$

$$= \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\cos\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \tag{2.14.10}$$

$$= \vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}^{\times 2}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}. \tag{2.14.11}$$

The next result expresses $\vec{R}_{\mathrm{B/A}}$ in terms of $\vec{\Theta}_{\mathrm{B/A}}$.

Fact 2.14.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in (0,\pi)$. Then,

$$\vec{R}_{\mathrm{B/A}} = \exp\left(\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \tag{2.14.12}$$

$$= (\cos\theta_{\mathrm{B/A}})\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} \tag{2.14.13}$$

$$= \frac{1}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\cos\theta_{\mathrm{B/A}})\left(\vec{I} - \frac{1}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}}\right) + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} \tag{2.14.14}$$

$$= \vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}} + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}}$$

$$= \vec{I} + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}_{\mathrm{B/A}}^{\times} + \frac{1}{2}\left(\frac{\sin\frac{1}{2}\theta_{\mathrm{B/A}}}{\frac{1}{2}\theta_{\mathrm{B/A}}}\right)^2 \vec{\Theta}_{\mathrm{B/A}}^{\times 2}. \tag{2.14.15}$$

Proof. The result follows from Fact 2.14.1 by setting $\vec{\theta}^{\times}_{\vec{y}/\vec{x}} = \vec{\Theta}_{\mathrm{B/A}}$, that is, by setting $\theta_{\vec{y}/\vec{x}} = \theta_{\mathrm{B/A}}$ and $\hat{\theta}_{\vec{y}/\vec{x}} = \hat{n}_{\mathrm{B/A}}$. □

Combining (2.11.1), (2.11.43), and (2.14.12) we have the identities

$$\begin{aligned}\vec{R}_{\mathrm{B/A}} &= \vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}})\\ &= (\cos\theta_{\mathrm{B/A}})\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}\\ &= \exp\left(\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\\ &= \exp\left[\frac{\theta_{\mathrm{B/A}}}{2\sin\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right)\right],\end{aligned} \tag{2.14.16}$$

where the last expression is valid for the case $\theta_{\mathrm{B/A}} \in (0,\pi)$. It follows from (2.14.16) that

$$\begin{aligned}\mathcal{R}_{\mathrm{B/A}} &= \mathcal{R}_{n_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}})\\ &= (\cos\theta_{\mathrm{B/A}})I_3 + (1-\cos\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}n^{\mathrm{T}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})n^{\times}_{\mathrm{B/A}}\\ &= \exp\left(\Theta^{\times}_{\mathrm{B/A}}\right)\\ &= \exp\left[\frac{\theta_{\mathrm{B/A}}}{2\sin\theta_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}^{\mathrm{T}}_{\mathrm{B/A}})\right].\end{aligned} \tag{2.14.17}$$

The following result expresses $\vec{R}_{\mathrm{C/A}} = \vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}}$ in terms of eigenaxis angle vectors.

Fact 2.14.3. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$\exp\left(\vec{\Theta}^{\times}_{\mathrm{C/A}}\right) = \exp\left(\vec{\Theta}^{\times}_{\mathrm{C/B}}\right)\exp\left(\vec{\Theta}^{\times}_{\mathrm{B/A}}\right). \tag{2.14.18}$$

The following result gives an alternative exponential representation of a physical rotation matrix.

Fact 2.14.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $\vec{\Theta}_{\mathrm{B/A}}$ as in Fact 2.14.2. Furthermore, let $\vec{v}_{\mathrm{B/A}}$ and $\vec{w}_{\mathrm{B/A}}$ satisfy $\vec{\Theta}_{\mathrm{B/A}} = \vec{v}_{\mathrm{B/A}} \times \vec{w}_{\mathrm{B/A}}$. Then,

$$\vec{R}_{\mathrm{B/A}} = \exp(\vec{w}_{\mathrm{B/A}}\vec{v}'_{\mathrm{B/A}} - \vec{v}_{\mathrm{B/A}}\vec{w}'_{\mathrm{B/A}}). \tag{2.14.19}$$

Proof. It follows from (2.14.12) and (2.9.19) that

$$\vec{R}_{\mathrm{B/A}} = \exp\left(\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) = \exp[(\vec{v}_{\mathrm{B/A}} \times \vec{w}_{\mathrm{B/A}})^{\times}] = \exp(\vec{w}_{\mathrm{B/A}}\vec{v}'_{\mathrm{B/A}} - \vec{v}_{\mathrm{B/A}}\vec{w}'_{\mathrm{B/A}}).$$ □

In Fact 2.14.4, the vectors $\vec{v}_{\mathrm{B/A}}$ and $\vec{w}_{\mathrm{B/A}}$ define a plane that is perpendicular to $\vec{\Theta}_{\mathrm{B/A}}$. This plane and the length of the cross product of $\vec{v}_{\mathrm{B/A}}$ and $\vec{w}_{\mathrm{B/A}}$ characterize the physical rotation matrix. An analogous idea is given by the concept of a rotor in Chapter 3.

2.15 Euler Parameters

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames with eigenangle $\theta_{\mathrm{B/A}} \in [0, \pi]$ and eigenaxis $n_{\mathrm{B/A}}$. Then, rewriting (2.11.39) as

$$\cos\theta_{\mathrm{B/A}} = 2\cos^2 \tfrac{1}{2}\theta_{\mathrm{B/A}} - 1 = \tfrac{1}{2}(\operatorname{tr}\mathcal{R}_{\mathrm{B/A}} - 1), \tag{2.15.1}$$

we define

$$a \triangleq \cos\tfrac{1}{2}\theta_{\mathrm{B/A}} = \tfrac{1}{2}\sqrt{1 + \operatorname{tr}\mathcal{R}_{\mathrm{B/A}}}, \tag{2.15.2}$$

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} \triangleq (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}. \tag{2.15.3}$$

Note that $a \geq 0$, and that $a = 1$ if and only if $\theta_{\mathrm{B/A}} = 0$ rad, whereas $a = 0$ if and only if $\theta_{\mathrm{B/A}} = \pi$ rad. Furthermore, $\theta_{\mathrm{B/A}} = 0$ rad if and only if

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} = 0, \tag{2.15.4}$$

whereas $\theta_{\mathrm{B/A}} = \pi$ rad if and only if

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} = n_{\mathrm{B/A}}. \tag{2.15.5}$$

Now, assume that $\theta_{\mathrm{B/A}} \in [0, \pi)$, so that $a > 0$. Then, it follows from (2.11.43) and the identity $\sin\theta_{\mathrm{B/A}} = 2(\sin\frac{1}{2}\theta_{\mathrm{B/A}})\cos\frac{1}{2}\theta_{\mathrm{B/A}}$ that

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix}^{\times} = (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times} = \frac{\sin\frac{1}{2}\theta_{\mathrm{B/A}}}{2\sin\theta_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}}) = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}}). \tag{2.15.6}$$

Therefore,

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}})^{-\times}, \tag{2.15.7}$$

that is,

$$b = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}(3,2)} - \mathcal{R}_{\mathrm{B/A}(2,3)}), \tag{2.15.8}$$

$$c = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}(1,3)} - \mathcal{R}_{\mathrm{B/A}(3,1)}), \tag{2.15.9}$$

$$d = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}(2,1)} - \mathcal{R}_{\mathrm{B/A}(1,2)}). \tag{2.15.10}$$

Fact 2.15.1. Define a, b, c, d by (2.15.2) and (2.15.3). Then,

$$a^2 + b^2 + c^2 + d^2 = 1. \tag{2.15.11}$$

Proof. Note that

$$a^2+b^2+c^2+d^2 = a^2 + \begin{bmatrix} b \\ c \\ d \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$
$$= \cos^2 \tfrac{1}{2}\theta_{\mathrm{B/A}} + (\sin^2 \tfrac{1}{2}\theta_{\mathrm{B/A}}) n_{\mathrm{B/A}}^{\mathrm{T}} n_{\mathrm{B/A}}$$
$$= \cos^2 \tfrac{1}{2}\theta_{\mathrm{B/A}} + \sin^2 \tfrac{1}{2}\theta_{\mathrm{B/A}} = 1. \qquad \square$$

The case $\theta_{\mathrm{B/A}} = \pi$ must be handled separately.

Fact 2.15.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} = \pi$. Then,

$$\mathcal{R}_{\mathrm{B/A}} = \begin{bmatrix} 2b^2-1 & 2bc & 2bd \\ 2bc & 2c^2-1 & 2cd \\ 2bd & 2cd & 2d^2-1 \end{bmatrix}. \tag{2.15.12}$$

Proof. It follows from (2.11.34) that

$$\mathcal{R}_{\mathrm{B/A}} = -I_3 + 2n_{\mathrm{B/A}} n_{\mathrm{B/A}}^{\mathrm{T}} = -I_3 + 2 \begin{bmatrix} b \\ c \\ d \end{bmatrix} \begin{bmatrix} b & c & d \end{bmatrix}. \qquad \square$$

Fact 2.15.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in (0, \pi)$. Then,

$$\mathcal{R}_{\mathrm{B/A}} = \begin{bmatrix} 2a^2+2b^2-1 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & 2a^2+2c^2-1 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & 2a^2+2d^2-1 \end{bmatrix} \tag{2.15.13}$$

$$= \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{bmatrix}. \tag{2.15.14}$$

Proof. It follows from (2.11.45) that

$$\mathcal{R}_{\mathrm{B/A}} = (\cos\theta_{\mathrm{B/A}})I_3 + (1-\cos\theta_{\mathrm{B/A}}) n_{\mathrm{B/A}} n_{\mathrm{B/A}}^{\mathrm{T}} + (\sin\theta_{\mathrm{B/A}}) n_{\mathrm{B/A}}^{\times}$$
$$= (2\cos^2 \tfrac{1}{2}\theta_{\mathrm{B/A}} - 1)I_3 + 2(\sin^2 \tfrac{1}{2}\theta_{\mathrm{B/A}}) n_{\mathrm{B/A}} n_{\mathrm{B/A}}^{\mathrm{T}} + 2(\cos \tfrac{1}{2}\theta_{\mathrm{B/A}})(\sin \tfrac{1}{2}\theta_{\mathrm{B/A}}) n_{\mathrm{B/A}}^{\times}$$
$$= (2a^2-1)I_3 + 2\begin{bmatrix} b \\ c \\ d \end{bmatrix} \begin{bmatrix} b & c & d \end{bmatrix} + 2a \begin{bmatrix} b \\ c \\ d \end{bmatrix}^{\times}. \qquad \square$$

Next, in analogy with the eigenaxis angle vector defined in (2.14.5), we define the *Euler vector*

$$\vec{\varepsilon}_{\mathrm{B/A}} \triangleq (\sin \tfrac{1}{2}\theta_{\mathrm{B/A}}) \hat{n}_{\mathrm{B/A}}. \tag{2.15.15}$$

This vector is uniquely defined for all $\theta_{\mathrm{B/A}} \in [0, \pi)$. It follows from (2.11.50) and (2.11.51) that

$$\vec{\varepsilon}_{\mathrm{B/A}} = -\vec{\varepsilon}_{\mathrm{A/B}}. \tag{2.15.16}$$

In the case $\theta_{\mathrm{B/A}} = \pi$, there are two possible choices of the eigenaxis $\hat{n}_{\mathrm{A/B}}$, and thus two possible choices of $\vec{\varepsilon}_{\mathrm{B/A}}$; these choices are related by the factor -1. Despite this ambiguity, we write $\vec{\varepsilon}_{\mathrm{A/B}} = -\vec{\varepsilon}_{\mathrm{B/A}}$ in all cases. Note that, if $\theta_{\mathrm{B/A}} \in (0, \pi]$, then

$$\hat{\varepsilon}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}. \tag{2.15.17}$$

Next, define

$$\eta_{\mathrm{B/A}} \triangleq a, \tag{2.15.18}$$

$$\varepsilon_{\mathrm{B/A}} \triangleq \vec{\varepsilon}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{\varepsilon}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = (\sin \tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}. \tag{2.15.19}$$

Then, the *Euler parameter vector* $q_{\mathrm{B/A}}$ of $\mathrm{F_B}$ relative to $\mathrm{F_A}$ is defined by

$$q_{\mathrm{B/A}} \triangleq \begin{bmatrix} \eta_{\mathrm{B/A}} \\ \varepsilon_{\mathrm{B/A}} \end{bmatrix} = \begin{bmatrix} \cos \frac{1}{2}\theta_{\mathrm{B/A}} \\ (\sin \frac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}. \tag{2.15.20}$$

Note that $\eta_{\mathrm{B/A}} \ge 0$, and that $\eta_{\mathrm{B/A}} = 0$ if and only if $\theta_{\mathrm{B/A}} = \pi$ rad. Furthermore, if $\theta_{\mathrm{B/A}} = 0$, then

$$q_{\mathrm{B/A}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.15.21}$$

whereas, if $\theta_{\mathrm{B/A}} = \pi$, then

$$q_{\mathrm{B/A}} = \begin{bmatrix} 0 \\ n_{\mathrm{B/A}} \end{bmatrix}. \tag{2.15.22}$$

The Euler parameter vector $q_{\mathrm{B/A}}$ provides a representation of the rotation matrix $\mathcal{R}_{\mathrm{B/A}}$. The components a, b, c, d of $q_{\mathrm{B/A}}$ are the *Euler parameters*. It follows from (2.11.50) and (2.11.51) that

$$\eta_{\mathrm{B/A}} = \eta_{\mathrm{A/B}}, \tag{2.15.23}$$

$$\varepsilon_{\mathrm{B/A}} = -\varepsilon_{\mathrm{A/B}}. \tag{2.15.24}$$

The following result shows that $q_{\mathrm{B/A}}$ is an element of the unit sphere in $\mathbb{R}^4$.

Fact 2.15.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\eta_{\mathrm{B/A}}^2 + \vec{\varepsilon}_{\mathrm{B/A}}^{\,\prime}\vec{\varepsilon}_{\mathrm{B/A}} = 1, \tag{2.15.25}$$

and thus

$$\eta_{\mathrm{B/A}}^2 + \varepsilon_{\mathrm{B/A}}^{\mathrm{T}}\varepsilon_{\mathrm{B/A}} = 1. \tag{2.15.26}$$

If $\theta_{\mathrm{B/A}} \in (-2\pi, 2\pi]$ so that $\frac{1}{2}\theta_{\mathrm{B/A}} \in (-\pi, \pi]$, then this representation of the rotation matrices is two-to-one since $q_{\mathrm{B/A}}$ and $-q_{\mathrm{B/A}}$ both represent $\mathcal{R}_{\mathrm{B/A}}$. With this increased range of $\theta_{\mathrm{B/A}}$, the values of $q_{\mathrm{B/A}}$ are in one-to-one correspondence with each point on the unit sphere in $\mathbb{R}^4$. However, unless stated otherwise, the eigenangle $\theta_{\mathrm{B/A}}$ is assumed to be an element of $[0, \pi]$.

Fact 2.15.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{R}_{\mathrm{B/A}} = (2\eta_{\mathrm{B/A}}^2 - 1)\vec{I} + 2\eta_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}}^{\times} + 2\vec{\varepsilon}_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}}^{\prime} \tag{2.15.27}$$

$$= \vec{I} + 2\eta_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}}^{\times} + 2\vec{\varepsilon}_{\mathrm{B/A}}^{\times 2}. \tag{2.15.28}$$

Proof. The result follows from Fact 2.11.8 using the identities

$$2(\cos^2 \tfrac{1}{2}\theta) - 1 = \cos\theta, \quad 2\cos\tfrac{1}{2}\theta = 1 - \cos\theta, \quad 2(\cos\tfrac{1}{2}\theta)\sin\tfrac{1}{2}\theta = \sin\theta. \qquad \square$$

Resolving (2.15.27) and (2.15.28) yields the following result.

Fact 2.15.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{R}_{\mathrm{B/A}} = (2\eta_{\mathrm{B/A}}^2 - 1)I_3 + 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\mathrm{T}} \tag{2.15.29}$$

$$= I_3 + 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}^{\times 2}. \tag{2.15.30}$$

Furthermore,

$$\mathcal{O}_{\mathrm{B/A}} = (2\eta_{\mathrm{B/A}}^2 - 1)I_3 - 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\mathrm{T}} \tag{2.15.31}$$

$$= I_3 - 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}^{\times 2}. \tag{2.15.32}$$

The following result determines the Euler-parameter vector for a rotation rising from a pair of rotation matrices expressed in terms of Euler parameters. In effect, this result provides an expression for the product of Euler-parameter vectors, corresponding to the identity $\mathcal{O}_{\mathrm{C/A}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}$.

For the next result, define

$$\mathcal{Q}(q) \triangleq \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}, \tag{2.15.33}$$

where $q \triangleq [a \;\; b \;\; c \;\; d]^{\mathrm{T}}$.

Fact 2.15.7. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$q_{\mathrm{C/A}} = \begin{bmatrix} \eta_{\mathrm{C/A}} \\ \varepsilon_{\mathrm{C/A}} \end{bmatrix} = \begin{bmatrix} \eta_{\mathrm{C/B}}\eta_{\mathrm{B/A}} - \varepsilon_{\mathrm{C/B}}^{\mathrm{T}}\varepsilon_{\mathrm{B/A}} \\ \eta_{\mathrm{B/A}}\varepsilon_{\mathrm{C/B}} + \eta_{\mathrm{C/B}}\varepsilon_{\mathrm{B/A}} + \varepsilon_{\mathrm{C/B}} \times \varepsilon_{\mathrm{B/A}} \end{bmatrix} = \mathcal{Q}(q_{\mathrm{C/B}})q_{\mathrm{B/A}}. \tag{2.15.34}$$

Proof. Define

$$\mathcal{R}_3 \triangleq \mathcal{R}_{\mathrm{C/A}}, \quad \mathcal{R}_2 \triangleq \mathcal{R}_{\mathrm{C/B}}, \quad \mathcal{R}_1 \triangleq \mathcal{R}_{\mathrm{B/A}},$$

$$q_3 \triangleq \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} \triangleq q_{\mathrm{C/A}}, \quad q_2 \triangleq \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \triangleq q_{\mathrm{C/B}}, \quad q_1 \triangleq \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \triangleq q_{\mathrm{B/A}}.$$

Then, for $i = 1, 2, 3$, it follows from (2.15.30) that

$$\mathcal{R}_i = (2a_i^2 - 1)I_3 + 2\begin{bmatrix} b_i \\ c_i \\ d_i \end{bmatrix}\begin{bmatrix} b_i & c_i & d_i \end{bmatrix} + 2a_i\begin{bmatrix} b_i \\ c_i \\ d_i \end{bmatrix}^{\times}.$$

Hence,

$$a_3 = \tfrac{1}{2}\sqrt{1+\operatorname{tr}\mathcal{R}_3} = \tfrac{1}{2}\sqrt{1+\operatorname{tr}\mathcal{R}_1\mathcal{R}_2} = a_2a_1 - \begin{bmatrix} b_2 \\ c_2 \\ d_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b_1 \\ c_1 \\ d_1 \end{bmatrix},$$

which confirms the expression for $\eta_{\mathrm{C/A}}$ in (2.15.34). Furthermore,

$$\begin{bmatrix} b_3 \\ c_3 \\ d_3 \end{bmatrix} = \frac{1}{4a_3}(\mathcal{R}_3 - \mathcal{R}_3^{\mathrm{T}}) = \frac{1}{4a_3}(\mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2^{\mathrm{T}}\mathcal{R}_1^{\mathrm{T}}) = a_1\begin{bmatrix} b_2 \\ c_2 \\ d_2 \end{bmatrix} + a_2\begin{bmatrix} b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} b_2 \\ c_2 \\ d_2 \end{bmatrix} \times \begin{bmatrix} b_1 \\ c_1 \\ d_1 \end{bmatrix},$$

which confirms the expression for $\varepsilon_{\mathrm{C/A}}$ in (2.15.34). The last expression for $q_{\mathrm{C/A}}$ can be confirmed directly. □

2.16 Quaternions

The unit quaternions $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ provide a representation of the Euler parameters. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \tag{2.16.1}$$

and define

$$\mathbb{H} \triangleq \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}\colon a, b, c, d \in \mathbb{R}\}. \tag{2.16.2}$$

Furthermore, for all $a, b, c, d \in \mathbb{R}$, define

$$\mathbf{q} \triangleq a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad \overline{\mathbf{q}} \triangleq a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}, \tag{2.16.3}$$

$$|\mathbf{q}| \triangleq \sqrt{\mathbf{q}\overline{\mathbf{q}}} = \sqrt{a^2+b^2+c^2+d^2} = |\overline{\mathbf{q}}|. \tag{2.16.4}$$

Then,

$$\mathbf{q}I_4 = U\mathcal{Q}(\mathbf{q})U, \tag{2.16.5}$$

where

$$\mathcal{Q}(\mathbf{q}) \triangleq \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}, \quad U \triangleq \frac{1}{2}\begin{bmatrix} 1 & \mathbf{i} & \mathbf{j} & k \\ -\mathbf{i} & 1 & \mathbf{k} & -\mathbf{j} \\ -\mathbf{j} & -\mathbf{k} & 1 & \mathbf{i} \\ -\mathbf{k} & \mathbf{j} & -\mathbf{i} & 1 \end{bmatrix}. \tag{2.16.6}$$

Note that $U^2 = I_4$. In addition,

$$\det \mathcal{Q}(\mathbf{q}) = (a^2+b^2+c^2+d^2)^2. \tag{2.16.7}$$

Furthermore, if $|\mathbf{q}| = 1$, then $\mathcal{Q}(\mathbf{q})$ is orthogonal.

For $i = 1, 2$, let $a_i, b_i, c_i, d_i \in \mathbb{R}$ and define $\mathbf{q}_i \triangleq a_i + b_i\mathbf{i} + c_i\mathbf{j} + d_i\mathbf{k}$ and $v_i \triangleq [b_i \;\; c_i \;\; d_i]^{\mathrm{T}}$. In addition, define $\mathbf{q}_3 \triangleq \mathbf{q}_2\mathbf{q}_1 = a_3 + b_3\mathbf{i} + c_3\mathbf{j} + d_3\mathbf{k}$. Then,

$$\mathcal{Q}(\mathbf{q}_3) = \mathcal{Q}(\mathbf{q}_2)\mathcal{Q}(\mathbf{q}_1), \quad \overline{\mathbf{q}_3} = \overline{\mathbf{q}_2}\,\overline{\mathbf{q}_1}, \tag{2.16.8}$$

$$|\mathbf{q}_3| = |\mathbf{q}_2\mathbf{q}_1| = |\mathbf{q}_1\mathbf{q}_2| = |\mathbf{q}_1\overline{\mathbf{q}_2}| = |\overline{\mathbf{q}_1}\mathbf{q}_2| = |\overline{\mathbf{q}_1}\,\overline{\mathbf{q}_2}| = |\mathbf{q}_1|\,|\mathbf{q}_2|. \tag{2.16.9}$$

Furthermore, it follows from (2.15.34) that

$$\begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \mathcal{Q}(\mathbf{q}_2) \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} a_2 a_1 - v_2^{\mathrm{T}} v_1 \\ a_1 v_2 + a_2 v_1 + v_2 \times v_1 \end{bmatrix}. \tag{2.16.10}$$

This result shows that the unit quaternions satisfy the multiplicative property (2.15.34) for the Euler parameters.

We now give multiplication tables to relate the complex numbers and the quaternions. These representations involve 2×2 matrices, such as the skew-symmetric matrix

$$J_2 \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.16.11}$$

and the *Pauli matrices* $\sigma_1, \sigma_2, \sigma_3$ and their products, which are given by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\jmath \\ \jmath & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.16.12}$$

Note that

$$\sigma_1\sigma_2 = \jmath\sigma_3 = -\sigma_2\sigma_1 = \begin{bmatrix} \jmath & 0 \\ 0 & -\jmath \end{bmatrix}, \tag{2.16.13}$$

$$\sigma_2\sigma_3 = \jmath\sigma_1 = -\sigma_3\sigma_2 = \begin{bmatrix} 0 & \jmath \\ \jmath & 0 \end{bmatrix}, \tag{2.16.14}$$

$$\sigma_3\sigma_1 = \jmath\sigma_2 = -\sigma_1\sigma_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.16.15}$$

$$\sigma_1\sigma_2\sigma_3 = \begin{bmatrix} \jmath & 0 \\ 0 & \jmath \end{bmatrix}. \tag{2.16.16}$$

Equivalent multiplication tables involving 2×2 real matrices and complex scalars are given in Table 2.16.1.

	I_2	J_2
I_2	I_2	J_2
J_2	J_2	$-I_2$

	I_2	$\sigma_1\sigma_2$
I_2	I_2	$\sigma_1\sigma_2$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$-I_2$

	1	$\jmath$
1	1	$\jmath$
$\jmath$	$\jmath$	-1

Figure 2.16.1: Equivalent multiplication tables for 2×2 matrices (a), (b), and the complex numbers (c).

	I_2	$\sigma_2\sigma_3$	$\sigma_1\sigma_3$	$\sigma_1\sigma_2$
I_2	I_2	$\sigma_2\sigma_3$	$\sigma_1\sigma_3$	$\sigma_1\sigma_2$
$\sigma_1\sigma_3$	$\sigma_1\sigma_3$	$-I_2$	$\sigma_1\sigma_2$	$-\sigma_1\sigma_3$
$\sigma_1\sigma_3$	$\sigma_1\sigma_3$	$-\sigma_1\sigma_2$	$-I_2$	$\sigma_2\sigma_3$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$\sigma_2\sigma_3$	$-\sigma_2\sigma_3$	$-I_2$

(a)

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

(b)

Figure 2.16.2: Equivalent multiplication tables for products of Pauli matrices (a) and the quaternions (b).

2.17 Gibbs Parameters

An alternative representation of rotation matrices is given in terms of the Gibbs vector. This representation is valid for all rotations except for the case where the eigenangle is π rad.

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in [0,\pi)$. In analogy with the Euler vector defined in (2.15.15) and the eigenaxis angle vector defined in (2.14.5), we define the *Gibbs vector*

$$\vec{g}_{\mathrm{B/A}} \triangleq (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}. \tag{2.17.1}$$

If $\theta_{\mathrm{B/A}} = 0$, then $\vec{g}_{\mathrm{A/B}} = 0$. If $\theta_{\mathrm{B/A}} \in (0,\pi)$, then $\theta_{\mathrm{B/A}} = \theta_{\mathrm{A/B}}$ and $\hat{n}_{\mathrm{B/A}} = -\hat{n}_{\mathrm{A/B}}$, and thus $\vec{g}_{\mathrm{A/B}} = -\vec{g}_{\mathrm{B/A}}$. In the case $\theta_{\mathrm{B/A}} = \pi$, $\vec{g}_{\mathrm{B/A}}$ is not defined. Note that, if $\theta_{\mathrm{B/A}} \in (0,\pi)$, then

$$\hat{g}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}. \tag{2.17.2}$$

Finally, define the *Gibbs parameter vector* by

$$g_{\mathrm{B/A}} \triangleq \vec{g}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{g}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}. \tag{2.17.3}$$

Hence,

$$g_{\mathrm{B/A}} = -g_{\mathrm{A/B}}. \tag{2.17.4}$$

Next, define the physical matrix

$$\vec{R}_{\vec{g}_{\mathrm{B/A}}} \triangleq \frac{1}{1+\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}}[(1-\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}})\vec{I} + 2\vec{g}_{\mathrm{B/A}}\vec{g}'_{\mathrm{B/A}} + 2\vec{g}^{\times}_{\mathrm{B/A}}]. \tag{2.17.5}$$

The following result shows that $\vec{R}_{\vec{g}_{\mathrm{B/A}}}$ is the physical rotation matrix whose eigenaxis is $\hat{n}_{\mathrm{B/A}}$ and eigenangle is $\theta_{\mathrm{B/A}}$.

Fact 2.17.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in [0,\pi)$. Then,

$$\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) = \vec{R}_{\vec{g}_{\mathrm{B/A}}}. \tag{2.17.6}$$

Furthermore,

$$\vec{R}_{\vec{g}_{\mathrm{B/A}}} = (\vec{I} - \vec{g}^{\times}_{\mathrm{B/A}})^{-1}(\vec{I} + \vec{g}^{\times}_{\mathrm{B/A}}) \tag{2.17.7}$$

$$= (\vec{I} + \vec{g}^{\times}_{\mathrm{B/A}})(\vec{I} - \vec{g}^{\times}_{\mathrm{B/A}})^{-1}. \tag{2.17.8}$$

Proof. To prove (2.17.6) note that it follows from (2.11.1) and the identities

$$\cos\theta = \frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}, \quad 1-\cos\theta = \frac{2\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}, \quad \sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$$

that

$$\begin{aligned}
\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) &= (\cos\theta)\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \\
&= \frac{1-\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}{1+\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}\vec{I} + \frac{2\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}{1+\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + \frac{2\tan\frac{1}{2}\theta_{\mathrm{B/A}}}{1+\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}\hat{n}^{\times}_{\mathrm{B/A}} \\
&= \frac{1}{1+\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}}[(1-\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}})\vec{I} + 2\vec{g}_{\mathrm{B/A}}\vec{g}'_{\mathrm{B/A}} + 2\vec{g}^{\times}_{\mathrm{B/A}}] = \vec{R}_{\vec{g}_{\mathrm{B/A}}}.
\end{aligned}$$

Next, to prove (2.17.7) note that

$$\begin{aligned}
(\vec{I} - \vec{g}_{\mathrm{B/A}}^{\times})^{-1}(\vec{I} + \vec{g}_{\mathrm{B/A}}^{\times}) &= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}^{\prime}\vec{g}_{\mathrm{B/A}}}(\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}^{\prime} + \vec{g}_{\mathrm{B/A}}^{\times})(\vec{I} + \vec{g}_{\mathrm{B/A}}^{\times}) \\
&= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}^{\prime}\vec{g}_{\mathrm{B/A}}}[\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}^{\prime} + 2\vec{g}_{\mathrm{B/A}}^{\times} + \vec{g}_{\mathrm{B/A}}^{\times 2}] \\
&= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}^{\prime}\vec{g}_{\mathrm{B/A}}}[\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}^{\prime} + 2\vec{g}_{\mathrm{B/A}}^{\times} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}^{\prime} - \vec{g}_{\mathrm{B/A}}^{\prime}\vec{g}_{\mathrm{B/A}}\vec{I}] \\
&= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}^{\prime}\vec{g}_{\mathrm{B/A}}}[(1 - \vec{g}_{\mathrm{B/A}}^{\prime}\vec{g}_{\mathrm{B/A}})\vec{I} + 2\vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}^{\prime} + 2\vec{g}_{\mathrm{B/A}}^{\times}] = \vec{R}_{\vec{g}_{\mathrm{B/A}}}. \quad \square
\end{aligned}$$

The following result relates the Gibbs parameter vector to the Euler parameter vector $q_{\mathrm{B/A}}$ defined by (2.15.20).

Fact 2.17.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \neq \pi$. Then, $\eta_{\mathrm{B/A}} \neq 0$. Furthermore,

$$g_{\mathrm{B/A}} = \frac{1}{\eta_{\mathrm{B/A}}}\varepsilon_{\mathrm{B/A}}. \tag{2.17.9}$$

Finally,

$$g_{\mathrm{B/A}} = \frac{1}{1 + \operatorname{tr}\mathcal{R}_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}})^{-\times}. \tag{2.17.10}$$

The following result determines the Gibbs parameter vector for a rotation arising from a pair of rotation matrices expressed in terms of Gibbs parameter vectors. In effect, this result provides an expression for the product of Gibbs-parameter vectors, corresponding to the identity $\mathcal{O}_{\mathrm{C/A}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}$.

Fact 2.17.3. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames, and assume that neither of the eigenangles $\theta_{\mathrm{C/A}}$, $\theta_{\mathrm{C/B}}$, nor $\theta_{\mathrm{B/A}}$ is equal to π rad. Then,

$$\vec{g}_{\mathrm{C/A}} = \frac{1}{1 - \vec{g}_{\mathrm{C/B}}^{\prime}\vec{g}_{\mathrm{B/A}}}(\vec{g}_{\mathrm{C/B}} + \vec{g}_{\mathrm{B/A}} - \vec{g}_{\mathrm{C/B}} \times \vec{g}_{\mathrm{B/A}}). \tag{2.17.11}$$

2.18 Summary of Rotation-Matrix Representations

Table 2.18.1 summarizes the existence and uniqueness of the eigenaxis, eigenaxis angle vector, Euler vector, and Gibbs vector in terms of the eigenangle.

2.19 Additivity of Angle Vectors

The following result concerns the additivity of angles for linearly independent physical vectors.

Fact 2.19.1. Suppose that the physical vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly dependent, and let $\hat{n}$ denote a unit vector that is orthogonal to $\vec{x}$, $\vec{y}$, and $\vec{z}$. Then, there exists $i \in \{-1, 0, 1\}$ such that

$$\theta_{\vec{z}/\vec{x}/\hat{n}} = \theta_{\vec{z}/\vec{y}/\hat{n}} + \theta_{\vec{y}/\vec{x}/\hat{n}} \pm 2i\pi. \tag{2.19.1}$$

$\operatorname{tr} \vec{R}_{\mathrm{B/A}}$	$\theta_{\mathrm{B/A}}$	$\hat{n}_{\mathrm{B/A}}$	$\vec{\Theta}_{\mathrm{B/A}} = \theta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}$	$\vec{\varepsilon}_{\mathrm{B/A}} = (\sin \frac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}$	$\vec{g}_{\mathrm{B/A}} = (\tan \frac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}$
$\operatorname{tr} \vec{R}_{\mathrm{B/A}} = -1$	$\theta_{\mathrm{B/A}} = \pi$	Two choices $\pm\hat{n}_{\mathrm{B/A}}$	Two choices $\vec{\Theta}_{\mathrm{B/A}} = \pm\pi\hat{n}_{\mathrm{B/A}}$	Unique $\vec{\varepsilon}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}$	Not Defined
$-1 < \operatorname{tr} \vec{R}_{\mathrm{B/A}} < 3$	$\theta_{\mathrm{B/A}} \in (0, \pi)$	Unique	Unique	Unique	Unique
$\operatorname{tr} \vec{R}_{\mathrm{B/A}} = 3$	$\theta_{\mathrm{B/A}} = 0$	Arbitrary	Unique $\vec{\Theta}_{\mathrm{B/A}} = 0$	Unique $\vec{\varepsilon}_{\mathrm{B/A}} = 0$	Unique $\vec{g}_{\mathrm{B/A}} = 0$

Figure 2.18.1: Existence and uniqueness of the eigenaxis, eigenaxis angle vector, Euler vector, and Gibbs vector in terms of the eigenangle.

In addition,

$$\vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{x}/\hat{n}}) = \vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{y}/\hat{n}})\vec{R}_{\hat{n}}(\theta_{\vec{y}/\vec{x}/\hat{n}}). \tag{2.19.2}$$

Now, assume that $\theta_{\vec{z}/\vec{x}/\hat{n}}, \theta_{\vec{z}/\vec{y}/\hat{n}}, \theta_{\vec{y}/\vec{x}/\hat{n}} \in (0, \pi)$, $\theta_{\vec{z}/\vec{y}/\hat{n}} \le \theta_{\vec{z}/\vec{x}/\hat{n}}$, and $\theta_{\vec{y}/\vec{x}/\hat{n}} \le \theta_{\vec{z}/\vec{x}/\hat{n}}$. Then,

$$\theta_{\vec{z}/\vec{x}} = \theta_{\vec{z}/\vec{y}} + \theta_{\vec{y}/\vec{x}}, \tag{2.19.3}$$

$$\vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{x}}) = \vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{y}})\vec{R}_{\hat{n}}(\theta_{\vec{y}/\vec{x}}). \tag{2.19.4}$$

The case corresponding to (2.19.3) is illustrated in Figure **??**.

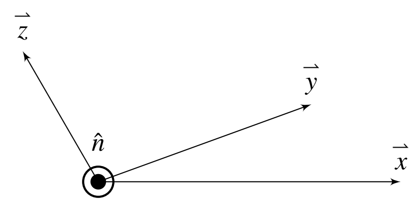


Figure 2.19.2: Angle additivity.

Fact 2.19.2. Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be unit dimensionless physical vectors. Then,

$$|\hat{x} \times \hat{y}|^2 \cos^2 \theta_{\hat{x}\times\hat{y}/\hat{z}} = 1 + 2(\cos\theta_{\hat{x}/\hat{y}})(\cos\theta_{\hat{y}/\hat{z}})(\cos\theta_{\hat{z}/\hat{x}}) - \cos^2\theta_{\hat{x}/\hat{y}} - \cos^2\theta_{\hat{y}/\hat{z}} - \cos^2\theta_{\hat{z}/\hat{x}}. \tag{2.19.5}$$

Proof. Using Fact 2.9.7 we have

$$|\hat{x} \times \hat{y}|^2 \cos^2 \theta_{\hat{x}\times\hat{y}/\hat{z}} = [(\hat{x} \times \hat{y})'\hat{z}]^2 = \det \begin{bmatrix} \vec{x}\big|_{\mathrm{A}}^{\mathrm{T}} \\ \vec{y}\big|_{\mathrm{A}}^{\mathrm{T}} \\ \vec{z}\big|_{\mathrm{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & \hat{x}'\hat{y} & \hat{x}'\hat{z} \\ \hat{x}'\hat{y} & 1 & \hat{y}'\hat{z} \\ \hat{x}'\hat{z} & \hat{y}'\hat{z} & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & \cos\theta_{\hat{x}/\hat{y}} & \cos\theta_{\hat{x}/\hat{z}} \\ \cos\theta_{\hat{x}/\hat{y}} & 1 & \cos\theta_{\hat{y}/\hat{z}} \\ \cos\theta_{\hat{x}/\hat{z}} & \cos\theta_{\hat{y}/\hat{z}} & 1 \end{bmatrix}. \qquad \square$$

The following result shows that angle vectors are additive if and only if both angles lie in the same plane.

Fact 2.19.3. Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be nonzero physical vectors, no two of which are parallel. Then, the following statements are equivalent:

i) $\hat{x}$, $\hat{y}$, and $\hat{z}$ are linearly dependent.

ii) $\hat{\theta}_{\hat{z}/\hat{x}}$, $\hat{\theta}_{\hat{x}/\hat{y}}$, and $\hat{\theta}_{\hat{y}/\hat{z}}$ are parallel.

iii) At least two of the vectors $\hat{\theta}_{\hat{z}/\hat{x}}$, $\hat{\theta}_{\hat{x}/\hat{y}}$, and $\hat{\theta}_{\hat{y}/\hat{z}}$ are parallel.

iv) Either $\theta_{\hat{x}/\hat{y}} + \theta_{\hat{y}/\hat{z}} + \theta_{\hat{z}/\hat{x}} = 2\pi$, $\theta_{\hat{z}/\hat{x}} = \theta_{\hat{z}/\hat{y}} + \theta_{\hat{y}/\hat{x}}$, $\theta_{\hat{y}/\hat{z}} = \theta_{\hat{y}/\hat{x}} + \theta_{\hat{x}/\hat{z}}$, or $\theta_{\hat{z}/\hat{y}} = \theta_{\hat{z}/\hat{x}} + \theta_{\hat{x}/\hat{y}}$.

v) $\cos\theta_{\hat{x}\times\hat{y}/\hat{z}} = 0$.

vi) $\cos^2\theta_{\hat{x}/\hat{y}} + \cos^2\theta_{\hat{y}/\hat{z}} + \cos^2\theta_{\hat{z}/\hat{x}} = 1 + 2(\cos\theta_{\hat{x}/\hat{y}})(\cos\theta_{\hat{y}/\hat{z}})(\cos\theta_{\hat{z}/\hat{x}})$.

vii) $\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}})\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})$.

The following result considers the case where the vectors do not necessarily lie in a single plane.

Fact 2.19.4. Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be unit dimensionless physical vectors, and assume that $\hat{x}$ and $\hat{y}$ are not parallel and $\hat{y}$ and $\hat{z}$ are not parallel. Then,

$$\begin{aligned}
\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}&(\theta_{\hat{z}/\hat{y}})\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) - \vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}})\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}}) \\
&= (\sin\theta_{\hat{z}/\hat{y}})(\sin\theta_{\hat{y}/\hat{x}})(\hat{\theta}_{\hat{z}/\hat{y}} \times \hat{\theta}_{\hat{y}/\hat{x}})^{\times} \\
&\quad + (\sin\theta_{\hat{z}/\hat{y}})(1-\cos\theta_{\hat{y}/\hat{x}})(\hat{\theta}^{\times}_{\hat{z}/\hat{y}}\hat{\theta}_{\hat{y}/\hat{x}}\hat{\theta}'_{\hat{y}/\hat{x}} - \hat{\theta}_{\hat{y}/\hat{x}}\hat{\theta}'_{\hat{y}/\hat{x}}\hat{\theta}^{\times}_{\hat{z}/\hat{y}}) \\
&\quad + (\sin\theta_{\hat{y}/\hat{x}})(1-\cos\theta_{\hat{z}/\hat{y}})(\hat{\theta}_{\hat{z}/\hat{y}}\hat{\theta}'_{\hat{z}/\hat{y}}\hat{\theta}^{\times}_{\hat{y}/\hat{x}} - \hat{\theta}^{\times}_{\hat{y}/\hat{x}}\hat{\theta}_{\hat{z}/\hat{y}}\hat{\theta}'_{\hat{z}/\hat{y}}) \\
&\quad + (1-\cos\theta_{\hat{z}/\hat{y}})(1-\cos\theta_{\hat{y}/\hat{x}})\hat{\theta}'_{\hat{z}/\hat{y}}\hat{\theta}_{\hat{y}/\hat{x}}(\hat{\theta}_{\hat{z}/\hat{y}}\hat{\theta}'_{\hat{y}/\hat{x}} - \hat{\theta}_{\hat{y}/\hat{x}}\hat{\theta}'_{\hat{z}/\hat{y}}).
\end{aligned} \tag{2.19.6}$$

Finally,

$$\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}})\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}}) \tag{2.19.7}$$

if and only if at least one of the following conditions holds:

i) $\hat{\theta}_{\hat{z}/\hat{y}}$ and $\hat{\theta}_{\hat{y}/\hat{x}}$ are parallel.

ii) Either $\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}}) = \vec{I}$ or $\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{I}$.

iii) $\vec{R}^{2}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}}) = \vec{I}$, $\vec{R}^{2}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{I}$, and $\hat{\theta}'_{\hat{y}/\hat{x}}\hat{\theta}_{\hat{z}/\hat{y}} = 0$.

Proof. See [1, p. 211]. □

2.20 Rotation of a Rigid Body about a Point

Since a frame has no location, rotation of a frame affects only the orientation of the frame. Rotation of a body, however, concerns a physical object, which has an orientation as well as a location in space. Consequently, rotation of a body is not fully defined by specifying only the physical rotation matrix; additional information is needed to characterize the effect of a rotation on the location of the body. Although time plays no explicit role in this chapter, we assume that the body is rigid in order to emphasize that the shape of the body is unchanged by the rotation.

In order to discuss the rotation of a rigid body, we define a point that is fixed in the body and that remains spatially fixed despite the rotation. For example, suppose that the rigid body $\mathcal{B}$ is a cube, let the point x be one of the vertices of $\mathcal{B}$, let $\vec{R}$ be a physical rotation matrix, let $\mathcal{B}'$ denote the body $\mathcal{B}$ rotated by $\vec{R}$, and let x' denote the point on $\mathcal{B}'$ corresponding to x on $\mathcal{B}$. We then assume that the rotation of $\mathcal{B}$ occurs so that x is spatially fixed in the sense that $\vec{r}_{x'/x} = \vec{0}$. In this case, we say that $\mathcal{B}$ *is rotated by* $\vec{R}$ *about* x. To be more precise, let $\mathcal{B} = \{y_1, \ldots, y_l\}$, where $y_1, \ldots, y_l$ are particles comprising $\mathcal{B}$, and let $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$, where $y'_1, \ldots, y'_l$ are the corresponding particles in $\mathcal{B}'$ after rotation. Note that, since mass plays no role here, the particles $y_1, \ldots, y_l$ can be viewed as points in space that define $\mathcal{B}$, while the particles $y'_1, \ldots, y'_l$ in $\mathcal{B}'$ can be viewed as the points in $\mathcal{B}'$ that correspond to $y_1, \ldots, y_l$ in $\mathcal{B}$. Since $\mathcal{B}$ is rotated by $\vec{R}$ about x, it follows that, for all $i = 1, \ldots, l$, $\vec{r}_{y'_i/x} = \vec{R}\vec{r}_{y_i/x}$.

Rotation of a body about a point is also possible in the case where the point x is not fixed in $\mathcal{B}$ but rather is arbitrary. In this case, we view x as connected to $\mathcal{B}$ by means of a rigid link. Consequently, x can be viewed as fixed in $\mathcal{B}$. With this extension, $\mathcal{B}$ can be rotated about an arbitrary point x, which remains spatially fixed under the rotation.

Definition 2.20.1. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ be a rigid body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$ be a rigid body with particles $y'_1, \ldots, y'_l$ whose masses are $m_1, \ldots, m_l$, respectively. Then, $\mathcal{B}$ and $\mathcal{B}'$ are *identical* if, for all $i, j = 1, \ldots, l$, $|\vec{r}_{y_i/y_j}| = |\vec{r}_{y'_i/y'_j}|$; $\mathcal{B}$ and $\mathcal{B}'$ have the *same orientation* if, for all $i, j = 1, \ldots, l$, $\vec{r}_{y_i/y_j} = \vec{r}_{y'_i/y'_j}$; and $\mathcal{B}$ and $\mathcal{B}'$ are *colocated* if, for all $i = 1, \ldots, l$, $\vec{r}_{y'_i/y_i} = \vec{0}$.

Note that, if $\mathcal{B}$ and $\mathcal{B}'$ have the same orientation, then they are identical. Furthermore, if $\mathcal{B}$ and $\mathcal{B}'$ are colocated, then, for all $i, j = 1, \ldots, l$,

$$\vec{r}_{y_i/y_j} = \vec{r}_{y_i/y'_i} + \vec{r}_{y'_i/y'_j} + \vec{r}_{y'_j/y_j} = \vec{r}_{y'_i/y'_j},$$

and thus they have the same orientation.

The following result shows that two bodies related by a rotation about a point are identical.

Fact 2.20.2. Let $\mathcal{B}$ be a rigid body, let x be a point, let $\vec{R}$ be a physical rotation matrix, and let $\mathcal{B}'$ denote $\mathcal{B}$ rotated by $\vec{R}$ about x. Then, $\mathcal{B}$ and $\mathcal{B}'$ are identical.

Proof. Note that, for all $i, j \in \{1, \ldots, l\}$,

$$|\vec{r}_{y'_i/y'_j}| = |\vec{r}_{y'_i/x} - \vec{r}_{y'_j/x}| = |\vec{R}(\vec{r}_{y_i/x} - \vec{r}_{y_j/x})| = |\vec{r}_{y_i/x} - \vec{r}_{y_j/x}| = |\vec{r}_{y_i/y_j}|. \qquad \square$$

As a special case, let $\mathcal{B}$ be a rigid body, let x be a point, and consider rotation of $\mathcal{B}$ by $\vec{R}_{\hat{n}}(\theta)$ about x. Then $\vec{R}_{\hat{n}}(\theta)$ rotates $\mathcal{B}$ about the line that is parallel to $\hat{n}$ and passes through x.

Fact 2.20.3. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ be a rigid body, let $\vec{R}_{\hat{n}}(\theta)$ be a physical rotation matrix, let x be a point, and let $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$ denote $\mathcal{B}$ rotated by $\vec{R}_{\hat{n}}(\theta)$ about x. Then, for all $i = 1, \ldots, l$, $\hat{n}'\vec{r}_{y'_i/y_i} = 0$.

Proof. Let $i \in \{1, \ldots, l\}$. Note that

$$\hat{n}'\vec{r}_{y_i'/y_i} = \hat{n}'(\vec{r}_{y_i'/x} + \vec{r}_{x/y_i}) = \hat{n}'(\vec{R}_{\hat{n}}(\theta)\vec{r}_{y_i/x} - \vec{r}_{y_i/x}) = \hat{n}'(\vec{R}_{\hat{n}}(\theta) - \vec{I})\vec{r}_{y_i/x} = 0. \qquad \square$$

2.21 Chasles's Theorem

In this section we consider a transformation of a rigid body that is more general than rotation about a point. This transformation involves rotation of a body about a point followed by a translation. The following result, which is an extension of Fact 2.20.2, shows that two bodies that are related by a rotation about a point followed by a translation are identical.

Fact 2.21.1. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y_1', \ldots, y_l'\}$ be rigid bodies, assume that, for all $i = 1, \ldots, l$, y_i and y_i' have the same mass, let x be a point, let $\vec{R}$ be a physical rotation matrix, let $\vec{r}$ be a physical position vector, and assume that, for all $i = 1, \ldots, l$, $\vec{r}_{y_i'/x} = \vec{R}\vec{r}_{y_i/x} + \vec{r}$. Then, $\mathcal{B}$ and $\mathcal{B}'$ are identical.

Proof. Note that, for all $i, j \in \{1, \ldots, l\}$,

$$|\vec{r}_{y_i'/y_j'}| = |\vec{r}_{y_i'/x} - \vec{r}_{y_j'/x}| = |\vec{R}\vec{r}_{y_i/x} + \vec{r} - (\vec{R}\vec{r}_{y_j/x} + \vec{r})| = |\vec{r}_{y_i/x} - \vec{r}_{y_j/x}| = |\vec{r}_{y_i/y_j}|. \qquad \square$$

The following result, which is the converse of Fact 2.21.1, shows that, for every point x, a pair of identical rigid bodies are related by a rotation about x followed by a translation.

Fact 2.21.2. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y_1', \ldots, y_l'\}$ be identical rigid bodies, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames that are fixed identically in $\mathcal{B}$ and $\mathcal{B}'$, respectively, let x be a point, and define the position vector

$$\vec{r} \triangleq \vec{r}_{y_1'/x} - \vec{R}_{\mathrm{B/A}}\vec{r}_{y_1/x}. \tag{2.21.1}$$

Then, for all $i = 1, \ldots, l$,

$$\vec{r}_{y_i'/x} = \vec{R}_{\mathrm{B/A}}\vec{r}_{y_i/x} + \vec{r}. \tag{2.21.2}$$

Proof. Since $\mathcal{B}$ and $\mathcal{B}'$ are identical and the frames $\mathrm{F_A}$ and $\mathrm{F_B}$ are fixed identically in $\mathcal{B}$ and $\mathcal{B}'$, respectively, it follows that, for all $i = 2, \ldots, l$, $\vec{r}_{y_i'/y_1'} = \vec{R}_{\mathrm{B/A}}\vec{r}_{y_i/y_1}$. Therefore, for all $i = 2, \ldots, l$,

$$\begin{aligned}
\vec{r}_{y_i'/x} - \vec{R}_{\mathrm{B/A}}\vec{r}_{y_i/x} &= \vec{r}_{y_i'/y_1'} + \vec{r}_{y_1'/x} - \vec{R}_{\mathrm{B/A}}(\vec{r}_{y_i/y_1} + \vec{r}_{y_1/x}) \\
&= \vec{r}_{y_1'/x} - \vec{R}_{\mathrm{B/A}}\vec{r}_{y_1/x} + \vec{r}_{y_i'/y_1'} - \vec{R}_{\mathrm{B/A}}\vec{r}_{y_i/y_1} \\
&= \vec{r}_{y_1'/x} - \vec{R}_{\mathrm{B/A}}\vec{r}_{y_1/x} \\
&= \vec{r}. \qquad \square
\end{aligned}$$

The following result extends Fact 2.20.3 to include a translation.

Fact 2.21.3. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y_1', \ldots, y_l'\}$ be rigid bodies, assume that, for all $i = 1, \ldots, l$, y_i and y_i' have the same mass, let x be a point, let $\vec{R}$ be a physical rotation matrix, let $\vec{r}$

be a physical position vector, and assume that, for all $i = 1, \ldots, l$, $\vec{r}_{y_i'/x} = \vec{R}\vec{r}_{y_i/x} + \vec{r}$. Then, for all $i = 1, \ldots, l$, $\hat{n}'(\vec{r}_{y_i'/y_i} - \vec{r}) = 0$.

Proof. Let $i \in \{1, \ldots, l\}$. Note that

$$\hat{n}'(\vec{r}_{y_i'/y_i} - \vec{r}) = \hat{n}'(\vec{r}_{y_i'/x} + \vec{r}_{x/y_i} - \vec{r}) = \hat{n}'(\vec{R}_{\hat{n}}(\theta) - \vec{I})\vec{r}_{y_i/x} = 0. \tag{2.21.3}$$

The next result, which is *Chasles's theorem*, is a stronger version of Fact 2.21.2. This result states that an arbitrary rotation and displacement of a rigid body can be expressed in terms of a rotation about an eigenaxis and a displacement along the eigenaxis.

Fact 2.21.4. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y_1', \ldots, y_l'\}$ be identical rigid bodies. Then, there exist a point z, an eigenaxis $\hat{n}$, an eigenangle θ, and a real number α such that, for all $i = 1, \ldots, l$,

$$\vec{r}_{y_i'/z} = \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_i/z} + \alpha\hat{n}. \tag{2.21.4}$$

Proof. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames that are fixed identically in $\mathcal{B}$ and $\mathcal{B}'$, respectively. Let $\hat{n}$ and θ be an eigenaxis and eigenangle such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. We first consider the case where $\theta \in (0, \pi)$.

Let $\mathcal{P}$ be the plane that is orthogonal to $\hat{n}$ and that contains y_1. Furthermore, consider the isosceles triangle $\mathcal{T}$ contained in $\mathcal{P}$ with vertices y_1, w_1, and z, where w_1 is the projection of y_1' onto $\mathcal{P}$, the angle between the sides (y_1, z) and (w_1, z) is θ, and the angles opposite the sides (y_1, z) and (w_1, z) are $\pi - \theta/2$. Analogously, let $\mathcal{P}'$ be the plane that is orthogonal to $\hat{n}$ and that contains y_1'. Furthermore, consider the isosceles triangle $\mathcal{T}'$ contained in $\mathcal{P}'$ with vertices y_1', w_1', and z', where w_1' is the projection of y_1 onto $\mathcal{P}'$, the angle between the sides (y_1', z') and (w_1', z') is θ, and the angles opposite the sides (y_1', z') and (w_1', z') are $\pi - \theta/2$. It can thus be seen that there exists a real number α such that $\vec{r}_{z'/z} = \alpha\hat{n}$. Since $\vec{r}_{y_1'/z'} = \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_1/z}$, it follows that

$$\begin{aligned}\vec{r}_{y_1'/z} &= \vec{r}_{y_1'/z'} + \vec{r}_{z'/z} \\ &= \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_1/z} + \alpha\hat{n}.\end{aligned} \tag{2.21.5}$$

By using (2.21.5), Fact 2.21.2 implies that (2.21.4) holds for all $i = 1, \ldots, l$. The case where $\theta \in (-\pi, 0)$ is proved by replacing θ with $|\theta|$. □

2.22 Geometry of a Chain of Rigid Bodies

Consider rigid bodies $\mathcal{B}_\mathrm{B}$, $\mathcal{B}_\mathrm{C}$, and $\mathcal{B}_\mathrm{D}$ connected to the base rigid body $\mathcal{B}_\mathrm{A}$ in the form of a chain with three links as shown in Figure 2.22.1. In particular, $\mathcal{B}_\mathrm{B}$ is connected to $\mathcal{B}_\mathrm{A}$ at the point z_A, $\mathcal{B}_\mathrm{C}$ is connected to $\mathcal{B}_\mathrm{B}$ at the point z_B, and $\mathcal{B}_\mathrm{D}$ is connected to $\mathcal{B}_\mathrm{C}$ at the point z_C. The point z_D is the *end effector*. The point z_A is fixed in $\mathcal{B}_\mathrm{B}$ and $\mathcal{B}_\mathrm{A}$; the point z_B is fixed in $\mathcal{B}_\mathrm{C}$ and $\mathcal{B}_\mathrm{B}$; the point z_C is fixed in $\mathcal{B}_\mathrm{D}$ and $\mathcal{B}_\mathrm{C}$; and the point z_D is fixed in $\mathcal{B}_\mathrm{D}$. Each attachment point is assumed to represent a rotational joint, such as a pin, universal joint, or ball joint. Note that

$$\vec{r}_{z_\mathrm{D}/z_\mathrm{A}} = \vec{r}_{z_\mathrm{D}/z_\mathrm{C}} + \vec{r}_{z_\mathrm{C}/z_\mathrm{B}} + \vec{r}_{z_\mathrm{B}/z_\mathrm{A}}. \tag{2.22.1}$$

We assume that $\vec{r}_{z_\mathrm{D}/z_\mathrm{C}}$ is known in $\mathrm{F_D}$, $\vec{r}_{z_\mathrm{C}/z_\mathrm{B}}$ is known in $\mathrm{F_C}$, and $\vec{r}_{z_\mathrm{B}/z_\mathrm{A}}$ is known in $\mathrm{F_B}$. Hence, $r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}}$, $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}}$, and $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}}$ are known. Furthermore, we assume that $\mathcal{O}_\mathrm{D/C}$, $\mathcal{O}_\mathrm{C/B}$, and $\mathcal{O}_\mathrm{B/A}$ are

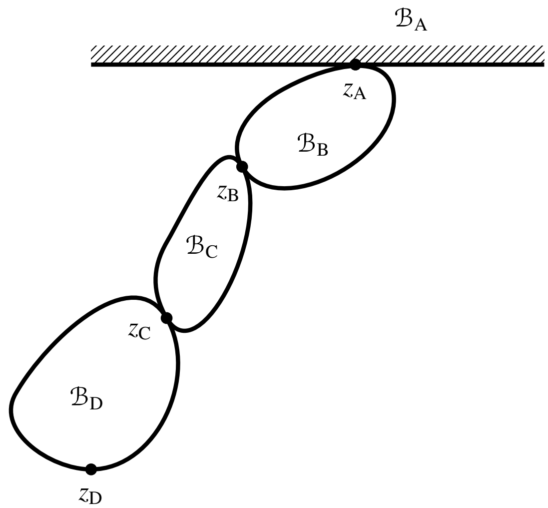


Figure 2.22.1: Chain of rigid bodies.

known. In order to express $r_{z_\mathrm{D}/z_\mathrm{A}|\mathrm{A}}$ in terms of the known quantities, note that

$$\begin{aligned}
r_{z_\mathrm{D}/z_\mathrm{A}|\mathrm{A}} &= r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{A}} + r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}} + r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}} \\
&= \mathcal{O}_{\mathrm{A/D}} r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}} + \mathcal{O}_{\mathrm{A/C}} r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}} + \mathcal{O}_{\mathrm{A/B}} r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}} \\
&= \mathcal{O}_{\mathrm{A/C}}\mathcal{O}_{\mathrm{C/D}} r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}} + \mathcal{O}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/C}} r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}} + \mathcal{O}_{\mathrm{A/B}} r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}}.
\end{aligned} \tag{2.22.2}$$

Note that (2.22.2) has the form of a recursive algorithm that proceeds from the top link of the chain to the bottom link of the chain, where $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}}$ and $\mathcal{O}_{\mathrm{A/B}}$ are used first, $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}}$ and $\mathcal{O}_{\mathrm{B/C}}$ are used second, where $\mathcal{O}_{\mathrm{B/C}}$ is multiplied by $\mathcal{O}_{\mathrm{A/B}}$ to obtain $\mathcal{O}_{\mathrm{A/C}}$, and $r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}}$ and $\mathcal{O}_{\mathrm{C/D}}$ are used last, where $\mathcal{O}_{\mathrm{C/D}}$ is multiplied by $\mathcal{O}_{\mathrm{A/C}}$ computed in the second step to obtain $\mathcal{O}_{\mathrm{A/D}}$. Alternatively, we can write

$$\begin{aligned}
r_{z_\mathrm{D}/z_\mathrm{A}|\mathrm{A}} &= r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{A}} + r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}} + r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}} \\
&= \mathcal{O}_{\mathrm{A/D}} r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}} + \mathcal{O}_{\mathrm{A/C}} r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}} + \mathcal{O}_{\mathrm{A/B}} r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}} \\
&= \mathcal{O}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/C}}\mathcal{O}_{\mathrm{C/D}} r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}} + \mathcal{O}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/C}} r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}} + \mathcal{O}_{\mathrm{A/B}} r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}} \\
&= \mathcal{O}_{\mathrm{A/B}}[\mathcal{O}_{\mathrm{B/C}}\mathcal{O}_{\mathrm{C/D}} r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}} + \mathcal{O}_{\mathrm{B/C}} r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}} + r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}}] \\
&= \mathcal{O}_{\mathrm{A/B}}[\mathcal{O}_{\mathrm{B/C}}(\mathcal{O}_{\mathrm{C/D}} r_{z_\mathrm{D}/z_\mathrm{D}|\mathrm{D}} + r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}}) + r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}}].
\end{aligned} \tag{2.22.3}$$

Note that (2.22.3) has the form of a recursive algorithm that proceeds from the bottom link of the chain to the top link of the chain, where $r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}}$ and $\mathcal{O}_{\mathrm{C/D}}$ are used first, $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}}$ and $\mathcal{O}_{\mathrm{B/C}}$ are used second, and $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}}$ and $\mathcal{O}_{\mathrm{A/B}}$ are used last. For a chain consisting of $n \ge 4$ rigid bodies, recursive equations of the form (2.22.2) and (2.22.3) can be derived.

2.23 Nonstandard Frames and Reciprocal Frames

Thus far, and throughout this book, all frames are assumed to be *standard frames*, which are orthogonal, right-handed frames with dimensionless, unit-length axes. In this section we develop properties of frames that consist of three linearly independent axes. These frames arise naturally in many applications.

A *nonstandard frame* is a collection of three linearly independent dimensionless physical vectors. Note that "nonstandard" means "not necessarily standard." Letting $\mathrm{F_A}$ be a nonstandard frame, we denote its dimensionless axes by $\vec{a}_1, \vec{a}_2, \vec{a}_3$. These vectors are not necessarily orthogonal and need not have unit length. Furthermore, it follows from Fact 2.4.5 that, for each physical vector $\vec{x}$, there exist unique real numbers x_1, x_2, x_3 such that

$$\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3. \tag{2.23.1}$$

We thus write

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{2.23.2}$$

Fact 2.23.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be nonstandard frames with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{b}_1, \vec{b}_2, \vec{b}_3$, respectively. Then

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}, \tag{2.23.3}$$

where $\mathcal{O}_{\mathrm{B/A}} \in \mathbb{R}^{3\times 3}$ is defined by

$$\mathcal{O}_{\mathrm{B/A}} \triangleq \frac{1}{\alpha} \begin{bmatrix} \vec{b}_1\cdot(\vec{a}_2\times\vec{a}_3) & \vec{b}_1\cdot(\vec{a}_3\times\vec{a}_1) & \vec{b}_1\cdot(\vec{a}_1\times\vec{a}_2) \\ \vec{b}_2\cdot(\vec{a}_2\times\vec{a}_3) & \vec{b}_2\cdot(\vec{a}_3\times\vec{a}_1) & \vec{b}_2\cdot(\vec{a}_1\times\vec{a}_2) \\ \vec{b}_3\cdot(\vec{a}_2\times\vec{a}_3) & \vec{b}_3\cdot(\vec{a}_3\times\vec{a}_1) & \vec{b}_3\cdot(\vec{a}_1\times\vec{a}_2) \end{bmatrix}, \tag{2.23.4}$$

where

$$\alpha \triangleq \vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3). \tag{2.23.5}$$

Furthermore, $\mathcal{O}_{\mathrm{B/A}}$ is nonsingular, and

$$\mathcal{O}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{B/A}}^{-1}. \tag{2.23.6}$$

Now, let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_{\mathrm{B}} = \mathcal{O}_{\mathrm{A/B}}^{\mathrm{T}}\ \vec{x}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}}^{-\mathrm{T}}\ \vec{x}\Big|_{\mathrm{A}}. \tag{2.23.7}$$

Finally, if $\mathrm{F_A}$ and $\mathrm{F_B}$ are standard frames, then $\mathcal{O}_{\mathrm{B/A}}$ is a rotation matrix and

$$\vec{x}\Big|_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_{\mathrm{A}}. \tag{2.23.8}$$

Proof. To verify (2.23.3), note that the first equation is given by

$$\alpha\vec{b}_1 = \vec{b}_1\cdot(\vec{a}_2\times\vec{a}_3)\vec{a}_1 + \vec{b}_1\cdot(\vec{a}_3\times\vec{a}_1)\vec{a}_2 + \vec{b}_1\cdot(\vec{a}_1\times\vec{a}_2)\vec{a}_3.$$

Resolving each vector in this equation yields *xlvii*) in [1, p. 386]. □

Let $\mathrm{F_A}$ be a nonstandard frame with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Now, define the dimensionless physical vectors

$$\vec{a}_{[1]} \triangleq \frac{1}{\alpha}\vec{a}_2\times\vec{a}_3, \tag{2.23.9}$$

$$\vec{a}_{[2]} \triangleq \frac{1}{\alpha}\vec{a}_3 \times \vec{a}_1, \tag{2.23.10}$$

$$\vec{a}_{[3]} \triangleq \frac{1}{\alpha}\vec{a}_1 \times \vec{a}_2, \tag{2.23.11}$$

where α is defined by (2.23.5).

Fact 2.23.2. Let $\mathrm{F_A}$ be a nonstandard frame with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Then, the physical vectors $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$ are linearly independent.

The physical vectors $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$ define the *reciprocal frame* $\mathrm{F_{[A]}}$. Using these definitions, (2.23.4) can be written as

$$\mathcal{O}_{[\mathrm{A}]/\mathrm{A}} = \begin{bmatrix} \vec{b}_1 \cdot \vec{a}_{[1]} & \vec{b}_1 \cdot \vec{a}_{[2]} & \vec{b}_1 \cdot \vec{a}_{[3]} \\ \vec{b}_2 \cdot \vec{a}_{[1]} & \vec{b}_2 \cdot \vec{a}_{[2]} & \vec{b}_2 \cdot \vec{a}_{[3]} \\ \vec{b}_3 \cdot \vec{a}_{[1]} & \vec{b}_3 \cdot \vec{a}_{[2]} & \vec{b}_3 \cdot \vec{a}_{[3]} \end{bmatrix}. \tag{2.23.12}$$

Fact 2.23.3. Let $\mathrm{F_A}$ be a nonstandard frame with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$, and let $\mathrm{F_{[A]}}$ denote the reciprocal frame with axes $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$. Then, for all $i, j = 1, 2, 3$,

$$\vec{a}_{[i]} \cdot \vec{a}_j = \delta_{i,j}. \tag{2.23.13}$$

Hence,

$$\begin{bmatrix} \vec{a}_{[1]} \\ \vec{a}_{[2]} \\ \vec{a}_{[3]} \end{bmatrix} = \mathcal{O}_{[\mathrm{A}]/\mathrm{A}} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}, \tag{2.23.14}$$

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \mathcal{O}_{\mathrm{A}/[\mathrm{A}]} \begin{bmatrix} \vec{a}_{[1]} \\ \vec{a}_{[2]} \\ \vec{a}_{[3]} \end{bmatrix}, \tag{2.23.15}$$

where

$$\mathcal{O}_{[\mathrm{A}]/\mathrm{A}} = \begin{bmatrix} \vec{a}_{[1]} \cdot \vec{a}_{[1]} & \vec{a}_{[1]} \cdot \vec{a}_{[2]} & \vec{a}_{[1]} \cdot \vec{a}_{[3]} \\ \vec{a}_{[2]} \cdot \vec{a}_{[1]} & \vec{a}_{[2]} \cdot \vec{a}_{[2]} & \vec{a}_{[2]} \cdot \vec{a}_{[3]} \\ \vec{a}_{[3]} \cdot \vec{a}_{[1]} & \vec{a}_{[3]} \cdot \vec{a}_{[2]} & \vec{a}_{[3]} \cdot \vec{a}_{[3]} \end{bmatrix}, \tag{2.23.16}$$

$$\mathcal{O}_{\mathrm{A}/[\mathrm{A}]} = \mathcal{O}_{[\mathrm{A}]/\mathrm{A}}^{-1} = \begin{bmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \vec{a}_1 \cdot \vec{a}_3 \\ \vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \vec{a}_2 \cdot \vec{a}_3 \\ \vec{a}_3 \cdot \vec{a}_1 & \vec{a}_3 \cdot \vec{a}_2 & \vec{a}_3 \cdot \vec{a}_3 \end{bmatrix}. \tag{2.23.17}$$

Now, let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_{[1]} \cdot \vec{x} \\ \vec{a}_{[2]} \cdot \vec{x} \\ \vec{a}_{[3]} \cdot \vec{x} \end{bmatrix}, \tag{2.23.18}$$

$$\vec{x}\Big|_{[\mathrm{A}]} = \begin{bmatrix} x_{[1]} \\ x_{[2]} \\ x_{[3]} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \vec{a}_3 \cdot \vec{x} \end{bmatrix}. \tag{2.23.19}$$

Furthermore,

$$\vec{x}\Big|_{[\mathrm{A}]} = \mathcal{O}_{\mathrm{A}/[\mathrm{A}]}\, \vec{x}\Big|_{\mathrm{A}}, \tag{2.23.20}$$

$$\vec{x}\Big|_{\mathrm{A}} = \mathcal{O}_{[\mathrm{A}]/\mathrm{A}}\, \vec{x}\Big|_{[\mathrm{A}]}. \tag{2.23.21}$$

Finally, if F_{A} is a standard frame, then $\mathrm{F}_{\mathrm{A}} = \mathrm{F}_{[\mathrm{A}]}$.

By rewriting (2.23.13) as

$$\vec{a}'_{[i]}\vec{a}_j = \delta_{i,j}, \tag{2.23.22}$$

the covectors $\vec{a}'_{[1]}, \vec{a}'_{[2]}, \vec{a}'_{[3]}$ corresponding to the axes $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$ of the reciprocal frame can be viewed as the axes of the *co-reciprocal frame* $\mathrm{F}_{[\mathrm{A}]'}$, which is a basis for the space $\mathcal{V}'$ of covectors. If F_{A} is a standard frame, then $\mathrm{F}_{[\mathrm{A}]} = \mathrm{F}_{\mathrm{A}}$, and thus $\mathrm{F}_{[\mathrm{A}]'} = \mathrm{F}_{\mathrm{A}'}$.

Using a frame and its reciprocal, it is possible to resolve a physical vector $\vec{x}$ in two ways, namely,

$$\vec{x} = \sum_{i=1}^{3} x_i \vec{a}_i = \sum_{i=1}^{3} x_{[i]} \vec{a}_{[i]}, \tag{2.23.23}$$

where

$$x_i = \vec{a}'_{[i]}\vec{x}, \quad x_{[i]} = \vec{a}'_i\vec{x}. \tag{2.23.24}$$

As a side remark, (2.23.23) and (2.23.24) are written traditionally as

$$\vec{x} = \sum_{i=1}^{3} x^i \vec{a}_i = \sum_{i=1}^{3} x_i \vec{a}^{\,i}, \tag{2.23.25}$$

where

$$x^i = \vec{a}^{\,i} \cdot \vec{x}, \quad x_i = \vec{a}_i \cdot \vec{x}. \tag{2.23.26}$$

Note that $\mathcal{O}_{\mathrm{A}/[\mathrm{A}]}$ and $\mathcal{O}_{[\mathrm{A}]/\mathrm{A}}$ are symmetric, and therefore the transpose in (2.23.7) does not appear in (2.23.20) and (2.23.21).

The following result shows that the reciprocal frame is the unique nonstandard frame satisfying (2.23.13).

Fact 2.23.4. Let F_{A} and F_{B} be nonstandard frames such that, for all $i, j = 1, 2, 3$, $\vec{b}_i \cdot \vec{a}_j = \delta_{i,j}$. Then, $\mathrm{F}_{\mathrm{B}} = \mathrm{F}_{[\mathrm{A}]}$.

Fact 2.23.5. Let F_{A} and F_{B} be nonstandard frames, and let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_{[\mathrm{B}]} = \mathcal{O}_{\mathrm{B}/\mathrm{A}}\, \vec{x}\Big|_{[\mathrm{A}]} = \mathcal{O}^{\mathrm{T}}_{[\mathrm{A}]/[\mathrm{B}]}\, \vec{x}\Big|_{[\mathrm{A}]}. \tag{2.23.27}$$

Furthermore,

$$\mathcal{O}_{\mathrm{B}/\mathrm{A}} = \mathcal{O}_{\mathrm{B}/[\mathrm{B}]}\mathcal{O}^{\mathrm{T}}_{\mathrm{A}/\mathrm{B}}\mathcal{O}_{[\mathrm{A}]/\mathrm{A}}. \tag{2.23.28}$$

Proof. Note that

$$\vec{x}\Big|_{[B]} = \begin{bmatrix} \vec{x}\cdot\vec{b}_1 \\ \vec{x}\cdot\vec{b}_2 \\ \vec{x}\cdot\vec{b}_3 \end{bmatrix} = \begin{bmatrix} \vec{x}\cdot\mathrm{row}_1(\mathcal{O}_{B/A})\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \\ \vec{x}\cdot\mathrm{row}_2(\mathcal{O}_{B/A})\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \\ \vec{x}\cdot\mathrm{row}_3(\mathcal{O}_{B/A})\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathrm{row}_1(\mathcal{O}_{B/A}) \\ \mathrm{row}_2(\mathcal{O}_{B/A}) \\ \mathrm{row}_3(\mathcal{O}_{B/A}) \end{bmatrix}\begin{bmatrix} \vec{x}\cdot\vec{a}_1 \\ \vec{x}\cdot\vec{a}_2 \\ \vec{x}\cdot\vec{a}_3 \end{bmatrix} = \mathcal{O}_{B/A}\,\vec{x}\Big|_{[A]}.$$

Next, using (2.23.7) and (2.23.20) it follows that

$$\mathcal{O}_{A/B}^{T}\,\vec{x}\Big|_{A} = \vec{x}\Big|_{B} = \mathcal{O}_{[B]/B}\,\vec{x}\Big|_{[B]} = \mathcal{O}_{[B]/B}\mathcal{O}_{B/A}\,\vec{x}\Big|_{[A]} = \mathcal{O}_{[B]/B}\mathcal{O}_{B/A}\mathcal{O}_{A/[A]}\,\vec{x}\Big|_{A}. \qquad \square$$

The components of $\vec{x}\Big|_{[A]}$ are the *reciprocal components* of $\vec{x}$.

The components of $\vec{x}\Big|_{A}$ are the *contravariant components* of $\vec{x}$, whereas the components of $\vec{x}\Big|_{[A]}$ are the *covariant components* of $\vec{x}$. The word contravariant reflects the reversal in the formulas

$$\begin{bmatrix}\vec{b}_1\\ \vec{b}_2\\ \vec{b}_3\end{bmatrix} = \mathcal{O}_{B/A}\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \tag{2.23.29}$$

and

$$\vec{x}\Big|_{A} = \mathcal{O}_{B/A}^{T}\,\vec{x}\Big|_{B}. \tag{2.23.30}$$

Fact 2.23.6. Let $\mathrm{F_A}$ be a nonstandard frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{x}\cdot\vec{y} = \vec{x}\Big|_{A}^{T}\,\vec{y}\Big|_{[A]} = \vec{x}\Big|_{A}^{T}\,\mathcal{O}_{A/[A]}\,\vec{y}\Big|_{A}. \tag{2.23.31}$$

In particular,

$$|\vec{x}|^2 = \vec{x}\cdot\vec{x} = \vec{x}\Big|_{A}^{T}\,\vec{x}\Big|_{[A]} = \vec{x}\Big|_{A}^{T}\,\mathcal{O}_{A/[A]}\,\vec{x}\Big|_{A}. \tag{2.23.32}$$

Proof. Let

$$\vec{x}\Big|_{A} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \quad \vec{y}\Big|_{A} = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}.$$

Then,

$$\vec{x}\cdot\vec{y} = (x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3)\cdot(y_1\vec{a}_{[1]} + y_2\vec{a}_{[2]} + y_3\vec{a}_{[3]})$$

$$= x_1y_1 + x_2y_2 + x_3y_3 = \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}} \vec{y}\Big|_{[\mathrm{A}]} = \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}} \mathcal{O}_{\mathrm{A}/[\mathrm{A}]} \vec{y}\Big|_{\mathrm{A}}. \qquad \square$$

It follows from (2.23.32) that, if $\mathrm{F_A}$ is nonstandard, then $|\vec{x}|$ and $\left\|\vec{x}\big|_{\mathrm{A}}\right\|$ may be different. For example, suppose that $\vec{a}_1 = \hat{\imath}_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}$, $\vec{a}_2 = \hat{\jmath}_{\mathrm{A}}$, and $\vec{a}_3 = \hat{k}_{\mathrm{A}}$, and let $\vec{x} = \hat{\imath}_{\mathrm{A}}$. Then, $|\vec{x}| = 1$, but $\left\|\vec{x}\big|_{\mathrm{A}}\right\| = \sqrt{3}$.

The following result follows from (2.23.32).

Fact 2.23.7. Let $\mathrm{F_A}$ be a nonstandard frame. Then, $\mathcal{O}_{\mathrm{A}/[\mathrm{A}]}$ is positive semidefinite.

Let $\mathrm{F_A}$ be a nonstandard frame. As in the case of orthogonal frames, define the row vector

$$\vec{x}'\Big|_{\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}}. \tag{2.23.33}$$

Furthermore, for the physical matrix

$$\vec{M} = \sum_{i=1}^{r} \vec{x}_i \vec{y}_i', \tag{2.23.34}$$

we define

$$\vec{M}\Big|_{\mathrm{A}} \triangleq \sum_{i=1}^{r} \vec{x}_i\Big|_{\mathrm{A}} \vec{y}_i\Big|_{\mathrm{A}}^{\mathrm{T}}. \tag{2.23.35}$$

Fact 2.23.8. Let $\mathrm{F_A}$ be a nonstandard frame, and let $\vec{M}$ be a physical matrix. Then,

$$\vec{M}\Big|_{\mathrm{A}} = \begin{bmatrix} \vec{a}_{[1]}'\vec{M}\vec{a}_{[1]} & \vec{a}_{[1]}'\vec{M}\vec{a}_{[2]} & \vec{a}_{[1]}'\vec{M}\vec{a}_{[3]} \\ \vec{a}_{[2]}'\vec{M}\vec{a}_{[1]} & \vec{a}_{[2]}'\vec{M}\vec{a}_{[2]} & \vec{a}_{[2]}'\vec{M}\vec{a}_{[3]} \\ \vec{a}_{[3]}'\vec{M}\vec{a}_{[1]} & \vec{a}_{[3]}'\vec{M}\vec{a}_{[2]} & \vec{a}_{[3]}'\vec{M}\vec{a}_{[3]} \end{bmatrix}. \tag{2.23.36}$$

Furthermore,

$$\vec{M} = \sum_{i,j=1}^{3} \left(\vec{a}_{[i]}'\vec{M}\vec{a}_{[j]}\right)\vec{a}_i\vec{a}_j'. \tag{2.23.37}$$

Proof. For simplicity, let $\vec{M} = \vec{x}\vec{y}'$, and write

$$\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3, \quad \vec{y} = y_1\vec{a}_1 + y_2\vec{a}_2 + y_3\vec{a}_3,$$

so that

$$\vec{M} = \sum_{i,j=1}^{3} x_i y_j \vec{a}_i \vec{a}_j'. \tag{2.23.38}$$

Hence,

$$\vec{M}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3. \end{bmatrix}.$$

It follows from (2.23.38) that, for all $i, j = 1, 2, 3$, $\vec{a}'_{[i]}\vec{M}\vec{a}_{[j]} = x_i y_j$, which yields (2.23.36) and (2.23.37). □

Defining

$$M_{ij} \triangleq \vec{a}'_{[i]}\vec{M}\vec{a}_{[j]}, \tag{2.23.39}$$

which, by (2.23.18), is the i, j entry of $\vec{M}\Big|_{\mathrm{A}}$, (2.23.37) can be written as

$$\vec{M} = \sum_{i,j=1}^{3} M_{ij}\vec{a}_i\vec{a}'_j. \tag{2.23.40}$$

Similarly, defining

$$M_{[i][j]} \triangleq \vec{a}'_i\vec{M}\vec{a}_j, \tag{2.23.41}$$

which, by (2.23.19), is the i, j entry of $\vec{M}\Big|_{[\mathrm{A}]}$, we have

$$\vec{M} = \sum_{i,j=1}^{3} M_{[i][j]}\vec{a}_{[i]}\vec{a}'_{[j]}. \tag{2.23.42}$$

Likewise, defining

$$M_{i[j]} \triangleq \vec{a}'_{[i]}\vec{M}\vec{a}_j, \quad M_{[i]j} \triangleq \vec{a}'_i\vec{M}\vec{a}_{[j]}, \tag{2.23.43}$$

it follows that

$$\vec{M} = \sum_{i,j=1}^{3} M_{i[j]}\vec{a}_i\vec{a}'_{[j]} = \sum_{i,j=1}^{3} M_{[i]j}\vec{a}_{[i]}\vec{a}'_j. \tag{2.23.44}$$

Consequently, by using a frame and the associated coframe, reciprocal, and co-reciprocal frames, a physical matrix $\vec{M}$ can be resolved in four ways, namely,

$$\vec{M} = \sum_{i,j=1}^{3} M_{ij}\vec{a}_i\vec{a}'_j = \sum_{i,j=1}^{3} M_{[i][j]}\vec{a}_{[i]}\vec{a}'_{[j]} = \sum_{i,j=1}^{3} M_{i[j]}\vec{a}_i\vec{a}'_{[j]} = \sum_{i,j=1}^{3} M_{[i]j}\vec{a}_{[i]}\vec{a}'_j. \tag{2.23.45}$$

Fact 2.23.9. Let $\vec{M}$ be a physical matrix, let $\vec{x}$ be a physical vector, and let $\mathrm{F_A}$ be a nonstandard frame. Then,

$$(\vec{M}\vec{x})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} \mathcal{O}_{\mathrm{A}/[\mathrm{A}]} \vec{x}\Big|_{\mathrm{A}}. \tag{2.23.46}$$

Fact 2.23.9 implies that

$$\vec{I}\Big|_{\mathrm{A}} = \mathcal{O}_{[\mathrm{A}]/\mathrm{A}}, \quad \vec{I}\Big|_{[\mathrm{A}]} = \mathcal{O}_{\mathrm{A}/[\mathrm{A}]}. \tag{2.23.47}$$

The following result extends (2.10.11) to nonstandard frames.

Fact 2.23.10. Let $\vec{M}$ be a physical matrix, and let F_A and F_B be nonstandard frames. Then,

$$\vec{M}\Big|_B = \mathcal{O}_{B/A}^{-T}\,\vec{M}\Big|_A\,\mathcal{O}_{A/B}. \tag{2.23.48}$$

Fact 2.23.11. Let $\vec{M}$ be a physical matrix, and let F_A be a nonstandard frame. Then,

$$\vec{M}\Big|_{[A]} = \mathcal{O}_{A/[A]}\,\vec{M}\Big|_A\,\mathcal{O}_{A/[A]}. \tag{2.23.49}$$

Fact 2.23.12. Let $\vec{M}$ be a physical matrix, let F_A be a nonstandard frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors satisfying $\vec{y} = \vec{M}\vec{x}$. Then,

$$\vec{y}\Big|_{[A]} = \mathcal{O}_{A/[A]}\,\vec{M}\Big|_A\,\mathcal{O}_{A/[A]}\,\vec{x}\Big|_{[A]}. \tag{2.23.50}$$

Consider the physical matrix

$$\vec{M} = \sum_{i=1}^{r}\vec{x}_i\vec{y}_i^{\,\prime}, \tag{2.23.51}$$

and let F_A and F_B be nonstandard frames. Then, we define

$$\vec{M}\Big|_{A,B} \triangleq \sum_{i=1}^{r}\vec{x}_i\Big|_A\,\vec{y}_i\Big|_B^{T}. \tag{2.23.52}$$

Fact 2.23.13. Let $\vec{M}$ be a physical matrix, let $\vec{x}$ be a physical vector, and let F_A and F_B be nonstandard frames. Then,

$$(\vec{M}\vec{x})\Big|_B = \vec{M}\Big|_{B,A}\,\vec{x}\Big|_{[A]}. \tag{2.23.53}$$

It follows from Fact 2.23.13 that

$$\vec{I}\Big|_{[A]/A} = \vec{I}\Big|_{A/[A]} = I_3. \tag{2.23.54}$$

Fact 2.23.14. Let $\vec{M}$ be a physical matrix, and let F_A be a nonstandard frame. Then,

$$\vec{M}\Big|_{[A],A} = \mathcal{O}_{A/[A]}\,\vec{M}\Big|_A, \tag{2.23.55}$$

$$\vec{M}\Big|_{A,[A]} = \vec{M}\Big|_A\,\mathcal{O}_{A/[A]}. \tag{2.23.56}$$

2.24 Partial Derivatives and Gradients

Let $f\colon \mathbb{R}^n \mapsto \mathbb{R}$. Then, with $x = (x_1,\ldots,x_n)$, the partial derivative of f with respect to x_i is denoted by $\partial_{x_i} f(x)$. Note that $\partial_{x_i} f(x)$ is the derivative of the function $g\colon \mathbb{R} \mapsto \mathbb{R}$ defined by $g(x_i) \triangleq f(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)$. The *gradient* $\partial_x f(x) \in \mathbb{R}^{1\times n}$ of f is defined by

$$\partial_x f(x) \triangleq [\partial_{x_1} f(x)\ \cdots\ \partial_{x_n} f(x)]. \tag{2.24.1}$$

Next, let $f\colon \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ so that f is a real-valued function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then, $\partial_x f(x,y) \in \mathbb{R}^{1\times n}$ denotes the gradient of f with respect to x. In addition, $\partial_x f(x,y)|_{x=\bar{x}}$ denotes the gradient of f with respect to x evaluated at $(\bar{x},y)$. If it is clear that x is the first argument of f, then we write $\partial_x f(\bar{x},y)$. Defining the function $g\colon \mathbb{R}^n \mapsto \mathbb{R}^{1\times n}$ by $g(x) \triangleq \partial_x f(x,y)$, it follows that $\partial_x f(\bar{x},y) = g(\bar{x})$.

As a special case, let $f\colon \mathbb{R}^3 \mapsto \mathbb{R}$, where an arbitrary argument of f is denoted by $r = [x\ y\ z]^{\mathrm{T}} \in \mathbb{R}$. Then, the *gradient* of f at $r \in \mathbb{R}^3$ is the row vector

$$\partial_r f(r) = [\partial_x f(r)\ \ \partial_y f(r)\ \ \partial_z f(r)] \in \mathbb{R}^{1\times 3}, \tag{2.24.2}$$

where $\partial_x f(r)$ is the partial derivative of f with respect to x evaluated at r.

Now, let f denote a mapping from physical vectors to real numbers, that is, for each physical vector $\vec{r}$, let $f(\vec{r}) \in \mathbb{R}$. Furthermore, for the frame $\mathrm{F_A}$, define $f_\mathrm{A}\colon \mathbb{R}^3 \mapsto \mathbb{R}$ by

$$f_\mathrm{A}(r) \triangleq f(\mathrm{F_A} r). \tag{2.24.3}$$

Defining $r_\mathrm{A} \triangleq \vec{r}\Big|_\mathrm{A} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, (2.24.3) can be written as

$$f(\vec{r}) = f_\mathrm{A}(r_\mathrm{A}). \tag{2.24.4}$$

Furthermore, if $\mathrm{F_B}$ is also a frame, then

$$f_\mathrm{A}(r_\mathrm{A}) = f(\mathrm{F_A} r_\mathrm{A}) = f(\mathrm{F_B}\mathcal{O}_{\mathrm{B/A}} r_\mathrm{A}) = f_\mathrm{B}(\mathcal{O}_{\mathrm{B/A}} r_\mathrm{A}) = f_\mathrm{B}(r_\mathrm{B}). \tag{2.24.5}$$

It thus follows from the chain rule that

$$\partial_{r_\mathrm{A}} f_\mathrm{A}(r_\mathrm{A}) = \partial_{r_\mathrm{A}} f_\mathrm{B}(\mathcal{O}_{\mathrm{B/A}} r_\mathrm{A}) = \partial_{\mathcal{O}_{\mathrm{B/A}} r_\mathrm{A}} f_\mathrm{B}(\mathcal{O}_{\mathrm{B/A}} r_\mathrm{A})\mathcal{O}_{\mathrm{B/A}} = \partial_{r_\mathrm{B}} f_\mathrm{B}(r_\mathrm{B})\mathcal{O}_{\mathrm{B/A}}. \tag{2.24.6}$$

Next, define the *physical gradient* $\vec{\partial} f(\vec{r})$ of f at $\vec{r}$ by

$$\vec{\partial} f(\vec{r}) \triangleq \partial_{r_\mathrm{A}} f_\mathrm{A}(r_\mathrm{A})\mathrm{F}_\mathrm{A}'^{\mathrm{T}}. \tag{2.24.7}$$

Hence,

$$\vec{\partial} f(\vec{r}) = \partial_{\bar{x}} f_\mathrm{A}(r_\mathrm{A})\hat{\imath}_\mathrm{A}' + \partial_{\bar{y}} f_\mathrm{A}(r_\mathrm{A})\hat{\jmath}_\mathrm{A}' + \partial_{\bar{z}} f_\mathrm{A}(r_\mathrm{A})\hat{k}_\mathrm{A}', \tag{2.24.8}$$

and thus

$$\vec{\partial} f(\vec{r})\Big|_\mathrm{A} = \partial_{r_\mathrm{A}} f_\mathrm{A}(r_\mathrm{A}) = \begin{bmatrix} \partial_{\bar{x}} f_\mathrm{A}(r_\mathrm{A}) & \partial_{\bar{y}} f_\mathrm{A}(r_\mathrm{A}) & \partial_{\bar{z}} f_\mathrm{A}(r_\mathrm{A}) \end{bmatrix}. \tag{2.24.9}$$

Note that the physical gradient is a covector.

Fact 2.24.1. $\vec{\partial} f(\vec{r})$ is independent of the choice of the frame $\mathrm{F_A}$ used in (2.24.7).

Proof. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $r_\mathrm{A} \triangleq \vec{r}\Big|_\mathrm{A}$ and $r_\mathrm{B} \triangleq \vec{r}\Big|_\mathrm{B}$. Then, it follows from (2.10.25) and (2.24.6) that

$$\begin{aligned}\partial_{r_\mathrm{A}} f_\mathrm{A}(r_\mathrm{A})\mathrm{F}_\mathrm{A}'^{\mathrm{T}} &= \partial_{r_\mathrm{B}} f_\mathrm{B}(r_\mathrm{B})\mathcal{O}_{\mathrm{B/A}}\mathrm{F}_\mathrm{A}'^{\mathrm{T}} \\ &= \partial_{r_\mathrm{B}} f_\mathrm{B}(r_\mathrm{B})\mathrm{F}_\mathrm{B}'^{\mathrm{T}}.\end{aligned}$$

□

For a scalar-valued function f whose domain is physical vectors $\vec{r}$, $\vec{\partial}_{\vec{r}} f(\vec{r})\Big|_{\vec{r}=\vec{r}_{y/z}}$ denotes the physical gradient with respect to $\vec{r}$ evaluated at $\vec{r}_{y/z}$. If $\vec{r}_{y/z}$ is nonzero, then

$$\vec{\partial}_{\vec{r}} |\vec{r}|\Big|_{\vec{r}=\vec{r}_{y/z}} = \frac{1}{|\vec{r}_{y/z}|} \vec{r}^{\,\prime}_{y/z} = \hat{r}'_{y/z}, \tag{2.24.10}$$

and, for all $n \geq 1$,

$$\vec{\partial}_{\vec{r}} \frac{1}{|\vec{r}|^n}\Big|_{\vec{r}=\vec{r}_{y/z}} = -\frac{n}{|\vec{r}_{z/y}|^{n+1}} \vec{\partial}_{\vec{r}} |\vec{r}|\Big|_{\vec{r}=\vec{r}_{y/z}} = -\frac{n}{|\vec{r}_{z/y}|^{n+2}} \vec{r}^{\,\prime}_{z/y}. \tag{2.24.11}$$

Furthermore, for all covectors $\vec{v}^{\,\prime}$,

$$\vec{\partial}_{\vec{r}} (\vec{v}^{\,\prime} \vec{r})\Big|_{\vec{r}=\vec{r}_{y/z}} = \vec{v}^{\,\prime}. \tag{2.24.12}$$

The operator $\vec{\nabla}$ is the vector version of the covector physical gradient $\vec{\partial}$; that is, $\vec{\partial} = \vec{\nabla}^{\prime}$. The divergence and curl of the vector field $\vec{M}$ are thus given by

$$\vec{\nabla}_{\vec{r}} \cdot \vec{M}(\vec{r}) = \operatorname{tr} \vec{\partial}_{\vec{r}} \vec{M}(\vec{r}), \tag{2.24.13}$$

$$\vec{\nabla}_{\vec{r}} \times \vec{M}(\vec{r}) = \vec{\nabla}^{\times}_{\vec{r}} \vec{M}(\vec{r}). \tag{2.24.14}$$

Finally, we note the Jacobian

$$\frac{\mathrm{d}}{\mathrm{d}\vec{r}} \vec{r} = \vec{I}. \tag{2.24.15}$$

2.25 Examples

Example 2.25.1. Consider the 3-bar linkage shown from above in Figure 2.25.1 with links of lengths $\ell_1 = 3$, $\ell_2 = 4$, and $\ell_3 = 2$, and pin joints a, b, and c labeled as shown. The links are initially aligned as shown, lying in a horizontal plane, that is, the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. The pins at joints a and c are vertical, that is, parallel with $\hat{k}_{\mathrm{A}}$, and the pin at joint b is horizontal, that is, lying in the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. The rotation angles at joints a, b, and c are ψ, θ, and ϕ, respectively, where a positive value of θ indicates that joint c moves in the negative $\hat{k}_{\mathrm{A}}$ direction. The joints are rotated such that $\psi = 30$ deg, $\theta = -20$ deg, and $\phi = 45$ deg. In terms of the frame $\mathrm{F_A}$, determine the position of the tip d of the linkage relative to a after these rotations.

Solution: Define the sequence of rotations

$$\mathrm{F_A} \xrightarrow[3]{\psi} \mathrm{F_B} \xrightarrow[2]{\theta} \mathrm{F_C} \xrightarrow[3]{\phi} \mathrm{F_D}.$$

The position of d relative to a is given by

$$\begin{aligned} \vec{r}_{d/a} &= \vec{r}_{d/c} + \vec{r}_{c/b} + \vec{r}_{b/a} \\ &= \ell_3 \hat{\imath}_{\mathrm{D}} + \ell_2 \hat{\imath}_{\mathrm{C}} + \ell_1 \hat{\imath}_{\mathrm{B}}. \end{aligned}$$

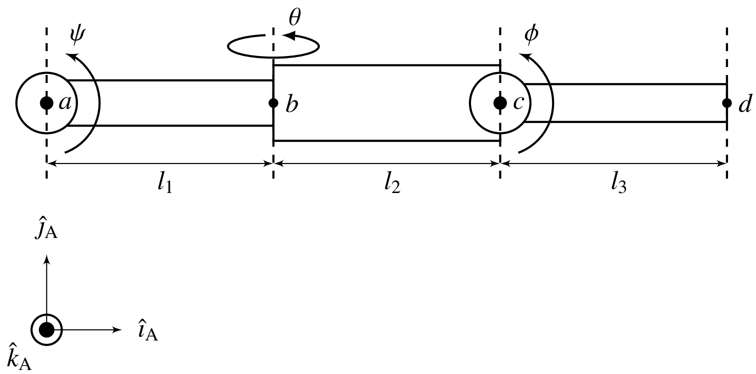


Figure 2.25.1: Example 2.25.1. Three-bar linkage. The shown orientation of the linkage corresponds to $\psi = \theta = \phi = 0$.

Resolving this equation in $\mathrm{F_A}$ yields

$$\begin{aligned}
\vec{r}_{d/a}\Big|_{\rm A} &= \ell_3\ \hat{\imath}_{\rm D}|_{\rm A} + \ell_2\ \hat{\imath}_{\rm C}|_{\rm A} + \ell_1\ \hat{\imath}_{\rm B}|_{\rm A} \\
&= \ell_3 \mathcal{O}_{\rm A/B}\mathcal{O}_{\rm B/C}\mathcal{O}_{\rm C/D}\ \hat{\imath}_{\rm D}|_{\rm D} + \ell_2 \mathcal{O}_{\rm A/B}\mathcal{O}_{\rm B/C}\ \hat{\imath}_{\rm C}|_{\rm C} + \ell_1 \mathcal{O}_{\rm A/B}\ \hat{\imath}_{\rm B}|_{\rm B} \\
&= [\ell_3 \mathcal{O}_3^{\rm T}(\psi)\mathcal{O}_2^{\rm T}(\theta)\mathcal{O}_3^{\rm T}(\phi) + \ell_2 \mathcal{O}_3^{\rm T}(\psi)\mathcal{O}_2^{\rm T}(\theta) + \ell_1 \mathcal{O}_3^{\rm T}(\psi)] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \mathcal{O}_3^{\rm T}(\psi)[\ell_3 \mathcal{O}_2^{\rm T}(\theta)\mathcal{O}_3^{\rm T}(\phi) + \ell_2 \mathcal{O}_2^{\rm T}(\theta) + \ell_1 I_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \mathcal{O}_3^{\rm T}(\psi)\left(\ell_3 \mathcal{O}_2^{\rm T}(\theta) \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{bmatrix} + \ell_2 \begin{bmatrix} \cos(\theta) \\ 0 \\ -\sin(\theta) \end{bmatrix} + \ell_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ell_3(\cos\theta)\cos\phi + \ell_2\cos\theta + \ell_1 \\ \ell_3 \sin\phi \\ -\ell_3(\sin\theta)\cos\phi - \ell_2\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} \ell_3(\cos\psi)(\cos\theta)\cos\phi + \ell_2(\cos\psi)\cos\theta + \ell_1\cos\psi - \ell_3(\sin\psi)\sin\phi \\ \ell_3(\sin\psi)(\cos\theta)\cos\phi + \ell_2(\sin\psi)\cos\theta + \ell_1\sin\psi + \ell_3(\cos\psi)\sin\phi \\ -\ell_3(\sin\theta)\cos\phi - \ell_2\sin\theta \end{bmatrix}.
\end{aligned}$$

Evaluating this expression at $\psi = 30$ deg, $\theta = -20$ deg, and $\phi = 45$ deg yields

$$\vec{r}_{d/a}\Big|_{\rm A} = \begin{bmatrix} 6.29 \\ 5.27 \\ 1.85 \end{bmatrix}. \qquad \diamond$$

2.26 Theoretical Problems

Problem 2.26.1. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Show that the following statements are equivalent:

i) $\theta_{\vec{y}/\vec{x}} = \pi/2$.

ii) $\vec{x} \cdot \vec{y} = 0$.

iii) $|\vec{x} \times \vec{y}| = |\vec{x}||\vec{y}|$.

Now, assume that $\vec{x}$ and $\vec{y}$ are nonzero. Then, show that the following statements are equivalent:

iv) $\theta_{\vec{y}/\vec{x}} = \pi$.

v) $|\hat{x} \cdot \hat{y}| = 1$.

vi) $\vec{x} \times \vec{y} = 0$.

vii) Either $\hat{x} = \hat{y}$ or $\hat{x} = -\hat{y}$.

Finally, assume that $\vec{x}$ and $\vec{y}$ are nonzero and not parallel, and let $\vec{z}$ be a physical vector. Show that, if $\vec{z} \times \vec{x} = 0$ and $\vec{z} \times \vec{y} = 0$, then $\vec{z} = 0$.

Problem 2.26.2. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors. Determine the length of the projection of $\vec{x}$ onto the line through $\vec{y}$ in terms of $|\vec{x} \cdot \vec{y}|$. Likewise, determine the length of the projection of $\vec{y}$ onto the line through $\vec{x}$. Separately consider the cases in which $\theta_{\vec{x}/\vec{y}} \in [0, \pi/2]$ and $\theta_{\vec{x}/\vec{y}} \in [\pi/2, \pi]$.

Problem 2.26.3. Define $\vec{x} \triangleq 3\hat{\imath}_{\rm A} - 4\hat{\jmath}_{\rm A}$, $\vec{y} \triangleq -\hat{\imath}_{\rm A} + 5\hat{\jmath}_{\rm A} - 2\hat{k}_{\rm A}$, $\vec{M} \triangleq \vec{x}\vec{y}'$, and $\vec{N} \triangleq \vec{y}\vec{x}'$. Then, do the following:

i) Resolve $\vec{M}$, $\vec{N}$, $\vec{M}\vec{N}$, and $\vec{M}\vec{x}$ in $\rm F_A$.

ii) Confirm that $(\vec{M}\vec{N})|_{\rm A} = \vec{M}|_{\rm A}\vec{N}|_{\rm A}$ and $(\vec{M}\vec{x})|_{\rm A} = \vec{M}|_{\rm A}\vec{x}|_{\rm A}$.

Problem 2.26.4. Let $\vec{M}$ and $\vec{N}$ be physical matrices, and let $\rm F_A$ be a frame. Show that if $\vec{M}\hat{\imath}_{\rm A} = \vec{N}\hat{\imath}_{\rm A}$, $\vec{M}\hat{\jmath}_{\rm A} = \vec{N}\hat{\jmath}_{\rm A}$, and $\vec{M}\hat{k}_{\rm A} = \vec{N}\hat{k}_{\rm A}$, then $\vec{M} = \vec{N}$.

Problem 2.26.5. Let $\rm F_A$ be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors lying in the plane spanned by $\hat{\imath}_{\rm A}$ and $\hat{\jmath}_{\rm A}$. Show that

$$x_1y_2 - x_2y_1 = |\vec{x}||\vec{y}|\sin\theta_{\vec{y}/\vec{x}/\hat{k}_{\rm A}},$$

where

$$\vec{x}\Big|_{\rm A} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \qquad \vec{y}\Big|_{\rm A} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}.$$

Problem 2.26.6. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors. Show that

$$\vec{x} \times (\vec{y} \times \vec{z}) + \vec{y} \times (\vec{z} \times \vec{x}) + \vec{z} \times (\vec{x} \times \vec{y}) = 0.$$

This is *Jacobi's identity*.

Problem 2.26.7. Let $\vec{y}$ and $\vec{z}$ be nonzero physical vectors that are not parallel. Show that

$$\vec{P}_{\vec{y}-\vec{P}_{\vec{z}}\vec{y}} + \vec{P}_{\vec{z}} = \frac{1}{|\vec{z}|^2|\vec{y}|^2 - (\vec{y}\,'\vec{z})^2}\left[|\vec{z}|^2\vec{y}\vec{y}\,' - \vec{y}\,'\vec{z}(\vec{y}\vec{z}\,' + \vec{z}\vec{y}\,') + |\vec{y}|^2\vec{z}\vec{z}\,'\right],$$

and thus

$$\vec{P}_{\vec{y}-\vec{P}_{\vec{z}}\vec{y}} + \vec{P}_{\vec{z}} = \vec{P}_{\vec{z}-\vec{P}_{\vec{y}}\vec{z}} + \vec{P}_{\vec{y}}.$$

Problem 2.26.8. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be linearly independent physical vectors, and define

$$\vec{u} \triangleq \vec{x},$$
$$\vec{v} \triangleq (\vec{I} - \vec{P}_{\vec{x}})\vec{y},$$
$$\vec{w} \triangleq (\vec{I} - \vec{P}_{\vec{x}} - \vec{P}_{\vec{y}})\vec{z}.$$

Show that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal. (Remark: This is *Gram-Schmidt orthogonalization*.)

Problem 2.26.9. Let $\mathrm{F_A}$ be a frame, and let the frame $\mathrm{F_B}$ be obtained by rotating $\mathrm{F_A}$ according to the right hand rule around the axis $\hat{k}_\mathrm{A}$ by the angle $\theta_{\mathrm{B/A}} = \pi/2$. Determine $\vec{R}_{\mathrm{B/A}}$, $\mathcal{R}_{\mathrm{B/A}}$ (by computing both $\vec{R}_{\mathrm{B/A}}\Big|_\mathrm{A}$ and $\vec{R}_{\mathrm{B/A}}\Big|_\mathrm{B}$), $n_{\mathrm{B/A}}$, and $q_{\mathrm{B/A}}$. Finally, verify (2.11.39), (2.11.45), and (2.15.30).

Problem 2.26.10. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Express $\mathcal{O}_{\mathrm{C/A}}$ in terms of a product of Euler orientation matrices for the following cases:

i) $\mathrm{F_B} = \vec{R}_{\hat{k}_\mathrm{A}}(\psi)\mathrm{F_A}$ and $\mathrm{F_C} = \vec{R}_{\hat{\jmath}_\mathrm{B}}(\theta)\mathrm{F_B}$.

ii) $\mathrm{F_B} = \vec{R}_{\hat{k}_\mathrm{A}}(\psi)\mathrm{F_A}$ and $\mathrm{F_C} = \vec{R}_{\hat{\jmath}_\mathrm{A}}(\theta)\mathrm{F_B}$.

Problem 2.26.11. Let $\mathrm{F_A}$ be a frame, and let $S \in \mathbb{R}^{3\times 3}$ be a rotation matrix. Show that there exists a frame $\mathrm{F_B}$ such that $\vec{R}_{\mathrm{A/B}}\Big|_\mathrm{B} = S$.

Problem 2.26.12. Let $\vec{x}$ be a physical vector, and let $\vec{R}$ be a physical rotation matrix. Use Fact 2.9.8 to show that $\vec{R}\vec{x} = \vec{x}$ if and only if $\vec{R}\vec{x}^{\times} = \vec{x}^{\times}\vec{R}$.

Problem 2.26.13. Let $\hat{n}$ be a unit dimensionless physical vector, let $\theta \in (-\pi, \pi]$, and let $\vec{S}$ be a physical rotation matrix. Show that

$$\vec{R}_{\vec{S}\hat{n}}(\theta) = \vec{S}\vec{R}_{\hat{n}}(\theta)\vec{S}\,'.$$

Problem 2.26.14. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $\theta_{\mathrm{B/A}}$ and $n_{\mathrm{B/A}}$ as in Fact 2.11.6. Then, show that

$$\mathcal{O}_{\mathrm{A/B}}n_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}n_{\mathrm{B/A}} = n_{\mathrm{B/A}},$$
$$\mathcal{O}_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\times} = n_{\mathrm{B/A}}^{\times}\mathcal{O}_{\mathrm{B/A}},$$
$$\mathcal{R}_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\times} = n_{\mathrm{B/A}}^{\times}\mathcal{R}_{\mathrm{B/A}}.$$

Problem 2.26.15. Let M be an $n \times n$ orthogonal matrix, that is, a nonsingular matrix that satisfies $M^{\mathrm{T}} = M^{-1}$. Show that $|\det M| = 1$ and that the absolute value of every eigenvalue of M is 1. Now, assume that M is a rotation matrix, that is, a real 3×3 orthogonal matrix whose determinant is 1. Show that the eigenvalues of M are given by $\{1, \lambda, \bar{\lambda}\}$, where λ is a complex (possibly real) number whose absolute value is 1. Next, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\hat{n}$ and $\theta \in (-\pi, \pi]$ be such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. Show that $\hat{n}|_{\mathrm{A}}$ is an eigenvector of $\mathcal{R}_{\mathrm{B/A}}$ corresponding to the eigenvalue 1, and show that $\lambda = (\cos\theta) + (\sin\theta)\jmath$. (Hint: Let $v \in \mathbb{C}^n$ be an eigenvector of M associated with the eigenvalue λ, and note that $v^* M^{\mathrm{T}} M v = \lambda v^* M^{\mathrm{T}} v$.)

Problem 2.26.16. Let R be a rotation matrix. Show that

$$(\operatorname{tr} R)^2 = \operatorname{tr} R^2 + 2\operatorname{tr} R,$$
$$(\operatorname{tr} R)^3 + 2\operatorname{tr} R^3 = 3(\operatorname{tr} R)(\operatorname{tr} R^2) + 6.$$

Problem 2.26.17. Let R be a rotation matrix. Show that

$$R^3 - (\operatorname{tr} R)R^2 + (\operatorname{tr} R)R - I_3 = 0.$$

Now, define $c \triangleq \frac{1}{2}(\operatorname{tr} R - 1)$. Show that

$$R^4 - (2 + 2c)R^3 + (2 + 4c)R^2 - (2 + 2c)R + I_3 = 0.$$

Furthermore, if $c \neq -1$, then show that

$$\frac{1 + 2c}{2(1 + c)} I_3 + \frac{1}{4(1 + c)}(R^2 + R^{2\mathrm{T}}) + \tfrac{1}{2}(R - R^{\mathrm{T}}) = R.$$

(Hint: For the first equality, use the Cayley-Hamilton theorem. For the second equality, use Problem 2.26.11 to express c in terms of the eigenvalues of R, and use an orthogonal transformation to diagonalize R.)

Problem 2.26.18. Let $\vec{R} = \vec{R}_{\hat{n}}(\theta)$ be a physical rotation matrix, where $\hat{n}$ is a unit dimensionless physical vector and $\theta \in [0, \pi]$. Show that the following statements are equivalent:

i) $\vec{R}$ is symmetric.

ii) Either $\operatorname{tr}\vec{R} = -1$ or $\operatorname{tr}\vec{R} = 3$.

iii) Either $\theta = 0$ or $\theta = \pi$.

Problem 2.26.19. Let $\vec{R}$ be a physical rotation matrix, let $\vec{w}$ be a physical vector, let α be a real number, and, for each physical vector $\vec{x}$, define the physical vector $\vec{y}$ by the combined rotation and translation transformation

$$\vec{y} = \vec{R}\vec{x} + \alpha\vec{w}.$$

Let $\mathrm{F_A}$ be a frame. Show that

$$\begin{bmatrix} \tilde{y} \\ \alpha \end{bmatrix} = \begin{bmatrix} \tilde{R} & \tilde{w} \\ 0_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \alpha \end{bmatrix}.$$

where $\tilde{y} \triangleq \vec{y}\Big|_{\mathrm{A}}$ and likewise for the remaining quantities.

Problem 2.26.20. Let F_A and F_B be frames with origins O_A and O_B, respectively, and let x be a point. Show that

$$\begin{bmatrix} \vec{r}_{x/O_B}\Big|_B \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{B/A} & \vec{r}_{O_A/O_B}\Big|_B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{r}_{x/O_A}\Big|_A \\ 1 \end{bmatrix}.$$

Furthermore, show that

$$\begin{bmatrix} \mathcal{O}_{B/A} & \vec{r}_{O_A/O_B}\Big|_B \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{O}_{A/B} & -\mathcal{O}_{A/B}\, \vec{r}_{O_A/O_B}\Big|_B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{A/B} & \vec{r}_{O_B/O_A}\Big|_A \\ 0 & 1 \end{bmatrix}.$$

Problem 2.26.21. Let F_A and F_B be frames. Show that

$$\vec{R}_{B/A}\Big|_B = \vec{R}_{B/A}\Big|_A$$

in two different ways, namely:

i) By resolving $\vec{R}_{B/A}$ in both F_A and F_B and using (2.10.31).

ii) By using Euler's theorem to express $\vec{R}_{B/A}$ in terms of Rodrigues's formula.

Problem 2.26.22. Let b, c, d be real numbers, and define the pure quaternion $\mathbf{q} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ be nonzero. Show that

$$e^{\frac{1}{2}\mathbf{q}} = \cos \tfrac{1}{2}|\mathbf{q}| + \frac{\sin \frac{1}{2}|\mathbf{q}|}{|\mathbf{q}|}\mathbf{q}$$

and thus

$$|e^{\frac{1}{2}\mathbf{q}}| = 1.$$

(Remark: See [7, p. 71].)

Problem 2.26.23. Let a, b, c, d be real numbers such that $\sqrt{a^2 + b^2 + c^2 + d^2} = 1$, and define the unit quaternion $\mathbf{q} \triangleq a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Furthermore, let F_A be a frame, let $\vec{x}$ be a physical vector, and let

$$\vec{x}\Big|_A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Finally, define the pure quaternion

$$\mathbf{x} \triangleq x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

Then, show that

$$\mathbf{y} \triangleq y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k} \triangleq \mathbf{q}\mathbf{x}\mathbf{q}^{-1}$$

is a pure quaternion and that the physical vector

$$\vec{y} \triangleq y_1\hat{\imath}_A + y_2\hat{\jmath}_A + y_3\hat{k}_A$$

satisfies

$$\vec{y} = \vec{R}\vec{x},$$

where $\vec{R}$ is the physical rotation matrix

$$\vec{R}\Big|_{\mathrm{A}} \triangleq \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{bmatrix}.$$

(Hint: Note that $\mathbf{q}^{-1} = \overline{\mathbf{q}} \triangleq a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.) (Remark: This result shows that the two-sided transformation $\mathbf{y} = \mathbf{q}\mathbf{x}\mathbf{q}^{-1}$, where $\mathbf{q}$ is a unit quaternion, represents a physical rotation matrix. Conversely, since $\mathbf{y} = (-\mathbf{q})\mathbf{x}(-\mathbf{q})^{-1}$, this result also shows that every physical rotation matrix that is not the identity can be represented by two distinct unit quaternions, namely, $\mathbf{q}$ and $-\mathbf{q}$.)

2.27 Applied Problems

Problem 2.27.1. Consider a 3-gimbal mechanism that emulates 3-2-1 Euler angles, that is, the outermost gimbal rotates around the $\hat{k}$-axis of the support frame, the intermediate axis rotates around the $\hat{\jmath}$-axis of a frame attached to the outermost gimbal, and the innermost gimbal rotates around the $\hat{\imath}$-axis of the intermediate gimbal. Physically, it can be seen that the rotation of each gimbal relative to its support can be performed in an arbitrary order without changing the final configuration. Prove this mathematically, that is, show that if the outermost gimbal is rotated first, followed by the intermediate gimbal, and, then, finally, by the innermost gimbal, then the configuration that results is the same as the configuration that results from rotating the gimbals in the reverse order. To do this, resolve two products of physical rotation matrices in the support frame.

Problem 2.27.2. Consider the box shown in Figure 2.27.1 with side lengths and vertices as shown. The box is rotated 30 degrees clockwise around its edge ab as viewed from a to b, that is, 30 degrees by the right-hand rule around $\hat{r}_{b/a}$. Next, the box is rotated 45 degrees counterclockwise around its edge ae as viewed from a to e. In terms of the frame $\mathrm{F_A}$, determine the position of g relative to a after both rotations.

Problem 2.27.3. Consider the box shown in Figure 2.27.1 with side lengths and vertices as shown. The box is rotated 60 degrees counterclockwise around the diagonal af as viewed from a to f, that is, -60 degrees by the right-hand rule around $\hat{r}_{f/a}$. In terms of the frame $\mathrm{F_A}$, determine the position of g relative to a after the rotation.

Problem 2.27.4. Consider the bent wire abc shown in Figure 2.27.2. This wire consists of two straight segments of length ℓ_1 and ℓ_2, and lies in the $\hat{\imath}_\mathrm{A}$-$\hat{k}_\mathrm{A}$ plane. The angle θ describes how much the wire is bent at the point b. The bent wire is rotated counterclockwise around the line passing through the points a and c (as seen looking from a to c) by the angle $\phi > 0$. Determine the distance from the original position of the point b to its final position after the rotation. Check your solution by specializing it to the case $\theta = 0$.

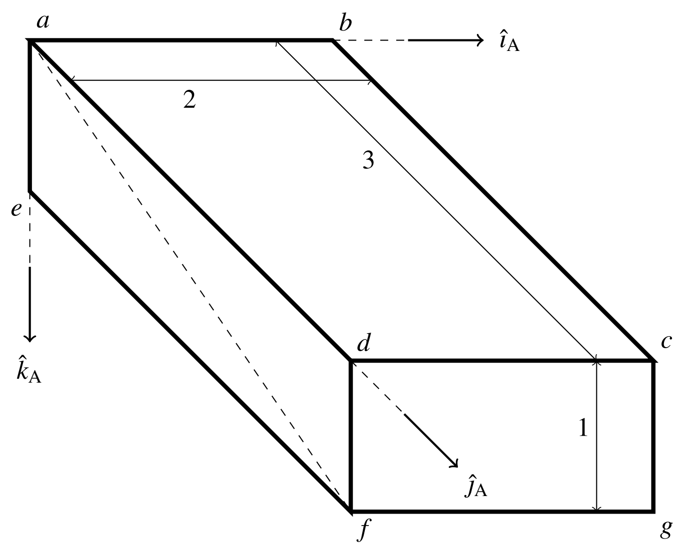


Figure 2.27.1: Box for Problem 2.27.2 and Problem 2.27.3.

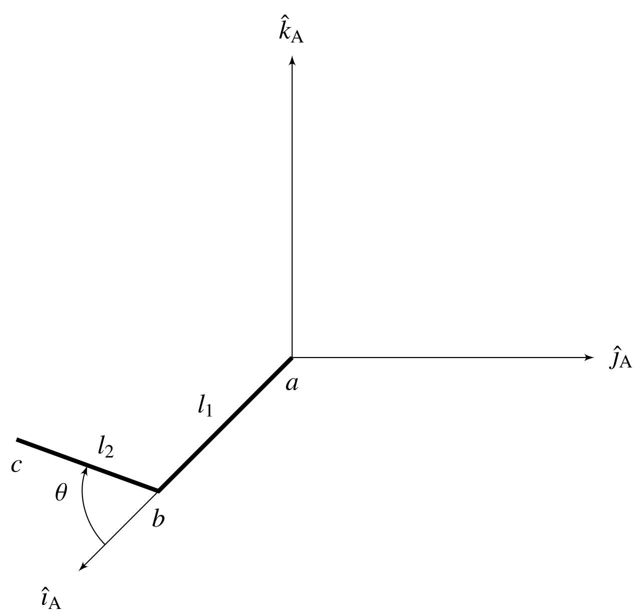


Figure 2.27.2: Bent wire for Problem 2.27.4.

Symbol	Definition
x	Point or particle x
$\mathcal{B}$	Body $\mathcal{B}$
$\vec{x}$	Physical vector
$\lvert\vec{x}\rvert$	Magnitude of physical vector $\vec{x}$
$\hat{\imath}, \hat{\jmath}, \hat{k}$	Unit dimensionless physical vectors
$\theta_{\vec{y}/\vec{x}}$	Angle in $[0, \pi]$ between $\vec{y}$ and $\vec{x}$
$\vec{\theta}_{\vec{y}/\vec{x}}$	Angle vector of $\vec{y}$ relative to $\vec{x}$
$\vec{r}_{y/x}$	Position of y relative to x
$\theta_{\vec{y}/\vec{x}/\hat{n}}$	Directed angle in $(-\pi, \pi]$ from $\vec{x}$ to $\vec{y}$ around $\hat{n}$
$\vec{\theta}_{\vec{y}/\vec{x}/\hat{n}}$	Directed angle vector of $\vec{y}$ relative to $\vec{x}$ around $\hat{n}$
$\vec{r}_{y/x}$	Position of y relative to x
F_{A}	Frame A written as a row vectrix
$\mathcal{F}_{\mathrm{A}}$	Frame A written as a column vectrix
$\vec{R}_{\mathrm{B/A}}$	Physical rotation matrix that rotates F_{A} to F_{B}
$\mathcal{R}_{\mathrm{B/A}}$	Rotation matrix from F_{A} to F_{B}
$\mathcal{O}_{\mathrm{B/A}}$	Orientation matrix of F_{B} relative to F_{A}
$q_{\mathrm{B/A}}$	Euler parameter vector of F_{B} relative to F_{A}
3-2-1 rotation: Ψ, Θ, Φ	Yaw, pitch, roll Euler angles
3-1-3 rotation: Φ, Θ, Ψ	Precession, nutation, spin Euler angles
3-1-3 rotation: Ω, i, ω	Right ascension of the ascending node, inclination, argument of periapsis Euler angles

Table 2.1: Symbols for Chapter 2.

Chapter Three
Tensors

The duality between covariance and contravariance arises whenever a vector or tensor quantity is represented by its components, although modern differential geometry uses more sophisticated index-free methods to represent tensors.

3.1 Tensors

A *tensor* is a real-valued function of physical covectors and physical vectors that is multilinear, that is, linear in each argument separately. For example, if $\vec{w}$ is a physical vector and $\vec{z}'$ is a physical covector, then the function $\vec{T}\colon \mathcal{V}' \times \mathcal{V} \mapsto \mathbb{R}$ defined by

$$\vec{T}(\vec{x}', \vec{y}) = (\vec{x}'\vec{w})(\vec{z}'\vec{y}) \tag{3.1.1}$$

is a tensor, where $\mathcal{V}$ denotes the set of physical vectors and $\mathcal{V}'$ denotes the set of physical covectors. Note that the value of $\vec{T}(\vec{x}', \vec{y})$ is a product of inner products. In this case we write

$$\vec{T}(\vec{x}', \vec{y}) = (\vec{w} \otimes \vec{z}')(\vec{x}', \vec{y}), \tag{3.1.2}$$

where the tensor $\vec{T}$ is represented by the tensor product notation

$$\vec{T} = \vec{w} \otimes \vec{z}'. \tag{3.1.3}$$

The physical vector $\vec{w}$ and the physical covector $\vec{z}'$ are the factors of $\vec{T}$, whereas the physical covector $\vec{x}'$ and the physical vector $\vec{y}$ are the arguments of $\vec{T}$.

The tensor $\vec{T}$ defined by (3.1.1) can be viewed as a physical matrix. To see this, define $\vec{M} = \vec{x}\vec{y}'$. Then, for every physical vector $\vec{z}$ and physical covector $\vec{w}'$,

$$\vec{w}'\vec{M}\vec{z} = \vec{w}'\vec{x}\vec{y}'\vec{z} = (\vec{x}'\vec{w})(\vec{z}'\vec{y}) = (w \otimes \vec{z}')(\vec{x}', \vec{y}) = \vec{T}(\vec{w}', \vec{z}). \tag{3.1.4}$$

We thus identify $\vec{x}\vec{y}'$ with $\vec{x} \otimes \vec{y}'$.

Like physical matrices, tensors of the form (3.1.2) can be added. For example, for physical vectors $\vec{w}_1, \vec{w}_2$ and physical covectors $\vec{z}'_1, \vec{z}'_2$, we can define the tensor

$$\vec{T} = \vec{w}_1 \otimes \vec{z}'_1 + \vec{w}_2 \otimes \vec{z}'_2, \tag{3.1.5}$$

which satisfies

$$\vec{T}(\vec{x}', \vec{y}) = (\vec{w}_1 \otimes \vec{z}'_1)(\vec{x}', \vec{y}) + (\vec{w}_2 \otimes \vec{z}'_2)(\vec{x}', \vec{y})$$

$$= (\vec{x}'\vec{w}_1)(\vec{z}_1'\vec{y}) + (\vec{x}'\vec{w}_2)(\vec{z}_2'\vec{y}). \tag{3.1.6}$$

As another example, the function $\vec{T}\colon \mathcal{V} \times \mathcal{V} \mapsto \mathbb{R}$ defined by

$$\vec{T}(\vec{x}, \vec{y}) = \vec{x}'\vec{y} \tag{3.1.7}$$

is a tensor. To see this, note that

$$\begin{aligned}(\hat{\imath}_{\mathrm{A}}' \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}}' \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}}' \otimes \hat{k}_{\mathrm{A}}')(\vec{x}, \vec{y}) &= (\hat{\imath}_{\mathrm{A}}'\vec{x})(\hat{\imath}_{\mathrm{A}}'\vec{y}) + (\hat{\jmath}_{\mathrm{A}}'\vec{x})(\hat{\jmath}_{\mathrm{A}}'\vec{y}) + (\hat{k}_{\mathrm{A}}'\vec{x})(\hat{k}_{\mathrm{A}}'\vec{y})\\ &= \vec{x}'\vec{y}.\end{aligned} \tag{3.1.8}$$

Hence,

$$\vec{T} = \hat{\imath}_{\mathrm{A}}' \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}}' \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}}' \otimes \hat{k}_{\mathrm{A}}'. \tag{3.1.9}$$

Alternatively, the function $\vec{T}\colon \mathcal{V}' \times \mathcal{V} \mapsto \mathbb{R}$ defined by

$$\vec{T}(\vec{x}', \vec{y}) = \vec{x}'\vec{y} \tag{3.1.10}$$

is a tensor since

$$\begin{aligned}(\hat{\imath}_{\mathrm{A}} \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}} \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}} \otimes \hat{k}_{\mathrm{A}}')(\vec{x}', \vec{y}) &= (\vec{x}'\hat{\imath}_{\mathrm{A}})(\hat{\imath}_{\mathrm{A}}'\vec{y}) + (\vec{x}'\hat{\jmath}_{\mathrm{A}})(\hat{\jmath}_{\mathrm{A}}'\vec{y}) + (\vec{x}'\hat{k}_{\mathrm{A}})(\hat{k}_{\mathrm{A}}'\vec{y})\\ &= (\hat{\imath}_{\mathrm{A}}'\vec{x})(\hat{\imath}_{\mathrm{A}}'\vec{y}) + (\hat{\jmath}_{\mathrm{A}}'\vec{x})(\hat{\jmath}_{\mathrm{A}}'\vec{y}) + (\hat{k}_{\mathrm{A}}'\vec{x})(\hat{k}_{\mathrm{A}}'\vec{y})\\ &= \vec{x}'\vec{y}.\end{aligned} \tag{3.1.11}$$

Hence,

$$\vec{T} = \hat{\imath}_{\mathrm{A}} \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}} \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}} \otimes \hat{k}_{\mathrm{A}}'. \tag{3.1.12}$$

The *type* of a tensor $\vec{T}$ is denoted by (p, q), where p is the number of physical vectors multiplied together (*contravariant factors*) to form $\vec{T}$, while q is the number of physical covectors that are multiplied together (*covariant factors*) to form $\vec{T}$. Equivalently, p is the number of physical covectors (*contravariant arguments*) that $\vec{T}$ operates on, and q is the number of physical vectors (*covariant* arguments) that $\vec{T}$ operates on.

The *order* of a tensor of type (p, q) is $p + q$. Therefore, a tensor of type $(2, 0)$, $(1, 1)$, and $(0, 2)$ is a second-order tensor. A scalar is a zeroth-order tensor.

If the factors of $\vec{T}$ include both physical vectors and physical covectors, then the physical vectors appear first, followed by the physical covectors. In this case, the arguments of $\vec{T}$ include both physical covectors and physical vectors, and thus the physical covectors are listed first followed by the physical vectors. Hence, if $\vec{T} = \vec{w} \otimes \vec{z}'$, then we write $\vec{T}(\vec{x}', \vec{y})$. The set of tensors of type (p, q) is denoted by $\mathcal{T}_{(p,q)}$, and thus, for $\vec{T} \in \mathcal{T}_{(p,q)}$, we write $\vec{T}\colon \mathcal{V}'^{p} \times \mathcal{V}^{q} \mapsto \mathbb{R}$.

Given p physical vectors and q physical covectors, it is possible to construct a tensor of type (p, q). Specifically, given physical vectors $\vec{w}_1, \ldots, \vec{w}_p$ and physical covectors $\vec{z}_1', \ldots, \vec{z}_q'$, we can construct the tensor $\vec{T}$ of type (p,q) given by

$$\vec{T} = \vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_q. \tag{3.1.13}$$

Then, $\vec{T}$ operates on the physical covectors $\vec{x}'_1, \ldots, \vec{x}'_p$ and the physical vectors $\vec{y}_1, \ldots, \vec{y}_q$ according to

$$\begin{aligned}\vec{T}(\vec{x}'_1, \ldots, \vec{x}'_p, \vec{y}_1, \ldots, \vec{y}_q) &= (\vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_q)(\vec{x}'_1, \ldots, \vec{x}'_p, \vec{y}_1, \ldots, \vec{y}_q) \\ &= (\vec{x}'_1\vec{w}_1) \cdots (\vec{x}'_p\vec{w}_p)(\vec{z}'_1\vec{y}_1) \cdots (\vec{z}'_q\vec{y}_q).\end{aligned} \tag{3.1.14}$$

Note that the value of $\vec{T}$ is the product of $p+q$ inner products. A sum of tensors of type (p,q) is also a tensor of type (p,q). That is, if $\vec{T}_1 \in \mathfrak{T}_{(p,q)}$ and $\vec{T}_2 \in \mathfrak{T}_{(p,q)}$, then $\vec{T}_1 + \vec{T}_2 \in \mathfrak{T}_{(p,q)}$.

Let $\vec{T} \in \mathfrak{T}_{(p,q)}$. Then, $\vec{T}$ is *wide* if $p < q$, *square* if $p = q$, and tall if $p > q$. Note that $\vec{T}$ is square if and only if $\vec{T}$ has an equal number of vector and covector factors.

The *coform* of the tensor $\vec{T}$ given by (3.1.13) is the (q,p) tensor

$$\vec{T}' = \vec{z}_1 \otimes \cdots \otimes \vec{z}_q \otimes \vec{w}'_1 \otimes \cdots \otimes \vec{w}'_p. \tag{3.1.15}$$

For example,

$$(\vec{w}_1 \otimes \vec{z}'_1)' = \vec{z}_1 \otimes \vec{w}'_1, \tag{3.1.16}$$

$$(\vec{w}_1 \otimes \vec{w}_2)' = \vec{w}'_1 \otimes \vec{w}'_2. \tag{3.1.17}$$

If $\vec{T}_1$ and $\vec{T}_2$ are tensors of the same type, then

$$(\vec{T}_1 + \vec{T}_2)' = \vec{T}'_1 + \vec{T}'_2. \tag{3.1.18}$$

Let $\vec{z}'$ be a physical covector. Then,

$$\vec{T} = \vec{z}' \tag{3.1.19}$$

is a first-order tensor of type $(0,1)$. Furthermore, for each physical vector $\vec{x}$, $\vec{T}(\vec{x})$ is given by

$$\vec{T}(\vec{x}) = \vec{z}'\vec{x}. \tag{3.1.20}$$

Similarly, let $\vec{w}$ be a physical vector. Then,

$$\vec{T} = \vec{w} \tag{3.1.21}$$

is a first-order tensor of type $(1,0)$. Furthermore, for each physical covector $\vec{x}'$, $\vec{T}(\vec{x}')$ is given by

$$\vec{T}(\vec{x}') = \vec{x}'\vec{w}. \tag{3.1.22}$$

The second-order tensor of type (1,1) given by $\vec{T} = \vec{w} \otimes \vec{z}'$ operates on the pair $(\vec{x}', \vec{y})$ to yield the real number

$$\vec{T}(\vec{x}', \vec{y}) = (\vec{w} \otimes \vec{z}')(\vec{x}', \vec{y}) = (\vec{x}'\vec{w})(\vec{z}'\vec{y}) = \vec{x}'\vec{M}\vec{y}, \tag{3.1.23}$$

where $\vec{M}$ is the physical matrix $\vec{M} = \vec{w}\vec{z}'$. For example, letting $\mathrm{F_A}$ be a frame, it follows that

$$\vec{T} = \hat{\imath}_{\mathrm{A}} \otimes \hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}} \otimes \hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{A}} \otimes \hat{k}'_{\mathrm{A}} = \hat{\imath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} = \vec{I}. \tag{3.1.24}$$

Therefore,

$$\vec{T}(\vec{x}', \vec{y}) = \vec{x}'\vec{I}\vec{y} = \vec{x}'\vec{y}. \tag{3.1.25}$$

In the case of a tensor of type (1,1) it is convenient to omit "⊗" and recognize that every tensor of type $(1,1)$ is a physical matrix, and vice versa. We thus write $\vec{T} = \vec{w} \otimes \vec{z}' = \vec{w}\vec{z}'$.

As a final example, let $\vec{z}'_1$ and $\vec{z}'_2$ be physical covectors. Then the second-order tensor

$$\vec{T} = \vec{z}'_1 \otimes \vec{z}'_2 \tag{3.1.26}$$

is of type $(0,2)$. In particular,

$$\vec{T}(\vec{y}_1, \vec{y}_2) = (\vec{z}'_1 \otimes \vec{z}'_2)(\vec{y}_1, \vec{y}_2) = (\vec{z}'_1\vec{y}_1)(\vec{z}'_2\vec{y}_2). \tag{3.1.27}$$

3.2 Tensor Contraction and Tensor Multiplication

Let

$$\vec{T} = \vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_q \tag{3.2.1}$$

be a tensor of type (p,q), let $1 \le i \le p$ and $1 \le j \le q$. Then a single *contraction* of $\vec{T}$ yields the tensor of type $(p-1, q-1)$ given by

$$\vec{T}_{(i,j)} = (\vec{z}'_j\vec{w}_i)\vec{w}_1 \otimes \cdots \otimes w_{i-1} \otimes w_{i+1} \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes z_{j-1} \otimes z_{j+1} \otimes \cdots \otimes \vec{z}'_q. \tag{3.2.2}$$

Note that $\vec{w}_i$ and $\vec{z}_j$ are removed from the tensor product and now appear as a scalar factor in an inner product. Likewise, if $1 \le i \le p$ and $1 \le k \le p$ are distinct and $1 \le j \le q$ and $1 \le l \le q$ are distinct, then a *double contraction* of $\vec{T}$ yields the tensor of type $(p-2, q-2)$ given by

$$\vec{T}_{(i,j),(k,l)} = (\vec{z}'_j\vec{w}_i)(\vec{z}'_l\vec{w}_k)\vec{T}_1, \tag{3.2.3}$$

where $\vec{T}_1$ is identical to $\vec{T}$ except that $\vec{w}_i, \vec{w}_k, \vec{z}_j, \vec{z}_l$ are removed. More generally, application of an r-contraction to a (p,q) tensor, where $r \le \min\{p,q\}$, yields a tensor of type $(p-r, q-r)$. Each contraction thus reduces the order of a tensor by 2.

As an extreme case of contraction, suppose that $\vec{T} \in \mathcal{T}_{(p,p)}$, and thus $\vec{T}$ is square. Then, applying p contractions to $\vec{T}$ yields a scalar, which is a product of p inner products. The p contractions constitute a *total contraction* of $\vec{T}$. Note that there are $p!$ different total contractions of $\vec{T}$. A *partial contraction* of $\vec{T} \in \mathcal{T}_{(p,q)}$ is a contraction that is not a total contraction. Note that, if $\vec{T}$ is not square, then $\vec{T}$ does not have a total contraction.

Next, to define tensor multiplication, let

$$\vec{T}_1 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_{p_1} \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_{q_1} \tag{3.2.4}$$

be a tensor of type (p_1, q_1), and let

$$\vec{T}_2 = \vec{w}_{p_1+1} \otimes \cdots \otimes \vec{w}_{p_1+p_2} \otimes \vec{z}'_{q_1+1} \otimes \cdots \otimes \vec{z}'_{q_1+q_2} \tag{3.2.5}$$

be a tensor of type (p_2, q_2). Then the *tensor product* $\vec{T}_1 \otimes \vec{T}_2$ of $\vec{T}_1$ and $\vec{T}_2$ is the tensor of type $(p_1 + p_2, q_1 + q_2)$ given by

$$\vec{T}_1 \otimes \vec{T}_2 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_{p_1+p_2} \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_{q_1+q_2}. \tag{3.2.6}$$

Next, consider the tensor of type (p, q) given by

$$\vec{T}_1 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_q \tag{3.2.7}$$

and evaluate $\vec{T}_1$ at the arguments $z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}$. This yields the real number

$$\begin{aligned} T_1(z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}) &= (\vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_q)(z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}) \\ &= (\vec{z}'_{q+1}\vec{w}_1) \cdots (\vec{z}'_{p+q}\vec{w}_p)(\vec{z}'_1\vec{w}_{p+1}) \cdots (\vec{z}'_q\vec{w}_{p+q}). \end{aligned} \tag{3.2.8}$$

Now, using the arguments $z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}$, construct the tensor

$$\vec{T}_2 \triangleq \vec{w}_{p+1} \otimes \cdots \otimes \vec{w}_{p+q} \otimes \vec{z}'_{q+1} \otimes \cdots \otimes \vec{z}'_{q+p}. \tag{3.2.9}$$

Then, the tensor product $\vec{T}_3 \triangleq \vec{T}_1 \otimes \vec{T}_2$ is given by

$$T_3 \triangleq \vec{T}_1 \otimes \vec{T}_2 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_{p+q} \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_{p+q}. \tag{3.2.10}$$

Note that $\vec{T}_3$ is a square tensor of type $(p+q, p+q)$. Furthermore, a total contraction of $\vec{T}_3$ is given by

$$\begin{aligned} (\vec{T}_3)_{(1,q+1),\ldots,(p,q+p),(p+1,1),\ldots,(p+q,q)} &= (\vec{z}'_{q+1}\vec{w}_1) \cdots (\vec{z}'_{p+q}\vec{w}_p)(\vec{z}'_1\vec{w}_{p+1}) \cdots (\vec{z}'_q\vec{w}_{p+q}) \\ &= T_1(z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}). \end{aligned} \tag{3.2.11}$$

Note that the value of this total contraction is precisely the value of $\vec{T}_1$ evaluated at the arguments $(\vec{z}'_{q+1}, \ldots, \vec{z}'_{q+p}, \vec{w}_{p+1}, \ldots, \vec{w}_{p+q})$. It thus follows that the evaluation of $\vec{T}_1$ at given arguments can be expressed as the total contraction of the tensor product of $\vec{T}_1$ and a tensor $\vec{T}_2$ whose factors are the arguments of $\vec{T}_1$. Conversely, every total contraction of the product of two tensors $\vec{T}_1$ and $\vec{T}_2$ can be expressed as the evaluation of $\vec{T}_1$ at arguments that are the factors of $\vec{T}_2$.

To illustrate this connection, consider the physical matrices $\vec{M} \triangleq \vec{x}\vec{y}' = \vec{x} \otimes \vec{y}'$ and $\vec{N} \triangleq \vec{w}\vec{z}' = \vec{w} \otimes \vec{z}'$. Then, a total contraction of the tensor product

$$\vec{T} \triangleq \vec{M} \otimes \vec{N} = (\vec{x}\vec{y}') \otimes (\vec{w}\vec{z}') = (\vec{x} \otimes \vec{y}') \otimes (\vec{w} \otimes \vec{z}') = \vec{x} \otimes \vec{w} \otimes \vec{y}' \otimes \vec{z}' \tag{3.2.12}$$

is given by

$$(\vec{T})_{(2,1),(1,4)} = (\vec{M} \otimes \vec{N})_{(2,1),(1,4)} = (\vec{y}'\vec{w})(\vec{z}'\vec{x}), \tag{3.2.13}$$

which is one of two possible total contractions of $\vec{T}$. On the other hand,

$$\vec{M}(\vec{z}', \vec{w}) = (\vec{z}'\vec{x})(\vec{y}'\vec{w}) = (\vec{M} \otimes \vec{N})_{(2,1),(1,4)}. \tag{3.2.14}$$

Consequently, evaluating the second-order tensor $\vec{M}$ at the arguments $\vec{z}', \vec{w}$ is equivalent to taking the tensor product $\vec{M}$ and $\vec{N}$ and then forming a total contraction of the product. Note, however, that the physical matrix $\vec{P} = \vec{M}\vec{N} = (\vec{x}\vec{y}')(\vec{w}\vec{z}') = (\vec{y}'\vec{w})\vec{x}\vec{z}'$ is not a scalar and thus is not a total contraction. In fact, $\vec{P}$ can be viewed as a partial contraction of $\vec{M} \otimes \vec{N}$, as discussed in the next section.

3.3 Partial Tensor Evaluation and the Contracted Tensor Product

In the previous section, we showed that evaluating a tensor at given arguments is equivalent to multiplying the tensor by another tensor whose factors are the arguments and then forming a total contraction of the product. In this case, if the first tensor is of type (p, q), then the second tensor is of type (q, p). In this section we generalize this idea by introducing partial evaluation of a tensor. In this case, the number of covectors and vectors comprising the arguments may be less than the order of the tensor. We then show that partial evaluation is equivalent to constructing a tensor from the given arguments, forming the tensor product, and then partially contracting the product. Both operations–partial evaluation versus tensor multiplication followed by partial tensor contraction—yield a tensor. In the case where the evaluation is performed with the maximum number of arguments, the result is equivalent to a total contraction, as discussed in the previous section. Partial contraction of the tensor product allows us to extend the notion of the tensor product to obtain a tensor of lower order than the tensor product defined above. For example, with this extension, the product of a fourth-order tensor and a second-order tensor may be a tensor of order 6, 4, or 2.

To illustrate the main idea, let $\vec{T}_1$ be the fifth-order tensor of type $(3, 2)$ given by

$$\vec{T}_1 = \vec{w}_1 \otimes \vec{w}_2 \otimes \vec{w}_3 \otimes z_1' \otimes z_2'. \tag{3.3.1}$$

Although $\vec{T}_1$ has five arguments, we partially evaluate $\vec{T}_1$ at the arguments $\vec{x}', \vec{y}_1, \vec{y}_2$ to obtain the third-order tensor $\vec{T}_3$ of type $(1, 2)$ given by

$$\vec{T}_3 \triangleq \vec{T}_1(\vec{x}', \vec{y}_1, \vec{y}_2; 3, 2, 1) = (\vec{x}'\vec{w}_3)(\vec{z}_2'\vec{y}_1)(\vec{z}_1'\vec{y}_2)\vec{w}_1 \otimes \vec{w}_2.. \tag{3.3.2}$$

Note that $(3, 2, 1)$ associates the first argument, which is a covector, with the third vector factor in $\vec{T}_1$; the first vector argument with the second covector factor in $\vec{T}_1$; and the second vector argument with the first covector factor in $\vec{T}_1$.

We can arrive at partial evaluation from a different direction. To do this, define the $(2, 1)$ tensor

$$\vec{T}_2 \triangleq \vec{y}_1 \otimes \vec{y}_2 \otimes \vec{x}' \tag{3.3.3}$$

and multiply it by $\vec{T}_1$ given by (3.3.1) to obtain the $(5,3)$ tensor

$$\vec{T}_1 \otimes \vec{T}_2 = \vec{w}_1 \otimes \vec{w}_2 \otimes \vec{w}_3 \otimes \vec{y}_1 \otimes \vec{y}_2 \otimes z_1' \otimes z_2' \otimes \vec{x}'. \tag{3.3.4}$$

It thus follows that

$$\begin{aligned}(\vec{T}_1 \otimes \vec{T}_2)_{(3,3),(4,2),(5,1)} &= (\vec{w}_1 \otimes \vec{w}_2 \otimes \vec{w}_3 \otimes \vec{y}_1 \otimes \vec{y}_2 \otimes z_1' \otimes z_2' \otimes \vec{x}')_{(3,3),(4,2),(5,1)} \\ &= (\vec{x}'\vec{w}_3)(\vec{z}_2'\vec{y}_1)(\vec{z}_1'\vec{y}_2)\vec{w}_1 \otimes \vec{w}_2 \\ &= \vec{T}_1(\vec{x}', \vec{y}_1, \vec{y}_2; 3, 2, 1) \\ &= \vec{T}_3,\end{aligned} \tag{3.3.5}$$

where $\vec{T}_3$ is defined by (3.3.2). Note that $\vec{T}_3$ arises in two different but equivalent ways, namely, from the partial evaluation $\vec{T}_1(\vec{x}', \vec{y}_1, \vec{y}_2; 3, 2, 1)$ of $\vec{T}_1$ as well as from the *contracted tensor product* $(\vec{T}_1 \otimes \vec{T}_2)_{(3,3),(4,2),(5,1)}$, both of which produce a tensor of lower rank than the tensor product $\vec{T}_1 \otimes \vec{T}_2$.

As another example, we compare "standard" multiplication and tensor multiplication of the physical matrix $\vec{M} = \vec{x} \otimes \vec{y}' = \vec{x}\vec{y}'$, which is a tensor of type $(1,1)$, with the physical vector $\vec{z}$, which is a tensor of type $(1,0)$. In this case, the tensor product of $\vec{M}$ and z is the $(2,1)$ tensor

$$\vec{T} \triangleq \vec{M} \otimes \vec{z} = \vec{x} \otimes \vec{z} \otimes \vec{y}'. \tag{3.3.6}$$

On the other hand, standard multiplication yields

$$\vec{M}\vec{z} = \vec{x}\vec{y}'\vec{z} = (\vec{y}'\vec{z})\vec{x} = \vec{M}(\vec{z}; 1) = (\vec{x} \otimes \vec{y}')(\vec{z}; 1) = (\vec{x} \otimes \vec{z} \otimes \vec{y}')_{(2,1)} = (M \otimes \vec{z})_{(2,1)}. \tag{3.3.7}$$

Next, consider the physical matrices $\vec{M} = \vec{w}\vec{z}'$ and $\vec{N} = \vec{x}\vec{y}'$. Their product can be written as the contracted tensor product

$$\vec{M}\vec{N} = (\vec{w} \otimes \vec{x} \otimes \vec{z}' \otimes \vec{y}')_{(2,1)} = (\vec{z}'\vec{x})\vec{w}\vec{y}'. \tag{3.3.8}$$

As another example, consider the *metric tensor*, which is the $(0,2)$ tensor

$$\vec{G} = \hat{\imath}_{\mathrm{A}}' \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}}' \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}}' \otimes \hat{k}_{\mathrm{A}}'. \tag{3.3.9}$$

Letting $\vec{x} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}$, it follows that

$$\begin{aligned}(\vec{G} \otimes \vec{x})_{(1,1)} &= ((\hat{\imath}_{\mathrm{A}}' \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}}' \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}}' \otimes \hat{k}_{\mathrm{A}}') \otimes (x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}))_{(1,1)} \\ &= x_1\hat{\imath}_{\mathrm{A}}' + x_2\hat{\jmath}_{\mathrm{A}}' + x_3\hat{k}_{\mathrm{A}}' \\ &= \vec{G}(\vec{x}; 1) \\ &= \vec{x}'.\end{aligned} \tag{3.3.10}$$

The metric tensor thus maps the vector $\vec{x}$ to the corresponding covector $\vec{x}'$. Likewise, the *cometric tensor* $\vec{G}' = \hat{\imath}_{\mathrm{A}} \otimes \hat{\imath}_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}} \otimes \hat{\jmath}_{\mathrm{A}} + \hat{k}_{\mathrm{A}} \otimes \hat{k}_{\mathrm{A}}$, which is of type $(2,0)$, converts the covector $\vec{x}'$ to the corresponding vector $\vec{x}$; that is, $(\vec{G}' \otimes \vec{x}')_{(1,1)} = \vec{x}$. Consequently,

$$(\vec{G}' \otimes (\vec{G} \otimes \vec{x})_{(1,1)})_{(1,1)} = (\vec{G}' \otimes \vec{x}')_{(1,1)} = \vec{x}, \tag{3.3.11}$$

$$(\vec{G} \otimes (\vec{G}' \otimes \vec{x}')_{(1,1)})_{(1,1)} = (\vec{G} \otimes \vec{x})_{(1,1)} = \vec{x}'. \tag{3.3.12}$$

Furthermore,

$$(\vec{G}' \otimes \vec{G})_{(1,1)} = (\vec{G} \otimes \vec{G}')_{(1,1)} = \vec{U}. \tag{3.3.13}$$

As another example, consider the identity (2.9.3) for the physical cross product matrix, that is,

$$(\hat{k}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}})\hat{\imath}'_{\mathrm{A}}\vec{x} + (\hat{\imath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}})\hat{\jmath}'_{\mathrm{A}}\vec{x} + (\hat{\jmath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}})\hat{k}'_{\mathrm{A}}\vec{x} = \vec{x}^{\times}. \tag{3.3.14}$$

Defining the $(1,2)$ tensor

$$\vec{T} = (\hat{k}_{\mathrm{A}} \otimes \hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}} \otimes \hat{k}'_{\mathrm{A}}) \otimes \hat{\imath}'_{\mathrm{A}} + (\hat{\imath}_{\mathrm{A}} \otimes \hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}} \otimes \hat{\imath}'_{\mathrm{A}}) \otimes \hat{\jmath}'_{\mathrm{A}} + (\hat{\jmath}_{\mathrm{A}} \otimes \hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}} \otimes \hat{\jmath}'_{\mathrm{A}}) \otimes \hat{k}'_{\mathrm{A}}, \tag{3.3.15}$$

it follows that

$$(\vec{T} \otimes \vec{x})_{(2,2)} = \vec{T}(\vec{x}; 2) = \vec{x}^{\times}. \tag{3.3.16}$$

A compact expression for the tensor (3.3.15) is given by

$$\vec{T} = -\sum_{i,j,k=1}^{n} \epsilon_{ijk}\hat{e}_i \otimes \hat{e}'_j \otimes e'_k, \tag{3.3.17}$$

where

$$\epsilon_{ijk} \triangleq \begin{cases} 1, & ijk \in \{123, 231, 312\}, \\ -1, & ijk \in \{321, 132, 213\}, \\ 0, & (i-j)(j-k)(k-i) = 0. \end{cases} \tag{3.3.18}$$

As another example, define the fourth-order tensor $\vec{U}_4 \triangleq \vec{U} \otimes \vec{U}$ of type $(2,2)$, and let $\vec{M} = \sum_{i,j=1}^{3} m_{ij}\hat{e}_i \otimes \hat{e}'_j$ be a physical matrix. Then,

$$\vec{U}_4 = \left(\sum_{i=1}^{3} \hat{e}_i \otimes \hat{e}'_i\right) \otimes \left(\sum_{j=1}^{3} \hat{e}_j \otimes \hat{e}'_j\right) = \sum_{i,j=1}^{3} \left(\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}'_i \otimes \hat{e}'_j\right). \tag{3.3.19}$$

Therefore,

$$\begin{aligned} \vec{U}_4 \otimes \vec{M} &= \sum_{i,j=1}^{3} \left(\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}'_i \otimes \hat{e}'_j\right) \otimes \sum_{k,l=1}^{3} m_{kl}\hat{e}_k \otimes \hat{e}_l \\ &= \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} m_{kl} \left(\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}'_i \otimes \hat{e}'_j \otimes e'_l\right) \end{aligned} \tag{3.3.20}$$

and thus

$$(\vec{U}_4 \otimes \vec{M})_{(2,3),(3,1)} = \sum_{i,j=1}^{3} m_{ij}\hat{e}_i \otimes \hat{e}'_j = \vec{M}. \tag{3.3.21}$$

Hence, with the contraction $(2,3),(3,1)$, $\vec{U}_4$ can be viewed as the identity tensor on the $\mathcal{T}_{(1,1)}$. Alternatively, note that

$$(\vec{U}_4 \otimes \vec{M})_{(2,3),(3,2)} = \sum_{i,j=1}^{3} m_{jj}\hat{e}_i \otimes \hat{e}_i' = (\operatorname{tr} \vec{M})\vec{U}. \tag{3.3.22}$$

3.4 Stress, Strain, and Elasticity Tensors

Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ denote the axes of an orthogonal frame, and consider stress and strain tensors, which are (1, 1) tensors, that is, physical matrices, of the form

$$\vec{\sigma} = \sum_{i,j=1}^{3} \sigma_{ij}\hat{e}_i\hat{e}_j', \tag{3.4.1}$$

$$\vec{\varepsilon} = \sum_{i,j=1}^{3} \varepsilon_{ij}\hat{e}_i\hat{e}_j'. \tag{3.4.2}$$

Next, define the *stiffness tensor* $\vec{\mathcal{K}}$, which is a fourth-order tensor of type (2, 2) of the form

$$\vec{\mathcal{K}} = \sum_{k,l,m,n=1}^{3} c_{klmn}\hat{e}_k \otimes \hat{e}_l \otimes \hat{e}_m' \otimes \hat{e}_n'.$$

The tensor product of $\vec{\mathcal{K}}$ and $\vec{\varepsilon}$ is given by

$$\vec{\mathcal{K}} \otimes \vec{\varepsilon} = \sum_{k,l,m,n=1}^{3} \sum_{i,j=1}^{3} c_{klmn}\varepsilon_{ij}(\hat{e}_k \otimes \hat{e}_l \otimes \hat{e}_i \otimes \hat{e}_m' \otimes \hat{e}_n' \otimes \hat{e}_j'). \tag{3.4.3}$$

Then, *Hooke's law* is the constitutive law given by the contracted tensor product

$$\vec{\sigma} = (\vec{\mathcal{K}} \otimes \vec{\varepsilon})_{(2,3),(3,1)}. \tag{3.4.4}$$

Therefore,

$$\begin{aligned}
\vec{\sigma} &= \sum_{k,l,m,n=1}^{3} \sum_{i,j=1}^{3} c_{klmn}\varepsilon_{ij}(\hat{e}_k \otimes \hat{e}_l \otimes \hat{e}_i \otimes \hat{e}_m' \otimes \hat{e}_n' \otimes \hat{e}_j')_{(2,3),(3,1)} \\
&= \sum_{k,l,m,n=1}^{3} c_{klmn} \sum_{i,j=1}^{3} \varepsilon_{ij}(\hat{e}_j'\hat{e}_l)(\hat{e}_m'\hat{e}_i)(\hat{e}_k \otimes \hat{e}_n') \\
&= \sum_{k,l,m,n=1}^{3} c_{klmn}\varepsilon_{ml}(\hat{e}_k \otimes \hat{e}_n') \\
&= \sum_{i,j=1}^{3} \sigma_{ij}\hat{e}_i \otimes \hat{e}_j',
\end{aligned}$$

where, for all $i, j = 1, 2, 3$,

$$\sigma_{ij} \triangleq \sum_{k,l=1}^{3} c_{iklj}\varepsilon_{lk}. \tag{3.4.5}$$

For physical reasons, the strain and stress tensors are symmetric, that is, $\vec{\sigma}' = \vec{\sigma}$ and $\vec{\varepsilon}' = \vec{\varepsilon}$. Therefore, for all $i, j = 1, 2, 3$, $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$. Since $\vec{\sigma}$ is symmetric, it follows that, for all $i, j = 1, 2, 3$, $c_{iklj} = c_{jkli}$. Consequently, $\mathcal{K}$ can be characterized by at most $3^4 - 3 \cdot 9 = 81 - 27 = 54$ parameters. Furthermore, since $\vec{\varepsilon}$ is symmetric, it follows that, for all $i, j = 1, 2, 3$, $c_{iklj} = c_{ilkj}$. Consequently, $\mathcal{K}$ can be characterized by at most $54 - 3 \cdot 6 = 54 - 18 = 36$ parameters. Alternatively, note that the symmetry of $\vec{\sigma}$ and $\vec{\varepsilon}$ implies that each of these tensors can be represented by a vector with six components. Consequently, (3.4.5) can be represented by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1113} & c_{1123} \\ c_{2211} & c_{2222} & c_{2233} & c_{2212} & c_{2213} & c_{2223} \\ c_{3311} & c_{3322} & c_{3333} & c_{3312} & c_{3313} & c_{3323} \\ c_{1211} & c_{1222} & c_{1233} & c_{1212} & c_{1213} & c_{1223} \\ c_{1311} & c_{1322} & c_{1333} & c_{1312} & c_{1313} & c_{1323} \\ c_{2311} & c_{2322} & c_{2333} & c_{2312} & c_{2313} & c_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}, \tag{3.4.6}$$

where the matrix in (3.4.6) has 36 entries.

Since the strain energy $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{1}{2} \sum_{i,j,k,l=1}^{3} c_{ikjl} \varepsilon_{ij} \varepsilon_{kl}, \tag{3.4.7}$$

it follows that, for all $i, j, k, l = 1, 2, 3$, $c_{ijkl} = c_{klij}$. The 6×6 matrix in (3.4.6) is thus characterized by $6 + 5 + 4 + 3 + 2 + 1 = 21$ (rather than 36) constants, and (3.4.6) can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1113} & c_{1123} \\ c_{1122} & c_{2222} & c_{2233} & c_{2212} & c_{2213} & c_{2223} \\ c_{1133} & c_{2233} & c_{3333} & c_{3312} & c_{3313} & c_{3323} \\ c_{1112} & c_{2212} & c_{3312} & c_{1212} & c_{1213} & c_{1223} \\ c_{1113} & c_{1322} & c_{3313} & c_{1213} & c_{1313} & c_{1323} \\ c_{1123} & c_{2322} & c_{3323} & c_{1223} & c_{1323} & c_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}. \tag{3.4.8}$$

For an isotropic material it can be shown that (3.4.8) can be characterized by 9 (rather than 21) constants, and (3.4.8) can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ c_{1122} & c_{2222} & c_{2233} & 0 & 0 & 0 \\ c_{1133} & c_{2233} & c_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}. \tag{3.4.9}$$

The 12 nonzero entries in (3.4.9) can be parameterized by two constants. Specifically,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}, \tag{3.4.10}$$

where the positive numbers μ and λ are the *Lame constants*. In tensor form, it follows from (3.4.10) that Hooke's law for isotropic materials has the form

$$\vec{\sigma} = 2\mu\vec{\varepsilon} + \lambda(\operatorname{tr}\vec{\varepsilon})\vec{I}, \tag{3.4.11}$$

which defines the fourth-order stiffness tensor $\vec{\mathcal{K}}$. In particular, it follows from (3.3.21) and (3.3.22) that

$$\begin{aligned}\vec{\sigma} &= (\vec{\mathcal{K}} \otimes \vec{\varepsilon})_{(2,3),(3,1)} \\ &= 2\mu\vec{\varepsilon} + \lambda(\operatorname{tr}\vec{\varepsilon})\vec{I} \\ &= 2\mu(\vec{U}_4 \otimes \vec{\varepsilon})_{(2,3),(3,1)} + \lambda(\vec{U}_4 \otimes \vec{\varepsilon})_{(2,3),(3,2)}.\end{aligned} \tag{3.4.12}$$

The inverse of the stiffness tensor is the *compliance tensor* $\vec{\mathcal{C}}$, which satisfies

$$\vec{\varepsilon} = (\vec{\mathcal{C}} \otimes \vec{\sigma})_{(2,3),(3,1)}. \tag{3.4.13}$$

The compliance tensor is represented by the relation

$$\vec{\varepsilon} = \frac{1+\nu}{E}\vec{\sigma} - \frac{\nu}{E}(\operatorname{tr}\vec{\sigma})\vec{I}, \tag{3.4.14}$$

where ν is *Poisson's ratio* and E is *Young's modulus*. In terms of μ and λ, the parameters ν and E are given by

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}. \tag{3.4.15}$$

Conversely, μ and λ are given in terms of ν and E by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{3.4.16}$$

3.5 Kronecker Algebra

For matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, we consider the Kronecker product $A \otimes B$. For details, see [1]. Note that "$\otimes$" is the same notation used for the tensor product. However, no confusion can occur since the tensor product is used for physical vectors and physical covectors, whereas the Kronecker product is used only for math vectors and math matrices. In this section we show that the Kronecker product can be used to resolve tensors.

For $A \in \mathbb{F}^{n\times m}$ define the *vec* operator as

$$\operatorname{vec} A \triangleq \begin{bmatrix} \operatorname{col}_1(A) \\ \vdots \\ \operatorname{col}_m(A) \end{bmatrix} \in \mathbb{F}^{nm}, \tag{3.5.1}$$

which is the column vector of size $nm \times 1$ obtained by stacking the columns of A. We recover A from $\operatorname{vec} A$ by writing

$$A = \operatorname{vec}^{-1}(\operatorname{vec} A). \tag{3.5.2}$$

Note that, if $x \in \mathbb{F}^n$, then $\operatorname{vec} x = \operatorname{vec} x^{\mathrm{T}} = x$.

Fact 3.5.1. Let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times n}$. Then,

$$\operatorname{tr} AB = (\operatorname{vec} A^{\mathrm{T}})^{\mathrm{T}} \operatorname{vec} B = (\operatorname{vec} B^{\mathrm{T}})^{\mathrm{T}} \operatorname{vec} A. \tag{3.5.3}$$

Next, we introduce the Kronecker product.

Definition 3.5.2. Let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{l\times k}$. Then, the *Kronecker product* $A \otimes B \in \mathbb{F}^{nl\times mk}$ of A and B is the partitioned matrix

$$A \otimes B \triangleq \begin{bmatrix} A_{(1,1)}B & A_{(1,2)}B & \cdots & A_{(1,m)}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{(n,1)}B & A_{(n,2)}B & \cdots & A_{(n,m)}B \end{bmatrix}. \tag{3.5.4}$$

Unlike matrix multiplication, the Kronecker product $A \otimes B$ does not entail a restriction on either the size of A or the size of B.

The following results are immediate consequences of the definition of the Kronecker product.

Fact 3.5.3. Let $\alpha \in \mathbb{F}$, $A \in \mathbb{F}^{n\times m}$, and $B \in \mathbb{F}^{l\times k}$. Then,

$$\alpha \otimes A = A \otimes \alpha = \alpha A, \quad A \otimes (\alpha B) = (\alpha A) \otimes B = \alpha (A \otimes B), \tag{3.5.5}$$

$$\overline{A \otimes B} = \overline{A} \otimes \overline{B}, \quad (A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}, \quad (A \otimes B)^{*} = A^{*} \otimes B^{*}. \tag{3.5.6}$$

Fact 3.5.4. Let $A, B \in \mathbb{F}^{n\times m}$ and $C \in \mathbb{F}^{l\times k}$. Then,

$$(A + B) \otimes C = A \otimes C + B \otimes C, \tag{3.5.7}$$

$$C \otimes (A + B) = C \otimes A + C \otimes B. \tag{3.5.8}$$

The next result shows that the Kronecker product is associative.

Fact 3.5.5. Let $A \in \mathbb{F}^{n\times m}$, $B \in \mathbb{F}^{l\times k}$, and $C \in \mathbb{F}^{p\times q}$. Then,

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C. \tag{3.5.9}$$

We thus write $A \otimes B \otimes C$ for $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$.

The next result shows how matrix multiplication interacts with the Kronecker product.

Fact 3.5.6. Let $A \in \mathbb{F}^{n\times m}$, $B \in \mathbb{F}^{l\times k}$, $C \in \mathbb{F}^{m\times q}$, and $D \in \mathbb{F}^{k\times p}$. Then,

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \tag{3.5.10}$$

Next, we consider the inverse of a Kronecker product.

Fact 3.5.7. Assume that $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{m\times m}$ are nonsingular. Then, $A \otimes B$ is nonsingular, and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{3.5.11}$$

Fact 3.5.8. Let $x \in \mathbb{F}^{n}$ and $y \in \mathbb{F}^{m}$. Then,

$$xy^{\mathrm{T}} = x \otimes y^{\mathrm{T}} = y^{\mathrm{T}} \otimes x, \tag{3.5.12}$$

$$\operatorname{vec} xy^{\mathrm{T}} = y \otimes x. \tag{3.5.13}$$

Fact 3.5.9. Let $A \in \mathbb{F}^{n\times m}$, $B \in \mathbb{F}^{m\times l}$, and $C \in \mathbb{F}^{l\times k}$. Then,

$$\operatorname{vec}(ABC) = (C^{\mathrm{T}} \otimes A)\operatorname{vec} B. \tag{3.5.14}$$

3.6 Composing Tensors

Let $\vec{T} \in \mathcal{T}_{(p,q)}$ be given by

$$\vec{T} = \sum_{i=1}^{r} \vec{w}_{1i} \otimes \cdots \otimes \vec{w}_{pi} \otimes \vec{z}\,'_{1i} \otimes \cdots \otimes \vec{z}\,'_{qi}, \tag{3.6.1}$$

and let F_{A} be a frame. Then we define

$$\vec{T}\Big|_{\mathrm{A}} \triangleq \sum_{i=1}^{r} \vec{w}_{1i}\Big|_{\mathrm{A}} \otimes \cdots \otimes \vec{w}_{pi}\Big|_{\mathrm{A}} \otimes \vec{z}_{1i}\Big|_{\mathrm{A}}^{\mathrm{T}} \otimes \cdots \otimes \vec{z}_{qi}\Big|_{\mathrm{A}}^{\mathrm{T}}. \tag{3.6.2}$$

Therefore, $\vec{T}\Big|_{\mathrm{A}} \in \mathbb{R}^{3^p\times 3^q}$.

The following result shows that the Kronecker product representation of tensors is compatible with tensor multiplication.

Fact 3.6.1. Let $\vec{T}_1 \in \mathcal{T}_{(p_1,q_1)}$ and $\vec{T}_2 \in \mathcal{T}_{(p_2,q_2)}$, and define $\vec{T}_3 \triangleq \vec{T}_1 \otimes \vec{T}_2 \in \mathcal{T}_{(p_1+p_2,q_1+q_2)}$. Then,

$$\vec{T}_3\Big|_{\mathrm{A}} = \vec{T}_1\Big|_{\mathrm{A}} \otimes \vec{T}_2\Big|_{\mathrm{A}}. \tag{3.6.3}$$

Finally, the Kronecker product representation of tensors is also compatible with restricted tensor multiplication. To show this, we need to introduce the *restricted Kronecker product*. Let $\vec{T}_1 \in \mathcal{T}_{(p_1,q_1)}$ and $\vec{T}_2 \in \mathcal{T}_{(p_2,q_2)}$, and define

$$T_1 \triangleq \vec{T}_1\Big|_{\mathrm{A}} = \sum \alpha_{j_1,\dots,j_p,k_1,\dots,k_q} e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes e_{k_1}^{\mathrm{T}} \otimes \cdots \otimes e_{k_q}^{\mathrm{T}} \tag{3.6.4}$$

and

$$T_2 \triangleq \vec{T}_2\Big|_{\mathrm{A}} = \sum \beta_{j_1,\dots,j_p,k_1,\dots,k_q} e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes e_{k_1}^{\mathrm{T}} \otimes \cdots \otimes e_{k_q}^{\mathrm{T}}. \tag{3.6.5}$$

where $e_i \triangleq \hat{e}_i|_{\mathrm{A}}$ for $i = 1,2,3$, and the summation is over all indices in the range $1,2,3$. Then, for example,

$$T_1 \otimes_{(j_p,k_q)} T_2 \triangleq \sum \alpha_{j_1,\dots,j_p,k_1,\dots,k_{q-1},j_p} \beta_{j_1,\dots,j_p,k_1,\dots,k_{q-1},j_p} e_{j_1} \otimes \cdots \otimes e_{j_{p-1}} \otimes e_{k_1}^{\mathrm{T}} \otimes \cdots \otimes e_{k_{q-1}}^{\mathrm{T}}. \tag{3.6.6}$$

3.7 Alternating Tensors and the Wedge Product

An *alternating tensor* (also called a *skew-symmetric tensor*) is a contravariant or covariant tensor whose sign changes when two of its arguments are interchanged. The set of alternating covariant tensors in $\mathcal{T}_{(0,q)}$ is denoted by $\hat{\mathcal{T}}_{(0,q)}$, while the set of alternating contravariant tensors in $\mathcal{T}_{(p,0)}$ is denoted by $\hat{\mathcal{T}}_{(p,0)}$. By definition, $\hat{\mathcal{T}}_{(0,1)} = \mathcal{T}_{(0,1)}$ and $\hat{\mathcal{T}}_{(1,0)} = \mathcal{T}_{(1,0)}$. The wedge product is used to construct alternating tensors.

3.7.1 Bivectors

Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then the *wedge product* (also called the *exterior product*) $\vec{x} \wedge \vec{y}$, of $\vec{x}$ and $\vec{y}$ is the $(2,0)$ tensor defined by

$$\vec{x} \wedge \vec{y} \triangleq \vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x}. \tag{3.7.1}$$

The wedge product satisfies the properties

$$\vec{x} \wedge \vec{x} = 0, \tag{3.7.2}$$

$$\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x}, \tag{3.7.3}$$

$$(\alpha\vec{x}) \wedge \vec{y} = \alpha(\vec{x} \wedge \vec{y}), \tag{3.7.4}$$

$$\vec{x} \wedge (\alpha\vec{y}) = \alpha(\vec{x} \wedge \vec{y}), \tag{3.7.5}$$

for all real numbers α. The identity (3.7.3) shows that $\vec{x} \wedge \vec{y}$ is an alternating tensor. The contravariant alternating tensor $\vec{x} \wedge \vec{y}$ is called a *bivector*.

The identity (3.7.1) is analogous to the physical cross-product matrix identity given by (2.9.19). In particular, note that

$$(\vec{x} \times \vec{y})^{\times} = \vec{y}\,\vec{x}\,' - \vec{x}\,\vec{y}\,' = \vec{y} \otimes \vec{x}\,' - \vec{x} \otimes \vec{y}\,', \tag{3.7.6}$$

which is a $(1,1)$ tensor.

Fact 3.7.1. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\begin{aligned}(\vec{x} \wedge \vec{y})\Big|_{\mathrm{A}} &= \operatorname{vec}\left[(\vec{x} \times \vec{y})^{\times}\Big|_{\mathrm{A}}\right] = \operatorname{vec}\left[(\vec{x} \times \vec{y})\Big|_{\mathrm{A}}^{\times}\right] \\ &= \left\|(\vec{x} \times \vec{y})^{\times}\Big|_{\mathrm{A}}\right\|_{\mathrm{F}} = \left\|(\vec{x} \times \vec{y})\Big|_{\mathrm{A}}^{\times}\right\|_{\mathrm{F}} \\ &= \operatorname{vec}\left[(\vec{x}\,\vec{y}\,' - \vec{y}\,\vec{x}\,')\Big|_{\mathrm{A}}\right] = \left\|(\vec{x}\,\vec{y}\,' - \vec{y}\,\vec{x}\,')\Big|_{\mathrm{A}}\right\|_{\mathrm{F}}.\end{aligned} \tag{3.7.7}$$

Now, define $x \triangleq \vec{x}\Big|_{\mathrm{A}}$ and $y \triangleq \vec{y}\Big|_{\mathrm{A}}$. Then,

$$\begin{aligned}x \otimes y - y \otimes x &= \operatorname{vec}(y \otimes x^{\mathrm{T}} - x \otimes y^{\mathrm{T}}) = \operatorname{vec}(yx^{\mathrm{T}} - xy^{\mathrm{T}}) \\ &= \operatorname{vec}[(x \times y)^{\times}] = \|(x \times y)^{\times}\|_{\mathrm{F}}.\end{aligned} \tag{3.7.8}$$

Proof. It follows from [1, p. 682] that $y \otimes x = \operatorname{vec}(x \otimes y^{\mathrm{T}}) = \operatorname{vec} xy^{\mathrm{T}}$, which proves the first and second equalities in (3.7.8). Next, it follows from (2.9.23) that $yx^{\mathrm{T}} - xy^{\mathrm{T}} = (x \times y)^{\times}$, which proves the third equality in (3.7.8). Finally, (3.7.7) is a restatement of the equality between the first and last terms in (3.7.8). □

As noted after Fact 2.9.6, the cross product $\vec{x} \times \vec{y}$ can be viewed as the directed area of a parallelogram. It thus follows from (3.7.7) that the bivector $\vec{x} \wedge \vec{y}$ can be viewed in the same way.

Fact 3.7.2. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors lying in the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. Then

$$\vec{x} \wedge \vec{y} = |\vec{x}||\vec{y}| \sin\theta_{\vec{y}/\vec{x}/\hat{k}_{\mathrm{A}}} \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}.$$

Proof. Let

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \quad \vec{y}\Big|_{\mathrm{A}} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}.$$

Then it follows from Problem 2.26.5 that

$$\begin{aligned} \vec{x} \wedge \vec{y} &= (x_1 \hat{\imath}_{\mathrm{A}} + x_2 \hat{\jmath}_{\mathrm{A}}) \wedge (y_1 \hat{\imath}_{\mathrm{A}} + y_2 \hat{\jmath}_{\mathrm{A}}) \\ &= (x_1 y_2 - x_2 y_1) \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \\ &= |\vec{x}||\vec{y}| \sin \theta_{\vec{y}/\vec{x}/\hat{k}_{\mathrm{A}}} \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}. \end{aligned}$$ □

The bivector $\vec{x} \wedge \vec{y}$ can be visualized as a planar region. This region can have the form of a parallelogram constructed by sweeping $\vec{y}$ along $\vec{x}$. The sides of the parallelogram are thus $\vec{x}$, $\vec{y}$, $-\vec{x}$, and $-\vec{y}$, and the magnitude of the bivector $\vec{x} \wedge \vec{y}$ is defined to be the area of the parallelogram, that is,

$$|\vec{x} \wedge \vec{y}| = |\vec{x}||\vec{y}| \sin \theta_{\vec{x}/\vec{y}} |\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}| = |\vec{x}||\vec{y}| \sin \theta_{\vec{x}/\vec{y}}, \tag{3.7.9}$$

where the area of the bivector $\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$ is the area of the unit square, namely, 1. Note that

$$|\vec{x} \wedge \vec{y}| = |\vec{x} \times \vec{y}|. \tag{3.7.10}$$

Therefore, the magnitude of $\vec{x} \wedge \vec{y}$ is the area of the bivector, its attitude is given by the plane within which the region lies, and its orientation is given by the direction determined by the right hand rule when $\vec{x}$ is rotated to $\vec{y}$, that is, the direction of $\vec{x} \times \vec{y}$. The shape of a bivector need not be a parallelogram, however; for example, it may be ellipsoidal. If the bivector is visualized as a square, then the length of each side is $\sqrt{|\vec{x}||\vec{y}| \sin \theta_{\vec{x}/\vec{y}}}$.

3.7.2 Trivectors

The tensor $\vec{T} = \vec{x} \wedge \vec{y}$ has the property that its sign changes when its arguments are interchanged. In particular, note that, if $\vec{u}'$ and $\vec{v}'$ are physical covectors, then

$$\begin{aligned} \vec{T}(\vec{u}', \vec{v}') &= (\vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x})(\vec{u}', \vec{v}') \\ &= \vec{u}'\vec{x}\vec{v}'\vec{y} - \vec{u}'\vec{y}\vec{v}'\vec{x} \\ &= -\vec{T}(\vec{v}', \vec{u}'). \end{aligned}$$

The wedge product can be applied to more than two physical vectors. For example, let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors. Then, $\vec{T} = \vec{x} \wedge \vec{y} \wedge \vec{z}$ is the $(3, 0)$ tensor defined by

$$\vec{x} \wedge \vec{y} \wedge \vec{z} \triangleq \vec{x} \otimes \vec{y} \otimes \vec{z} + \vec{y} \otimes \vec{z} \otimes \vec{x} + \vec{z} \otimes \vec{x} \otimes \vec{y} - \vec{x} \otimes \vec{z} \otimes \vec{y} - \vec{y} \otimes \vec{x} \otimes \vec{z} - \vec{z} \otimes \vec{y} \otimes \vec{x}. \tag{3.7.11}$$

Note that

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = -\vec{y} \wedge \vec{x} \wedge \vec{z} = \vec{y} \wedge \vec{z} \wedge \vec{x} = -\vec{z} \wedge \vec{y} \wedge \vec{x} = \vec{z} \wedge \vec{x} \wedge \vec{y} = -\vec{x} \wedge \vec{z} \wedge \vec{y}. \tag{3.7.12}$$

Consequently, if $\vec{u}'$, $\vec{v}'$, and $\vec{w}'$ are physical covectors, then

$$\vec{T}(\vec{u}', \vec{v}', \vec{w}') = -\vec{T}(\vec{v}', \vec{u}', \vec{w}') = \vec{T}(\vec{v}', \vec{w}', \vec{u}') = -\vec{T}(\vec{w}', \vec{v}', \vec{u}') = \vec{T}(\vec{w}', \vec{u}', \vec{v}') = -\vec{T}(\vec{u}', \vec{w}', \vec{v}'). \tag{3.7.13}$$

Let $\vec{T}$ denote a $(p, 0)$ tensor, let $\vec{w}'_1, \ldots, \vec{w}'_p$ be physical covectors, and let σ denote a permutation of the integers $1, \ldots, p$. Then, we define the *σ-permutation* $\vec{T}_\sigma$ of $\vec{T}$ by

$$\vec{T}_\sigma(\vec{w}'_1, \ldots, \vec{w}'_p) \triangleq \vec{T}(\vec{w}'_{\sigma(1)}, \ldots, \vec{w}'_{\sigma(p)}). \tag{3.7.14}$$

The parity $\operatorname{sign}(\sigma)$ of σ is either 1 or -1 depending on whether the number of transpositions of $\sigma(1), \ldots, \sigma(p)$ needed to reach $1, \ldots, p$ is even or odd, respectively. It is useful to note that, if $\vec{T} = \vec{x}_1 \otimes \cdots \otimes \vec{x}_p$, then $\vec{T}_\sigma = \vec{x}_{\sigma(1)} \otimes \cdots \otimes \vec{x}_{\sigma(p)}$.

The $(p, 0)$ tensor $\vec{T}$ is an *alternating tensor* if, for every permutation σ of the integers $1, \ldots, p$,

$$\vec{T}_\sigma = \operatorname{sign}(\sigma)\vec{T}. \tag{3.7.15}$$

Consequently, if $\vec{T}$ is an alternating tensor, then, for all physical covectors $\vec{w}'_1, \ldots, \vec{w}'_p$, it follows that

$$\vec{T}(\vec{w}'_{\sigma(1)}, \ldots, \vec{w}'_{\sigma(p)}) = \operatorname{sign}(\sigma)\vec{T}(\vec{w}'_1, \ldots, \vec{w}'_p). \tag{3.7.16}$$

Hence the sign of $\vec{T}(\vec{w}'_1, \ldots, \vec{w}'_p)$ changes whenever any two of its arguments are interchanged. As a special case of this definition, every physical vector is an alternating tensor.

The wedge product can be used to construct alternating tensors from each pair of alternating tensors. Let $\vec{T}_1$ and $\vec{T}_2$ be alternating tensors of orders $(p_1, 0)$ and $(p_2, 0)$, respectively. Then,

$$\vec{T}_1 \wedge \vec{T}_2 \triangleq \frac{1}{p_1!p_2!} \sum \operatorname{sign}(\sigma)(\vec{T}_1 \otimes \vec{T}_2)_\sigma, \tag{3.7.17}$$

where the summation is taken over all permutations σ of $1, \ldots, p_1 + p_2$.

To illustrate (3.7.17), let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors, and define $\vec{T}_1 = \vec{x}$ and $\vec{T}_2 = \vec{y} \wedge \vec{z}$. Then,

$$\begin{aligned}
\vec{x} \wedge (\vec{y} \wedge \vec{z}) &= \vec{T}_1 \wedge \vec{T}_2 \\
&= \frac{1}{2} \sum \operatorname{sign}(\sigma)[\vec{x} \otimes (\vec{y} \otimes \vec{z} - \vec{z} \otimes \vec{y})]_\sigma \\
&= \frac{1}{2} \sum \operatorname{sign}(\sigma)[\vec{x} \otimes \vec{y} \otimes \vec{z} - \vec{x} \otimes \vec{z} \otimes \vec{y}]_\sigma \\
&= \frac{1}{2}[\vec{x} \otimes \vec{y} \otimes \vec{z} - \vec{x} \otimes \vec{z} \otimes \vec{y} - (\vec{y} \otimes \vec{x} \otimes \vec{z} - \vec{z} \otimes \vec{x} \otimes \vec{y}) \\
&\quad - (\vec{z} \otimes \vec{y} \otimes \vec{x} - \vec{y} \otimes \vec{z} \otimes \vec{x}) - (\vec{x} \otimes \vec{z} \otimes \vec{y} - \vec{x} \otimes \vec{y} \otimes \vec{z}) \\
&\quad + \vec{y} \otimes \vec{z} \otimes \vec{x} - \vec{z} \otimes \vec{y} \otimes \vec{x} + \vec{z} \otimes \vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x} \otimes \vec{z}] \\
&= \vec{x} \wedge \vec{y} \wedge \vec{z}.
\end{aligned}$$

Hence the wedge product is associative for physical vectors, and we can write $\vec{x} \wedge \vec{y} \wedge \vec{z}$ without ambiguity. However, the cross product is not associative. Likewise, for alternating tensors $\vec{T}_1$, $\vec{T}_2$, and $\vec{T}_3$ of orders $(p_1, 0)$, $(p_2, 0)$, and $(p_3, 0)$, respectively, it follows that

$$\vec{T}_1 \wedge (\vec{T}_2 \wedge \vec{T}_3) = (\vec{T}_1 \wedge \vec{T}_2) \wedge \vec{T}_3. \tag{3.7.18}$$

The wedge product is thus associative for tensors, and we can write $\vec{T}_1 \wedge \vec{T}_2 \wedge \vec{T}_3$. In fact,

$$\vec{T}_1 \wedge \vec{T}_2 \wedge \vec{T}_3 = \frac{1}{p_1!p_2!p_3!} \sum \mathrm{sign}(\sigma)(\vec{T}_1 \otimes \vec{T}_2 \otimes \vec{T}_3)_\sigma. \tag{3.7.19}$$

For details, see [2, p. 260], [3, p. 278].

The following observation is useful.

Fact 3.7.3. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors. Then $\vec{x}$ and $\vec{y}$ are colinear if and only if $\vec{x} \wedge \vec{y} = 0$.

The following result shows that wedge products of three physical vectors are closely related to determinants.

Fact 3.7.4. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be nonzero physical vectors, and let $\mathrm{F_A}$ be a frame. Then

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = \det \begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix} \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} \tag{3.7.20}$$

Consequently, $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly dependent if and only if $\vec{x} \wedge \vec{y} \wedge \vec{z} = 0$.

Note that (3.7.20) can be written as

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = (\vec{x} \times \vec{y})' \vec{z} (\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}). \tag{3.7.21}$$

Fact 3.7.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then

$$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} = \hat{\imath}_{\mathrm{B}} \wedge \hat{\jmath}_{\mathrm{B}} \wedge \hat{k}_{\mathrm{B}}. \tag{3.7.22}$$

Proof. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be such that $(\vec{x} \times \vec{y})' \vec{z} \neq 0$. Then it follows from (3.7.21) that $\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} = \frac{1}{(\vec{x} \times \vec{y})' \vec{z}} \vec{x} \wedge \vec{y} \wedge \vec{z}$. Likewise, $\hat{\imath}_{\mathrm{B}} \wedge \hat{\jmath}_{\mathrm{B}} \wedge \hat{k}_{\mathrm{B}} = \frac{1}{(\vec{x} \times \vec{y})' \vec{z}} \vec{x} \wedge \vec{y} \wedge \vec{z}$. Hence (3.7.22) is satisfied. $\square$

The quantity $\vec{x} \wedge \vec{y} \wedge \vec{z}$ is called a *trivector*. The trivector $\vec{x} \wedge \vec{y} \wedge \vec{z}$ can be visualized as a parallelepiped three of whose edges are $\vec{x}$, $\vec{y}$, $\vec{z}$. This parallelepiped is constructed by sweeping the bivector $\vec{x} \wedge \vec{y}$ along $\vec{z}$. The magnitude of $\vec{x} \wedge \vec{y} \wedge \vec{z}$ is given by its volume

$$|\vec{x} \wedge \vec{y} \wedge \vec{z}| \triangleq \left| \det \begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix} \right|, \tag{3.7.23}$$

while its orientation is given by the sign of the determinant in (3.7.23).

Let $x, y, z \in \mathbb{R}^3$. Then, we define

$$x \wedge y \wedge z \triangleq x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y - x \otimes z \otimes y - y \otimes x \otimes z - z \otimes y \otimes x. \tag{3.7.24}$$

Fact 3.7.6. Let $x, y, z \in \mathbb{R}^3$, and let e_1, e_2, e_3 denote the columns of I_3. Then,

$$x \wedge y \wedge z = \det\,[x \;\; y \;\; z](e_1 \wedge e_2 \wedge e_3). \tag{3.7.25}$$

Therefore,

$$|\det[x\ \ y\ \ z]| = \frac{1}{\sqrt{6}}\|x \wedge y \wedge z\|. \tag{3.7.26}$$

3.7.3 Bicovectors, Tricovectors, and Forms

We define the *physical bicovector*

$$\vec{x}' \wedge \vec{y}' \triangleq \vec{x}' \otimes \vec{y}' - \vec{y}' \otimes \vec{x}'. \tag{3.7.27}$$

Note that

$$\vec{x}' \wedge \vec{y}' = (\vec{y} \otimes \vec{x} - \vec{x} \otimes \vec{y})' = (\vec{y} \wedge \vec{x})' = -(\vec{x} \wedge \vec{y})'. \tag{3.7.28}$$

Likewise, we define the *physical tricovector*

$$\begin{aligned}&\vec{x}' \wedge \vec{y}' \wedge \vec{z}'\\ &\quad\triangleq \vec{x}' \otimes \vec{y}' \otimes \vec{z}' + \vec{y}' \otimes \vec{z}' \otimes \vec{x}' + \vec{z}' \otimes \vec{x}' \otimes \vec{y}' - \vec{z}' \otimes \vec{y}' \otimes \vec{x}' - \vec{x}' \otimes \vec{z}' \otimes \vec{y}' - \vec{y}' \otimes \vec{x}' \otimes \vec{z}'\end{aligned} \tag{3.7.29}$$

Note that

$$\begin{aligned}\vec{x}' \wedge \vec{y}' \wedge \vec{z}' &= (\vec{z} \otimes \vec{y} \otimes \vec{x} + \vec{x} \otimes \vec{z} \otimes \vec{y} + \vec{y} \otimes \vec{x} \otimes \vec{z} - \vec{x} \otimes \vec{y} \otimes \vec{z} - \vec{y} \otimes \vec{z} \otimes \vec{x} - \vec{z} \otimes \vec{x} \otimes \vec{y})'\\ &= -(\vec{x} \otimes \vec{y} \otimes \vec{z} + \vec{y} \otimes \vec{z} \otimes \vec{x} + \vec{z} \otimes \vec{x} \otimes \vec{y} - \vec{z} \otimes \vec{y} \otimes \vec{x} - \vec{x} \otimes \vec{z} \otimes \vec{y} - \vec{y} \otimes \vec{x} \otimes \vec{z})'\\ &= -(\vec{x} \wedge \vec{y} \wedge \vec{z})'.\end{aligned} \tag{3.7.30}$$

A 0-form $\vec{\varphi}$ is a real-valued function $\vec{\varphi}\colon \mathcal{V} \to \mathbb{R}$. A 0-form is also called a *scalar field*.

A *1-form* $\vec{\varphi}$ is a mapping from the physical vectors to the set $\hat{\mathfrak{T}}_{(0,1)}$ of alternating covariant tensors, that is, $\vec{\varphi}\colon \mathcal{V} \to \hat{\mathfrak{T}}_{(0,1)}$. In particular, if $\vec{\varphi}$ is a 1-form, then, given the frame $\mathrm{F_A}$, there exist real-valued functions $\varphi_1, \varphi_2, \varphi_3$ on $\mathcal{V}$ such that

$$\vec{\varphi}(\vec{x}) = \varphi_1(\vec{x})\hat{\imath}'_{\mathrm{A}} + \varphi_2(\vec{x})\hat{\jmath}'_{\mathrm{A}} + \varphi_3(\vec{x})\hat{k}'_{\mathrm{A}}. \tag{3.7.31}$$

Note that every physical covector is a 1-form. A 1-form is also called a *covector field*.

A *2-form* $\vec{\varphi}$ is a mapping from the physical vectors to the set $\hat{\mathfrak{T}}_{(0,2)}$ of alternating covariant tensors, that is, $\vec{\varphi}\colon \mathcal{V} \to \hat{\mathfrak{T}}_{(0,2)}$. In particular, if $\vec{\varphi}$ is a 2-form, then, given the frame $\mathrm{F_A}$, there exist real-valued functions $\varphi_1, \varphi_2, \varphi_3$ on $\mathcal{V}$ such that

$$\vec{\varphi}(\vec{x}) = \varphi_1(\vec{x})\hat{\imath}'_{\mathrm{A}} \wedge \hat{\jmath}'_{\mathrm{A}} + \varphi_2(\vec{x})\hat{\jmath}'_{\mathrm{A}} \wedge \hat{k}'_{\mathrm{A}} + \varphi_3(\vec{x})\hat{k}'_{\mathrm{A}} \wedge \hat{\imath}'_{\mathrm{A}}. \tag{3.7.32}$$

Note that every physical bicovector is a 2-form. A 2-form is also called a *bicovector field*.

Finally, a *3-form* $\vec{\varphi}$ is a mapping from the physical vectors to the set $\hat{\mathfrak{T}}_{(0,3)}$ of alternating covariant tensors, that is, $\vec{\varphi}\colon \mathcal{V} \to \hat{\mathfrak{T}}_{(0,3)}$. In particular, if $\vec{\varphi}$ is a 3-form, then, given the frame $\mathrm{F_A}$, there exists a real-valued function φ_1 on $\mathcal{V}$ such that

$$\vec{\varphi}(\vec{x}) = \varphi_1(\vec{x})\hat{\imath}'_{\mathrm{A}} \wedge \hat{\jmath}'_{\mathrm{A}} \wedge \hat{k}'_{\mathrm{A}}. \tag{3.7.33}$$

Note that every physical tricovector is a 3-form. A 3-form is also called a *tricovector field*.

3.8 Multivectors

The real scalars α, vectors $\vec{x}$, bivectors $\vec{x} \wedge \vec{y}$, and trivectors $\vec{x} \wedge \vec{y} \wedge \vec{z}$ can be used to represent 0-1-, 2-, and 3-dimensional objects in 3-dimensional space. Since 3-dimensional space is spanned by the basis vectors $\hat{\imath}_{\mathrm{A}}$, $\hat{\jmath}_{\mathrm{A}}$, $\hat{k}_{\mathrm{A}}$ of the frame F_{A}, we can view 3-dimensional space as an 8-dimensional linear vector space spanned by 1, $\hat{\imath}_{\mathrm{A}}$, $\hat{\jmath}_{\mathrm{A}}$, $\hat{k}_{\mathrm{A}}$, $\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$, $\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$, $\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$, and $\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$. The elements of this space are *multivectors* $\overset{\twoheadrightarrow}{S}$, which have the form

$$\begin{aligned}\overset{\twoheadrightarrow}{S} = {} & \alpha + \beta\hat{\imath}_{\mathrm{A}} + \gamma\hat{\jmath}_{\mathrm{A}} + \delta\hat{k}_{\mathrm{A}} + \varepsilon\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} + \phi\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} + \psi\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}} \\ & + \rho\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}},\end{aligned} \tag{3.8.1}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \phi, \psi, \rho$ are real numbers. The trivector

$$\vec{\mathcal{I}} \triangleq \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$$

is called the *pseudoscalar*. Fact 3.7.5 shows that $\vec{\mathcal{I}}$ is independent of the choice of frame. In terms of $\vec{\mathcal{I}}$, (3.7.20) can be rewritten as

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = \det\begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix}\vec{\mathcal{I}}. \tag{3.8.2}$$

Multivectors can be multiplied by introducing a suitable multiplication operation. In particular, the *geometric product* of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x}\vec{y} \triangleq \vec{x} \cdot \vec{y} + \vec{x} \wedge \vec{y}. \tag{3.8.3}$$

We can thus write (3.8.3) as

$$\vec{x}\vec{y} = \vec{x} \cdot \vec{y} + \vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x}. \tag{3.8.4}$$

Note that $\vec{x}\vec{y}$ is a multivector since it is a linear combination of a scalar and a bivector.

Since $\vec{y} \wedge \vec{x} = -\vec{x} \wedge \vec{y}$, we have the identity

$$\vec{y}\vec{x} = \vec{x} \cdot \vec{y} - \vec{x} \wedge \vec{y}. \tag{3.8.5}$$

Hence,

$$\vec{x}\vec{y} + \vec{y}\vec{x} = 2\vec{x} \cdot \vec{y}, \tag{3.8.6}$$

$$\vec{x}\vec{y} - \vec{y}\vec{x} = 2\vec{x} \wedge \vec{y}. \tag{3.8.7}$$

Therefore,

$$\vec{x} \cdot \vec{y} = \tfrac{1}{2}(\vec{x}\vec{y} + \vec{y}\vec{x}), \tag{3.8.8}$$

$$\vec{x} \wedge \vec{y} = \tfrac{1}{2}(\vec{x}\vec{y} - \vec{y}\vec{x}). \tag{3.8.9}$$

Furthermore,

$$(\vec{x} \wedge \vec{y})^2 = (\vec{x} \cdot \vec{y})^2 - \vec{y}^2\vec{x}^2. \tag{3.8.10}$$

Note that, if $\vec{x}$ and $\vec{y}$ are orthogonal, then

$$\vec{x}\vec{y} = \vec{x} \wedge \vec{y}. \tag{3.8.11}$$

The geometric product can be applied to an arbitrary collection of multivectors, for example, $\vec{\vec{S}}_1\vec{\vec{S}}_2\cdots\vec{\vec{S}}_r$. This product is associative, that is, $\vec{\vec{S}}_1(\vec{\vec{S}}_2\vec{\vec{S}}_3) = (\vec{\vec{S}}_1\vec{\vec{S}}_2)\vec{\vec{S}}_3$, which thus can be written as $\vec{\vec{S}}_1\vec{\vec{S}}_2\vec{\vec{S}}_3$. In particular, the geometric product of three physical vectors is given by

$$\vec{x}\vec{y}\vec{z} = (\vec{y}\cdot\vec{z})\vec{x} - (\vec{x}\cdot\vec{z})\vec{y} + (\vec{x}\cdot\vec{y})\vec{z} + \vec{x}\wedge\vec{y}\wedge\vec{z}. \tag{3.8.12}$$

We thus have the identities

$$\vec{x}\vec{y}\vec{x} = 2(\vec{x}\cdot\vec{y})\vec{x} - |\vec{x}|^2\vec{y}, \tag{3.8.13}$$

$$\vec{x}\vec{y}\vec{z} + \vec{y}\vec{x}\vec{z} = 2(\vec{x}\cdot\vec{y})\vec{z}, \tag{3.8.14}$$

$$\vec{x}\vec{y}\vec{z} + \vec{z}\vec{y}\vec{x} = 2[(\vec{y}\cdot\vec{z})\vec{x} - (\vec{x}\cdot\vec{z})\vec{y} + (\vec{x}\cdot\vec{y})\vec{z}], \tag{3.8.15}$$

$$\vec{x}\vec{y}\vec{z} - \vec{y}\vec{z}\vec{x} = 2[(\vec{x}\cdot\vec{y})\vec{z} - (\vec{x}\cdot\vec{z})\vec{y}], \tag{3.8.16}$$

$$\begin{aligned}\vec{x}(\vec{y}\wedge\vec{z}) &= \tfrac{1}{2}(\vec{x}\vec{y}\vec{z} - \vec{x}\vec{z}\vec{y}) && (3.8.17)\\ &= (\vec{x}\cdot\vec{y})\vec{z} - (\vec{x}\cdot\vec{z})\vec{y} + \tfrac{1}{2}(\vec{z}\vec{x}\vec{y} - \vec{y}\vec{x}\vec{z}), && (3.8.18)\end{aligned}$$

$$\begin{aligned}(\vec{x}\wedge\vec{y})\vec{z} &= \tfrac{1}{2}(\vec{x}\vec{y}\vec{z} - \vec{y}\vec{x}\vec{z}) && (3.8.19)\\ &= (\vec{y}\cdot\vec{z})\vec{x} - (\vec{x}\cdot\vec{z})\vec{y} + \tfrac{1}{2}(\vec{y}\vec{z}\vec{x} - \vec{x}\vec{z}\vec{y}), && (3.8.20)\end{aligned}$$

$$\vec{x}(\vec{y}\wedge\vec{z}) - (\vec{y}\wedge\vec{z})\vec{x} = 2(\vec{x}\cdot\vec{y})\vec{z} - 2(\vec{x}\cdot\vec{z})\vec{y}, \tag{3.8.21}$$

$$\begin{aligned}\vec{x}\wedge\vec{y}\wedge\vec{z} &= \tfrac{1}{2}[\vec{x}(\vec{y}\wedge\vec{z}) + (\vec{y}\wedge\vec{z})\vec{x}] && (3.8.22)\\ &= \tfrac{1}{2}(\vec{x}\vec{y}\vec{z} - \vec{z}\vec{y}\vec{x}) && (3.8.23)\\ &= \tfrac{1}{4}(\vec{x}\vec{y}\vec{z} + \vec{y}\vec{z}\vec{x} - \vec{x}\vec{z}\vec{y} - \vec{z}\vec{y}\vec{x}) && (3.8.24)\\ &= \tfrac{1}{6}(\vec{x}\vec{y}\vec{z} + \vec{y}\vec{z}\vec{x} + \vec{z}\vec{x}\vec{y} - \vec{x}\vec{z}\vec{y} - \vec{y}\vec{x}\vec{z} - \vec{z}\vec{y}\vec{z}). && (3.8.25)\end{aligned}$$

For comparison, note that (3.7.11) states that

$$\vec{x}\wedge\vec{y}\wedge\vec{z} = \vec{x}\otimes\vec{y}\otimes\vec{z} + \vec{y}\otimes\vec{z}\otimes\vec{x} + \vec{z}\otimes\vec{x}\otimes\vec{y} - \vec{x}\otimes\vec{z}\otimes\vec{y} - \vec{y}\otimes\vec{x}\otimes\vec{z} - \vec{z}\otimes\vec{y}\otimes\vec{x}. \tag{3.8.26}$$

Furthermore,

$$\vec{\mathfrak{I}}^2 = -1. \tag{3.8.27}$$

For the case of four physical vectors we have

$$\vec{x}\wedge\vec{y}\wedge\vec{z}\wedge\vec{w} = 0, \tag{3.8.28}$$

$$\begin{aligned}\vec{x}\vec{y}\vec{z}\vec{w} = {}&(\vec{x}\cdot\vec{y})(\vec{z}\cdot\vec{w}) - (\vec{x}\cdot\vec{z})(\vec{y}\cdot\vec{w}) + (\vec{x}\cdot\vec{w})(\vec{y}\cdot\vec{z})\\ &+ (\vec{z}\cdot\vec{w})\vec{x}\wedge\vec{y} - (\vec{y}\cdot\vec{w})\vec{x}\wedge\vec{z} + (\vec{y}\cdot\vec{z})\vec{x}\wedge\vec{w}\\ &+ (\vec{x}\cdot\vec{w})\vec{y}\wedge\vec{z} - (\vec{x}\cdot\vec{z})\vec{y}\wedge\vec{w} + (\vec{x}\cdot\vec{y})\vec{z}\wedge\vec{w}.\end{aligned} \tag{3.8.29}$$

Consequently, for each frame $\mathrm{F_A}$,

$$\vec{\mathfrak{I}}\vec{x} = (\hat{\imath}_\mathrm{A} \cdot \vec{x})\hat{\jmath}_\mathrm{A} \wedge \hat{k}_\mathrm{A} + (\hat{\jmath}_\mathrm{A} \cdot \vec{x})\hat{k}_\mathrm{A} \wedge \hat{\imath}_\mathrm{A} + (\hat{k}_\mathrm{A} \cdot \vec{x})\hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A}. \tag{3.8.30}$$

The following equalities connect the geometric product with the cross product.

Fact 3.8.1. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{x} \times \vec{y} = -\vec{\mathfrak{I}}(\vec{x} \wedge \vec{y}), \tag{3.8.31}$$

$$\vec{x} \wedge \vec{y} = \vec{\mathfrak{I}}(\vec{x} \times \vec{y}). \tag{3.8.32}$$

Furthermore,

$$\vec{x} \wedge \vec{y} = \begin{bmatrix} \hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A} & \hat{\jmath}_\mathrm{A} \wedge \hat{k}_\mathrm{A} & \hat{k}_\mathrm{A} \wedge \hat{\imath}_\mathrm{A} \end{bmatrix} (\vec{x} \times \vec{y})\Big|_\mathrm{A}. \tag{3.8.33}$$

Moreover,

$$\vec{x} \times \vec{y} = \tfrac{1}{2}(\vec{y}\vec{\mathfrak{I}}\vec{x} - \vec{\mathfrak{I}}\vec{x}\vec{y}). \tag{3.8.34}$$

Finally, let $\vec{z}$ be a physical vector. Then

$$\vec{x} \cdot (\vec{y} \times \vec{z}) = \det\begin{bmatrix} \vec{x}\big|_\mathrm{A} & \vec{y}\big|_\mathrm{A} & \vec{z}\big|_\mathrm{A} \end{bmatrix} = -(\vec{x} \wedge \vec{y} \wedge \vec{z})\vec{\mathfrak{I}}, \tag{3.8.35}$$

$$\vec{x} \times (\vec{y} \times \vec{z}) = \tfrac{1}{2}[(\vec{y} \wedge \vec{z})\vec{x} - \vec{x}(\vec{y} \wedge \vec{z})]. \tag{3.8.36}$$

Proof. Note that

$$\begin{aligned}
-\vec{\mathfrak{I}}(\vec{x} \wedge \vec{y}) &= -\vec{\mathfrak{I}}[(x_1\hat{\imath}_\mathrm{A} + x_2\hat{\jmath}_\mathrm{A} + x_3\hat{k}_\mathrm{A}) \wedge (y_1\hat{\imath}_\mathrm{A} + y_2\hat{\jmath}_\mathrm{A} + y_3\hat{k}_\mathrm{A})] \\
&= -\vec{\mathfrak{I}}[(x_1y_2 - y_1x_2)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A} + (x_1y_3 - y_1x_3)\hat{\imath}_\mathrm{A}\hat{k}_\mathrm{A} + (x_2y_3 - y_2x_3)\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}] \\
&= (y_1x_2 - x_1y_2)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A} + (y_1x_3 - x_1y_3)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}\hat{\imath}_\mathrm{A}\hat{k}_\mathrm{A} + (y_2x_3 - x_2y_3)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A} \\
&= -(y_1x_2 - x_1y_2)\hat{k}_\mathrm{A} + (y_1x_3 - x_1y_3)\hat{\jmath}_\mathrm{A} - (y_2x_3 - x_2y_3)\hat{\imath}_\mathrm{A} \\
&= (x_2y_3 - y_2x_3)\hat{\imath}_\mathrm{A} + (y_1x_3 - x_1y_3)\hat{\jmath}_\mathrm{A} + (x_1y_2 - y_1x_2)\hat{k}_\mathrm{A} \\
&= \vec{x} \times \vec{y}.
\end{aligned}$$

□

The matrix in (3.8.33) is called a *bivectrix*.

Let $\mathrm{F_A}$ be a frame, and consider the multivector $\vec{\vec{S}}$

$$\vec{\vec{S}} = \alpha + \beta\hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A}, \tag{3.8.37}$$

where $\vec{\vec{S}}$ is a linear combination of a scalar and a bivector. Since $\hat{\imath}_\mathrm{A} \cdot \hat{\jmath}_\mathrm{A} = 0$, it follows that $\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A} = \hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A} = -\hat{\jmath}_\mathrm{A} \wedge \hat{\imath}_\mathrm{A} = -\hat{\jmath}_\mathrm{A}\hat{\imath}_\mathrm{A}$. Therefore, since $\hat{\imath}_\mathrm{A}\hat{\imath}_\mathrm{A} = \hat{\jmath}_\mathrm{A}\hat{\jmath}_\mathrm{A} = 1$, it follows that

$$(\hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A})^2 = (\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A})^2 = \hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A} = -\hat{\imath}_\mathrm{A}\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{\jmath}_\mathrm{A} = -1. \tag{3.8.38}$$

Therefore, $\hat{\imath}_{\rm A}\hat{\jmath}_{\rm A}$ behaves like $\jmath = \sqrt{-1}$. It is therefore useful to define the conjugate $\overline{\overset{\twoheadrightarrow}{S}}$ of $\overset{\twoheadrightarrow}{S}$ as

$$\overline{\overset{\twoheadrightarrow}{S}} = \alpha - \beta\hat{\imath}_{\rm A} \wedge \hat{\jmath}_{\rm A}. \tag{3.8.39}$$

We thus have

$$\overline{\overset{\twoheadrightarrow}{S}}\overset{\twoheadrightarrow}{S} = \overset{\twoheadrightarrow}{S}\overline{\overset{\twoheadrightarrow}{S}} = \alpha^2 + \beta^2. \tag{3.8.40}$$

More generally, for $\overset{\twoheadrightarrow}{S}$ given by (3.8.1), the conjugate of $\overset{\twoheadrightarrow}{S}$ is given by

$$\overline{\overset{\twoheadrightarrow}{S}} \triangleq \alpha - \beta\hat{\imath}_{\rm A} - \gamma\hat{\jmath}_{\rm A} - \delta\hat{k}_{\rm A} - \varepsilon\hat{\imath}_{\rm A} \wedge \hat{\jmath}_{\rm A} - \phi\hat{\jmath}_{\rm A} \wedge \hat{k}_{\rm A} - \psi\hat{k}_{\rm A} \wedge \hat{\imath}_{\rm A} - \rho\hat{\imath}_{\rm A} \wedge \hat{\jmath}_{\rm A} \wedge \hat{k}_{\rm A}. \tag{3.8.41}$$

We now give multiplication tables to illustrate the geometric product. Subsets of the basis multivectors have multiplication tables that are equivalent to those of alternative objects, such as the complex numbers and the quaternions. These objects involve 2×2 matrices, such as the skew-symmetric matrix

$$J_2 \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the *Pauli matrices* $\sigma_1, \sigma_2, \sigma_3$ and their products, which are given by

$$1 \longleftrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.8.42}$$

$$\hat{\imath}_{\rm A} \longleftrightarrow \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.8.43}$$

$$\hat{\jmath}_{\rm A} \longleftrightarrow \sigma_2 = \begin{bmatrix} 0 & -\jmath \\ \jmath & 0 \end{bmatrix}, \tag{3.8.44}$$

$$\hat{k}_{\rm A} \longleftrightarrow \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.8.45}$$

$$\hat{\imath}_{\rm A} \wedge \hat{\jmath}_{\rm A} \longleftrightarrow \sigma_1\sigma_2 = \jmath\sigma_3 = -\sigma_2\sigma_1 = \begin{bmatrix} \jmath & 0 \\ 0 & -\jmath \end{bmatrix}, \tag{3.8.46}$$

$$\hat{\jmath}_{\rm A} \wedge \hat{k}_{\rm A} \longleftrightarrow \sigma_2\sigma_3 = \jmath\sigma_1 = -\sigma_3\sigma_2 = \begin{bmatrix} 0 & \jmath \\ \jmath & 0 \end{bmatrix}, \tag{3.8.47}$$

$$\hat{k}_{\rm A} \wedge \hat{\imath}_{\rm A} \longleftrightarrow \sigma_3\sigma_1 = \jmath\sigma_2 = -\sigma_1\sigma_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.8.48}$$

$$\hat{\imath}_{\rm A} \wedge \hat{\jmath}_{\rm A} \wedge \hat{k}_{\rm A} \longleftrightarrow \sigma_1\sigma_2\sigma_3 = \begin{bmatrix} \jmath & 0 \\ 0 & \jmath \end{bmatrix}. \tag{3.8.49}$$

The equivalence of these tables shows that multivectors provide a unified description of diverse mathematical structures.

Multiplication tables involving two multivectors are given in Table 3.8.1. Equivalent tables are given in terms of 2×2 real matrices as well as the complex scalars.

A multiplication table involving four multivectors is given in Table 3.8.2 together with an equivalent multiplication table involving 2×2 matrices. This table extends the multiplication table of Table 2.16.1(a).

A multiplication table involving four multivectors is given in Table 3.8.2 together with equivalent multiplication tables involving 2×2 matrices and the basic quaternions. Note the correspondence

$$\hat{j}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} \longleftrightarrow \sigma_2\sigma_3 \longleftrightarrow \mathbf{i}, \tag{3.8.50}$$

$$\hat{\imath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} \longleftrightarrow \sigma_1\sigma_3 \longleftrightarrow \mathbf{j}, \tag{3.8.51}$$

$$\hat{\imath}_{\mathrm{A}} \wedge \hat{j}_{\mathrm{A}} \longleftrightarrow \sigma_1\sigma_2 \longleftrightarrow \mathbf{k}. \tag{3.8.52}$$

A multiplication table involving eight multivectors is given in Table 3.8.4. Table 3.8.5 is an equivalent multiplication table given in terms of the 2×2 Pauli matrices and their products.

	1	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$
1	1	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$
$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	-1

(a)

	1	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$
1	1	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$
$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	-1

(b)

	I_2	J_2
I_2	I_2	J_2
J_2	J_2	$-I_2$

(c)

	I_2	$\sigma_1\sigma_2$
I_2	I_2	$\sigma_1\sigma_2$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$-I_2$

(d)

	1	$\jmath$
1	1	$\jmath$
$\jmath$	$\jmath$	-1

(e)

Figure 3.8.1: Equivalent multiplication tables for two multivectors (a), (b), 2×2 matrices (c), (d), and the complex numbers (e).

	1	$\hat{\imath}_A$	$\hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$
1	1	$\hat{\imath}_A$	$\hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$
$\hat{\imath}_A$	$\hat{\imath}_A$	1	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\jmath}_A$
$\hat{\jmath}_A$	$\hat{\jmath}_A$	$-\hat{\imath}_A \wedge \hat{\jmath}_A$	1	$-\hat{\imath}_A$
$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$-\hat{\jmath}_A$	$\hat{\imath}_A$	-1

(a)

	I_2	σ_1	σ_2	$\sigma_1\sigma_2$
I_2	I_2	σ_1	σ_2	$\sigma_1\sigma_2$
σ_1	σ_1	I_2	$\sigma_1\sigma_2$	σ_2
σ_2	σ_2	$-\sigma_1\sigma_2$	I_2	$-\sigma_1$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$-\sigma_2$	σ_1	$-I_2$

(b)

Figure 3.8.2: Equivalent multiplication tables involving four multivectors (a) and Pauli matrices (b).

	1	$\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{\imath}_A \wedge \hat{k}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$
1	1	$\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{\imath}_A \wedge \hat{k}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$
$\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{\jmath}_A \wedge \hat{k}_A$	-1	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$-\hat{\imath}_A \wedge \hat{k}_A$
$\hat{\imath}_A \wedge \hat{k}_A$	$\hat{\imath}_A \wedge \hat{k}_A$	$-\hat{\imath}_A \wedge \hat{\jmath}_A$	-1	$\hat{\jmath}_A \wedge \hat{k}_A$
$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\jmath}_A \wedge \hat{k}_A$	$-\hat{\jmath}_A \wedge \hat{k}_A$	-1

(a)

	I_2	$\sigma_2\sigma_3$	$\sigma_1\sigma_3$	$\sigma_1\sigma_2$
I_2	I_2	$\sigma_2\sigma_3$	$\sigma_1\sigma_3$	$\sigma_1\sigma_2$
$\sigma_1\sigma_3$	$\sigma_1\sigma_3$	$-I_2$	$\sigma_1\sigma_2$	$-\sigma_1\sigma_3$
$\sigma_1\sigma_3$	$\sigma_1\sigma_3$	$-\sigma_1\sigma_2$	$-I_2$	$\sigma_2\sigma_3$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$\sigma_2\sigma_3$	$-\sigma_2\sigma_3$	$-I_2$

(b)

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

(c)

Figure 3.8.3: Equivalent multiplication tables for multivectors (a), products of Pauli matrices (b), and the quaternions (c).

	1	$\hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\vec{\jmath}$
1	1	$\hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\vec{\jmath}$
$\hat{\imath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}}$	1	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\vec{\jmath}$	$-\hat{k}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$
$\hat{\jmath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	1	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\vec{\jmath}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$
$\hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	1	$\vec{\jmath}$	$-\hat{\jmath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$
$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}}$	$\vec{\jmath}$	-1	$-\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$-\hat{k}_{\mathrm{A}}$
$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\vec{\jmath}$	$-\hat{k}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	-1	$-\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}}$
$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\vec{\jmath}$	$-\hat{\imath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	-1	$-\hat{\jmath}_{\mathrm{A}}$
$\vec{\jmath}$	$\vec{\jmath}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{k}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}}$	-1

Figure 3.8.4: Multiplication table involving eight multivectors.

	I_2	σ_1	σ_2	σ_3	$\sigma_1\sigma_2$	$\sigma_2\sigma_3$	$\sigma_3\sigma_1$	$\sigma_1\sigma_2\sigma_3$
I_2	I_2	σ_1	σ_2	σ_3	$\sigma_1\sigma_2$	$\sigma_2\sigma_3$	$\sigma_3\sigma_1$	$\sigma_1\sigma_2\sigma_3$
σ_1	σ_1	I_2	$\sigma_1\sigma_2$	$-\sigma_3\sigma_1$	$\hat{j}_{\mathrm{A}}$	$\sigma_1\sigma_2\sigma_3$	$-\sigma_3$	$\sigma_2\sigma_3$
σ_2	σ_2	$-\sigma_1\sigma_2$	I_2	$\sigma_2\sigma_3$	$-\sigma_1$	σ_3	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_1$
σ_3	σ_3	$\sigma_3\sigma_1$	$-\sigma_2\sigma_3$	I_2	$\sigma_1\sigma_2\sigma_3$	$-\sigma_2$	σ_1	$\sigma_1\sigma_2$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$-\sigma_2$	σ_1	$\sigma_1\sigma_2\sigma_3$	$-I_2$	$-\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$-\hat{k}_{\mathrm{A}}$
$\sigma_2\sigma_3$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$-\sigma_3$	σ_2	$\sigma_3\sigma_1$	$-I_2$	$-\sigma_1\sigma_2$	$-\sigma_1$
$\sigma_3\sigma_1$	$\sigma_3\sigma_1$	σ_3	$\sigma_1\sigma_2\sigma_3$	$-\sigma_1$	$-\sigma_2\sigma_3$	$\sigma_1\sigma_2$	$-I_2$	$-\sigma_2$
$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_2\sigma_3$	$\sigma_3\sigma_1$	$\sigma_1\sigma_2$	$-\sigma_3$	$-\sigma_1$	$-\sigma_2$	$-I_2$

Figure 3.8.5: Multiplication table for the Pauli matrices and their products, which is equivalent to the multiplication table in Table 3.8.4.

3.9 Rotations and Reflections

Let $\hat{v}$ and $\hat{w}$ be orthogonal vectors, and let θ denote an angle. Then the *rotor* $\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)$ is the multivector

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta) \triangleq \cos\theta/2 - (\sin\theta/2)(\hat{v}\wedge\hat{w}), \tag{3.9.1}$$

which is a linear combination of a scalar and a bivector. Therefore,

$$\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta) = \cos\theta/2 + (\sin\theta/2)(\hat{v}\wedge\hat{w}). \tag{3.9.2}$$

The following result shows that the rotor $\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)$ rotates each vector by the angle θ around the direction orthogonal to the plane spanned by $\hat{v}$ and $\hat{w}$ and in the direction determined by the orientation of the bivector $\hat{v}\wedge\hat{w}$.

Since $(\hat{v}\wedge\hat{w})^2 = -1$, (3.9.1) can be written as

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta) = \exp[-(\hat{v}\wedge\hat{w})\theta/2]. \tag{3.9.3}$$

In other words, (3.9.1) is analogous to Euler's formula $e^{\jmath\theta} = \cos\theta + \jmath\sin\theta$. Furthermore, note that

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta) = \overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta)\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta) = 1. \tag{3.9.4}$$

Fact 3.9.1. Let θ be an angle, let $\hat{v}$ and $\hat{w}$ denote orthogonal vectors, and let $\hat{n} = \hat{v}\times\hat{w}$. Then, for every physical vector $\vec{x}$,

$$\vec{R}_{\hat{n}}(\theta)\vec{x} = \overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)\vec{x}\,\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta). \tag{3.9.5}$$

Proof. Note that

$$\begin{aligned}
&\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)\vec{x}\,\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta)\\
&\quad= [\cos\theta/2 - (\sin\theta/2)(\hat{v}\wedge\hat{w})]\vec{x}[\cos\theta/2 + (\sin\theta/2)(\hat{v}\wedge\hat{w})] \qquad (3.9.6)\\
&\quad= [\cos\theta/2 - (\sin\theta/2)\hat{v}\hat{w}]\vec{x}[\cos\theta/2 + (\sin\theta/2)\hat{v}\hat{w}]\\
&\quad= (\cos^2\theta/2)\vec{x} + \tfrac{1}{2}(\sin\theta)(\hat{x}\hat{v}\hat{w} - \hat{v}\hat{w}\hat{x}) + (\sin^2\theta/2)\hat{v}\hat{w}\hat{x}\hat{w}\hat{v}\\
&\quad= (\cos^2\theta/2)\vec{x} + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}](\sin^2\theta/2)[\vec{x} - 2(\hat{v}\cdot\hat{x})\hat{v} - 2(\hat{w}\cdot\hat{x})\hat{w}]\\
&\quad= \vec{x} + [(\cos\theta) - 1][(\hat{v}\cdot\hat{x})\hat{v} + (\hat{w}\cdot\hat{x})\hat{w}] + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)(\hat{w}\otimes\hat{v}' - \hat{v}\otimes\hat{w}')\otimes_{(2,1)}[(\hat{v}\cdot\vec{x})\hat{w} - (\hat{w}\cdot\vec{x})\hat{v}] + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)(\hat{w}\otimes\hat{v}' - \hat{v}\otimes\hat{w}')^2\otimes_{(2,1)}\vec{x} + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)(\hat{v}\times\hat{w})^{\times 2}\otimes_{(2,1)}\vec{x} + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)\hat{n}^{\times 2}\otimes_{(2,1)}\vec{x} + (\sin\theta)(\hat{w}\otimes\hat{v}' - \hat{v}\otimes\hat{w}')\otimes_{(2,1)}\vec{x}\\
&\quad= \vec{x} + (1-\cos\theta)\hat{n}^{\times 2}\otimes_{(2,1)}\vec{x} + (\sin\theta)\hat{n}^{\times}\otimes_{(2,1)}\vec{x}\\
&\quad= \vec{R}_{\hat{n}}(\theta)\otimes_{(2,1)}\vec{x}
\end{aligned}$$

$$= \vec{R}_{\hat{n}}(\theta)\vec{x}. \qquad \square$$

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then we let $\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}$ denote the rotor that corresponds to $\vec{R}_{\mathrm{B/A}}$ in the sense that, for every physical vector $\vec{x}$,

$$\vec{R}_{\mathrm{B/A}}\vec{x} = \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}. \tag{3.9.7}$$

Fact 3.9.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then

$$\vec{x}\cdot(\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}) = (\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{x}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\cdot\vec{y} \tag{3.9.8}$$

and

$$\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}(\vec{x}\wedge\vec{y})\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}} = (\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{x}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\wedge(\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{y}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}). \tag{3.9.9}$$

Proof. To prove (3.9.8) note that

$$\vec{x}\cdot(\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}) = \vec{x}\cdot(\vec{R}_{\mathrm{B/A}}\vec{y}) = (\vec{R}'_{\mathrm{B/A}}\vec{x})\cdot\vec{y} = (\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{x}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\cdot\vec{y}.$$

Next, to prove (3.9.9) note that

$$\begin{aligned}
\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}(\vec{x}\wedge\vec{y})\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}} &= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}(\vec{x}\vec{y}-\vec{x}\cdot\vec{y})\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\\
&= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\vec{y}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}} - \vec{x}\cdot\vec{y}\\
&= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}} - \vec{x}\cdot\vec{y}\\
&= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}} - (\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{x}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\cdot(\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{y}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\\
&= (\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{x}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\wedge(\overline{\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}}\vec{y}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}). \qquad \square
\end{aligned}$$

Let $\hat{n}$ and $\vec{x}$ be physical vectors. Then $\vec{x}$ can be written as the sum

$$\vec{x} = \vec{x}_{\mathrm{perp},\hat{n}} + \vec{x}_{\mathrm{par},\hat{n}}, \tag{3.9.10}$$

where $\vec{x}_{\mathrm{perp},\hat{n}}$ is the component of $\vec{x}$ in the plane orthogonal to $\hat{n}$ and $\vec{x}_{\mathrm{par},\hat{n}}$ is the component of $\vec{x}$ in the direction of $\hat{n}$. Furthermore, the reflection of $\vec{x}$ in the plane orthogonal to $\hat{n}$ is defined by

$$\vec{x}_{\mathrm{refl}} \triangleq \vec{x}_{\mathrm{perp},\hat{n}} - \vec{x}_{\mathrm{par},\hat{n}}. \tag{3.9.11}$$

Fact 3.9.3. Let $\hat{n}$ be a physical vector and let $\vec{x}$ be a physical vector. Then,

$$\vec{x}_{\mathrm{perp},\hat{n}} = \hat{n}(\hat{n}\wedge\vec{x}) \tag{3.9.12}$$

$$= (\hat{n}\cdot\vec{x})\hat{n} - \hat{n}\vec{x}\hat{n} \tag{3.9.13}$$

$$= \tfrac{1}{2}(\vec{x} - \hat{n}\vec{x}\hat{n}) \tag{3.9.14}$$

$$= \vec{x} - (\hat{n}\cdot\vec{x})\hat{n}, \tag{3.9.15}$$

$$\vec{x}_{\mathrm{par},\hat{n}} = (\hat{n}\cdot\vec{x})\hat{n}, \tag{3.9.16}$$

$$\begin{aligned}\vec{x}_{\mathrm{refl},\hat{n}} &= -\hat{n}\vec{x}\hat{n} &(3.9.17)\\ &= \vec{x} - 2(\hat{n}\cdot\vec{x})\hat{n}. &(3.9.18)\end{aligned}$$

Proof. Since $\hat{n}^2 = 1$, it follows that

$$\vec{x} = \hat{n}^2\vec{x} = \hat{n}(\hat{n}\wedge\vec{x} + \hat{n}\cdot\vec{x}) = \vec{x}_{\mathrm{perp},\hat{n}} + \vec{x}_{\mathrm{par},\hat{n}}. \tag{3.9.19}$$

Furthermore, using (3.8.17) and (3.8.19) we have

$$\begin{aligned}\vec{x}_{\mathrm{refl},\hat{n}} &= \hat{n}(\hat{n}\wedge\vec{x}) - (\hat{n}\cdot\vec{x})\hat{n} = -(\hat{n}\wedge\vec{x})\hat{n} - (\hat{n}\cdot\vec{x})\hat{n} = -\hat{n}\vec{x}\hat{n}\\ &= \hat{n}(\hat{n}\vec{x} - \hat{n}\cdot\vec{x}) - (\hat{n}\cdot\vec{x})\hat{n} = \hat{n}^2\vec{x} - 2(\hat{n}\cdot\vec{x})\hat{n} = \vec{x} - 2(\hat{n}\cdot\vec{x})\hat{n}.\end{aligned}$$ □

The following result shows that the rotation of the vector $\vec{x}$ around the normal to the plane spanned by two physical vectors $\hat{v}$ and $\hat{w}$ through twice the angle between $\hat{v}$ and $\hat{w}$ is equivalent to reflecting $\vec{x}$ successively in the planes orthogonal to $\hat{v}$ and $\hat{w}$. Note that $\hat{v}$ and $\hat{w}$ are not necessarily orthogonal. This result is illustrated in [11, pp. 133–137].

Fact 3.9.4. Let $\hat{v}$ and $\hat{w}$ be physical vectors that are not colinear, and define $\hat{n} \triangleq \frac{1}{|\hat{v}\times\hat{w}|}\hat{v}\times\hat{w}$ and $\hat{z} \triangleq \hat{n}\times\hat{v}$. Then

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{z}}(2\theta_{\hat{v}/\hat{w}}) = \hat{w}\hat{v}. \tag{3.9.20}$$

Consequently, for every physical vector $\vec{x}$,

$$\vec{R}_{\hat{n}}(2\theta_{\hat{v}/\hat{w}})\vec{x} = \hat{w}\hat{v}\vec{x}\hat{v}\hat{w}. \tag{3.9.21}$$

Proof. Note that

$$(\sin\theta_{\hat{v}/\hat{w}})\hat{z} = \hat{w} - (\cos\theta_{\hat{v}/\hat{w}})\hat{v}.$$

Therefore,

$$\begin{aligned}\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{z}}(2\theta_{\hat{v}/\hat{w}}) &= \cos\theta_{\hat{v}/\hat{w}} - (\sin\theta_{\hat{v}/\hat{w}})(\hat{v}\wedge\hat{z})\\ &= \cos\theta_{\hat{v}/\hat{w}} - \hat{v}\wedge\hat{w}\\ &= \cos\theta_{\hat{v}/\hat{w}} + \hat{w}\wedge\hat{v}\\ &= \cos\theta_{\hat{v}/\hat{w}} + \hat{w}\hat{v} - \hat{w}\cdot\hat{v}\\ &= \hat{w}\hat{v}.\end{aligned}$$ □

Note that (3.9.20) implies

$$\begin{aligned}\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{z}}(2\theta_{\hat{v}/\hat{w}}) &= \hat{w}\hat{v}\\ &= \exp[-(\hat{v}\wedge\hat{w})\theta_{\hat{v}/\hat{w}}/|\hat{v}\times\hat{w}|]\\ &= \exp[-(\hat{v}\wedge\hat{w})\theta_{\hat{v}/\hat{w}}/\sin\theta_{\hat{v}/\hat{w}}]. &(3.9.22)\end{aligned}$$

3.10 Problems

Problem 3.10.1. Let $\vec{T}_1, \vec{T}_2, \vec{T}_3 \in \mathcal{T}_{(p,p)}$, and let $\mathcal{S} = \{(i_1, j_1), \ldots, (i_p, j_p)\}$, where $i_1, \ldots, i_p$ are distinct integers in $\{1, \ldots, 2p\}$ and $j_1, \ldots, j_p$ are distinct integers in $\{1, \ldots, 2p\}$. Show that

$$((\vec{T}_1 \otimes \vec{T}_2)_{\mathcal{S}} \otimes \vec{T}_3)_{\mathcal{S}} = (\vec{T}_1 \otimes (\vec{T}_2 \otimes \vec{T}_3)_{\mathcal{S}})_{\mathcal{S}}.$$

Problem 3.10.2. Let $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{w}$ be physical vectors. Show that

$$(\vec{x} \wedge \vec{y})'(\vec{z}, \vec{w}) = (\vec{x} \times \vec{y}) \cdot (\vec{z} \times \vec{w}) = (\vec{x} \cdot \vec{z})(\vec{y} \cdot \vec{w}) - (\vec{y} \cdot \vec{z})(\vec{x} \cdot \vec{y}).$$

Problem 3.10.3. Let $\vec{T}_1$ and $\vec{T}_2$ be tensors of order $(p_1, 0)$ and $(p_2, 0)$, respectively. Show that

$$\vec{T}_1 \wedge \vec{T}_2 = (-1)^{p_1 p_2} \vec{T}_2 \wedge \vec{T}_1.$$

Problem 3.10.4. Let $\vec{w}'_1$, $\vec{w}'_2$, $\vec{w}'_3$ be physical covectors, and let $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ be physical vectors. Show that

$$(\vec{w}'_1 \wedge \vec{w}'_1 \wedge \vec{w}'_1)(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \det \begin{bmatrix} \vec{w}'_1 \vec{x}_1 & \vec{w}'_1 \vec{x}_2 & \vec{w}'_1 \vec{x}_3 \\ \vec{w}'_2 \vec{x}_1 & \vec{w}'_2 \vec{x}_2 & \vec{w}'_2 \vec{x}_3 \\ \vec{w}'_3 \vec{x}_1 & \vec{w}'_3 \vec{x}_2 & \vec{w}'_3 \vec{x}_3 \end{bmatrix}.$$

Problem 3.10.5. Let x, y be real numbers, let θ be an angle, and define u, v by

$$u + v\jmath = e^{\theta\jmath}(x + y\jmath) = [\cos\theta + (\sin\theta)\jmath](x + y\jmath).$$

Show that

$$u\mathbf{i} + v\mathbf{j} = e^{\frac{1}{2}\theta\mathbf{k}}(x\mathbf{i} + y\mathbf{j})e^{-\frac{1}{2}\theta\mathbf{k}} = [(\cos\tfrac{1}{2}\theta) + (\sin\tfrac{1}{2}\theta)\mathbf{k}](x\mathbf{i} + y\mathbf{j})[(\cos\tfrac{1}{2}\theta) - (\sin\tfrac{1}{2}\theta)\mathbf{k}].$$

Chapter Four
Kinematics

4.1 Frame Derivatives

Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ be a physical vector expressed as

$$\vec{x} = x_1\hat{\imath}_\mathrm{A} + x_2\hat{\jmath}_\mathrm{A} + x_3\hat{k}_\mathrm{A}. \tag{4.1.1}$$

We can thus write

$$\vec{x}\Big|_\mathrm{A} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{4.1.2}$$

The *derivative of* $\vec{x}$ *with respect to* $\mathrm{F_A}$ is defined by

$$\begin{aligned} \overset{\mathrm{A}\bullet}{\vec{x}} &\triangleq \dot{x}_1\hat{\imath}_\mathrm{A} + x_1 \overset{\mathrm{A}\bullet}{\hat{\imath}}_\mathrm{A} + \dot{x}_2\hat{\jmath}_\mathrm{A} + x_2 \overset{\mathrm{A}\bullet}{\hat{\jmath}}_\mathrm{A} + \dot{x}_3\hat{k}_\mathrm{A} + x_3 \overset{\mathrm{A}\bullet}{\hat{k}}_\mathrm{A} \\ &= \dot{x}_1\hat{\imath}_\mathrm{A} + \dot{x}_2\hat{\jmath}_\mathrm{A} + \dot{x}_3\hat{k}_\mathrm{A}. \end{aligned} \tag{4.1.3}$$

Note that $\overset{\mathrm{A}\bullet}{\hat{\imath}}_\mathrm{A} = \overset{\mathrm{A}\bullet}{\hat{\jmath}}_\mathrm{A} = \overset{\mathrm{A}\bullet}{\hat{k}}_\mathrm{A} = 0$ since the axes of $\mathrm{F_A}$ are constant with respect to $\mathrm{F_A}$. Resolving $\overset{\mathrm{A}\bullet}{\vec{r}}$ in $\mathrm{F_A}$ yields

$$\overset{\mathrm{A}\bullet}{\vec{x}}\Bigg|_\mathrm{A} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}, \tag{4.1.4}$$

that is,

$$\overset{\mathrm{A}\bullet}{\vec{x}}\Bigg|_\mathrm{A} = \overset{\cdot}{\overbrace{\vec{x}\Big|_\mathrm{A}}}. \tag{4.1.5}$$

If, for all time, $\overset{\mathrm{A}\bullet}{\vec{x}}\Big|_\mathrm{A} = 0$, then the components of $\overset{\mathrm{A}\bullet}{\vec{x}}\Big|_\mathrm{A}$ are constant; these constants are called *constants of the motion with respect to* $\mathrm{F_A}$.

Let x and y be points, and let $\mathrm{F_A}$ be a frame. Then, the *velocity of* y *relative to* x *with respect to* $\mathrm{F_A}$ is defined by

$$\vec{v}_{y/x/\mathrm{A}} \triangleq \overset{\mathrm{A}\bullet}{\vec{r}}_{y/x}, \tag{4.1.6}$$

and the *acceleration of y relative to x with respect to* $\mathrm{F_A}$ is defined by

$$\vec{a}_{y/x/\mathrm{A}} \triangleq \overset{\mathrm{A}\bullet}{\vec{v}}_{y/x/\mathrm{A}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/x} . \tag{4.1.7}$$

Note that, if

$$\vec{r}_{y/x}\Big|_{\mathrm{A}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{4.1.8}$$

then

$$\vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{bmatrix}, \quad \vec{a}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \\ \ddot{r}_3 \end{bmatrix}. \tag{4.1.9}$$

More generally, if $\mathrm{F_B}$ is a frame, then each component of $\vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}}$ is a *signed speed*.

Let x and y be frames, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then we define the notation

$$r_{y/x|\mathrm{B}} \triangleq \vec{r}_{y/x}\Big|_{\mathrm{B}} . \tag{4.1.10}$$

Note that

$$r_{y/x|\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{r}_{y/x}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}} r_{y/x|\mathrm{A}}. \tag{4.1.11}$$

Next, define

$$v_{y/x/\mathrm{A|B}} \triangleq \vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{y/x}\Bigg|_{\mathrm{B}} . \tag{4.1.12}$$

Note that

$$v_{y/x/\mathrm{A|B}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}} v_{y/x/\mathrm{A|A}} = \mathcal{O}_{\mathrm{B/A}} \dot{r}_{y/x|\mathrm{A}}. \tag{4.1.13}$$

Furthermore, define

$$a_{y/x/\mathrm{A|B}} \triangleq \vec{a}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}} = \overset{\mathrm{A}\bullet}{\vec{v}}_{y/x|\mathrm{A}}\Bigg|_{\mathrm{B}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/x}\Bigg|_{\mathrm{B}} . \tag{4.1.14}$$

Note that

$$a_{y/x/\mathrm{A|B}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{a}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}} a_{y/x/\mathrm{A|A}} = \mathcal{O}_{\mathrm{B/A}} \dot{v}_{y/x/\mathrm{A|A}} = \mathcal{O}_{\mathrm{B/A}} \ddot{r}_{y/x|\mathrm{A}}. \tag{4.1.15}$$

The *mixed acceleration* is defined by

$$\vec{a}_{y/x/\mathrm{A/B}} \triangleq \overset{\mathrm{B}\bullet}{\vec{v}}_{y/x/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{r}}}_{y/x} \tag{4.1.16}$$

As a special case, $\vec{a}_{y/x/\mathrm{A/A}} = \vec{a}_{y/x/\mathrm{A}}$. Now let $\mathrm{F_C}$ be a frame. Then, the mixed acceleration can be resolved as

$$a_{y/x/\mathrm{A/B|C}} \triangleq \vec{a}_{y/x/\mathrm{A/B}}\Big|_{\mathrm{C}} . \tag{4.1.17}$$

As a special case, $a_{y/x/\mathrm{A}/\mathrm{A}|\mathrm{C}} = a_{y/x/\mathrm{A}|\mathrm{C}}$. Note that

$$\begin{aligned}
\vec{a}_{y/x/\mathrm{A}/\mathrm{B}|\mathrm{C}} &= \mathcal{O}_{\mathrm{C}/\mathrm{B}} \left.\vec{a}_{y/x/\mathrm{A}/\mathrm{B}}\right|_{\mathrm{B}} = \mathcal{O}_{\mathrm{C}/\mathrm{B}} \left.\overset{\mathrm{B}\bullet}{\vec{v}}_{y/x/\mathrm{A}}\right|_{\mathrm{B}} \\
&= \mathcal{O}_{\mathrm{C}/\mathrm{B}}\dot{v}_{y/x/\mathrm{A}|\mathrm{B}} = \mathcal{O}_{\mathrm{C}/\mathrm{B}}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{O}_{\mathrm{B}/\mathrm{A}}\dot{r}_{y/x|\mathrm{A}}) \\
&= \mathcal{O}_{\mathrm{C}/\mathrm{B}}(\dot{\mathcal{O}}_{\mathrm{B}/\mathrm{A}}\dot{r}_{y/x|\mathrm{A}} + \mathcal{O}_{\mathrm{B}/\mathrm{A}}\ddot{r}_{y/x|\mathrm{A}}) \\
&= \mathcal{O}_{\mathrm{C}/\mathrm{B}}\dot{\mathcal{O}}_{\mathrm{B}/\mathrm{A}}\dot{r}_{y/x|\mathrm{A}} + \mathcal{O}_{\mathrm{C}/\mathrm{A}}\ddot{r}_{y/x|\mathrm{A}}.
\end{aligned} \tag{4.1.18}$$

Note that, if $\mathrm{F_B} = \mathrm{F_A}$, then (4.1.18) yields (4.1.15).

Fact 4.1.1. Let $\mathrm{F_A}$ be a frame, and let x, y, and z be points. Then,

$$\vec{v}_{z/x/\mathrm{A}} = \vec{v}_{z/y/\mathrm{A}} + \vec{v}_{y/x/\mathrm{A}}, \tag{4.1.19}$$

$$\vec{a}_{z/x/\mathrm{A}} = \vec{a}_{z/y/\mathrm{A}} + \vec{a}_{y/x/\mathrm{A}}. \tag{4.1.20}$$

Fact 4.1.1 is based on the assumptions of Newtonian mechanics and thus is not valid within the context of relativity. For a particle of light in a medium such as vacuum or water, the speed of light c is independent of the velocity of all other bodies. Therefore, if x denotes a particle of light, w denotes a particle, and $\mathrm{F_A}$ is a frame, then

$$|\vec{v}_{x/w/\mathrm{A}}| = c. \tag{4.1.21}$$

Therefore, (4.1.19) does not hold for light particles.

The following result shows that the velocity of y relative to x with respect to $\mathrm{F_A}$ is zero if and only if the position of y relative to x is constant with respect to $\mathrm{F_A}$. Likewise, the acceleration of y relative to x with respect to $\mathrm{F_A}$ is zero if and only if y moves in a straight line at constant speed relative to x and with respect to $\mathrm{F_A}$.

Fact 4.1.2. Let $\mathrm{F_A}$ be a frame, let y and x be points, and let $t_1 < t_2$. Then, the following statements hold:

i) $\vec{v}_{y/x/\mathrm{A}}(t) = 0$ for all $t \in [t_1, t_2]$ if and only if there exist real numbers $\alpha_1, \alpha_2, \alpha_3$ such that, for all $t \in [t_1, t_2]$,

$$\left.\vec{r}_{y/x}(t)\right|_{\mathrm{A}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. \tag{4.1.22}$$

ii) $\vec{a}_{y/x/\mathrm{A}}(t) = 0$ for all $t \in [t_1, t_2]$ if and only if there exist real numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, such that, for all $t \in [t_1, t_2]$,

$$\left.\vec{r}_{y/x}(t)\right|_{\mathrm{A}} = \begin{bmatrix} \alpha_1 t + \beta_1 \\ \alpha_2 t + \beta_2 \\ \alpha_3 t + \beta_3 \end{bmatrix}. \tag{4.1.23}$$

Fact 4.1.3. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{x} \cdot \vec{y}) = \overset{\mathrm{A}\bullet}{\vec{x}} \cdot \vec{y} + \vec{x} \cdot \overset{\mathrm{A}\bullet}{\vec{y}}. \tag{4.1.24}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{x}\cdot\vec{x}) = 2\overset{\mathrm{A}\bullet}{\vec{x}}\cdot\vec{x}. \tag{4.1.25}$$

Proof. Write

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{y}\Big|_{\mathrm{A}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\vec{x}\cdot\vec{y}) &= \frac{\mathrm{d}}{\mathrm{d}t}(x_1y_1 + x_2y_2 + x_3y_3) \\
&= \dot{x}_1y_1 + \dot{x}_2y_2 + \dot{x}_3y_3 + x_1\dot{y}_1 + x_2\dot{y}_2 + x_3\dot{y}_3 \\
&= \overbrace{\vec{x}\Big|_{\mathrm{A}}}^{\cdot}{}^{\mathrm{T}}\,\vec{y}\Big|_{\mathrm{A}} + \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}}\,\overbrace{\vec{y}\Big|_{\mathrm{A}}}^{\cdot} \\
&= \overset{\mathrm{A}\bullet}{\vec{x}}\Big|_{\mathrm{A}}^{\mathrm{T}}\,\vec{y}\Big|_{\mathrm{A}} + \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}}\,\overset{\mathrm{A}\bullet}{\vec{y}}\Big|_{\mathrm{A}} \\
&= \overset{\mathrm{A}\bullet}{\vec{x}}\cdot\vec{y} + \vec{x}\cdot\overset{\mathrm{A}\bullet}{\vec{y}}.
\end{aligned}$$

□

Let $\mathrm{F_A}$ be a frame, let $\vec{x}$ and $\vec{y}$ be physical vectors, and let $\vec{M} = \vec{x}\,\vec{y}\,'$. Then, we define

$$\overset{\mathrm{A}\bullet}{\vec{y}\,'} \triangleq \overset{\mathrm{A}\bullet}{\vec{y}}{}', \tag{4.1.26}$$

$$\overset{\mathrm{A}\bullet}{\vec{M}} \triangleq \overset{\mathrm{A}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{A}\bullet}{\vec{y}}{}'. \tag{4.1.27}$$

Fact 4.1.4. Let $\mathrm{F_A}$ be a frame, let $\vec{M}$ and $\vec{N}$ be physical matrices, and let $\vec{x}$ be a physical vector. Then,

$$\overbrace{\vec{M}\vec{x}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{M}}\,\vec{x} + \vec{M}\,\overset{\mathrm{A}\bullet}{\vec{x}}, \tag{4.1.28}$$

$$\overbrace{\vec{M}\vec{N}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{M}}\,\vec{N} + \vec{M}\,\overset{\mathrm{A}\bullet}{\vec{N}}. \tag{4.1.29}$$

Furthermore,

$$\overset{\mathrm{A}\bullet}{\vec{M}}\Big|_{\mathrm{A}} = \overbrace{\vec{M}\Big|_{\mathrm{A}}}^{\cdot}. \tag{4.1.30}$$

Proof. For convenience, assume that $\vec{M} = \vec{y}\,\vec{z}\,'$. Using Fact 4.1.3 it follows that

$$\overbrace{\vec{M}\vec{x}}^{\mathrm{A}\bullet} = \overbrace{(\vec{y}\,\vec{z}\,')\vec{x}}^{\mathrm{A}\bullet}$$

$$
\begin{aligned}
&= \overbrace{(\vec{z}\cdot\vec{x})\vec{y}}^{\mathrm{A}\bullet}\\
&= \overbrace{(\vec{z}\cdot\vec{x})}^{\cdot}\,\vec{y} + (\vec{z}\cdot\vec{x})\overset{\mathrm{A}\bullet}{\vec{y}}\\
&= (\overset{\mathrm{A}\bullet}{\vec{z}}\cdot\vec{x})\vec{y} + (\vec{z}\cdot\overset{\mathrm{A}\bullet}{\vec{x}})\vec{y} + (\vec{z}\cdot\vec{x})\overset{\mathrm{A}\bullet}{\vec{y}}\\
&= (\vec{y}\overset{\mathrm{A}\bullet\prime}{\vec{z}})\vec{x} + (\overset{\mathrm{A}\bullet}{\vec{y}}\vec{z}^{\,\prime})\vec{x} + (\vec{y}\vec{z}^{\,\prime})\overset{\mathrm{A}\bullet}{\vec{x}}\\
&= \overset{\mathrm{A}\bullet}{\vec{M}}\vec{x} + \vec{M}\overset{\mathrm{A}\bullet}{\vec{x}}.
\end{aligned}
$$

Next, let $\vec{N} = \vec{w}\vec{u}^{\,\prime}$ and note that

$$
\begin{aligned}
\overbrace{\vec{M}\vec{N}}^{\mathrm{A}\bullet} &= \overbrace{(\vec{y}\vec{z}^{\,\prime})(\vec{w}\vec{u}^{\,\prime})}^{\mathrm{A}\bullet}\\
&= \overbrace{\vec{z}\cdot\vec{w}(\vec{y}\vec{u}^{\,\prime})}^{\mathrm{A}\bullet}\\
&= \vec{z}\cdot\vec{w}\,\overbrace{(\vec{y}\vec{u}^{\,\prime})}^{\mathrm{A}\bullet} + \overbrace{\vec{z}\cdot\vec{w}}^{\cdot}(\vec{y}\vec{u}^{\,\prime})\\
&= \vec{z}\cdot\vec{w}(\overset{\mathrm{A}\bullet}{\vec{y}}\vec{u}^{\,\prime} + \vec{y}\overset{\mathrm{A}\bullet\prime}{\vec{u}}) + (\overset{\mathrm{A}\bullet}{\vec{z}}\cdot\vec{w} + \vec{z}\cdot\overset{\mathrm{A}\bullet}{\vec{w}})\vec{y}\vec{u}^{\,\prime}\\
&= (\overset{\mathrm{A}\bullet}{\vec{y}}\vec{z}^{\,\prime} + \vec{y}\overset{\mathrm{A}\bullet\prime}{\vec{z}})\vec{w}\vec{u}^{\,\prime} + \vec{y}\vec{z}^{\,\prime}(\overset{\mathrm{A}\bullet}{\vec{w}}\vec{u}^{\,\prime} + \vec{w}\overset{\mathrm{A}\bullet\prime}{\vec{u}})\\
&= \overset{\mathrm{A}\bullet}{\vec{M}}\vec{N} + \vec{M}\overset{\mathrm{A}\bullet}{\vec{N}}.
\end{aligned}
$$

□

Fact 4.1.5. Let F_A be a frame, and let $\vec{x}$ be a physical vector. Then,

$$\overbrace{\vec{x}^{\times}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{x}}{}^{\times}, \tag{4.1.31}$$

$$\overset{\mathrm{A}\bullet}{\vec{x}}\times\vec{x} = -\vec{x}\times\overset{\mathrm{A}\bullet}{\vec{x}}, \tag{4.1.32}$$

$$(\vec{x}\times\overset{\mathrm{A}\bullet}{\vec{x}})^{\times} = \overset{\mathrm{A}\bullet}{\vec{x}}\vec{x}^{\,\prime} - \vec{x}\overset{\mathrm{A}\bullet\prime}{\vec{x}}. \tag{4.1.33}$$

If, in addition, $\vec{y}$ is a physical vector, then

$$\overbrace{\vec{x}\times\vec{y}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{x}}\times\vec{y} + \vec{x}\times\overset{\mathrm{A}\bullet}{\vec{y}}. \tag{4.1.34}$$

Fact 4.1.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{I}} = 0, \tag{4.1.35}$$

$$\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm B}}\;\hat\imath'_{\mathrm B} + \overset{\mathrm{A}\bullet}{\hat\jmath_{\mathrm B}}\;\hat\jmath'_{\mathrm B} + \overset{\mathrm{A}\bullet}{\hat k_{\mathrm B}}\;\hat k'_{\mathrm B} = -(\overset{\mathrm{A}\bullet\prime}{\hat\imath_{\mathrm B}\hat\imath_{\mathrm B}} + \overset{\mathrm{A}\bullet\prime}{\hat\jmath_{\mathrm B}\hat\jmath_{\mathrm B}} + \overset{\mathrm{A}\bullet\prime}{\hat k_{\mathrm B}\hat k_{\mathrm B}}). \tag{4.1.36}$$

Fact 4.1.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{A/B}} = -\vec R_{\mathrm{A/B}}\;\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{B/A}}\;\vec R_{\mathrm{A/B}}, \tag{4.1.37}$$

$$\overset{\mathrm{A}\bullet}{\vec R}_{\mathrm{A/B}} = -\vec R_{\mathrm{A/B}}\;\overset{\mathrm{A}\bullet}{\vec R}_{\mathrm{B/A}}\;\vec R_{\mathrm{A/B}}. \tag{4.1.38}$$

Proof. Differentiating $\vec R_{\mathrm{A/B}}\vec R_{\mathrm{B/A}} = \vec I$ and using (4.1.29) yields

$$\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{A/B}}\;\vec R_{\mathrm{B/A}} + \vec R_{\mathrm{A/B}}\;\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{B/A}} = 0,$$

which implies (4.1.37). □

Fact 4.1.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec R_{\mathrm{A/B}}\;\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec R}_{\mathrm{B/A}}\;\vec R_{\mathrm{A/B}} = -\;\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{A/B}}\;\vec R_{\mathrm{B/A}} = -\vec R_{\mathrm{B/A}}\;\overset{\mathrm{A}\bullet}{\vec R}_{\mathrm{A/B}}\;. \tag{4.1.39}$$

Proof. Note that

$$\begin{aligned}\left.\left(\vec R_{\mathrm{A/B}}\;\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{B/A}}\right)\right|_{\mathrm B} &= \vec R_{\mathrm{A/B}}\Big|_{\mathrm B}\;\overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{B/A}}\Big|_{\mathrm B} = \mathcal R_{\mathrm{A/B}}\dot{\mathcal R}_{\mathrm{B/A}} = \mathcal R_{\mathrm{A/B}}\dot{\mathcal R}_{\mathrm{B/A}}\mathcal O_{\mathrm{A/B}}\mathcal R_{\mathrm{A/B}}\\ &= \mathcal O_{\mathrm{B/A}}\;\overset{\mathrm{A}\bullet}{\vec R}_{\mathrm{B/A}}\Big|_{\mathrm A}\;\mathcal O_{\mathrm{A/B}}\;\vec R_{\mathrm{A/B}}\Big|_{\mathrm B} = \overset{\mathrm{A}\bullet}{\vec R}_{\mathrm{B/A}}\Big|_{\mathrm B}\;\vec R_{\mathrm{A/B}}\Big|_{\mathrm B} = \left.\left(\overset{\mathrm{A}\bullet}{\vec R}_{\mathrm{B/A}}\;\vec R_{\mathrm{A/B}}\right)\right|_{\mathrm B}.\end{aligned}$$

□

Fact 4.1.9. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet\prime}{\vec R}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{A/B}}\;. \tag{4.1.40}$$

Proof. Note that

$$\begin{aligned}\overset{\mathrm{B}\bullet\prime}{\vec R}_{\mathrm{B/A}} &= \left(\overbrace{(\hat\imath_{\mathrm B}\hat\imath'_{\mathrm A} + \hat\jmath_{\mathrm B}\hat\jmath'_{\mathrm A} + \hat k_{\mathrm B}\hat k'_{\mathrm A})}^{\mathrm{B}\bullet}\right)' = \left(\hat\imath_{\mathrm B}\overset{\mathrm{B}\bullet\prime}{\hat\imath_{\mathrm A}} + \hat\jmath_{\mathrm B}\overset{\mathrm{B}\bullet\prime}{\hat\jmath_{\mathrm A}} + \hat k_{\mathrm B}\overset{\mathrm{B}\bullet\prime}{\hat k_{\mathrm A}}\right)'\\ &= \overset{\mathrm{B}\bullet}{\hat\imath_{\mathrm A}}\hat\imath'_{\mathrm B} + \overset{\mathrm{B}\bullet}{\hat\jmath_{\mathrm A}}\hat\jmath'_{\mathrm B} + \overset{\mathrm{B}\bullet}{\hat k_{\mathrm A}}\hat k'_{\mathrm B} = \overset{\mathrm{B}\bullet}{\vec R}_{\mathrm{A/B}}\;.\end{aligned}$$

□

4.2 The Mixed-Dot Identity and the Physical Angular Velocity Matrix

Let F_A and F_B be frames, and let $\vec{x}$ be a physical vector. Then, the frame derivatives $\overset{A\bullet}{\vec{x}}$ and $\overset{B\bullet}{\vec{x}}$ may be different if F_A and F_B are rotating relative to each other. The following result, called the *mixed-dot identity*, shows how these derivatives are related.

Fact 4.2.1. Let F_A and F_B be frames, let $\vec{x}$ be a physical vector. Then,

$$\overset{B\bullet}{\overbrace{\vec{R}_{B/A}\vec{x}}} = \vec{R}_{B/A}\overset{A\bullet}{\vec{x}}. \tag{4.2.1}$$

Proof. Defining $\vec{y} \triangleq \vec{R}_{B/A}\vec{x}$, note that

$$\left(\vec{R}_{A/B}\overset{B\bullet}{\vec{y}}\right)\bigg|_A = \vec{R}_{A/B}\Big|_A \overset{B\bullet}{\vec{y}}\Big|_A = \mathcal{R}_{A/B}\mathcal{O}_{A/B}\overset{B\bullet}{\vec{y}}\Big|_B = \overset{B\bullet}{\overbrace{\vec{R}_{B/A}\vec{x}}}\bigg|_B = \overset{\cdot}{\overbrace{\vec{R}_{B/A}\vec{x}\Big|_B}}$$

$$= \overset{\cdot}{\overbrace{\mathcal{O}_{B/A}\vec{R}_{B/A}\vec{x}\Big|_A}} = \overset{\cdot}{\overbrace{\mathcal{O}_{B/A}\mathcal{R}_{B/A}\vec{x}\Big|_A}} = \overset{\cdot}{\overbrace{\vec{x}\Big|_A}} = \overset{A\bullet}{\vec{x}}\bigg|_A. \qquad \square$$

Fact 4.2.2. Let F_A and F_B be frames. Then,

$$\overset{A\bullet}{\vec{x}} = \overset{B\bullet}{\vec{x}} + \vec{R}_{A/B}\overset{B\bullet}{\vec{R}}_{B/A}\vec{x}. \tag{4.2.2}$$

Proof. Using (4.2.1) we have

$$\begin{aligned}
\overset{A\bullet}{\vec{x}} &= \vec{R}_{A/B}\overset{B\bullet}{\overbrace{\vec{R}_{B/A}\vec{x}}} \\
&= \vec{R}_{A/B}\left(\vec{R}_{B/A}\overset{B\bullet}{\vec{x}} + \overset{B\bullet}{\vec{R}}_{B/A}\vec{x}\right) \\
&= \overset{B\bullet}{\vec{x}} + \vec{R}_{A/B}\overset{B\bullet}{\vec{R}}_{B/A}\vec{x}.
\end{aligned} \qquad \square$$

Since $\overset{A\bullet}{\vec{x}}$ and $\overset{B\bullet}{\vec{x}}$ differ due to the relative rotation of F_A and F_B, we have the following definition.

Definition 4.2.3. Let F_A and F_B be frames. Then, the *physical angular velocity matrix* of F_B *relative* to F_A is defined by

$$\vec{\Omega}_{B/A} \triangleq \vec{R}_{A/B}\overset{B\bullet}{\vec{R}}_{B/A}. \tag{4.2.3}$$

It follows from (4.1.37) that

$$\vec{\Omega}_{B/A} = -\overset{B\bullet}{\vec{R}}_{A/B}\vec{R}_{B/A}. \tag{4.2.4}$$

Furthermore, (4.2.2) can be written as

$$\overset{\mathrm{A}\bullet}{\vec{x}} = \overset{\mathrm{B}\bullet}{\vec{x}} + \vec{\Omega}_{\mathrm{B/A}}\vec{x}. \tag{4.2.5}$$

Fact 4.2.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, $\vec{\Omega}_{\mathrm{B/A}}$ is skew symmetric, that is,

$$\vec{\Omega}'_{\mathrm{B/A}} = -\vec{\Omega}_{\mathrm{B/A}}. \tag{4.2.6}$$

Proof. Using (4.1.40) and (4.2.4), it follows that

$$\vec{\Omega}'_{\mathrm{B/A}} = \left(\vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\right)' = \overset{\mathrm{B}\bullet\prime}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}\,\vec{R}_{\mathrm{B/A}} = -\vec{\Omega}_{\mathrm{B/A}}. \qquad \square$$

Fact 4.2.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\vec{\Omega}_{\mathrm{B/A}}, \tag{4.2.7}$$

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} = -\,\vec{\Omega}_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}}. \tag{4.2.8}$$

Proof. (4.2.7) follows from (4.2.3), and (4.2.8) follows from (4.2.4). $\square$

Fact 4.2.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{\Omega}_{\mathrm{A/B}} = \hat{\imath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\imath}}_{\mathrm{B}} + \hat{\jmath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}}_{\mathrm{B}} + \hat{k}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{k}}_{\mathrm{B}}, \tag{4.2.9}$$

$$\vec{\Omega}_{\mathrm{A/B}} = -\left(\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}}\,\hat{\imath}'_{\mathrm{B}} + \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}}\,\hat{\jmath}'_{\mathrm{B}} + \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}}\,\hat{k}'_{\mathrm{B}}\right), \tag{4.2.10}$$

$$\vec{\Omega}_{\mathrm{B/A}} = \hat{\imath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\imath}}_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\jmath}}_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{k}}_{\mathrm{A}}, \tag{4.2.11}$$

$$\vec{\Omega}_{\mathrm{B/A}} = -\left(\overset{\mathrm{B}\bullet}{\hat{\imath}}_{\mathrm{A}}\,\hat{\imath}'_{\mathrm{A}} + \overset{\mathrm{B}\bullet}{\hat{\jmath}}_{\mathrm{A}}\,\hat{\jmath}'_{\mathrm{A}} + \overset{\mathrm{B}\bullet}{\hat{k}}_{\mathrm{A}}\,\hat{k}'_{\mathrm{A}}\right). \tag{4.2.12}$$

Proof. Note that

$$\begin{aligned}
\vec{\Omega}_{\mathrm{A/B}} &= \vec{R}_{\mathrm{B/A}}\,\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} \\
&= (\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{A}})(\hat{\imath}_{\mathrm{A}}\overset{\mathrm{A}\bullet\prime}{\hat{\imath}}_{\mathrm{B}} + \hat{\jmath}_{\mathrm{A}}\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}}_{\mathrm{B}} + \hat{k}_{\mathrm{A}}\overset{\mathrm{A}\bullet\prime}{\hat{k}}_{\mathrm{B}}) \\
&= \hat{\imath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\imath}}_{\mathrm{B}} + \hat{\jmath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}}_{\mathrm{B}} + \hat{k}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{k}}_{\mathrm{B}},
\end{aligned}$$

which proves (4.2.9). Using (4.1.36) yields (4.2.10). $\square$

Fact 4.2.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} = \vec{\Omega}_{\mathrm{B/A}}\hat{\imath}_{\mathrm{B}}, \tag{4.2.13}$$

$$\overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} = \vec{\Omega}_{\mathrm{B/A}}\hat{\jmath}_{\mathrm{B}}, \tag{4.2.14}$$

$$\overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} = \vec{\Omega}_{\mathrm{B/A}}\hat{k}_{\mathrm{B}}. \tag{4.2.15}$$

Proof. Using (4.2.11) and Fact 4.2.8, we have

$$\begin{aligned}
\vec{\Omega}_{\mathrm{B/A}}\hat{\imath}_{\mathrm{B}} &= (\hat{\imath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\imath}_{\mathrm{A}}} + \hat{\jmath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\jmath}_{\mathrm{A}}} + \hat{k}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{k}_{\mathrm{A}}})\hat{\imath}_{\mathrm{B}} \\
&= (\overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}} \cdot \hat{\imath}_{\mathrm{B}})\hat{\imath}_{\mathrm{A}} + (\overset{\mathrm{B}\bullet}{\hat{\jmath}_{\mathrm{A}}} \cdot \hat{\imath}_{\mathrm{B}})\hat{\jmath}_{\mathrm{A}} + (\overset{\mathrm{B}\bullet}{\hat{k}_{\mathrm{A}}} \cdot \hat{\imath}_{\mathrm{B}})\hat{k}_{\mathrm{A}} \\
&= (\hat{\imath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}})\hat{\imath}_{\mathrm{A}} + (\hat{\jmath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}})\hat{\jmath}_{\mathrm{A}} + (\hat{k}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}})\hat{k}_{\mathrm{A}} \\
&= (\hat{\imath}_{\mathrm{A}}\hat{\imath}_{\mathrm{A}}')\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} + (\hat{\jmath}_{\mathrm{A}}\hat{\jmath}_{\mathrm{A}}')\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} + (\hat{k}_{\mathrm{A}}\hat{k}_{\mathrm{A}}')\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} \\
&= \vec{I}\,\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} = \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}}. \qquad \square
\end{aligned}$$

Fact 4.2.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\hat{\imath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}} \cdot \hat{\imath}_{\mathrm{B}}, \quad \hat{\imath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}} \cdot \hat{\jmath}_{\mathrm{B}}, \quad \hat{\imath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}} \cdot \hat{k}_{\mathrm{B}}, \tag{4.2.16}$$

$$\hat{\jmath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{\jmath}_{\mathrm{A}}} \cdot \hat{\imath}_{\mathrm{B}}, \quad \hat{\jmath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{\jmath}_{\mathrm{A}}} \cdot \hat{\jmath}_{\mathrm{B}}, \quad \hat{\jmath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{\jmath}_{\mathrm{A}}} \cdot \hat{k}_{\mathrm{B}}, \tag{4.2.17}$$

$$\hat{k}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{k}_{\mathrm{A}}} \cdot \hat{\imath}_{\mathrm{B}}, \quad \hat{k}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{k}_{\mathrm{A}}} \cdot \hat{\jmath}_{\mathrm{B}}, \quad \hat{k}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}} = \overset{\mathrm{B}\bullet}{\hat{k}_{\mathrm{A}}} \cdot \hat{k}_{\mathrm{B}}. \tag{4.2.18}$$

Proof. Note that $\hat{\imath}_{\mathrm{A}} \cdot \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} = \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\imath}_{\mathrm{A}} \cdot \hat{\imath}_{\mathrm{B}}) = \overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}} \cdot \hat{\imath}_{\mathrm{B}}$. $\square$

Fact 4.2.9. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{\Omega}_{\mathrm{A/B}} = -\vec{\Omega}_{\mathrm{B/A}}, \tag{4.2.19}$$

$$\vec{\Omega}_{\mathrm{A/B}} = \vec{\Omega}_{\mathrm{B/A}}'. \tag{4.2.20}$$

Proof. (4.2.19) follows from the equality of the first, second, and fourth terms in (4.1.39). Alternatively, Fact 4.2.6 and (4.1.36) imply that

$$\begin{aligned}
\vec{\Omega}_{\mathrm{A/B}} &= -(\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}}\,\hat{\imath}_{\mathrm{B}}' + \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}}\,\hat{\jmath}_{\mathrm{B}}' + \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}}\,\hat{k}_{\mathrm{B}}') \\
&= -\left[\left(\vec{\Omega}_{\mathrm{B/A}}\hat{\imath}_{\mathrm{B}}\right)\hat{\imath}_{\mathrm{B}}' + \left(\vec{\Omega}_{\mathrm{B/A}}\hat{\jmath}_{\mathrm{B}}\right)\hat{\jmath}_{\mathrm{B}}' + \left(\vec{\Omega}_{\mathrm{B/A}}\hat{k}_{\mathrm{B}}\right)\hat{k}_{\mathrm{B}}'\right] \\
&= -\vec{\Omega}_{\mathrm{B/A}}\vec{I} = -\vec{\Omega}_{\mathrm{B/A}}. \qquad \square
\end{aligned}$$

It follows from (4.2.19) that

$$\vec{\Omega}_{\mathrm{B/A}} = -\vec{R}_{\mathrm{B/A}}\,\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}}, \tag{4.2.21}$$

$$\vec{\Omega}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{A/B}}. \tag{4.2.22}$$

Fact 4.2.10. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define

$$\Omega_{\mathrm{B/A|B}} \triangleq \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}. \tag{4.2.23}$$

Then,

$$\Omega_{\mathrm{B/A|B}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}} = -\dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}, \tag{4.2.24}$$

$$\Omega_{\mathrm{B/A|B}} = -\Omega^{\mathrm{T}}_{\mathrm{B/A|B}}. \tag{4.2.25}$$

Furthermore,

$$\Omega_{\mathrm{A/B|A}} \triangleq \vec{\Omega}_{\mathrm{A/B}}\Big|_{\mathrm{A}} = -\vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = -\mathcal{O}_{\mathrm{A/B}}\Omega_{\mathrm{B/A|B}}\mathcal{O}_{\mathrm{B/A}} = -\dot{\mathcal{O}}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/A}}. \tag{4.2.26}$$

Proof. It follows from (4.2.3) that

$$\Omega_{\mathrm{B/A|B}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}},$$

which proves (4.2.24). Next, it follows from (4.2.6) that

$$\Omega_{\mathrm{B/A|B}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = -\vec{\Omega}'_{\mathrm{B/A}}\Big|_{\mathrm{B}} = -\vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}^{\mathrm{T}} = -\Omega^{\mathrm{T}}_{\mathrm{B/A|B}},$$

which proves (4.2.25). Finally, (4.2.26) follows from (4.2.19). □

Note that (4.2.26) shows that, except in special cases, $\Omega_{\mathrm{B/A|B}} \neq -\Omega_{\mathrm{A/B}}$.

4.3 Physical Angular Velocity Vector and Poisson's Equation

To understand the meaning of $\vec{\Omega}_{\mathrm{B/A}}$, let $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$ so that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)$ on a time interval, where $\theta = \theta_{\hat{\imath}_{\mathrm{B}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}$. Therefore,

$$\vec{\Omega}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(-\theta) \overset{\mathrm{B}\bullet}{\vec{R}}_{\hat{k}_{\mathrm{A}}}(\theta). \tag{4.3.1}$$

Then, it follows that

$$\vec{\Omega}_{\mathrm{B/A}} = \dot{\theta}\hat{k}_{\mathrm{A}}^{\times}. \tag{4.3.2}$$

To see this in terms of components, we resolve (4.3.1) to obtain

$$\Omega_{\mathrm{B/A|B}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}}. \tag{4.3.3}$$

Since $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)$ represents a rotation around $\hat{k}_{\mathrm{A}}$ through the angle θ, it follows that $\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_3(\theta)$. Hence,

$$\Omega_{\mathrm{B/A|B}} = \mathcal{O}_3(\theta)\dot{\mathcal{O}}_3^{\mathrm{T}}(\theta) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}^{\times}. \tag{4.3.4}$$

Consequently, $\Omega_{\mathrm{B/A|B}}$ is associated with a physical vector that is aligned with the axis of rotation and whose length represents the rate of rotation around that axis.

The next result shows that the physical angular velocity matrix can be written as a physical cross product matrix.

Fact 4.3.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, there exists a physical vector $\vec{\omega}_{\mathrm{B/A}}$ such that

$$\vec{\Omega}_{\mathrm{B/A}} = \vec{\omega}_{\mathrm{B/A}}^{\times}. \tag{4.3.5}$$

Furthermore,

$$\Omega_{\mathrm{B/A|B}} = \omega_{\mathrm{B/A|B}}^{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{4.3.6}$$

where

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \triangleq \omega_{\mathrm{B/A|B}} = \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}. \tag{4.3.7}$$

In addition,

$$\vec{\omega}_{\mathrm{A/B}} = -\vec{\omega}_{\mathrm{B/A}}. \tag{4.3.8}$$

Finally,

$$\omega_{\mathrm{B/A|B}} = -\mathcal{O}_{\mathrm{B/A}}\omega_{\mathrm{A/B|A}} = -\mathcal{R}_{\mathrm{A/B}}\omega_{\mathrm{A/B|A}}, \tag{4.3.9}$$

$$\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = -\omega_{\mathrm{A/B|A}} = \mathcal{O}_{\mathrm{A/B}}\omega_{\mathrm{B/A|B}}. \tag{4.3.10}$$

Proof. It follows from Fact 4.2.4 that $\vec{\Omega}_{\mathrm{B/A}}$ is skew symmetric. Fact 2.9.3 thus implies that there exists a physical vector $\vec{\omega}_{\mathrm{B/A}}$ satisfying (4.3.5). Next, (4.3.8) follows from (4.2.19). Finally, (4.3.9) follows from (4.2.26). □

It follows from (4.2.21), (4.2.22), (4.2.3), and (4.2.4) that

$$\vec{\omega}_{\mathrm{B/A}}^{\times} = -\vec{R}_{\mathrm{B/A}} \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} \vec{R}_{\mathrm{A/B}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = -\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} \vec{R}_{\mathrm{B/A}}. \tag{4.3.11}$$

Suppose, for example, $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)$ on a time interval, so that $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$, where $\theta = \theta_{\hat{\imath}_{\mathrm{B}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}$. Then, it follows from (4.3.2) that

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}\hat{k}_{\mathrm{A}} = \dot{\theta}_{\hat{\imath}_{\mathrm{B}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}\hat{k}_{\mathrm{A}}. \tag{4.3.12}$$

Note that (4.3.9) shows that $\omega_{\mathrm{B/A|B}} = -\omega_{\mathrm{A/B|A}}$ may not be true. The following result gives conditions under which $\omega_{\mathrm{B/A|B}} = -\omega_{\mathrm{A/B}}$.

Fact 4.3.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time, the following statements are equivalent:

i) $\vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}$.

ii) $\Omega_{\mathrm{A/B}} = -\Omega_{\mathrm{B/A}}$.

iii) $\dot{\mathcal{O}}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}}$.

iv) $\dot{\mathcal{O}}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/A}} = -\dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}$.

v) $\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}$.

vi) $\omega_{\mathrm{A/B|A}} = -\omega_{\mathrm{B/A|B}}$.

vii) $\mathcal{O}_{\mathrm{B/A}}\omega_{\mathrm{A/B|A}} = \omega_{\mathrm{A/B|A}}$.

viii) $\mathcal{O}_{\mathrm{A/B}}\omega_{\mathrm{A/B|A}} = \omega_{\mathrm{A/B|A}}$.

Fact 4.3.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time, the following statements are equivalent:

i) $n_{\mathrm{B/A}} \times \omega_{\mathrm{B/A|B}} = 0$.

ii) $n_{\mathrm{B/A}} \times \omega_{\mathrm{A/B|A}} = 0$.

Fact 4.3.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time, the following statements are equivalent:

i) $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} = 0$.

ii) $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} = 0$.

iii) $\dot{\omega}_{\mathrm{B/A|B}} = 0$

iv) $\dot{\omega}_{\mathrm{A/B|A}} = 0$.

Proof. To prove that *iii)* implies *iv)*, note that

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} = \dot{\overbrace{\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}}} = \dot{\omega}_{\mathrm{B/A|B}} = 0.$$

Hence, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$, and thus $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$. Therefore,

$$\dot{\omega}_{\mathrm{A/B}} = \dot{\overbrace{\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}}} = \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{A}} = 0. \qquad \square$$

Using (4.3.5) we can rewrite Fact 4.2.7 as follows.

Fact 4.3.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} = \vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}}, \tag{4.3.13}$$

$$\overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} = \vec{\omega}_{\mathrm{B/A}} \times \hat{\jmath}_{\mathrm{B}}, \tag{4.3.14}$$

$$\overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} = \vec{\omega}_{\mathrm{B/A}} \times \hat{k}_{\mathrm{B}}. \tag{4.3.15}$$

The following result gives *Poisson's* equation (4.3.17).

Fact 4.3.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}^{\times}, \tag{4.3.16}$$

$$\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}\omega_{\mathrm{B/A|B}}^{\times}. \tag{4.3.17}$$

Furthermore,

$$\dot{\mathcal{O}}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{A/B}}\omega_{\mathrm{B/A|B}}^{\times}, \tag{4.3.18}$$

and thus

$$\dot{\mathcal{O}}_{\mathrm{B/A}} = -\omega_{\mathrm{B/A|B}}^{\times}\mathcal{O}_{\mathrm{B/A}}. \tag{4.3.19}$$

Hence,

$$\omega_{\mathrm{B/A|B}}^{\times} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}} = -\dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}} = -\dot{\mathcal{R}}_{\mathrm{A/B}}\mathcal{R}_{\mathrm{B/A}}. \tag{4.3.20}$$

Proof. Resolving (4.2.7) in $\mathrm{F_B}$ yields (4.3.17). □

The following result provides a vectrix version of Poisson's equation.

Fact 4.3.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} \end{bmatrix} = -\omega_{\mathrm{B/A|B}}^{\times}\begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix}. \tag{4.3.21}$$

Proof. It follows from (2.10.13) and (4.3.20) that

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} \end{bmatrix} = \dot{\mathcal{O}}_{\mathrm{B/A}}\begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix} = \dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}\begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix} = -\omega_{\mathrm{B/A|B}}^{\times}\begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix}.$$ □

Resolving (4.3.21) in $\mathrm{F_A}$ yields (4.3.17). Furthermore, (4.3.21) can be written as

$$\overset{\mathrm{A}\bullet}{\mathcal{F}}_{\mathrm{B}} = -\omega_{\mathrm{B/A|B}}^{\times}\mathcal{F}_{\mathrm{B}}. \tag{4.3.22}$$

4.4 Transport Theorem

The following result is the *transport theorem*. Note the "ABBA" pattern.

Fact 4.4.1. Let $\vec{x}$ be a physical vector, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{x}} = \overset{\mathrm{B}\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{x}. \tag{4.4.1}$$

Proof. The result follows from (4.3.5) and (4.2.5). □

The following result is an immediate consequence of the transport theorem.

Fact 4.4.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} . \tag{4.4.2}$$

Proof.

$$\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} . \qquad \square$$

The *angular acceleration of* $\mathrm{F_B}$ *relative to* $\mathrm{F_A}$ is defined by

$$\vec{\alpha}_{\mathrm{B/A}} \triangleq \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} . \tag{4.4.3}$$

Now, let C be a frame, Then, the *mixed angular acceleration* is defined by

$$\vec{\alpha}_{\mathrm{B/A/C}} \triangleq \overset{\mathrm{C}\bullet}{\vec{\omega}}_{\mathrm{B/A}} . \tag{4.4.4}$$

Note that $\vec{\alpha}_{\mathrm{B/A/A}} = \vec{\alpha}_{\mathrm{B/A/B}} = \vec{\alpha}_{\mathrm{B/A}}$. Furthermore, let $\mathrm{F_D}$ be a frame. Then, the angular acceleration is resolved as

$$\alpha_{\mathrm{B/A|D}} \triangleq \vec{\alpha}_{\mathrm{B/A}}\Big|_{\mathrm{D}} . \tag{4.4.5}$$

Note that

$$\alpha_{\mathrm{B/A|D}} = \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{D}} = \mathcal{O}_{\mathrm{D/A}}\dot{\omega}_{\mathrm{B/A|A}} = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{D}} = \mathcal{O}_{\mathrm{D/B}}\dot{\omega}_{\mathrm{B/A|B}}. \tag{4.4.6}$$

Finally, the mixed angular acceleration is resolved as

$$\alpha_{\mathrm{B/A/C|D}} \triangleq \vec{\alpha}_{\mathrm{B/A/C}}\Big|_{\mathrm{D}} \triangleq \overset{\mathrm{C}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{D}} . \tag{4.4.7}$$

Note that

$$\alpha_{\mathrm{B/A/C|D}} = \mathcal{O}_{\mathrm{D/C}}\, \vec{\alpha}_{\mathrm{B/A/C}}\Big|_{\mathrm{C}} = \mathcal{O}_{\mathrm{D/C}}\, \overset{\mathrm{C}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{C}} = \mathcal{O}_{\mathrm{D/C}}\dot{\omega}_{\mathrm{B/A|C}}. \tag{4.4.8}$$

The following result is an immediate consequence of the transport theorem.

Fact 4.4.3. Let $\mathrm{F_A}$ be a frame, let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let x, y, z be points, and assume that y and z are fixed in $\mathcal{B}$. Then,

$$\vec{v}_{z/x/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{z/y} + \vec{v}_{y/x/\mathrm{A}}. \tag{4.4.9}$$

The following result is based on Figure 4.4.1.

Fact 4.4.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames with origins $\mathrm{O_A}$ and $\mathrm{O_B}$, respectively, and let x be a point. Then,

$$\vec{v}_{x/\mathrm{O_A}/\mathrm{A}} = \vec{v}_{x/\mathrm{O_A}/\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{O_A}}, \tag{4.4.10}$$

$$\vec{v}_{x/\mathrm{O_A}/\mathrm{A}} = \vec{v}_{x/\mathrm{O_B}/\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{O_B}} + \vec{v}_{\mathrm{O_B}/\mathrm{O_A}/\mathrm{A}}. \tag{4.4.11}$$

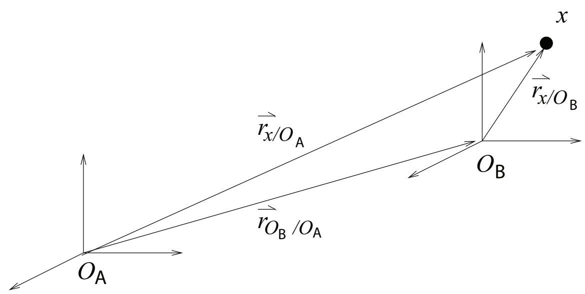


Figure 4.4.1: Geometry for relative motion.

Fact 4.4.5. Let $\vec{M}$ be a physical matrix, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{M}} = \overset{\mathrm{B}\bullet}{\vec{M}} + \vec{\Omega}_{\mathrm{B/A}}\vec{M} - \vec{M}\vec{\Omega}_{\mathrm{B/A}}. \tag{4.4.12}$$

Proof. For convenience, assume that $\vec{M} = \vec{x}\,\vec{y}\,'$. Note that

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{M}} &= \overset{\mathrm{A}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{A}\bullet}{\vec{y}}{}'\\
&= (\overset{\mathrm{B}\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}}\times\vec{x})\vec{y}\,' + \vec{x}(\overset{\mathrm{B}\bullet}{\vec{y}} + \vec{\omega}_{\mathrm{B/A}}\times\vec{y})'\\
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet}{\vec{y}}{}' + (\vec{\omega}_{\mathrm{B/A}}\times\vec{x})\vec{y}\,' + \vec{x}(\vec{\omega}_{\mathrm{B/A}}\times\vec{y})'\\
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet}{\vec{y}}{}' + (\vec{\omega}_{\mathrm{B/A}}^{\times}\vec{x})\vec{y}\,' + \vec{x}(\vec{\omega}_{\mathrm{B/A}}^{\times}\vec{y})'\\
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet}{\vec{y}}{}' + \vec{\omega}_{\mathrm{B/A}}^{\times}\vec{x}\,\vec{y}\,' + \vec{x}\,\vec{y}\,'\vec{\omega}_{\mathrm{B/A}}^{\times\prime}\\
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet}{\vec{y}}{}' + \vec{\omega}_{\mathrm{B/A}}^{\times}\vec{x}\,\vec{y}\,' - \vec{x}\,\vec{y}\,'\vec{\omega}_{\mathrm{B/A}}^{\times}\\
&= \overset{\mathrm{B}\bullet}{\vec{M}} + \vec{\Omega}_{\mathrm{B/A}}\vec{M} - \vec{M}\vec{\Omega}_{\mathrm{B/A}}.
\end{aligned}$$

□

Fact 4.4.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} + \vec{\Omega}_{\mathrm{B/A}}\vec{R}_{\mathrm{B/A}} - \vec{R}_{\mathrm{B/A}}\vec{\Omega}_{\mathrm{B/A}}, \tag{4.4.13}$$

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} + \vec{\Omega}_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}} - \vec{R}_{\mathrm{A/B}}\vec{\Omega}_{\mathrm{B/A}}. \tag{4.4.14}$$

4.5 Double Transport Theorem

Applying the transport theorem twice yields the *double transport theorem*.

Fact 4.5.1. Let F_{A} and F_{B} be frames, and let $\vec{x}$ be a physical vector. Then,

$$\overset{\mathrm{A}\bullet\bullet}{\vec{x}} = \overset{\mathrm{B}\bullet\bullet}{\vec{x}} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}). \tag{4.5.1}$$

Proof.

$$\begin{aligned}
\overset{\mathrm{A}\bullet\bullet}{\vec{x}} &= \overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{x}}} + \overbrace{\vec{\omega}_{\mathrm{B/A}} \times \vec{x}}^{\mathrm{A}\bullet} \\
&= \overset{\mathrm{B}\bullet\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\vec{x}} \\
&= \overset{\mathrm{B}\bullet\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times (\overset{\mathrm{B}\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{x}) \\
&= \overset{\mathrm{B}\bullet\bullet}{\vec{x}} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}).
\end{aligned}$$

□

If $\vec{x}$ is the position vector $\vec{r}_{x/\mathrm{O_B}}$, (4.5.1) can be written as

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{x/\mathrm{O_B}} = \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{x/\mathrm{O_B}} + \underbrace{2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{r}}_{x/\mathrm{O_B}}}_{\text{Coriolis acceleration}} + \underbrace{\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{O_B}}}_{\substack{\text{angular-acceleration} \\ (\mathrm{A}^2)\text{ acceleration}}} + \underbrace{\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{O_B}})}_{\text{centripetal acceleration}}. \tag{4.5.2}$$

This identity can be written in terms of position, velocity, and acceleration vectors, as shown by the following result and illustrated in Figure 4.4.1. Note that, if $\vec{\omega}_{\mathrm{B/A}}$ is perpendicular to a plane containing x and $\mathrm{O_B}$, then the centripetal acceleration is given by $-|\vec{\omega}_{\mathrm{B/A}}|^2\vec{r}_{x/\mathrm{O_B}}$, whose direction is opposite to the direction of $\vec{r}_{x/\mathrm{O_B}}$.

Fact 4.5.2. Let F_{A} and F_{B} be frames with origins $\mathrm{O_A}$ and $\mathrm{O_B}$, respectively, and let x be a point. Then,

$$\vec{a}_{x/\mathrm{O_A}/\mathrm{A}} = \vec{a}_{x/\mathrm{O_B}/\mathrm{B}} + 2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/\mathrm{O_B}/\mathrm{B}} + \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{O_B}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{O_B}}) + \vec{a}_{\mathrm{O_B}/\mathrm{O_A}/\mathrm{A}}. \tag{4.5.3}$$

Using (4.4.10), it follows that (4.5.3) can be written as

$$\vec{a}_{x/\mathrm{O_A}/\mathrm{A}} = \vec{a}_{x/\mathrm{O_B}/\mathrm{B}} + 2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/\mathrm{O_B}/\mathrm{B}} + \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{O_B}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{v}_{x/\mathrm{O_B}/\mathrm{A}} - \vec{v}_{x/\mathrm{O_B}/\mathrm{B}}) + \vec{a}_{\mathrm{O_B}/\mathrm{O_A}/\mathrm{A}}. \tag{4.5.4}$$

4.6 Summation of Angular Velocities and Angular Accelerations

Fact 4.6.1. Let F_{A}, F_{B}, and F_{C} be frames. Then,

$$\vec{\Omega}_{\mathrm{C/A}} = \vec{\Omega}_{\mathrm{C/B}} + \vec{\Omega}_{\mathrm{B/A}}, \tag{4.6.1}$$

$$\vec{\omega}_{\mathrm{C/A}} = \vec{\omega}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}}. \tag{4.6.2}$$

Furthermore,

$$\Omega_{\mathrm{C/A|C}} = \Omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}}\Omega_{\mathrm{B/A|B}}\mathcal{O}_{\mathrm{B/C}}, \tag{4.6.3}$$

$$\omega_{\mathrm{C/A|C}} = \omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}}\omega_{\mathrm{B/A|B}}. \tag{4.6.4}$$

Proof. Using the transport theorem and (4.2.19), we have

$$
\begin{aligned}
\vec{\Omega}_{\mathrm{C/A}} &= \hat{\imath}_{\mathrm{A}} \overset{\mathrm{C}\bullet\prime}{\hat{\imath}}_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}} \overset{\mathrm{C}\bullet\prime}{\hat{\jmath}}_{\mathrm{A}} + \hat{k}_{\mathrm{A}} \overset{\mathrm{C}\bullet\prime}{\hat{k}}_{\mathrm{A}} \\
&= \hat{\imath}_{\mathrm{A}} \left(\overset{\mathrm{B}\bullet}{\hat{\imath}}_{\mathrm{A}} + \vec{\Omega}_{\mathrm{B/C}} \hat{\imath}_{\mathrm{A}} \right)' + \hat{\jmath}_{\mathrm{A}} \left(\overset{\mathrm{B}\bullet}{\hat{\jmath}}_{\mathrm{A}} + \vec{\Omega}_{\mathrm{B/C}} \hat{\jmath}_{\mathrm{A}} \right)' + \hat{k}_{\mathrm{A}} \left(\overset{\mathrm{B}\bullet}{\hat{k}}_{\mathrm{A}} + \vec{\Omega}_{\mathrm{B/C}} \hat{k}_{\mathrm{A}} \right)' \\
&= \hat{\imath}_{\mathrm{A}} (\vec{\Omega}_{\mathrm{B/C}} \hat{\imath}_{\mathrm{A}})' + \hat{\jmath}_{\mathrm{A}} \left(\vec{\Omega}_{\mathrm{B/C}} \hat{\jmath}_{\mathrm{A}} \right)' + \hat{k}_{\mathrm{A}} \left(\vec{\Omega}_{\mathrm{B/C}} \hat{k}_{\mathrm{A}} \right)' + \hat{\imath}_{\mathrm{A}} \overset{\mathrm{B}\bullet\prime}{\hat{\imath}}_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}} \overset{\mathrm{B}\bullet\prime}{\hat{\jmath}}_{\mathrm{A}} + \hat{k}_{\mathrm{A}} \overset{\mathrm{B}\bullet\prime}{\hat{k}}_{\mathrm{A}} \\
&= \hat{\imath}_{\mathrm{A}} \hat{\imath}_{\mathrm{A}}' \vec{\Omega}_{\mathrm{C/B}} + \hat{\jmath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}}' \vec{\Omega}_{\mathrm{C/B}} + \hat{k}_{\mathrm{A}} \hat{k}_{\mathrm{A}}' \vec{\Omega}_{\mathrm{C/B}} + \vec{\Omega}_{\mathrm{B/A}} \\
&= \vec{\Omega}_{\mathrm{C/B}} + \vec{\Omega}_{\mathrm{B/A}}.
\end{aligned}
$$

Finally, to prove (4.6.3), note that

$$
\begin{aligned}
\Omega_{\mathrm{C/A|C}} &= \vec{\Omega}_{\mathrm{C/A}} \Big|_{\mathrm{C}} \\
&= \Omega_{\mathrm{C/B|C}} + \vec{\Omega}_{\mathrm{B/A}} \Big|_{\mathrm{C}} \\
&= \Omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}} \vec{\Omega}_{\mathrm{B/A}} \Big|_{\mathrm{B}} \mathcal{O}_{\mathrm{B/C}} \\
&= \Omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}} \Omega_{\mathrm{B/A|B}} \mathcal{O}_{\mathrm{B/C}}.
\end{aligned}
$$

□

Fact 4.6.2. Let F_{A}, F_{B}, and F_{C} be frames. Then,

$$
\vec{\alpha}_{\mathrm{C/A}} = \vec{\alpha}_{\mathrm{C/B}} + \vec{\alpha}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{C/B}} \times \vec{\omega}_{\mathrm{B/A}}. \tag{4.6.5}
$$

Fact 4.6.3. Let F_{A}, F_{B}, F_{C}, and F_{D} be frames. Then,

$$
\vec{\alpha}_{\mathrm{D/A}} = \vec{\alpha}_{\mathrm{D/C}} + \vec{\alpha}_{\mathrm{C/B}} + \vec{\alpha}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{D/C}} \times \vec{\omega}_{\mathrm{C/B}} - \vec{\omega}_{\mathrm{D/B}} \times \vec{\omega}_{\mathrm{B/A}} \tag{4.6.6}
$$

$$
= \vec{\alpha}_{\mathrm{D/C}} + \vec{\alpha}_{\mathrm{C/B}} + \vec{\alpha}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{D/C}} \times \vec{\omega}_{\mathrm{C/A}} - \vec{\omega}_{\mathrm{C/B}} \times \vec{\omega}_{\mathrm{B/A}}. \tag{4.6.7}
$$

4.7 Angular Velocity Vector and the Eigenaxis Derivative

In this section we relate the angular velocity vector to the derivative of the eigenaxis and eigenangle, which appear in Rodrigues's formula.

Fact 4.7.1. Let F_{A} be a frame, and let $\hat{n}$ be a unit dimensionless physical vector. Then, for all time,

$$
\overset{\mathrm{A}\bullet\prime}{\hat{n}}\, \hat{n} = \hat{n}'\, \overset{\mathrm{A}\bullet}{\hat{n}} = 0, \tag{4.7.1}
$$

$$
\hat{n}'\, \overset{\mathrm{A}\bullet\times}{\hat{n}} = -\overset{\mathrm{A}\bullet\prime}{\hat{n}}\, \hat{n}^{\times}, \tag{4.7.2}
$$

$$
\hat{n}'(\hat{n} \times \overset{\mathrm{A}\bullet}{\hat{n}}) = 0, \tag{4.7.3}
$$

$$
\overset{\mathrm{A}\bullet\prime}{\hat{n}}\, (\hat{n} \times \overset{\mathrm{A}\bullet}{\hat{n}}) = 0, \tag{4.7.4}
$$

$$
\hat{n} \times (\hat{n} \times \overset{\mathrm{A}\bullet}{\hat{n}}) = -\overset{\mathrm{A}\bullet}{\hat{n}}, \tag{4.7.5}
$$

$$\hat{n}\overset{\mathrm{A}\bullet\prime}{\hat{n}}\hat{n}^{\times} + \hat{n}^{\times}\overset{\mathrm{A}\bullet}{\hat{n}}\hat{n}' = -\overset{\mathrm{A}\bullet\times}{\hat{n}}, \tag{4.7.6}$$

$$\overset{\mathrm{A}\bullet\times}{\hat{n}}\hat{n}^{\times} = \hat{n}\overset{\mathrm{A}\bullet\prime}{\hat{n}}, \tag{4.7.7}$$

$$\hat{n}^{\times}\overset{\mathrm{A}\bullet\times}{\hat{n}} = \overset{\mathrm{A}\bullet}{\hat{n}}\hat{n}', \tag{4.7.8}$$

$$(\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}})^{\times} = \overset{\mathrm{A}\bullet}{\hat{n}}\hat{n}' - \hat{n}\overset{\mathrm{A}\bullet\prime}{\hat{n}}, \tag{4.7.9}$$

$$|\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}}| = |\overset{\mathrm{A}\bullet}{\hat{n}}|, \tag{4.7.10}$$

$$\overset{\mathrm{A}\bullet\bullet\prime}{\hat{n}}\hat{n} + \overset{\mathrm{A}\bullet\prime}{\hat{n}}\overset{\mathrm{A}\bullet}{\hat{n}} = 0. \tag{4.7.11}$$

Furthermore, at each instant of time, the following conditions are equivalent:

i) $\overset{\mathrm{A}\bullet}{\hat{n}} = 0$.

ii) $\overset{\mathrm{A}\bullet}{\hat{n}}\hat{n}' + \hat{n}\overset{\mathrm{A}\bullet\prime}{\hat{n}} = 0$.

iii) $\hat{n}\times(\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}}) = 0$.

iv) $\hat{n}\overset{\mathrm{A}\bullet\prime}{\hat{n}}\hat{n}^{\times} + \hat{n}^{\times}\overset{\mathrm{A}\bullet}{\hat{n}}\hat{n}' = 0$.

v) $\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}} = 0$.

Finally, if, for all $t\in[t_1,t_2]$, $\overset{\mathrm{A}\bullet\bullet}{\hat{n}}(t) = 0$, then, for all $t\in[t_1,t_2]$, $\overset{\mathrm{A}\bullet}{\hat{n}}(t) = 0$.

Proof. The proof is left to the reader. □

For the following result note that

$$\cot\tfrac{1}{2}\theta_{\mathrm{B/A}} = \frac{\sin\theta_{\mathrm{B/A}}}{1-\cos\theta_{\mathrm{B/A}}} = \frac{1+\cos\theta_{\mathrm{B/A}}}{\sin\theta_{\mathrm{B/A}}}. \tag{4.7.12}$$

Fact 4.7.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}}\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.13}$$

$$= (\cos\theta_{\mathrm{B/A}})\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} - (\sin\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\times\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.14}$$

$$= \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}}\times\hat{n}, \tag{4.7.15}$$

$$\begin{aligned}\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} = {} & [(\sin\theta_{\mathrm{B/A}})(\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - \vec{I}) + (\cos\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]\dot{\theta}_{\mathrm{B/A}}\\ & + (1-\cos\theta_{\mathrm{B/A}})(\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + \hat{n}_{\mathrm{B/A}}\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}) + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{A}\bullet\times}{\hat{n}}_{\mathrm{B/A}},\end{aligned} \tag{4.7.16}$$

$$\begin{aligned}\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = {} & [(\sin\theta_{\mathrm{B/A}})(\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - \vec{I}) + (\cos\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]\dot{\theta}_{\mathrm{B/A}}\\ & + (1-\cos\theta_{\mathrm{B/A}})(\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + \hat{n}_{\mathrm{B/A}}\overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}) + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{B}\bullet\times}{\hat{n}}_{\mathrm{B/A}},\end{aligned} \tag{4.7.17}$$

$$
\begin{aligned}
\vec{\omega}_{\mathrm{B/A}} &= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \qquad (4.7.18)\\
&= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} - (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}, \qquad (4.7.19)
\end{aligned}
$$

$$
\begin{aligned}
|\vec{\omega}_{\mathrm{B/A}}| &= \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(1-\cos\theta_{\mathrm{B/A}})\,|\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2} \qquad (4.7.20)\\
&= \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(1-\cos\theta_{\mathrm{B/A}})\,|\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2}, \qquad (4.7.21)
\end{aligned}
$$

$$
\dot{\theta}_{\mathrm{B/A}} = \hat{n}'_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}, \qquad (4.7.22)
$$

$$
\begin{aligned}
\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\,\vec{\omega}_{\mathrm{B/A}} &= (\sin\theta_{\mathrm{B/A}})\,\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \qquad (4.7.23)\\
&= -(\sin\theta_{\mathrm{B/A}})\,\overset{\mathrm{A}\bullet\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\,\hat{n}_{\mathrm{B/A}}, \qquad (4.7.24)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\,\vec{\omega}_{\mathrm{B/A}} &= (\sin\theta_{\mathrm{B/A}})\,\overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \qquad (4.7.25)\\
&= -(\sin\theta_{\mathrm{B/A}})\,\overset{\mathrm{B}\bullet\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\,\hat{n}_{\mathrm{B/A}}. \qquad (4.7.26)
\end{aligned}
$$

Finally, if $\theta_{\mathrm{B/A}} \neq 0$, then

$$
\begin{aligned}
|\vec{\omega}_{\mathrm{B/A}}| &= \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\,\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\,\vec{\omega}_{\mathrm{B/A}}} \qquad (4.7.27)\\
&= \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\,\overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\,\vec{\omega}_{\mathrm{B/A}}}, \qquad (4.7.28)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} &= \tfrac{1}{2}[(\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) - \hat{n}^{\times}_{\mathrm{B/A}}]\vec{\omega}_{\mathrm{B/A}} \qquad (4.7.29)\\
&= \tfrac{1}{2}[\dot{\theta}_{\mathrm{B/A}}(\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}}] \qquad (4.7.30)
\end{aligned}
$$

and

$$
\begin{aligned}
\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} &= \tfrac{1}{2}[(\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) + \hat{n}^{\times}_{\mathrm{B/A}}]\vec{\omega}_{\mathrm{B/A}} \qquad (4.7.31)\\
&= \tfrac{1}{2}[\dot{\theta}_{\mathrm{B/A}}(\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}}]. \qquad (4.7.32)
\end{aligned}
$$

Proof. Using Fact 2.11.8 and Fact 4.7.1, recalling that

$$
\vec{R}_{\mathrm{A/B}} = (\cos\theta_{\mathrm{B/A}})\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}},
$$

it follows that

$$
\begin{aligned}
\vec{\omega}^{\times}_{\mathrm{B/A}} &= \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{A/B}}\\
&= \dot{\theta}_{\mathrm{B/A}}\hat{n}^{\times}_{\mathrm{B/A}} + (1-\cos\theta_{\mathrm{B/A}})(\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}})^{\times} + (\sin\theta_{\mathrm{B/A}})\,\overset{\mathrm{A}\bullet\times}{\hat{n}}_{\mathrm{B/A}}\,.
\end{aligned}
$$

Next, sufficiency in *i*) is immediate. To prove necessity, suppose that $\vec{\omega}_{\mathrm{B/A}} = 0$. Then, $\dot{\theta}_{\mathrm{B/A}} = \hat{n}'_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}} = 0$. Furthermore, note that $(\sin\theta_{\mathrm{B/A}})|\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2 = \overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}} = 0$. Therefore, either $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$ or $\theta_{\mathrm{B/A}} = 0$ or $\theta_{\mathrm{B/A}} = \pi$. In the case that $\theta_{\mathrm{B/A}} = \pi$, it follows that $\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$. Finally, substituting (4.7.18) into the right hand side of (4.7.29) yields $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}$.

To derive (4.7.29), multiply (4.7.18) by $\hat{n}^{\times}_{\mathrm{B/A}}$, which yields

$$\hat{n}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} = (\cos\theta_{\mathrm{B/A}} - 1)[\vec{I} - (\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]\ \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}.$$

Hence,

$$\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \frac{1}{\cos\theta_{\mathrm{B/A}} - 1}[\vec{I} - (\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]^{-1}(\hat{n}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}}).$$

Now, using (2.9.8) yields (4.7.29). □

Equation (4.7.18) gives an expression for $\vec{\omega}_{\mathrm{B/A}}$ in terms of the eigenaxis $\hat{n}_{\mathrm{B/A}}$ and its derivative. If the eigenaxis is constant, then $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}$, which includes an Euler rotation as a special case.

Fact 4.7.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time the following statements are equivalent:

i) $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}$.

ii) $\vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}} = 0$.

iii) $\omega_{\mathrm{B/A}} \times n_{\mathrm{B/A}} = 0$.

iv) $\omega_{\mathrm{A/B}} \times n_{\mathrm{B/A}} = 0$.

v) $\mathcal{O}_{\mathrm{A/B}}\dot{n}_{\mathrm{B/A}} = \dot{n}_{\mathrm{B/A}}$.

vi) $\mathcal{O}_{\mathrm{B/A}}\dot{n}_{\mathrm{B/A}} = \dot{n}_{\mathrm{B/A}}$.

Furthermore, at each instant of time the following statements hold:

vii) If *i*)–*vi*) are satisfied and $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \neq 0$, then $\theta_{\mathrm{B/A}}(t) = 0$.

viii) If $\theta_{\mathrm{B/A}} = 0$, then *i*)–*vi*) are satisfied.

ix) If *i*)–*vi*) are satisfied and $\theta_{\mathrm{B/A}} \neq 0$, then $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

x) The following conditions are equivalent:

a) $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

b) $\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

xi) The following conditions are equivalent:

a) Either $\theta_{\mathrm{B/A}} = 0$ or $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

b) $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}$.

xii) If $\theta_{\mathrm{B/A}} = 0$ and $\dot{\theta}_{\mathrm{B/A}} = 0$, then $\vec{\omega}_{\mathrm{B/A}} = 0$.

xiii) The following conditions are equivalent:

a) $\vec{\omega}_{\mathrm{B/A}} = 0$.

b) $\dot{\theta}_{\mathrm{B/A}} = 0$ and either $\theta_{\mathrm{B/A}} = 0$ or $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

c) $\dot{\theta}_{\mathrm{B/A}} = 0$ and either $\theta_{\mathrm{B/A}} = 0$ or $\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

xiv) The following conditions are equivalent:

a) $\vec{\omega}_{\mathrm{B/A}} \neq 0$.

b) Either $\dot{\theta}_{\mathrm{B/A}} \neq 0$ or both $\theta_{\mathrm{B/A}} \neq 0$ and $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \neq 0$.

c) Either $\dot{\theta}_{\mathrm{B/A}} \neq 0$ or both $\theta_{\mathrm{B/A}} \neq 0$ and $\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \neq 0$.

xv) If $\vec{\omega}_{\mathrm{B/A}} \neq 0$, then either $\theta_{\mathrm{B/A}} \neq 0$ or $\dot{\theta}_{\mathrm{B/A}} \neq 0$.

Fact 4.7.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\omega_{\mathrm{B/A}} \neq 0$ and, for all time $t \in (t_1, t_2)$, $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) = 0$. Then, the following conditions are equivalent:

i) For all time $t \in (t_1, t_2)$, $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = 0$.

ii) For all time $t \in (t_1, t_2)$, $\vec{\omega}_{\mathrm{B/A}}(t) \times \hat{n}_{\mathrm{B/A}}(t) = 0$.

Proof. The result follows from Problem 4.20.3. □

For small angles $\theta_{\mathrm{B/A}}$ we have

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + [\tfrac{1}{2}\theta^2_{\mathrm{B/A}} + O(\theta^4_{\mathrm{B/A}})]\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + [\theta_{\mathrm{B/A}} + O(\theta^3_{\mathrm{B/A}})]\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.33}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} - [\tfrac{1}{2}\theta^2_{\mathrm{B/A}} + O(\theta^4_{\mathrm{B/A}})]\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + [\theta_{\mathrm{B/A}} + O(\theta^3_{\mathrm{B/A}})]\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}, \tag{4.7.34}$$

$$|\vec{\omega}_{\mathrm{B/A}}| = \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + [\theta^2_{\mathrm{B/A}} + O(\theta^4_{\mathrm{B/A}})]\,|\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2}. \tag{4.7.35}$$

where $\lim_{x\to 0} O(x)/x$ exists. We thus have the approximations

$$\vec{\omega}_{\mathrm{B/A}} \approx \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \tfrac{1}{2}\theta^2_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}}\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}, \tag{4.7.36}$$

$$\vec{\omega}_{\mathrm{B/A}} \approx \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} - \tfrac{1}{2}\theta^2_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}}\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}, \tag{4.7.37}$$

$$|\vec{\omega}_{\mathrm{B/A}}| \approx \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + \theta^2_{\mathrm{B/A}}|\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2}. \tag{4.7.38}$$

Consequently,

$$\lim_{\theta_{\mathrm{B/A}}\to 0} \vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}, \tag{4.7.39}$$

$$\lim_{\theta_{\mathrm{B/A}}\to 0} |\vec{\omega}_{\mathrm{B/A}}| = |\dot{\theta}_{\mathrm{B/A}}|. \tag{4.7.40}$$

Therefore, for an infinitisimal rotation from $\mathrm{F_A}$ to $\mathrm{F_B}$, the angular velocity vector $\vec{\omega}_{\mathrm{B/A}}$ can be viewed as the eigenaxis of rotation scaled by the rate of rotation. In other words, the physical vector $\vec{\omega}_{\mathrm{B/A}}$ can be viewed as the instantaneous axis of rotation, where the rate of rotation is given by $|\vec{\omega}_{\mathrm{B/A}}|$ and the direction of rotation is given by the curled fingers of the right hand with the thumb pointing in the direction of $\vec{\omega}_{\mathrm{B/A}}$.

4.8 Angular Acceleration Vector and the Eigenaxis and Eigenangle

Differentiating (4.7.18) and (4.7.19) yields

$$\begin{aligned}
\vec{\alpha}_{\mathrm{B/A}} &= \ddot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \dot{\theta}_{\mathrm{B/A}}[(1+\cos\theta_{\mathrm{B/A}})\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\times\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&\quad + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\times\overset{\mathrm{A}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{A}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}} && (4.8.1)\\
&= \ddot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \dot{\theta}_{\mathrm{B/A}}[(1+\cos\theta_{\mathrm{B/A}})\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} - (\sin\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\times\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&\quad - (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\times\overset{\mathrm{B}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{B}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}}. && (4.8.2)
\end{aligned}$$

Fact 4.8.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let t be an instant of time. Then, the following statements hold:

i) If $\theta_{\mathrm{B/A}}(t) = 0$, then $\vec{\alpha}_{\mathrm{B/A}}(t) = \ddot{\theta}_{\mathrm{B/A}}(t)\hat{n}_{\mathrm{B/A}}(t) + 2\dot{\theta}_{\mathrm{B/A}}(t)\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t)$.

ii) If $\vec{\alpha}_{\mathrm{B/A}}(t) = 0$, then

$$\begin{aligned}
\ddot{\theta}_{\mathrm{B/A}}(t) &= \overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}(t)\vec{\omega}_{\mathrm{B/A}}(t) && (4.8.3)\\
&= [\sin\theta_{\mathrm{B/A}}(t)]\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}(t)\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t) && (4.8.4)\\
&= -[\sin\theta_{\mathrm{B/A}}(t)]\hat{n}'_{\mathrm{B/A}}(t)\overset{\mathrm{A}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}}(t), && (4.8.5)
\end{aligned}$$

$$|\vec{\omega}_{\mathrm{B/A}}(t)|^2 = \dot{\theta}^2_{\mathrm{B/A}}(t) + 2[\tan\tfrac{1}{2}\theta_{\mathrm{B/A}}(t)]\ddot{\theta}_{\mathrm{B/A}}(t). \tag{4.8.6}$$

iii) If $\theta_{\mathrm{B/A}}(t) = 0$ and $\vec{\alpha}_{\mathrm{B/A}}(t) = 0$, then $|\vec{\omega}_{\mathrm{B/A}}(t)| = |\dot{\theta}_{\mathrm{B/A}}|$, $\ddot{\theta}_{\mathrm{B/A}}(t) = 0$, $\dot{\theta}_{\mathrm{B/A}}(t)\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = 0$, and $\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}(t)\vec{\omega}_{\mathrm{B/A}}(t) = 0$.

iv) Assume that $\vec{\alpha}_{\mathrm{B/A}}(t) = 0$ and $\sin\theta_{\mathrm{B/A}}(t) \neq 0$. Then, $\ddot{\theta}_{\mathrm{B/A}}(t) = 0$ if and only if $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = 0$.

v) Assume that $\overset{\mathrm{A}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = 0$. Then, $\vec{\alpha}_{\mathrm{B/A}}(t) = 0$ if and only if $\ddot{\theta}_{\mathrm{B/A}}(t) = 0$ and $\dot{\theta}_{\mathrm{B/A}}[(1+\cos\theta_{\mathrm{B/A}})\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\times\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}] = 0$.

vi) Assume that $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = 0$ and $\overset{\mathrm{A}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = 0$. Then, $\vec{\alpha}_{\mathrm{B/A}}(t) = 0$ if and only if $\ddot{\theta}_{\mathrm{B/A}}(t) = 0$.

Proof. All statements follow from (4.8.1). $\square$

Fact 4.8.2. Let F_A and F_B be frames, assume that there exists an instant of time t such that $\omega_{B/A}(t) \neq 0$, assume that, for all t, $\overset{A\bullet}{\vec{\omega}}_{B/A}(t) = 0$, and assume that there exists an instant of time t_0 such that $\mathcal{R}_{B/A}(t_0) = I_3$. Then, for all $t \geq t_0$ such that $\sin\theta_{B/A}(t) \neq 0$, it follows that $\overset{A\bullet}{\hat{n}}_{B/A}(t) = \overset{B\bullet}{\hat{n}}_{B/A}(t) = 0$ and $\ddot{\theta}_{B/A}(t) = 0$.

Proof. For convenience, let $t_0 = 0$. It follows from Fact 4.3.4 that both $\omega_{B/A|B}$ and $\omega_{A/B|A}$ are constant. Furthermore, since $\mathcal{O}_{A/B}(0) = I_3$, it follows from (4.3.10) that $\omega_{B/A|B} = \mathcal{O}_{A/B}(0)\omega_{B/A|B} = -\omega_{A/B|A} \neq 0$.

Next, it follows from (4.3.17) that

$$\dot{\mathcal{R}}_{A/B}(t) = -\omega^{\times}_{B/A|B}\mathcal{R}_{A/B}(t),$$
$$\dot{\mathcal{R}}_{B/A}(t) = -\omega^{\times}_{A/B|A}\mathcal{R}_{B/A}(t).$$

Therefore,

$$\mathcal{R}_{A/B}(t) = e^{-\omega^{\times}_{B/A|B}t}\mathcal{R}_{A/B}(0) = \mathcal{R}_{A/B}(0)e^{\omega^{\times}_{A/B|A}t},$$
$$\mathcal{R}_{B/A}(t) = e^{-\omega^{\times}_{A/B|A}t}\mathcal{R}_{B/A}(0) = \mathcal{R}_{B/A}(0)e^{\omega^{\times}_{B/A|B}t}.$$

Hence, it follows from (2.11.43) that, for all $t \geq 0$,

$$\begin{aligned}
2\sin\theta_{B/A}(t)n_{B/A}(t) \times \omega_{B/A|B} &= [\mathcal{R}_{B/A}(t) - \mathcal{R}_{A/B}(t)]\omega_{B/A|B}\\
&= \mathcal{R}_{B/A}(0)e^{\omega^{\times}_{B/A|B}t}\omega_{B/A|B} - \mathcal{R}_{A/B}(0)e^{\omega^{\times}_{A/B|A}t}\omega_{B/A|rmB}\\
&= \mathcal{R}_{B/A}(0)e^{\omega^{\times}_{B/A|B}t}\omega_{B/A|B} - \mathcal{R}_{A/B}(0)e^{-\omega^{\times}_{B/A|B}t}\omega_{B/A|B}\\
&= \mathcal{R}_{B/A}(0)\omega_{B/A|B} - \mathcal{R}_{A/B}(0)\omega_{B/A|B}\\
&= [\mathcal{R}_{B/A}(0) - \mathcal{R}_{A/B}(0)]\omega_{B/A|B}\\
&= 0.
\end{aligned}$$

Therefore, for all $t \geq 0$ such that $\sin\theta_{B/A}(t) \neq 0$, it follows that $\hat{n}_{B/A} \times \vec{\omega}_{B/A} = 0$, and thus $\overset{A\bullet}{\hat{n}}_{B/A} \times \vec{\omega}_{B/A} = 0$. It thus follows from Problem 4.20.4 that, for all $t \geq 0$ such that $\sin\theta_{B/A}(t) \neq 0$, it follows that $\overset{A\bullet}{\hat{n}}_{B/A} = \overset{B\bullet}{\hat{n}}_{B/A} = 0$. Finally, for all $t \geq 0$ such that $\sin\theta_{B/A}(t) \neq 0$, it follows from (4.8.3) that $\ddot{\theta}_{B/A}(t) = 0$. □

4.9 Angular Velocity Vector and the Eigenaxis-Angle-Vector Derivative

Recall from (2.14.5) that

$$\vec{\Theta}_{B/A} \triangleq \theta_{B/A}\hat{n}_{B/A}. \tag{4.9.1}$$

Fact 4.9.1. Let F_A and F_B be frames. Then,

$$\overset{A\bullet}{\vec{\Theta}}_{B/A} = \vec{R}_{A/B}\overset{B\bullet}{\vec{\Theta}}_{B/A}. \tag{4.9.2}$$

Proof. It follows from (2.11.42) and (4.7.13) that

$$\overset{A\bullet}{\vec{\Theta}}_{B/A} = \dot{\theta}_{B/A}\hat{n}_{B/A} + \theta_{B/A}\overset{A\bullet}{\hat{n}}_{B/A}$$

$$
\begin{aligned}
&= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \vec{R}_{\mathrm{A/B}}(\dot{\theta}_{\mathrm{B/A}}\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}) \\
&= \vec{R}_{\mathrm{A/B}}(\dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}) \\
&= \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} . \qquad \square
\end{aligned}
$$

Fact 4.9.2. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, and assume that the orientation of $\mathrm{F_B}$ relative to $\mathrm{F_A}$ is given by $\vec{R}_{\mathrm{B/A}} = \exp(\vec{\Theta}^{\times}_{\mathrm{B/A}})$. Then,

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \overset{\mathrm{A}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}} \exp\left((1-\tau)\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\mathrm{d}\tau, \tag{4.9.3}$$

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{A/B}}\right) \overset{\mathrm{A}\bullet\times}{\vec{\Theta}}_{\mathrm{A/B}} \exp\left((1-\tau)\vec{\Theta}^{\times}_{\mathrm{A/B}}\right)\mathrm{d}\tau, \tag{4.9.4}$$

and thus

$$\vec{\omega}^{\times}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} \vec{R}_{\mathrm{A/B}} \tag{4.9.5}$$

$$= \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \overset{\mathrm{A}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}} \exp\left(\tau\vec{\Theta}^{\times\prime}_{\mathrm{B/A}}\right)\mathrm{d}\tau. \tag{4.9.6}$$

Consequently,

$$\vec{\omega}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\mathrm{d}\tau \overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} . \tag{4.9.7}$$

Furthermore,

$$\vec{\omega}_{\mathrm{B/A}} = \frac{1}{\theta^2_{\mathrm{B/A}}}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{4.9.8}$$

$$= \frac{1}{\theta^2_{\mathrm{B/A}}}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} - (\vec{I} - \vec{R}_{\mathrm{A/B}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{4.9.9}$$

$$= \left(\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta^3_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right) \overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{4.9.10}$$

$$= \left(\vec{I} - \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta^3_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right) \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} . \tag{4.9.11}$$

Equivalently,

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.9.12}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{A/B}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.9.13}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}}) \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.9.14}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} - (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}. \tag{4.9.15}$$

In addition,

$$\overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \left(\vec{I} - \frac{1}{2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{2-\theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}, \tag{4.9.16}$$

$$\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \left(\vec{I} + \frac{1}{2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{2-\theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}. \tag{4.9.17}$$

Finally, if $\vec{\Theta}_{\mathrm{B/A}}$ and $\overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}$ are parallel, then $\vec{\omega}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}$.

Proof. Fact 11.14.3 of [1] yields (4.9.3). It then follows that

$$\vec{\omega}^{\times}_{\mathrm{B/A}} = -\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}\,\vec{R}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times\prime}_{\mathrm{B/A}}\right) \overset{\mathrm{B}\bullet}{\vec{\Theta}}{}^{\times}_{\mathrm{B/A}} \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\mathrm{d}\tau.$$

Therefore, it follows from Fact 2.9.8 that

$$\vec{\omega}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times\prime}_{\mathrm{B/A}}\right)\mathrm{d}\tau\, \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}.$$

Using Fact 3.10.1, Fact 6.6.9, and Fact 11.13.14 of [1] yields (4.9.9). Using Fact 2.9.4, (4.9.10), and (4.9.11) yields (4.9.16) and (4.9.17). □

Note that

$$\lim_{\theta_{\mathrm{B/A}}\to 0} \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}} = \frac{1}{2}, \tag{4.9.18}$$

$$\lim_{\theta_{\mathrm{B/A}}\to 0} \frac{\theta_{\mathrm{B/A}}-\sin\theta_{\mathrm{B/A}}}{\theta^3_{\mathrm{B/A}}} = \frac{1}{6}, \tag{4.9.19}$$

$$\lim_{\theta_{\mathrm{B/A}}\to 0} \frac{2-\theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta^2_{\mathrm{B/A}}} = \frac{1}{12}. \tag{4.9.20}$$

However, note that, concerning (4.7.29) and (4.7.31),

$$\lim_{\theta_{\mathrm{B/A}}\to 0} \cot\tfrac{1}{2}\theta_{\mathrm{B/A}} = \infty. \tag{4.9.21}$$

The nonexistence of this limit reflects the fact that, if $\theta_{\mathrm{B/A}} = 0$, then (4.7.18) and (4.7.19) become $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}$, which is independent of $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}$ and $\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}$.

The equivalence of (4.9.10) and (4.9.16) follows the equality

$$\left(\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{\theta_{\mathrm{B/A}}-\sin\theta_{\mathrm{B/A}}}{\theta^3_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\left(\vec{I} - \frac{1}{2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{2-\theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right) = \vec{I}. \tag{4.9.22}$$

Likewise, the equivalence of (4.9.11) and (4.9.17) follows the equality

$$\left(\vec{I} - \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^2}\vec{\Theta}_{\mathrm{B/A}}^{\times} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^3}\vec{\Theta}_{\mathrm{B/A}}^{\times 2}\right)\left(\vec{I} + \frac{1}{2}\vec{\Theta}_{\mathrm{B/A}}^{\times} + \frac{2-\theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta_{\mathrm{B/A}}^2}\vec{\Theta}_{\mathrm{B/A}}^{\times 2}\right) = \vec{I}. \tag{4.9.23}$$

Both (4.9.22) and (4.9.23) are consequences of (2.9.4) with $\theta_{\mathrm{B/A}} = \|\vec{\Theta}_{\mathrm{B/A}}\|_2$.

4.10 Angular Velocity Vector and Euler-Angle Derivatives

The angular velocity vector can be related to the derivatives of the Euler angles. For 3-2-1 (yaw-pitch-roll) Euler angles Ψ, Θ, Φ (see (2.12.21)), we have

$$\vec{\omega}_{\mathrm{D/A}} = \vec{\omega}_{\mathrm{D/C}} + \vec{\omega}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}} \tag{4.10.1}$$

$$= \dot{\Phi}\hat{\imath}_{\mathrm{C}} + \dot{\Theta}\hat{\jmath}_{\mathrm{B}} + \dot{\Psi}\hat{k}_{\mathrm{A}}. \tag{4.10.2}$$

Since $\hat{\imath}_{\mathrm{D}} = \hat{\imath}_{\mathrm{C}}$, $\hat{\jmath}_{\mathrm{C}} = \hat{\jmath}_{\mathrm{B}}$, and $\hat{k}_{\mathrm{B}} = \hat{k}_{\mathrm{A}}$, resolving $\vec{\omega}_{\mathrm{D/A}}$ in $\mathrm{F_D}$ yields

$$\vec{\omega}_{\mathrm{D/A}} = \dot{\Phi}\hat{\imath}_{\mathrm{D}} + \dot{\Theta}\hat{\jmath}_{\mathrm{C}} + \dot{\Psi}\hat{k}_{\mathrm{B}} \tag{4.10.3}$$

$$= \dot{\Phi}\hat{\imath}_{\mathrm{D}} + \dot{\Theta}[(\cos\Phi)\hat{\jmath}_{\mathrm{D}} - (\sin\Phi)\hat{k}_{\mathrm{D}}] + \dot{\Psi}[(\cos\Theta)\hat{k}_{\mathrm{C}} - (\sin\Theta)\hat{\imath}_{\mathrm{C}}] \tag{4.10.4}$$

$$= \dot{\Phi}\hat{\imath}_{\mathrm{D}} + \dot{\Theta}(\cos\Phi)\hat{\jmath}_{\mathrm{D}} - \dot{\Theta}(\sin\Phi)\hat{k}_{\mathrm{D}} + \dot{\Psi}(\cos\Theta)[(\cos\Phi)\hat{k}_{\mathrm{D}} + (\sin\Phi)\hat{\jmath}_{\mathrm{D}}] - \dot{\Psi}(\sin\Theta)\hat{\imath}_{\mathrm{D}} \tag{4.10.5}$$

$$= [-\dot{\Psi}(\sin\Theta) + \dot{\Phi}]\hat{\imath}_{\mathrm{D}} + [\dot{\Psi}(\sin\Phi)\cos\Theta + \dot{\Theta}\cos\Phi]\hat{\jmath}_{\mathrm{D}} + [\dot{\Psi}(\cos\Phi)\cos\Theta - \dot{\Theta}(\sin\Phi)]\hat{k}_{\mathrm{D}}. \tag{4.10.6}$$

Hence

$$\omega_{\mathrm{D/A|D}} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}. \tag{4.10.7}$$

We rewrite (4.10.7) as

$$\omega_{\mathrm{D/A|D}} = S(\Phi,\Theta)\dot{\theta}, \tag{4.10.8}$$

where

$$S(\Phi,\Theta) \triangleq \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}, \quad \theta \triangleq \begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix}. \tag{4.10.9}$$

Note that $S(\Phi,\Theta)$ is independent of Ψ and $\det S(\Phi,\Theta) = \cos\Theta$. Assuming that $S(\Phi,\Theta)$ is nonsingular, solving (4.10.8) for $\dot{\theta}$ yields

$$\dot{\theta} = S(\Phi,\Theta)^{-1}\omega_{\mathrm{D/A|D}}, \tag{4.10.10}$$

where

$$S(\Phi,\Theta)^{-1} \triangleq \begin{bmatrix} 1 & (\sin\Phi)\tan\Theta & (\cos\Phi)\tan\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & (\sin\Phi)\sec\Theta & (\cos\Phi)\sec\Theta \end{bmatrix}. \tag{4.10.11}$$

For 3-1-3 (precession, nutation, spin) Euler angles Φ, Θ, Ψ (see (2.12.35)), we have

$$\vec{\omega}_{\mathrm{D/A}} = \vec{\omega}_{\mathrm{D/C}} + \vec{\omega}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}} \tag{4.10.12}$$

$$= \dot{\Psi}\hat{k}_{\mathrm{C}} + \dot{\Theta}\hat{\imath}_{\mathrm{B}} + \dot{\Phi}\hat{k}_{\mathrm{A}}. \tag{4.10.13}$$

Since $\hat{k}_{\mathrm{A}} = \hat{k}_{\mathrm{B}}$, $\hat{\imath}_{\mathrm{B}} = \hat{\imath}_{\mathrm{C}}$, and $\hat{k}_{\mathrm{C}} = \hat{k}_{\mathrm{D}}$, resolving $\vec{\omega}_{\mathrm{D/A}}$ in $\mathrm{F_D}$ yields

$$\begin{aligned}
\vec{\omega}_{\mathrm{D/A}} &= \dot{\Psi}\hat{k}_{\mathrm{D}} + \dot{\Theta}\hat{\imath}_{\mathrm{C}} + \dot{\Phi}\hat{k}_{\mathrm{B}} \\
&= \dot{\Psi}\hat{k}_{\mathrm{D}} + \dot{\Theta}[(\cos\Psi)\hat{\imath}_{\mathrm{D}} - (\sin\Psi)\hat{\jmath}_{\mathrm{D}}] + \dot{\Phi}[(\cos\Theta)\hat{k}_{\mathrm{C}} + (\sin\Theta)\hat{\jmath}_{\mathrm{C}}] \\
&= \dot{\Psi}\hat{k}_{\mathrm{D}} + \dot{\Theta}(\cos\Psi)\hat{\imath}_{\mathrm{D}} - \dot{\Theta}(\sin\Psi)\hat{\jmath}_{\mathrm{D}} + \dot{\Phi}(\cos\Theta)\hat{k}_{\mathrm{D}} + \dot{\Phi}(\sin\Theta)[(\cos\Psi)\hat{\jmath}_{\mathrm{D}} + (\sin\Psi)\hat{\imath}_{\mathrm{D}}] \\
&= [\dot{\Theta}(\cos\Psi) + \dot{\Phi}(\sin\Psi)\sin\Theta]\hat{\imath}_{\mathrm{D}} + [\dot{\Phi}(\cos\Psi)\sin\Theta - \dot{\Theta}(\sin\Psi)]\hat{\jmath}_{\mathrm{D}} + [\dot{\Psi} + \dot{\Phi}(\cos\Theta)]\hat{k}_{\mathrm{D}}.
\end{aligned}$$

Hence,

$$\omega_{\mathrm{D/A|D}} = \begin{bmatrix} 0 & \cos\Psi & (\sin\Psi)\sin\Theta \\ 0 & -\sin\Psi & (\cos\Psi)\sin\Theta \\ 1 & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} \dot{\Psi} \\ \dot{\Theta} \\ \dot{\Phi} \end{bmatrix}. \tag{4.10.14}$$

We rewrite (4.10.14) as

$$\omega_{\mathrm{D/A|D}} = S(\Psi,\Theta)\dot{\theta}, \tag{4.10.15}$$

where

$$S(\Psi,\Theta) \triangleq \begin{bmatrix} 0 & \cos\Psi & (\sin\Psi)\sin\Theta \\ 0 & -\sin\Psi & (\cos\Psi)\sin\Theta \\ 1 & 0 & \cos\Theta \end{bmatrix}, \quad \theta \triangleq \begin{bmatrix} \Psi \\ \Theta \\ \Phi \end{bmatrix}. \tag{4.10.16}$$

Note that $S(\Psi,\Theta)$ is independent of Φ and $\det S(\Phi,\Theta) = \sin\Theta$. Assuming that $S(\Psi,\Theta)$ is nonsingular, solving (4.10.15) for $\dot{\theta}$ yields

$$\dot{\theta} = S(\Phi,\Theta)^{-1}\omega_{\mathrm{D/A|D}}, \tag{4.10.17}$$

where

$$S(\Psi,\Theta)^{-1} \triangleq \begin{bmatrix} -(\sin\Psi)\cot\Theta & -(\cos\Psi)\cot\Theta & 1 \\ \cos\Psi & -\sin\Psi & 0 \\ (\sin\Psi)\csc\Theta & (\cos\Psi)\csc\Theta & 0 \end{bmatrix}. \tag{4.10.18}$$

4.11 Angular Velocity Vector and Euler-Vector Derivative

Recall from (2.15.15) that the eigenaxis angle vector is given by

$$\vec{\varepsilon}_{\mathrm{B/A}} = (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}. \tag{4.11.1}$$

Fact 4.11.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}}\,. \tag{4.11.2}$$

Proof. It follows from (2.11.42) and (4.7.13) that

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} &= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\,\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \vec{R}_{\mathrm{A/B}}[\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\,\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}]
\end{aligned}$$

$$
\begin{aligned}
&= \vec{R}_{\rm A/B}[\tfrac{1}{2}\dot{\theta}_{\rm B/A}(\cos\tfrac{1}{2}\theta_{\rm B/A})\hat{n}_{\rm B/A} + (\sin\tfrac{1}{2}\theta_{\rm B/A})\overset{\rm B\bullet}{\hat{n}}_{\rm B/A}]\\
&= \vec{R}_{\rm A/B}\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A}. \qquad \square
\end{aligned}
$$

The following result relates the derivative of the Euler vector to the angular velocity vector.

Fact 4.11.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\eta_{\rm B/A}\dot{\eta}_{\rm B/A} + \vec{\varepsilon}^{\,\prime}_{\rm B/A}\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A} = 0, \tag{4.11.3}$$

$$\dot{\eta}_{\rm B/A} = -\tfrac{1}{2}\vec{\varepsilon}^{\,\prime}_{\rm B/A}\vec{\omega}_{\rm B/A}, \tag{4.11.4}$$

$$\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A} = \tfrac{1}{2}(\eta_{\rm B/A}\vec{\omega}_{\rm B/A} + \vec{\varepsilon}_{\rm B/A}\times\vec{\omega}_{\rm B/A}). \tag{4.11.5}$$

$$\vec{\omega}_{\rm B/A} = 2(\eta_{\rm B/A}\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A} - \dot{\eta}_{\rm B/A}\vec{\varepsilon}_{\rm B/A} - \vec{\varepsilon}_{\rm B/A}\times\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A}). \tag{4.11.6}$$

Proof. To prove (4.11.3), differentiate (2.15.25) with respect to $\mathrm{F_A}$.

To prove (4.11.4), note that it follows from (4.7.18) that

$$
\begin{aligned}
-\tfrac{1}{2}\vec{\varepsilon}^{\,\prime}_{\rm B/A}\vec{\omega}_{\rm B/A} &= -\tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\rm B/A})\hat{n}'_{\rm B/A}[\dot{\theta}_{\rm B/A}\hat{n}_{\rm B/A} + (1-\cos\theta_{\rm B/A})\hat{n}_{\rm B/A}\times\overset{\rm A\bullet}{\hat{n}}_{\rm B/A} + (\sin\theta_{\rm B/A})\overset{\rm A\bullet}{\hat{n}}_{\rm B/A}]\\
&= -\tfrac{1}{2}\dot{\theta}_{\rm B/A}\sin\tfrac{1}{2}\theta_{\rm B/A}\\
&= \dot{\eta}_{\rm B/A}.
\end{aligned}
$$

Next, to prove (4.11.5), note that it follows from (4.7.31) and (4.7.22) that

$$
\begin{aligned}
\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A} &= \tfrac{1}{2}\dot{\theta}_{\rm B/A}\eta_{\rm B/A}\hat{n}_{\rm B/A} + (\sin\tfrac{1}{2}\theta_{\rm B/A})\overset{\rm B\bullet}{\hat{n}}_{\rm B/A}\\
&= \tfrac{1}{2}\dot{\theta}_{\rm B/A}\eta_{\rm B/A}\hat{n}_{\rm B/A} + \tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\rm B/A})[\hat{n}^{\times}_{\rm B/A} + (\cot\tfrac{1}{2}\theta_{\rm B/A})(\vec{I} - \hat{n}_{\rm B/A}\hat{n}'_{\rm B/A})]\vec{\omega}_{\rm B/A}\\
&= \tfrac{1}{2}\dot{\theta}_{\rm B/A}\eta_{\rm B/A}\hat{n}_{\rm B/A} + [\tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A} + \tfrac{1}{2}\eta_{\rm B/A}(\vec{I} - \hat{n}_{\rm B/A}\hat{n}'_{\rm B/A})]\vec{\omega}_{\rm B/A}\\
&= \tfrac{1}{2}\dot{\theta}_{\rm B/A}\eta_{\rm B/A}\hat{n}_{\rm B/A} + [\tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A} + \tfrac{1}{2}\eta_{\rm B/A}\vec{I}\,]\vec{\omega}_{\rm B/A} - \tfrac{1}{2}\dot{\theta}_{\rm B/A}\eta_{\rm B/A}\hat{n}_{\rm B/A}\\
&= \tfrac{1}{2}[\eta_{\rm B/A}\vec{I} + (\sin\tfrac{1}{2}\theta_{\rm B/A})\hat{n}^{\times}_{\rm B/A}]\vec{\omega}_{\rm B/A}\\
&= \tfrac{1}{2}(\eta_{\rm B/A}\vec{I} + \vec{\varepsilon}^{\,\times}_{\rm B/A})\vec{\omega}_{\rm B/A}.
\end{aligned}
$$

Finally, to prove (4.11.6), multiply both sides of (4.11.5) by $2\vec{\varepsilon}^{\,\times}_{\rm B/A}$ to obtain

$$
\begin{aligned}
2\vec{\varepsilon}_{\rm B/A}\times\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A} &= \eta_{\rm B/A}\vec{\varepsilon}_{\rm B/A}\times\vec{\omega}_{\rm B/A} + \vec{\varepsilon}_{\rm B/A}\times(\vec{\varepsilon}_{\rm B/A}\times\vec{\omega}_{\rm B/A})\\
&= \eta_{\rm B/A}(2\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A} - \eta_{\rm B/A}\vec{\omega}_{\rm B/A}) + \vec{\varepsilon}^{\,\prime}_{\rm B/A}\vec{\omega}_{\rm B/A}\vec{\varepsilon}_{\rm B/A} - |\vec{\varepsilon}_{\rm B/A}|^2\vec{\omega}_{\rm B/A}\\
&= \eta_{\rm B/A}(2\overset{\rm B\bullet}{\vec{\varepsilon}}_{\rm B/A} - \eta_{\rm B/A}\vec{\omega}_{\rm B/A}) + \vec{\varepsilon}^{\,\prime}_{\rm B/A}\vec{\omega}_{\rm B/A}\vec{\varepsilon}_{\rm B/A} - (1-\eta^2_{\rm B/A})\vec{\omega}_{\rm B/A}
\end{aligned}
$$

$$= 2\eta_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} - 2\dot{\eta}_{\mathrm{B/A}} \vec{\varepsilon}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{B/A}}. \qquad \square$$

Fact 4.11.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $\varepsilon_{\mathrm{B/A|B}} \triangleq \vec{\varepsilon}_{\mathrm{B/A}}\Big|_{\mathrm{B}}$. Then,

$$\eta_{\mathrm{B/A}}\dot{\eta}_{\mathrm{B/A}} + \varepsilon^{\mathrm{T}}_{\mathrm{B/A|B}}\dot{\varepsilon}_{\mathrm{B/A|B}} = 0, \tag{4.11.7}$$

$$\dot{\eta}_{\mathrm{B/A}} = -\tfrac{1}{2}\varepsilon^{\mathrm{T}}_{\mathrm{B/A|B}}\omega_{\mathrm{B/A|B}}, \tag{4.11.8}$$

$$\dot{\varepsilon}_{\mathrm{B/A|B}} = \tfrac{1}{2}(\eta_{\mathrm{B/A}}\omega_{\mathrm{B/A|B}} + \varepsilon_{\mathrm{B/A|B}} \times \omega_{\mathrm{B/A|B}}), \tag{4.11.9}$$

$$\omega_{\mathrm{B/A|B}} = 2(\eta_{\mathrm{B/A}}\dot{\varepsilon}_{\mathrm{B/A|B}} - \dot{\eta}_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A|B}} - \varepsilon_{\mathrm{B/A|B}} \times \dot{\varepsilon}_{\mathrm{B/A|B}}). \tag{4.11.10}$$

In terms of the Euler parameter vector

$$q_{\mathrm{B/A}} = \begin{bmatrix} \eta_{\mathrm{B/A}} \\ \varepsilon_{\mathrm{B/A|B}} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{4.11.11}$$

defined by (2.15.20), it follows from (4.11.10) that $\omega_{\mathrm{B/A|B}}$ can be written in terms of $\dot{q}_{\mathrm{B/A}}$ as

$$\omega_{\mathrm{B/A|B}} = 2\begin{bmatrix} -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}\dot{q}_{\mathrm{B/A}}. \tag{4.11.12}$$

Conversely, it follows from (4.11.8) and (4.11.9) that $\dot{q}_{\mathrm{B/A}}$ can be written in terms of $\omega_{\mathrm{B/A|B}}$ as

$$\dot{q}_{\mathrm{B/A}} = \frac{1}{2}\begin{bmatrix} -\varepsilon^{\mathrm{T}}_{\mathrm{B/A|B}} \\ \eta_{\mathrm{B/A}}I_3 + \varepsilon^{\times}_{\mathrm{B/A|B}} \end{bmatrix}\omega_{\mathrm{B/A|B}} = \frac{1}{2}\begin{bmatrix} -b & -c & -d \\ a & -d & c \\ d & a & -b \\ -c & b & a \end{bmatrix}\omega_{\mathrm{B/A|B}}. \tag{4.11.13}$$

Equations (4.11.8) and (4.11.9) can be written as

$$\dot{q}_{\mathrm{B/A}} = Q(\omega_{\mathrm{B/A|B}})q_{\mathrm{B/A}}, \tag{4.11.14}$$

where

$$Q(\omega_{\mathrm{B/A|B}}) \triangleq \frac{1}{2}\begin{bmatrix} 0 & -\omega^{\mathrm{T}}_{\mathrm{B/A|B}} \\ \omega_{\mathrm{B/A|B}} & -\omega^{\times}_{\mathrm{B/A|B}} \end{bmatrix}. \tag{4.11.15}$$

Furthermore, if $\omega_{\mathrm{B/A|B}}$ is constant, then, for all $t \geq 0$,

$$q_{\mathrm{B/A}}(t) = e^{Q(\omega_{\mathrm{B/A|B}})t}q_{\mathrm{B/A}}(0), \tag{4.11.16}$$

where

$$e^{Q(\omega_{\mathrm{B/A|B}})t} = \cos(\tfrac{1}{2}\|\omega_{\mathrm{B/A|B}}\|t)I_4 + \frac{2\sin(\tfrac{1}{2}\|\omega_{\mathrm{B/A|B}}\|t)}{\|\omega_{\mathrm{B/A|B}}\|}Q(\omega_{\mathrm{B/A|B}}). \tag{4.11.17}$$

4.12 Angular Velocity Vector and Gibbs-Vector Derivative

Recall from (2.17.1) that

$$\vec{g}_{\mathrm{B/A}} \triangleq (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}. \tag{4.12.1}$$

Therefore,

$$\vec{g}_{\mathrm{B/A}} = \frac{1}{\eta_{\mathrm{B/A}}} \vec{\varepsilon}_{\mathrm{B/A}}. \tag{4.12.2}$$

Note that $\vec{g}_{\mathrm{B/A}}$ is defined only if $\theta_{\mathrm{B/A}} \neq \pi$, that is, only if $\vec{\theta}_{\mathrm{B/A}} \in [0, \pi)$.

Fact 4.12.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{g}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}}. \tag{4.12.3}$$

Proof. It follows from (2.11.42) and (4.7.13) that

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{g}}_{\mathrm{B/A}} &= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2 \tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\tan \tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2 \tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\tan \tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \vec{R}_{\mathrm{A/B}}[\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2 \tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\tan \tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&= \vec{R}_{\mathrm{A/B}}[\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2 \tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\tan \tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&= \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}}. \qquad \square
\end{aligned}$$

Fact 4.12.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \neq \pi$. Then,

$$\overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}} = \tfrac{1}{2}\left(\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}'_{\mathrm{B/A}} + \vec{g}^{\times}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}. \tag{4.12.4}$$

Furthermore,

$$\vec{\omega}_{\mathrm{B/A}} = \frac{2}{1 + |\vec{g}_{\mathrm{B/A}}|^2}(\vec{I} - \vec{g}^{\times}_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}}. \tag{4.12.5}$$

Proof. It follows from (2.11.42) and (4.7.13) that

$$\begin{aligned}
\overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}} &= -\frac{\dot{\eta}_{\mathrm{B/A}}}{\eta^2_{\mathrm{B/A}}}\vec{\varepsilon}_{\mathrm{B/A}} + \frac{1}{\eta_{\mathrm{B/A}}} \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} \\
&= -\frac{\dot{\eta}_{\mathrm{B/A}}}{\eta^2_{\mathrm{B/A}}}\vec{\varepsilon}_{\mathrm{B/A}} + \frac{1}{2\eta_{\mathrm{B/A}}}(\eta_{\mathrm{B/A}}\vec{I} + \vec{\varepsilon}^{\times}_{\mathrm{B/A}}) \\
&= -\frac{\dot{\eta}_{\mathrm{B/A}}}{\eta_{\mathrm{B/A}}}\vec{g}_{\mathrm{B/A}} + \tfrac{1}{2}(\vec{I} + \vec{g}^{\times}_{\mathrm{B/A}}) \\
&= \tfrac{1}{2}\frac{\vec{\varepsilon}'_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}}{\eta_{\mathrm{B/A}}}\vec{g}_{\mathrm{B/A}} + \tfrac{1}{2}(\vec{I} + \vec{g}^{\times}_{\mathrm{B/A}}) \\
&= \tfrac{1}{2}\vec{g}'_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}} + \tfrac{1}{2}(\vec{I} + \vec{g}^{\times}_{\mathrm{B/A}}) \\
&= \tfrac{1}{2}\left(\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}'_{\mathrm{B/A}} + \vec{g}^{\times}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}.
\end{aligned}$$

Finally, it follows from (2.9.8) that

$$\begin{aligned}\vec{\omega}_{\text{B/A}} &= 2\left(\vec{I} + \vec{g}_{\text{B/A}}\vec{g}'_{\text{B/A}} + \vec{g}^{\times}_{\text{B/A}}\right)^{-1} \overset{\text{B}\bullet}{\vec{g}}_{\text{B/A}} \\ &= \frac{2}{1 + |\vec{g}_{\text{B/A}}|^2}(\vec{I} - \vec{g}^{\times}_{\text{B/A}}) \overset{\text{B}\bullet}{\vec{g}}_{\text{B/A}}.\end{aligned}$$

□

4.13 6D Velocity Kinematics of a Chain of Rigid Bodies

Consider the chain of rigid bodies shown in Figure 2.22.1. We assume that $\omega_{\text{B/A|B}}(t)$, $\omega_{\text{C/A|C}}(t)$, and $\omega_{\text{D/A|D}}(t)$ are known for all time t and that the initial orientations $\mathcal{O}_{\text{A/B}}(0)$, $\mathcal{O}_{\text{B/C}}(0)$, and $\mathcal{O}_{\text{C/D}}(0)$ and the initial displacements $r_{z_{\text{B}}/z_{\text{A}}|\text{A}}(0)$, $r_{z_{\text{C}}/z_{\text{B}}|\text{A}}(0)$, and $r_{z_{\text{D}}/z_{\text{C}}|\text{A}}(0)$ are known. The goal is to determine $r_{z_{\text{B}}/z_{\text{A}}|\text{A}}(t)$, $r_{z_{\text{C}}/z_{\text{B}}|\text{A}}(t)$, and $r_{z_{\text{D}}/z_{\text{C}}|\text{A}}(t)$ as functions of time.

First, to determine the position of z_{B} relative to z_{A} in F_{A}, note that

$$\overset{\text{A}\bullet}{\vec{r}}_{z_{\text{B}}/z_{\text{A}}} = \vec{\omega}_{\text{B/A}} \times \vec{r}_{z_{\text{B}}/z_{\text{A}}}. \tag{4.13.1}$$

Resolving (4.13.1) in F_{A} yields

$$\begin{aligned}\dot{r}_{z_{\text{B}}/z_{\text{A}}|\text{A}} &= \omega_{\text{B/A|A}} \times r_{z_{\text{B}}/z_{\text{A}}|\text{A}} &(4.13.2)\\ &= (\mathcal{O}_{\text{A/B}}\omega_{\text{B/A|B}}) \times r_{z_{\text{B}}/z_{\text{A}}|\text{A}}. &(4.13.3)\end{aligned}$$

Since $\omega_{\text{B/A|B}}(t)$ is known for all t and $\mathcal{O}_{\text{A/B}}(0)$ is known, integrating Poisson's equation (4.3.20) in the form

$$\dot{\mathcal{O}}_{\text{A/B}} = \mathcal{O}_{\text{A/B}}\omega^{\times}_{\text{B/A|B}} \tag{4.13.4}$$

yields $\mathcal{O}_{\text{A/B}}(t)$. Since $\mathcal{O}_{\text{A/B}}(t)\omega_{\text{B/A|B}}(t)$ is known for all t and $r_{z_{\text{B}}/z_{\text{A}}|\text{A}}(0)$ is known, integrating (4.13.3) yields $r_{z_{\text{B}}/z_{\text{A}}|\text{A}}(t)$.

Next, to determine the position of z_{C} relative to z_{B} in F_{A}, note that

$$\overset{\text{A}\bullet}{\vec{r}}_{z_{\text{C}}/z_{\text{B}}} = \vec{\omega}_{\text{C/A}} \times \vec{r}_{z_{\text{C}}/z_{\text{B}}}. \tag{4.13.5}$$

Resolving (4.13.5) in F_{A} yields

$$\begin{aligned}\dot{r}_{z_{\text{C}}/z_{\text{B}}|\text{A}} &= \omega_{\text{C/A|A}} \times r_{z_{\text{C}}/z_{\text{B}}|\text{A}} &(4.13.6)\\ &= (\mathcal{O}_{\text{A/C}}\omega_{\text{C/A|C}}) \times r_{z_{\text{C}}/z_{\text{B}}|\text{A}}. &(4.13.7)\end{aligned}$$

Since $\omega_{\text{C/A|C}}(t)$ is known for all t and $\mathcal{O}_{\text{A/C}}(0)$ is known, integrating Poisson's equation (4.3.20) in the form

$$\dot{\mathcal{O}}_{\text{A/C}} = \mathcal{O}_{\text{A/C}}\omega^{\times}_{\text{C/A|C}} \tag{4.13.8}$$

yields $\mathcal{O}_{\text{A/C}}(t)$. Since $\mathcal{O}_{\text{A/C}}(t)\omega_{\text{C/A|C}}(t)$ is known for all t and $r_{z_{\text{C}}/z_{\text{B}}|\text{A}}(0)$ is known, integrating (4.13.7) yields $r_{z_{\text{C}}/z_{\text{B}}|\text{A}}(t)$.

Next, to determine the position of z_{D} relative to z_{C} in F_{A}, note that

$$\overset{\text{A}\bullet}{\vec{r}}_{z_{\text{D}}/z_{\text{C}}} = \vec{\omega}_{\text{D/A}} \times \vec{r}_{z_{\text{D}}/z_{\text{C}}}. \tag{4.13.9}$$

Resolving (4.13.9) in F_A yields

$$\dot{r}_{z_D/z_C|A} = \omega_{D/A|A} \times r_{z_D/z_C|A} \tag{4.13.10}$$
$$= (\mathcal{O}_{A/D}\omega_{D/A|D}) \times r_{z_D/z_C|A}. \tag{4.13.11}$$

Since $\omega_{D/A|D}(t)$ is known for all t and $\mathcal{O}_{A/D}(0)$ is known, integrating Poisson's equation (4.3.20) in the form

$$\dot{\mathcal{O}}_{A/D} = \mathcal{O}_{A/D}\omega^{\times}_{D/A|D} \tag{4.13.12}$$

yields $\mathcal{O}_{A/D}(t)$. Since $\mathcal{O}_{A/D}(t)\omega_{D/A|D}(t)$ is known for all t and $r_{z_D/z_C|A}(0)$ is known, integrating (4.13.11) yields $r_{z_D/z_C|A}(t)$. Finally, since $r_{z_D/z_C|A}$, $r_{z_C/z_B|A}$, and $r_{z_B/z_A|A}$ are known, it follows that $r_{z_D/z_A|A} = r_{z_D/z_C|A} + r_{z_C/z_B|A} + r_{z_B/z_A|A}$ is known.

As an alternative approach, note that

$$\vec{r}_{z_C/z_A} = \vec{r}_{z_C/z_B} + \vec{r}_{z_B/z_A}, \tag{4.13.13}$$
$$\vec{r}_{z_D/z_A} = \vec{r}_{z_D/z_C} + \vec{r}_{z_C/z_A}. \tag{4.13.14}$$

Since

$$\overset{\mathrm{B}\bullet}{\vec{r}}_{z_B/z_A} = \overset{\mathrm{C}\bullet}{\vec{r}}_{z_C/z_B} = \overset{\mathrm{D}\bullet}{\vec{r}}_{z_D/z_C} = 0, \tag{4.13.15}$$

it follows that

$$\vec{v}_{z_B/z_A/A} = \vec{\omega}_{B/A} \times \vec{r}_{z_B/z_A}, \tag{4.13.16}$$
$$\vec{v}_{z_C/z_A/A} = \vec{\omega}_{C/A} \times \vec{r}_{z_C/z_B} + \vec{v}_{z_B/z_A/A}, \tag{4.13.17}$$
$$\vec{v}_{z_D/z_A/A} = \vec{\omega}_{D/A} \times \vec{r}_{z_D/z_C} + \vec{v}_{z_C/z_A/A}. \tag{4.13.18}$$

Combining (4.13.16)–(4.13.18) with angular velocities yields

$$\begin{bmatrix} \vec{v}_{z_B/z_A/A} \\ \vec{\omega}_{C/A} \end{bmatrix} = \begin{bmatrix} 0 & -\vec{r}^{\,\times}_{z_B/z_A} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} 0 \\ \vec{\omega}_{B/A} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{\omega}_{C/B} \end{bmatrix}, \tag{4.13.19}$$

$$\begin{bmatrix} \vec{v}_{z_C/z_A/A} \\ \vec{\omega}_{D/A} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\,\times}_{z_C/z_B} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{v}_{z_B/z_A/A} \\ \vec{\omega}_{C/A} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{\omega}_{D/C} \end{bmatrix}, \tag{4.13.20}$$

$$\begin{bmatrix} \vec{v}_{z_D/z_A/A} \\ \vec{\omega}_{D/A} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\,\times}_{z_D/z_C} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{v}_{z_C/z_A/A} \\ \vec{\omega}_{D/A} \end{bmatrix}. \tag{4.13.21}$$

Resolving (4.13.16)–(4.13.18) yields

$$v_{z_B/z_A/A|B} = \omega_{B/A|B} \times r_{z_B/z_A|B}, \tag{4.13.22}$$
$$v_{z_C/z_A/A|C} = \omega_{C/A|C} \times r_{z_C/z_B|C} + v_{z_B/z_A/A|C}, \tag{4.13.23}$$
$$v_{z_D/z_A/A|D} = \omega_{D/A|D} \times r_{z_D/z_C|D} + v_{z_C/z_A/A|D}, \tag{4.13.24}$$

and thus resolving (4.13.19)–(4.13.21) yields

$$\begin{bmatrix} v_{z_B/z_A/A|B} \\ \omega_{C/A|B} \end{bmatrix} = \begin{bmatrix} 0 & -r^{\times}_{z_B/z_A|B} \\ 0 & I_3 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_{B/A|B} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{O}_{B/C}\omega_{C/B|C} \end{bmatrix}, \tag{4.13.25}$$

$$\begin{bmatrix} v_{z_C/z_A/A|C} \\ \omega_{D/A|C} \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{C/B} & -r^{\times}_{z_C/z_B|C}\mathcal{O}_{C/B} \\ 0 & \mathcal{O}_{C/B} \end{bmatrix} \begin{bmatrix} v_{z_B/z_A/A|B} \\ \omega_{C/A|B} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{O}_{C/D}\omega_{D/C|D} \end{bmatrix}, \tag{4.13.26}$$

$$\begin{bmatrix} v_{z_D/z_A/A|D} \\ \omega_{D/A|D} \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{D/C} & -r^{\times}_{z_D/z_C|D}\mathcal{O}_{D/C} \\ 0 & \mathcal{O}_{D/C} \end{bmatrix} \begin{bmatrix} v_{z_C/z_A/A|C} \\ \omega_{D/A|C} \end{bmatrix}. \tag{4.13.27}$$

Now, assume that, for all time t, the angular velocities $\omega_{B/A|B}(t)$, $\omega_{C/B|C}(t)$, and $\omega_{D/C|D}(t)$, the orientation matrices $\mathcal{O}_{C/B}(t)$ and $\mathcal{O}_{D/C}(t)$, and the position vectors $r_{z_B/z_A|B}(t)$, $r_{z_C/z_B|C}(t)$, and $r_{z_D/z_C|D}(t)$ are known. Then (4.13.25)–(4.13.27) yield $v_{z_B/z_A/A|B}(t)$, $v_{z_C/z_A/A|C}(t)$, $v_{z_D/z_A/A|D}(t)$, $\omega_{C/A|B}(t)$, $\omega_{D/A|C}(t)$, and $\omega_{D/A|D}(t)$ as functions of time. Note that $\mathcal{O}_{B/A}(0)$ is not needed for these computations.

For a chain consisting of $n \geq 4$ rigid bodies, n equations of the form (4.13.25)–(4.13.27) can be derived, where one equation is of the form (4.13.25) for the top link of the chain, $n-2$ equations are of the form (4.13.26) for the intermediate links of the chain, and one equation is of the form (4.13.27) for the bottom link of the chain.

Next, defining the column vectrices

$$\vec{\mathcal{V}}_A \triangleq \begin{bmatrix} 0 \\ \vec{\omega}_{B/A} \end{bmatrix}, \quad \vec{\mathcal{V}}_B \triangleq \begin{bmatrix} \vec{v}_{z_B/z_A/A} \\ \vec{\omega}_{C/A} \end{bmatrix}, \quad \vec{\mathcal{V}}_C \triangleq \begin{bmatrix} \vec{v}_{z_C/z_A/A} \\ \vec{\omega}_{D/A} \end{bmatrix}, \quad \vec{\mathcal{V}}_D \triangleq \begin{bmatrix} \vec{v}_{z_D/z_A/A} \\ \vec{\omega}_{D/A} \end{bmatrix} \tag{4.13.28}$$

and the physical matrices

$$\vec{\mathcal{T}}_{B/A} \triangleq \begin{bmatrix} 0 & -\vec{r}^{\times}_{z_B/z_A} \\ 0 & \vec{I} \end{bmatrix}, \quad \vec{\mathcal{T}}_{C/B} \triangleq \begin{bmatrix} \vec{I} & -\vec{r}^{\times}_{z_C/z_B} \\ 0 & \vec{I} \end{bmatrix}, \quad \vec{\mathcal{T}}_{D/C} \triangleq \begin{bmatrix} \vec{I} & -\vec{r}^{\times}_{z_D/z_C} \\ 0 & \vec{I} \end{bmatrix}, \tag{4.13.29}$$

(4.13.19)–(4.13.21) can be written as

$$\vec{\mathcal{V}}_B = \vec{\mathcal{T}}_{B/A}\vec{\mathcal{V}}_A + \begin{bmatrix} 0 \\ \vec{\omega}_{C/B} \end{bmatrix}, \tag{4.13.30}$$

$$\vec{\mathcal{V}}_C = \vec{\mathcal{T}}_{C/B}\vec{\mathcal{V}}_B + \begin{bmatrix} 0 \\ \vec{\omega}_{D/C} \end{bmatrix}, \tag{4.13.31}$$

$$\vec{\mathcal{V}}_D = \vec{\mathcal{T}}_{D/C}\vec{\mathcal{V}}_C. \tag{4.13.32}$$

Furthermore, defining the 6×1 vectors

$$\mathcal{V}_A \triangleq \begin{bmatrix} 0 \\ \omega_{B/A|B} \end{bmatrix}, \quad \mathcal{V}_B \triangleq \begin{bmatrix} v_{z_B/z_A/A|B} \\ \omega_{C/A|B} \end{bmatrix}, \quad \mathcal{V}_C \triangleq \begin{bmatrix} v_{z_C/z_A/A|C} \\ \omega_{D/A|C} \end{bmatrix}, \quad \mathcal{V}_D \triangleq \begin{bmatrix} v_{z_D/z_A/A|D} \\ \omega_{D/A|D} \end{bmatrix} \tag{4.13.33}$$

and the 6×6 matrices

$$\mathcal{T}_{B/A} \triangleq \begin{bmatrix} 0 & -r^{\times}_{z_B/z_A|B} \\ 0 & I_3 \end{bmatrix}, \quad \mathcal{T}_{C/B} \triangleq \begin{bmatrix} \mathcal{O}_{C/B} & -r^{\times}_{z_C/z_B|C}\mathcal{O}_{C/B} \\ 0 & \mathcal{O}_{C/B} \end{bmatrix}, \tag{4.13.34}$$

$$\mathcal{T}_{D/C} \triangleq \begin{bmatrix} \mathcal{O}_{D/C} & -r^{\times}_{z_D/z_C|D}\mathcal{O}_{D/C} \\ 0 & \mathcal{O}_{D/C} \end{bmatrix}, \tag{4.13.35}$$

(4.13.25)–(4.13.27) can be written as

$$\mathcal{V}_B = \mathcal{T}_{B/A}\mathcal{V}_A + \begin{bmatrix} 0 \\ \omega_{C/B|B} \end{bmatrix}, \tag{4.13.36}$$

$$\mathcal{V}_C = \mathcal{T}_{C/B}\mathcal{V}_B + \begin{bmatrix} 0 \\ \omega_{D/C|C} \end{bmatrix}, \tag{4.13.37}$$

$$\mathcal{V}_D = \mathcal{T}_{D/C}\mathcal{V}_C. \tag{4.13.38}$$

4.14 6D Acceleration Kinematics of a Chain of Rigid Bodies

Differentiating (4.13.19), (4.13.20), and (4.13.21) with respect to $\mathrm{F_B}$, $\mathrm{F_C}$, and $\mathrm{F_D}$, respectively, yields

$$\begin{bmatrix} \vec{a}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}/\mathrm{B}} \\ \vec{\alpha}_{\mathrm{C/A}} \end{bmatrix} = \begin{bmatrix} 0 & -\vec{r}^{\times}_{z_\mathrm{B}/z_\mathrm{A}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{B/A}} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{C/B}} \end{bmatrix}, \tag{4.14.1}$$

$$\begin{bmatrix} \vec{a}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}/\mathrm{C}} \\ \vec{\alpha}_{\mathrm{D/A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\times}_{z_\mathrm{C}/z_\mathrm{B}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{a}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}/\mathrm{B}} \\ \vec{\alpha}_{\mathrm{C/A}} \end{bmatrix} + \begin{bmatrix} \vec{\omega}_{\mathrm{B/C}} \times \vec{v}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}} \\ \vec{\alpha}_{\mathrm{D/C}} + \vec{\omega}_{\mathrm{C/A}} \times \vec{\omega}_{\mathrm{D/C}} \end{bmatrix}, \tag{4.14.2}$$

$$\begin{bmatrix} \vec{a}_{z_\mathrm{D}/z_\mathrm{A}/\mathrm{A}/\mathrm{D}} \\ \vec{\alpha}_{\mathrm{D/A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\times}_{z_\mathrm{D}/z_\mathrm{C}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{a}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}/\mathrm{C}} \\ \vec{\alpha}_{\mathrm{D/A}} \end{bmatrix} + \begin{bmatrix} \vec{\omega}_{\mathrm{C/D}} \times \vec{v}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}} \\ 0 \end{bmatrix}. \tag{4.14.3}$$

Defining

$$\vec{\mathcal{A}}_\mathrm{A} \triangleq \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{B/A}} \end{bmatrix}, \quad \vec{\mathcal{A}}_\mathrm{B} \triangleq \begin{bmatrix} \vec{a}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}/\mathrm{B}} \\ \vec{\alpha}_{\mathrm{C/A}} \end{bmatrix}, \quad \vec{\mathcal{A}}_\mathrm{C} \triangleq \begin{bmatrix} \vec{a}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}/\mathrm{C}} \\ \vec{\alpha}_{\mathrm{D/A}} \end{bmatrix}, \quad \vec{\mathcal{A}}_\mathrm{D} \triangleq \begin{bmatrix} \vec{a}_{z_\mathrm{D}/z_\mathrm{A}/\mathrm{A}/\mathrm{D}} \\ \vec{\alpha}_{\mathrm{D/A}} \end{bmatrix}, \tag{4.14.4}$$

and using (4.13.29), (4.14.1)–(4.14.3) can be written as

$$\vec{\mathcal{A}}_\mathrm{B} = \vec{\mathfrak{T}}_{\mathrm{B/A}} \vec{\mathcal{A}}_\mathrm{A} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{C/B}} \end{bmatrix}, \tag{4.14.5}$$

$$\vec{\mathcal{A}}_\mathrm{C} = \vec{\mathfrak{T}}_{\mathrm{C/B}} \vec{\mathcal{A}}_\mathrm{B} + \begin{bmatrix} \vec{\omega}_{\mathrm{B/C}} \times \vec{v}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}} \\ \vec{\alpha}_{\mathrm{D/C}} + \vec{\omega}_{\mathrm{C/A}} \times \vec{\omega}_{\mathrm{D/C}} \end{bmatrix}, \tag{4.14.6}$$

$$\vec{\mathcal{A}}_\mathrm{D} = \vec{\mathfrak{T}}_{\mathrm{D/C}} \vec{\mathcal{A}}_\mathrm{C} + \begin{bmatrix} \vec{\omega}_{\mathrm{C/D}} \times \vec{v}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}} \\ 0 \end{bmatrix}. \tag{4.14.7}$$

Next, by defining the acceleration vectrices that depend on angular accelerations

$$\vec{\mathcal{A}}_{\mathrm{B},\alpha} \triangleq \vec{\mathfrak{T}}_{\mathrm{B/A}} \vec{\mathcal{A}}_\mathrm{A} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{C/B}} \end{bmatrix}, \tag{4.14.8}$$

$$\vec{\mathcal{A}}_{\mathrm{C},\alpha} \triangleq \vec{\mathfrak{T}}_{\mathrm{C/B}} \vec{\mathcal{A}}_{\mathrm{B},\alpha} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{D/C}} \end{bmatrix}, \tag{4.14.9}$$

$$\vec{\mathcal{A}}_{\mathrm{D},\alpha} \triangleq \vec{\mathfrak{T}}_{\mathrm{D/C}} \vec{\mathcal{A}}_{\mathrm{C},\alpha} \tag{4.14.10}$$

and the acceleration vectrices that depend on angular velocities

$$\vec{\mathcal{A}}_{\mathrm{B},\omega} \triangleq \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{C/B}} \end{bmatrix}, \tag{4.14.11}$$

$$\vec{\mathcal{A}}_{\mathrm{C},\omega} \triangleq \vec{\mathfrak{T}}_{\mathrm{C/B}} \vec{\mathcal{A}}_{\mathrm{B},\omega} + \begin{bmatrix} \vec{\omega}_{\mathrm{B/C}} \times \vec{v}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{C/A}} \times \vec{\omega}_{\mathrm{D/C}} \end{bmatrix}, \tag{4.14.12}$$

$$\vec{\mathcal{A}}_{\mathrm{D},\omega} \triangleq \vec{\mathfrak{T}}_{\mathrm{D/C}} \vec{\mathcal{A}}_{\mathrm{C},\omega} + \begin{bmatrix} \vec{\omega}_{\mathrm{C/D}} \times \vec{v}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}} \\ 0 \end{bmatrix}, \tag{4.14.13}$$

it follows that

$$\vec{\mathcal{A}}_\mathrm{B} = \vec{\mathcal{A}}_{\mathrm{B},\alpha} + \vec{\mathcal{A}}_{\mathrm{B},\omega}, \tag{4.14.14}$$

$$\vec{\mathcal{A}}_{\mathrm{C}} = \vec{\mathcal{A}}_{\mathrm{C},\alpha} + \vec{\mathcal{A}}_{\mathrm{C},\omega}, \tag{4.14.15}$$

$$\vec{\mathcal{A}}_{\mathrm{D}} = \vec{\mathcal{A}}_{\mathrm{D},\alpha} + \vec{\mathcal{A}}_{\mathrm{D},\omega}. \tag{4.14.16}$$

4.15 Instantaneous Velocity Center of Rotation

Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p be a point fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$. Then p is an *instantaneous velocity center of rotation* (IVCR) at time t if $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$ and $\vec{v}_{p/\mathrm{O_A}/\mathrm{A}}(t) = 0$. The motion of $\mathcal{B}$ can be viewed as instantaneously rotating around p. See Figure 4.15.1.

Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p be a point fixed in $\mathcal{B}$, let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$, and let q be a point fixed in $\mathcal{B}$. Then,

$$\vec{v}_{p/\mathrm{O_A}/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{O_A}/\mathrm{A}}, \tag{4.15.1}$$

and thus

$$\vec{\omega}_{\mathrm{B/A}} \cdot \vec{v}_{p/\mathrm{O_A}/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \cdot \vec{v}_{q/\mathrm{O_A}/\mathrm{A}}. \tag{4.15.2}$$

Fact 4.15.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$, assume that, at time t, p is an IVCR. Then,

$$\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{O_A}/\mathrm{A}} = 0, \tag{4.15.3}$$

$$\vec{\omega}_{\mathrm{B/A}} \cdot \vec{v}_{p/\mathrm{O_A}/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \cdot \vec{v}_{q/\mathrm{O_A}/\mathrm{A}} = 0. \tag{4.15.4}$$

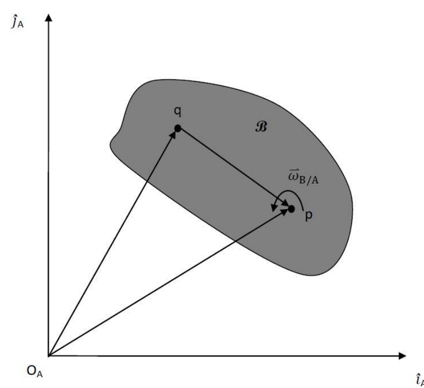


Figure 4.15.1: Instantaneous velocity center of rotation. At time t, $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$. At the same time t, the point p, which is fixed in $\mathcal{B}$, satisfies $\vec{v}_{p/\mathrm{O_A}/\mathrm{A}}(t) = 0$. Hence, $\mathcal{B}$ is instantaneously rotating around p.

The following result follows from Fact 4.15.1.

Fact 4.15.2. Let $\mathcal{B}$ be a rigid body, and let p be a point fixed in $\mathcal{B}$. Then, at time t, the following statements are equivalent:

i) p is an IVCR.

ii) $\vec{\omega}_{\mathrm{B/A}} \neq 0$, and, for every point q fixed in $\mathcal{B}$,

$$\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{O_A/A}} = 0. \tag{4.15.5}$$

iii) $\vec{\omega}_{\mathrm{B/A}} \neq 0$, and there exists a point q fixed in $\mathcal{B}$ such that (4.15.5) is satisfied.

If, at time t, $\vec{\omega}_{\mathrm{B/A}}$ is nonzero, then Fact 4.15.2 implies that $\mathcal{B}$ has no IVCR if and only if there exists a point q fixed in $\mathcal{B}$ such that $\vec{\omega}_{\mathrm{B/A}} \cdot \vec{v}_{q/\mathrm{O_A/A}} \neq 0$. This situation occurs, for example, when the translational velocity of q is parallel to the angular velocity.

Fact 4.15.3. Let $\mathcal{B}$ be a rigid body, and let p be a point fixed in $\mathcal{B}$. Then, at time t, p is an IVCR if and only if $\vec{\omega}_{\mathrm{B/A}} \neq 0$ and there exists a point q fixed in $\mathcal{B}$ such that the following conditions are satisfied:

i) $\vec{\omega}_{\mathrm{B/A}} \cdot \vec{v}_{q/\mathrm{O_A/A}} = 0$.

ii) $\vec{\omega}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \dfrac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} \vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{q/\mathrm{O_A/A}} \right) = 0$.

If these conditions are satisfied, then

$$(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{I} - \vec{\omega}_{\mathrm{B/A}} \vec{\omega}'_{\mathrm{B/A}}) \vec{r}_{p/q} = \vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{q/\mathrm{O_A/A}}. \tag{4.15.6}$$

Proof. Assume that p is an IVCR. Then, Fact 4.15.1 implies *i*). To prove *ii*), (4.15.5) and *i*) imply that

$$\vec{\omega}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} \vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{q/\mathrm{O_A/A}} \right) = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{O_A/A}} = 0.$$

Conversely, (4.15.1), *i*), and *ii*) yield

$$\begin{aligned}
\vec{v}_{p/\mathrm{O_A/A}} &= \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{O_A/A}} \\
&= \vec{\omega}_{\mathrm{B/A}} \times \left(\frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} \vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{q/\mathrm{O_A/A}} \right) + \vec{v}_{q/\mathrm{O_A/A}} \\
&= -\vec{v}_{q/\mathrm{O_A/A}} + \vec{v}_{q/\mathrm{O_A/A}} \\
&= 0.
\end{aligned}$$

To obtain (4.15.6), note that (4.15.3) implies

$$\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{O_A/A}}) = 0,$$

and thus

$$(\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{p/q}) \vec{\omega}_{\mathrm{B/A}} - |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{q/\mathrm{O_A/A}} = 0. \tag{4.15.7}$$

Solving (4.15.7) for $\vec{r}_{p/q}$ yields (4.15.6). $\square$

4.16 Instantaneous Acceleration Center of Rotation

Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p be a point fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$. Then p is an *instantaneous acceleration center of rotation* (IACR) at time t if either $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$ or $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \neq 0$ and $\vec{a}_{p/\mathrm{O_A}/\mathrm{A}}(t) = 0$. The motion of $\mathcal{B}$ can be viewed as instantaneously rotating or accelerating around p.

Fact 4.16.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$. Then, at time t, p is an IACR if and only if either $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$ or $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \neq 0$ and

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0. \tag{4.16.1}$$

Fact 4.16.2. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$. Assume that, at time t, $\vec{\omega}_{\mathrm{B/A}} = 0$, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \neq 0$, and $\vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \neq 0$. Then, p is an IACR of $\mathcal{B}$ if and only if the following conditions are satisfied:

i) $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0.$

ii) $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \dfrac{1}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) = 0.$

If these conditions are satisfied, then

$$\left(|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2 \vec{I} - \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}} \right) \vec{r}_{p/q} = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}. \tag{4.16.2}$$

Proof. Assume that, at time t, p is an IACR of $\mathcal{B}$. Since $\vec{\omega}_{\mathrm{B/A}} = 0$, (4.16.1) implies

$$\begin{aligned}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \left(- \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) \right) \\ &= - \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \left(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} \right) = 0,\end{aligned}$$

which proves *i*). To prove *ii*), it follows from *i*) and (4.16.1) that

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \frac{1}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0.$$

Conversely, *i*) implies

$$\vec{a}_{p/\mathrm{O_A}/\mathrm{A}} = \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{p/q} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{r}}_{p/q} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}$$

$$= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\frac{1}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}$$

$$= \frac{\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0.$$

Hence, p is an IACR at time t.

To obtain (4.16.2), note that (4.16.1) implies

$$\begin{aligned} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{p/\mathrm{O_A}/\mathrm{A}} &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times (\vec{a}_{p/q/\mathrm{A}} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}) \\ &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) = 0. \end{aligned}$$

Hence,

$$(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - (\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}) \vec{r}_{p/q} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0,$$

which yields (4.16.2). □

Fact 4.16.3. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$. Assume that, at time t, $\vec{\omega}_{\mathrm{B/A}} \neq 0$, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$, and $\vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \neq 0$. Then, p is an IACR if and only if the following conditions are satisfied:

i) $\vec{\omega}_{\mathrm{B/A}} \cdot \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0$.

ii) $\vec{\omega}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \dfrac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) = 0$.

If these conditions are satisfied, then

$$(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{\vec{I}} - \vec{\omega}_{\mathrm{B/A}} \vec{\omega}'_{\mathrm{B/A}}) \vec{r}_{p/q} = \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}. \tag{4.16.3}$$

Proof. Assume that, at time t, p is an IACR of $\mathcal{B}$. Since $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$, (4.16.1) implies

$$\begin{aligned} \vec{\omega}_{\mathrm{B/A}} \cdot \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} &= \vec{\omega}_{\mathrm{B/A}} \cdot [- \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q})] \\ &= -\vec{\omega}_{\mathrm{B/A}} \cdot [\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q})] = 0, \end{aligned}$$

which proves *i)*. To prove *ii)*, it follows from (4.16.1) that

$$\begin{aligned} &\vec{\omega}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) \\ &\quad = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} [\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q})] \end{aligned}$$

$$= \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} - \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} = 0.$$

Conversely, *i*) implies

$$\begin{aligned}
\vec{a}_{p/\mathrm{O_A}/\mathrm{A}} &= \vec{a}_{p/q/\mathrm{A}} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&= \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&= \vec{\omega}_{\mathrm{B/A}} \times \left(\vec{\omega}_{\mathrm{B/A}} \times \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}\right) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&= -\vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0.
\end{aligned}$$

Hence, p is an IACR at time t.

To obtain (4.16.3), note that (4.16.1) implies

$$\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0.$$

Hence,

$$(\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} - (\vec{\omega}_{\mathrm{B/A}} \cdot \vec{\omega}_{\mathrm{B/A}})\vec{r}_{p/q} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0,$$

which yields (4.16.3). □

Fact 4.16.4. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$. Assume that, at time t, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}$ and $\vec{\omega}_{\mathrm{B/A}}$ are nonzero and parallel. Then, p is an IACR if and only if the following conditions are satisfied:

i) $\vec{\omega}_{\mathrm{B/A}} \cdot \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0.$

ii) $\vec{\omega}_{\mathrm{B/A}}$ and $\vec{r}_{p/q} - \dfrac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2}\left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}\right)$ are parallel.

If these conditions are satisfied, then

$$\begin{aligned}
&[(|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2)\vec{I} - \kappa\vec{\omega}_{\mathrm{B/A}} \overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}} - |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{\omega}_{\mathrm{B/A}} \vec{\omega}'_{\mathrm{B/A}}]\vec{r}_{p/q} \\
&\qquad\qquad = |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}},
\end{aligned} \tag{4.16.4}$$

where $\kappa \triangleq \dfrac{\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{\omega}_{\mathrm{B/A}}}{|\vec{\omega}_{\mathrm{B/A}}|^2}$.

Proof. Assume that, at time t, p is an IACR. Then, it follows from (4.16.1) that $\vec{\omega}_{\mathrm{B/A}} \cdot \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0$, which proves *i*). To prove *ii*), note that (4.16.1) implies

$$\begin{aligned}
0 &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + (\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} - |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{r}_{p/q} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}.
\end{aligned} \tag{4.16.5}$$

Therefore,

$$
\begin{aligned}
0 &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + (\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} - |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{r}_{p/q} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) \\
&= (\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - |\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2 \vec{r}_{p/q} - |\vec{\omega}_{\mathrm{B/A}}|^2 (\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}.
\end{aligned} \tag{4.16.6}
$$

Furthermore, (4.16.5) implies

$$
\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} = -(\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} + |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{r}_{p/q} - \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}. \tag{4.16.7}
$$

Substituting (4.16.7) into (4.16.6) and using $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \kappa \vec{\omega}_{\mathrm{B/A}}$ yields

$$
\begin{aligned}
0 &= (\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - |\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2 \vec{r}_{p/q} + |\vec{\omega}_{\mathrm{B/A}}|^2 (\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} \\
&\quad - |\vec{\omega}_{\mathrm{B/A}}|^4 \vec{r}_{p/q} + |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&= [\kappa\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \cdot \vec{r}_{p/q} + |\vec{\omega}_{\mathrm{B/A}}|^2 (\vec{\omega}_{\mathrm{B/A}} \cdot \vec{r}_{p/q})]\vec{\omega}_{\mathrm{B/A}} + |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&\quad + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} - (|\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2 + |\vec{\omega}_{\mathrm{B/A}}|^4)\vec{r}_{p/q},
\end{aligned} \tag{4.16.8}
$$

which implies *ii*).

Conversely, *ii*) implies that there exists $\alpha \in \mathbb{R}$ such that

$$
\vec{r}_{p/q} = \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2} \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) + \alpha \vec{\omega}_{\mathrm{B/A}}.
$$

Using *i*) and *ii*), it follows that

$$
\begin{aligned}
\vec{a}_{p/\mathrm{O_A}/\mathrm{A}} &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left[\frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2} \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) + \alpha \vec{\omega}_{\mathrm{B/A}} \right] \\
&\quad + \vec{\omega}_{\mathrm{B/A}} \times \left(\vec{\omega}_{\mathrm{B/A}} \times \left[\frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2} \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) + \alpha \vec{\omega}_{\mathrm{B/A}} \right] \right) \\
&\quad + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \\
&= \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2}\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) \\
&\quad + \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\, |^2} \vec{\omega}_{\mathrm{B/A}} \times \left[\vec{\omega}_{\mathrm{B/A}} \times \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} \right) \right]
\end{aligned}
$$

$$+ \vec{a}_{q/\mathrm{O_A}/\mathrm{A}}$$
$$= -\vec{a}_{q/\mathrm{O_A}/\mathrm{A}} + \vec{a}_{q/\mathrm{O_A}/\mathrm{A}} = 0.$$

Hence, p is an IACR at time t.

Finally, (4.16.8) implies (4.16.4). □

4.17 Kinematics Based on Chasle's Theorem

Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y_1', \ldots, y_l'\}$ be identical rigid bodies. Then, Chasle's theorem given by Fact 2.21.4 implies that there exist a point z, an eigenaxis $\hat{n}$, an eigenangle θ, and a real number α such that, for all $i = 1, \ldots, l$,

$$\vec{r}_{y_i'/z} = \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_i/z} + \alpha\hat{n}. \tag{4.17.1}$$

Letting $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames that are fixed identically in $\mathcal{B}$ and $\mathcal{B}'$, respectively, and differentiating (4.17.1) yields

$$\vec{v}_{y_i'/z/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{R}_{\hat{n}}}(\theta)\vec{r}_{y_i/z} + \vec{\xi}, \tag{4.17.2}$$

where

$$\vec{\xi} \triangleq \dot{\alpha}\hat{n} + \alpha \overset{\mathrm{A}\bullet}{\hat{n}}. \tag{4.17.3}$$

Solving (4.17.1) for $\vec{r}_{y_i/z}$ and substituting into (4.17.2) yields

$$\vec{v}_{y_i'/z/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{R}_{\hat{n}}}(\theta)\vec{R}_{\hat{n}}'(\theta)(\vec{r}_{y_i'/z} - \alpha\hat{n}) + \vec{\xi}. \tag{4.17.4}$$

Noting from (4.2.22) that

$$\overset{\mathrm{A}\bullet}{\vec{R}_{\hat{n}}}(\theta)\vec{R}_{\hat{n}}'(\theta) = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}} = \vec{\Omega}_{\mathrm{B/A}}, \tag{4.17.5}$$

(4.17.4) can be written as

$$\vec{v}_{y_i'/z/\mathrm{A}} = \vec{\Omega}_{\mathrm{B/A}}(\vec{r}_{y_i'/z} - \alpha\hat{n}) + \vec{\xi}. \tag{4.17.6}$$

Using $\vec{\Omega}_{\mathrm{B/A}} = \vec{\omega}_{\mathrm{B/A}}^{\times}$ and rearranging (4.17.6) yields

$$\vec{v}_{y_i'/z/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i'/z} + \alpha\hat{n} \times \vec{\omega}_{\mathrm{B/A}} + \vec{\xi}. \tag{4.17.7}$$

4.18 Rolling With and Without Slipping

When a disk of radius r is moving in contact with a flat surface, the absence of slipping can be determined by comparing the velocity of its center to its angular velocity, that is, $v = r\omega$, where ω is the rate of rotation of the disk relative to the surface. The relation $v = r\omega$ equates the speed of the center of the disk to the rate of arc length along the path of motion. However, for more general bodies, it is difficult to determine the arc length. We therefore adopt a more general approach, which involves the relative velocity between two points that are in contact.

At a given instant of time, the points x_1 and x_2 are *colocated* if x_1 and x_2 are at the same location. The following result shows that, if two bodies are in contact, then the relative velocity of a pair of colocated body-fixed points is independent of the frame with respect to which the velocity is determined.

Fact 4.18.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let x_1 and x_2 be points that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in contact with x_1 and x_2 colocated. Then,

$$\vec{v}_{x_1/x_2/\mathrm{A}} = \vec{v}_{x_1/x_2/\mathrm{B}}. \tag{4.18.1}$$

Proof. Note that

$$\vec{v}_{x_1/x_2/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{x_1/x_2} = \overset{\mathrm{B}\bullet}{\vec{r}}_{x_1/x_2} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x_1/x_2} = \vec{v}_{x_1/x_2/\mathrm{B}}. \qquad \square$$

Fact 4.18.1 shows that the choice of frame in the following definition is irrelevant.

Definition 4.18.2. Let $\mathrm{F_A}$ be a frame, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let x_1 and x_2 be points that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in contact with x_1 and x_2 colocated. Then, $\mathcal{B}_1$ and $\mathcal{B}_2$ are *rolling without slipping* if $\vec{v}_{x_1/x_2/\mathrm{A}} = 0$. Otherwise, $\mathcal{B}_1$ and $\mathcal{B}_2$ are *slipping*.

Fact 4.18.3. Let $\mathrm{F_A}$ be a frame, let w be a point, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let x_1 and x_2 be points that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in contact with x_1 and x_2 colocated, and assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are rolling without slipping. Then,

$$\vec{v}_{x_1/w/\mathrm{A}} = \vec{v}_{x_2/w/\mathrm{A}}. \tag{4.18.2}$$

Proof. Note that

$$\vec{v}_{x_1/w/\mathrm{A}} = \vec{v}_{x_1/x_2/\mathrm{A}} + \vec{v}_{x_2/w/\mathrm{A}} = \vec{v}_{x_2/w/\mathrm{A}}. \qquad \square$$

Example 4.18.4. Consider a wheel that rolls without slipping in a straight line on a flat surface, let p denote a point fixed on the circumference of the disk, let $\mathrm{F_A}$ denote a body-fixed frame in the surface, and let x and q denote points that are fixed in the plane. When p is in contact with q, then $\vec{v}_{\mathrm{P}/x/\mathrm{A}} = \vec{v}_{\mathrm{Q}/x/\mathrm{A}} = 0$.

4.19 Examples

Example 4.19.1. Consider a small disk with radius r that rolls without slipping inside a large hoop of radius R as shown in Figure 4.19.1. Frame $\mathrm{F_A}$ is fixed to the hoop, frame $\mathrm{F_B}$ is fixed to the arm that connects the disk to the center of the hoop, and frame $\mathrm{F_C}$ is fixed to the disk. Point a is the center of the hoop, point b is the center of the disk, and point c is fixed in the disk. The distance from b to c is r_0. Point d is fixed to the edge of the disk and point e is fixed on the hoop. Define angles θ and ϕ as shown. Determine $\vec{a}_{c/a/\mathrm{A}}$ in terms of $r, R, r_0, \phi, \dot{\phi}, \ddot{\phi}$ resolved in $\mathrm{F_B}$ at the instant that d and e are colocated.

Solution: The frames are related by

$$\mathrm{F_C} \xrightarrow[3]{\phi} \mathrm{F_B} \xrightarrow[3]{\theta} \mathrm{F_A},$$

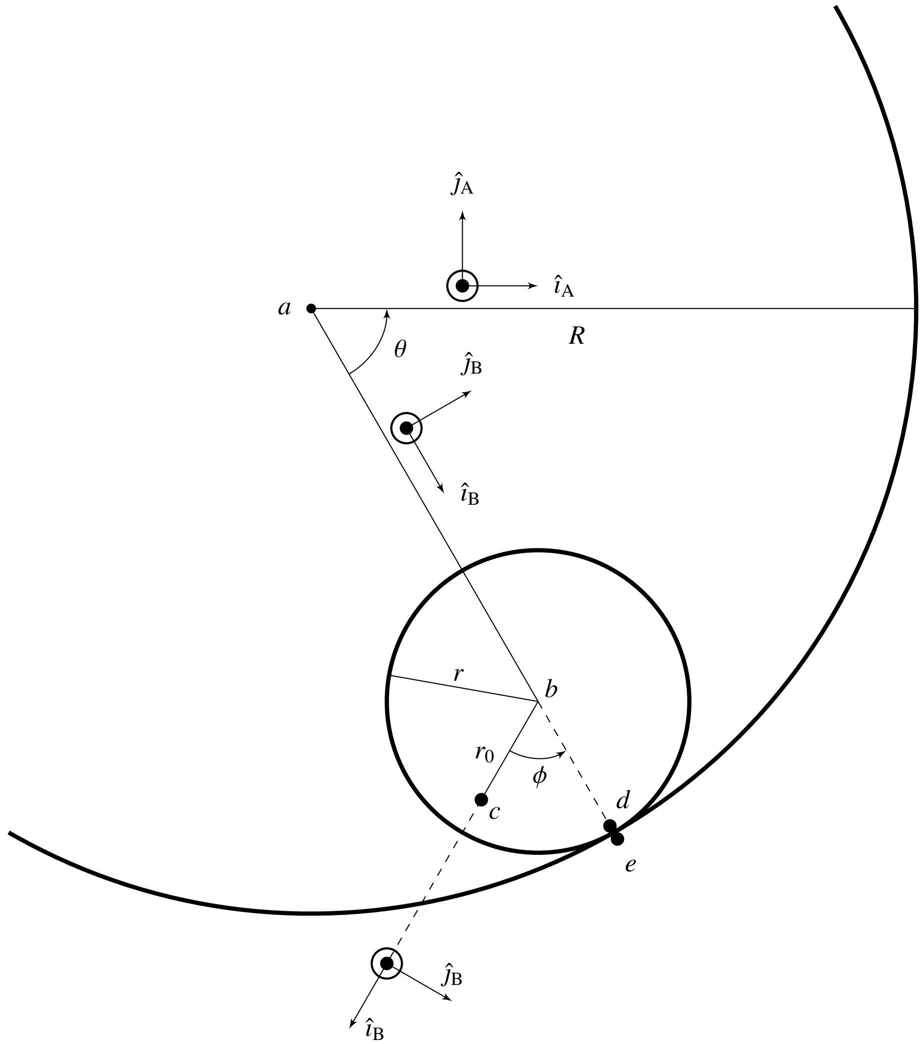


Figure 4.19.1: Example 4.19.1. Disk rolling inside a circle.

with angular velocities

$$\vec{\omega}_{\mathrm{A/B}} = \dot{\theta}\hat{k}_{\mathrm{A}}, \qquad \vec{\omega}_{\mathrm{B/C}} = \dot{\phi}\hat{k}_{\mathrm{A}}.$$

so that

$$\vec{\omega}_{\mathrm{A/C}} = (\dot{\theta} + \dot{\phi})\hat{k}_{\mathrm{A}}.$$

Furthermore, the position vectors are

$$\vec{r}_{c/b} = r_0\hat{\imath}_{\mathrm{C}}, \qquad \vec{r}_{b/a} = (R - r)\hat{\imath}_{\mathrm{B}},$$

so that

$$\vec{r}_{c/a} = r_0\hat{\imath}_{\mathrm{C}} + (R - r)\hat{\imath}_{\mathrm{B}}.$$

Hence,

$$\vec{v}_{c/a/\mathrm{A}} = r_0 \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{C}} + (R - r) \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}}$$

$$
\begin{aligned}
&= r_0(\vec{\omega}_{\mathrm{C/A}} \times \hat{\imath}_{\mathrm{C}}) + (R-r)(\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}})\\
&= r_0(-\dot{\phi} - \dot{\theta})\hat{k}_{\mathrm{C}} \times \hat{\imath}_{\mathrm{C}} + (R-r)(-\dot{\theta})\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{B}}\\
&= -r_0(\dot{\phi} + \dot{\theta})\hat{\jmath}_{\mathrm{C}} - (R-r)\dot{\theta}\hat{\jmath}_{\mathrm{B}}\\
&= -r_0(\dot{\phi} + \dot{\theta})[(\sin\phi)\hat{\imath}_{\mathrm{B}} + (\cos\phi)\hat{\jmath}_{\mathrm{B}}] - (R-r)\dot{\theta}\hat{\jmath}_{\mathrm{B}}\\
&= -r_0(\dot{\phi} + \dot{\theta})(\sin\phi)\hat{\imath}_{\mathrm{B}} - [r_0(\dot{\phi} + \dot{\theta})\cos\phi + (R-r)\dot{\theta}]\hat{\jmath}_{\mathrm{B}}.
\end{aligned}
$$

Next, the no-slip condition implies

$$0 = \vec{v}_{d/e/\mathrm{A}} = \vec{v}_{d/a/\mathrm{A}} - \vec{v}_{e/a/\mathrm{A}}.$$

Since, in addition, $\vec{r}_{e/a}$ is fixed in $\mathrm{F_A}$, it follows that

$$\vec{v}_{d/a/\mathrm{A}} = \vec{v}_{e/a/\mathrm{A}} = 0.$$

Therefore, when d and e are colocated so that $\vec{r}_{d/b} = r\hat{\imath}_{\mathrm{B}}$, it follows that

$$
\begin{aligned}
0 &= \vec{v}_{d/a/\mathrm{A}}\\
&= \vec{v}_{d/b/\mathrm{A}} + \vec{v}_{b/a/\mathrm{A}}\\
&= \overset{\mathrm{A}\bullet}{\vec{r}}_{d/b} + (R-r)\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}}\\
&= \overset{\mathrm{C}\bullet}{\vec{r}}_{d/b} + \vec{\omega}_{\mathrm{C/A}} \times \vec{r}_{d/b} + (R-r)\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}}\\
&= \vec{0} + (-\dot{\phi} - \dot{\theta})\hat{k}_{\mathrm{A}} \times r\hat{\imath}_{\mathrm{B}} + (R-r)(-\dot{\theta})\hat{k}_{\mathrm{A}} \times \hat{\imath}_{\mathrm{B}}\\
&= -r(\dot{\phi} + \dot{\theta})\hat{\jmath}_{\mathrm{B}} - (R-r)\dot{\theta}\hat{\jmath}_{\mathrm{B}}\\
&= -(r\dot{\phi} + R\dot{\theta})\hat{\jmath}_{\mathrm{B}}.
\end{aligned}
$$

Therefore,

$$\dot{\theta} = -\frac{r}{R}\dot{\phi},$$

and thus

$$\vec{\omega}_{\mathrm{A/B}} = -\frac{r}{R}\dot{\phi}\hat{k}_{\mathrm{A}}, \qquad \vec{\omega}_{\mathrm{C/A}} = \rho\dot{\phi}\hat{k}_{\mathrm{A}}.$$

where

$$\rho \triangleq \frac{r}{R} - 1.$$

Consequently,

$$\vec{v}_{c/a/\mathrm{A}} = \alpha\hat{\imath}_{\mathrm{B}} + \beta\hat{\jmath}_{\mathrm{B}},$$

where

$$\alpha \triangleq \rho r_0\dot{\phi}\sin\phi, \qquad \beta \triangleq \rho r_0\dot{\phi}\cos\phi - \rho r\dot{\phi}.$$

Finally,

$$
\begin{aligned}
\vec{\alpha}_{c/a/\mathrm{A}} &= \dot{\alpha}\hat{\imath}_{\mathrm{B}} + \alpha\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} + \dot{\beta}\hat{\jmath}_{\mathrm{B}} + \beta\overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}}\\
&= \dot{\alpha}\hat{\imath}_{\mathrm{B}} + \alpha\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}} + \dot{\beta}\hat{\jmath}_{\mathrm{B}} + \beta\vec{\omega}_{\mathrm{B/A}} \times \hat{\jmath}_{\mathrm{B}}\\
&= \dot{\alpha}\hat{\imath}_{\mathrm{B}} + \alpha\frac{r}{R}\dot{\phi}\hat{k}_{\mathrm{A}} \times \hat{\imath}_{\mathrm{B}} + \dot{\beta}\hat{\jmath}_{\mathrm{B}} + \beta\frac{r}{R}\dot{\phi}\hat{k}_{\mathrm{A}} \times \hat{\jmath}_{\mathrm{B}}
\end{aligned}
$$

$$= \left(\dot{\alpha} - \beta\frac{r}{R}\dot{\phi}\right)\hat{\imath}_{\rm B} + \left(\dot{\beta} + \alpha\frac{r}{R}\dot{\phi}\right)\hat{\jmath}_{\rm B}$$

$$= \rho[r_0(\sin\phi)\ddot{\phi} + (r^2/R - \rho r_0\cos\phi)\dot{\phi}^2]\hat{\imath}_{\rm B} + \rho[(r_0\cos\phi - r)\ddot{\phi} + \rho r_0(\sin\phi)\dot{\phi}^2]\hat{\jmath}_{\rm B}. \qquad \diamond$$

Example 4.19.2. The wheel shown in Figure 4.19.2 has radius R and rotates clockwise at a constant rate around its center point b, which is pinned to a fixed point in the ground. The point P denotes a pin located on the circumference of the wheel; this pin slides along the slot in the arm as shown. The arm is pinned to the ground at point a, which is fixed in the ground, and the distance from a to b is L. Using your intuition only, make a rough sketch of θ over the interval $[t_0, t_{\rm f}]$, where $\phi = 0$ deg at $t = 0$, and $\phi = 180$ deg at $t_{\rm f}$. Mark the points on the plot at which *i*) θ achieves its maximum value, and *ii*) $\phi = 90$ deg. Next, derive expressions for $\dot{\theta}$ and $\ddot{\theta}$, and specialize these expressions to the case $\phi = 90$ deg. Finally, check the signs of $\dot{\theta}$ and $\ddot{\theta}$ and determine whether those signs are consistent with your sketch of θ versus t.

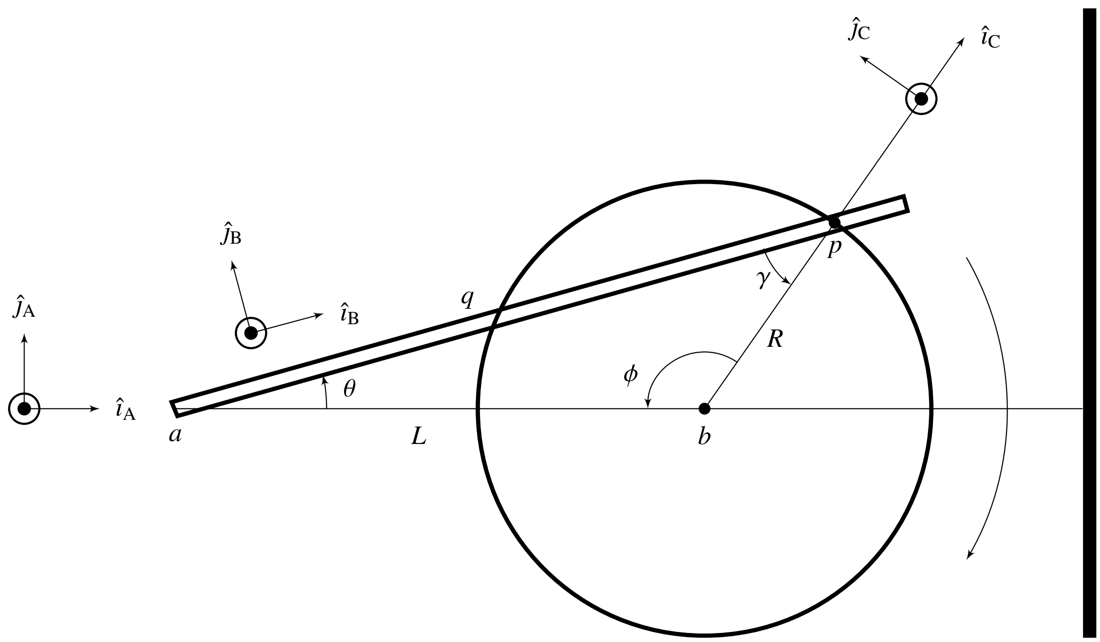


Figure 4.19.2: Example 4.19.2. Rotating wheel with slotted arm.

Solution: The frames are related by

$$\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B} \xrightarrow[3]{\gamma} \mathrm{F_C},$$

with angular velocities

$$\vec{\omega}_{\rm B/A} = \dot{\theta}\hat{k}_{\rm A}, \qquad \vec{\omega}_{\rm C/B} = \dot{\gamma}\hat{k}_{\rm A}.$$

Since $\theta + \gamma + \phi = \pi$, it follows that

$$\vec{\omega}_{\rm C/A} = -\dot{\phi}\hat{k}_{\rm A}.$$

Next, let $q \triangleq |\vec{r}_{P/a}|$ so that $\vec{r}_{P/a} = q\hat{\imath}_{\mathrm{B}}$. Therefore,

$$\begin{aligned}\vec{v}_{P/a/\mathrm{A}} &= \dot{q}\hat{\imath}_{\mathrm{B}} + q\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}}\\ &= \dot{q}\hat{\imath}_{\mathrm{B}} + q\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}}\\ &= \dot{q}\hat{\imath}_{\mathrm{B}} + q\dot{\theta}\hat{\jmath}_{\mathrm{B}}.\end{aligned}$$

On the other hand, defining $v \triangleq R\dot{\phi}$, it follows that

$$\begin{aligned}\vec{v}_{P/a/\mathrm{A}} &= \vec{v}_{P/b/\mathrm{A}} + \vec{v}_{b/a/\mathrm{A}}\\ &= \overset{\mathrm{A}\bullet}{\vec{r}}_{P/b}\\ &= R\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{C}}\\ &= R\vec{\omega}_{\mathrm{C/A}} \times \hat{\imath}_{\mathrm{C}}\\ &= -v\hat{\jmath}_{\mathrm{C}}\\ &= v(\sin\gamma)\hat{\imath}_{\mathrm{B}} - v(\cos\gamma)\hat{\jmath}_{\mathrm{B}}.\end{aligned}$$

Equating the expressions for $\vec{v}_{P/a/\mathrm{A}}$ yields

$$\dot{\theta} = -\frac{v\cos\gamma}{q}.$$

The law of sines implies

$$\frac{\sin\theta}{R} = \frac{\sin\gamma}{L} = \frac{\sin\phi}{q},$$

while the law of cosines implies

$$R^2 = q^2 + L^2 - 2qL\cos\theta$$

and

$$q^2 = R^2 + L^2 - 2RL\cos\phi,$$

and thus

$$\cos\theta = \frac{L - R\cos\phi}{q} = \frac{L - R\cos\phi}{\sqrt{R^2 + L^2 - 2RL\cos\phi}}.$$

It can be shown that θ is maximized when $\cos\phi = R/L$. Furthermore,

$$\dot{\theta} = \frac{(RL\cos\phi - R^2)\dot{\phi}}{R^2 + L^2 - 2RL\cos\phi},$$

and thus

$$\ddot{\theta} = \frac{RL(R^2 - L^2)(\sin\phi)\dot{\phi}^2}{(R^2 + L^2 - 2RL\cos\phi)^2}.$$

Therefore, when $\phi = 90$ deg it follows that

$$\dot{\theta} = -\frac{R^2\dot{\phi}}{R^2 + L^2}$$

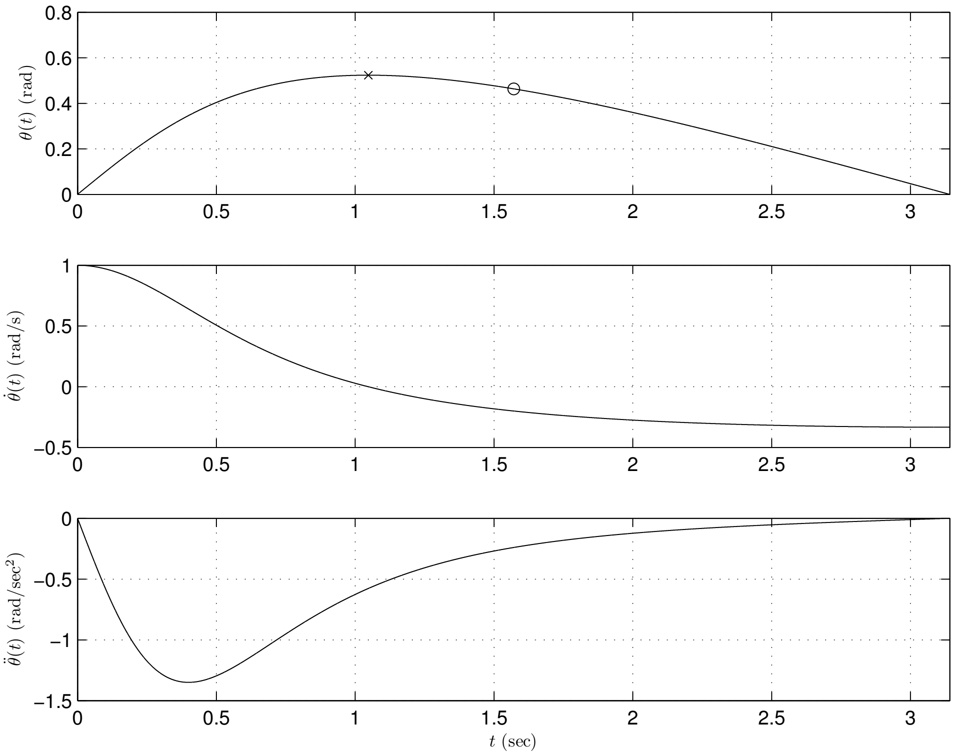


Figure 4.19.3: Example 4.19.2. Rotating wheel with slotted arm. The angle θ and angle rates $\dot{\theta}$ and $\ddot{\theta}$ are plotted for $0 \le t \le \pi$ sec assuming that $R = 2$ m, $L = 4$ m, and $\dot{\phi} = 1$ rad/sec. The points marked ‘x’ and ‘o’ denote $\theta(t_1)$, where $\phi(t_1) = 60$ deg and thus the arm is tangent to the wheel, and $\theta(t_2)$, where $\phi(t_2) = 90$ deg, respectively.

and

$$\ddot{\theta} = \frac{RL(R^2 - L^2)\dot{\phi}^2}{(R^2 + L^2)^2}.$$

Note that, when $\phi = 90$ deg, $\dot{\phi}$ is positive, and thus $\dot{\theta}$ is negative. Furthermore, since $R < L$, it follows that $\ddot{\theta}$ is negative. Figure 4.19.3 shows that the angle θ is maximized at a time t_1 at which the arm is tangent to the wheel, that is, when $\phi = 60$ deg. In addition, at time $t_2 > t_1$, $\phi(t_2) = \pi/2$ rad, where θ is decreasing, and the function $\theta(t)$ is convex. ◇

4.20 Theoretical Problems

Problem 4.20.1. Prove (4.6.1) by using (4.2.3) to represent each physical angular velocity matrix and by resolving each term in $\mathrm{F_C}$.

Problem 4.20.2. Let $\vec{x} = \vec{x}(t)$ be a physical vector that is nonzero on the time interval $[t_1, t_2]$, and let $\mathrm{F_A}$ be a frame. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}| = \frac{1}{|\vec{x}|}\overset{\mathrm{A}\bullet\prime}{\vec{x}}\ \vec{x}.$$

Furthermore, show that, if $|\vec{x}(t)|$ is constant, then $\vec{x}(t)$ and $\overset{\mathrm{A}\bullet}{\vec{x}}(t)$ are mutually orthogonal. Finally, apply this result to the following two cases:

i) The position of a point moving in a circle.

ii) The velocity of a point moving with constant speed in a circle.

Problem 4.20.3. Let $\mathrm{F_A}$ be a frame, and, for all $t \in (t_1, t_2)$, let $\hat{x}(t)$ be a unit dimensionless vector and let $\vec{y}(t)$ be a vector such that $\hat{x}(t) \times \vec{y}(t) = 0$ and $\overset{\mathrm{A}\bullet}{\vec{y}}(t) = 0$. Furthermore, assume that there exists $t_0 \in (t_1, t_2)$ such that $\vec{y}(t_0) \neq 0$. Show that $\overset{\mathrm{A}\bullet}{\hat{x}}(t_0) = 0$. (Hint: Apply Problem 2.26.1.)

Problem 4.20.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, assume that, for all $t \in (t_1, t_2)$, $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) = 0$ and $\vec{\omega}_{\mathrm{B/A}}(t) \times \hat{n}_{\mathrm{B/A}}(t) = 0$, and assume that there exists $t_0 \in (t_1, t_2)$ such that $\vec{\omega}_{\mathrm{B/A}}(t_0) \neq 0$. Then, $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t_0) = \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}(t_0) = 0$. (Hint: Apply Problem 4.20.3.)

Problem 4.20.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Show that

$$\vec{\omega}_{\mathrm{B/A}} = \tfrac{1}{2}(\hat{\imath}_{\mathrm{B}} \times \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} + \hat{\jmath}_{\mathrm{B}} \times \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} + \hat{k}_{\mathrm{B}} \times \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}}),$$

$$\vec{\omega}_{\mathrm{B/A}} = (\overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} \cdot \hat{k}_{\mathrm{B}})\hat{\imath}_{\mathrm{B}} + (\overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} \cdot \hat{\imath}_{\mathrm{B}})\hat{\jmath}_{\mathrm{B}} + (\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \cdot \hat{\jmath}_{\mathrm{B}})\hat{k}_{\mathrm{B}}.$$

Problem 4.20.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ and $\vec{y}$ be position vectors that are constant with respect to $\mathrm{F_B}$. Show that

$$\overset{\mathrm{A}\bullet\prime}{\vec{x}}\;\vec{y}\vec{\omega}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{x}} \times \overset{\mathrm{A}\bullet}{\vec{y}},$$

that is,

$$(\vec{\omega}_{\mathrm{B/A}} \times \vec{x})'\vec{y}\vec{\omega}_{\mathrm{B/A}} = (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}) \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{y}).$$

Problem 4.20.7. Consider 3-2-1 Euler angles Ψ, Θ, and Φ that transform $\mathrm{F_A}$ to $\mathrm{F_D}$.

i) Determine all values of the Euler angles such that *not* all angular velocities $\vec{\omega}_{\mathrm{D/A}}$ can be attained by Euler-angle derivatives $\dot{\Psi}$, $\dot{\Theta}$, and $\dot{\Phi}$. In particular, show that not all angular velocities $\vec{\omega}_{\mathrm{D/A}}$ can be attained by Euler-angle derivatives if and only if $\Theta = \pm\pi/2$.

ii) Show that, if $\Theta = \pm\pi/2$ and $\omega \neq 0$, then $\vec{\omega}_{\mathrm{D/A}} = \omega\hat{k}_{\mathrm{D}}$ is attainable by Euler-angle derivatives if and only if $\Phi = \pm\pi/2$.

(Remark: *ii*) illustrates *gimbal lock*.)

Problem 4.20.8. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ be a physical vector. Show that

$$\overset{\mathrm{A}\bullet\bullet}{\vec{x}}\Bigg|_{\mathrm{A}} = \overset{\cdot\cdot}{\overbrace{\vec{x}\Big|_{\mathrm{A}}}}.$$

Problem 4.20.9. Show that the double transport theorem (4.5.1) can be rewritten so that it has the same form as (4.5.1) but with A and B interchanged. (Hint: Move the last three terms to the left hand side and use the transport theorem.)

Problem 4.20.10. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Show that

$$\overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}}, \quad \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}},$$

$$\overset{\mathrm{A}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}, \quad \overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}.$$

Problem 4.20.11. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Show that

$$\overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{x}}} + \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{x}}} = \overset{\mathrm{A}\bullet\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet\bullet}{\vec{x}} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}),$$

$$\overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{x}}} = \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{x}}} + \vec{x} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}.$$

If, in addition, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$, then

$$\overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{x}}} = \overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{x}}} = \tfrac{1}{2}[\overset{\mathrm{A}\bullet\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet\bullet}{\vec{x}} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x})].$$

Finally, if $\vec{\omega}_{\mathrm{B/A}} = 0$, then

$$\overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{x}}} = \overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{x}}} = \tfrac{1}{2}(\overset{\mathrm{A}\bullet\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet\bullet}{\vec{x}}).$$

Confirm these identities for the example $\vec{x} = \hat{\imath}_{\mathrm{B}}$ and $\vec{\omega}_{\mathrm{B/A}} = \hat{\imath}_{\mathrm{A}}$. (Hint: To prove the second equality, write $\vec{x} = x_1\hat{\imath}_{\mathrm{B}} + x_2\hat{\jmath}_{\mathrm{B}} + x_3\hat{k}_{\mathrm{B}}$.)

Problem 4.20.12. Derive the triple transport theorem

$$\begin{aligned}\overset{\mathrm{A}\bullet\bullet\bullet}{\vec{x}} = {} & \overset{\mathrm{B}\bullet\bullet\bullet}{\vec{x}} + 3\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet\bullet}{\vec{x}} + 3\,\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}} + 3\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}}) \\ & + (\overset{\mathrm{B}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}) \times \vec{x} + 2\,\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}) \\ & + \vec{\omega}_{\mathrm{B/A}} \times (\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x}) + \vec{\omega}_{\mathrm{B/A}} \times [\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x})].\end{aligned}$$

4.21 Applied Problems

Problem 4.21.1. For all parts below, assume that the given numbers are exact. In addition, assume that the Sun, Earth, and Moon all rotate and travel counterclockwise as viewed from “above” the solar system (that is, looking down on the North Pole of the Earth), and that all orbits are circular and lie in the same plane. Finally, assume that all stars (including the Sun) do not move relative to each other.

i) Assume that the length of the solar day on Earth is 24 hours, and assume that the Earth completes one orbit around the Sun in relation to the stars every 365.25 solar days (the sidereal year). Determine the length of the sidereal day, that is, the time it takes for the Earth to rotate around its axis once relative to a star frame, that is, a frame whose axes have fixed directions relative to the stars. State your solution in hours and minutes.

ii) In addition to the assumptions in *i*), assume that the Sun rotates around its axis relative to the star frame once every 27 solar days. Determine the length of time that it takes the Earth to rotate around its axis once relative to a Sun body-fixed frame.

iii) Assume that the Moon completes one orbit around the Earth in relation to the stars every 27.3 solar days (which is the Moon's sidereal period, also called the lunar month). For a given initial relative configuration, determine the time it takes for the Moon, Earth, and Sun to return to the same configuration (which is the Moon's synodic period).

(Hint: Create several frames, including a star frame and frames that are attached to the Sun, Earth, and Moon as well as the "invisible" arms connecting these bodies. Then, use sums of angular velocities to determine relationships between the periods.)

Problem 4.21.2. Let F_A be a frame that is fixed to a horizontal plane with origin O_A. The axis $\hat{\imath}_A$ points to the right, and the axis $\hat{\jmath}_A$ points downward. A disk $\mathcal{B}$ of radius R rolls in a straight path to the right, which is in the direction of $\hat{\imath}_A$. The center of $\mathcal{B}$ is the point O_B, and F_B is fixed to the disk. The speed $v \geq 0$ of O_B along the path is not necessarily constant, and the angle ϕ of $\mathcal{B}$ relative to its starting angle has the rate $\dot{\phi} > 0$, which is not necessarily constant. As $\mathcal{B}$ rolls, it may also slip, and thus v is not necessarily equal to $R\dot{\phi}$. Let p denote a point fixed on the circumference of $\mathcal{B}$, and let q denote a point fixed in the path. Define $x \triangleq |\vec{r}_{q/O_A}|$. Let c denote the instantaneous contact point between $\mathcal{B}$ and the path.

i) Determine the velocity of q relative to O_A with respect to F_A resolved in F_A.

ii) Determine the velocity of q relative to O_A with respect to F_B resolved in F_A.

iii) Determine the velocity of p relative to O_A with respect to F_A resolved in F_A when p is at the 9:00, 12:00, and 3:00 positions.

iv) Determine the velocity of p relative to O_A with respect to F_A resolved in F_A when p and q are colocated.

v) Determine the velocity of p relative to O_A with respect to F_B resolved in F_A when p and q are colocated.

vi) Determine the acceleration of p relative to O_A with respect to F_A resolved in F_A when p and q are colocated. If v is constant, what is the direction of this acceleration?

Specialize the solutions to *i*)–*v*) to the case where $\mathcal{B}$ rolls without slipping, that is, $v = R\dot{\phi}$. Now, assume that $\mathcal{B}$ rolls without slipping.

vii) Show that $\vec{v}_{c/O_A/A} = v\hat{\imath}_A = \vec{v}_{c/O_B/B} = R\dot{\phi}\hat{\imath}_A$.

viii) Determine the velocity of c relative to O_A with respect to F_B resolved in F_A.

Problem 4.21.3. Consider the gimbal mechanism shown in Figure 4.21.1. Assume that the angle of the outer gimbal relative to the support is given by $\psi(t) = 0.6\sin(20\pi t)$, the angle of the inner gimbal relative to the outer gimbal is $\theta(t) = 0.3\sin(90\pi t)$, and the disk supported by the inner gimbal spins at 700 rpm with zero initial angle. Resolve the angular velocity and angular acceleration of the disk relative to the support in a frame attached to the disk, and compute these vectors at time $t = .01$ sec.

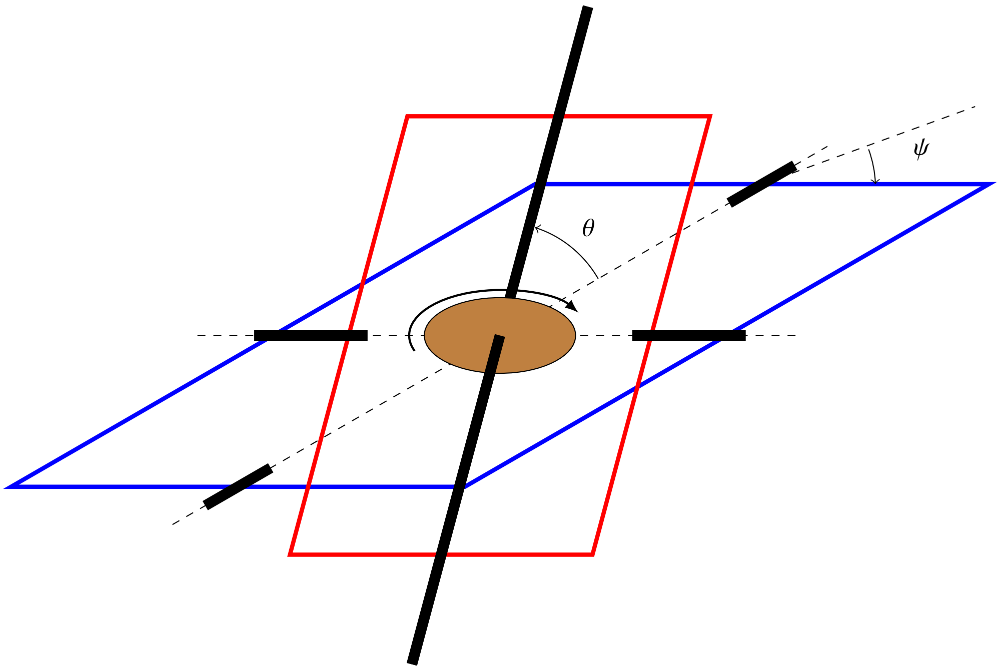


Figure 4.21.1: Gimbal mechanism for Problem 4.21.3.

Problem 4.21.4. Consider a horizontal platform connected to the horizontal ground by a vertical pin joint. The platform rotates at a constant rate relative to the ground. Let x and y be distinct points on the platform.

i) Show that the acceleration of y relative to x with respect to a frame fixed to the ground is independent of the location of the pin joint relative to x and y.

ii) Now, suppose that the pin joint is mounted on a horizontal X-Y table that can move the pin joint along an arbitrary path, for example, a straight line or a curve. How does this additional motion affect the acceleration of y relative to x with respect to a frame fixed to the ground?

Problem 4.21.5. Points a and b are connected by a rigid bar with pin joints at both a and b as shown in Figure 4.21.2. The pin joint at a moves horizontally to the right with constant velocity v. The pin joint at b is connected to a disk with center c that rolls clockwise without slipping. The length of the bar is $2R$, and the radius of the disk is R. At the time instant shown, the vector $\vec{r}_{c/b}$ is parallel to the horizontal surface. For the configuration shown, determine the velocity and acceleration of c relative to a point fixed in the horizontal surface and with respect to a frame that is also fixed to the horizontal surface. Resolve your solution in the frame that is fixed to the horizontal surface.

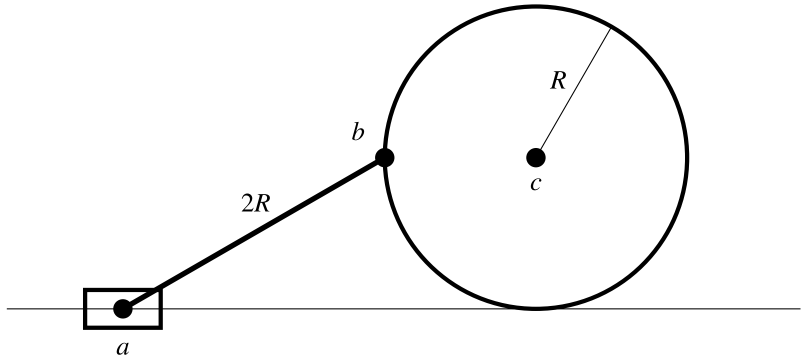


Figure 4.21.2: Bar and disk linkage for Problem 4.21.5.

Problem 4.21.6. The horizontal rod in Figure 4.21.3 moves to the right with constant speed v with respect to the ground. A pin at the end of the rod slides within the slot that passes through the center of the wheel. The radius of the wheel is R, and the distance from the center of the wheel to the rod is d. The wheel rolls clockwise in a straight line without slipping. Let θ denote the angle between the vertical direction and the direction of the slot. Determine $\dot{\theta}$ and $\ddot{\theta}$ as functions of d, R, v, and θ.

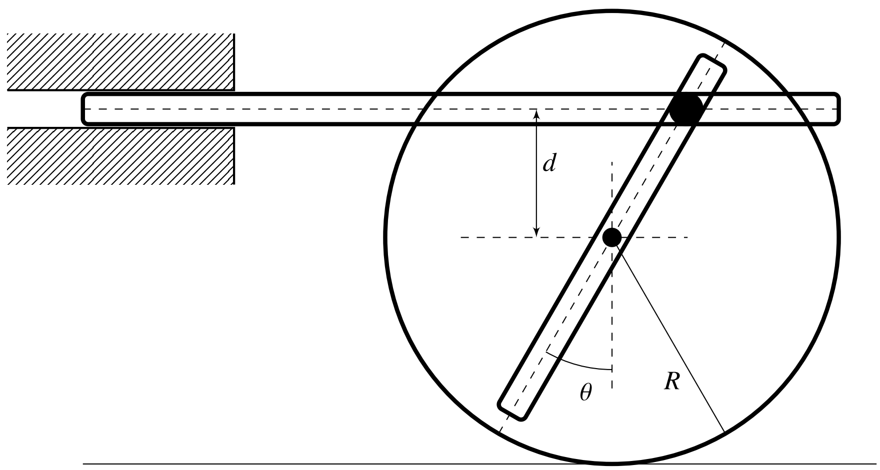


Figure 4.21.3: Wheel with slot and horizontal rod for Problem 4.21.6.

Problem 4.21.7. The wheel of radius r in Figure 4.21.4 is attached at its center b to an axle, whose other end is connected to a central hub at the point a. The wheel rolls without slipping along a circular path on the ground and with radius R. The constant spin rate of the wheel is $\omega > 0$, and the constant spin rate of the hub is $\Omega > 0$. The directions of rotation are noted in the figure. The point c is fixed on the edge of the wheel. Determine the velocity of c relative to a with respect to a frame fixed to the ground at the instant at which c touches the ground and resolved in the ground frame. Next, derive an equation that relates r, R, ω, and Ω. Finally, determine the acceleration of c relative to a with respect to the ground frame at the instant at which c touches the ground and resolved in a frame fixed to the hub.

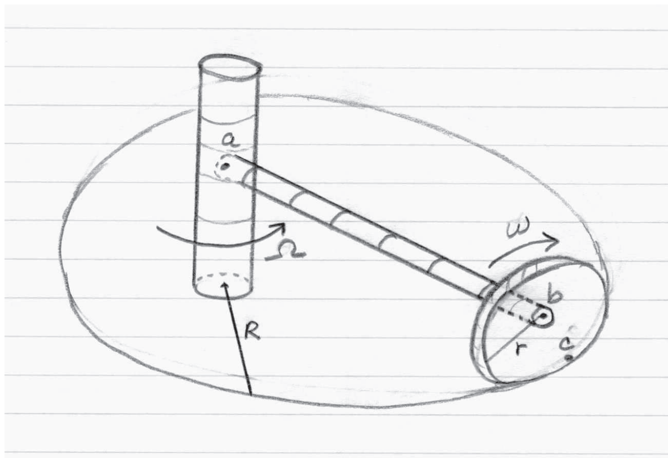


Figure 4.21.4: Wheel with hub for Problem 4.21.7.

Symbol	Definition
$\overset{\mathrm{A}\bullet}{\vec{x}}$	Derivative of $\vec{x}$ with respect to frame A
$\vec{v}_{y/x/\mathrm{A}}$	Velocity vector $\overset{\mathrm{A}\bullet}{\vec{r}}_{y/x}$
$\vec{a}_{y/x/\mathrm{A}}$	Acceleration vector $\overset{\mathrm{A}\bullet}{\vec{v}}_{y/x/\mathrm{A}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/x}$
$\vec{\omega}_{\mathrm{B/A}}$	Angular velocity vector
$\vec{\Omega}_{\mathrm{B/A}}$	Angular velocity matrix
$\vec{\alpha}_{\mathrm{B/A}}$	Angular acceleration vector
$\vec{\alpha}_{\mathrm{B/A/C}}$	Angular acceleration vector

Table 4.1: Symbols for Chapter 4.

Chapter Five

Geometry and Kinematics in Alternative Frames

If the orientation of a frame depends on the position of a point x, then the frame is a *position-dependent frame associated with* x. The cylindrical and spherical frames are position-dependent frames.

5.1 Cylindrical Frame

The cylindrical frame $\mathrm{F}_{\mathrm{cyl}}$ associated with the point x is obtained by rotating a given frame F_{A} around the vector $\hat{k}_{\mathrm{A}}$ until the vector $\hat{\imath}_{\mathrm{A}}$ is aligned with the projection of $\vec{r}_{x/\mathrm{O_A}}$ onto the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. Consequently, the cylindrical frame is related to F_{A} by

$$\mathrm{F}_{\mathrm{cyl}} = \vec{R}_{\mathrm{cyl/A}}\mathrm{F}_{\mathrm{A}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\mathrm{F}_{\mathrm{A}}, \tag{5.1.1}$$

that is,

$$\mathrm{F}_{\mathrm{A}} \xrightarrow[3]{\theta} \mathrm{F}_{\mathrm{cyl}}, \tag{5.1.2}$$

where the *azimuthal angle* $\theta \in (-\pi, \pi]$ is the signed angle from $\hat{\imath}_{\mathrm{A}}$ to $\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{O_A}}$ around $\hat{k}_{\mathrm{A}}$, that is,

$$\theta \triangleq \theta_{\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{O_A}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}. \tag{5.1.3}$$

If $\vec{r}_{x/\mathrm{O_A}}$ is parallel with $\hat{k}_{\mathrm{A}}$, then θ is defined to be 0. The *radial*, *tangential*, and *axial* axes of the cylindrical frame $\mathrm{F}_{\mathrm{cyl}}$ associated with x are denoted by $\hat{e}_{\mathrm{r}}$, $\hat{e}_{\mathrm{t}}$, and $\hat{e}_{\mathrm{a}}$, respectively, and defined by

$$\hat{e}_{\mathrm{r}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{\imath}_{\mathrm{A}}, \tag{5.1.4}$$

$$\hat{e}_{\mathrm{t}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{\jmath}_{\mathrm{A}}, \tag{5.1.5}$$

$$\hat{e}_{\mathrm{a}} = \hat{k}_{\mathrm{A}}. \tag{5.1.6}$$

The *cylindrical frame* is thus given by

$$\mathrm{F}_{\mathrm{cyl}} = [\hat{e}_{\mathrm{r}} \;\; \hat{e}_{\mathrm{t}} \;\; \hat{e}_{\mathrm{a}}]. \tag{5.1.7}$$

It follows from (5.1.3) and (5.1.4) that

$$\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{O_A}} = |\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{O_A}}|\hat{e}_{\mathrm{r}}. \tag{5.1.8}$$

See Figure 5.1.1.

The cylindrical frame can be viewed as a body-fixed frame. In particular, consider a shaft

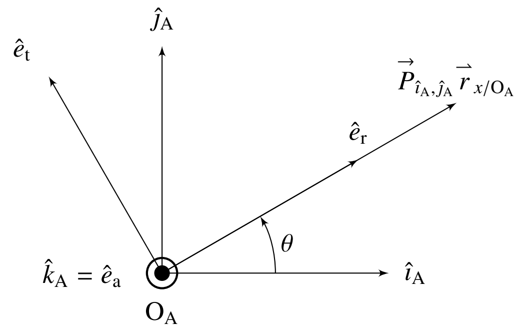


Figure 5.1.1: The signed angle θ that defines the cylindrical frame is the angle around $\hat{k}_A$ from $\hat{\imath}_A$ to the projection $\vec{P}_{\hat{\imath}_A,\hat{\jmath}_A}\vec{r}_{x/O_A}$ of $\vec{r}_{x/O_A}$ onto the plane spanned by $\hat{\imath}_A$ and $\hat{\jmath}_A$.

aligned with the axis $\hat{k}_A$ of the frame F_A attached to the base of the shaft. The shaft has a collar. A telescoping arm with a sleeve joint is attached to the collar at a right angle to the shaft. A rigid body $\mathcal{B}$ is rigidly attached to the tip of the telescoping arm. A body-fixed frame F_B is attached to $\mathcal{B}$ such that $\hat{k}_B$ is aligned with $\hat{k}_A$ and $\hat{\imath}_B$ is parallel to the telescoping arm. The angle θ is defined to be $\theta \triangleq \theta_{\hat{\imath}_B/\hat{\imath}_A/\hat{k}_A}$. Therefore, if $\theta = 0$, then F_A and F_B are aligned. Consequently, the body-fixed frame F_B is the cylindrical frame F_{cyl}. See Figure 5.1.2.

Next, it follows from (5.1.6) that

$$\vec{P}_{\hat{\imath}_A,\hat{\jmath}_A} = \vec{P}_{\hat{e}_r,\hat{e}_t}, \tag{5.1.9}$$

that is,

$$\hat{\imath}_A\hat{\imath}_A + \hat{\jmath}_A\hat{\jmath}_A = \hat{e}_r\hat{e}_r + \hat{e}_t\hat{e}_t. \tag{5.1.10}$$

Using (5.1.8) and (5.1.9), it follows that

$$\vec{P}_{\hat{e}_r,\hat{e}_t}\vec{r}_{x/O_A} = \vec{P}_{\hat{\imath}_A,\hat{\jmath}_A}\vec{r}_{x/O_A} = |\vec{P}_{\hat{\imath}_A,\hat{\jmath}_A}\vec{r}_{x/O_A}|\hat{e}_r. \tag{5.1.11}$$

Next, since $\vec{R}_{cyl/A} = \vec{R}_{\hat{k}_A}(\theta)$, it follows from (5.1.4), (5.1.5), (5.1.6), and (2.10.13) that

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_t \\ \hat{e}_a \end{bmatrix} = \mathcal{O}_{cyl/A} \begin{bmatrix} \hat{\imath}_A \\ \hat{\jmath}_A \\ \hat{k}_A \end{bmatrix}, \tag{5.1.12}$$

where, using (2.12.12),

$$\mathcal{O}_{cyl/A} = \vec{R}_{\hat{k}_A}(\theta)\Big|_A^T = \mathcal{O}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.1.13}$$

Consequently,

$$\hat{e}_r = (\cos\theta)\hat{\imath}_A + (\sin\theta)\hat{\jmath}_A, \tag{5.1.14}$$

$$\hat{e}_t = -(\sin\theta)\hat{\imath}_A + (\cos\theta)\hat{\jmath}_A, \tag{5.1.15}$$

$$\hat{e}_a = \hat{k}_A. \tag{5.1.16}$$

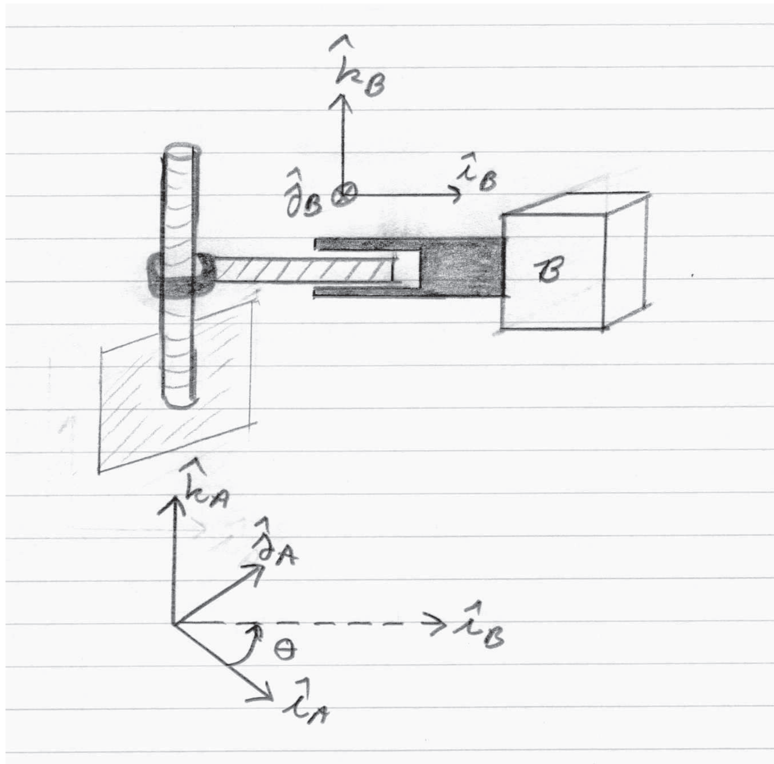


Figure 5.1.2: Cylindrical frame given as a body-fixed frame.

Next, write

$$r_{x/\mathrm{O_A}|\mathrm{A}} = \vec{r}_{x/\mathrm{O_A}}\Big|_{\mathrm{A}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{5.1.17}$$

$$r_{x/\mathrm{O_A}|\mathrm{cyl}} = \vec{r}_{x/\mathrm{O_A}}\Big|_{\mathrm{cyl}} = \begin{bmatrix} r_\mathrm{r} \\ r_\mathrm{t} \\ r_\mathrm{a} \end{bmatrix}. \tag{5.1.18}$$

Then,

$$\begin{bmatrix} r_\mathrm{r} \\ r_\mathrm{t} \\ r_\mathrm{a} \end{bmatrix} = \mathcal{O}_{\mathrm{cyl/A}} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} (\cos\theta)r_1 + (\sin\theta)r_2 \\ -(\sin\theta)r_1 + (\cos\theta)r_2 \\ r_3 \end{bmatrix}. \tag{5.1.19}$$

Furthermore, with this notation (5.1.11) can be rewritten as

$$r_\mathrm{r}\hat{e}_\mathrm{r} + r_\mathrm{t}\hat{e}_\mathrm{t} = r_1\hat{\imath}_\mathrm{A} + r_2\hat{\jmath}_\mathrm{A} = |\vec{P}_{\hat{\imath}_\mathrm{A},\hat{\jmath}_\mathrm{A}}\vec{r}_{x/\mathrm{O_A}}|\hat{e}_\mathrm{r}, \tag{5.1.20}$$

which implies that

$$r_\mathrm{t} = 0, \tag{5.1.21}$$

$$r_\mathrm{r}\hat{e}_\mathrm{r} = r_1\hat{\imath}_\mathrm{A} + r_2\hat{\jmath}_\mathrm{A}, \tag{5.1.22}$$

$$\vec{r}_{x/\mathrm{O_A}} = r_1\hat{\imath}_\mathrm{A} + r_2\hat{\jmath}_\mathrm{A} + r_3\hat{k}_\mathrm{A} = r_\mathrm{r}\hat{e}_\mathrm{r} + r_\mathrm{a}\hat{e}_\mathrm{a}, \tag{5.1.23}$$

$$r_\mathrm{r} = |\vec{P}_{\hat{\imath}_\mathrm{A},\hat{\jmath}_\mathrm{A}}\vec{r}_{x/\mathrm{O_A}}| = \sqrt{r_1^2 + r_2^2}. \tag{5.1.24}$$

Next, it follows from the second equation in (5.1.19) that

$$(\cos\theta)r_2 = (\sin\theta)r_1, \tag{5.1.25}$$

If $r_1 = r_2 = 0$, then $\vec{r}_{x/\mathrm{O_A}}$ is parallel with $\hat{k}_\mathrm{A}$, and thus, by definition, $\theta = 0$. On the other hand, if $r_1 = 0$ and $r_2 \neq 0$, then $\cos\theta = 0$. In this case, $\theta = -\pi/2$ if and only if $r_2 < 0$, and $\theta = \pi/2$ if and only if $r_2 > 0$. In the case $r_1 \neq 0$, we have

$$\tan\theta = \frac{r_2}{r_1}. \tag{5.1.26}$$

Since θ is the angle of the complex number $r_1 + r_2\jmath$, it follows that

$$\theta = \mathrm{atan2}(r_2, r_1), \tag{5.1.27}$$

which determines θ for all values of r_1 and r_2. See (2.3.9). Finally, dotting (5.1.22) with $\hat{\imath}_\mathrm{A}$ yields

$$r_1 = (\cos\theta)r_\mathrm{r}, \tag{5.1.28}$$

while using (5.1.25) yields

$$r_2 = (\sin\theta)r_\mathrm{r}. \tag{5.1.29}$$

The *cylindrical coordinates* $(r_\mathrm{r}, \theta, r_\mathrm{a})$ associated with x are thus given by

$$r_\mathrm{r} = \sqrt{r_1^2 + r_2^2}, \tag{5.1.30}$$

$$\theta = \mathrm{atan2}(r_2, r_1), \tag{5.1.31}$$

$$r_\mathrm{a} = r_3. \tag{5.1.32}$$

5.2 Kinematics in the Cylindrical Frame

It follows from (5.1.2) that

$$\vec{\omega}_{\mathrm{cyl/A}} = \dot{\theta}\hat{e}_\mathrm{a}, \tag{5.2.1}$$

and thus

$$\omega_{\mathrm{cyl/A|cyl}} = \vec{\omega}_{\mathrm{cyl/A}}\Big|_{\mathrm{cyl}} = \dot{\theta}\,\hat{e}_\mathrm{a}|_{\mathrm{cyl}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}. \tag{5.2.2}$$

Alternatively, using (4.3.20) it follows that

$$\begin{aligned} \omega^{\times}_{\mathrm{cyl/A|cyl}} &= -\dot{\mathcal{O}}_{\mathrm{cyl/A}}\mathcal{O}_{\mathrm{A/cyl}} \\ &= -\begin{bmatrix} -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta & 0 \\ -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{5.2.3}$$

Next, it follows from (4.3.21) that

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{r}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{a}} \end{bmatrix} = -\omega^{\times}_{\mathrm{cyl/A|cyl}} \begin{bmatrix} \hat{e}_{\mathrm{r}} \\ \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{a}} \end{bmatrix}. \tag{5.2.4}$$

Therefore,

$$\overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{r}} = \dot{\theta}\hat{e}_{\mathrm{t}} = \vec{\omega}_{\mathrm{cyl/A}} \times \hat{e}_{\mathrm{r}}, \tag{5.2.5}$$

$$\overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{t}} = -\dot{\theta}\hat{e}_{\mathrm{r}} = \vec{\omega}_{\mathrm{cyl/A}} \times \hat{e}_{\mathrm{t}}, \tag{5.2.6}$$

$$\overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{a}} = 0 = \vec{\omega}_{\mathrm{cyl/A}} \times \hat{e}_{\mathrm{a}}. \tag{5.2.7}$$

Now, let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$, and let x be a point. Then, $\vec{r}_{x/\mathrm{O_A}}$ can be expressed in the cylindrical frame as

$$\vec{r}_{x/\mathrm{O_A}} = r_{\mathrm{r}}\hat{e}_{\mathrm{r}} + r_{\mathrm{a}}\hat{e}_{\mathrm{a}}. \tag{5.2.8}$$

Therefore,

$$\begin{aligned}
\vec{v}_{x/\mathrm{O_A/A}} &= \overset{\mathrm{A}\bullet}{\vec{r}}_{x/\mathrm{O_A}} \\
&= \dot{r}_{\mathrm{r}}\hat{e}_{\mathrm{r}} + r_{\mathrm{r}} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{r}} + \dot{r}_{\mathrm{a}}\hat{e}_{\mathrm{a}} + r_{\mathrm{a}} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{a}} \\
&= \dot{r}_{\mathrm{r}}\hat{e}_{\mathrm{r}} + \dot{\theta} r_{\mathrm{r}}\hat{e}_{\mathrm{t}} + \dot{r}_{\mathrm{a}}\hat{e}_{\mathrm{a}} \\
&= \dot{r}_{\mathrm{r}}\hat{e}_{\mathrm{r}} + \dot{r}_{\mathrm{a}}\hat{e}_{\mathrm{a}} + \dot{\theta} r_{\mathrm{r}}\hat{e}_{\mathrm{t}} \\
&= \overset{\mathrm{cyl}\bullet}{\vec{r}}_{x/\mathrm{O_A}} + \vec{\omega}_{\mathrm{cyl/A}} \times \vec{r}_{x/\mathrm{O_A}},
\end{aligned} \tag{5.2.9}$$

which is the transport theorem for the cylindrical frame. Furthermore,

$$\begin{aligned}
\vec{a}_{x/\mathrm{O_A/A}} &= \overset{\mathrm{A}\bullet}{\vec{v}}_{x/\mathrm{O_A/A}} \\
&= \ddot{r}_{\mathrm{r}}\hat{e}_{\mathrm{r}} + \dot{r}_{\mathrm{r}} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{r}} + \ddot{\theta} r_{\mathrm{r}}\hat{e}_{\mathrm{t}} + \dot{\theta}\dot{r}_{\mathrm{r}}\hat{e}_{\mathrm{t}} + \dot{\theta} r_{\mathrm{r}} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{t}} + \ddot{r}_{\mathrm{a}}\hat{e}_{\mathrm{a}} + \dot{r}_{\mathrm{a}} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{a}} \\
&= \ddot{r}_{\mathrm{r}}\hat{e}_{\mathrm{r}} + \dot{\theta}\dot{r}_{\mathrm{r}}\hat{e}_{\mathrm{t}} + \ddot{\theta} r_{\mathrm{r}}\hat{e}_{\mathrm{t}} + \dot{\theta}\dot{r}_{\mathrm{r}}\hat{e}_{\mathrm{t}} - \dot{\theta}^2 r_{\mathrm{r}}\hat{e}_{\mathrm{r}} + \ddot{r}_{\mathrm{a}}\hat{e}_{\mathrm{a}} \\
&= (\ddot{r}_{\mathrm{r}} - \dot{\theta}^2 r_{\mathrm{r}})\hat{e}_{\mathrm{r}} + (2\dot{\theta}\dot{r}_{\mathrm{r}} + \ddot{\theta} r_{\mathrm{r}})\hat{e}_{\mathrm{t}} + \ddot{r}_{\mathrm{a}}\hat{e}_{\mathrm{a}} \\
&= \ddot{r}_{\mathrm{r}}\hat{e}_{\mathrm{r}} + \ddot{r}_{\mathrm{a}}\hat{e}_{\mathrm{a}} + \underbrace{2\dot{\theta}\dot{r}_{\mathrm{r}}\hat{e}_{\mathrm{t}}}_{\substack{\text{Coriolis} \\ \text{acceleration}}} + \underbrace{\ddot{\theta} r_{\mathrm{r}}\hat{e}_{\mathrm{t}}}_{\substack{\mathrm{A}^2 \\ \text{acceleration}}} + \underbrace{-\dot{\theta}^2 r_{\mathrm{r}}\hat{e}_{\mathrm{r}}}_{\substack{\text{centripetal} \\ \text{acceleration}}} \\
&= \overset{\mathrm{cyl}\bullet\bullet}{\vec{r}}_{x/\mathrm{O_A}} + 2\vec{\omega}_{\mathrm{cyl/A}} \times \overset{\mathrm{cyl}\bullet}{\vec{r}}_{x/\mathrm{O_A}} + \overset{\mathrm{cyl}\bullet}{\vec{\omega}}_{\mathrm{cyl/A}} \times \vec{r}_{x/\mathrm{O_A}} + \vec{\omega}_{\mathrm{cyl/A}} \times (\vec{\omega}_{\mathrm{cyl/A}} \times \vec{r}_{x/\mathrm{O_A}}),
\end{aligned} \tag{5.2.10}$$

which is the double transport theorem for the cylindrical frame.

5.3 Spherical Frame

The spherical frame $\mathrm{F_{sph}}$ associated with the point x is obtained by rotating the cylindrical frame around its tangential vector until the radial vector is aligned with the position of x relative to the origin of $\mathrm{F_A}$. Consequently, the spherical frame is related to the cylindrical frame $\mathrm{F_{cyl}}$ by

$$\mathrm{F_{sph}} = \vec{R}_{\mathrm{sph/cyl}}\mathrm{F_{cyl}} = \vec{R}_{\mathrm{sph/A}}\mathrm{F_A}, \tag{5.3.1}$$

that is,

$$\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_{cyl}} \xrightarrow[2]{\phi} \mathrm{F_{sph}}, \tag{5.3.2}$$

where

$$\vec{R}_{\mathrm{sph/cyl}} = \vec{R}_{\hat{e}_\mathrm{t}}(\phi), \tag{5.3.3}$$

$\hat{e}_\mathrm{t}$ is the tangential vector (5.1.5) of the cylindrical frame with $\theta \in (-\pi, \pi]$ the azimuthal angle (5.1.3), and the *elevation angle* $\phi \in [-\pi/2, \pi/2]$ is the signed angle from the radial vector $\hat{e}_\mathrm{r}$ (5.1.4) of the cylindrical frame to $\vec{r}_{x/\mathrm{O_A}}$ around $\hat{e}_\mathrm{t}$, that is,

$$\phi \triangleq \theta_{\vec{r}_{x/\mathrm{O_A}}/\hat{e}_\mathrm{r}/\hat{e}_\mathrm{t}}. \tag{5.3.4}$$

Note that $\vec{R}_{\hat{e}_\mathrm{t}}(\phi)$ denotes a right-hand-rule clockwise rotation around $\hat{e}_\mathrm{t}$ for positive values of ϕ. Hence, if ϕ is positive, then the component $\vec{r}_{x/\mathrm{O_A}} \cdot \hat{k}_\mathrm{A}$ of $\vec{r}_{x/\mathrm{O_A}}$ in the direction of $\hat{k}_\mathrm{A}$ is negative. The *spherical frame* is thus given by

$$\mathrm{F_{sph}} = \vec{R}_{\hat{e}_\mathrm{t}}(\phi)\vec{R}_{\hat{k}_\mathrm{A}}(\theta)\mathrm{F_A}. \tag{5.3.5}$$

Furthermore,

$$\vec{r}_{x/\mathrm{O_A}} = |\vec{r}_{x/\mathrm{O_A}}|\vec{R}_{\hat{e}_\mathrm{t}}(\phi)\vec{R}_{\hat{k}_\mathrm{A}}(\theta)\hat{\imath}_\mathrm{A}. \tag{5.3.6}$$

The spherical frame can be viewed as a body-fixed frame. In particular, consider a rotating shaft whose axis is aligned with $\hat{k}_\mathrm{A}$. A telescoping arm is connected to the shaft by means of a pin. The telescoping arm has a sleeve, and a rigid body $\mathcal{B}$ is rigidly attached to the end of the arm opposite to the pin. Now, consider a body-fixed frame $\mathrm{F_B}$ attached to $\mathcal{B}$ such that $\hat{\imath}_\mathrm{B}$ is aligned with the arm. As the shaft rotates, the arm rotates at the pin joint, and the arm extends and retracts, the frame $\mathrm{F_B}$ coincides with the spherical frame. See Figure 5.3.3.

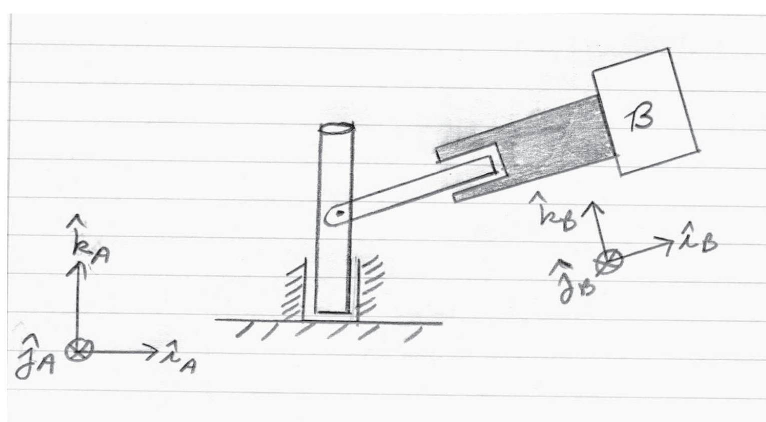


Figure 5.3.3: Spherical frame given as a body-fixed frame.

The axes of the spherical frame $\mathrm{F_{sph}}$ associated with x are *up*, *east*, and *north*, denoted by $\hat{e}_\mathrm{u}$, $\hat{e}_\mathrm{e}$, and $\hat{e}_\mathrm{n}$, respectively, where

$$\hat{e}_\mathrm{u} = \vec{R}_{\hat{e}_\mathrm{t}}(\phi)\vec{R}_{\hat{k}_\mathrm{A}}(\theta)\hat{\imath}_\mathrm{A} = \vec{R}_{\hat{e}_\mathrm{t}}(\phi)\hat{e}_\mathrm{r}, \tag{5.3.7}$$

$$\hat{e}_\mathrm{e} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\hat{\jmath}_\mathrm{A} = \hat{e}_\mathrm{t}, \tag{5.3.8}$$

$$\hat{e}_\mathrm{n} = \vec{R}_{\hat{e}_\mathrm{t}}(\phi)\hat{k}_\mathrm{A}. \tag{5.3.9}$$

Hence,

$$\mathrm{F_{sph}} = [\hat{e}_\mathrm{u} \ \ \hat{e}_\mathrm{e} \ \ \hat{e}_\mathrm{n}]. \tag{5.3.10}$$

Furthermore, it follows from (5.3.6) and (5.3.7) that

$$\vec{r}_{x/\mathrm{O_A}} = |\vec{r}_{x/\mathrm{O_A}}|\hat{e}_\mathrm{u}. \tag{5.3.11}$$

Next, it follows from (5.3.5) that

$$\begin{bmatrix} \hat{e}_\mathrm{u} \\ \hat{e}_\mathrm{e} \\ \hat{e}_\mathrm{n} \end{bmatrix} = \mathcal{O}_\mathrm{sph/A} \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix}, \tag{5.3.12}$$

where

$$\begin{aligned}
\mathcal{O}_\mathrm{sph/A} &= \left(\vec{R}_{\hat{e}_\mathrm{t}}(\phi)\vec{R}_{\hat{k}_\mathrm{A}}(\theta)\right)\Big|_\mathrm{A}^\mathrm{T} \\
&= \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\Big|_\mathrm{A}^\mathrm{T} \vec{R}_{\hat{e}_\mathrm{t}}(\phi)\Big|_\mathrm{A}^\mathrm{T} \\
&= \mathcal{O}_\mathrm{cyl/A}\mathcal{O}_\mathrm{A/cyl} \ \vec{R}_{\hat{e}_\mathrm{t}}(\phi)\Big|_\mathrm{cyl}^\mathrm{T} \mathcal{O}_\mathrm{cyl/A} \\
&= \mathcal{O}_2(\phi)\mathcal{O}_3(\theta) \\
&= \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (\cos\phi)\cos\theta & (\cos\phi)\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ (\sin\phi)\cos\theta & (\sin\phi)\sin\theta & \cos\phi \end{bmatrix}.
\end{aligned} \tag{5.3.13}$$

Consequently,

$$\hat{e}_\mathrm{u} = (\cos\phi)(\cos\theta)\hat{\imath}_\mathrm{A} + (\cos\phi)(\sin\theta)\hat{\jmath}_\mathrm{A} - (\sin\phi)\hat{k}_\mathrm{A}, \tag{5.3.14}$$

$$\hat{e}_\mathrm{e} = -(\sin\theta)\hat{\imath}_\mathrm{A} + (\cos\theta)\hat{\jmath}_\mathrm{A}, \tag{5.3.15}$$

$$\hat{e}_\mathrm{n} = (\sin\phi)(\cos\theta)\hat{\imath}_\mathrm{A} + (\sin\phi)(\sin\theta)\hat{\jmath}_\mathrm{A} + (\cos\phi)\hat{k}_\mathrm{A}. \tag{5.3.16}$$

Likewise,

$$\mathcal{O}_\mathrm{A/sph} = \begin{bmatrix} (\cos\phi)\cos\theta & -\sin\theta & (\sin\phi)\cos\theta \\ (\cos\phi)\sin\theta & \cos\theta & (\sin\phi)\sin\theta \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}, \tag{5.3.17}$$

and thus

$$\hat{\imath}_{\rm A} = (\cos\phi)(\cos\theta)\hat{e}_{\rm u} - (\sin\theta)\hat{e}_{\rm e} + (\sin\phi)(\cos\theta)\hat{e}_{\rm n}, \tag{5.3.18}$$

$$\hat{\jmath}_{\rm A} = (\cos\phi)(\sin\theta)\hat{e}_{\rm u} + (\cos\theta)\hat{e}_{\rm e} + (\sin\phi)(\sin\theta)\hat{e}_{\rm n}, \tag{5.3.19}$$

$$\hat{k}_{\rm A} = -(\sin\phi)\hat{e}_{\rm u} + (\cos\phi)\hat{e}_{\rm n}. \tag{5.3.20}$$

Next, write

$$r_{x/{\rm O_A}|{\rm A}} = \vec{r}_{x/{\rm O_A}}\Big|_{\rm A} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{5.3.21}$$

$$r_{x/{\rm O_A}|{\rm sph}} \ \vec{r}_{x/{\rm O_A}}\Big|_{\rm sph} = \begin{bmatrix} r_{\rm u} \\ r_{\rm e} \\ r_{\rm n} \end{bmatrix}. \tag{5.3.22}$$

Thus,

$$\begin{bmatrix} r_{\rm u} \\ r_{\rm e} \\ r_{\rm n} \end{bmatrix} = \mathcal{O}_{\rm sph/A} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} (\cos\phi)(\cos\theta)r_1 + (\cos\phi)(\sin\theta)r_2 - (\sin\phi)r_3 \\ -(\sin\theta)r_1 + (\cos\theta)r_2 \\ (\sin\phi)(\cos\theta)r_1 + (\sin\phi)(\sin\theta)r_2 + (\cos\phi)r_3 \end{bmatrix}. \tag{5.3.23}$$

Next, it follows from (5.3.11) that

$$r_{\rm u}\hat{e}_{\rm u} + r_{\rm e}\hat{e}_{\rm e} + r_{\rm n}\hat{e}_{\rm n} = |\vec{r}_{x/{\rm O_A}}|\hat{e}_{\rm u}. \tag{5.3.24}$$

Thus,

$$r_{\rm e} = 0, \tag{5.3.25}$$

$$r_{\rm n} = 0, \tag{5.3.26}$$

$$\vec{r}_{x/{\rm O_A}} = r_1\hat{\imath}_{\rm A} + r_2\hat{\jmath}_{\rm A} + r_3\hat{k}_{\rm A} = r_{\rm u}\hat{e}_{\rm u}, \tag{5.3.27}$$

$$r_{\rm u} = |\vec{r}_{x/{\rm O_A}}| = \sqrt{r_1^2 + r_2^2 + r_3^2}. \tag{5.3.28}$$

Now, it follows from (5.3.16) and (5.3.27) that

$$r_1 = (\hat{e}_{\rm u} \cdot \hat{\imath}_{\rm A})r_{\rm u} = (\cos\phi)(\cos\theta)r_{\rm u}, \tag{5.3.29}$$

$$r_2 = (\hat{e}_{\rm u} \cdot \hat{\jmath}_{\rm A})r_{\rm u} = (\cos\phi)(\sin\theta)r_{\rm u}, \tag{5.3.30}$$

$$r_3 = (\hat{e}_{\rm u} \cdot \hat{k}_{\rm A})r_{\rm u} = -(\sin\phi)r_{\rm u}, \tag{5.3.31}$$

where the minus sign in (5.3.31) indicates that points for which ϕ is positive are located in the southern hemisphere.

Since $r_{\rm e} = 0$, it follows from the second equation in (5.3.23) that

$$(\sin\theta)r_1 - (\cos\theta)r_2 = 0, \tag{5.3.32}$$

and thus, as in the case of the cylindrical frame,

$$\theta = {\rm atan2}(r_2, r_1). \tag{5.3.33}$$

Next, since $r_{\rm n} = 0$ it follows from the third equation in (5.3.23) that

$$(\sin\phi)(\cos\theta)r_1 + (\sin\phi)(\sin\theta)r_2 + (\cos\phi)r_3 = 0. \tag{5.3.34}$$

Recall that $\phi \in [-\pi/2, \pi/2]$. Now, suppose that $r_1 = r_2 = 0$. Then, assuming that x is not located at $\mathrm{O_A}$, it follows from (5.3.34) that $\cos\phi = 0$, and thus either $\phi = \pi/2$ or $\phi = -\pi/2$. Conversely, suppose that either $\phi = \pi/2$ or $\phi = -\pi/2$. Then, it follows from (5.3.34) that

$$(\cos\theta)r_1 + (\sin\theta)r_2 = 0. \tag{5.3.35}$$

It now follows from (5.3.32) and (5.3.35) that $r_1 = r_2 = 0$. Consequently, either $\phi = \pi/2$ or $\phi = -\pi/2$ if and only if $r_1 = r_2 = 0$.

Now, assume that r_1 and r_2 are not both zero. Then, $\cos\phi \neq 0$, and thus we can write

$$\tan\phi = -\frac{r_3}{(\cos\theta)r_1 + (\sin\theta)r_2}. \tag{5.3.36}$$

Hence, using (5.3.32) it follows that

$$\tan\phi = -\frac{r_3}{\sqrt{r_1^2 + r_2^2}} \in (-\pi/2, \pi/2), \tag{5.3.37}$$

and thus

$$\phi = -\tan^{-1}\frac{r_3}{\sqrt{r_1^2 + r_2^2}} \tag{5.3.38}$$

Note that $r_3 > 0$ if and only if $\phi < 0$. Finally, including the case $r_1 = r_2 = 0$, we have

$$\phi = \begin{cases} -\pi/2, & r_1 = r_2 = 0, r_3 > 0, \\ \pi/2, & r_1 = r_2 = 0, r_3 < 0, \\ -\tan^{-1}\frac{r_3}{\sqrt{r_1^2+r_2^2}}, & r_1^2 + r_2^2 > 0, \end{cases} \tag{5.3.39}$$

or, equivalently,

$$\phi = -\mathrm{atan2}\left(r_3, \sqrt{r_1^2 + r_2^2}\right). \tag{5.3.40}$$

The *spherical coordinates* $(r_\mathrm{u}, \theta, \phi)$ associated with x are thus given by

$$r_\mathrm{u} = \sqrt{r_1^2 + r_2^2 + r_3^2}, \tag{5.3.41}$$

$$\theta = \mathrm{atan2}(r_2, r_1), \tag{5.3.42}$$

$$\phi = -\mathrm{atan2}\left(r_3, \sqrt{r_1^2 + r_2^2}\right). \tag{5.3.43}$$

5.4 Kinematics in the Spherical Frame

It follows from (5.3.2) that

$$\vec{\omega}_{\mathrm{sph/A}} = \dot\theta\hat{e}_\mathrm{a} + \dot\phi\hat{e}_\mathrm{e}, \tag{5.4.1}$$

and thus

$$\begin{aligned} \omega_{\mathrm{sph/A|sph}} &= \vec{\omega}_{\mathrm{sph/A}}\Big|_{\mathrm{sph}} \\ &= \dot\theta\ \hat{e}_\mathrm{a}\big|_{\mathrm{sph}} + \dot\phi\ \hat{e}_\mathrm{e}\big|_{\mathrm{sph}} \\ &= \dot\theta\mathcal{O}_{\mathrm{sph/cyl}}\ \hat{e}_\mathrm{a}\big|_{\mathrm{cyl}} + \dot\phi\ \hat{e}_\mathrm{e}\big|_{\mathrm{sph}} \end{aligned}$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\phi \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\dot\theta\sin\phi \\ \dot\phi \\ \dot\theta\cos\phi \end{bmatrix}. \tag{5.4.2}$$

Alternatively, using (4.3.20) it follows that

$$\omega^{\times}_{\mathrm{sph/A|sph}} = \mathcal{O}_{\mathrm{sph/A}}\dot{\mathcal{O}}_{\mathrm{A/sph}}$$

$$= \begin{bmatrix} (\cos\phi)\cos\theta & (\cos\phi)\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ (\sin\phi)\cos\theta & (\sin\phi)\sin\theta & \cos\phi \end{bmatrix}$$

$$\times \begin{bmatrix} -\dot\phi(\sin\phi)\cos\theta - \dot\theta(\cos\phi)\sin\theta & -\dot\theta\cos\theta & \dot\phi(\cos\phi)\cos\theta - \dot\theta(\sin\phi)\sin\theta \\ -\dot\phi(\sin\phi)\sin\theta + \dot\theta(\cos\phi)\cos\theta & -\dot\theta\sin\theta & \dot\phi(\cos\phi)\sin\theta + \dot\theta(\sin\phi)\cos\theta \\ -\dot\phi\cos\phi & 0 & -\dot\phi\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dot\theta\cos\phi & \dot\phi \\ \dot\theta\cos\phi & 0 & \dot\theta\sin\phi \\ -\dot\phi & -\dot\theta\sin\phi & 0 \end{bmatrix}. \tag{5.4.3}$$

Next, it follows from (4.3.21) that

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{u}} \\ \overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{e}} \\ \overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{n}} \end{bmatrix} = -\omega^{\times}_{\mathrm{sph/A|sph}} \begin{bmatrix} \hat e_{\mathrm{u}} \\ \hat e_{\mathrm{e}} \\ \hat e_{\mathrm{n}} \end{bmatrix}. \tag{5.4.4}$$

Therefore,

$$\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{u}} = \dot\theta(\cos\phi)\hat e_{\mathrm{e}} - \dot\phi\hat e_{\mathrm{n}} = \vec\omega_{\mathrm{sph/A}} \times \hat e_{\mathrm{u}}, \tag{5.4.5}$$

$$\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{e}} = -\dot\theta(\cos\phi)\hat e_{\mathrm{u}} - \dot\theta(\sin\phi)\hat e_{\mathrm{n}} = \vec\omega_{\mathrm{sph/A}} \times \hat e_{\mathrm{e}}, \tag{5.4.6}$$

$$\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{n}} = \dot\phi\hat e_{\mathrm{u}} + \dot\theta(\sin\phi)\hat e_{\mathrm{e}} = \vec\omega_{\mathrm{sph/A}} \times \hat e_{\mathrm{n}}. \tag{5.4.7}$$

Now, let $\mathrm{F_A}$ be a frame with origin $\mathrm{O_A}$, and let x be a point. Then, $\vec r_{x/\mathrm{O_A}}$ can be expressed in the spherical frame as

$$\vec r_{x/\mathrm{O_A}} = r_{\mathrm{u}}\hat e_{\mathrm{u}}. \tag{5.4.8}$$

Therefore,

$$\begin{aligned} \vec v_{x/\mathrm{O_A/A}} &= \overset{\mathrm{A}\bullet}{\vec r}_{x/\mathrm{O_A}} \\ &= \dot r_{\mathrm{u}}\hat e_{\mathrm{u}} + r_{\mathrm{u}}\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{u}} \\ &= \dot r_{\mathrm{u}}\hat e_{\mathrm{u}} + r_{\mathrm{u}}\dot\theta(\cos\phi)\hat e_{\mathrm{e}} - r_{\mathrm{u}}\dot\phi\hat e_{\mathrm{n}} \\ &= \overset{\mathrm{sph}\bullet}{\vec r}_{x/\mathrm{O_A}} + \vec\omega_{\mathrm{sph/A}} \times \vec r_{x/\mathrm{O_A}}, \end{aligned} \tag{5.4.9}$$

which is the transport theorem for the spherical frame. Furthermore,

$$\begin{aligned}
\vec{a}_{x/\mathrm{O_A}/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec{v}}_{x/\mathrm{O_A}/\mathrm{A}} \\
&= \ddot{r}_\mathrm{u}\hat{e}_\mathrm{u} + \dot{r}_\mathrm{u}\overset{\mathrm{A}\bullet}{\hat{e}}_\mathrm{u} + [\dot{r}_\mathrm{u}\dot{\theta}(\cos\phi) + r_\mathrm{u}\ddot{\theta}(\cos\phi) - r_\mathrm{u}\dot{\theta}\dot{\phi}\sin\phi]\hat{e}_\mathrm{e} + r_\mathrm{u}\dot{\theta}(\cos\phi)\overset{\mathrm{A}\bullet}{\hat{e}}_\mathrm{e} \\
&\quad - (\dot{r}_\mathrm{u}\dot{\phi} + r_\mathrm{u}\ddot{\phi})\hat{e}_\mathrm{n} - r_\mathrm{u}\dot{\phi}\overset{\mathrm{A}\bullet}{\hat{e}}_\mathrm{n} \\
&= \ddot{r}_\mathrm{u}\hat{e}_\mathrm{u} + \dot{r}_\mathrm{u}\dot{\theta}(\cos\phi)\hat{e}_\mathrm{e} - \dot{r}_\mathrm{u}\dot{\phi}\hat{e}_\mathrm{n} + [\dot{r}_\mathrm{u}\dot{\theta}(\cos\phi) + r_\mathrm{u}\ddot{\theta}(\cos\phi) - r_\mathrm{u}\dot{\theta}\dot{\phi}\sin\phi]\hat{e}_\mathrm{e} \\
&\quad - r_\mathrm{u}\dot{\theta}(\cos\phi)[\dot{\theta}(\cos\phi)\hat{e}_\mathrm{u} + \dot{\theta}(\sin\phi)\hat{e}_\mathrm{n}] - (\dot{r}_\mathrm{u}\dot{\phi} + r_\mathrm{u}\ddot{\phi})\hat{e}_\mathrm{n} - r_\mathrm{u}\dot{\phi}[\dot{\phi}\hat{e}_\mathrm{u} + \dot{\theta}(\sin\phi)\hat{e}_\mathrm{e}] \\
&= [\ddot{r}_\mathrm{u} - r_\mathrm{u}\dot{\theta}^2(\cos^2\phi) - r_\mathrm{u}\dot{\phi}^2]\hat{e}_\mathrm{u} + [2\dot{r}_\mathrm{u}\dot{\theta}(\cos\phi) + r_\mathrm{u}\ddot{\theta}(\cos\phi) - 2r_\mathrm{u}\dot{\phi}\dot{\theta}\sin\phi]\hat{e}_\mathrm{e} \\
&\quad + [r_\mathrm{u}\dot{\theta}^2(\cos\phi)(\sin\phi) - 2\dot{r}_\mathrm{u}\dot{\phi} - r_\mathrm{u}\ddot{\phi}]\hat{e}_\mathrm{n} \\
&= \ddot{r}_\mathrm{u}\hat{e}_\mathrm{u} + \underbrace{2\dot{r}_\mathrm{u}[\dot{\theta}(\cos\phi)\hat{e}_\mathrm{e} - \dot{\phi}\hat{e}_\mathrm{n}]}_{\text{Coriolis acceleration}} + \underbrace{r_\mathrm{u}([\ddot{\theta}(\cos\phi) - \dot{\phi}\dot{\theta}\sin\phi]\hat{e}_\mathrm{e} - \ddot{\phi}\hat{e}_\mathrm{n})}_{\mathrm{A}^2\text{ acceleration}} \\
&\quad + \underbrace{-r_\mathrm{u}[(\dot{\phi}^2 + \dot{\theta}^2\cos^2\phi)\hat{e}_\mathrm{u} + \dot{\phi}\dot{\theta}(\sin\phi)\hat{e}_\mathrm{e} + \dot{\theta}^2(\cos\phi)(\sin\phi)\hat{e}_\mathrm{n}]}_{\text{centripetal acceleration}} \\
&= \overset{\mathrm{sph}\bullet\bullet}{\vec{r}}_{x/\mathrm{O_A}} + 2\vec{\omega}_{\mathrm{sph}/\mathrm{A}} \times \overset{\mathrm{sph}\bullet}{\vec{r}}_{x/\mathrm{O_A}} + \overset{\mathrm{sph}\bullet}{\vec{\omega}}_{\mathrm{sph}/\mathrm{A}} \times \vec{r}_{x/\mathrm{O_A}} + \vec{\omega}_{\mathrm{sph}/\mathrm{A}} \times (\vec{\omega}_{\mathrm{sph}/\mathrm{A}} \times \vec{r}_{x/\mathrm{O_A}}),
\end{aligned} \tag{5.4.10}$$

which is the double transport theorem for the spherical frame. Note that the Coriolis acceleration is zero if r_u is constant.

Note that the Coriolis acceleration is given by

$$\vec{a}_\mathrm{Cor} = 2\vec{\omega}_{\mathrm{sph}/\mathrm{A}} \times \overset{\mathrm{sph}\bullet}{\vec{r}}_{x/\mathrm{O_A}} = 2\dot{r}_\mathrm{u}[\dot{\theta}(\cos\phi)\hat{e}_\mathrm{e} - \dot{\phi}\hat{e}_\mathrm{n}]. \tag{5.4.11}$$

5.5 Frenet-Serret Frame

Let x be a point whose position relative to a point w is parameterized by the real number α. The position of x relative to w can thus be written as $\vec{r}_{x/w}(\alpha)$. The Frenet-Serret frame is a frame that depends on the location of x and thus on α. The set of all locations of x over all possible values of α constitutes the curve $\mathcal{C}$. The parameter α may represent time, in which case the curve can be viewed as growing in time. Alternatively, α may represent arc length, in which case the location of x can be found by specifying the distance along the curve from w. Finally, the parameter α can itself be parameterized by time by writing $\alpha = \alpha(t)$.

This notation is incomplete, however, since it does not indicate which parameterization is chosen when α is set to a numerical value. For example, $\vec{r}_{x/w}(3)$ is ambiguous. To remove this ambiguity when confusion can arise, we may write $\vec{r}_{\alpha,x/w}(\beta)$, where the additional subscript α identifies the parameter that parameterizes the curve and β denotes a value of α. When an additional subscript is not used, the choice of parameterization is assumed to be inferred by the argument α in $\vec{r}_{x/w}(\alpha)$.

Resolving $\vec{r}_{x/w}(\alpha)$ in $\mathrm{F_A}$, we have

$$\vec{r}_{x/w}(\alpha) = r_1(\alpha)\hat{\imath}_\mathrm{A} + r_2(\alpha)\hat{\jmath}_\mathrm{A} + r_3(\alpha)\hat{k}_\mathrm{A}, \tag{5.5.1}$$

where

$$r_1(\alpha) \triangleq \vec{r}_{x/w}(\alpha) \cdot \hat{\imath}_{\mathrm{A}}, \tag{5.5.2}$$

$$r_2(\alpha) \triangleq \vec{r}_{x/w}(\alpha) \cdot \hat{\jmath}_{\mathrm{A}}, \tag{5.5.3}$$

$$r_3(\alpha) \triangleq \vec{r}_{x/w}(\alpha) \cdot \hat{k}_{\mathrm{A}}. \tag{5.5.4}$$

The *path derivative* $\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)$ is defined by

$$\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \triangleq r_1'(\alpha)\hat{\imath}_{\mathrm{A}} + r_2'(\alpha)\hat{\jmath}_{\mathrm{A}} + r_3'(\alpha)\hat{k}_{\mathrm{A}}. \tag{5.5.5}$$

Hence,

$$\left.\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)\right|_{\mathrm{A}} = \begin{bmatrix} r_1'(\alpha) \\ r_2'(\alpha) \\ r_3'(\alpha) \end{bmatrix}. \tag{5.5.6}$$

Note that, if α denotes time t, then $\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)$ is the usual frame derivative $\overset{\mathrm{A}\bullet}{\vec{r}}_{x/w}$. Consequently, the path derivative is a generalization of the frame derivative.

Next, let $s(\alpha)$ denote the length of the path from $\vec{r}_{x/w}(0)$ to $\vec{r}_{x/w}(\alpha)$. Then,

$$s(\alpha) = \int_0^\alpha |\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\sigma)|\, \mathrm{d}\sigma, \tag{5.5.7}$$

which states that path length is the integral of the parametric speed along the path. Therefore,

$$s'(\alpha) = |\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)| = \sqrt{r_1'^2(\alpha) + r_2'^2(\alpha) + r_3'^2(\alpha)} \tag{5.5.8}$$

and, assuming that $s'(\alpha)$ is nonzero,

$$\begin{aligned} s''(\alpha) &= \frac{r_1'(\alpha)r_1''(\alpha) + r_2'(\alpha)r_2''(\alpha) + r_3'(\alpha)r_3''(\alpha)}{s'(\alpha)} \\ &= \frac{\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \cdot \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)}{s'(\alpha)}. \end{aligned} \tag{5.5.9}$$

It follows from (5.5.8) that $s'(\alpha)$ is nonzero if and only if $\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)$ is nonzero.

Next, suppose that α is parameterized by s, that is, $\alpha = \alpha(s)$. Then, $\vec{r}_{s,x/w}(s) = \vec{r}_{\alpha,x/w}(\alpha(s))$, where the additional subscripts denote different parameterizations. Then, it follows from the chain rule that the *path-length derivative* $\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(\alpha)$ is given by

$$\begin{aligned} \overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s) &= \overset{\mathrm{A}s\bullet}{\vec{r}}_{s,x/w}(s) \\ &= \overset{\mathrm{A}s\bullet}{\vec{r}}_{\alpha,x/w}(\alpha(s)) \\ &= \alpha'(s)\, \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha). \end{aligned} \tag{5.5.10}$$

Since $\alpha'(s)s'(\alpha) = 1$, we have

$$\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(\alpha) = \frac{1}{s'(\alpha)} \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha). \tag{5.5.11}$$

As a special case, suppose that $\alpha(s) = s$. Then, the length of the path from $\vec{r}_{x/w}(0)$ to $\vec{r}_{x/w}(\alpha)$ is s, and thus $s(\alpha) = \alpha$. Therefore, $s'(\alpha) = 1$ and $s''(\alpha) = 0$, and thus it follows from (5.5.5) with $\alpha = s$ and (5.5.8) that

$$|\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s)| = 1. \tag{5.5.12}$$

Hence, $\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s)$ is a unit vector. Therefore,

$$\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s) \cdot \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s) = 0. \tag{5.5.13}$$

The axes of the Frenet-Serret frame F_{FS} associated with x are *tangential*, *normal*, and *binormal*, denoted by $(\hat{e}_{\mathrm{t}}, \hat{e}_{\mathrm{n}}, \hat{e}_{\mathrm{b}})$, respectively. The tangential vector $\hat{e}_{\mathrm{t}}$ is the unit tangent vector (see (5.5.12)) to the curve at the location of the point, that is,

$$\hat{e}_{\mathrm{t}} \triangleq \overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s). \tag{5.5.14}$$

Using (5.5.10) and (5.5.11) it follows that

$$\hat{e}_{\mathrm{t}} = \alpha'(s) \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) = \frac{1}{s'(\alpha)} \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha). \tag{5.5.15}$$

Furthermore,

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} = \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s), \tag{5.5.16}$$

$$\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = \frac{-s''(\alpha)}{s'^2(\alpha)} \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) + \frac{1}{s'(\alpha)} \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha). \tag{5.5.17}$$

The normal vector $\hat{e}_{\mathrm{n}}$ is defined to be the unit vector in the direction of $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}}$, that is,

$$\hat{e}_{\mathrm{n}} \triangleq \frac{\rho(\alpha)}{s'(\alpha)} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}}, \tag{5.5.18}$$

where

$$\rho(\alpha) \triangleq \frac{|s'(\alpha)|}{|\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}}|} > 0 \tag{5.5.19}$$

is the *radius of curvature*. However, $\hat{e}_{\mathrm{n}}$ is not defined when $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = 0$, that is, when $\hat{e}_{\mathrm{t}}$ is not changing, for example, if the curve is a straight line or at an inflection point. Furthermore, using (5.5.11) yields

$$\hat{e}_{\mathrm{n}} = \frac{\rho(\alpha)}{s'^3(\alpha)} \left(s'(\alpha) \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - s''(\alpha) \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \right), \tag{5.5.20}$$

which, using (5.5.9), can be written as

$$\hat{e}_{\mathrm{n}} = \frac{\rho(\alpha)}{s'^4(\alpha)}\left[s'^2(\alpha) \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - \left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \cdot \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right)\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)\right]. \tag{5.5.21}$$

Note that it follows from (5.5.10) that

$$\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) = s'(\alpha)\hat{e}_{\mathrm{t}} \tag{5.5.22}$$

and thus using (5.5.18) we have

$$\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) = s''(\alpha)\hat{e}_{\mathrm{t}} + \frac{s'^2(\alpha)}{\rho(\alpha)}\hat{e}_{\mathrm{n}}. \tag{5.5.23}$$

In the special case $\alpha = s$, (5.5.20) becomes

$$\hat{e}_{\mathrm{n}} = \rho(s) \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}}, \tag{5.5.24}$$

where

$$\rho(s) \triangleq \frac{1}{|\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}}|} > 0. \tag{5.5.25}$$

Hence, it follows from (5.5.24) and (5.5.13) that

$$\hat{e}_{\mathrm{n}} = \rho(s) \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s). \tag{5.5.26}$$

The vectors $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{n}}$ are orthogonal since $\hat{e}_{\mathrm{t}} \cdot \hat{e}_{\mathrm{t}} = 1$, and thus $\hat{e}_{\mathrm{t}} \cdot \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = 0$. For a circular path, $\hat{e}_{\mathrm{n}}$ points toward the center of the circle. The plane spanned by $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{n}}$ is called the *osculating plane*.

To complete the Frenet-Serret frame, the binormal vector is defined by

$$\hat{e}_{\mathrm{b}} \triangleq \hat{e}_{\mathrm{t}} \times \hat{e}_{\mathrm{n}}. \tag{5.5.27}$$

Hence, using (5.5.14), (5.5.26), (5.5.10), and (5.5.21), it follows that

$$\begin{aligned} \hat{e}_{\mathrm{b}} &= \rho(s)\left(\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s) \times \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s)\right) \\ &= \frac{\rho(\alpha)}{s'^3(\alpha)}\left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right). \end{aligned} \tag{5.5.28}$$

The plane spanned by $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{b}}$ is called the *rectifying plane*, while the plane spanned by $\hat{e}_{\mathrm{n}}$ and $\hat{e}_{\mathrm{b}}$ is called the *normal plane*. The *Frenet-Serret frame* is thus given by

$$\mathrm{F}_{\mathrm{FS}} = [\hat{e}_{\mathrm{t}}\ \hat{e}_{\mathrm{n}}\ \hat{e}_{\mathrm{b}}] = \vec{R}_{\mathrm{FS/A}}\mathrm{F}_{\mathrm{A}}, \tag{5.5.29}$$

so that

$$\begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix} = \mathcal{O}_{\mathrm{FS/A}} \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix}. \tag{5.5.30}$$

Next, we find the path-length derivatives of the unit vectors $(\hat{e}_{\rm t}, \hat{e}_{\rm n}, \hat{e}_{\rm b})$. It follows from (5.5.18) that

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm t} = \kappa(s)\hat{e}_{\rm n}, \tag{5.5.31}$$

and thus

$$\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm t} = \kappa(\alpha)s'(\alpha)\hat{e}_{\rm n}, \tag{5.5.32}$$

where $\kappa(\alpha) \stackrel{\triangle}{=} 1/\rho(\alpha)$ is the *curvature*. It follows from (5.5.28) that

$$\kappa(\alpha) = \frac{|\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)|}{|\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)|^3}. \tag{5.5.33}$$

Thus, if $\alpha = s$, then

$$\kappa(s) = |\overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s)|. \tag{5.5.34}$$

If $\kappa(\alpha) = 0$, then $\rho(\alpha) = \infty$. Taking the magnitude of (5.5.32) and using (5.5.11) yields

$$\begin{aligned}
\kappa(\alpha) &= \frac{|\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm t}|}{s'(\alpha)} \\
&= \frac{1}{s'^3(\alpha)}\left|s'(\alpha)\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - s''(\alpha)\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)\right| \\
&= \frac{1}{s'^3(\alpha)}\sqrt{s'^2(\alpha)|\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)|^2 - (\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)\cdot\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha))^2}.
\end{aligned} \tag{5.5.35}$$

Note that, since $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm t} = \kappa(\alpha)s'(\alpha)\hat{e}_{\rm n}$ and $\hat{e}_{\rm b}$ are perpendicular, it follows that

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}(\hat{e}_{\rm t}\cdot\hat{e}_{\rm b}) \\
&= \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm t}\cdot\hat{e}_{\rm b} + \hat{e}_{\rm t}\cdot\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b} \\
&= \hat{e}_{\rm t}\cdot\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b}.
\end{aligned} \tag{5.5.36}$$

In addition, since $\hat{e}_{\rm b}\cdot\hat{e}_{\rm b} = 1$, it follows that $\hat{e}_{\rm b}\cdot\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b} = 0$. Consequently, $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b}$ is orthogonal to both $\hat{e}_{\rm t}$ and $\hat{e}_{\rm b}$. Since $\hat{e}_{\rm n}$ is also orthogonal to both $\hat{e}_{\rm t}$ and $\hat{e}_{\rm b}$, it follows that $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b}$ and $\hat{e}_{\rm n}$ are parallel. Consequently, $\overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm b} = \frac{1}{s'(\alpha)}\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b}$ and $\hat{e}_{\rm n}$ are parallel. We thus define the *torsion* $\tau(\alpha)$ such that

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm b} = -\tau(\alpha)\hat{e}_{\rm n}. \tag{5.5.37}$$

Therefore,

$$\tau(\alpha) = -\hat{e}_{\rm n}\cdot\overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm b}$$

$$
\begin{aligned}
&= -\frac{1}{s'(\alpha)}\hat{e}_{\mathrm{n}} \cdot \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}} \\
&= -\frac{\rho(\alpha)}{s'^{10}(\alpha)}\left(s'(\alpha)\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - s''(\alpha)\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}\right) \\
&\quad \cdot \left[[s'^3(\alpha)\rho'(\alpha) - 3s'^2(\alpha)s''(\alpha)\rho(\alpha)]\left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right)\right. \\
&\qquad \left. + s'^3(\alpha)\rho(\alpha)\left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right)\right] \\
&= -\frac{\rho^2(\alpha)}{s'^6(\alpha)}\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) \cdot \left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right).
\end{aligned}
\tag{5.5.38}
$$

Next, since $\hat{e}_{\mathrm{n}} \cdot \hat{e}_{\mathrm{n}} = 1$, it follows that $\hat{e}_{\mathrm{n}} \cdot \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} = 0$. Therefore, there exist real numbers γ and δ such that

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} = \gamma\hat{e}_{\mathrm{t}} + \delta\hat{e}_{\mathrm{b}}. \tag{5.5.39}$$

Using (5.5.37) and taking the path-length derivative of (5.5.27) yields

$$
\begin{aligned}
-\tau(\alpha)\hat{e}_{\mathrm{n}} &= \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} \\
&= \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} \times \hat{e}_{\mathrm{n}} + \hat{e}_{\mathrm{t}} \times \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} \\
&= \kappa(\alpha)\hat{e}_{\mathrm{n}} \times \hat{e}_{\mathrm{n}} + \hat{e}_{\mathrm{t}} \times (\gamma\hat{e}_{\mathrm{t}} + \delta\hat{e}_{\mathrm{b}}) \\
&= -\delta\hat{e}_{\mathrm{n}}.
\end{aligned}
\tag{5.5.40}
$$

Therefore, $\delta = \tau(\alpha)$. Next, since $\hat{e}_{\mathrm{n}} \cdot \hat{e}_{\mathrm{t}} = 0$, it follows that

$$
\begin{aligned}
\gamma = \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} \cdot \hat{e}_{\mathrm{t}} &= -\hat{e}_{\mathrm{n}} \cdot \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} \\
&= -\hat{e}_{\mathrm{n}} \cdot [\kappa(\alpha)\hat{e}_{\mathrm{n}}] \\
&= -\kappa(\alpha).
\end{aligned}
\tag{5.5.41}
$$

Therefore,

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} = -\kappa(\alpha)\hat{e}_{\mathrm{t}} + \tau(\alpha)\hat{e}_{\mathrm{b}}. \tag{5.5.42}$$

In summary, the *Frenet-Serret relations* (5.5.31), (5.5.42), and (5.5.37) are given by the vectrix equation

$$
\begin{bmatrix} \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix},
\tag{5.5.43}
$$

which can be written as

$$\begin{bmatrix} \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = -\begin{bmatrix} \tau(s) \\ 0 \\ \kappa(s) \end{bmatrix}^{\times} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}. \tag{5.5.44}$$

In terms of an arbitrary parameterization involving the parameter α, the Frenet-Serret relations are given by

$$\begin{bmatrix} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} 0 & s'(\alpha)\kappa(\alpha) & 0 \\ -s'(\alpha)\kappa(\alpha) & 0 & s'(\alpha)\tau(\alpha) \\ 0 & -s'(\alpha)\tau(\alpha) & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}, \tag{5.5.45}$$

which can be written as

$$\begin{bmatrix} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = -\begin{bmatrix} s'(\alpha)\tau(\alpha) \\ 0 \\ s'(\alpha)\kappa(\alpha) \end{bmatrix}^{\times} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}. \tag{5.5.46}$$

In the special case where α denotes time, we have

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = -\begin{bmatrix} s'(t)\tau(t) \\ 0 \\ s'(t)\kappa(t) \end{bmatrix}^{\times} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}. \tag{5.5.47}$$

To illustrate the meaning of the Frenet-Serret relations, consider the case in which the curve lies in a plane. Then, $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{n}}$ both lie in the plane, while $\hat{e}_{\mathrm{b}}$ is perpendicular to the plane, and thus $\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} = 0$. In this case, the curvature $\kappa(s)$ is the rate with respect to s of the rotation of $\hat{e}_{\mathrm{t}}$ around $\hat{e}_{\mathrm{b}}$. If, however, the curve is not confined to a plane, then the torsion $\tau(s)$ is the rate with respect to s of the rotation of $\hat{e}_{\mathrm{b}}$ around $\hat{e}_{\mathrm{n}}$.

The Frenet-Serret relations can be viewed as analogous to the vectrix form of Poisson's equation given by (4.3.21), that is,

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} \end{bmatrix} = -\omega_{\mathrm{B/A|B}}^{\times} \begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix}. \tag{5.5.48}$$

Recall that Poisson's equation applies to a rigid body and is parameterized by time. It is thus convenient to define the *angular velocity* of F_{FS} relative to F_{A} as

$$\vec{\omega}_{s,\mathrm{FS/A}}(s) \stackrel{\triangle}{=} \tau(s)\hat{e}_{\mathrm{t}} + \kappa(s)\hat{e}_{\mathrm{b}}, \tag{5.5.49}$$

which lies in the rectifying plane. Defining

$$\omega_{s,\mathrm{FS/A|FS}} \stackrel{\triangle}{=} \vec{\omega}_{s,\mathrm{FS/A}}\Big|_{\mathrm{FS}} = \begin{bmatrix} \tau(s) \\ 0 \\ \kappa(s) \end{bmatrix}, \tag{5.5.50}$$

we can rewrite (5.5.44) as

$$\begin{bmatrix} \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = -\omega^{\times}_{s,\mathrm{FS/A|FS}} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}. \tag{5.5.51}$$

For an arbitrary parameter α, we define

$$\vec{\omega}_{\alpha,\mathrm{FS/A}}(\alpha) \triangleq \tau(\alpha)\hat{e}_{\mathrm{t}} + \kappa(\alpha)\hat{e}_{\mathrm{b}}, \tag{5.5.52}$$

$$\omega_{\alpha,\mathrm{FS/A|FS}} \triangleq \vec{\omega}_{\alpha,\mathrm{FS/A}}\Big|_{\mathrm{FS}} = \begin{bmatrix} s'(\alpha)\tau(\alpha) \\ 0 \\ s'(\alpha)\kappa(\alpha) \end{bmatrix}. \tag{5.5.53}$$

We thus have

$$\begin{bmatrix} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = -\omega^{\times}_{\alpha,\mathrm{FS/A|FS}} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}. \tag{5.5.54}$$

Finally, in terms of time, we define

$$\vec{\omega}_{t,\mathrm{FS/A}}(t) \triangleq s'(t)\tau(t)\hat{e}_{\mathrm{t}} + s'(t)\kappa(t)\hat{e}_{\mathrm{b}}, \tag{5.5.55}$$

$$\omega_{t,\mathrm{FS/A|FS}} \triangleq \vec{\omega}_{t,\mathrm{FS/A}}\Big|_{\mathrm{FS}} = \begin{bmatrix} s'(t)\tau(t) \\ 0 \\ s'(t)\kappa(t) \end{bmatrix}. \tag{5.5.56}$$

We thus have

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = -\omega^{\times}_{t,\mathrm{FS/A|FS}} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}. \tag{5.5.57}$$

Comparing (5.5.57) with (4.3.21), it can be seen that $\vec{\omega}_{t,\mathrm{FS/A}}$ is the angular velocity of $\mathrm{F_{FS}}$ relative to $\mathrm{F_A}$.

To obtain a matrix form of the Frenet-Serret relations, define

$$x_{\mathrm{t}} \triangleq \hat{e}_{\mathrm{t}}|_{\mathrm{A}}\,, \tag{5.5.58}$$

$$x_{\mathrm{n}} \triangleq \hat{e}_{\mathrm{n}}|_{\mathrm{A}}\,, \tag{5.5.59}$$

$$x_{\mathrm{b}} \triangleq \hat{e}_{\mathrm{b}}|_{\mathrm{A}}\,, \tag{5.5.60}$$

$$X \triangleq \begin{bmatrix} x_{\mathrm{t}} & x_{\mathrm{n}} & x_{\mathrm{b}} \end{bmatrix}. \tag{5.5.61}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}X(\alpha) = X(\alpha)\omega^{\times}_{\alpha,\mathrm{FS/A|FS}}, \tag{5.5.62}$$

which is analogous to the matrix form of Poisson's equation given by (4.3.17), that is,

$$\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}\omega^{\times}_{\mathrm{B/A|B}}. \tag{5.5.63}$$

Let the Frenet-Serret frame be defined by the position vector $\vec{r}_{x/w}(\alpha)$, where α is a parameter, and let $\vec{x}(\alpha)$ be a physical vector that depends on α. Then, the transport theorem in terms of α has the form

$$\overset{\mathrm{FS}\alpha\bullet}{\vec{x}}(\alpha) = \overset{\mathrm{A}\alpha\bullet}{\vec{x}}(\alpha) + \vec{\omega}_{\alpha,\mathrm{FS/A}} \times \vec{x}(\alpha). \tag{5.5.64}$$

Finally, let α be a function of time t, and note that

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{x/w}(\alpha(t)) = \dot{\alpha}(t)\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \tag{5.5.65}$$

Therefore,

$$\vec{v}_{x/w/\mathrm{A}}(\alpha(t)) = s'(\alpha(t))\dot{\alpha}(t)\hat{e}_{\mathrm{t}}, \tag{5.5.66}$$

$$\vec{a}_{x/w/\mathrm{A}}(\alpha(t)) = [s'(\alpha(t))\ddot{\alpha}(t) + s''(\alpha(t))\dot{\alpha}^2(t)]\hat{e}_{\mathrm{t}} + \frac{s'^2(\alpha(t))\dot{\alpha}^2(t)}{\rho(\alpha(t))}\hat{e}_{\mathrm{n}}. \tag{5.5.67}$$

Equivalently, we can write

$$\vec{v}_{x/w/\mathrm{A}}(\alpha(t)) = v(t)\hat{e}_{\mathrm{t}}, \tag{5.5.68}$$

$$\vec{a}_{x/w/\mathrm{A}}(\alpha(t)) = \dot{v}(t)\hat{e}_{\mathrm{t}} + \frac{v^2(t)}{\rho(\alpha(t))}\hat{e}_{\mathrm{n}}, \tag{5.5.69}$$

where $v(t) \triangleq s'(\alpha(t))\dot{\alpha}(t)$ is the speed.

The following result considers the rotational path of a rigid body whose attitude is given by a parameterization of (2.14.12). This result is analogous to Fact 4.9.2, where now the angular velocity is parameterized by α instead of time.

Fact 5.5.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, and assume that the physical rotation matrix that transforms $\mathrm{F_A}$ to $\mathrm{F_B}$ is given by $\vec{R}_{\mathrm{B/A}}(\alpha) = \exp(\vec{\Theta}^{\times}_{\mathrm{B/A}}(\alpha))$, where $\theta_{\mathrm{B/A}}(\alpha) \triangleq |\vec{\Theta}_{\mathrm{B/A}}(\alpha)| \in [0,\pi]$. Furthermore, define $\hat{n}_{\mathrm{B/A}}(\alpha) \triangleq \hat{\Theta}_{\mathrm{B/A}}(\alpha)$. Then,

$$\vec{\omega}_{\mathrm{B/A}}(\alpha) = \frac{1}{\theta^2_{\mathrm{B/A}}}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\overset{\mathrm{A}\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{5.5.70}$$

$$= \left(\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta^3_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\overset{\mathrm{A}\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}}. \tag{5.5.71}$$

Furthermore,

$$\overset{\mathrm{A}\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \left(\vec{I} - \frac{1}{2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{2 - \theta_{\mathrm{B/A}}\cot\frac{\theta_{\mathrm{B/A}}}{2}}{2\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}. \tag{5.5.72}$$

5.6 Theoretical Problems

Problem 5.6.1. Let x and w be points, and let $\vec{r}_{x/w}(\alpha)$ depend on a parameter α, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Show that, if

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{O}_{\mathrm{B/A}} = 0,$$

then

$$\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w} = \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}.$$

Problem 5.6.2. Consider a helix with radius r and distance between turns h, wrapped around a cylinder whose axis is aligned with the $\hat{k}_{\mathrm{A}}$ axis. Show that this curve is parameterized in terms of arc length s by

$$\vec{r}_{x/w}(s) = \left(r \cos \frac{s}{\sqrt{r^2+h^2}}\right)\hat{\imath}_{\mathrm{A}} + \left(r \sin \frac{s}{\sqrt{r^2+h^2}}\right)\hat{\jmath}_{\mathrm{A}} + \left(\frac{hs}{\sqrt{r^2+h^2}}\right)\hat{k}_{\mathrm{A}}.$$

Furthermore, determine $\hat{e}_{\mathrm{t}}$, $\hat{e}_{\mathrm{n}}$, and $\hat{e}_{\mathrm{b}}$, and show that

$$\kappa(s) = \frac{r}{r^2+h^2}, \quad \tau(s) = \frac{h}{r^2+h^2}.$$

Problem 5.6.3. Consider the logarithmic spiral curve parameterized by

$$\vec{r}_{x/w}(\alpha) = (e^{\alpha}\cos\alpha)\hat{\imath}_{\mathrm{A}} + (e^{\alpha}\sin\alpha)\hat{\jmath}_{\mathrm{A}}.$$

Show that

$$\kappa(s) = \frac{1}{s}.$$

Problem 5.6.4. Consider the catenary curve parameterized by

$$\vec{r}_{x/w}(\alpha) = \alpha\hat{\imath}_{\mathrm{A}} + \frac{a}{2}(e^{\alpha/a} + e^{-\alpha/a})\hat{\jmath}_{\mathrm{A}},$$

where a is a positive constant. Show that

$$\kappa(s) = \frac{a}{s^2+a}.$$

Problem 5.6.5. Show that if $\tau \neq 0$, then $\kappa \neq 0$.

Problem 5.6.6. Let $\mathrm{F_A}$ be a frame, let w be a point, let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let x and y be points that are fixed in $\mathcal{B}$, and let $\mathrm{F}_{\mathrm{FS}_x}$ and $\mathrm{F}_{\mathrm{FS}_y}$ be the Frenet-Serret frames associated with the curves generated by x and y, respectively. Find a relationship between $\vec{R}_{\mathrm{FS}_y/\mathrm{A}}$ and $\vec{R}_{\mathrm{FS}_x/\mathrm{A}}$.

5.7 Applied Problems

Problem 5.7.1. Consider a bowling alley located at λ degrees north latitude with bowling lanes that are l feet long. A ball is rolled down the center of the lane with speed v. Determine the Coriolis acceleration due to the rotation of the Earth and the resulting lateral position of the ball at the end of the lane. The bowling alley may be oriented in an arbitrary horizontal direction.

Symbol	Definition
$\hat{e}_{\mathrm{r}}, \hat{e}_{\mathrm{t}}, \hat{e}_{\mathrm{a}}$	Cylindrical (radial, tangential, axial) frame
$\hat{e}_{\mathrm{u}}, \hat{e}_{\mathrm{e}}, \hat{e}_{\mathrm{n}}$	Spherical (up, east, north) frame
$\hat{e}_{\mathrm{t}}, \hat{e}_{\mathrm{n}}, \hat{e}_{\mathrm{b}}$	Tangential, normal, binormal (Frenet-Serret) frame

Table 5.1: Symbols for Chapter 5.

Chapter Six

Statics

A *body* is a finite collection of particles, not necessarily rigid. A body is rigid if its shape does not change, that is, if the distance between each pair of particles is constant. An *inertia point* is either a particle in a body, a point on a rigid massless link connecting two particles, or a point along a rigid massless link in a (nontrivial) rigid body. In the last case, the link may be rigidly attached to a single particle without connecting two particles.

6.1 Zeroth and First Moments of Mass

Definition 6.1.1. Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively. Then, the *zeroth moment of mass* of $\mathcal{B}$ is the *total mass* $m_{\mathcal{B}}$ of the body, that is,

$$m_{\mathcal{B}} \triangleq \sum_{i=1}^{l} m_i. \tag{6.1.1}$$

Now, let w be a point. Then, the *center of mass* c of $\mathcal{B}$ is the point c defined by the position vector

$$\vec{r}_{\mathrm{c}/w} \triangleq \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w}, \tag{6.1.2}$$

and the *first moment of mass* of $\mathcal{B}$ is the physical vector

$$m_{\mathcal{B}} \vec{r}_{\mathrm{c}/w} = \sum_{i=1}^{l} m_i \vec{r}_{y_i/w}. \tag{6.1.3}$$

The following result shows that the location of the center of mass is independent of the choice of the point w.

Fact 6.1.2. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w and w' be points, and define the points c and c′ by

$$\vec{r}_{\mathrm{c}/w} \triangleq \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w}, \tag{6.1.4}$$

$$\vec{r}_{\mathrm{c}'/w'} \triangleq \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w'}. \tag{6.1.5}$$

Then, c and c′ are colocated.

Proof. Note that

$$\begin{aligned}
\vec{r}_{c'/c} &= \vec{r}_{c'/w'} + \vec{r}_{w'/w} + \vec{r}_{w/c}\\
&= \vec{r}_{c'/w'} - \vec{r}_{c/w} + \vec{r}_{w'/w}\\
&= \frac{1}{m_{\mathcal{B}}}\sum_{i=1}^{l} m_i \vec{r}_{y_i/w'} - \frac{1}{m_{\mathcal{B}}}\sum_{i=1}^{l} m_i \vec{r}_{y_i/w} + \vec{r}_{w'/w}\\
&= \frac{1}{m_{\mathcal{B}}}\sum_{i=1}^{l} m_i(\vec{r}_{y_i/w'} - \vec{r}_{y_i/w}) + \vec{r}_{w'/w}\\
&= \frac{1}{m_{\mathcal{B}}}\sum_{i=1}^{l} m_i(\vec{r}_{y_i/w'} + \vec{r}_{w/y_i}) + \vec{r}_{w'/w}\\
&= \frac{1}{m_{\mathcal{B}}}\sum_{i=1}^{l} m_i \vec{r}_{w/w'} + \vec{r}_{w'/w}\\
&= \vec{r}_{w/w'} + \vec{r}_{w'/w} = \vec{r}_{w/w} = \vec{0}.
\end{aligned}$$

□

Fact 6.1.3. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$. Then, c satisfies

$$\sum_{i=1}^{l} m_i \vec{r}_{y_i/c} = 0. \tag{6.1.6}$$

Proof. Set $w = \mathrm{c}$ in (6.1.2). □

6.2 Second Moment of Mass

The *second moment of mass* of a body is given by the physical inertia matrix $\vec{J}_{\mathcal{B}/z}$, which characterizes the mass distribution of a body $\mathcal{B}$ relative to a reference point z.

Definition 6.2.1. Let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let z be a point. Then, the *physical inertia matrix* $\vec{J}_{\mathcal{B}/z}$ of $\mathcal{B}$ relative to z is defined by

$$\vec{J}_{\mathcal{B}/z} \triangleq \sum_{i=1}^{l} m_i \vec{r}^{\,\times\prime}_{y_i/z} \vec{r}^{\,\times}_{y_i/z}. \tag{6.2.1}$$

Fact 6.2.2. Let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let z be a point. Then,

$$\vec{J}_{\mathcal{B}/z} = -\sum_{i=1}^{l} m_i \vec{r}^{\,\times 2}_{y_i/z} \tag{6.2.2}$$

$$= \sum_{i=1}^{l} m_i \left(|\vec{r}_{y_i/z}|^2 \vec{I} - \vec{r}_{y_i/z} \vec{r}^{\,\prime}_{y_i/z} \right). \tag{6.2.3}$$

Now, let $\mathrm{F_B}$ be a frame. Then,

$$\vec{J}_{\mathcal{B}/z} = J_{xx/z|\mathrm{B}}\hat{\imath}_\mathrm{B}\hat{\imath}'_\mathrm{B} + J_{yy/z|\mathrm{B}}\hat{\jmath}_\mathrm{B}\hat{\jmath}'_\mathrm{B} + J_{zz/z|\mathrm{B}}\hat{k}_\mathrm{B}\hat{k}'_\mathrm{B} - J_{xy/z|\mathrm{B}}(\hat{\imath}_\mathrm{B}\hat{\jmath}'_\mathrm{B} + \hat{\jmath}_\mathrm{B}\hat{\imath}'_\mathrm{B})$$
$$- J_{xz/z|\mathrm{B}}(\hat{\imath}_\mathrm{B}\hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B}\hat{\imath}'_\mathrm{B}) - J_{yz/z|\mathrm{B}}(\hat{\jmath}_\mathrm{B}\hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B}\hat{\jmath}'_\mathrm{B}). \tag{6.2.4}$$

Hence,

$$J_{\mathcal{B}/z|\mathrm{B}} = \begin{bmatrix} J_{xx/z|\mathrm{B}} & -J_{xy/z|\mathrm{B}} & -J_{xz/z|\mathrm{B}} \\ -J_{xy/z|\mathrm{B}} & J_{yy/z|\mathrm{B}} & -J_{yz/z|\mathrm{B}} \\ -J_{xz/z|\mathrm{B}} & -J_{yz/z|\mathrm{B}} & J_{zz/z|\mathrm{B}} \end{bmatrix}, \tag{6.2.5}$$

where

$$J_{xx/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i(\bar{y}_i^2 + \bar{z}_i^2), \qquad J_{xy/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i\bar{x}_i\bar{y}_i, \tag{6.2.6}$$

$$J_{yy/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i(\bar{x}_i^2 + \bar{z}_i^2), \qquad J_{xz/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i\bar{x}_i\bar{z}_i, \tag{6.2.7}$$

$$\underbrace{J_{zz/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i(\bar{x}_i^2 + \bar{y}_i^2),}_{\text{moments of inertia}} \qquad \underbrace{J_{yz/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i\bar{y}_i\bar{z}_i,}_{\text{products of inertia}} \tag{6.2.8}$$

and where

$$r_{y_i/z|\mathrm{B}} = \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{bmatrix}. \tag{6.2.9}$$

Proof. For $i = 1, \ldots, l$, $\vec{r}_{y_i/z}^{\,\times'} = \vec{r}_{y_i/z}^{\,\times}$, which implies that (6.2.1) can be written as (6.2.2). Furthermore, (2.9.13) implies that (6.2.1) can be written as (6.2.3). Next, resolving $\vec{J}_{\mathcal{B}/z}$ in $\mathrm{F_B}$ yields

$$J_{\mathcal{B}/z|\mathrm{B}} = \sum_{i=1}^{l} m_i\left(|\vec{r}_{y_i/z}|^2 I_3 - r_{y_i/z|\mathrm{B}}r_{y_i/z|\mathrm{B}}^{\mathrm{T}}\right)$$
$$= \sum_{i=1}^{l} m_i\left(\begin{bmatrix} \bar{x}_i^2+\bar{y}_i^2+\bar{z}_i^2 & 0 & 0 \\ 0 & \bar{x}_i^2+\bar{y}_i^2+\bar{z}_i^2 & 0 \\ 0 & 0 & \bar{x}_i^2+\bar{y}_i^2+\bar{z}_i^2 \end{bmatrix} - \begin{bmatrix} \bar{x}_i^2 & \bar{x}_i\bar{y}_i & \bar{x}_i\bar{z}_i \\ \bar{y}_i\bar{x}_i & \bar{y}_i^2 & \bar{y}_i\bar{z}_i \\ \bar{z}_i\bar{x}_i & \bar{z}_i\bar{y}_i & \bar{z}_i^2 \end{bmatrix}\right),$$

where

$$\vec{r}_{y_i/z} = \bar{x}_i\hat{\imath}_\mathrm{B} + \bar{y}_i\hat{\jmath}_\mathrm{B} + \bar{z}_i\hat{k}_\mathrm{B}.$$

Hence, $J_{\mathcal{B}/z|\mathrm{B}}$ is given by (6.2.5)–(6.2.8). □

The diagonal entries $J_{xx/z|\mathrm{B}}, J_{yy/z|\mathrm{B}}, J_{zz/z|\mathrm{B}}$ of $J_{\mathcal{B}/z|\mathrm{B}}$ are the *moments of inertia of* $\mathcal{B}$ *relative to* z *determined by* $\mathrm{F_B}$, whereas the off-diagonal entries $J_{xy/z|\mathrm{B}}, J_{xz/z|\mathrm{B}}, J_{yz/z|\mathrm{B}}$ of $J_{\mathcal{B}/z|\mathrm{B}}$ are the *products of inertia of* $\mathcal{B}$ *relative to* z *determined by* $\mathrm{F_A}$.

The following result relates the physical inertia matrix resolved in different frames.

Fact 6.2.3. Let $\mathcal{B}$ be a body, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let z be a point. Then,

$$J_{\mathcal{B}/z|\mathrm{A}} = \mathcal{O}_{\mathrm{A/B}} J_{\mathcal{B}/z|\mathrm{B}} \mathcal{O}_{\mathrm{B/A}}. \tag{6.2.10}$$

Proof. The result follows from Fact 2.10.11. □

The following result relates the physical inertia matrix of a body to the physical inertia matrix of the rotated body.

Fact 6.2.4. Let $\mathcal{B}$ be a body, let z be point in $\mathcal{B}$, let $\vec{R}$ be a physical rotation matrix, let $\mathcal{B}'$ be the body $\mathcal{B}$ rotated by $\vec{R}$, and let z' denote the point in $\mathcal{B}'$ corresponding to z in $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}'/z'} = \vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}'. \tag{6.2.11}$$

Proof. Note that

$$\begin{aligned}
\vec{J}_{\mathcal{B}'/z'} &= -\sum_{i=1}^{l} m_i \vec{r}_{y_i'/z'}^{\times 2} \\
&= -\sum_{i=1}^{l} m_i \left(\vec{R}\vec{r}_{y_i/z}\right)^{\times 2} \\
&= -\sum_{i=1}^{l} m_i \left(\vec{R}\vec{r}_{y_i/z}^{\times}\vec{R}'\right)^{2} \\
&= -\sum_{i=1}^{l} m_i \vec{R}\vec{r}_{y_i/z}^{\times 2}\vec{R}' \\
&= \vec{R}\left(-\sum_{i=1}^{l} m_i \vec{r}_{y_i/z}^{\times 2}\right)\vec{R}' \\
&= \vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}'.
\end{aligned}$$

□

Fact 6.2.5. Let $\mathcal{B}$ be a body, let z be a point in $\mathcal{B}$, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\mathcal{B}'$ be the body $\mathcal{B}$ rotated by $\vec{R}_{\mathrm{A/B}}$, and let z' be the point in $\mathcal{B}'$ corresponding to z in $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}'/z'} = \vec{R}_{\mathrm{A/B}}\vec{J}_{\mathcal{B}/z}\vec{R}_{\mathrm{B/A}}. \tag{6.2.12}$$

Consequently,

$$J_{\mathcal{B}'/z'|\mathrm{A}} = J_{\mathcal{B}/z|\mathrm{B}}. \tag{6.2.13}$$

Proof. The equality (6.2.12) follows from (6.2.11). Furthermore,

$$J_{\mathcal{B}'/z'|\mathrm{A}} = \left(\vec{R}_{\mathrm{A/B}}\vec{J}_{\mathcal{B}/z}\vec{R}_{\mathrm{B/A}}\right)\Big|_{\mathrm{A}} = \mathcal{R}_{\mathrm{A/B}} J_{\mathcal{B}/z|\mathrm{A}} \mathcal{R}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}} J_{\mathcal{B}/z|\mathrm{A}} \mathcal{O}_{\mathrm{A/B}} = J_{\mathcal{B}/z|\mathrm{B}}.$$

□

If $J_{\mathcal{B}/z|\mathrm{B}}$ is diagonal, that is,

$$J_{\mathcal{B}/z|\mathrm{B}} = \begin{bmatrix} J_{xx/z|\mathrm{B}} & 0 & 0 \\ 0 & J_{yy/z|\mathrm{B}} & 0 \\ 0 & 0 & J_{zz/z|\mathrm{B}} \end{bmatrix}, \tag{6.2.14}$$

then the axes of $\mathrm{F_B}$ are *principal axes* of $\mathcal{B}$ *relative to* z, $\mathrm{F_B}$ is a *principal-axis frame of* $\mathcal{B}$ *relative to* z, and $J_{xx/z|\mathrm{B}}$, $J_{yy/z|\mathrm{B}}$, $J_{zz/z|\mathrm{B}}$ are the *principal moments of inertia of* $\mathcal{B}$ *relative to* z.

The following result shows that every body $\mathcal{B}$ has a principal-axis frame relative to every point z.

Fact 6.2.6. Let $\mathcal{B}$ be a body, let z be a point, and let $\mathrm{F_B}$ be a frame. Then, the following statements hold:

i) There exists a rotation matrix S such that $SJ_{\mathcal{B}/z|\mathrm{B}}S^{\mathrm{T}}$ is diagonal.

ii) There exists a frame $\mathrm{F_A}$ such that $J_{\mathcal{B}/z|\mathrm{A}}$ is diagonal.

iii) There exists a physical rotation matrix $\vec{R}$ such that $(\vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}')\big|_{\mathrm{B}}$ is diagonal.

iv) The following statements are equivalent:

a) $\mathrm{F_B}$ is a principal-axis frame of $\mathcal{B}$ relative to z.

b) The axes of $\mathrm{F_B}$ are physical eigenvectors of $\vec{J}_{\mathcal{B}/z}$.

c) The vectors e_1, e_2, e_3 are eigenvectors of $J_{\mathcal{B}/z|\mathrm{B}}$.

v) The principal moments of inertia of $\mathcal{B}$ relative to z are the eigenvalues of $J_{\mathcal{B}/z|\mathrm{B}}$.

Proof. To prove *i*), note that it follows from the Schur decomposition given by Corollary 5.4.5 given in [1, p. 320] that there exists an orthogonal matrix $S \in \mathbb{R}^{3\times3}$ such that the positive-semidefinite matrix $D \triangleq SJ_{\mathcal{B}/z|\mathrm{B}}S^{\mathrm{T}}$ is diagonal. If $\det S = 1$, then S is a rotation matrix. If $\det S = -1$, then S can be replaced by $-S$, which is a rotation matrix that satisfies $D = (-S)J_{\mathcal{B}/z|\mathrm{B}}(-S)^{\mathrm{T}}$.

To prove *ii*), let S be given by *i*). Then, it follows from Problem 2.26.11 that there exists a frame $\mathrm{F_A}$ such that $\mathcal{O}_{\mathrm{A/B}} = S$. Then

$$J_{\mathcal{B}/z|\mathrm{A}} = \mathcal{O}_{\mathrm{A/B}}J_{\mathcal{B}/z|\mathrm{B}}\mathcal{O}_{\mathrm{B/A}} = SJS^{\mathrm{T}}$$

is diagonal.

To prove *iii*), let the frame $\mathrm{F_A}$ be given by statement *ii*), and define $\vec{R} \triangleq \vec{R}_{\mathrm{B/A}}$. Then

$$\begin{aligned}(\vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}')\big|_{\mathrm{B}} &= (\vec{R}_{\mathrm{B/A}}\vec{J}_{\mathcal{B}/z}\vec{R}'_{\mathrm{B/A}})\big|_{\mathrm{B}}\\ &= \mathcal{O}_{\mathrm{A/B}}J_{\mathcal{B}/z|\mathrm{B}}\mathcal{O}_{\mathrm{B/A}}\\ &= J_{\mathcal{B}/z|\mathrm{A}}\end{aligned}$$

is diagonal. □

The following two results summarize properties of the inertia matrix.

Fact 6.2.7. Let $\mathcal{B}$ be a body, let z be a point, and let $\mathrm{F_B}$ be a frame.

Then, the following statements hold:

i) $J_{\mathcal{B}/z|\mathrm{B}}$ is positive semidefinite.

ii) $J_{\mathcal{B}/z|\mathrm{B}} = 0$ if and only if $\mathcal{B}$ consists of a single particle colocated with z.

iii) $\operatorname{rank} J_{\mathcal{B}/z|\mathrm{B}} \neq 1$.

iv) rank $J_{\mathcal{B}/z|\mathrm{B}} = 2$ if and only if z and all of the particles of $\mathcal{B}$ are colinear.

v) $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite if and only if $\mathcal{B}$ contains at least two particles y_i and y_j such that $\vec{r}_{y_i/z}$ and $\vec{r}_{y_j/z}$ are linearly independent.

vi) If $\mathcal{B}$ contains three particles that are not colinear, then $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite.

vii) If z and $\mathcal{B}$ are not coplanar, then $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite.

Proof. To prove *i*), let $\vec{x}$ be a physical vector. Then, it follows from (7.8.2) that

$$\begin{aligned}\vec{x}\Big|_{\mathrm{B}}^{\mathrm{T}} J_{\mathcal{B}/z|\mathrm{B}}\, \vec{x}\Big|_{\mathrm{B}} &= \vec{x}'\vec{J}_{\mathcal{B}/z}\vec{x} = \vec{x}'\sum_{i=1}^{l} m_i \vec{r}_{y_i/z}^{\,\times\prime}\vec{r}_{y_i/z}^{\,\times}\vec{x}\\ &= \sum_{i=1}^{l} m_i \vec{x}'\vec{r}_{y_i/z}^{\,\times\prime}\vec{r}_{y_i/z}^{\,\times}\vec{x} = \sum_{i=1}^{l} m_i (\vec{r}_{y_i/z}^{\,\times}\vec{x})'\vec{r}_{y_i/z}^{\,\times}\vec{x}\\ &= \sum_{i=1}^{l} m_i |\vec{r}_{y_i/z}\times\vec{x}|^2 \ge 0.\end{aligned}$$

Statements *ii*)–*iv*) follow from Fact 8.93 and Fact 8.94 given in [1, p. 495].

To prove *v*), let $\vec{x}$ be a nonzero physical vector. Then, since $\vec{r}_{y_i/z}$ and $\vec{r}_{y_j/z}$ are linearly independent, it follows $\vec{r}_{y_i/z}\times\vec{x}$ and $\vec{r}_{y_y/z}\times\vec{x}$ are not both zero. Consequently,

$$\vec{x}\Big|_{\mathrm{B}}^{\mathrm{T}} J_{\mathcal{B}/z|\mathrm{B}}\, \vec{x}\Big|_{\mathrm{B}} = \sum_{i=1}^{l} m_i |\vec{r}_{y_i/z}\times\vec{x}|^2 > 0,$$

which implies that $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite.

To prove *vi*), note that, let y_i, y_j, and y_k be particles of $\mathcal{B}$ that are not colinear.

To prove *vii*), note that, since z and $\mathcal{B}$ are not coplanar, it follows that $\mathcal{B}$ must have at least three particles that are not colinear. It thus follows from *vi*) that $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite. □

For the following result, let J_1, J_2, J_3 be the moments of inertia of $\mathcal{B}$ relative to z determined by $\mathrm{F_B}$, that is, the diagonal entries of $J_{\mathcal{B}/z|\mathrm{B}}$. If, in addition, $\mathrm{F_B}$ is a principal-axis frame, then let $\lambda_1, \lambda_2, \lambda_3$ be the principal moments of inertia of $\mathcal{B}$ relative to z, that is, the diagonal entries of $J_{\mathcal{B}/z|\mathrm{B}}$. This result shows that the moments of inertia of $\mathcal{B}$ may represent the sides of a triangle.

Fact 6.2.8. Let $\mathcal{B}$ be a body, and let z be a point. Then, the following statements hold:

i) The moments of inertia $J_1 \ge J_2 \ge J_3 \ge 0$ of $\mathcal{B}$ determined by $\mathrm{F_B}$ satisfy

$$J_1 \le J_2 + J_3. \tag{6.2.15}$$

ii) If $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite, then the moments of inertia $J_1 \ge J_2 \ge J_3 > 0$ of $\mathcal{B}$ determined by $\mathrm{F_B}$ satisfy

$$1 \le \min\left\{\frac{J_2}{J_3}, \frac{J_1}{J_2}\right\} \le \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.16}$$

iii) If z and $\mathcal{B}$ do not lie in a single plane, then the moments of inertia $J_1 \ge J_2 \ge J_3 > 0$ of $\mathcal{B}$

determined by $\mathrm{F_B}$ satisfy

$$J_1 < J_2 + J_3 \tag{6.2.17}$$

and

$$1 \le \min\left\{\frac{J_2}{J_3}, \frac{J_1}{J_2}\right\} < \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.18}$$

iv) The principal moments of inertia $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge 0$ of $\mathcal{B}$ satisfy

$$\lambda_1 \le \lambda_2 + \lambda_3. \tag{6.2.19}$$

v) If $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite, then the principal moments of inertia $\lambda_1 \ge \lambda_2 \ge \lambda_3 > 0$ of $\mathcal{B}$ determined by $\mathrm{F_B}$ satisfy

$$1 \le \min\left\{\frac{J_2}{J_3}, \frac{J_1}{J_2}\right\} \le \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.20}$$

vi) If z and $\mathcal{B}$ do not lie in a single plane, then the principal moments of inertia $\lambda_1 \ge \lambda_2 \ge \lambda_3 > 0$ of $\mathcal{B}$ satisfy

$$\lambda_1 < \lambda_2 + \lambda_3 \tag{6.2.21}$$

and

$$1 \le \min\left\{\frac{\lambda_2}{\lambda_3}, \frac{\lambda_1}{\lambda_2}\right\} < \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.22}$$

Proof. To prove *i*), assume for convenience that $J_1 = J_{xx}$, $J_2 = J_{yy}$, and $J_3 = J_{zz}$. Then,

$$\begin{aligned} J_1 &= \sum_{i=1}^{l} m_i(\bar{y}_i^2 + \bar{z}_i^2) \\ &\le \sum_{i=1}^{l} m_i(\bar{y}_i^2 + 2\bar{x}_i^2 + \bar{z}_i^2) \\ &= \sum_{i=1}^{l} m_i(\bar{x}_i^2 + \bar{z}_i^2) + \sum_{i=1}^{l} m_i(\bar{x}_i^2 + \bar{y}_i^2) \\ &= J_2 + J_3. \end{aligned}$$

To prove *iii*), note that, since z and $\mathcal{B}$ do not lie in a single plane, there exist three particles y_i, y_j, y_k such that $\vec{r}_{y_i/z}, \vec{r}_{y_j/z}, \vec{r}_{y_k/z}$ are linearly independent. Therefore, it follows from Fact 6.2.7 that $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite. Furthermore, $\bar{x}_m$ is nonzero for some particle y_m,, and thus the inequality in the proof of *i*) is strict, which proves (6.2.17). The right-hand inequality in (6.2.18) is a property of triangles given in [9, p. 145]. Statement *ii*) is a limiting case of *iii*).

Finally, statements *iv*)–*vi*) follow from statements *i*)–*iii*) by choosing $\mathrm{F_B}$ to be a principal-axis frame. □

Figure 6.2.1 shows the triangular region of feasible principal moments of inertia of a rigid body. There are five cases that are highlighted for principal moments of inertia $\lambda_1 \ge \lambda_2 \ge \lambda_3 > 0$, where $\lambda_1, \lambda_2, \lambda_3$ satisfy the triangle inequality $\lambda_1 < \lambda_2 + \lambda_3$. Let m be the mass of the rigid body. The point $\lambda_1 = \lambda_2 = \lambda_3$ corresponds to a sphere of radius $R = \sqrt{\frac{5\lambda_1}{2m}}$, a cube whose sides have length

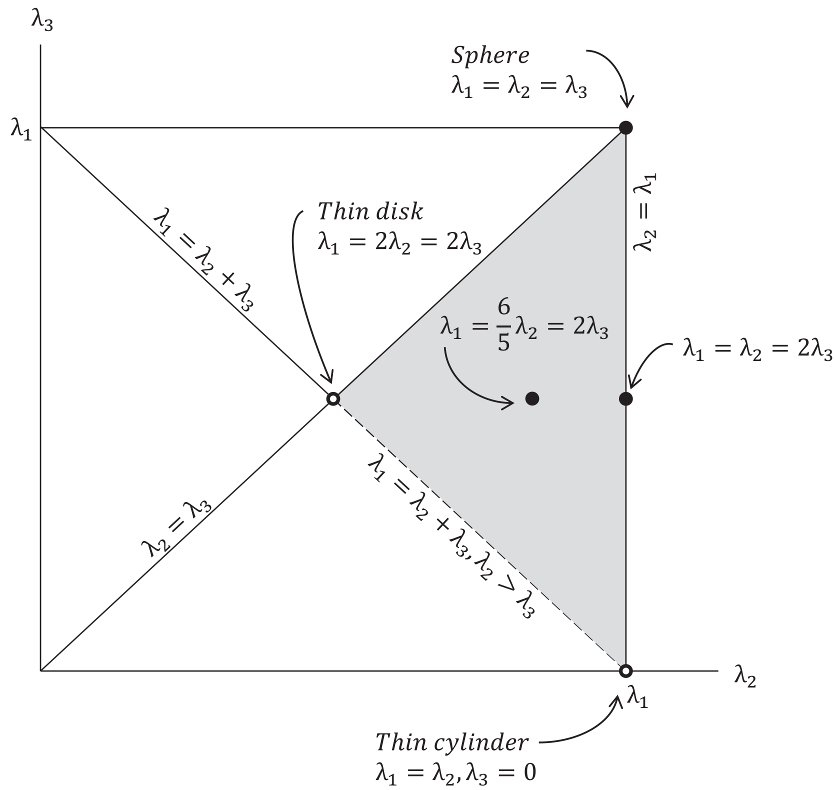


Figure 6.2.1: Feasible region of the principal moments of inertia $\lambda_1, \lambda_2, \lambda_3$ of a rigid body satisfying $0 < \lambda_3 \le \lambda_2 \le \lambda_1$, where $\lambda_1 < \lambda_2 + \lambda_3$. The shaded region shows all feasible values of λ_2 and λ_3 in terms of the largest principal moment of inertia λ_1. The open dots and dashed line segment indicate nonphysical, limiting cases.

$L = \sqrt{\frac{6\lambda_1}{m}}$, and a cylinder of length L and radius R, where $L/R = \sqrt{3}$ and $R = \sqrt{\frac{2\lambda_1}{m}}$. The point $\lambda_1 = \lambda_2 = 2\lambda_3$ corresponds to a cylinder of length L and radius R, where $L/R = 3$ and $R = \sqrt{\frac{2\lambda_1}{m}}$. The point $\lambda_1 = \frac{6}{5}\lambda_2 = 2\lambda_3$, located at the centroid of the triangular region, corresponds to a solid rectangular body with sides $L_1 = \sqrt{\frac{8\lambda_1}{m}} > L_2 = \sqrt{\frac{4\lambda_1}{m}} > L_3 = \sqrt{\frac{2\lambda_1}{m}}$.

The remaining cases in Figure 6.2.1 are limiting cases. The point $\lambda_1 = 2\lambda_2 = 2\lambda_3$ corresponds to a thin disk of radius $R = \sqrt{\frac{2\lambda_1}{m}}$. The point $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$ corresponds to a thin cylinder of radius $R = 0$ and length $L = \sqrt{\frac{12\lambda_1}{m}}$. Finally, points on the line segment $\lambda_1 = \lambda_2 + \lambda_3$, where $\lambda_2 > \lambda_3$ correspond to a thin rectangular plate with sides of length $L_1 = \sqrt{\frac{12\lambda_2}{m}} > L_2 = \sqrt{\frac{12\lambda_3}{m}}$.

The following result shows that the inertia of a body is the sum of the inertias of the components of the body.

Fact 6.2.9. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let $\mathcal{B}_3$ be the union of $\mathcal{B}_1$ and $\mathcal{B}_2$, and let z be a point. Then,

$$\vec{J}_{\mathcal{B}_3/z} = \vec{J}_{\mathcal{B}_1/z} + \vec{J}_{\mathcal{B}_2/z}. \tag{6.2.23}$$

The following result is an immediate consequence of Fact 6.2.9.

Fact 6.2.10. Let $\mathcal{B}_2$ be a body, let $\mathcal{B}_1$ be a body contained in $\mathcal{B}_2$, let $\mathcal{B}_3$ be the body $\mathcal{B}_2$ with the body $\mathcal{B}_1$ removed, and let z be a point. Then,

$$\vec{J}_{\mathcal{B}_3/z} = \vec{J}_{\mathcal{B}_2/z} - \vec{J}_{\mathcal{B}_1/z}. \tag{6.2.24}$$

For symmetric matrices $A, B \in \mathbb{R}^{3\times 3}$, the notation "$A \le B$" means that $B - A$ is positive semidefinite. The following result is a consequence of Fact 6.2.9.

Fact 6.2.11. Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be bodies, assume that $\mathcal{B}$ is contained in $\tilde{\mathcal{B}}$, and let z be a point. Then,

$$J_{\mathcal{B}/z|\mathrm{B}} \le J_{\tilde{\mathcal{B}}/z|\mathrm{B}}. \tag{6.2.25}$$

The following result shows how the physical inertia matrix can be shifted from the center of mass to an arbitrary point. This result shows that every shift away from the center of mass increases the inertia of the body in the sense that $\vec{J}_{\mathcal{B}/\mathrm{c}} \le \vec{J}_{\mathcal{B}/z}$, that is, $\vec{J}_{\mathcal{B}/z} - \vec{J}_{\mathcal{B}/\mathrm{c}}$ is positive semidefinite. The physical inertia matrix relative to z turns out to be equivalent to the physical inertia matrix of a modified body $\mathcal{B}'$ consisting of the original body $\mathcal{B}$ and a particle of mass $m_\mathcal{B}$ located at z relative to the center of mass of $\mathcal{B}$.

Fact 6.2.12. Let $\mathcal{B}$ be a body, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, let z be a point, and let c be the center of mass of $\mathcal{B}$. Then,

$$\begin{aligned}
\vec{J}_{\mathcal{B}/z} &= \vec{J}_{\mathcal{B}/\mathrm{c}} - m_\mathcal{B} \vec{r}_{z/\mathrm{c}}^{\,\times 2} &(6.2.26)\\
&= \vec{J}_{\mathcal{B}/\mathrm{c}} + m_\mathcal{B} \vec{r}_{z/\mathrm{c}}^{\,\times\prime} \vec{r}_{z/\mathrm{c}}^{\,\times} &(6.2.27)\\
&= \vec{J}_{\mathcal{B}/\mathrm{c}} + m_\mathcal{B}(|\vec{r}_{z/\mathrm{c}}|^2 \vec{I} - \vec{r}_{z/\mathrm{c}} \vec{r}_{z/\mathrm{c}}^{\,\prime}). &(6.2.28)
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
\vec{J}_{\mathcal{B}/z} &= -\sum_{i=1}^{l} m_i \vec{r}_{y_i/z}^{\,\times 2} = -\sum_{i=1}^{l} m_i \left(\vec{r}_{y_i/\mathrm{c}}^{\,\times} + \vec{r}_{\mathrm{c}/z}^{\,\times}\right)^2\\
&= -\sum_{i=1}^{l} m_i \left[\vec{r}_{y_i/\mathrm{c}}^{\,\times 2} + \vec{r}_{y_i/\mathrm{c}}^{\,\times}\vec{r}_{\mathrm{c}/z}^{\,\times} + \vec{r}_{\mathrm{c}/z}^{\,\times}\vec{r}_{y_i/\mathrm{c}}^{\,\times} + \vec{r}_{\mathrm{c}/z}^{\,\times 2}\right]\\
&= -\sum_{i=1}^{l} m_i \vec{r}_{y_i/\mathrm{c}}^{\,\times 2} - \left(\sum_{i=1}^{l} m_i \vec{r}_{y_i/\mathrm{c}}\right)^{\times} \vec{r}_{\mathrm{c}/z}^{\,\times} - \vec{r}_{\mathrm{c}/z}^{\,\times}\left(\sum_{i=1}^{l} m_i \vec{r}_{y_i/\mathrm{c}}\right)^{\times} - m_\mathcal{B} \vec{r}_{\mathrm{c}/z}^{\,\times 2}\\
&= \vec{J}_{\mathcal{B}/\mathrm{c}} - m_\mathcal{B} \vec{r}_{z/\mathrm{c}}^{\,\times 2} = \vec{J}_{\mathcal{B}/\mathrm{c}} + m_\mathcal{B} \vec{r}_{z/\mathrm{c}}^{\,\times\prime}\vec{r}_{z/\mathrm{c}}^{\,\times} = \vec{J}_{\mathcal{B}/\mathrm{c}} + m_\mathcal{B}(|\vec{r}_{z/\mathrm{c}}|^2 \vec{I} - \vec{r}_{z/\mathrm{c}}\vec{r}_{z/\mathrm{c}}^{\,\prime}).
\end{aligned}$$

□

The next result follows from Fact 6.2.12.

Fact 6.2.13. Let $\mathcal{B}$ be a body, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, and let z_1 and z_2 be points. Then,

$$\vec{J}_{\mathcal{B}/z_2} = \vec{J}_{\mathcal{B}/z_1} + m_{\mathcal{B}}[\vec{r}^{\times}_{z_2/z_1}\vec{r}^{\times}_{c/z_1} + \vec{r}^{\times}_{c/z_1}\vec{r}^{\times}_{z_2/z_1} - \vec{r}^{\times 2}_{z_2/z_1}]. \tag{6.2.29}$$

Fact 6.2.12 yields the *parallel axis theorem* given by the following result.

Fact 6.2.14. Let $\mathcal{B}$ be a body, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, and let z be a point fixed in $\mathcal{B}$, and let c be the center of mass of $\mathcal{B}$, assume that the frame axis $\hat{\imath}_{\rm A}$ is perpendicular to $\vec{r}_{z/c}$. Then,

$$J_{xx/z|\rm B} = J_{xx/c|\rm B} + m_{\mathcal{B}}|\vec{r}_{z/c}|^2. \tag{6.2.30}$$

Proof. Multiplying (6.2.29) on the left and the right by $\hat{\imath}_{\rm A}$ yields

$$J_{xx/z|\rm B} = \hat{\imath}'_{\rm A}\vec{J}_{\mathcal{B}/z}\hat{\imath}_{\rm A} = \hat{\imath}'_{\rm A}[\vec{J}_{\mathcal{B}/c} + m_{\mathcal{B}}(|\vec{r}_{z/c}|^2\vec{I} - \vec{r}_{z/c}\vec{r}^{\,\prime}_{z/c})]\hat{\imath}_{\rm A} = J_{xx/c|\rm B} + m_{\mathcal{B}}|\vec{r}_{z/c}|^2. \qquad \square$$

A *plane of symmetry* of a body $\mathcal{B}$ is a plane that divides $\mathcal{B}$ into two "mirror-image" parts, which are identical in both geometry and mass properties. Note that the center of mass of $\mathcal{B}$ lies on every plane of symmetry of $\mathcal{B}$.

Fact 6.2.15. Let $\mathcal{B}$ be a body, let z be a point, and let $\rm F_B$ be a frame. Then, the following statements hold:

i) If z is an element of a plane of symmetry of $\mathcal{B}$ that is parallel to the $\hat{\imath}_{\rm B}$-$\hat{\jmath}_{\rm B}$ plane, then $J_{xz/z|\rm B} = J_{yz/z|\rm B} = 0$.

ii) If z is an element of a plane of symmetry of $\mathcal{B}$ that is parallel to the $\hat{\imath}_{\rm B}$-$\hat{k}_{\rm B}$ plane, then $J_{xy/z|\rm B} = J_{yz/z|\rm B} = 0$.

iii) If z is an element of a plane of symmetry of $\mathcal{B}$ that is parallel to the $\hat{\jmath}_{\rm B}$-$\hat{k}_{\rm B}$ plane, then $J_{xy/z|\rm B} = J_{xz/z|\rm B} = 0$.

iv) If $\mathcal{B}$ has orthogonal planes of symmetry P_1 and P_2 that are spanned by pairs of frame axes of $\rm F_B$, then $\rm F_B$ is a principal-axis frame of $\mathcal{B}$ relative to every point $z \in P_1 \cup P_2$.

Proof. To prove *i*), note that $\mathcal{B}$ contains an even number of particles $y_1, \ldots, y_{2r}$ whose masses are $m_1, \ldots, m_r, m_1, \ldots, m_r$ and whose locations relative to z and resolved in $\rm F_B$ have components $(\bar{x}_1, \bar{y}_1, \bar{z}_1), \ldots, (\bar{x}_r, \bar{y}_r, \bar{z}_r), (\bar{x}_1, \bar{y}_1, -\bar{z}_1), \ldots, (\bar{x}_r, \bar{y}_r, -\bar{z}_r)$, respectively, where $\bar{z}_1, \ldots, \bar{z}_r$ are nonzero. It thus follows from (6.2.8) that

$$J_{xz/z|\rm B} = \sum_{i=1}^{2r} m_i\bar{x}_i\bar{z}_i = \sum_{i=1}^{r} m_i\bar{x}_i\bar{z}_i + \sum_{i=r+1}^{2r} m_i\bar{x}_i\bar{z}_i = \sum_{i=1}^{r} m_i\bar{x}_i\bar{z}_i + \sum_{i=1}^{r} m_i\bar{x}_i(-\bar{z}_i) = 0.$$

Likewise, $J_{yz} = 0$. *iv*) follows from Fact 6.2.12. $\square$

A body that has multiple planes of symmetry need not have two orthogonal planes of symmetry. For example, a bar whose cross section is an equilateral triangle whose size varies along the length of the bar has exactly three planes of symmetry, but no pair of these planes of symmetry is orthogonal.

6.3 The Physical Inertia Matrix for Continuum Bodies

For continuum bodies, we replace the finite sums in (6.2.6)–(6.2.8) with integrals.

Fact 6.3.1. Let $\mathcal{B}$ be a continuum body and let z be a point. Then,

$$\vec{J}_{\mathcal{B}/z} = -\int_{\mathcal{B}} \vec{r}^{\times 2}_{\mathrm{d}m/z}\,\mathrm{d}m \tag{6.3.1}$$

$$= \int_{\mathcal{B}} \vec{r}^{\times\prime}_{\mathrm{d}m/z}\vec{r}^{\times}_{\mathrm{d}m/z}\,\mathrm{d}m \tag{6.3.2}$$

$$= \int_{\mathcal{B}} |\vec{r}_{\mathrm{d}m/z}|^2\vec{I} - \vec{r}_{\mathrm{d}m/z}\vec{r}^{\,\prime}_{\mathrm{d}m/z}\,\mathrm{d}m. \tag{6.3.3}$$

Now, let $\mathrm{F_B}$ be a frame. Then,

$$\begin{aligned}\vec{J}_{\mathcal{B}/z} = {}& J_{xx/z|\mathrm{B}}\hat{\imath}_\mathrm{B}\hat{\imath}'_\mathrm{B} + J_{yy/z|\mathrm{B}}\hat{\jmath}_\mathrm{B}\hat{\jmath}'_\mathrm{B} + J_{zz/z|\mathrm{B}}\hat{k}_\mathrm{B}\hat{k}'_\mathrm{B} - J_{xy/z|\mathrm{B}}(\hat{\imath}_\mathrm{B}\hat{\jmath}'_\mathrm{B} + \hat{\jmath}_\mathrm{B}\hat{\imath}'_\mathrm{B})\\ &- J_{xz/z|\mathrm{B}}(\hat{\imath}_\mathrm{B}\hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B}\hat{\imath}'_\mathrm{B}) - J_{yz/z|\mathrm{B}}(\hat{\jmath}_\mathrm{B}\hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B}\hat{\jmath}'_\mathrm{B}),\end{aligned} \tag{6.3.4}$$

that is,

$$J_{\mathcal{B}/z|\mathrm{B}} = \begin{bmatrix} J_{xx/z|\mathrm{B}} & -J_{xy/z|\mathrm{B}} & -J_{xz/z|\mathrm{B}} \\ -J_{yx/z|\mathrm{B}} & J_{yy/z|\mathrm{B}} & -J_{yz/z|\mathrm{B}} \\ -J_{zx/z|\mathrm{B}} & -J_{zy/z|\mathrm{B}} & J_{zz/z|\mathrm{B}} \end{bmatrix}, \tag{6.3.5}$$

where

$$J_{xx/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} (y^2+z^2)\,\mathrm{d}m, \qquad J_{xy/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} xy\,\mathrm{d}m, \tag{6.3.6}$$

$$J_{yy/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} (x^2+z^2)\,\mathrm{d}m, \qquad J_{xz/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} xz\,\mathrm{d}m, \tag{6.3.7}$$

$$\underbrace{J_{zz/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} (x^2+y^2)\,\mathrm{d}m,}_{\text{moments of inertia}} \qquad \underbrace{J_{yz/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} yz\,\mathrm{d}m}_{\text{products of inertia}}. \tag{6.3.8}$$

If the density ρ of the material is constant, then $\mathrm{d}m = \rho\mathrm{d}V$, and each integral can be written as a volume integral. For example,

$$J_{xx/z|\mathrm{B}} = \rho\int_{\mathcal{B}} (y^2+z^2)\,\mathrm{d}V. \tag{6.3.9}$$

For a flat plate, this integral becomes an integral over an area, and ρ is the area density, that is, mass per area. For a thin body, this integral becomes an integral over a length, and ρ is the linear density, that is, mass per length.

For a continuum body, the center of mass is the unique point fixed in the body and satisfies

$$\int_{\mathcal{B}} \vec{r}_{\mathrm{d}m/\mathrm{c}}\,\mathrm{d}m = 0. \tag{6.3.10}$$

Example 6.3.2. Let $\mathcal{B}$ be a homogeneous sphere of mass m and radius r, and let $\mathrm{F_B}$ be a frame. Then, the inertia matrix of the sphere relative to its center of mass c determined by $\mathrm{F_B}$ is given by

$$J_{\mathcal{B}/\mathrm{c}|\mathrm{B}} = \begin{bmatrix} J_{xx/\mathrm{c}|\mathrm{B}} & 0 & 0 \\ 0 & J_{yy/\mathrm{c}|\mathrm{B}} & 0 \\ 0 & 0 & J_{zz/\mathrm{c}|\mathrm{B}} \end{bmatrix}, \tag{6.3.11}$$

where $J_{xx/\mathrm{c}|\mathrm{B}} = J_{yy/\mathrm{c}|\mathrm{B}} = J_{zz/\mathrm{c}|\mathrm{B}} = \frac{2}{5}mr^2$ are the moments of inertia of $\mathcal{B}$ relative to the center of mass c determined by F_B. Therefore,

$$J_{\mathcal{B}/c|B} = \frac{2}{5}mr^2 I_3. \tag{6.3.12}$$

◇

Example 6.3.3. Let $\mathcal{B}$ be a homogeneous rectangular solid, and let F_B be a frame whose axes $\hat{\imath}_B$, $\hat{\jmath}_B$, and $\hat{k}_B$ are parallel to the sides of length a, b, and c, respectively. Then,

$$J_{\mathcal{B}/c|B} = \begin{bmatrix} J_{xx/c|B} & 0 & 0 \\ 0 & J_{yy/c|B} & 0 \\ 0 & 0 & J_{zz/c|B} \end{bmatrix}, \tag{6.3.13}$$

where $J_{xx/c|B} = \frac{1}{12}m(b^2 + c^2)$, $J_{yy/c|B} = \frac{1}{12}m(a^2 + c^2)$, and $J_{zz/c|B} = \frac{1}{12}m(a^2 + b^2)$. If $a > b > c$, then $J_{zz/c|B} > J_{yy/c|B} > J_{xx/c|B}$, where $J_{zz/c|B}$, $J_{yy/c|B}$, and $J_{xx/c|B}$ are the major, intermediate, and minor principal moments of inertia, respectively, of $\mathcal{B}$ relative to the center of mass c determined by F_B. If $b = c \approx 0$, then the rectangular solid approximates a thin bar, in which case $J_{xx/c|B} \approx 0$ and $J_{yy/c|B} = J_{zz/c|B} \approx \frac{1}{12}ma^2$. If, in addition, $c \approx 0$, then the rectangular solid approximates a rectangular plate with sides a and b, in which case $J_{xx/c|B} \approx \frac{1}{12}mb^2$, $J_{yy/c|B} \approx \frac{1}{12}ma^2$, and $J_{zz/c|B} = \frac{1}{12}m(a^2 + b^2)$. ◇

Example 6.3.4. Let $\mathcal{B}$ be a homogeneous cylinder of length l and radius r, or, equivalently, a homogeneous disk of thickness l and radius r. Let F_B be a frame such that $\hat{\imath}_B$ is parallel to the longitudinal axis of the cylinder. Then

$$J_{\mathcal{B}/c|B} = \begin{bmatrix} J_{xx/c|B} & 0 & 0 \\ 0 & J_{yy/c|B} & 0 \\ 0 & 0 & J_{zz/c|B} \end{bmatrix}, \tag{6.3.14}$$

where $J_{xx/c|B} = \frac{1}{2}mr^2$ and $J_{yy/c|B} = J_{zz/c|B} = \frac{1}{12}m(3r^2 + l^2)$ are the principal moments of inertia of $\mathcal{B}$ relative to the center of mass c determined by F_B. The cylinder approximates a thin bar if $r \approx 0$, in which case $J_{xx/c|B} \approx 0$ and $J_{yy/c|B} = J_{zz/c|B} \approx \frac{1}{12}ml^2$. The cylinder approximates a circular plate if $l \approx 0$, in which case $J_{xx/c|B} = \frac{1}{2}mr^2$ and $J_{yy/c|B} = J_{zz/c|B} \approx \frac{1}{4}mr^2$. Finally, if $l = \sqrt{3}r$, then

$$J_{\mathcal{B}/c|B} = \frac{1}{5}mr^2 I_3. \tag{6.3.15}$$

◇

6.4 Moments, Balanced Forces, and Torques

Let y be an inertia point, let w be a point, and let $\vec{f}_y$ be the force applied to y. Then, the *moment* $\vec{M}_{y/w}$ *on y relative to w due to* $\vec{f}_y$ is defined by

$$\vec{M}_{y/w} \triangleq \vec{r}_{y/w} \times \vec{f}_y. \tag{6.4.1}$$

The moment $\vec{M}_{y/w}$ is illustrated by Figure 6.4.1.

We may consider a moment on an inertia point without first introducing a force. Let y be an inertia point and let w be a point. Then, the physical vector $\vec{M}$ is a *moment on y relative to w* if there exists a force $\vec{f}_y$ such that $\vec{M} = \vec{r}_{y/w} \times \vec{f}_y$. Note that the force vector $\vec{f}_y$ is not unique.

Intuitively speaking, the moment $\vec{M} = \vec{r}_{y/w} \times \vec{f}_y$ induces a rotation of y around w in the direction given by the direction of the curled fingers of the right hand, where the right-hand thumb is pointing in the direction of $\vec{M}$.

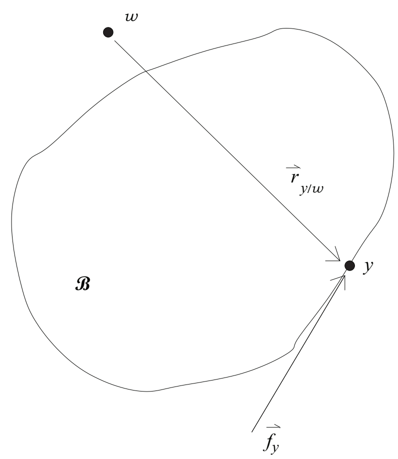


Figure 6.4.1: Representation of the moment $\vec{M}_{y/w} = \vec{r}_{y/w} \times \vec{f}_y$ on the inertia point y in the body $\mathcal{B}$ relative to the point w due to the force $\vec{f}_y$ applied to y.

Next, let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force applied to y_i, and let w be a point. Then, the *moment* $\vec{M}_{\mathcal{B}/w}$ *on* $\mathcal{B}$ *relative to* w *due to* $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ is defined by

$$\vec{M}_{\mathcal{B}/w} \triangleq \sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{6.4.2}$$

As in the case of an inertia point, we may consider a moment on a body without first introducing forces. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, and let w be a point. Then, the physical vector $\vec{M}$ is a *moment on* $\mathcal{B}$ *relative to* w if there exist forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ applied to inertia points $y_1, \ldots, y_l$, respectively, such that $\vec{M}$ is the moment on $\mathcal{B}$ relative to w due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$. Note that the forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are not unique.

Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, and, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force applied to y_i. Then, the *total force on* $\mathcal{B}$ *due to* $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ is defined by

$$\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}. \tag{6.4.3}$$

The forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are *balanced* if $\vec{f}_{\mathcal{B}} = 0$.

Fact 6.4.1. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force applied to y_i, and let w and z be points. Then,

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}}, \tag{6.4.4}$$

where the total force $\vec{f}_{\mathcal{B}}$ is given by (6.4.3). If, in addition, the forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, then

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/z}. \tag{6.4.5}$$

Proof.

$$\begin{aligned}\vec{M}_{\mathcal{B}/w} &= \sum_{i=1}^{l}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) = \sum_{i=1}^{l}(\vec{r}_{y_i/z} + \vec{r}_{z/w}) \times \vec{f}_{y_i} \\ &= \sum_{i=1}^{l}(\vec{r}_{y_i/z} \times \vec{f}_{y_i}) + \sum_{i=1}^{l}(\vec{r}_{z/w} \times \vec{f}_{y_i}) \\ &= \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \sum_{i=1}^{l}\vec{f}_{y_i} = \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}}.\end{aligned}$$

□

Fact 6.4.1 shows that, if the forces on the body $\mathcal{B}$ are balanced, then the moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ relative to w is independent of the point w. The next two results focus on this case.

Fact 6.4.2. Let $\mathcal{B}$ be a body, let w be a point, let $\vec{f}_x$ and $\vec{f}_y$ be forces applied to inertia points x and y in $\mathcal{B}$, respectively, assume that $\vec{f}_y = -\vec{f}_x$, and assume that $\vec{f}_x$ and $\vec{f}_y$ are the only forces applied to $\mathcal{B}$. Then,

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{y/x} = \vec{M}_{x/y}. \tag{6.4.6}$$

Proof. Since the forces are balanced, it follows from Fact 6.4.1 that $\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/x}$. Therefore,

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/x} = \vec{r}_{x/x} \times \vec{f}_x + \vec{r}_{y/x} \times \vec{f}_y = \vec{M}_{y/x}.$$

Likewise, it follows from Fact 6.4.1 that $\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/y}$. Therefore,

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/y} = \vec{r}_{x/y} \times \vec{f}_x + \vec{r}_{y/y} \times \vec{f}_y = \vec{M}_{x/y}.$$

□

Fact 6.4.3. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force applied to y_i, let w be a point, and assume that $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced. Then, for all $j = 1, \ldots, l$,

$$\vec{M}_{\mathcal{B}/w} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \vec{M}_{y_i/y_j}. \tag{6.4.7}$$

Proof. For convenience, set $j = l$. Then, note that

$$\begin{aligned}\vec{M}_{\mathcal{B}/w} &= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} = \sum_{i=1}^{l-1}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) + \vec{r}_{y_l/w} \times \vec{f}_{y_l} \\ &= \sum_{i=1}^{l-1}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) + \vec{r}_{y_l/w} \times \left(-\sum_{i=1}^{l-1}\vec{f}_{y_i}\right) \\ &= \sum_{i=1}^{l-1}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) + \sum_{i=1}^{l-1}(\vec{r}_{w/y_l} \times \vec{f}_{y_i}) \\ &= \sum_{i=1}^{l-1}(\vec{r}_{y_i/y_l} \times \vec{f}_{y_i}) = \sum_{\substack{i=1\\ i\neq l}}^{n} \vec{M}_{y_i/y_l}. \qquad \square\end{aligned}$$

Assume that $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced. Then, Fact 6.4.3 shows that the moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ is independent of w. In this case, we write $\vec{M}_{\mathcal{B}}$ instead of $\vec{M}_{\mathcal{B}/w}$, and we call $\vec{M}_{\mathcal{B}}$ the *torque on* $\mathcal{B}$ *due to* $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$.

The following result considers forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ and the resulting moment $\vec{M}_{\mathcal{B}/w}$. If w is an inertia point and the additional force $-\vec{f}_{\mathcal{B}}$ is applied to w, then the forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ are balanced, and thus the resulting moment $\vec{M}_{\mathcal{B}}$ is a torque. In addition, since $-\vec{f}_{\mathcal{B}}$ is applied to w, it does not contribute to the moment relative to w, and thus $\vec{M}_{\mathcal{B}} = \vec{M}_{\mathcal{B}/w}$. Since the forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ are balanced, $\vec{M}_{\mathcal{B}}$ is independent of the choice of reference point. However, $\vec{M}_{\mathcal{B}}$ does depend on the reference point w used to define $\vec{M}_{\mathcal{B}/w}$. If w is not an inertia point, then the result holds with $-\vec{f}_{\mathcal{B}}$ applied to an inertia point z such that $\vec{r}_{w/z}$ is parallel to $\vec{f}_{\mathcal{B}}$.

Fact 6.4.4. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ be forces applied to $y_1, \ldots, y_l$, respectively, let w be a point, let $\vec{M}_{\mathcal{B}/w}$ be the moment on $\mathcal{B}$ relative to w due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$, and define $\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}$. Furthermore, assume there exists an inertia point z in $\mathcal{B}$ such that $\vec{r}_{z/w}$ is parallel to $\vec{f}_{\mathcal{B}}$, and let $\vec{M}_{\mathcal{B}}$ denote the torque on $\mathcal{B}$ due to the balanced forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ applied to $y_1, \ldots, y_l, z$, respectively. Then, $\vec{M}_{\mathcal{B}} = \vec{M}_{\mathcal{B}/w}$.

Proof. Since $\vec{r}_{z/w}$ is parallel to $\vec{f}_{\mathcal{B}}$ and $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ are balanced, it follows that the torque $\vec{M}_{\mathcal{B}}$ on $\mathcal{B}$ due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ applied to $y_1, \ldots, y_l, z$, respectively, is given by

$$\vec{M}_{\mathcal{B}} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} + \vec{r}_{z/w} \times (-\vec{f}_{\mathcal{B}}) = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} = \vec{M}_{\mathcal{B}/w}. \qquad \square$$

6.5 Laws of Statics

For a body $\mathcal{B}$, the laws of statics are as follows:

i) The total force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ is zero.

ii) The torque $\vec{M}_{\mathcal{B}}$ on $\mathcal{B}$ due to $\vec{f}_{\mathcal{B}}$ is zero.

Note that, since the forces on $\mathcal{B}$ are balanced, it follows that, for all points w, the moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ relative to w is a torque and thus is independent of w.

6.6 Moment Due to Uniform Gravity

We now consider the total force and moment on a body due to uniform gravity.

Fact 6.6.1. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$. Then, the total force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ due to gravity is given by

$$\vec{f}_{\mathcal{B}} = m_{\mathcal{B}}\vec{g}. \tag{6.6.1}$$

Fact 6.6.2. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, and let w be a point. Then, the moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ relative to w due to gravity is given by

$$\vec{M}_{\mathcal{B}/w} = \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{g}. \tag{6.6.2}$$

In particular,

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = 0. \tag{6.6.3}$$

Proof. Note that

$$\begin{aligned}
\vec{M}_{\mathcal{B}/w} &= \sum_{i=1}^{l}(\vec{r}_{y_i/w} \times m_i\vec{g}) = \left(\sum_{i=1}^{l} m_i\vec{r}_{y_i/w}\right) \times \vec{g} \\
&= \left(\frac{1}{m_{\mathcal{B}}}\sum_{i=1}^{l} m_i\vec{r}_{y_i/w}\right) \times m_{\mathcal{B}}\vec{g} = \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{g}.
\end{aligned}$$
□

Note that, although the location of the center of mass c is independent of the choice of w, the physical vector $\vec{r}_{\mathrm{c}/w}$ depends on w, and thus the moment $\vec{M}_{\mathcal{B}/w}$ depends on w.

6.7 Forces and Torques Due to Springs and Rotational Springs

Consider a spring connecting inertia points y and w, where the relaxed length of the spring is $d \geq 0$ and the stiffness of the spring is $k > 0$. The spring may be in either compression or extension, in which case its length $|\vec{r}_{y/w}|$ is either less than d or greater than d, respectively. Then, the force $\vec{f}_{y/w}$ applied to y by the spring is given by

$$\vec{f}_{y/w} = -k(|\vec{r}_{y/w}| - d)\hat{r}_{y/w}. \tag{6.7.1}$$

Note that $\vec{f}_{y/w}$ is aligned with the line passing through y and w. Furthermore, $\vec{f}_{y/w}$ pushes y in the direction $\hat{r}_{y/w}$ when the spring is in compression, and pushes y in the direction $-\hat{r}_{y/w}$ when the spring is in extension. Note that the forces applied to y and w are equal and opposite, that is,

$$\vec{f}_{w/y} = -\vec{f}_{y/w}. \tag{6.7.2}$$

Finally, if $d = 0$, then (6.7.1) becomes

$$\vec{f}_{y/w} = -k\vec{r}_{y/w}. \tag{6.7.3}$$

Next, consider rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ that are connected by a pin joint at a point that is fixed in both bodies. Let $\hat{z}$ be a unit dimensionless vector that is parallel with the pin joint. A rotational spring applies torques to $\mathcal{B}_1$ and $\mathcal{B}_2$ that are parallel with $\hat{z}$. Let $\hat{x}_1$ and $\hat{x}_2$ be unit dimensionless vectors that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that are orthogonal to $\hat{z}$. Assume that the rotational spring is relaxed when $\hat{x}_1$ and $\hat{x}_2$ are parallel, and assume that the rotational stiffness of the rotational spring is $\kappa > 0$. Then, the torque $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}$ applied to $\mathcal{B}_1$ by the rotational spring is given by

$$\vec{M}_{\mathcal{B}_1/\mathcal{B}_2} = -\kappa(\theta_{\hat{x}_1/\hat{x}_2/\hat{z}} - \theta_0)\hat{z}. \tag{6.7.4}$$

where θ_0 is the relaxed angle. Note that the torque $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}$ applied to $\mathcal{B}_1$ by the rotational spring is clockwise around $\hat{z}$ when the rotational spring is wound up counterclockwise around $\hat{z}$, and vice versa. Furthermore, the torques applied to $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal and opposite, that is,

$$\vec{M}_{\mathcal{B}_2/\mathcal{B}_1} = -\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}. \tag{6.7.5}$$

6.8 Forces and Torques Due to Dashpots and Rotational Dashpots

Consider a dashpot connecting inertia points y and w, where the damping coefficient of the dashpot is $c > 0$. Then, the force $\vec{f}_{y/w}$ applied to y by the dashpot is given by

$$\vec{f}_{y/w} = -c\frac{\mathrm{d}}{\mathrm{d}t}|\vec{r}_{y/w}|\hat{r}_{y/w}. \tag{6.8.1}$$

Note that $\vec{f}_{y/w}$ is aligned with the line passing through y and w. Furthermore, $\vec{f}_{y/w}$ pushes y in the direction $\hat{r}_{y/w}$ when the dashpot is compressing, and pushes y in the direction $-\hat{r}_{y/w}$ when the dashpot is extending. Note that the forces applied to y and w are equal and opposite, that is,

$$\vec{f}_{w/y} = -\vec{f}_{y/w}. \tag{6.8.2}$$

Next, consider rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ that are connected by a pin joint at a point that is fixed in both bodies. Let $\hat{z}$ be a unit dimensionless vector that is parallel with the pin joint. A rotational dashpot applies torques to $\mathcal{B}_1$ and $\mathcal{B}_2$ that are parallel with $\hat{z}$. Let $\hat{x}_1$ and $\hat{x}_2$ be unit dimensionless vectors that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that are orthogonal to $\hat{z}$. Let $\gamma > 0$ denote the rotational damping coefficient. Then, the torque $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}$ applied to $\mathcal{B}_1$ by the rotational dashpot is given by

$$\vec{M}_{\mathcal{B}_1/\mathcal{B}_2} = -\gamma\frac{\mathrm{d}}{\mathrm{d}t}|\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}|\hat{z}. \tag{6.8.3}$$

Note that the torque $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}$ applied to $\mathcal{B}_1$ by the rotational spring is clockwise around $\hat{z}$ when the rotational dashpot is rotating counterclockwise around $\hat{z}$, and vice versa. Furthermore, the torques applied to $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal and opposite, that is,

$$\vec{M}_{\mathcal{B}_2/\mathcal{B}_1} = -\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}. \tag{6.8.4}$$

6.9 Newton's Third Law

Two inertia points can exert forces on each other in various ways. For example, the inertia points may be in direct contact, or they may be in indirect contact due to connection by a rigid massless link, spring, dashpot, or inerter. In addition, two particles can exert noncontacting forces on each other, such as gravitational forces (which are attractive) or electrostatic forces (which may be attractive or repulsive). All of these forces are *reaction forces*. The following fundamental result is *Newton's third law*.

Fact 6.9.1. Let x and y be inertia points that are either in direct contact, in indirect contact due to connection by a rigid massless link, spring, dashpot, or inerter, or noncontacting through gravitational or electrostatic forces. Then, the force $\vec{f}_{y/x}$ on y due to x is equal and opposite in direction to the force $\vec{f}_{x/y}$ on x due to y, that is, $\vec{f}_{y/x} = -\vec{f}_{x/y}$. Furthermore, $\vec{f}_{y/x}$ and $\vec{f}_{x/y}$ are parallel with $\vec{r}_{y/x}$.

Magnetic forces and electrodynamic forces also give rise to reaction forces. However, magnetic forces involve field lines and polarity, and thus Newton's third law is more complicated. In addition, electrodynamic forces do not satisfy Newton's third law. Henceforth, we consider only reaction forces that satisfy Newton's third law stated by Fact 6.9.1. These forces have equal magnitude, are opposite in direction, and are parallel with the line that passes through the inertia points or particles.

Consider a body consisting of n bodies $\mathcal{B}_1, \ldots, \mathcal{B}_n$, some of which may be rigid bodies. These bodies may interact with each other through direct contact (for example, through collisions, rolling, sliding (with or without friction), and pivoting (with or without friction)), indirect contact (for example, through rigid massless links, springs, dashpots, and inerters), or noncontacting forces due to gravitational or electrostatic forces. Newton's third law states that each type of interaction produces reaction forces that are equal in magnitude and opposite in direction. For cases of direct or indirect contact, the reaction forces are applied to each body at the points of contact or attachment. For noncontacting reaction forces, the reaction forces are applied to each particle. These reaction forces depend on the relative displacement, velocity, or acceleration of the bodies at each instant of time. For $i = 1, \ldots, n$, the total force $\vec{f}_{\mathcal{B}_i}$ on $\mathcal{B}_i$ is due to the reaction forces on $\mathcal{B}_i$ due to its interaction with all of the remaining bodies.

Fact 6.9.2. Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be bodies, and, for all $i = 1, \ldots, n$, let $\vec{f}_{\mathcal{B}_i}$ denote the total reaction force on $\mathcal{B}_i$ due to direct contact, indirect contact, or noncontacting interaction with the remaining bodies. Then, $\sum_{i=1}^{n} \vec{f}_{\mathcal{B}_i} = \vec{0}$. If, in particular, $n = 2$, then $\vec{f}_{\mathcal{B}_2} = -\vec{f}_{\mathcal{B}_1}$.

Note that Fact 6.9.2 does not specify the directions of the total force vectors $\vec{f}_{\mathcal{B}_i}$. Problem 6.14.4 shows that, for two bodies subject to gravitational reaction forces, the total force vectors need not be parallel with the lines connecting either the centers of mass or the centers of gravity of the bodies.

For a body consisting of n bodies $\mathcal{B}_1, \ldots, \mathcal{B}_n$, the reaction forces on $\mathcal{B}_i$ produce a *reaction* *moment* on $\mathcal{B}_i$ relative to a point w. If the reaction forces on $\mathcal{B}_i$ are balanced, then the reaction moment is a *reaction torque*. The following result is *Newton's third law for moments*.

Fact 6.9.3. Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be bodies, let w be a point, and, for all $i = 1, \ldots, n$, let $\vec{M}_{\mathcal{B}_i/w}$ denote the moment on $\mathcal{B}_i$ relative to w arising from all reaction forces on $\mathcal{B}_i$ due to direct contact, indirect contact, and noncontacting interaction with the remaining bodies. Then $\sum_{i=1}^n \vec{M}_{\mathcal{B}_i/w} = \vec{0}$. If, in particular, $n = 2$, then $\vec{M}_{\mathcal{B}_2/w} = -\vec{M}_{\mathcal{B}_1/w}$.

Proof. First, we consider the case $n = 2$. Let l denote the number of points at which a reaction force is applied to $\mathcal{B}_1$ due to the interaction between $\mathcal{B}_1$ and $\mathcal{B}_2$. For all $j = 1, \ldots, l$, let $z_{1,j}$ denote a point in $\mathcal{B}_1$ at which a reaction force is applied to $\mathcal{B}_1$ due to interaction with $\mathcal{B}_2$, and let $z_{2,j}$ denote the corresponding point in $\mathcal{B}_2$. Furthermore, for all $j = 1, \ldots, l$, let $\vec{f}_{1,j}$ denote the reaction force on $\mathcal{B}_1$ at $z_{1,j}$, and let $\vec{f}_{2,j}$ denote the reaction force on $\mathcal{B}_2$ at $z_{2,j}$. Therefore, for all $j = 1, \ldots, l$, the reaction forces $\vec{f}_{1,j}$ on $\mathcal{B}_1$ and $\vec{f}_{2,j}$ on $\mathcal{B}_2$ at the points $z_{1,j}$ and $z_{2,j}$, respectively, satisfy $\vec{f}_{2,j} = -\vec{f}_{1,j}$. Furthermore, Newton's third law implies that, for all $j = 1, \ldots, l$, $\vec{r}_{z_{1,j},z_{2,j}}$ and $\vec{f}_{1,j}$ are parallel. It thus follows that

$$\begin{aligned}
\sum_{i=1}^{2} \vec{M}_{\mathcal{B}_i/w} &= \vec{M}_{\mathcal{B}_1/w} + \vec{M}_{\mathcal{B}_2/w} \\
&= \sum_{j=1}^{l} r_{z_{1,j},w} \times \vec{f}_{1,j} + \sum_{j=1}^{l} r_{z_{2,j},w} \times \vec{f}_{2,j} \\
&= \sum_{j=1}^{l} (\vec{r}_{z_{1,j},z_{2,j}} + \vec{r}_{z_{2,j},w}) \times \vec{f}_{1,j} - \sum_{j=1}^{l} r_{z_{2,j},w} \times \vec{f}_{1,j} \\
&= \sum_{j=1}^{l} \vec{r}_{z_{2,j},w} \times \vec{f}_{1,j} - \sum_{j=1}^{l} \vec{r}_{z_{2,j},w} \times \vec{f}_{1,j} \\
&= \vec{0}.
\end{aligned}$$

Now, we consider the case $n = 3$. Let $l_{1,2}$ denote the number of points at which a reaction force is applied to $\mathcal{B}_1$ due to the interaction between $\mathcal{B}_1$ and $\mathcal{B}_2$. For all $j = 1, \ldots, l_{1,2}$, let $z_{1,2;j}$ denote a point in $\mathcal{B}_1$ at which a reaction force is applied to $\mathcal{B}_1$ due to interaction with $\mathcal{B}_2$, and let $z_{2,1;j}$ denote the corresponding point in $\mathcal{B}_2$. Furthermore, for all $j = 1, \ldots, l_{1,2}$, let $\vec{f}_{1,2;j}$ denote the reaction force on $\mathcal{B}_1$ at $z_{1,2;j}$, and let $\vec{f}_{2,1;j}$ denote the reaction force on $\mathcal{B}_2$ at $z_{2,1;j}$. Therefore, for all $j = 1, \ldots, l_{1,2}$, the reaction forces $\vec{f}_{1,2;j}$ on $\mathcal{B}_1$ and $\vec{f}_{2,1;j}$ on $\mathcal{B}_2$ at the reaction point $z_{1,2;j}$ and $z_{2,1;j}$, respectively, satisfy $\vec{f}_{2,1;j} = -\vec{f}_{1,2;j}$. Furthermore, Newton's third law implies that, for all $j = 1, \ldots, l_{1,2}$, $\vec{r}_{z_{1,2;j},z_{2,1;j}}$ and $\vec{f}_{1,2;j}$ are parallel. Similar notation applies to the interactions between $\mathcal{B}_1$ and $\mathcal{B}_2$ as well as between $\mathcal{B}_2$ and $\mathcal{B}_2$. It thus follows that

$$\sum_{i=1}^{3} \vec{M}_{\mathcal{B}_i/w} = \vec{M}_{\mathcal{B}_1/w} + \vec{M}_{\mathcal{B}_2/w} + \vec{M}_{\mathcal{B}_3/w}$$

$$
\begin{aligned}
&= \sum_{j=1}^{l_{1,2}} r_{z_{1,2;j},w} \times \vec{f}_{1,2;j} + \sum_{j=1}^{l_{1,3}} r_{z_{1,3;j},w} \times \vec{f}_{1,3;j} \\
&\quad + \sum_{j=1}^{l_{1,2}} r_{z_{2,1;j},w} \times \vec{f}_{2,1;j} + \sum_{j=1}^{l_{2,3}} r_{z_{2,3;j},w} \times \vec{f}_{2,3;j} \\
&\quad + \sum_{j=1}^{l_{1,3}} r_{z_{3,1;j},w} \times \vec{f}_{3,1;j} + \sum_{j=1}^{l_{2,3}} r_{z_{3,2;j},w} \times \vec{f}_{3,2;j} \\
&= \sum_{j=1}^{l_{1,2}} r_{z_{1,2;j},w} \times \vec{f}_{1,2;j} + \sum_{j=1}^{l_{1,2}} r_{z_{2,1;j},w} \times \vec{f}_{2,1;j} \\
&\quad \sum_{j=1}^{l_{1,3}} r_{z_{1,3;j},w} \times \vec{f}_{1,3;j} + \sum_{j=1}^{l_{1,3}} r_{z_{3,1;j},w} \times \vec{f}_{3,1;j} \\
&\quad \sum_{j=1}^{l_{2,3}} r_{z_{2,3;j},w} \times \vec{f}_{2,3;j} + \sum_{j=1}^{l_{2,3}} r_{z_{3,2;j},w} \times \vec{f}_{3,2;j} \\
&= \sum_{j=1}^{l_{1,2}} (r_{z_{1,2;j},z_{2,1;j}} + r_{z_{2,1;j},w}) \times \vec{f}_{1,2;j} - \sum_{j=1}^{l_{1,2}} r_{z_{2,1;j},w} \times \vec{f}_{1,2;j} \\
&\quad \sum_{j=1}^{l_{1,3}} (r_{z_{1,3;j},z_{3,1;j}} + r_{z_{3,1;j},w}) \times \vec{f}_{1,3;j} - \sum_{j=1}^{l_{1,3}} r_{z_{3,1;j},w} \times \vec{f}_{1,3;j} \\
&\quad \sum_{j=1}^{l_{2,3}} (r_{z_{2,3;j},z_{3,2;j}} + r_{z_{3,2;j},w}) \times \vec{f}_{2,3;j} - \sum_{j=1}^{l_{2,3}} r_{z_{3,2;j},w} \times \vec{f}_{2,3;j} \\
&= \sum_{j=1}^{l_{1,2}} r_{z_{2,1;j},w} \times \vec{f}_{1,2;j} - \sum_{j=1}^{l_{1,2}} r_{z_{2,1;j},w} \times \vec{f}_{1,2;j} \\
&\quad \sum_{j=1}^{l_{1,3}} r_{z_{3,1;j},w} \times \vec{f}_{1,3;j} - \sum_{j=1}^{l_{1,3}} r_{z_{3,1;j},w} \times \vec{f}_{1,3;j} \\
&\quad \sum_{j=1}^{l_{2,3}} r_{z_{3,2;j},w} \times \vec{f}_{2,3;j} - \sum_{j=1}^{l_{2,3}} r_{z_{3,2;j},w} \times \vec{f}_{2,3;j} \\
&= \vec{0}.
\end{aligned}
$$

A similar argument can be used in the case $n \geq 4$. □

The following result is *Newton's third law for torques*.

Fact 6.9.4. Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be bodies, for all $i = 1, \ldots, n$, assume that the total reaction force $\vec{f}_{\mathcal{B}_i}$ on $\mathcal{B}_i$ is zero, and let $\vec{M}_{\mathcal{B}_i}$ denote the torque on $\mathcal{B}_i$ due to all reaction forces on $\mathcal{B}_i$ arising from direct contact, indirect contact, and noncontacting interaction with the remaining bodies. Then $\sum_{i=1}^{n} \vec{M}_{\mathcal{B}_i} = \vec{0}$. If, in particular, $n = 2$, then $\vec{M}_{\mathcal{B}_2} = -\vec{M}_{\mathcal{B}_1}$.

An example of a reaction torque is the case of a frictionless pin joint connecting two rigid bodies, where the bodies undergo twisting motion that induces a reaction torque that is orthogonal to the axis of the pin. Reaction torques also arise due to rotational springs, rotational dashpots, and rotational inerters. See Section 7.4 and Section 8.11.

6.10 Free-Body Analysis

Consider a quasi-rigid body comprised of multiple interacting rigid bodies. The dynamics of the quasi-rigid body can be analyzed by considering the dynamics of each rigid body separately, that is, as a collection of free (unconstrained) bodies subject to external forces and moments as well as reaction forces and moments. Unlike external forces and moments, the reaction forces and moments must be determined through simultaneous analysis of the statics or dynamics of each rigid body.

If the reaction forces and moments are conservative, then Lagrangian dynamics can be used to derive the equations of motion without determining the reaction forces and moments. The conservative reaction forces and moments can be found subsequently by applying Newton-Euler dynamics. If, however, the reaction forces and torques dissipate energy, then Lagrangian dynamics cannot be used, and Newton-Euler methods must be used exclusively.

For a degenerate rigid body, such as a massless link with a single particle attached to one of its ends, all components of the total force and total torque (arising from either external or reaction forces and torques) that can produce infinite translational or rotational acceleration must be zero. Therefore, in the special case of a massless rigid body, such as a massless rigid link, both the total force and the total torque must be zero. In particular, since springs, dashpots, and inerter are massless, the total force and total torque on these components must be zero. However, the points on the massless rigid body are not necessarily colocated with unforced particles.

6.11 Newtonian Bodies

Let $\mathcal{B}$ be a body. The force on each inertia point in $\mathcal{B}$ is due to a combination of *external forces* and *internal forces*, where the internal forces in $\mathcal{B}$ are due to the reaction forces between inertia points in $\mathcal{B}$. If there are no external forces on $\mathcal{B}$, then an inertia point in $\mathcal{B}$ may still be subject to internal forces. An *internal torque* in $\mathcal{B}$ is a torque on a rigid massless link in $\mathcal{B}$ due to the interactions with another rigid massless link.

The body $\mathcal{B}$ is *Newtonian* if the internal forces between every pair of inertia points in $\mathcal{B}$ are reaction forces, that is, the forces have equal magnitude, are opposite in direction, and are parallel with the line that passes through the particles, and if the internal torques between every pair of rigid massless links are equal in magnitude, are opposite in direction, and are parallel. If the interaction between every pair of inertia points satisfied Newton's third law, then the body is Newtonian.

Figure 6.11.1 shows a rigid body $\mathcal{B}$ composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots,$ m_l, respectively.

Fact 6.11.1. Let $\mathcal{B}$ be a body, and assume that $\mathcal{B}$ is Newtonian. Then, the total force on $\mathcal{B}$ due to all internal forces is zero, that is, the internal forces are balanced, and the total torque on $\mathcal{B}$ due to all internal forces is zero.

Proof. Let $y_1, \ldots, y_l$ denote the particles of $\mathcal{B}$, and, for all $i, j = 1, \ldots, l$, let $\vec{f}_{y_{ij}}$ denote the internal force on particle y_i due to particle y_j. Since $\mathcal{B}$ is Newtonian, it follows that, for all $i =$ $1, \ldots, j$, $\vec{f}_{y_{ij}} = -\vec{f}_{y_{ji}}$ and $\vec{f}_{y_{ii}} = 0$. Consequently, the total force $\vec{f}_{\mathcal{B},\mathrm{int}}$ on $\mathcal{B}$ due to internal forces is

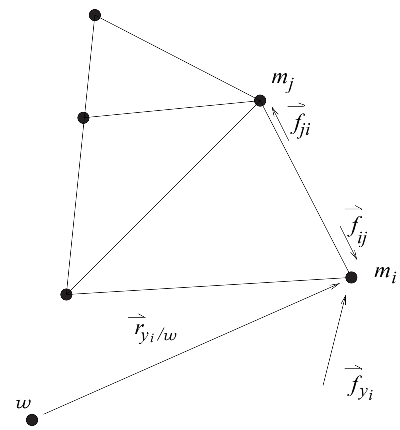


Figure 6.11.1: Newtonian rigid body $\mathcal{B}$, where the internal forces between each pair of particles are equal in magnitude, opposite in direction, and parallel with the line that passes through the particles.

given by

$$\overrightarrow{f}_{\mathcal{B},\mathrm{int}} = \sum_{\substack{i,j=1,\ldots,l\\ i\neq j}} \overrightarrow{f}_{ij} = 0.$$

Next, let w be a point. Since $\overrightarrow{f}_{\mathcal{B},\mathrm{int}} = 0$, it follows that the resulting moment $\overrightarrow{M}_{\mathcal{B},\mathrm{int}}$ on $\mathcal{B}$ due to internal forces is independent of w. Since $\overrightarrow{f}_{ij}$ is parallel with $\overrightarrow{r}_{y_i/y_j}$, it follows that

$$\begin{aligned}
\overrightarrow{M}_{\mathcal{B},\mathrm{int}} &= \sum_{\substack{i,j=1,\ldots,l\\ i\neq j}} \overrightarrow{r}_{y_i/w} \times \overrightarrow{f}_{ij}\\
&= \sum_{\substack{i,j=1,\ldots,l\\ i<j}} \overrightarrow{r}_{y_i/w} \times \overrightarrow{f}_{ij} + \sum_{\substack{i,j=1,\ldots,l\\ i>j}} \overrightarrow{r}_{y_i/w} \times \overrightarrow{f}_{ij}\\
&= \sum_{\substack{i,j=1,\ldots,l\\ i<j}} \overrightarrow{r}_{y_i/w} \times \overrightarrow{f}_{ij} + \sum_{\substack{i,j=1,\ldots,l\\ i<j}} \overrightarrow{r}_{y_j/w} \times \overrightarrow{f}_{ji}\\
&= \sum_{\substack{i,j=1,\ldots,l\\ i<j}} \overrightarrow{r}_{y_i/w} \times \overrightarrow{f}_{ij} - \sum_{\substack{i,j=1,\ldots,l\\ i<j}} \overrightarrow{r}_{y_j/w} \times \overrightarrow{f}_{ij}\\
&= \sum_{\substack{i,j=1,\ldots,l\\ i<j}} \left(\overrightarrow{r}_{y_i/w} - \overrightarrow{r}_{y_j/w}\right) \times \overrightarrow{f}_{ij}\\
&= \sum_{\substack{i,j=1,\ldots,l\\ i<j}} \overrightarrow{r}_{y_i/y_j} \times \overrightarrow{f}_{ij}\\
&= 0.
\end{aligned}$$

□

6.12 Center of Gravity and Central Gravity

In this section we do not assume that gravity is uniform, but rather consider central gravity. In this case, we define the center of gravity, which may be different from the center of mass. We also consider conditions under which the center of gravity coincides with the center of mass.

The following result is *Newton's law of universal gravitation*.

Fact 6.12.1. Let x be a particle whose mass is M, and let y be a particle whose mass is m. Then, the force on y due to x is given by

$$\vec{f}_{y/x} = \frac{GMm}{|\vec{r}_{y/x}|^2}\hat{r}_{x/y}, \tag{6.12.1}$$

where the universal gravitational constant G is given by

$$G = 6.67428 \text{ N-m}^2/\text{kg}^2. \tag{6.12.2}$$

Let x and y be particles whose masses are M and m, respectively. Then, the force $\vec{f}_{y/x}$ is the *central gravitational force* on y due to x, and the *weight* of y relative to x is given by

$$w_{y/x} \triangleq |\vec{f}_{y/x}| = \frac{GMm}{|\vec{r}_{y/x}|^2}. \tag{6.12.3}$$

Now, let $\mathcal{B}_1$ be a body consisting of particles $x_1, \ldots, x_m$ whose masses are $M_1, \ldots, M_j$, respectively. Then, the central gravitational force on y due to $\mathcal{B}_1$ is defined by

$$\vec{f}_{y/\mathcal{B}_1} \triangleq \sum_{j=1}^{m} \vec{f}_{y/x_j} = \sum_{j=1}^{m} \frac{GM_jm}{|\vec{r}_{y/x_j}|^2}\hat{r}_{y/x_j}, \tag{6.12.4}$$

and the *weight* of y relative to $\mathcal{B}_1$ is defined by

$$w_{y/\mathcal{B}_1} \triangleq \sum_{j=1}^{m} w_{y/x_j} = \sum_{j=1}^{m} |\vec{f}_{y/x_j}| = \sum_{j=1}^{m} \frac{GM_jm}{|\vec{r}_{y/x_j}|^2}. \tag{6.12.5}$$

Finally, let $\mathcal{B}_2$ be a body consisting of particles $y_1, \ldots, y_l$. Then, the central gravitational force on $\mathcal{B}_2$ due to $\mathcal{B}_1$ is defined by

$$\vec{f}_{\mathcal{B}_2/\mathcal{B}_1} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i/\mathcal{B}_1} = \sum_{i=1}^{l}\sum_{j=1}^{m} \vec{f}_{y_i/x_j} = \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{GM_jm_i}{|\vec{r}_{y_i/x_j}|^2}\hat{r}_{y_i/x_j}, \tag{6.12.6}$$

and the *weight* of $\mathcal{B}_2$ relative to $\mathcal{B}_1$ is defined by

$$w_{\mathcal{B}_2/\mathcal{B}_1} \triangleq \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} = \sum_{i=1}^{l}\sum_{j=1}^{m} w_{y_i/x_j} = \sum_{i=1}^{l}\sum_{j=1}^{m} |\vec{f}_{y_i/x_j}| = \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{GM_jm_i}{|\vec{r}_{y_i/x_j}|^2}. \tag{6.12.7}$$

Fact 6.12.2. Let $\mathcal{B}_1$ be a body, let $\mathcal{B}_2$ be a body consisting of particles $y_1, \ldots, y_l$, let w and w' be points, and define the points g and g′ by

$$\vec{r}_{\mathrm{g}/w} \triangleq \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w}, \tag{6.12.8}$$

$$\vec{r}_{\mathrm{g}'/w'} \triangleq \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w'}. \tag{6.12.9}$$

Then, g and g′ are colocated.

Proof. Note that

$$\begin{aligned}
\vec{r}_{\mathrm{g}'/\mathrm{g}} &= \vec{r}_{\mathrm{g}'/w'} + \vec{r}_{w'/w} + \vec{r}_{w/\mathrm{g}} = \vec{r}_{\mathrm{g}'/w'} - \vec{r}_{\mathrm{g}/w} + \vec{r}_{w'/w} \\
&= \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w'} - \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w} + \vec{r}_{w'/w} \\
&= \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} (\vec{r}_{y_i/w'} - \vec{r}_{y_i/w}) + \vec{r}_{w'/w} \\
&= \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} (\vec{r}_{y_i/w'} + \vec{r}_{w/y_i}) + \vec{r}_{w'/w} \\
&= \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{w/w'} + \vec{r}_{w'/w} \\
&= \left(\frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \right) \vec{r}_{w/w'} + \vec{r}_{w'/w} = \vec{r}_{w/w'} + \vec{r}_{w'/w} = 0.
\end{aligned}$$

□

Fact 6.12.2 shows that the point g is uniquely defined irrespective of the reference point w.

The following definition is analogous to Definition 6.1.2.

Definition 6.12.3. Let $\mathcal{B}_1$ be a body, let $\mathcal{B}_2$ be a body composed of particles $y_1, \ldots, y_l$, and let w be a point. Then, the *center of gravity* $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}$ of $\mathcal{B}_2$ relative to $\mathcal{B}_1$ is defined by

$$\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} \triangleq \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w}. \tag{6.12.10}$$

The following result is analogous to Fact 6.1.3.

Fact 6.12.4. Let $\mathcal{B}_1$ be a body, let $\mathcal{B}_2$ be a body composed of particles $y_1, \ldots, y_l$, and let w be a point. Then, $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}$ satisfies

$$\sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} = 0. \tag{6.12.11}$$

Proof. It follows from (6.12.10) that

$$\begin{aligned}
\frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} &= \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} (\vec{r}_{y_i/w} + \vec{r}_{w/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}}) \\
&= \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w} + \frac{1}{w_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} w_{y_i/\mathcal{B}_1} \vec{r}_{w/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} \\
&= \vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} + \vec{r}_{w/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} = \vec{r}_{w/w} = 0.
\end{aligned}$$

□

The following result gives conditions under which g is independent of G and M and under which $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}$ is colocated with the center of mass of $\mathcal{B}_2$.

Fact 6.12.5. Let $\mathcal{B}_1$ be a body with particles $x_1,\ldots,x_m$ whose masses are all equal to M, let $\mathcal{B}_2$ be a body composed of particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let w be a point, and let c_2 denote the center of mass of $\mathcal{B}_2$. Then,

$$w_{\mathcal{B}_2/\mathcal{B}_1} = GM\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{|\vec{r}_{y_i/x_j}|^2}, \tag{6.12.12}$$

$$\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} = \frac{1}{\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{|\vec{r}_{y_i/x_j}|^2}}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{|\vec{r}_{y_i/x_j}|^2}\vec{r}_{y_i/w}. \tag{6.12.13}$$

Now, assume that $|\vec{r}_{y_i/x_j}|$ is independent of i and j. Then, $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1} = \mathrm{c}_2$.

Proof. To prove the second statement, define $\alpha \triangleq |\vec{r}_{y_i/x_j}|^2$. Then,

$$\begin{aligned}\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} &= \frac{1}{\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{\alpha}}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{\alpha}\vec{r}_{y_i/w}\\ &= \frac{1}{\sum_{i=1}^{l}\frac{mm_i}{\alpha}}\sum_{i=1}^{l}\frac{mm_i}{\alpha}\vec{r}_{y_i/w} = \frac{1}{\sum_{i=1}^{l}m_i}\sum_{i=1}^{l}m_i\vec{r}_{y_i/w}\\ &= \frac{1}{m_{\mathcal{B}_2}}\sum_{i=1}^{l}m_i\vec{r}_{y_i/w} = \vec{r}_{\mathrm{c}_2/w}.\end{aligned}$$

□

The following result shows that, if the distance between two bodies is large, then the center of gravity of each body is approximately colocated with its center of mass.

Fact 6.12.6. Let $\mathcal{B}_1$ be a body with particles $x_1,\ldots,x_m$ whose masses are $M_1,\ldots,M_m$, respectively, let $\mathcal{B}_2$ be a body composed of particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, and let c_2 denote the center of mass of $\mathcal{B}_2$. Furthermore, let γ be a parameter such that, for all $i = 1,\ldots,l$ and for all $j = 1,\ldots,m$, it follows that $|\vec{r}_{y_i/x_j}| \to \infty$ as $\gamma \to \infty$ and, for all $i_1, i_2 = 1,\ldots,l$ and for all $j_1, j_2 = 1,\ldots,m$, it follows that $\frac{|\vec{r}_{y_{i_1}/x_{j_1}}|}{|\vec{r}_{y_{i_2}/x_{j_2}}|} \to 1$ as $\gamma\to\infty$. Then,

$$\lim_{\gamma\to\infty}\vec{r}_{\mathrm{c}_2/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} = 0. \tag{6.12.14}$$

Proof. For convenience, define $\alpha_{ij} \triangleq |\vec{r}_{y_i/x_j}|^2$. Then,

$$\begin{aligned}\lim_{\gamma\to\infty}\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} &= \lim_{\gamma\to\infty}\frac{1}{\sum_{i'=1}^{l}\sum_{j'=1}^{m}\frac{M_{j'}m_{i'}}{\alpha_{i'j'}}}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{M_jm_i}{\alpha_{ij}}\vec{r}_{y_i/w}\\ &= \lim_{\gamma\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{M_jm_i}{\alpha_{ij}\sum_{i'=1}^{l}\sum_{j'=1}^{m}\frac{M_{j'}m_{i'}}{\alpha_{i'j'}}}\vec{r}_{y_i/w}\end{aligned}$$

$$= \lim_{\gamma\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{M_j m_i}{\sum_{i'=1}^{l}\sum_{j'=1}^{m} \frac{\alpha_{ij}M_{j'}m_{i'}}{\alpha_{i'j'}}} \vec{r}_{y_i/w}$$

$$= \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{M_j m_i}{\sum_{i'=1}^{l}\sum_{j'=1}^{m} M_{j'}m_{i'}} \vec{r}_{y_i/w}$$

$$= \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{M_j m_i}{m_{\mathcal{B}_1}m_{\mathcal{B}_2}} \vec{r}_{y_i/w} = \sum_{i=1}^{l} \frac{m_i}{m_{\mathcal{B}_2}} \vec{r}_{y_i/w} = \vec{r}_{\mathrm{c}_2/w}. \qquad \square$$

6.13 Newton's Third Law for Magnetic Forces and Torques

Let $\vec{m}_y$ and $\vec{m}_z$ be magnetic dipole moments located at the distinct points y and z, respectively. Let $\vec{r}_{z/y}$ denote the position of z relative to y. At the point z, the magnetic field $\vec{B}_y$ due to the magnetic dipole $\vec{m}_y$ is given by [6, p. 186]

$$\vec{B}_y(\vec{r}_{z/y}) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}_{z/y}\cdot\vec{m}_y}{|\vec{r}_{z/y}|^5}\vec{r}_{z/y} - \frac{1}{|\vec{r}_{z/y}|^3}\vec{m}_y\right), \tag{6.13.1}$$

where μ_0 is the permeability of free space. For convenience, we adopt Gaussian units such that $\mu_0 = 4\pi$.

According to Maxwell's equations, the field $\vec{B}_y$ due to the dipole $\vec{m}_y$ is divergence free. To show this for (6.13.1), note that the Jacobian of $\vec{B}_y$ at $\vec{r}_{z/y}$ is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{B}_y(\vec{r})\bigg|_{\vec{r}=\vec{r}_{y/z}} &= \frac{\mathrm{d}}{\mathrm{d}\vec{r}}\left(\frac{3}{|\vec{r}|^5}\vec{m}_y{}'\vec{r}\,\vec{r} - \frac{1}{|\vec{r}|^3}\vec{m}_y\right)\bigg|_{\vec{r}=\vec{r}_{y/z}} \\
&= 3\vec{m}_y{}'\vec{r}_{y/z}\vec{r}_{y/z}\left(\vec{\partial}_{\vec{r}}\frac{1}{|\vec{r}|^5}\right)\bigg|_{\vec{r}=\vec{r}_{y/z}} + \frac{3}{|\vec{r}_{y/z}|^5}\vec{r}_{y/z}\,(\vec{\partial}_{\vec{r}}\vec{m}_y{}'\vec{r})\Big|_{\vec{r}=\vec{r}_{y/z}} \\
&\quad + \frac{3}{|\vec{r}_{y/z}|^5}\vec{m}_y{}'\vec{r}_{y/z}\left(\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{r}\right)\bigg|_{\vec{r}=\vec{r}_{y/z}} - \vec{m}_y\left(\vec{\partial}_{\vec{r}}\frac{1}{|\vec{r}|^3}\right)\bigg|_{\vec{r}=\vec{r}_{y/z}} \\
&= 3\vec{m}_y{}'\vec{r}_{z/y}\vec{r}_{z/y}\left(\frac{-5}{|\vec{r}_{z/y}|^7}\vec{r}_{z/y}{}'\right) + \frac{3}{|\vec{r}_{z/y}|^5}\vec{r}_{z/y}\vec{m}_y{}' \\
&\quad + \frac{3}{|\vec{r}_{z/y}|^5}\vec{m}_y{}'\vec{r}_{z/y}\vec{I} - \vec{m}_y\left(\frac{-3}{|\vec{r}_{z/y}|^5}\vec{r}_{z/y}{}'\right) \\
&= \frac{3}{|\vec{r}_{z/y}|^5}\left[\vec{m}_y\vec{r}_{z/y}{}' + \vec{r}_{z/y}\vec{m}_y{}' + \vec{m}_y{}'\vec{r}_{z/y}\left(\vec{I} - \frac{5}{|\vec{r}_{z/y}|^2}\vec{r}_{z/y}\vec{r}_{z/y}{}'\right)\right].
\end{aligned} \tag{6.13.2}$$

Consequently, the divergence of $\vec{B}_y$ is given by

$$\vec{\nabla}_{\vec{r}}\cdot\vec{B}_y(\vec{r}) = \operatorname{tr}\vec{\partial}_{\vec{r}}\vec{B}_y(\vec{r}) = \operatorname{tr}\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{B}_y(\vec{r}) = 0. \tag{6.13.3}$$

Since $\vec{B}_y$ is divergence free, it can be written as the curl of a vector potential, namely,

$$\vec{A}_y(\vec{r}) = \frac{1}{|\vec{r}|^3}\vec{m}_y \times \vec{r}. \tag{6.13.4}$$

To confirm this, note first that

$$\vec{\nabla}_{\vec{r}} \cdot \left(\frac{1}{|\vec{r}|^3}\vec{r}\right) = 0. \tag{6.13.5}$$

It thus follows that

$$\begin{aligned}
\vec{\nabla}_{\vec{r}} \times \vec{A}_y(\vec{r}) &= \vec{\nabla}_{\vec{r}} \times \left(\vec{m}_y \times \frac{1}{|\vec{r}|^3}\vec{r}\right) \\
&= \vec{\nabla}_{\vec{r}} \cdot \left(\frac{1}{|\vec{r}|^3}\vec{r}\right)\vec{m}_y - (\vec{m}_y'\vec{\nabla}_{\vec{r}})\frac{1}{|\vec{r}|^3}\vec{r} \\
&= -\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\left(\frac{1}{|\vec{r}|^3}\vec{r}\right)\vec{m}_y \\
&= \left(\frac{3}{|\vec{r}|^5}\vec{r}\,\vec{r}' - \frac{1}{|\vec{r}|^3}\vec{I}\right)\vec{m}_y \\
&= \frac{3\vec{r}'\vec{m}_y}{|\vec{r}|^5}\vec{r} - \frac{1}{|\vec{r}|^3}\vec{m}_y \\
&= \vec{B}_y(\vec{r}).
\end{aligned} \tag{6.13.6}$$

Note that (6.13.1) can be rewritten as

$$\vec{B}_y(\vec{r}_{z/y}) = \vec{J}(\vec{r}_{z/y})\vec{m}_y, \tag{6.13.7}$$

where the second-order tensor $\vec{J}(\vec{r}_{z/y})$ is defined by

$$\vec{J}(\vec{r}_{z/y}) \triangleq \frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}_{z/y}\hat{r}_{z/y}' - \vec{I}). \tag{6.13.8}$$

Note that $\operatorname{tr}\vec{J}(\vec{r}_{z/y}) = 0$, which reflects the divergence-free condition. Note that $\vec{J}(\vec{r}_{y/z}) = \vec{J}(\vec{r}_{z/y}) = \vec{J}'(\vec{r}_{y/z})$.

The force $\vec{f}_{z/y}$ on the magnetic dipole $\vec{m}_z$ due to the magnetic dipole $\vec{m}_y$ is given by [6, p. 189]

$$\vec{f}_{z/y} = \vec{\nabla}_{\vec{r}}(\vec{m}_z'\vec{B}_y(\vec{r}))\Big|_{\vec{r}=\vec{r}_{y/z}}. \tag{6.13.9}$$

Likewise, the torque $\vec{\tau}_{z/y}$ on the magnetic dipole $\vec{m}_z$ due to the magnetic dipole $\vec{m}_y$ is given by [6, p. 190]

$$\vec{\tau}_{z/y} = \vec{m}_z \times \vec{B}_y(\vec{r}_{z/y}). \tag{6.13.10}$$

6.13.1 Newton's Third Law for Magnetic Forces

The following result provides an expression for $\vec{f}_{z/y}$ and states Newton's third law for forces arising from a pair of magnetic dipoles.

Fact 6.13.1. $\vec{f}_{z/y}$ is given by

$$\vec{f}_{z/y} = \frac{3}{|\vec{r}_{z/y}|^4}(\vec{m}_y'\hat{r}_{z/y}\vec{m}_z + \vec{m}_z'\hat{r}_{z/y}\vec{m}_y + \vec{m}_z'\vec{m}_y\hat{r}_{z/y} - 5\vec{m}_y'\hat{r}_{z/y}\vec{m}_z'\hat{r}_{z/y}\hat{r}_{z/y}). \tag{6.13.11}$$

Consequently, $\vec{f}_{z/y}$ and $\vec{f}_{y/z}$ satisfy

$$\vec{f}_{z/y} = -\vec{f}_{y/z}. \tag{6.13.12}$$

Proof. Using (2.24.11) it follows that, for all vectors $\vec{w}$ and $\vec{v}$,

$$\begin{aligned}
\vec{\partial}_{\vec{r}}(\vec{w}'\vec{J}(\vec{r})\vec{v})\Big|_{\vec{r}=\vec{r}_{y/z}} &= \vec{\partial}_{\vec{r}}\left(\frac{3}{|\vec{r}|^5}\vec{w}'\vec{r}\vec{v}'\vec{r} - \frac{1}{|\vec{r}|^3}\vec{w}'\vec{v}\right)\Bigg|_{\vec{r}=\vec{r}_{y/z}}\\
&= -\frac{15}{|\vec{r}_{z/y}|^7}\vec{w}'\vec{r}_{z/y}\vec{v}'\vec{r}_{z/y}\vec{r}_{z/y}' + \frac{3}{|\vec{r}_{z/y}|^5}\vec{v}'\vec{r}_{z/y}\vec{w}'\\
&\quad + \frac{3}{|\vec{r}_{z/y}|^5}\vec{w}'\vec{r}_{z/y}\vec{v}' + \frac{3}{|\vec{r}_{z/y}|^5}\vec{w}'\vec{v}\vec{r}_{z/y}'\\
&= \frac{3}{|\vec{r}_{z/y}|^4}(-5\vec{w}'\hat{r}_{z/y}\vec{v}'\hat{r}_{z/y}\hat{r}_{z/y}' + \vec{v}'\hat{r}_{z/y}\vec{w}' + \vec{w}'\hat{r}_{z/y}\vec{v}' + \vec{w}'\vec{v}\hat{r}_{z/y}').
\end{aligned} \tag{6.13.13}$$

Finally, setting $\vec{v} = \vec{m}_y$ and $\vec{w} = \vec{m}_z$ yields

$$\begin{aligned}
\vec{f}_{z/y} &= \vec{\nabla}_{\vec{r}}(\vec{m}_z'\vec{J}(\vec{r})\vec{m}_y))\Big|_{\vec{r}=\vec{r}_{y/z}}\\
&= \frac{3}{|\vec{r}_{z/y}|^4}(\vec{m}_y'\hat{r}_{z/y}\vec{m}_z + \vec{m}_z'\hat{r}_{z/y}\vec{m}_y + \vec{m}_z'\vec{m}_y\hat{r}_{z/y} - 5\vec{m}_y'\hat{r}_{z/y}\vec{m}_z'\hat{r}_{z/y}\hat{r}_{z/y})\\
&= -\vec{f}_{y/z}.
\end{aligned}$$

□

It can be seen from (6.13.11) that the direction of the magnetic force $\vec{f}_{z/y}$ on the dipole $\vec{m}_z$ due to $\vec{m}_y$ is not necessarily aligned with $\hat{r}_{z/y}$. This means that Newton's third law for magnetic forces does not share the alignment property that holds for mechanical forces, gravity, and electrostatics. As shown below, this misalignment accounts for an additional contribution to the torque on each magnetic dipole.

Next, note that the directional derivative of the magnetic field vector $\vec{B}_y$ in the direction of the magnetic dipole $\vec{m}_z$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\vec{B}_y(\vec{r}_{y/z} + \alpha\vec{m}_z)\Big|_{\alpha=0} = \frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{B}_y(\vec{r})\Big|_{\vec{r}=\vec{r}_{y/z}}\vec{m}_z = \vec{f}_{z/y}. \tag{6.13.14}$$

This shows that the misalignment between $\vec{f}_{z/y}$ and $\hat{r}_{z/y}$ arises from the fact that the force $\vec{f}_{z/y}$ on $\vec{m}_z$ due to $\vec{m}_y$ is parallel to the directional derivative of the magnetic field vector $\vec{B}_y$ in the direction of the magnetic dipole $\vec{m}_z$.

Example 6.13.2. Consider the dipoles $\vec{m}_y = m_y\hat{\imath}_{\rm A}$ and $\vec{m}_z = m_z\hat{\jmath}_{\rm A}$, where $\vec{r}_{z/y} = r\hat{\imath}_{\rm A}$. Then

$$\vec{J}(\vec{r}_{z/y}) = \frac{1}{r^3}(3\hat{\imath}_{\rm A}\hat{\imath}'_{\rm A} - \vec{I}) = \frac{1}{r^3}(2\hat{\imath}_{\rm A}\hat{\imath}'_{\rm A} - \hat{\jmath}_{\rm A}\hat{\jmath}'_{\rm A} - \hat{k}_{\rm A}\hat{k}'_{\rm A}), \tag{6.13.15}$$

and thus

$$\vec{B}_y(\vec{r}_{z/y}) = \frac{2m_y}{r^3}\hat{\imath}_{\rm A}. \tag{6.13.16}$$

It follows from (6.13.11) that

$$\vec{f}_{z/y} = \frac{3m_y m_z}{r^4}\hat{\jmath}_{\rm A}. \tag{6.13.17}$$

Alternatively, to obtain (6.13.17) directly without using (6.13.11), note that it follows from (6.13.7) and (6.13.9) that

$$\begin{aligned}
\vec{f}'_{z/y} &= \vec{\partial}_{\vec{r}}(\vec{m}'_z\vec{J}(\vec{r})\vec{m}_y)\Big|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= m_z m_y\,\vec{\partial}_{\vec{r}}\left(\frac{1}{|\vec{r}|^3}\hat{\jmath}'_{\rm A}(3\hat{r}\hat{r}' - \vec{I})\hat{\imath}_{\rm A}\right)\Bigg|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= 3m_z m_y\,\vec{\partial}_{\vec{r}}\left(\frac{1}{|\vec{r}|^5}\hat{\jmath}'_{\rm A}\vec{r}\hat{\imath}'_{\rm A}\vec{r}\right)\Bigg|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= 3m_z m_y\left(\frac{-5}{|\vec{r}|^7}\hat{\jmath}'_{\rm A}\vec{r}\hat{\imath}'_{\rm A}\vec{r}\vec{r}' + \frac{1}{|\vec{r}|^5}\hat{\imath}'_{\rm A}\vec{r}\hat{\jmath}'_{\rm A} + \frac{1}{|\vec{r}|^5}\hat{\jmath}'_{\rm A}\vec{r}\hat{\imath}'_{\rm A}\right)\Bigg|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= \frac{3m_z m_y}{r^4}\hat{\jmath}'_{\rm A}.
\end{aligned}$$

◇

The following result, which is an immediate consequence of Fact 6.13.1, shows that the net force arising from an arbitrary collection of magnetic dipoles is zero.

Fact 6.13.3. Suppose that a body consists of magnetic dipoles $\vec{m}_1, \ldots, \vec{m}_n$. Then the net magnetic force on the body is zero.

6.13.2 Newton's Third Law for Magnetic Torques

The following result provides an expression for $\vec{\tau}_{z/y}$ and states Newton's third law for torques arising from a pair of magnetic dipoles.

Fact 6.13.4. $\vec{\tau}_{z/y}$ is given by

$$\vec{\tau}_{z/y} = \frac{1}{|\vec{r}_{z/y}|^3}[3\hat{r}'_{z/y}\vec{m}_y(\vec{m}_z \times \hat{r}_{z/y}) - \vec{m}_z \times \vec{m}_y]. \tag{6.13.18}$$

Furthermore,

$$\vec{\tau}_{z/y} + \vec{\tau}_{y/z} + \vec{r}_{z/y} \times \vec{f}_{z/y} = 0, \tag{6.13.19}$$

where

$$\vec{r}_{z/y} \times \vec{f}_{z/y} = \frac{3}{|\vec{r}_{z/y}|^5}[\vec{r}\,'_{z/y}\vec{m}_y(\vec{r}_{z/y} \times \vec{m}_z) + \vec{r}\,'_{y/z}\vec{m}_z(\vec{r}_{y/z} \times \vec{m}_y)]. \tag{6.13.20}$$

Proof. It follows from (6.13.7), (6.13.8), and (6.13.10) that

$$\begin{aligned}
\vec{\tau}_{z/y} &= \vec{m}_z \times \vec{B}_y(\vec{r}_{z/y}) \\
&= \vec{m}_z \times \vec{J}(\vec{r}_{z/y})\vec{m}_y \\
&= \vec{m}_z \times \left(\frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}_{z/y}\hat{r}'_{z/y} - \vec{I})\right)\vec{m}_y \\
&= \frac{1}{|\vec{r}_{z/y}|^3}[3\hat{r}'_{z/y}\vec{m}_y(\vec{m}_z \times \hat{r}_{z/y}) - \vec{m}_z \times \vec{m}_y].
\end{aligned} \tag{6.13.21}$$

It thus follows from (6.13.11) and (6.13.21) that

$$\begin{aligned}
\vec{r}_{z/y} \times \vec{f}_{z/y} &= \frac{3}{|\vec{r}_{z/y}|^4}[\vec{m}'_y\hat{r}_{z/y}(\vec{r}_{z/y} \times \vec{m}_z) + \vec{m}'_z\hat{r}_{z/y}(\vec{r}_{z/y} \times \vec{m}_y)] \\
&= -\frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}'_{z/y}\vec{m}_y(\vec{m}_z \times \hat{r}_{z/y}) - \vec{m}_z \times \vec{m}_y) \\
&\quad - \frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}'_{y/z}\vec{m}_z(\vec{m}_y \times \hat{r}_{y/z}) - \vec{m}_y \times \vec{m}_z) \\
&= -(\vec{\tau}_{z/y} + \vec{\tau}_{y/z}).
\end{aligned}$$

□

Equation (6.13.19) involves the torque $\vec{\tau}_{z/y}$, which is applied to $\vec{m}_z$ due to the magnetic field generated by $\vec{m}_y$, and the torque $\vec{\tau}_{y/z}$, which is applied to $\vec{m}_y$ due to the magnetic field generated by $\vec{m}_z$. Furthermore, note that, since $f_{y/z} = -f_{z/y}$ and, as noted above, $f_{z/y}$ and $f_{y/z}$ are not aligned with $\vec{r}_{z/y}$, these forces create an additional torque $\tau_{\rm f}$; this torque can be computed relative to an arbitrary point. Choosing this point to be y, it follows that

$$\vec{\tau}_{\rm f} = \vec{r}_{z/y} \times \vec{f}_{z/y} + \vec{r}_{y/y} \times \vec{f}_{y/z} = \vec{r}_{z/y} \times \vec{f}_{z/y}. \tag{6.13.22}$$

Consequently, $\vec{r}_{z/y} \times \vec{f}_{z/y}$ in (6.13.19) is the torque due to $f_{z/y}$ and $f_{y/z}$.

The following result provides a symmetric version of Fact 6.13.4, where the torque $\vec{\tau}_{\rm f}$ is evaluated relative to an arbitrary point x.

Fact 6.13.5. Let x be a point. Then,

$$\vec{\tau}_{z/y} + \vec{r}_{z/x} \times \vec{f}_{z/y} = -(\vec{\tau}_{y/z} + \vec{r}_{y/x} \times \vec{f}_{y/z}). \tag{6.13.23}$$

Proof. Using $\vec{r}_{z/y} = \vec{r}_{z/x} + \vec{r}_{x/y}$, (6.13.19) implies

$$\begin{aligned}\vec{\tau}_{z/y} + \vec{\tau}_{y/z} &= -\vec{r}_{z/y} \times \vec{f}_{z/y} \\ &= -(\vec{r}_{z/x} + \vec{r}_{x/y}) \times \vec{f}_{z/y} \\ &= -\vec{r}_{z/x} \times \vec{f}_{z/y} - \vec{r}_{x/y} \times \vec{f}_{z/y} \\ &= -\vec{r}_{z/x} \times \vec{f}_{z/y} - \vec{r}_{y/x} \times \vec{f}_{y/z}.\end{aligned}$$ □

Setting $x = y$ in (6.13.23) recovers (6.13.19).

The following result, which is an immediate consequence of Fact 6.13.4, shows that the net torque arising from an arbitrary collection of magnetic dipoles is zero.

Fact 6.13.6. Suppose that a body consists of magnetic dipoles $\vec{m}_1, \ldots, \vec{m}_n$. Then the net magnetic torque on the body is zero.

Example 6.13.7. (Example 6.13.2 continued.) Using (6.13.18) it follows that

$$\vec{\tau}_{z/y} = \frac{1}{r^3}[3\imath'_{\mathrm{A}} m_y \hat{\imath}_{\mathrm{A}}(m_z \hat{\jmath}_{\mathrm{A}} \times \hat{\imath}_{\mathrm{A}}) - m_z \hat{\jmath}_{\mathrm{A}} \times m_y \hat{\imath}_{\mathrm{A}}] = -\frac{2m_y m_z}{r^3}\hat{k}_{\mathrm{A}}, \tag{6.13.24}$$

$$\vec{\tau}_{y/z} = \frac{1}{r^3}[-3\imath'_{\mathrm{A}} m_z \hat{\jmath}_{\mathrm{A}}(m_y \hat{\imath}_{\mathrm{A}} \times -\hat{\imath}_{\mathrm{A}}) - m_y \hat{\imath}_{\mathrm{A}} \times m_z \hat{\jmath}_{\mathrm{A}}] = -\frac{m_y m_z}{r^3}\hat{k}_{\mathrm{A}}, \tag{6.13.25}$$

$$\vec{\tau}_{\mathrm{f}} = r\hat{\imath}_{\mathrm{A}} \times \frac{3m_y m_z}{r^4}\hat{\jmath}_{\mathrm{A}} = \frac{3m_y m_z}{r^3}\hat{k}_{\mathrm{A}}. \tag{6.13.26}$$

Summing $\vec{\tau}_{z/y}$, $\vec{\tau}_{y/z}$, and $\vec{\tau}_{\mathrm{f}}$ verifies (6.13.19). ◇

Analogous results hold for electric dipoles.

6.14 Theoretical Problems

Problem 6.14.1. Let $\mathcal{B}$ be a rigid body consisting of particles $y_1, \ldots, y_l$ with masses $m_1, \ldots, m_l$, respectively, and let $\mathrm{F_B}$ be a body-fixed frame. Show that, for all $i = 1, \ldots, l$, $\vec{v}_{y_i/\mathrm{c/B}} = 0$.

Problem 6.14.2. Consider a triangle $\mathcal{T}$ with vertices a, b, c, and define the following bodies:

i) $\mathcal{B}_1$ consists of three identical particles located at a, b, c.

ii) $\mathcal{B}_2$ consists of three thin homogeneous rigid links connecting a, b, c.

iii) $\mathcal{B}_3$ is a thin homogeneous triangular-shaped plate with vertices a, b, c.

Show that all three bodies have the same center of mass, which is located at the centroid of $\mathcal{T}$.

Problem 6.14.3. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, and let $\mathcal{B}_3$ denote the union of $\mathcal{B}_1$ and $\mathcal{B}_2$, that is, $\mathcal{B}_3$ is the body whose particles include all of the particles of $\mathcal{B}_1$ and $\mathcal{B}_2$. Let w be a point. Show that the center of mass of $\mathcal{B}_3$ relative to w lies on the line segment connecting the centers of mass of $\mathcal{B}_1$ and $\mathcal{B}_2$ relative to w. In particular, show that the location of the center of mass of $\mathcal{B}_3$ relative to w coincides with the center of mass of two "virtual" particles, namely, a particle y_1 located at the center of mass of $\mathcal{B}_1$ and whose mass is m_1, and a particle y_2 located at the center of mass of $\mathcal{B}_2$ and whose mass is m_2.

Problem 6.14.4. Consider a particle y whose mass is m and a body $\mathcal{B}$ that consists of two particles y_1 and y_2 with masses m_1 and m_2, respectively, connected by a massless rigid link of length ℓ. The distance from y to y_1 is ℓ_1, and the distance from y to y_2 is ℓ_2, where $\ell_1^2 + \ell_2^2 = \ell^2$. Aside from the constraint forces on y_1 and y_2 due to the link, all forces on y, y_1, and y_2 are due to central gravitational attraction. Show that Newton's third law holds in the sense that the force $\vec{f}_y$ on y due to $\mathcal{B}$ is equal and opposite in direction to the total force on $\mathcal{B}$ due to y, but that the force $\vec{f}_y$ on y is not necessarily aligned with either the center of mass of $\mathcal{B}$ or the center of gravity of $\mathcal{B}$. In particular, show that the following statements are equivalent:

i) $\ell_1 = \ell_2$.

ii) The center of mass of $\mathcal{B}$ coincides with the center of gravity of $\mathcal{B}$.

iii) The force $\vec{f}_y$ on y is aligned with the center of mass of $\mathcal{B}$.

iv) The force $\vec{f}_y$ on y is aligned with the center of gravity of $\mathcal{B}$.

Problem 6.14.5. Let $\mathcal{B}$ be a body subject to at least three forces that do not lie in a single plane, let z and w be distinct points, and assume that the moments $\vec{M}_{\mathcal{B}/w}$ and $\vec{M}_{\mathcal{B}/z}$ due to the forces are equal and nonzero. Does it follow that the forces are balanced? (Note that the converse is true due to the fact that, if the forces are balanced, then $\vec{M}_{\mathcal{B}/w}$ and $\vec{M}_{\mathcal{B}/z}$ are independent of w and z and thus are equal.)

Problem 6.14.6. Let $\mathcal{B}$ be a body, let z be a point, and let $\mathrm{F_B}$ be a frame. Show that $J_{xx} = \hat{\imath}_{\mathrm{B}}' \vec{J}_{\mathcal{B}/z} \hat{\imath}_{\mathrm{B}}$, and provide similar expressions for the remaining components of $\vec{J}_{\mathcal{B}/z}\Big|_{\mathrm{B}}$. Furthermore, let $\hat{n}$ be a unit vector. Under what conditions is $\hat{n}' \vec{J}_{\mathcal{B}/z} \hat{n}$ a moment of inertia of $\mathcal{B}$? Show that $\hat{n}' \vec{J}_{\mathcal{B}/z} \hat{n}$ is a principal moment of inertia of $\mathcal{B}$ if and only if $\hat{n}$ is an eigenvector of $\vec{J}_{\mathcal{B}/z}$.

Problem 6.14.7. Let $\mathcal{B}$ be a homogeneous cube, and let c be its center of mass. Show that every frame is a principal-axis frame relative to c.

Problem 6.14.8. Let $\mathcal{B}$ be a homogeneous rectangular solid whose mass is m, and let $\mathrm{F_B}$ be a frame whose axes $\hat{\imath}_{\mathrm{B}}$, $\hat{\jmath}_{\mathrm{B}}$, and $\hat{k}_{\mathrm{B}}$ are parallel to the sides of length a, b, and c, respectively, where $a > b > c$. Determine $J_{\mathcal{B}/z|\mathrm{B}}$ in the following cases:

i) z is the center of a face of $\mathcal{B}$ whose sides have lengths a and b.

ii) z is the center of an edge of $\mathcal{B}$ of length a.

iii) z is a vertex of $\mathcal{B}$.

Specialize *iii*) to the case where the rectangular solid approximates a thin bar, that is, $b = c \approx 0$.

Problem 6.14.9. Consider a rectangular plate whose sides have lengths $a > b > 0$, where

$a = \sqrt{\frac{1+\sqrt{5}}{2}}b$. Show that $J_2 = \sqrt{J_1 J_3}$, and that the right-hand inequality in (6.2.16) holds as an equality.

Problem 6.14.10. Let $\mathcal{B}$ be an annulus, that is, a flat circular ring, with mass m, outer radius R, and inner radius r. Show that the principal moments of inertia of $\mathcal{B}$ relative to it center of mass are given by $m(R^2 + r^2)/4$, $m(R^2 + r^2)/4$, and $m(R^2 + r^2)/2$. Furthermore, show that if the annulus is thin, that is, $R \approx r$, then the principal moments of inertia of $\mathcal{B}$ are given approximately by $\frac{1}{2}mR^2$, $\frac{1}{2}mR^2$, and mR^2. (Hint: Express the moments of inertia of the annulus in terms of the area density of the material.)

Problem 6.14.11. Let $\mathcal{B}$ be a spherical shell with mass m, outer radius R, and inner radius r. Show that the principal moments of inertia of $\mathcal{B}$ relative to it center of mass are given by $\frac{2m(R^5-r^5)}{5(R^3-r^3)}$. Furthermore, show that if the annulus is thin, that is, $R \approx r$, then the principal moments of inertia of $\mathcal{B}$ are given approximately by $2mR^2/3$.

Problem 6.14.12. Determine the physical inertia matrix of a triangular plate relative to its center of mass. Separately consider the cases where the triangle is a right triangle and the triangle is isosceles.

Problem 6.14.13. Let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, and let c be the center of mass of $\mathcal{B}$. Furthermore, let z be a point, and let $\mathcal{B}'$ be the body consisting of $\mathcal{B}$ and a particle y_{l+1} of mass $m_\mathcal{B}$ located at z. Finally, let c$'$ be the center of mass of $\mathcal{B}'$. Show that

$$\vec{J}_{\mathcal{B}/z} = \vec{J}_{\mathcal{B}'/\mathrm{c}} = 2\vec{J}_{\mathcal{B}'/\mathrm{c}'} - \vec{J}_{\mathcal{B}/\mathrm{c}}.$$

6.15 Applied Problems

Problem 6.15.1. Consider the bar shown in Figure 6.15.1, which has a pin joint at a and whose opposite end c is attached to a linear spring connected to b. The stiffness of the linear spring is k, and the relaxed length of the linear spring is zero. In addition, a torsional spring, whose relaxed angle is zero and whose torsional stiffness κ, is attached to the bar around the pin joint. Determine the moment on the bar relative to a and due to the force applied to the bar by the linear spring. Furthermore, determine the longitudinal force on the bar due to the linear spring. Finally, determine κ.

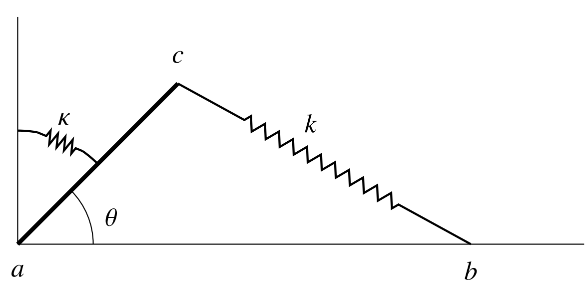


Figure 6.15.1: Bar with linear and torsional springs for Problem 6.15.1.

Problem 6.15.2. The planar triangle shown in Figure 6.15.2 is attached to a vertical wall by

Symbol	Definition
c	Center of mass of $\mathcal{B}$
$\vec{f}$	Force vector
$\vec{M}_{x/y}$	Moment on x relative to y
$\vec{M}_{\mathcal{B}/y}$	Moment on $\mathcal{B}$ relative to y

Table 6.1: Symbols for Chapter 6.

the pin joint at P, and has point masses m_1 and m_2 at the remaining vertices y_1 and y_2, respectively. There is no mass at the pin joint P, and all links are massless. The angles $\theta_1, \theta_2, \theta_3$ and opposite sides ℓ_1, ℓ_2, ℓ_3 are defined in the figure. The side of length ℓ_3 is horizontal, and the direction of gravity is vertical as shown. A third mass M is connected to y_1 by a rope that passes around a small wheel at the point z, which is at the same height as y_1. Determine the location of the center of mass of the triangle. Furthermore, assuming that the triangle and mass M are in equilibrium, determine M. Finally, assume that the wheel at z and the mass M are moved horizontally to the right side of the triangle. Show that the triangle is not in equilibrium when the side of length ℓ_3 is horizontal.

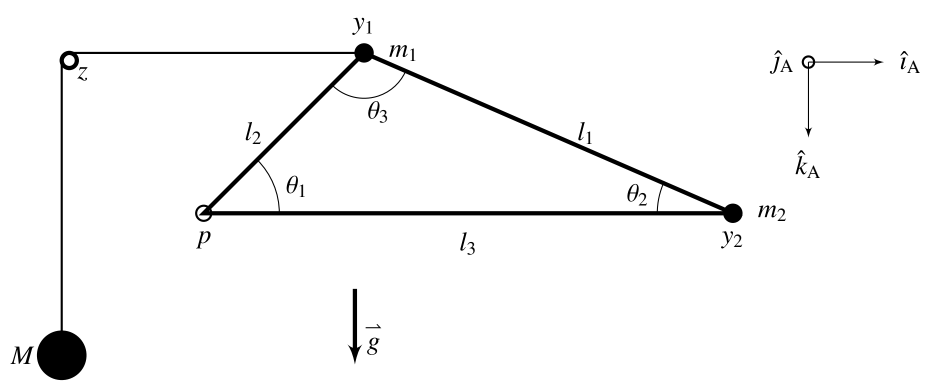


Figure 6.15.2: Triangular structure for Problem 6.15.2.

Problem 6.15.3. Consider three identical metal spheres that fit exactly inside a metal ring on a horizontal surface so that each sphere contacts the other two spheres as well as the inner surface of the ring. A fourth identical metal sphere is placed on top of the three spheres in a pyramid configuration. Determine the force that each of the three spheres applies to the ring due to the weight of the fourth sphere.

Chapter Seven
Newton-Euler Dynamics

Forces and moments can be applied to particles and bodies, resulting in changes in translational momentum and angular momentum. The basis for these changes is Newton's laws.

7.1 Newton's First Law for Particles

An *unforced particle* is a particle that has no forces applied to it. We use the concept of an unforced particle to define the concept of an inertial frame.

Definition 7.1.1. A frame $\mathrm{F_A}$ is an *inertial frame* if, for all unforced particles y and w,

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0. \tag{7.1.1}$$

Equation (7.1.1) can be written as

$$\vec{a}_{y/w/\mathrm{A}} = 0. \tag{7.1.2}$$

The following result is *Newton's first law*. This statement is an axiom that concerns the existence of an inertial frame.

Fact 7.1.2. There exists an inertial frame.

Newton's first law cannot be proved mathematically; in fact, it is an approximation to actual motion. Assuming that Newton's first law is valid, it is shown in the following section that the stars provide an approximate inertial frame.

The following result shows that, for each pair of unforced particles y and w, the velocity of y relative to w with respect to an inertial frame $\mathrm{F_A}$ is constant. This means that the motion of y relative to w with respect to an inertial frame $\mathrm{F_A}$ is along a straight line whose direction is fixed with respect to $\mathrm{F_A}$ and with constant speed.

Fact 7.1.3. Let y and w be points, and let $\mathrm{F_A}$ be a frame. Then, (7.1.1) is satisfied if and only if there exist physical vectors $\vec{\alpha}$ and $\vec{\beta}$ such that $\overset{\mathrm{A}\bullet}{\vec{\alpha}} = 0$, $\overset{\mathrm{A}\bullet}{\vec{\beta}} = 0$, and

$$\vec{r}_{y/w} = t\vec{\alpha} + \vec{\beta}. \tag{7.1.3}$$

Proof. To prove sufficiency, note that it follows from (7.1.3) that the velocity of y relative to w

with respect to $\mathrm{F_A}$ is given by

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{y/w} = \vec{\alpha},$$

while the acceleration of y relative to w with respect to $\mathrm{F_A}$ is given by

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0,$$

which verifies (7.1.1).

Conversely, resolving (7.1.1) in $\mathrm{F_A}$ yields

$$\ddot{\overbrace{\vec{r}_{y/w}\Big|_{\mathrm{A}}}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w}\Bigg|_{\mathrm{A}} = 0,$$

which implies that there exist $\alpha, \beta \in \mathbb{R}^3$ such that

$$\vec{r}_{y/w}\Big|_{\mathrm{A}} = t\alpha + \beta.$$

Therefore, (7.1.3) is satisfied with $\vec{\alpha} \triangleq \mathrm{F_A}\alpha$ and $\vec{\beta} \triangleq \mathrm{F_A}\beta$. □

Note that $\vec{\alpha}$ and $\vec{\beta}$ are the physical velocity and physical position vectors given, respectively, by

$$\vec{\alpha} = \overset{\mathrm{A}\bullet}{\vec{r}}_{y/w}(t) \tag{7.1.4}$$

and

$$\vec{\beta} = \vec{r}_{y/w}(0). \tag{7.1.5}$$

Therefore, $\vec{\alpha}$ represents the velocity of y relative to w with respect to $\mathrm{F_A}$, while $\vec{\beta}$ represents the initial position of of y relative to w. Since both vectors are constant with respect to $\mathrm{F_A}$, the motion of y relative to w has the form of a straight line whose direction is constant with respect to $\mathrm{F_A}$.

The following result shows that all pairs of inertial frames have zero relative angular velocity.

Fact 7.1.4. Let $\mathrm{F_B}$ be an inertial frame, and let $\mathrm{F_A}$ be a frame. Then, $\mathrm{F_A}$ is an inertial frame if and only if $\vec{\omega}_{\mathrm{B/A}} = 0$.

Proof. Assume that $\vec{\omega}_{\mathrm{B/A}} = 0$, and let w and y be unforced particles. Since $\mathrm{F_B}$ is an inertial frame, it follows that $\overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w} = 0$. It then follows from (4.5.1) that $\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0$. Consequently, $\mathrm{F_A}$ is an inertial frame.

Conversely, let y and w be distinct unforced particles. Since $\mathrm{F_A}$ and $\mathrm{F_B}$ are inertial frames, it follows that $\overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0$. It then follows from (4.5.1) that

$$2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{r}}_{y/w} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{y/w} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y/w}) = 0. \tag{7.1.6}$$

Now, choose distinct particles y and w such that $\vec{r}_{y/w} = \vec{\beta}$, where $\overset{\mathrm{B}\bullet}{\vec{\beta}} = 0$. Then, it follows from

(7.1.6) that

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{\beta} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{\beta}) = 0. \tag{7.1.7}$$

Next, choosing y and w such that, at time t, $\vec{\omega}_{\mathrm{B/A}}$ and $\vec{\beta}$ are parallel, it follows from (7.1.7) that, at time t,

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} = 0. \tag{7.1.8}$$

Alternatively, choosing y and w such that, at time t, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}$ and $\vec{\beta}$ are parallel, it follows from (7.1.7) and (7.1.8) that, at time t,

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0. \tag{7.1.9}$$

Hence, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$. Therefore, for all choices of y and w, it follows from (7.1.7) that, at time t,

$$\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{\beta}) = 0, \tag{7.1.10}$$

and thus

$$(\vec{\beta}{}' \vec{\omega}_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}} = |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{\beta}. \tag{7.1.11}$$

Finally, choosing distinct y and w such that, at time t, $\vec{\beta}$ and $\vec{\omega}_{\mathrm{B/A}}$ are mutually orthogonal, it follows from (7.1.11) that $\vec{\omega}_{\mathrm{B/A}} = 0$. □

Fact 7.1.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be inertial frames, and let $\vec{x}(t)$ be a physical vector. Then,

$$\overset{\mathrm{A}\bullet}{\vec{x}}(t) = \overset{\mathrm{B}\bullet}{\vec{x}}(t), \tag{7.1.12}$$

$$\overset{\mathrm{A}\bullet\bullet}{\vec{x}}(t) = \overset{\mathrm{B}\bullet\bullet}{\vec{x}}(t). \tag{7.1.13}$$

7.2 Why the Stars Approximate an Inertial Frame

Next, we explain why the distant stars approximate an inertial frame. Intuitively, the distant stars, although visible to us, are at such a great distance that the angle between every pair does not change of short time periods. Therefore, the motion of these stars can be viewed as the motion of unforced particles. Consequently, Newton's first law implies that the relative motion of an unforced particle is along a straight line relative to the stars. This motion gives us the impression that the stars define an inertial frame. Notice that this argument assumes that Newton's first law is valid; in other words, the presence of visible stars per se does not imply the validity or approximate validity of Newton's first law.

Let $\mathrm{F_A}$ be an inertial frame. Let w be an unforced particle, and let y_1, y_2, y_3 be distant stars that form three mutually orthogonal directions as viewed from w starting at time $t = 0$. Assuming that the stars are unforced particles, there exist velocity vectors $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$ and position vectors $\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3$

that are constant with respect to F_A and satisfy, for all $i = 1, 2, 3$,

$$\vec{r}_{y_i/w} = t\vec{\alpha}_i + \vec{\beta}_i. \tag{7.2.1}$$

Since $\vec{\beta}_1 \cdot \vec{\beta}_2 = \vec{\beta}_2 \cdot \vec{\beta}_3 = \vec{\beta}_3 \cdot \vec{\beta}_1 = 0$, the angle $\theta_{i,j}$ formed by the ith and jth stars satisfies

$$\cos\theta_{i,j} = \frac{\vec{\alpha}_i \cdot \vec{\alpha}_j t^2 + (\vec{\alpha}_i \cdot \vec{\beta}_j + \vec{\alpha}_j \cdot \vec{\beta}_i)t}{\|t\vec{\alpha}_i + \vec{\beta}_i\|\,\|t\vec{\alpha}_j + \vec{\beta}_j\|}. \tag{7.2.2}$$

Because the distances are large, $\|t\vec{\alpha}_i + \vec{\beta}_i\|$ and $\|t\vec{\alpha}_j + \vec{\beta}_j\|$ are large compared to the numerator of (7.2.2) over short time intervals. Hence, $\theta_{i,j}$ remains approximately $\pi/2$ rad over short time intervals.

Next, let F_S be the frame $[\hat{r}_{y_1/w}\ \hat{r}_{y_2/w}\ \hat{r}_{y_3/w}]$, which, over short time intervals, is given approximately by $[\hat{\beta}_1\,\hat{\beta}_2\,\hat{\beta}_3]$. Using (4.2.9), it follows that the physical angular velocity matrix of F_A relative to F_S is given by

$$\begin{aligned}
\vec{\Omega}_{S/A} &= -\vec{\Omega}_{A/S} && (7.2.3)\\
&= -[\hat{\imath}_S \overset{A\bullet\prime}{\hat{\imath}}_S + \hat{\jmath}_S \overset{A\bullet\prime}{\hat{\jmath}}_S + \hat{k}_S \overset{A\bullet\prime}{\hat{k}}_S]\\
&= -[\hat{r}_{y_1/w} \overset{A\bullet\prime}{\hat{r}}_{y_1/w} + \hat{r}_{y_2/w} \overset{A\bullet\prime}{\hat{r}}_{y_2/w} + \hat{r}_{y_3/w} \overset{A\bullet\prime}{\hat{r}}_{y_3/w}]\\
&\approx -[\hat{\beta}_1 \overset{A\bullet\prime}{\hat{\beta}}_1 + \hat{\beta}_2 \overset{A\bullet\prime}{\hat{\beta}}_2 + \hat{\beta}_3 \overset{A\bullet\prime}{\hat{\beta}}_3]\\
&= 0, && (7.2.4)
\end{aligned}$$

where the approximation holds over short time intervals and the last equality follows from the fact that $\vec{\beta}_i$ and thus $\hat{\beta}_i$ are constant with respect to F_A. Finally, since F_A is an inertial frame, it follows from Fact 7.1.4 that F_S is approximately an inertial frame.

The observation that the stars determine an approximate inertial frame provides a practical framework for Newton's first law. These observations are consistent with the everyday experience that unforced motion evolves in straight lines with respect to the stars but not with respect to frames that are rotating relative to the stars.

7.3 Newton's Second Law for Particles

The following result is *Newton's second law*. This statement is an axiom that concerns the effect of forces on particles.

Fact 7.3.1. Let F_A be an inertial frame, let y be a particle with mass m, let $\vec{f}_y$ be the force acting on y, and let w be an unforced particle. Then,

$$m \overset{A\bullet\bullet}{\vec{r}}_{y/w} = \vec{f}_y. \tag{7.3.1}$$

Fact 7.3.2. Let F_A and F_B be inertial frames, let y be a particle, let $\vec{f}_y$ be a force acting on y, and let w be an unforced particle. Then,

$$\overset{A\bullet\bullet}{\vec{r}}_{y/w} = \overset{B\bullet\bullet}{\vec{r}}_{y/w}. \tag{7.3.2}$$

Proof. Since $\mathrm{F_A}$ and $\mathrm{F_B}$ are inertial frames, the result follows from Fact 7.1.5. Alternatively, since $\mathrm{F_A}$ and $\mathrm{F_B}$ are inertial frames, it follows from Fact 7.3.1 that $m\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = \vec{f}_y$ and $m\overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w} = \vec{f}_y$. Hence, $\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = (1/m)\vec{f}_y = \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w}$. □

The following result considers the acceleration of one particle relative to another particle in the case where forces are acting on both particles.

Fact 7.3.3. Let $\mathrm{F_A}$ be an inertial frame, let y_1 and y_2 be particles whose masses are m_1 and m_2, respectively, and let $\vec{f}_{y_1}$ and $\vec{f}_{y_2}$ be the forces acting on y_1 and y_2, respectively. Then,

$$m_1\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} = \vec{f}_{y_1} - \frac{m_1}{m_2}\vec{f}_{y_2}. \tag{7.3.3}$$

Note that, if either y_1 or y_2 is unforced, then this result specializes to Fact 7.3.1. However, although (7.3.3) superficially has the form of (7.3.1), it is not a statement of Newton's second law since y_2 is forced. The distinction between (7.3.3) and (7.3.1) is due to the fact that the force on the right hand side of (7.3.1) is the force applied to y, whereas the force on the right hand side of (7.3.1) is not the force applied to y_1. Finally, note that, in the case where $m_2\vec{f}_{y_1} = m_1\vec{f}_{y_2}$, (7.3.3) becomes

$$m_1\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} = 0. \tag{7.3.4}$$

Although (7.3.4) superficially has the form of (7.1.1), it is not a statement of Newton's second law since both y_1 and y_2 are forced.

It is often the case that a body is constrained in its motion due to its connection with a plane, which may represent a floor, ceiling, wall, or the ground. For example, a body may rotate around a pin joint connected to a wall. To address these problems, it is convenient to view the plane as if it is unaffected by reaction forces. A *massive particle* is a particle whose mass is infinite and thus is unaffected by all forces except gravity. In the absence of gravity, the motion of a massive particle is identical to the motion of an unforced particle. A body that contains at least one massive particle is a *massive body*. Consequently, a massive body that contains at least three massive particles that are not colinear is unaffected by all forces except gravity itself. Therefore, in the absence of gravity, the motion of every point in a massive body is identical to the motion of an unforced particle. Finally, for a massive body, every body-fixed frame is an inertial frame.

It is useful to rewrite (7.3.3) as

$$\frac{m_1 m_2}{m_1+m_2}\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} = \frac{m_2}{m_1+m_2}\vec{f}_{y_1} - \frac{m_1}{m_1+m_2}\vec{f}_{y_2}, \tag{7.3.5}$$

where $\frac{m_1 m_2}{m_1+m_2}$ is the *reduced mass*. Now, assume that the forces $\vec{f}_{y_1}$ and $\vec{f}_{y_2}$ have approximately the same magnitude. Then, it can be seen from (7.3.5) that, if m_1 is much larger than m_2, then

$$m_2\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_2/y_1} \approx \vec{f}_{y_2}, \tag{7.3.6}$$

whereas, if m_2 is much larger than m_1, then

$$m_1\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \approx \vec{f}_{y_1}. \tag{7.3.7}$$

Consequently, the particle with significantly larger mass approximately plays the role of an unforced particle, and thus a massive particle plays the role of an unforced particle.

7.4 Translational Momentum of Particles and Bodies

Let $\mathrm{F_A}$ be a frame, let y be a particle with mass m, and let w be a point. Then, the *translational momentum* $\vec{p}_{y/w/\mathrm{A}}$ *of* y *relative to* w *with respect to* $\mathrm{F_A}$ is defined by

$$\vec{p}_{y/w/\mathrm{A}} \triangleq m\vec{v}_{y/w/\mathrm{A}} = m \overset{\mathrm{A}\bullet}{\vec{r}}_{y/w}. \tag{7.4.1}$$

We can thus restate Newton's second law as follows.

Fact 7.4.1. Let $\mathrm{F_A}$ be an inertial frame, let y be a particle with mass m, let $\vec{f}_y$ be the force acting on y, and let w be an unforced particle. Then,

$$\overset{\mathrm{A}\bullet}{\vec{p}}_{y/w} = \vec{f}_y. \tag{7.4.2}$$

We now apply Newton's second law to each particle in a body $\mathcal{B}$. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$ as defined by (6.1.2), let $\mathrm{F_A}$ be a frame, and let w be a point. Then, the velocity and acceleration of the center of mass of $\mathcal{B}$ relative to w and with respect to $\mathrm{F_A}$ are given by

$$\vec{v}_{\mathrm{c}/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{\mathrm{c}/w} = \frac{1}{m_\mathcal{B}} \sum_{i=1}^{l} m_i \vec{v}_{y_i/w/\mathrm{A}}, \tag{7.4.3}$$

$$\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{v}}_{\mathrm{c}/w} = \frac{1}{m_\mathcal{B}} \sum_{i=1}^{l} m_i \vec{a}_{y_i/w/\mathrm{A}}. \tag{7.4.4}$$

Hence,

$$m_\mathcal{B} \vec{v}_{\mathrm{c}/w/\mathrm{A}} = \sum_{i=1}^{l} m_i \vec{v}_{y_i/w/\mathrm{A}}, \tag{7.4.5}$$

$$m_\mathcal{B} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \sum_{i=1}^{l} m_i \vec{a}_{y_i/w/\mathrm{A}}. \tag{7.4.6}$$

Recall that external forces applied to a particle in a body include all forces that are not due to interactions with other particles in the body. Note that the mass of an inertia point y_i is defined to be zero.

Fact 7.4.2. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let c be the center of mass of $\mathcal{B}$. Furthermore, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, and let w be an unforced particle. Then,

$$m_\mathcal{B} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_\mathcal{B}, \tag{7.4.7}$$

where the mass $m_{\mathcal{B}}$ of $\mathcal{B}$ is defined by

$$m_{\mathcal{B}} \triangleq \sum_{i=1}^{l} m_i \tag{7.4.8}$$

and the total force $\vec{f}_{\mathcal{B}}$ applied to $\mathcal{B}$ is defined by

$$\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}. \tag{7.4.9}$$

Now, let z be a point. Then,

$$m_{\mathcal{B}} \vec{a}_{\mathrm{c}/z/\mathrm{A}} + m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}. \tag{7.4.10}$$

Finally, if the external forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, then

$$\vec{a}_{\mathrm{c}/w/\mathrm{A}} = 0, \tag{7.4.11}$$

and thus

$$\vec{a}_{\mathrm{c}/z/\mathrm{A}} + \vec{a}_{z/w/\mathrm{A}} = 0. \tag{7.4.12}$$

Proof. Let $\vec{f}_{ij}$ be the internal force on y_i due to y_j. Since $\mathcal{B}$ is a Newtonian body, it follows that $\sum_{j=1,\ldots,l,j\neq i} \vec{f}_{ij} = 0$. Since w is an unforced particle, we thus have

$$m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \sum_{i=1}^{l} m_i \vec{a}_{y_i/w/\mathrm{A}} = \sum_{i=1}^{l} m_i \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w} = \sum_{i=1}^{l} \left(\vec{f}_{y_i} + \sum_{\substack{j=1,\ldots,l \\ j\neq i}} \vec{f}_{ij} \right) = \sum_{i=1}^{l} \vec{f}_{y_i} = \vec{f}_{\mathcal{B}},$$

which proves (7.4.7). Furthermore,

$$\vec{f}_{\mathcal{B}} = \sum_{i=1}^{l} m_i (\vec{a}_{y_i/z/\mathrm{A}} + \vec{a}_{z/w/\mathrm{A}}),$$

which implies (7.4.10). □

Let $\mathcal{B}$ be a body with mass $m_{\mathcal{B}}$, and let c be the center of mass of $\mathcal{B}$. Then, the *translational momentum* $\vec{p}_{\mathcal{B}/w/\mathrm{A}}$ *of* $\mathcal{B}$ *relative to* w *with respect to* $\mathrm{F_A}$ is defined by

$$\vec{p}_{\mathcal{B}/w/\mathrm{A}} \triangleq m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}} = m_{\mathcal{B}} \overset{\mathrm{A}\bullet}{\vec{r}}_{\mathrm{c}/w}. \tag{7.4.13}$$

The following result restates Fact 7.4.2 in terms of the translational momentum of $\mathcal{B}$.

Fact 7.4.3. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let c be the center of mass of $\mathcal{B}$. Furthermore, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, and let w be an unforced particle, Then,

$$\overset{\mathrm{A}\bullet}{\vec{p}}_{\mathcal{B}/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}, \tag{7.4.14}$$

where the mass $m_{\mathcal{B}}$ and the total force $\vec{f}_{\mathcal{B}}$ are defined by (7.4.8) and (7.4.9), respectively. Finally, if $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, then

$$\overset{\mathrm{A}\bullet}{\vec{p}}_{\mathcal{B}/w/\mathrm{A}} = 0. \tag{7.4.15}$$

Fact 7.4.3 shows that the acceleration of a body due to all external forces applied to the body can be viewed as the acceleration of an equivalent particle located at the center of mass of the body, where the mass of the equivalent particle is equal to the mass of the body and where the applied force is given by the total force, that is, the sum of all of the forces applied to all of the particles in the body. Note that the internal forces do not contribute to the total force since the body is assumed to be Newtonian. If the total force is zero, then the equivalent particle has constant inertial velocity, that is, constant velocity relative to an unforced particle and with respect to an inertial frame. The external forces can also cause the body to rotate, as discussed in the following sections.

Suppose that $\vec{f}_{\mathcal{B}}\Big|_{\mathrm{A}}$ has a component that is identically zero. Then, it follows from (7.4.14) that the corresponding component of $\overset{\mathrm{A}\bullet}{\vec{p}}_{\mathcal{B}/w/\mathrm{A}}\Big|_{\mathrm{A}}$ is identically zero. In this case, *conservation of momentum* holds along one of the axes of $\mathrm{F_A}$, and the corresponding component of the translational momentum is a constant of the motion relative to $\mathrm{F_A}$. If $\vec{f}_{\mathcal{B}} = 0$, then translational momentum is conserved along all three axes of $\mathrm{F_A}$, and all three components of the momentum are constants of the motion relative to $\mathrm{F_A}$.

The following result provides an alternative version of Fact 7.4.2 involving the first moment of inertia.

Fact 7.4.4. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body with mass $m_{\mathcal{B}}$, let c be the center of mass of $\mathcal{B}$, let $\vec{f}_{\mathcal{B}}$ be the total force on $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let w be an unforced particle, and let z be a point. Then,

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{\mathrm{c}/z/\mathrm{A}} + m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}. \tag{7.4.16}$$

If, in addition, $\mathcal{B}$ is a rigid body and z is fixed in $\mathcal{B}$, then

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\alpha}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}) = \vec{f}_{\mathcal{B}}. \tag{7.4.17}$$

Proof. Rewriting (7.4.7) as

$$m_{\mathcal{B}}\overset{\mathrm{A}\bullet}{\vec{v}}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}$$

and applying the transport theorem yields

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{\mathrm{c}/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_{\mathcal{B}},$$

which implies (7.4.16). Now, consider the case where $\mathcal{B}$ is a rigid body and z is fixed in $\mathcal{B}$. Then, using $\overset{\mathrm{B}\bullet}{\vec{r}}_{z/\mathrm{c}} = 0$ and $\overset{\mathrm{B}\bullet}{\vec{v}}_{\mathrm{c}/z/\mathrm{A}} = \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}/z}$, (7.4.16) implies (7.4.17). $\square$

7.5 Dynamics of Interconnected Particles

Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ with masses $m_1, \ldots, m_l$, respectively, for all $i = 1, \ldots, l$, let $\vec{f}_i$ be the external force applied to y_i, let w be an unforced particle, and let $\mathrm{F_A}$ be an inertial frame. Furthermore, assume that, for all distinct $i, j \in \{1, \ldots, l\}$, the particles y_i and y_j are connected by a dashpot with viscosity $c_{ij} \geq 0$ and a spring with stiffness $k_{ij} \geq 0$. Note that, for all $i, j = 1, \ldots, l$, $c_{ij} = c_{ji}$, $c_{ii} = 0$, $k_{ij} = k_{ji}$, and $k_{ii} = 0$. Then, for all $i \in \{1, \ldots, l\}$, it follows that

$$m_i \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w} + \sum_{j=1}^{l} c_{ij} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/y_j} + \sum_{j=1}^{l} k_{ij} \vec{r}_{y_i/y_j} = \vec{f}_i, \tag{7.5.1}$$

and thus

$$M \begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + C \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + K \begin{bmatrix} \vec{r}_{y_1/w} \\ \vdots \\ \vec{r}_{y_l/w} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_l \end{bmatrix}, \tag{7.5.2}$$

where $M \triangleq \operatorname{diag}(m_1, \ldots, m_l)$,

$$C \triangleq \begin{bmatrix} \sum_{j=1}^{l} c_{1j} & -c_{12} & -c_{13} & \cdots & -c_{1l} \\ -c_{12} & \sum_{j=1}^{l} c_{2j} & -c_{23} & \cdots & -c_{2l} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -c_{1l} & -c_{2l} & -c_{3l} & \cdots & \sum_{j=1}^{l} c_{lj} \end{bmatrix}, \tag{7.5.3}$$

$$K \triangleq \begin{bmatrix} \sum_{j=1}^{l} k_{1j} & -k_{12} & -k_{13} & \cdots & -k_{1l} \\ -k_{12} & \sum_{j=1}^{l} k_{2j} & -k_{23} & \cdots & -k_{2l} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -k_{1l} & -k_{2l} & -k_{3l} & \cdots & \sum_{j=1}^{l} k_{lj} \end{bmatrix}. \tag{7.5.4}$$

As a special case, assume that the motion of $y_1, \ldots, y_l$ is confined to a single line in the direction $\hat{n}$, where the direction of $\hat{n}$ is fixed with respect to the inertial frame $\mathrm{F_A}$, and that the external forces $\vec{f}_1, \ldots, \vec{f}_l$ are parallel with $\hat{n}$. Then, for all $i = 1, \ldots, l$, it follows that

$$\vec{r}_{y_i/w} \triangleq q_i \hat{n}, \tag{7.5.5}$$

$$\vec{f}_i \triangleq f_i \hat{n}. \tag{7.5.6}$$

Then, (7.5.2) can be written as

$$M\ddot{q} + C\dot{q} + Kq = f, \tag{7.5.7}$$

where

$$q \triangleq \begin{bmatrix} q_1 \\ \vdots \\ q_l \end{bmatrix}, \quad f \triangleq \begin{bmatrix} f_1 \\ \vdots \\ f_l \end{bmatrix}. \tag{7.5.8}$$

Next, defining

$$\Gamma \triangleq \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \\ m_1 & m_2 & m_3 & \cdots & m_n \end{bmatrix}, \tag{7.5.9}$$

(7.5.2) can be rewritten as

$$\tilde{M}\Gamma \begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + \tilde{C}\Gamma \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + \tilde{K}\Gamma \begin{bmatrix} \vec{r}_{y_1/w} \\ \vdots \\ \vec{r}_{y_l/w} \end{bmatrix} = \Gamma^{-\mathrm{T}} \begin{bmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_l \end{bmatrix}, \tag{7.5.10}$$

where

$$\tilde{M} \triangleq \Gamma^{-\mathrm{T}} M \Gamma^{-1}, \quad \tilde{C} \triangleq \Gamma^{-\mathrm{T}} C \Gamma^{-1}, \quad \tilde{K} \triangleq \Gamma^{-\mathrm{T}} K \Gamma^{-1}. \tag{7.5.11}$$

Note that

$$\Gamma \begin{bmatrix} \vec{r}_{y_1/w} \\ \vdots \\ \vec{r}_{y_l/w} \end{bmatrix} = \begin{bmatrix} \vec{r}_{y_1/y_2} \\ \vdots \\ \vec{r}_{y_{l-1}/y_l} \\ \sum_{i=1}^{l} m_i \vec{r}_{y_i/w} \end{bmatrix}. \tag{7.5.12}$$

Furthermore,

$$\det \Gamma = m_{\mathcal{B}} \triangleq \sum_{i=1}^{l} m_i, \tag{7.5.13}$$

$$\Gamma^{-\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \\ \frac{1}{m_{\mathcal{B}}} & \frac{1}{m_{\mathcal{B}}} & \frac{1}{m_{\mathcal{B}}} & \cdots & \frac{1}{m_{\mathcal{B}}} \end{bmatrix} - \frac{1}{m_{\mathcal{B}}} \begin{bmatrix} m_1 & m_1 & \cdots & m_1 \\ m_1 + m_2 & m_1 + m_2 & \cdots & m_1 + m_2 \\ \vdots & \vdots & \vdots & \vdots \\ m_{\mathcal{B}} - m_l & m_{\mathcal{B}} - m_l & \cdots & m_{\mathcal{B}} - m_l \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{7.5.14}$$

and thus

$$\Gamma^{-\mathrm{T}}\begin{bmatrix}\vec{f}_1\\ \vdots\\ \vec{f}_l\end{bmatrix}=\begin{bmatrix}\vec{f}_1\\ \vec{f}_1+\vec{f}_2\\ \vdots\\ \vec{f}_{\mathcal{B}}-\vec{f}_l\\ \frac{1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\end{bmatrix}-\begin{bmatrix}m_1\\ m_1+m_2\\ \vdots\\ m_{\mathcal{B}}-m_l\\ 0\end{bmatrix}\frac{1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}. \tag{7.5.15}$$

Therefore, (7.5.10) can be written as

$$\tilde{M}\begin{bmatrix}\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2}\\ \vdots\\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_{l-1}/y_l}\\ \sum_{i=1}^{l}m_i\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w}\end{bmatrix}+\tilde{C}\begin{bmatrix}\overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2}\\ \vdots\\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_{l-1}/y_l}\\ \sum_{i=1}^{l}m_i\overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w}\end{bmatrix}+\tilde{K}\begin{bmatrix}\vec{r}_{y_1/y_2}\\ \vdots\\ \vec{r}_{y_{l-1}/y_l}\\ \sum_{i=1}^{l}m_i\vec{r}_{y_i/w}\end{bmatrix}=\begin{bmatrix}\vec{f}_1-\frac{m_1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\\ \vec{f}_1+\vec{f}_2-\frac{m_1+m_2}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\\ \vdots\\ -\vec{f}_l+\frac{m_l}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\\ \frac{1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\end{bmatrix}. \tag{7.5.16}$$

In particular, if $l=2$, then

$$\begin{aligned}&\frac{1}{m_1+m_2}\begin{bmatrix}m_1m_2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2}\\ \sum_{i=1}^{2}m_i\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w}\end{bmatrix}+\frac{1}{(m_1+m_2)^2}\begin{bmatrix}-2m_1m_2c_{12} & (m_2-m_1)c_{12}\\ (m_2-m_1)c_{12} & 2c_{12}\end{bmatrix}\begin{bmatrix}\overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2}\\ \sum_{i=1}^{2}m_i\overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w}\end{bmatrix}\\ &\quad+\frac{1}{(m_1+m_2)^2}\begin{bmatrix}-2m_1m_2k_{12} & (m_2-m_1)k_{12}\\ (m_2-m_1)k_{12} & 2k_{12}\end{bmatrix}\begin{bmatrix}\vec{r}_{y_1/y_2}\\ \sum_{i=1}^{2}m_i\vec{r}_{y_i/w}\end{bmatrix}=\begin{bmatrix}\frac{m_2}{m_1+m_2}\vec{f}_1-\frac{m_1}{m_1+m_2}\vec{f}_2\\ \frac{1}{m_1+m_2}(\vec{f}_1+\vec{f}_2)\end{bmatrix}.\end{aligned} \tag{7.5.17}$$

Furthermore, if $l=3$, then

$$\begin{aligned}&\frac{1}{m_{\mathcal{B}}}\begin{bmatrix}m_1(m_2+m_3) & m_1m_3 & 0\\ m_1m_3 & m_3(m_1+m_2) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2}\\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_2/y_3}\\ \sum_{i=1}^{3}m_i\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w}\end{bmatrix}+\frac{1}{m_{\mathcal{B}}^2}\begin{bmatrix}\tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13}\\ \tilde{c}_{12} & \tilde{c}_{22} & \tilde{c}_{23}\\ \tilde{c}_{13} & \tilde{c}_{23} & \tilde{c}_{33}\end{bmatrix}\begin{bmatrix}\overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2}\\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_2/y_3}\\ \sum_{i=1}^{3}m_i\overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w}\end{bmatrix}\\ &\quad+\frac{1}{m_{\mathcal{B}}^2}\begin{bmatrix}\tilde{k}_{11} & \tilde{k}_{12} & \tilde{k}_{13}\\ \tilde{k}_{12} & \tilde{k}_{22} & \tilde{k}_{23}\\ \tilde{k}_{13} & \tilde{k}_{23} & \tilde{k}_{33}\end{bmatrix}\begin{bmatrix}\vec{r}_{y_1/y_2/\mathrm{A}}\\ \vec{r}_{y_2/y_3}\\ \sum_{i=1}^{3}m_i\vec{r}_{y_i/w}\end{bmatrix}=\begin{bmatrix}\vec{f}_1-\frac{m_1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\\ -\vec{f}_3+\frac{m_3}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\\ \frac{1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}}\end{bmatrix}.\end{aligned} \tag{7.5.18}$$

where

$$\tilde{c}_{11}\triangleq-2m_1(m_2+m_3)(c_{12}+c_{13})+2m_1^2c_{23},$$

$$\tilde{c}_{12}\triangleq c_{23}m_{\mathcal{B}}^2-m_3m_{\mathcal{B}}(c_{12}+2c_{13}+3c_{23})-m_2m_{\mathcal{B}}(c_{13}+c_{23})+2m_3(m_2+m_3)(c_{12}+c_{13}+c_{23}),$$

$$\tilde{c}_{13}\triangleq-m_1(c_{12}+c_{13}+m_2(c_{12}+c_{13})+2c_{23})+m_3(c_{12}+c_{13}),$$

$$\tilde{c}_{22}\triangleq 2m_3(m_2-m_1)(c_{13}+c_{23})+2m_3^2c_{12},$$

$$\tilde{c}_{23}\triangleq-m_1(c_{13}+c_{23})+m_3(2c_{12}+c_{13}+c_{23})-m_2(c_{13}+c_{23}),$$

$$\tilde{c}_{33}\triangleq 2(c_{12}+c_{13}+c_{23}),$$

$$\begin{aligned}
\tilde{k}_{11} &\triangleq -2m_1(m_2+m_3)(k_{12}+k_{13}) + 2m_1^2 k_{23},\\
\tilde{k}_{12} &\triangleq k_{23}m_{\mathcal{B}}^2 - m_3 m_{\mathcal{B}}(k_{12}+2k_{13}+3k_{23}) - m_2 m_{\mathcal{B}}(k_{13}+k_{23}) + 2m_3(m_2+m_3)(k_{12}+k_{13}+k_{23}),\\
\tilde{k}_{13} &\triangleq -m_1(k_{12}+k_{13}+m_2(k_{12}+k_{13})+2k_{23}) + m_3(k_{12}+k_{13}),\\
\tilde{k}_{22} &\triangleq 2m_3(m_2-m_1)(k_{13}+k_{23}) + 2m_3^2 k_{12},\\
\tilde{k}_{23} &\triangleq -m_1(k_{13}+k_{23}) + m_3(2k_{12}+k_{13}+k_{23}) - m_2(k_{13}+k_{23}),\\
\tilde{k}_{33} &\triangleq 2(k_{12}+k_{13}+k_{23}).
\end{aligned}$$

Again, assume, as a special case, that the motion of $y_1,\ldots,y_l$ is confined to a single line in the direction $\hat{n}$, where the direction of $\hat{n}$ is fixed with respect to the inertial frame F_A, and that the external forces $\vec{f}_1,\ldots,\vec{f}_l$ are parallel with $\hat{n}$. Then, for all $i=1,\ldots,l-1$, it follows that

$$\vec{r}_{y_i/y_{i+1}} \triangleq (q_i - q_{i+1})\hat{n}. \tag{7.5.19}$$

Now, defining

$$\tilde{q} \triangleq \Gamma q = \begin{bmatrix} q_1 - q_2 \\ \vdots \\ q_{l-1} - q_l \\ \sum_{i=1}^{l} m_i q_i \end{bmatrix} \tag{7.5.20}$$

and, with $f_{\mathcal{B}} \triangleq \sum_{i=1}^{l} f_i$,

$$\tilde{f} \triangleq \begin{bmatrix} f_1 - \frac{m_1}{m_{\mathcal{B}}} f_{\mathcal{B}} \\ f_1 + f_2 - \frac{m_1+m_2}{m_{\mathcal{B}}} f_{\mathcal{B}} \\ \vdots \\ -f_l + \frac{m_l}{m_{\mathcal{B}}} f_{\mathcal{B}} \\ \frac{1}{m_{\mathcal{B}}} f_{\mathcal{B}} \end{bmatrix}, \tag{7.5.21}$$

it follows from (7.5.16) that

$$\tilde{M}\ddot{\tilde{q}} + \tilde{C}\dot{\tilde{q}} + \tilde{K}\tilde{q} = \tilde{f}. \tag{7.5.22}$$

Finally, assume that $\vec{f}_{\mathcal{B}} = 0$, and thus the center of mass c of $\mathcal{B}$ is unforced. Then, choosing $w = \mathrm{c}$, it follows that $\sum_{i=1}^{l} m_i \vec{r}_{y_i/w} = 0$. Hence, (7.5.16) becomes

$$\tilde{M}\begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_{l-1}/y_l} \\ 0 \end{bmatrix} + \tilde{C}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_{l-1}/y_l} \\ 0 \end{bmatrix} + \tilde{K}\begin{bmatrix} \vec{r}_{y_1/y_2} \\ \vdots \\ \vec{r}_{y_{l-1}/y_l} \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_1 + \vec{f}_2 \\ \vdots \\ -\vec{f}_l \\ 0 \end{bmatrix}. \tag{7.5.23}$$

Now, truncating the last row and last column of $\tilde{M}$, $\tilde{C}$, and $\tilde{K}$, yields

$$\tilde{M}_{[l,l]}\begin{bmatrix}\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2}\\ \vdots\\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_{l-1}/y_l}\end{bmatrix} + \tilde{C}_{[l,l]}\begin{bmatrix}\overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2}\\ \vdots\\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_{l-1}/y_l}\end{bmatrix} + \tilde{K}_{[l,l]}\begin{bmatrix}\vec{r}_{y_1/y_2}\\ \vdots\\ \vec{r}_{y_{l-1}/y_l}\end{bmatrix} = \begin{bmatrix}\vec{f}_1\\ \vec{f}_1+\vec{f}_2\\ \vdots\\ -\vec{f}_l\end{bmatrix}, \tag{7.5.24}$$

where $\tilde{M}_{[l,l]}$, $\tilde{C}_{[l,l]}$, and $\tilde{K}_{[l,l]}$ denote $\tilde{M}$, $\tilde{C}$, and $\tilde{K}$, respectively, after deleting the last row and last column of each matrix.

7.6 Angular Momentum of Particles and Bodies

Let y be a particle with mass m, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *angular momentum of y relative to w with respect to* $\mathrm{F_A}$ is defined by

$$\vec{H}_{y/w/\mathrm{A}} \triangleq \vec{r}_{y/w} \times m\vec{v}_{y/w/\mathrm{A}}, \tag{7.6.1}$$

where

$$\vec{v}_{y/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{y/w}\,. \tag{7.6.2}$$

Now, let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *angular momentum of $\mathcal{B}$ relative to w with respect to* $\mathrm{F_A}$ is defined by

$$\vec{H}_{\mathcal{B}/w/\mathrm{A}} \triangleq \sum_{i=1}^{l} \vec{H}_{y_i/w/\mathrm{A}}, \tag{7.6.3}$$

where

$$\vec{H}_{y_i/w/\mathrm{A}} = \vec{r}_{y_i/w} \times m_i\vec{v}_{y_i/w/\mathrm{A}}. \tag{7.6.4}$$

The following result relates the inertial change in the angular momentum of a body to the moment on the body. For this result, an inertia point is either a particle or a point in a body where a force is applied. For example, an inertia point may be a point on a massless link connecting two particles. If no particle is located at the inertia point, then the mass of the inertia point is defined to be zero.

Fact 7.6.1. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let w be an unforced particle, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.6.5}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.6.6}$$

Proof. For $i = 1, \ldots, l$, let m_i be the mass of y_i, and, for $i, j = 1, \ldots, l$, let $\vec{f}_{ij}$ be the internal force on y_i due to y_j. Since w is an unforced particle, it follows from Fact 7.3.1 that, for all $i = 1, \ldots, l$,

$$m_i \vec{a}_{y_i/w/\mathrm{A}} = \vec{f}_{y_i} + \sum_{j=1}^{l} \vec{f}_{ij}. \tag{7.6.7}$$

Using (7.6.7) and Fact 4.1.5, it follows that the derivative of the angular momentum with respect to $\mathrm{F_A}$ is given by

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{H}}_{y_i/w/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w} \times m_i \vec{v}_{y_i/w/\mathrm{A}} + \vec{r}_{y_i/w} \times m_i \overset{\mathrm{A}\bullet}{\vec{v}}_{y_i/w/\mathrm{A}} \\
&= \vec{v}_{y_i/w/\mathrm{A}} \times m_i \vec{v}_{y_i/w/\mathrm{A}} + \vec{r}_{y_i/w} \times m_i \vec{a}_{y_i/w/\mathrm{A}} \\
&= \vec{r}_{y_i/w} \times m_i \vec{a}_{y_i/w/\mathrm{A}} \\
&= \vec{r}_{y_i/w} \times \left(\vec{f}_{y_i} + \sum_{j=1}^{l} \vec{f}_{ij} \right).
\end{aligned}$$

Summing over the particles $y_1, \ldots, y_l$ yields

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} &= \sum_{i=1}^{l} \overset{\mathrm{A}\bullet}{\vec{H}}_{y_i/w/\mathrm{A}} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \left(\vec{f}_{y_i} + \sum_{j=1}^{l} \vec{f}_{ij} \right) \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} + \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \sum_{j=1}^{l} \vec{f}_{ij} \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} = \vec{M}_{\mathcal{B}/w}.
\end{aligned}$$

Note that, since $\mathcal{B}$ is Newtonian, Fact, 6.11.1 implies that the internal forces are balanced and the torque $\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \sum_{j=1}^{l} \vec{f}_{ij}$ on $\mathcal{B}$ due to all of the internal forces is zero. Under the stronger assumption that $\mathcal{B}$ is Newtonian, this term is zero due to the fact that, for all distinct i and j, $\vec{f}_{ij} = -\vec{f}_{ji}$ and $\vec{f}_{ij}$ and $\vec{r}_{y_i/y_j}$ are parallel. □

Note that the change in angular momentum given by Fact 7.6.1 is not affected by internal forces since the body is assumed to be Newtonian.

Fact 7.6.1 relates the inertial change of the angular momentum of a body to the moment on the body. This result is a direct consequence of Newton's second law, and thus the angular momentum of the body and the moment on the body are both defined relative to the unforced particle w, while the angular momentum and its derivative are defined with respect to an inertial frame. The moment on the body is due to all of the external forces applied to inertia points in the body. Equation (7.6.5) is applicable to a body that rotates around a pivot point w relative to which all moments are determined and whose motion coincides with the motion of an unforced particle.

Since the moment $\vec{M}_{\mathcal{B}/w}$ does not depend on the inertial frame $\mathrm{F_A}$, it follows that the change in angular momentum $\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}}$ is independent of the choice of the inertial frame.

Fact 7.6.2. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let w be an unforced particle, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be inertial frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} \,. \tag{7.6.8}$$

If the external forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, then the moment $\vec{M}_{\mathcal{B}/w}$ is independent of the point w. Consequently, the change in angular momentum $\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}}$ is independent of the choice of the unforced particle w.

Fact 7.6.3. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, assume that $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, let w and w' be unforced particles, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w'/\mathrm{A}} \,. \tag{7.6.9}$$

The following result considers the change in angular momentum of a body relative to an arbitrary point z.

Fact 7.6.4. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let m_B be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, let w and z be points, and let $\mathrm{F_A}$ be a frame. Then,

$$\vec{H}_{\mathcal{B}/w/\mathrm{A}} = \vec{H}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{z/w} \times m_\mathcal{B} \vec{v}_{\mathrm{c}/w/\mathrm{A}}. \tag{7.6.10}$$

Now, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, assume that w is an unforced particle, and assume that $\mathrm{F_A}$ is an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.6.11}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.6.12}$$

Proof. For $i = 1, \ldots, l$, let m_i be the mass of y_i. To derive (7.6.10), note that

$$\begin{aligned}
\vec{H}_{\mathcal{B}/w/\mathrm{A}} &= \sum_{i=1}^{l} \vec{H}_{y_i/w/\mathrm{A}} \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times m_i \vec{v}_{y_i/w/\mathrm{A}} = \sum_{i=1}^{l} (\vec{r}_{y_i/z} + \vec{r}_{z/w}) \times m_i (\vec{v}_{y_i/z/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}}) \\
&= \sum_{i=1}^{l} \vec{H}_{y_i/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{z/w} \times m_\mathcal{B} \vec{v}_{\mathrm{c}/w/\mathrm{A}} \\
&= \vec{H}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{z/w} \times m_\mathcal{B} \vec{v}_{\mathrm{c}/w/\mathrm{A}}.
\end{aligned}$$

Now, assume that F_A is an inertial frame and w is an unforced particle. Then,

$$\begin{aligned}\vec{M}_{\mathcal{B}/w} &= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} = \sum_{i=1}^{l} (\vec{r}_{y_i/z} + \vec{r}_{z/w}) \times \vec{f}_{y_i} \\ &= \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i} \right) + \vec{r}_{z/w} \times \sum_{i=1}^{l} \vec{f}_{y_i} = \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}},\end{aligned}$$

where the total force on the body is

$$\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}.$$

Using (7.6.5), differentiating (7.6.10), and using (7.4.7) we thus have

$$\begin{aligned}\vec{M}_{\mathcal{B}/z} &= \vec{M}_{\mathcal{B}/w} - \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}} = \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} - \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}} \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{v}_{\mathrm{c}/z/\mathrm{A}} \times m_{\mathcal{B}} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} \\
&\quad + \vec{v}_{z/w/\mathrm{A}} \times m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}} + \vec{r}_{z/w} \times m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} - \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}} \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}}(\vec{v}_{\mathrm{c}/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}} \times \vec{v}_{\mathrm{c}/w/\mathrm{A}}) \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}}(\vec{v}_{\mathrm{c}/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}} - \vec{v}_{\mathrm{c}/w/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}}) \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}}(\vec{v}_{\mathrm{c}/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}} + \vec{v}_{w/\mathrm{c}/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}}) \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}}(\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \vec{v}_{w/\mathrm{c}/\mathrm{A}}) \times \vec{v}_{z/w/\mathrm{A}} \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}}(\vec{v}_{w/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}}) \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}}.\end{aligned}$$

□

If the point z in Fact 7.6.4 is colocated with an unforced particle w', then it follows from Newton's first law that $\vec{a}_{z/w/\mathrm{A}} = \vec{a}_{w'/w/\mathrm{A}} = 0$, and thus (7.6.11) specializes to (7.6.5). This situation can occur when z is the pivot point of a rotating body, where the pivot point is colocated with an unforced particle.

By choosing z in Fact 7.6.4 to be the center of mass, we obtain the following result.

Fact 7.6.5. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let m_{B} be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, let w be a point, and let F_A be a frame. Then,

$$\vec{H}_{\mathcal{B}/w/\mathrm{A}} = \vec{H}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}}. \tag{7.6.13}$$

Now, assume that F_A is an inertial frame, and, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i. Then,

$$\overset{\text{A}\bullet}{\vec{H}}_{\mathcal{B}/\text{c}/\text{A}} = \vec{M}_{\mathcal{B}/\text{c}}, \tag{7.6.14}$$

where the moment on $\mathcal{B}$ relative to c is given by

$$\vec{M}_{\mathcal{B}/\text{c}} = \sum_{i=1}^{l} \vec{M}_{y_i/\text{c}} = \sum_{i=1}^{l} \vec{r}_{y_i/\text{c}} \times \vec{f}_{y_i}. \tag{7.6.15}$$

Note that the form of (7.6.14) is identical to the form of (7.6.5) with w replaced by c whether or not c is an unforced particle. In fact, (7.6.14) does not involve an unforced particle, which shows that the center of mass plays a special role with regard to the change in angular momentum.

According to Newton's third law, reaction forces and moments arise from the interaction of rigid bodies. Note that the forces at the point z do not contribute to the moment $\vec{M}_{\mathcal{B}/z}$. Therefore, if z is chosen to be a point at which reaction forces occur (such as a pin joint), then $\vec{M}_{\mathcal{B}/z}$ can be determined without knowledge of the reaction forces at z.

The following result expresses the moment on a body relative to an arbitrary point in terms of the moment on the body relative to the center of mass of the body.

Fact 7.6.6. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let w be an unforced particle, let z be a point, and let F_A be an inertial frame. Then,

$$\vec{M}_{\mathcal{B}/z} = \vec{M}_{\mathcal{B}/\text{c}} + \vec{r}_{\text{c}/z} \times m_{\mathcal{B}} \vec{a}_{\text{c}/w/\text{A}}. \tag{7.6.16}$$

Proof. Using Fact 7.4.2 it follows that

$$\begin{aligned}
\vec{M}_{\mathcal{B}/z} &= \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i} = \sum_{i=1}^{l} (\vec{r}_{y_i/\text{c}} + \vec{r}_{\text{c}/z}) \times \vec{f}_{y_i} \\
&= \left(\sum_{i=1}^{l} \vec{r}_{y_i/\text{c}} \times \vec{f}_{y_i} \right) + \vec{r}_{\text{c}/z} \times \sum_{i=1}^{l} \vec{f}_{y_i} \\
&= \vec{M}_{\mathcal{B}/\text{c}} + \vec{r}_{\text{c}/z} \times \vec{f}_{\mathcal{B}} = \vec{M}_{\mathcal{B}/\text{c}} + \vec{r}_{\text{c}/z} \times m_{\mathcal{B}} \vec{a}_{\text{c}/w/\text{A}}.
\end{aligned}$$ □

Applying Fact 7.6.6 to Fact 7.6.5 yields the following result.

Fact 7.6.7. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let w be an unforced particle, let z be a point, and let F_A be an inertial frame. Then,

$$\overset{\text{A}\bullet}{\vec{H}}_{\mathcal{B}/\text{c}/\text{A}} + \vec{r}_{\text{c}/z} \times m_{\mathcal{B}} \vec{a}_{\text{c}/w/\text{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.6.17}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.6.18}$$

Setting $z = w$ in Fact (7.6.7) yields the following result.

Fact 7.6.8. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let w be an unforced particle, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.6.19}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.6.20}$$

Let $\mathcal{B}$ be a body, and let w be either an unforced particle or the center of mass of $\mathcal{B}$. Then, it follows from Fact 7.6.1 and Fact 7.6.5 that

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.6.21}$$

Now, suppose that $\vec{M}_{\mathcal{B}/w}\Big|_{\mathrm{A}}$ has a component that is identically zero. Then, it follows from (7.6.21) that the corresponding component of $\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}}\Big|_{\mathrm{A}}$ is identically zero. In this case, *conservation of angular momentum* holds along one of the axes of $\mathrm{F_A}$, and the corresponding component of the angular momentum is a constant of the motion relative to $\mathrm{F_A}$. If $\vec{M}_{\mathcal{B}/w} = 0$, then angular momentum is conserved along all three axes of $\mathrm{F_A}$, and all three components of the momentum are constants of the motion relative to $\mathrm{F_A}$.

7.7 Effect of Gravity on Translational Momentum and Angular Momentum

If the bodies in facts 7.4.2, 7.6.1, 7.6.4, 7.6.5, 7.6.7, and 7.6.8 are subject to gravity, then the external force $\vec{f}_{y_i}$ on y_i includes the force $m_i\vec{g}$ due to gravity, where $\vec{g}$ is the acceleration due to gravity. For the case where the body is subject to gravity, we now consider the effect of gravity separately from the remaining external forces. Throughout this chapter we assume that gravity is uniform over the body.

The following result restates Fact 7.4.2 with gravity separated from the remaining external forces.

Fact 7.7.1. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$ let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to gravity, for $i = 1, \ldots, l$, let $\vec{f}_{y_i,\mathrm{ng}}$ be the nongravitational external force applied to the particle m_i, let $\mathrm{F_A}$ be an inertial frame, let c be the center of mass of $\mathcal{B}$, and let w be an unforced particle. Then,

$$m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}, \tag{7.7.1}$$

where the total force $\vec{f}_{\mathcal{B}}$ applied to $\mathcal{B}$ is given by

$$\vec{f}_{\mathcal{B}} \triangleq \left(\sum_{i=1}^{l} \vec{f}_{y_i,\mathrm{ng}}\right) + m_{\mathcal{B}}\vec{g}. \tag{7.7.2}$$

Note that it follows from (7.7.1) and (7.7.2) that, if the only external force present is gravity, then $\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{g}$.

The following result restates Fact 7.6.1 with gravity separated from the remaining external forces. This result is based on Fact 6.6.2.

Fact 7.7.2. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to gravity, for $i = 1, \ldots, l$, let $\vec{f}_{y_i,\mathrm{ng}}$ be the nongravitational external force applied to y_i, let w be an unforced particle, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.7.3}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i,\mathrm{ng}}\right) + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{g}. \tag{7.7.4}$$

The following result follows from Fact 7.6.4 with gravity separated from the remaining external forces.

Fact 7.7.3. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to gravity, for $i = 1, \ldots, l$, let $\vec{f}_{y_i,\mathrm{ng}}$ be the nongravitational external force applied to y_i, let w be an unforced particle, let z be a point, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.7.5}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i,\mathrm{ng}}\right) + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\vec{g}. \tag{7.7.6}$$

The following result follows from Fact 7.6.5 with gravity separated from the remaining external forces. In this case, gravity has no effect on the change in angular momentum.

Fact 7.7.4. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to gravity, for $i = 1, \ldots, l$, let $\vec{f}_{y_i,\mathrm{ng}}$ be the nongravitational external force applied to y_i, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = \vec{M}_{\mathcal{B}/\mathrm{c}}, \tag{7.7.7}$$

where the moment on $\mathcal{B}$ relative to c is given by

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i,\mathrm{ng}}. \tag{7.7.8}$$

Proof. For $i = 1, \ldots, l$, let m_i be the mass of y_i. Note that

$$\begin{aligned}\vec{M}_{\mathcal{B}/\mathrm{c}} &= \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} (\vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i} + \vec{r}_{y_i/\mathrm{c}} \times m_i \vec{g}) \\ &= \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i} + \left(\sum_{i=1}^{l} m_i \vec{r}_{y_i/\mathrm{c}} \right) \times \vec{g} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i}. \qquad \square\end{aligned}$$

The following result restates Fact 7.6.7 with gravity separated from the remaining external forces.

Fact 7.7.5. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i,\mathrm{ng}}$ be the nongravitational external force applied to y_i, let w be an unforced particle, let z be a point, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.7.9}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i,\mathrm{ng}} \right) + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{g}. \tag{7.7.10}$$

The following result restates Fact 7.6.8 with gravity separated from the remaining external forces.

Fact 7.7.6. Let $\mathcal{B}$ be a body with inertia points $y_1, \ldots, y_l$, let $\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i,\mathrm{ng}}$ be the nongravitational external force applied to y_i, let w be an unforced particle, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.7.11}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i,\mathrm{ng}} \right) + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}} \vec{g}. \tag{7.7.12}$$

7.8 Euler's Equation for the Rotational Dynamics of a Rigid Body

The following result expresses the angular momentum of a body in terms of the physical inertia matrix defined by Definition 6.2.1.

Fact 7.8.1. Let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let z be a point. Then,

$$\vec{H}_{\mathcal{B}/z/\mathrm{A}} = \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + \vec{H}_{\mathcal{B}/z/\mathrm{B}}, \tag{7.8.1}$$

where

$$\vec{J}_{\mathcal{B}/z} = \sum_{i=1}^{l} m_i \vec{r}^{\times\prime}_{y_i/z}\vec{r}^{\times}_{y_i/z}. \tag{7.8.2}$$

Proof. Using (7.6.4) and the transport theorem it follows that

$$\begin{aligned}
\vec{H}_{y_i/z/\mathrm{A}} &= \vec{r}_{y_i/z} \times m_i \vec{v}_{y_i/z/\mathrm{A}} \\
&= \vec{r}_{y_i/z} \times m_i \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/z} \\
&= \vec{r}_{y_i/z} \times m_i \left(\overset{\mathrm{B}\bullet}{\vec{r}}_{y_i/z} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z}\right) \\
&= \vec{r}_{y_i/z} \times m_i \left(\vec{v}_{y_i/z/\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z}\right).
\end{aligned}$$

Summing over the particles in the body and using (2.9.7) yields

$$\begin{aligned}
\vec{H}_{\mathcal{B}/z/\mathrm{A}} &= \sum_{i=1}^{l} \vec{r}_{y_i/z} \times m_i \left(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z}\right) + \sum_{i=1}^{l} \vec{r}_{y_i/z} \times m_i \vec{v}_{y_i/z/\mathrm{B}} \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/z} \times m_i \left(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z}\right) + \vec{H}_{\mathcal{B}/z/\mathrm{B}} \\
&= -\sum_{i=1}^{l} m_i \vec{r}_{y_i/z} \times \left(\vec{r}_{y_i/z} \times \vec{\omega}_{\mathrm{B/A}}\right) + \vec{H}_{\mathcal{B}/z/\mathrm{B}} \\
&= \sum_{i=1}^{l} m_i \vec{r}^{\times\prime}_{y_i/z}\vec{r}^{\times}_{y_i/z}\vec{\omega}_{\mathrm{B/A}} + \vec{H}_{\mathcal{B}/z/\mathrm{B}} \\
&= \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + \vec{H}_{\mathcal{B}/z/\mathrm{B}}.
\end{aligned}$$

□

The following result specializes Fact 7.8.1 to the case where $\mathcal{B}$ is a rigid body and the point z is fixed in $\mathcal{B}$.

Fact 7.8.2. Let $\mathcal{B}$ be a rigid body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $\mathrm{F_A}$ be a frame, let $\mathrm{F_B}$ be a body-fixed frame, and let z be a point fixed in $\mathcal{B}$. Then,

$$\vec{H}_{\mathcal{B}/z/\mathrm{B}} = 0, \tag{7.8.3}$$

and thus

$$\vec{H}_{\mathcal{B}/z/\mathrm{A}} = \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}}. \tag{7.8.4}$$

Furthermore,

$$\overset{\mathrm{B}\bullet}{\vec{J}}_{\mathcal{B}/z} = 0, \tag{7.8.5}$$

and thus

$$\overset{\mathrm{B}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} = \vec{J}_{\mathcal{B}/z}\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}}\,. \tag{7.8.6}$$

Proof. Since $\mathcal{B}$ is a rigid body and z is fixed in $\mathcal{B}$, it follows that, for all $i = 1, \ldots, l$, $\overset{\mathrm{B}\bullet}{\vec{r}}_{y_i/z} = 0$. Hence, (7.8.3) is satisfied, and thus (7.8.1) implies (7.8.4). Furthermore, differentiating (**??**) yields (7.8.5). Finally, (7.8.4), (7.8.5), and Fact 4.1.4 yield (7.8.6). □

The following result follows from Fact 7.6.4 and Fact 7.7.3 in the case where $\mathcal{B}$ is a rigid body. This result is *Euler's equation*.

Fact 7.8.3. Let $\mathcal{B}$ be a rigid body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, let w be an unforced particle, let c be the center of mass of $\mathcal{B}$, and let z be a point fixed in $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}/z}\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.8.7}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.8.8}$$

If, in addition, $\mathcal{B}$ is subject to gravity and, for $i = 1, \ldots, l$, $\vec{f}_{y_i,\mathrm{ng}}$ is the nongravitational external force applied to y_i, then $\vec{M}_{\mathcal{B}/z}$ is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{g}. \tag{7.8.9}$$

Proof. Using (7.6.11), (7.8.4), and (7.8.6), it follows that

$$\begin{aligned}
\vec{M}_{\mathcal{B}/z} &= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}} \\
&= \overbrace{\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}}}^{\mathrm{A}\bullet} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}} \\
&= \overbrace{\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}}}^{\mathrm{B}\bullet} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}} \\
&= \vec{J}_{\mathcal{B}/z}\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}}.
\end{aligned}$$
□

The following result specializes Fact 7.8.3 to the case where the point z, which is fixed in $\mathcal{B}$, is colocated with an unforced particle. In this case, $\vec{a}_{z/w/\mathrm{A}} = 0$.

Fact 7.8.4. Let $\mathcal{B}$ be a rigid body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, and let w be an unforced particle that is fixed in $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}/w}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.8.10}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.8.11}$$

If, in addition, $\mathcal{B}$ is subject to gravity and, for $i = 1, \ldots, l$, $\vec{f}_{y_i,\mathrm{ng}}$ is the nongravitational external force applied to y_i, then $\vec{M}_{\mathcal{B}/w}$ is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i,\mathrm{ng}} \right) + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{g}. \tag{7.8.12}$$

Alternatively, the following result specializes Fact 7.8.3 to the case where the point z is the center of mass. In this case, $\vec{r}_{\mathrm{c}/z} = 0$, and gravity, if present, does not appear in the equations of motion.

Fact 7.8.5. Let $\mathcal{B}$ be a rigid body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, and let c be the center of mass of $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}/\mathrm{c}}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}} = \vec{M}_{\mathcal{B}/\mathrm{c}}, \tag{7.8.13}$$

where the moment on $\mathcal{B}$ relative to c is given by

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i}. \tag{7.8.14}$$

If, in addition, $\mathcal{B}$ is subject to gravity and, for $i = 1, \ldots, l$, $\vec{f}_{y_i,\mathrm{ng}}$ is the nongravitational external force applied to y_i, then $\vec{M}_{\mathcal{B}/\mathrm{c}}$ is given by (7.8.14).

Note that (7.8.13) has the same form as (7.10.2) except that the unforced particle w in the latter equation is replaced by the center of mass c. This may seem surprising since there is no assumption in Fact 7.8.5 that the center of mass is unforced. However, notice that (7.10.2) concerns only the rotational dynamics of the rigid body. If an additional force $-\vec{f}_{\mathcal{B}}$ is applied to $\mathcal{B}$, then the total force is balanced and the resulting torque $\vec{M}_{\mathcal{B}}$ is equal to the original moment $\vec{M}_{\mathcal{B}/w}$. Consequently, the rotational dynamics are unchanged. In addition, since the total force is balanced, it follows from Fact 7.4.2 that the center of mass is unforced. Consequently, from the point of view of the rotational dynamics, the center of mass plays the role of an unforced particle.

Next, the following result specializes Fact 7.6.7 and Fact 7.7.5 to the case where $\mathcal{B}$ is a rigid body.

Fact 7.8.6. Let $\mathcal{B}$ be a rigid body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, let c be the center of mass of $\mathcal{B}$, let w be an unforced particle, and let z be a point.

Then,

$$\vec{J}_{\mathcal{B}/\mathrm{c}} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.8.15}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.8.16}$$

If, in addition, $\mathcal{B}$ is subject to gravity and, for $i = 1, \ldots, l$, $\vec{f}_{y_i,\mathrm{ng}}$ is the nongravitational external force applied to y_i, then

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i,\mathrm{ng}}\right) + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\vec{g}. \tag{7.8.17}$$

Finally, the following result specializes Fact 7.6.8 and Fact 7.7.6 to the case where $\mathcal{B}$ is a rigid body.

Fact 7.8.7. Let $\mathcal{B}$ be a rigid body with inertia points $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force applied to y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, let c be the center of mass of $\mathcal{B}$, and let w be an unforced particle. Then,

$$\vec{J}_{\mathcal{B}/\mathrm{c}} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}} + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.8.18}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.8.19}$$

If, in addition, $\mathcal{B}$ is subject to gravity and, for $i = 1, \ldots, l$, $\vec{f}_{y_i,\mathrm{ng}}$ is the nongravitational external force applied to y_i, then

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i,\mathrm{ng}}\right) + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{g}. \tag{7.8.20}$$

Example 7.8.8. Consider the rigid body $\mathcal{B}$ shown in Figure 7.8.1 consisting of particles y_1 and y_2 with masses m_1 and m_2, respectively, connected by a rigid massless link $\mathcal{L}$ of length ℓ. The body-fixed frame $\mathrm{F_B}$ is aligned with $\mathcal{B}$ such that $\vec{r}_{y_2/y_1} = \ell\hat{\imath}_{\mathrm{B}}$. An external force $\vec{f}$ is applied to the link at the point z, whose distance from y_1 is $\ell_0 > 0$. Hence, $\vec{r}_{z/y_1} = \ell_0\hat{\imath}_{\mathrm{B}}$. $\mathcal{B}$ translates and rotates such that it remains in the $\hat{\imath}_{\mathrm{A}}$-$\hat{\jmath}_{\mathrm{A}}$ plane of the inertial frame $\mathrm{F_A}$. The point w is colocated with an unforced particle. The location of the center of mass c of $\mathcal{B}$ is given by $\vec{r}_{\mathrm{c}/y_1} = \frac{m_2}{m_{\mathcal{B}}}\ell\hat{\imath}_{\mathrm{B}}$, where $m_{\mathcal{B}} = m_1 + m_2$. Note that $\vec{r}_{y_2/\mathrm{c}} = \frac{m_1}{m_{\mathcal{B}}}\ell\hat{\imath}_{\mathrm{B}}$. The rotation angle θ is defined such that $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$, so that $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}\hat{k}_{\mathrm{A}}$.

Using (7.4.7) given by

$$m_{\mathcal{B}}\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{f} \tag{7.8.21}$$

and defining the notation

$$\vec{r}_{\mathrm{c}/w/\mathrm{A}}\Big|_{\mathrm{A}} = \begin{bmatrix} x_\mathrm{c} \\ y_\mathrm{c} \\ z_\mathrm{c} \end{bmatrix}, \quad \vec{f}\Big|_{\mathrm{B}} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}, \tag{7.8.22}$$

it follows that

$$m_{\mathcal{B}} \begin{bmatrix} \ddot{x}_\mathrm{c} \\ \ddot{y}_\mathrm{c} \\ \ddot{z}_\mathrm{c} \end{bmatrix} = \mathcal{O}_{\mathrm{A/B}} \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}, \tag{7.8.23}$$

where

$$\mathcal{O}_{\mathrm{A/B}} = \mathcal{O}_3(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.8.24}$$

Next, the physical inertia matrix of $\mathcal{B}$ relative to c is given by

$$\vec{J}_{\mathcal{B}/\mathrm{c}} = \left[m_1 \left(\frac{m_2}{m_{\mathcal{B}}} \ell \right)^2 + m_2 \left(\frac{m_1}{m_{\mathcal{B}}} \ell \right)^2 \right] (\hat{\jmath}_\mathrm{B} \hat{\jmath}'_\mathrm{B} + \hat{k}_\mathrm{B} \hat{k}'_\mathrm{B}) = \left(\frac{m_1 m_2}{m_{\mathcal{B}}} \ell^2 \right) (\hat{\jmath}_\mathrm{B} \hat{\jmath}'_\mathrm{B} + \hat{k}_\mathrm{B} \hat{k}'_\mathrm{B}), \tag{7.8.25}$$

and the moment on $\mathcal{B}$ relative to c is given by

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \vec{r}_{z/\mathrm{c}} \times \vec{f} = (\vec{r}_{z/y_1} + \vec{r}_{y_1/\mathrm{c}}) \times \vec{f} = \left(\ell_0 - \frac{m_2}{m_{\mathcal{B}}} \ell \right) f_2 \hat{k}_\mathrm{B}. \tag{7.8.26}$$

It thus follows from Euler's equation (7.8.13) that

$$m_1 m_2 \ell^2 \ddot{\theta} = (m_{\mathcal{B}} \ell_0 - m_2 \ell) f_2. \tag{7.8.27}$$

Equations (7.8.23) and (7.8.27) provide a complete description of the dynamics of $\mathcal{B}$.

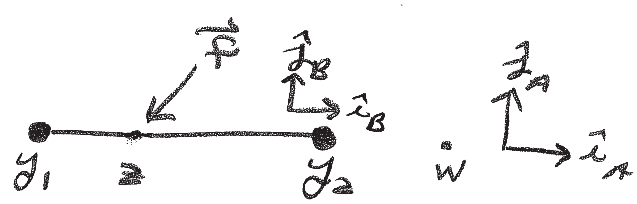


Figure 7.8.1: Rigid body consisting of two particles connected by a massless link for Example 7.8.8.

Next, we use free-body analysis to determine the reaction forces on the particles and the endpoints of $\mathcal{L}$. Let $\vec{f}_{\mathrm{R1}}$ and $\vec{f}_{\mathrm{R2}}$ denote the reaction forces on y_1 and y_2, respectively, so that $-\vec{f}_{\mathrm{R1}}$ and $-\vec{f}_{\mathrm{R2}}$ are the reaction forces applied to the endpoints x_1 and x_2, respectively, of $\mathcal{L}$. As shown in Figure 7.8.2, $\mathcal{B}$ can be decomposed in three ways. Considering the decomposition in Figure 7.8.2(a), it follows that

$$m_1 \vec{a}_{y_1/w/\mathrm{A}} = \vec{f}_{\mathrm{R1}}, \tag{7.8.28}$$

and thus

$$m_1 \vec{a}_{y_1/\mathrm{c}/\mathrm{A}} = -m_1 \vec{a}_{\mathrm{c}/w/\mathrm{A}} + \vec{f}_{\mathrm{R1}} = -\frac{m_1}{m_{\mathcal{B}}} \vec{f} + \vec{f}_{\mathrm{R1}}. \tag{7.8.29}$$

Therefore,

$$\begin{aligned}\vec{f}_{\mathrm{R1}} &= \frac{m_1}{m_{\mathcal{B}}}\vec{f} + m_1[\alpha_{\mathrm{B/A}} \times \vec{r}_{y_1/\mathrm{c}} + \omega_{\mathrm{B/A}} \times (\omega_{\mathrm{B/A}} \times \vec{r}_{y_1/\mathrm{c}})] \\ &= \frac{m_1}{m_{\mathcal{B}}}\vec{f} + \frac{m_1 m_2 \ell\dot{\theta}^2}{m_{\mathcal{B}}}\hat{\imath}_{\mathrm{B}} - \frac{m_1 m_2 \ell\ddot{\theta}}{m_{\mathcal{B}}}\hat{\jmath}_{\mathrm{B}}.\end{aligned} \tag{7.8.30}$$

Writing $\vec{f}_{\mathrm{R1}} = f_{\mathrm{R11}}\hat{\imath}_{\mathrm{B}} + f_{\mathrm{R12}}\hat{\jmath}_{\mathrm{B}}$, it follows that

$$f_{\mathrm{R11}} = \frac{m_1}{m_{\mathcal{B}}}f_1 + \frac{m_1 m_2 \ell\dot{\theta}^2}{m_{\mathcal{B}}}, \tag{7.8.31}$$

$$f_{\mathrm{R12}} = \frac{m_1}{m_{\mathcal{B}}}f_2 - \frac{m_1 m_2 \ell\ddot{\theta}}{m_{\mathcal{B}}}. \tag{7.8.32}$$

Using (7.8.27), it follows from (7.8.32) that

$$f_{\mathrm{R12}} = \frac{m_1}{m_{\mathcal{B}}}f_2 - \frac{m_{\mathcal{B}}\ell_0 - m_2\ell}{m_{\mathcal{B}}\ell}f_2 = \frac{\ell - \ell_0}{\ell}f_2. \tag{7.8.33}$$

Next, considering the decomposition in Figure 7.8.2(b), it follows that

$$m_2\vec{a}_{y_2/w/\mathrm{A}} = \vec{f}_{\mathrm{R2}}, \tag{7.8.34}$$

and thus

$$m_2\vec{a}_{y_2/\mathrm{c/A}} = -m_2\vec{a}_{\mathrm{c}/w/\mathrm{A}} + \vec{f}_{\mathrm{R2}} = -\frac{m_2}{m_{\mathcal{B}}}\vec{f} + \vec{f}_{\mathrm{R2}}. \tag{7.8.35}$$

Therefore,

$$\begin{aligned}\vec{f}_{\mathrm{R2}} &= \frac{m_2}{m_{\mathcal{B}}}\vec{f} + m_2[\alpha_{\mathrm{B/A}} \times \vec{r}_{y_2/\mathrm{c}} + \omega_{\mathrm{B/A}} \times (\omega_{\mathrm{B/A}} \times \vec{r}_{y_2/\mathrm{c}})] \\ &= \frac{m_2}{m_{\mathcal{B}}}\vec{f} - \frac{m_1 m_2 \ell\dot{\theta}^2}{m_{\mathcal{B}}}\hat{\imath}_{\mathrm{B}} + \frac{m_1 m_2 \ell\ddot{\theta}}{m_{\mathcal{B}}}\hat{\jmath}_{\mathrm{B}}.\end{aligned} \tag{7.8.36}$$

Writing $\vec{f}_{\mathrm{R2}} = f_{\mathrm{R21}}\hat{\imath}_{\mathrm{B}} + f_{\mathrm{R22}}\hat{\jmath}_{\mathrm{B}}$, it follows that

$$f_{\mathrm{R21}} = \frac{m_2}{m_{\mathcal{B}}}f_1 - \frac{m_1 m_2 \ell\dot{\theta}^2}{m_{\mathcal{B}}}, \tag{7.8.37}$$

$$f_{\mathrm{R22}} = \frac{m_2}{m_{\mathcal{B}}}f_2 + \frac{m_1 m_2 \ell\ddot{\theta}}{m_{\mathcal{B}}}. \tag{7.8.38}$$

Using (7.8.27), it follows from (7.8.39) that

$$f_{\mathrm{R22}} = \frac{m_2}{m_{\mathcal{B}}}f_2 + \frac{m_{\mathcal{B}}\ell_0 - m_2\ell}{m_{\mathcal{B}}\ell}f_2 = \frac{\ell_0}{\ell}f_2. \tag{7.8.39}$$

Finally, considering the decomposition in Figure 7.8.2(c), it follows that the total force and torque on the massless link must be zero. Therefore,

$$\vec{f} = \vec{f}_{\mathrm{R1}} + \vec{f}_{\mathrm{R2}}, \tag{7.8.40}$$

which implies that

$$f_1 = f_{\mathrm{R11}} + f_{\mathrm{R21}}, \tag{7.8.41}$$

$$f_2 = f_{\mathrm{R12}} + f_{\mathrm{R22}}. \tag{7.8.42}$$

Furthermore,

$$\vec{M}_{\mathcal{L}/y_1} = \vec{r}_{z/y_2} \times \vec{f} + \vec{r}_{y_2/y_1} \times (-\vec{f}_{\mathrm{R2}}) = (\ell_0 f_2 - \ell f_{\mathrm{R22}})\hat{k}_{\mathrm{B}} = 0, \tag{7.8.43}$$

and thus it follows from (7.8.42) that

$$f_{\mathrm{R12}} = \frac{\ell - \ell_0}{\ell} f_2, \quad f_{\mathrm{R22}} = \frac{\ell_0}{\ell} f_2, \tag{7.8.44}$$

which agrees with (7.8.32) and (7.8.39).

Next, we consider the case where the external force $\vec{f}$ is applied to the particle y_1. In this case, $\ell_0 = 0$, and the equations of motion are given by (7.8.23) and by (7.8.27) with $\ell_0 = 0$, that is,

$$m_1 \ell \ddot{\theta} = -f_2. \tag{7.8.45}$$

To determine the reaction forces, the external force $\vec{f}$ can be applied to either the endpoint x_1 of $\mathcal{L}$ or to y_1. In the first approach, the free-body analysis and resulting reaction forces are identical to the free-body analysis with $\ell_0 > 0$ except that now $\ell_0 = 0$. Hence, f_{R11} and f_{R21} are given by (7.8.31) and (7.8.37), respectively, and it follows from (7.8.33) and (7.8.39) that f_{R12} and f_{R22} are given by

$$f_{\mathrm{R12}} = f_2, \quad f_{\mathrm{R22}} = 0. \tag{7.8.46}$$

Therefore, the total force $\vec{f}_{x_1}$ at x_1 is given by

$$\begin{aligned} \vec{f}_{x_1} &= \vec{f} - \vec{f}_{\mathrm{R1}} \\ &= f_1 \hat{\imath}_{\mathrm{B}} + f_2 \hat{\jmath}_{\mathrm{B}} - (f_{\mathrm{R11}} \hat{\imath}_{\mathrm{B}} + f_{\mathrm{R12}} \hat{\jmath}_{\mathrm{B}}) \\ &= \frac{m_2}{m_{\mathcal{B}}} (f_1 - m_1 \ell \dot{\theta}^2) \hat{\imath}_{\mathrm{B}}. \end{aligned} \tag{7.8.47}$$

In the second approach, free-body analysis is based on Figure 7.8.3. It follows from Figure 7.8.3(a) that the moment $\vec{M}_{[\mathcal{L},y_2]/y_2}$ relative to y_2 applied to the body $[\mathcal{L}, y_2]$ consisting of $\mathcal{L}$ and y_2 must be zero. Hence,

$$\vec{M}_{[\mathcal{L},y_2]/y_2} = \vec{r}_{x_1/y_2} \times (-\vec{f}_{\mathrm{R1}}) = -\ell \hat{\imath}_{\mathrm{B}} \times (-f_{\mathrm{R11}} \hat{\imath}_{\mathrm{B}} - f_{\mathrm{R12}} \hat{\jmath}_{\mathrm{B}}) = \ell f_{\mathrm{R12}} \hat{k}_{\mathrm{B}} = 0, \tag{7.8.48}$$

and thus $f_{\mathrm{R12}} = 0$. Next, it follows from Figure 7.8.3(b) that f_{R21} is given by (7.8.37) and that $f_{\mathrm{R22}} = 0$. Furthermore, it follows from Figure 7.8.3(c) that $\vec{f}_{\mathrm{R1}} + \vec{f}_{\mathrm{R2}} = 0$, and thus $f_{\mathrm{R11}} + f_{\mathrm{R21}} = 0$ and $f_{\mathrm{R12}} + f_{\mathrm{R22}} = 0$. Hence,

$$f_{\mathrm{R11}} = -f_{\mathrm{R21}} = \frac{m_2}{m_{\mathcal{B}}} (m_1 \ell \dot{\theta}^2 - f_1), \tag{7.8.49}$$

and thus the total force $\vec{f}_{x_1}$ at x_1 is given by

$$\vec{f}_{x_1} = -f_{\mathrm{R11}} \hat{\imath}_{\mathrm{B}} = \frac{m_2}{m_{\mathcal{B}}} (f_1 - m_1 \ell \dot{\theta}^2) \hat{\imath}_{\mathrm{B}}, \tag{7.8.50}$$

which agrees with (7.8.47).

Example 7.8.9. Consider two rigid bodies connected by a massless link. Centers of mass of the bodies can be offset from the ends of the link. Motion is 3D. External force is applied to one of the bodies. Determine equations of motion and reaction forces.

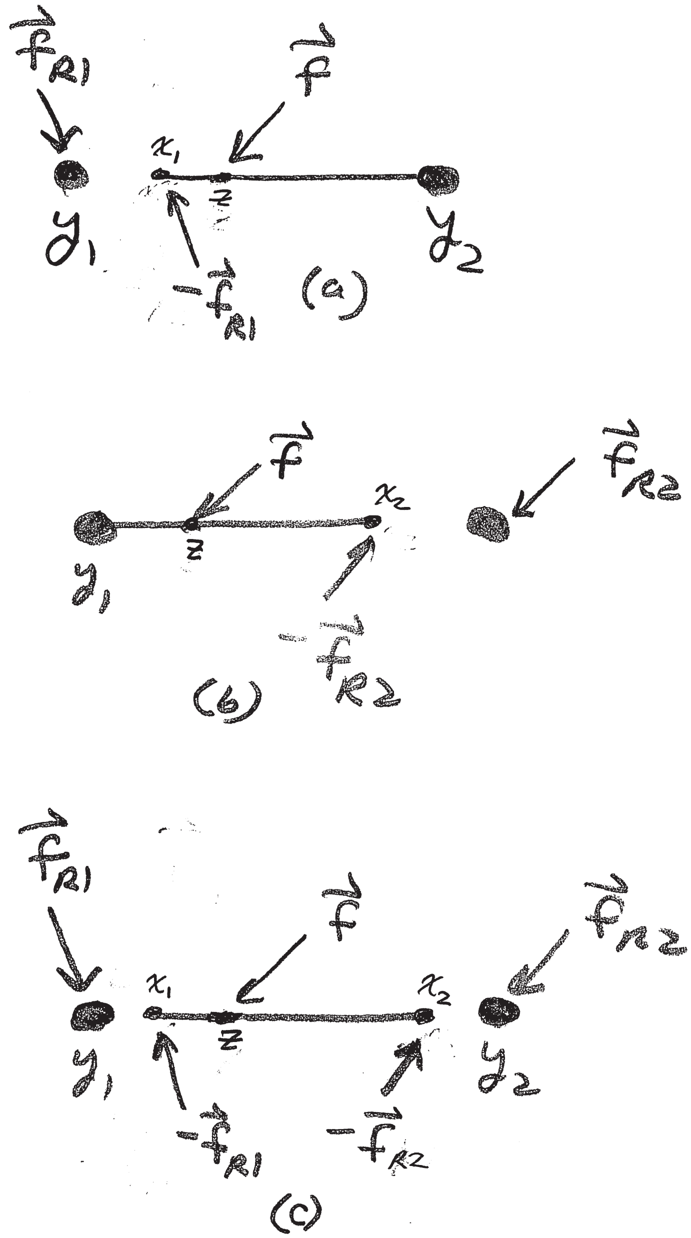


Figure 7.8.2: Free-body analysis of the rigid body consisting of two particles connected by a massless link for Example 7.8.8.

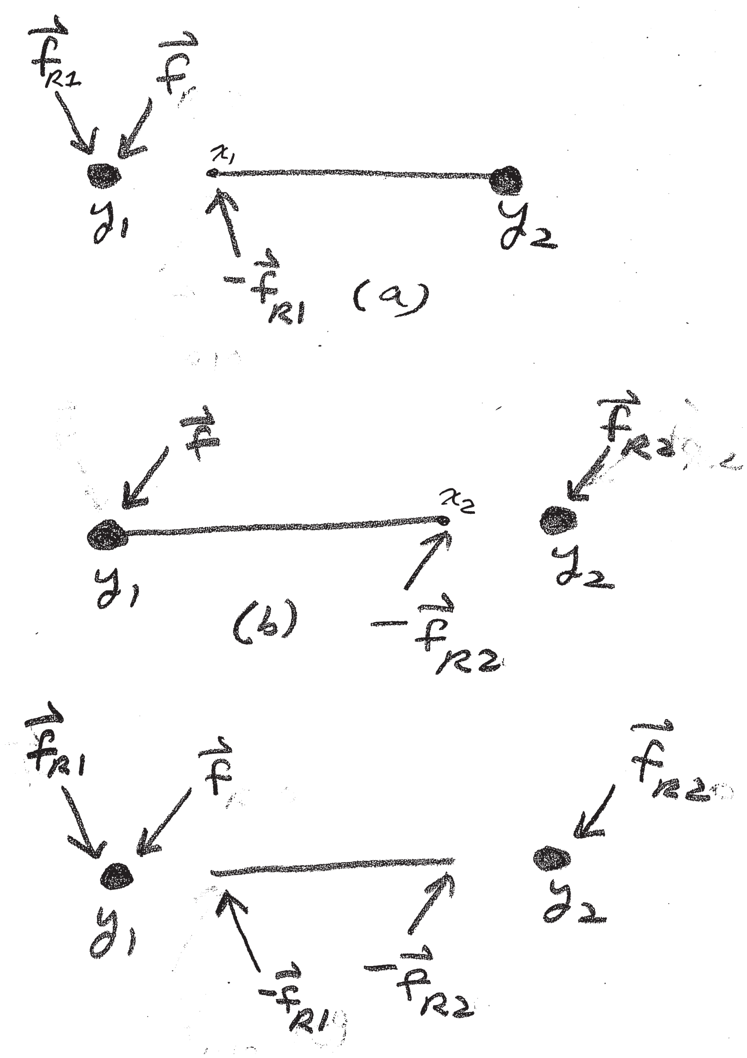


Figure 7.8.3: Free-body analysis of the rigid body consisting of two particles connected by a massless link for Example 7.8.8. Note that the external force $\vec{f}$ is applied to the particle y_1.

Example 7.8.10. Consider two particles connected by massless links to a pin joint. Motion is planar. External force is applied to one of the particles. Determine equations of motion and reaction forces.

Example 7.8.11. Consider two rigid bodies connected by massless links to a pin joint. Motion is 3D. External force is applied along one of the links or to one of the bodies. Determine equations of motion and reaction forces.

7.9 Euler's Equation and the Eigenaxis Angle Vector

Let F_A and F_B be frames, and recall from (4.9.11) that

$$\vec{\omega}_{B/A} = \vec{S}\,\overset{B\bullet}{\vec{\Theta}}_{B/A}, \tag{7.9.1}$$

where

$$\vec{S} \triangleq \alpha\vec{\Theta}^{\times 2}_{B/A} + \beta\vec{\Theta}^{\times}_{B/A} + \vec{I}, \tag{7.9.2}$$

$$\alpha \triangleq \frac{\theta_{B/A} - \sin\theta_{B/A}}{\theta^3_{B/A}}, \quad \beta \triangleq \frac{\cos\theta_{B/A} - 1}{\theta^2_{B/A}}. \tag{7.9.3}$$

Consequently,

$$\overset{B\bullet}{\vec{\omega}}_{B/A} = \vec{S}\,\overset{B\bullet\bullet}{\vec{\Theta}}_{B/A} + \overset{B\bullet}{\vec{S}}\,\overset{B\bullet}{\vec{\Theta}}_{B/A}, \tag{7.9.4}$$

where

$$\overset{B\bullet}{\vec{S}} = \dot{\alpha}\vec{\Theta}^{\times 2}_{B/A} + \alpha\Big(\overset{B\bullet}{\vec{\Theta}}{}^{\times}_{B/A}\,\vec{\Theta}^{\times}_{B/A} + \vec{\Theta}^{\times}_{B/A}\,\overset{B\bullet}{\vec{\Theta}}{}^{\times}_{B/A}\Big) + \dot{\beta}\vec{\Theta}^{\times}_{B/A} + \beta\,\overset{B\bullet}{\vec{\Theta}}{}^{\times}_{B/A}\,. \tag{7.9.5}$$

Now, substituting (7.9.1) and (7.9.4) into (7.8.5) yields

$$\vec{J}_{\mathcal{B}/c}\Big(\vec{S}\,\overset{B\bullet\bullet}{\vec{\Theta}}_{B/A} + \overset{B\bullet}{\vec{S}}\,\overset{B\bullet}{\vec{\Theta}}_{B/A}\Big) + \Big(\vec{S}\,\overset{B\bullet}{\vec{\Theta}}_{B/A}\Big)\times\vec{J}_{\mathcal{B}/c}\Big(\vec{S}\,\overset{B\bullet}{\vec{\Theta}}_{B/A}\Big) = \vec{M}_{\mathcal{B}/c}. \tag{7.9.6}$$

Next, note that

$$\vec{S}\vec{\Theta}_{B/A} = \vec{\Theta}_{B/A}, \tag{7.9.7}$$

and, thus,

$$\overset{B\bullet}{\vec{S}}\,\vec{\Theta}_{B/A} + \vec{S}\,\overset{B\bullet}{\vec{\Theta}}_{B/A} = \overset{B\bullet}{\vec{\Theta}}_{B/A}\,. \tag{7.9.8}$$

Furthermore,

$$\overset{B\bullet}{\vec{S}}\,\vec{\Theta}_{B/A} = \alpha\vec{\Theta}_{B/A}\times(\overset{B\bullet}{\vec{\Theta}}_{B/A}\times\vec{\Theta}_{B/A}) + \beta\,\overset{B\bullet}{\vec{\Theta}}_{B/A}\times\vec{\Theta}_{B/A}. \tag{7.9.9}$$

Hence,

$$\vec{S}\,\overset{B\bullet}{\vec{\Theta}}_{B/A} = \overset{B\bullet}{\vec{\Theta}}_{B/A} - \alpha\vec{\Theta}_{B/A}\times(\overset{B\bullet}{\vec{\Theta}}_{B/A}\times\vec{\Theta}_{B/A}) - \beta\,\overset{B\bullet}{\vec{\Theta}}_{B/A}\times\vec{\Theta}_{B/A}. \tag{7.9.10}$$

In addition,

$$\overset{\mathrm{B}\bullet\mathrm{B}\bullet}{\vec{S}\,\vec{\Theta}}_{\mathrm{B/A}} = \dot{\alpha}\vec{\Theta}_{\mathrm{B/A}}^{\times 2}\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} + \alpha\overset{\mathrm{B}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}}\vec{\Theta}_{\mathrm{B/A}}^{\times}\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} + \dot{\beta}\vec{\Theta}_{\mathrm{B/A}}^{\times}\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}. \tag{7.9.11}$$

7.10 6D Dynamics of a Rigid Body

In this section we express the translational and rotational dynamics of a rigid body in terms of 6D vectors. In the next section, this formulation is applied to the chain of rigid bodies considered in Section 2.22, Section 4.13, and Section 4.14.

Let $\mathcal{B}$ be a rigid body, let c denote the center of mass of $\mathcal{B}$, let z be a point fixed in $\mathcal{B}$, let w be an unforced particle, let $\mathrm{F_B}$ be a body-fixed frame, and let $\mathrm{F_A}$ be an inertial frame. In addition, let $\vec{f}_{\mathcal{B}}$ denote the total force applied to $\mathcal{B}$, and let $\vec{M}_{\mathcal{B}/z}$ denote the total moment applied to $\mathcal{B}$ relative to z. It follows from (7.4.17) that the translational dynamics of $\mathcal{B}$ are given by

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\alpha}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}) = \vec{f}_{\mathcal{B}}. \tag{7.10.1}$$

and from (7.8.7) that the rotational dynamics of $\mathcal{B}$ are given by

$$\vec{J}_{\mathcal{B}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.10.2}$$

which can be rewritten as

$$\vec{J}_{\mathcal{B}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{z/w/\mathrm{A}}) = \vec{M}_{\mathcal{B}/z}, \tag{7.10.3}$$

To construct a single equation for both the translational and rotational dynamics, note that (7.10.1) and (7.10.3) can be written as the single equation

$$\begin{bmatrix} m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\alpha}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}) \\ \vec{J}_{\mathcal{B}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{z/w/\mathrm{A}}) \end{bmatrix} = \begin{bmatrix} \vec{f}_{\mathcal{B}} \\ \vec{M}_{\mathcal{B}/z} \end{bmatrix}. \tag{7.10.4}$$

Using Jacobi's identity given by Problem 2.26.6 in the form

$$m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{z/w/\mathrm{A}}) = \vec{\omega}_{\mathrm{B/A}} \times (m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \vec{v}_{z/w/\mathrm{A}}) - \vec{v}_{z/w/\mathrm{A}} \times (m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \vec{\omega}_{\mathrm{B/A}}), \tag{7.10.5}$$

(7.10.4) can be rewritten as

$$\begin{bmatrix} m_{\mathcal{B}}\vec{I} & -m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\times} \\ m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\times} & \vec{J}_{\mathcal{B}/z} \end{bmatrix} \overbrace{\begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}}^{\mathrm{B}\bullet} + \begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}^{\times} \begin{bmatrix} m_{\mathcal{B}}\vec{I} & -m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\times} \\ m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\times} & \vec{J}_{\mathcal{B}/z} \end{bmatrix} \begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix} = \begin{bmatrix} \vec{f}_{\mathcal{B}} \\ \vec{M}_{\mathcal{B}/z} \end{bmatrix}, \tag{7.10.6}$$

where

$$\begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}^{\times} \triangleq \begin{bmatrix} \vec{\omega}_{\mathrm{B/A}}^{\times} & 0 \\ \vec{v}_{z/w/\mathrm{A}}^{\times} & \vec{\omega}_{\mathrm{B/A}}^{\times} \end{bmatrix}. \tag{7.10.7}$$

By defining

$$\vec{\mathcal{J}}_{\mathcal{B}/z} \triangleq \begin{bmatrix} m_{\mathcal{B}}\vec{I} & -m_{\mathcal{B}}\vec{r}^{\times}_{\mathrm{c}/z} \\ m_{\mathcal{B}}\vec{r}^{\times}_{\mathrm{c}/z} & \vec{J}_{\mathcal{B}/z} \end{bmatrix}, \quad \vec{\mathcal{V}}_{\mathcal{B}} \triangleq \begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}, \quad \vec{\mathcal{F}}_{\mathcal{B}/z} \triangleq \begin{bmatrix} \vec{f}_{\mathcal{B}} \\ \vec{M}_{\mathcal{B}/z} \end{bmatrix}, \tag{7.10.8}$$

(7.10.6) can be written as

$$\vec{\mathcal{J}}_{\mathcal{B}/z}\vec{\mathcal{A}}_{\mathcal{B}} + \vec{\mathcal{V}}^{\times}_{\mathcal{B}}\vec{\mathcal{J}}_{\mathcal{B}/z}\vec{\mathcal{V}}_{\mathcal{B}} = \vec{\mathcal{F}}_{\mathcal{B}/z}. \tag{7.10.9}$$

where

$$\vec{\mathcal{A}}_{\mathcal{B}} \triangleq \overset{\mathrm{B}\bullet}{\vec{\mathcal{V}}_{\mathcal{B}}}. \tag{7.10.10}$$

Resolving (7.10.9) in $\mathrm{F_B}$ yields

$$\mathcal{J}_{\mathcal{B}/z|\mathrm{B}}\mathcal{A}_{\mathcal{B}|\mathrm{B}} + \mathcal{V}^{\times}_{\mathcal{B}|\mathrm{B}}\mathcal{J}_{\mathcal{B}/z|\mathrm{B}}\mathcal{V}_{\mathcal{B}|\mathrm{B}} = \mathcal{F}_{\mathcal{B}/z|\mathrm{B}}, \tag{7.10.11}$$

where

$$\mathcal{J}_{\mathcal{B}/z|\mathrm{B}} \triangleq \begin{bmatrix} m_{\mathcal{B}}I & -m_{\mathcal{B}}r^{\times}_{\mathrm{c}/z|\mathrm{B}} \\ m_{\mathcal{B}}r^{\times}_{\mathrm{c}/z|\mathrm{B}} & J_{\mathcal{B}/z|\mathrm{B}} \end{bmatrix}, \quad \mathcal{V}_{\mathcal{B}|\mathrm{B}} \triangleq \begin{bmatrix} v_{z/w/\mathrm{A|B}} \\ \omega_{\mathrm{B/A|B}} \end{bmatrix}, \quad \mathcal{A}_{\mathcal{B}|\mathrm{B}} \triangleq \dot{\mathcal{V}}_{\mathcal{B}|\mathrm{B}} = \begin{bmatrix} \dot{v}_{z/w/\mathrm{A|B}} \\ \alpha_{\mathrm{B/A|B}} \end{bmatrix}, \tag{7.10.12}$$

$$\mathcal{V}^{\times}_{\mathcal{B}|\mathrm{B}} \triangleq \begin{bmatrix} \omega^{\times}_{\mathrm{B/A|B}} & 0 \\ v^{\times}_{z/w/\mathrm{A|B}} & \omega^{\times}_{\mathrm{B/A|B}} \end{bmatrix}, \quad \mathcal{F}_{\mathcal{B}/z|\mathrm{B}} \triangleq \begin{bmatrix} f_{\mathcal{B}|\mathrm{B}} \\ M_{\mathcal{B}/z|\mathrm{B}} \end{bmatrix}. \tag{7.10.13}$$

7.11 6D Dynamics of a Chain of Rigid Bodies

For the chain of rigid bodies shown in Figure 2.22.1, assume that z_{A} is colocated with an unforced particle, let $\mathrm{c_B}$, $\mathrm{c_C}$, and $\mathrm{c_D}$ denote the center of mass of $\mathcal{B}_{\mathrm{B}}$, $\mathcal{B}_{\mathrm{C}}$, and $\mathcal{B}_{\mathrm{D}}$, respectively. and define

$$\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}} \triangleq \begin{bmatrix} m_{\mathrm{B}}\vec{I} & -m_{\mathrm{B}}\vec{r}^{\times}_{\mathrm{c_B}/z_{\mathrm{B}}} \\ m_{\mathrm{B}}\vec{r}^{\times}_{\mathrm{c_B}/z_{\mathrm{B}}} & \vec{J}_{\mathrm{B}/z_{\mathrm{B}}} \end{bmatrix}, \quad \vec{\mathcal{V}}_{\mathrm{B}} \triangleq \begin{bmatrix} \vec{v}_{z_{\mathrm{B}}/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}, \quad \vec{\mathcal{F}}_{\mathrm{B}/z_{\mathrm{B}}} \triangleq \begin{bmatrix} \vec{f}_{\mathrm{B}} \\ \vec{M}_{\mathrm{B}/z_{\mathrm{B}}} \end{bmatrix}, \tag{7.11.1}$$

and likewise for $\mathcal{B}_{\mathrm{C}}$ and $\mathcal{B}_{\mathrm{D}}$. Note that the moment $\vec{M}_{\mathrm{B}/z_{\mathrm{B}}}$ on $\mathcal{B}_{\mathrm{B}}$ relative to z_{B} includes external moments on $\mathcal{B}_{\mathrm{B}}$ relative to z_{B}, reaction moments on $\mathcal{B}_{\mathrm{B}}$ due forces applied to $\mathcal{B}_{\mathrm{B}}$ at the joints connecting $\mathcal{B}$ to $\mathcal{A}$ and $\mathcal{C}$, and reaction torques on $\mathcal{B}_{\mathrm{B}}$ due to the joints connecting $\mathcal{B}$ to $\mathcal{A}$ and $\mathcal{C}$. It thus follows from (7.10.9) that

$$\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B}} + \vec{\mathcal{V}}^{\times}_{\mathrm{B}}\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{V}}_{\mathrm{B}} = \vec{\mathcal{F}}_{\mathrm{B}/z_{\mathrm{B}}}, \tag{7.11.2}$$

$$\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C}} + \vec{\mathcal{V}}^{\times}_{\mathrm{C}}\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{V}}_{\mathrm{C}} = \vec{\mathcal{F}}_{\mathrm{C}/z_{\mathrm{C}}}, \tag{7.11.3}$$

$$\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D}} + \vec{\mathcal{V}}^{\times}_{\mathrm{D}}\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{V}}_{\mathrm{D}} = \vec{\mathcal{F}}_{\mathrm{D}/z_{\mathrm{D}}}. \tag{7.11.4}$$

$$\tag{7.11.5}$$

where

$$\vec{\mathcal{A}}_{\mathrm{B}} \triangleq \overset{\mathrm{B}\bullet}{\vec{\mathcal{V}}_{\mathrm{B}}}, \quad \vec{\mathcal{A}}_{\mathrm{C}} \triangleq \overset{\mathrm{C}\bullet}{\vec{\mathcal{V}}_{\mathrm{C}}}, \quad \vec{\mathcal{A}}_{\mathrm{D}} \triangleq \overset{\mathrm{D}\bullet}{\vec{\mathcal{V}}_{\mathrm{D}}}. \tag{7.11.6}$$

The forces and moments can be written as

$$\vec{\mathcal{F}}_{\mathrm{B}/z_{\mathrm{B}}} = \vec{\mathcal{F}}_{\mathrm{ext,B}/z_{\mathrm{B}}} + \vec{\mathcal{F}}_{\mathrm{int,B}}, \tag{7.11.7}$$

$$\vec{\mathcal{F}}_{\mathrm{C}/z_{\mathrm{C}}} = \vec{\mathcal{F}}_{\mathrm{ext,C}/z_{\mathrm{C}}} + \vec{\mathcal{F}}_{\mathrm{int,C}}, \tag{7.11.8}$$

$$\vec{\mathcal{F}}_{\mathrm{D}/z_{\mathrm{D}}} = \vec{\mathcal{F}}_{\mathrm{ext,D}/z_{\mathrm{D}}} + \vec{\mathcal{F}}_{\mathrm{int,D}}, \tag{7.11.9}$$

where "ext" denotes externally applied forces and moments, and "int" denotes reaction forces and torques applied at the joints. Hence,

$$\vec{\mathcal{F}}_{\mathrm{int,B}} = \vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{A}}} + \vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{B}}} = \vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{A}}} - \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}}, \tag{7.11.10}$$

$$\vec{\mathcal{F}}_{\mathrm{int,C}} = -\vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{B}}} + \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{C}}} = \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}} - \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}, \tag{7.11.11}$$

$$\vec{\mathcal{F}}_{\mathrm{int,D}} = -\vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{C}}} = \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}. \tag{7.11.12}$$

Combining (7.11.2)–(7.11.4), (7.11.7)–(7.11.9), and (7.11.10)–(7.11.12) yields

$$\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B}} + \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{B}}\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{V}}_{\mathrm{B}} = \vec{\mathcal{F}}_{\mathrm{ext,B}/z_{\mathrm{B}}} + \vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{A}}} - \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}}, \tag{7.11.13}$$

$$\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C}} + \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{C}}\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{V}}_{\mathrm{C}} = \vec{\mathcal{F}}_{\mathrm{ext,C}/z_{\mathrm{C}}} + \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}} - \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}, \tag{7.11.14}$$

$$\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D}} + \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{D}}\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{V}}_{\mathrm{D}} = \vec{\mathcal{F}}_{\mathrm{ext,D}/z_{\mathrm{D}}} + \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}. \tag{7.11.15}$$

Next, writing $\vec{\mathcal{A}}_{\mathrm{B}}$, $\vec{\mathcal{A}}_{\mathrm{B}}$, and $\vec{\mathcal{A}}_{\mathrm{B}}$ as

$$\vec{\mathcal{A}}_{\mathrm{B}} = \vec{\mathcal{A}}_{\mathrm{B},\alpha} + \vec{\mathcal{A}}_{\mathrm{B},\omega}, \quad \vec{\mathcal{A}}_{\mathrm{C}} = \vec{\mathcal{A}}_{\mathrm{C},\alpha} + \vec{\mathcal{A}}_{\mathrm{C},\omega}, \quad \vec{\mathcal{A}}_{\mathrm{D}} = \vec{\mathcal{A}}_{\mathrm{D},\alpha} + \vec{\mathcal{A}}_{\mathrm{D},\omega}. \tag{7.11.16}$$

where $\vec{\mathcal{A}}_{\mathrm{B},\alpha}$, $\vec{\mathcal{A}}_{\mathrm{C},\alpha}$, $\vec{\mathcal{A}}_{\mathrm{D},\alpha}$, $\vec{\mathcal{A}}_{\mathrm{B},\omega}$, $\vec{\mathcal{A}}_{\mathrm{C},\omega}$, $\vec{\mathcal{A}}_{\mathrm{D},\omega}$ are defined by (4.14.8)–(4.14.13), it follows that (7.11.13)–(7.11.15) can be written as

$$\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\alpha} = -\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\omega} - \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{B}}\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{V}}_{\mathrm{B}} + \vec{\mathcal{F}}_{\mathrm{ext,B}/z_{\mathrm{B}}} + \vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{A}}} - \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}}, \tag{7.11.17}$$

$$\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\alpha} = -\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\omega} - \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{C}}\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{V}}_{\mathrm{C}} + \vec{\mathcal{F}}_{\mathrm{ext,C}/z_{\mathrm{C}}} + \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}} - \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}, \tag{7.11.18}$$

$$\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\alpha} = -\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\omega} - \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{D}}\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{V}}_{\mathrm{D}} + \vec{\mathcal{F}}_{\mathrm{ext,D}/z_{\mathrm{D}}} + \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}. \tag{7.11.19}$$

Finally, we reorder and rewrite (7.11.17)–(7.11.19) as

$$\vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}} = \vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\alpha} + \vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\omega} + \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{D}}\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{V}}_{\mathrm{D}} - \vec{\mathcal{F}}_{\mathrm{ext,D}/z_{\mathrm{D}}}, \tag{7.11.20}$$

$$\vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}} = \vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\alpha} + \vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\omega} + \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{C}}\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{V}}_{\mathrm{C}} - \vec{\mathcal{F}}_{\mathrm{ext,C}/z_{\mathrm{C}}} + \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}, \tag{7.11.21}$$

$$\vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{A}}} = \vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\alpha} + \vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\omega} + \overset{\times}{\vec{\mathcal{V}}}_{\mathrm{B}}\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{V}}_{\mathrm{B}} - \vec{\mathcal{F}}_{\mathrm{ext,B}/z_{\mathrm{B}}} + \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}}. \tag{7.11.22}$$

Note that, assuming that the velocities and accelerations are known, (7.11.20)–(7.11.22) can be solved recursively for the reaction torques and moments. The same technique can be applied to a chain consisting of $n \geq 4$ rigid bodies, where the recursion proceeds from the bottom link of the chain to the top link of the chain.

7.12 Forces and Moments Due to Springs, Dashpots, and Inerters

Forces can be exerted on a pair of particles or inertia points by a spring, dashpot, or inerter. These forces are internal forces that satisfy Newton's third law. Likewise, a moment can be exerted on a pair of rigid bodies by a rotational spring, rotational dashpot, or rotational inerter. Springs and rotational springs were discussed in Section 7.4. A Newtonian body may possess springs, dashpots, and inerters as well as rotational springs, rotational dashpots, and rotational inerters.

Consider a dashpot connected to the particles y and w, and assume that the viscosity of the dashpot is c. Then, the force $\vec{f}_{y/w}$ applied to y by the dashpot is given by

$$\vec{f}_{y/w} = -c\left(\frac{\mathrm{d}}{\mathrm{d}t}|\vec{r}_{y/w}|\right)\hat{r}_{y/w}. \tag{7.12.1}$$

Furthermore,

$$\vec{f}_{w/y} = -\vec{f}_{y/w}. \tag{7.12.2}$$

Consider an inerter connected to the particles y and w, and assume that the inertance of the inerter is b. Then, the force $\vec{f}_{y/w}$ applied to y by the inerter is given by

$$\vec{f}_{y/w} = -b\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}|\vec{r}_{y/w}|\right)\hat{r}_{y/w}. \tag{7.12.3}$$

Furthermore,

$$\vec{f}_{w/y} = -\vec{f}_{y/w}. \tag{7.12.4}$$

Next, consider rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ that are connected by a pin joint at a point fixed in both bodies. Let $\hat{z}$ be a unit dimensionless vector that is parallel with the pin joint. A rotational dashpot applies torques to $\mathcal{B}_1$ and $\mathcal{B}_2$ that are parallel with $\hat{z}$. Let $\hat{x}_1$ and $\hat{x}_2$ be unit dimensionless vectors that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that are orthogonal to $\hat{z}$. Assume that the rotational viscosity of the rotational spring is $\xi > 0$. Then, the torque $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}$ applied to $\mathcal{B}_1$ by the rotational dashpot is given by

$$\vec{M}_{\mathcal{B}_1/\mathcal{B}_2} = -\xi\dot{\theta}_{\vec{x}_1/\vec{x}_2/\hat{z}}\hat{z}. \tag{7.12.5}$$

Furthermore, the torques applied to $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal and opposite, that is,

$$\vec{M}_{\mathcal{B}_2/\mathcal{B}_1} = -\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}. \tag{7.12.6}$$

Finally, consider rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ that are connected by a pin joint at a point fixed in both bodies. Let $\hat{z}$ be a unit dimensionless vector that is parallel with the pin joint. A rotational dashpot applies torques to $\mathcal{B}_1$ and $\mathcal{B}_2$ that are parallel with $\hat{z}$. Let $\hat{x}_1$ and $\hat{x}_2$ be unit dimensionless vectors that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that are orthogonal to $\hat{z}$. Assume that the rotational inertance of the rotational spring is $\beta > 0$. Then, the torque $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}$ applied to $\mathcal{B}_1$ by the rotational inerter is given by

$$\vec{M}_{\mathcal{B}_1/\mathcal{B}_2} = -\beta\ddot{\theta}_{\vec{x}_1/\vec{x}_2/\hat{z}}\hat{z}. \tag{7.12.7}$$

Furthermore, the torques applied to $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal and opposite, that is,

$$\vec{M}_{\mathcal{B}_2/\mathcal{B}_1} = -\vec{M}_{\mathcal{B}_1/\mathcal{B}_2}. \tag{7.12.8}$$

7.13 Collisions

Consider a body $\mathcal{B}$ consisting of particles y_1 and y_2 whose masses are m_1 and m_2, respectively, let c be the center of mass of $\mathcal{B}$, let w be an unforced particle, and let $\mathrm{F_A}$ be an inertial frame. It thus follows from (7.4.5) and (7.4.6) that

$$(m_1 + m_2)\vec{v}_{\mathrm{c}/w/\mathrm{A}} = m_1 \vec{v}_{y_1/w/\mathrm{A}} + m_2 \vec{v}_{y_2/w/\mathrm{A}}, \tag{7.13.1}$$

$$(m_1 + m_2)\vec{a}_{\mathrm{c}/w/\mathrm{A}} = m_1 \vec{a}_{y_1/w/\mathrm{A}} + m_2 \vec{a}_{y_2/w/\mathrm{A}}. \tag{7.13.2}$$

Assuming that the external force applied to $\mathcal{B}$ is zero, it follows that $\vec{a}_{\mathrm{c}/w/\mathrm{A}} = 0$. It thus follows from (7.13.1) and (7.13.2) that

$$\overbrace{m_1 \vec{v}_{y_1/w/\mathrm{A}} + m_2 \vec{v}_{y_2/w/\mathrm{A}}}^{\mathrm{A}\bullet} = 0, \tag{7.13.3}$$

which shows that the translational momentum of $\mathcal{B}$ is conserved with respect to $\mathrm{F_A}$.

Next, writing (7.13.3) as

$$\overbrace{m_1 \vec{v}_{y_1/w/\mathrm{A}}\Big|_{\mathrm{A}} + m_2 \vec{v}_{y_2/w/\mathrm{A}}\Big|_{\mathrm{A}}}^{\cdot} = 0 \tag{7.13.4}$$

and letting

$$\vec{v}_{y_1/w/\mathrm{A}}(t)\Big|_{\mathrm{A}} = \begin{bmatrix} u_1(t) \\ v_1(t) \\ w_1(t) \end{bmatrix}, \qquad \vec{v}_{y_2/w/\mathrm{A}}(t)\Big|_{\mathrm{A}} = \begin{bmatrix} u_2(t) \\ v_2(t) \\ w_2(t) \end{bmatrix}, \tag{7.13.5}$$

it follows that, for all times t_1 and t_2,

$$m_1 \begin{bmatrix} u_1(t_1) \\ v_1(t_1) \\ w_1(t_1) \end{bmatrix} + m_2 \begin{bmatrix} u_2(t_1) \\ v_2(t_1) \\ w_2(t_1) \end{bmatrix} = m_1 \begin{bmatrix} u_1(t_2) \\ v_1(t_2) \\ w_1(t_2) \end{bmatrix} + m_2 \begin{bmatrix} u_2(t_2) \\ v_2(t_2) \\ w_2(t_2) \end{bmatrix}. \tag{7.13.6}$$

Now, we view y_1 and y_2 as small spheres, and assume that y_1 and y_2 collide and thus are in contact during a short time interval centered at time t_c, where $t_1 < t_\mathrm{c} < t_2$. When y_1 and y_2 are in contact, it follows from Newton's third law that equal and opposite reaction forces $\vec{f}_\mathrm{R}$ and $-\vec{f}_\mathrm{R}$ are applied to the particles y_1 and y_2, respectively, at the point of contact. Therefore, during contact, it follows that

$$m_1 \vec{a}_{y_1/w/\mathrm{A}} = \vec{f}_\mathrm{R}, \tag{7.13.7}$$

$$m_2 \vec{a}_{y_2/w/\mathrm{A}} = -\vec{f}_\mathrm{R}. \tag{7.13.8}$$

For convenience, we assume that these reaction forces are parallel to $\hat{\imath}_\mathrm{B}$, and thus the force applied to the particles in the directions of $\hat{\jmath}_\mathrm{A}$ and $\hat{k}_\mathrm{A}$ are zero. Therefore, the components of $\vec{v}_{y_1/w/\mathrm{A}}$ and $\vec{v}_{y_2/w/\mathrm{A}}$ in the directions $\hat{\jmath}_\mathrm{A}$ and $\hat{k}_\mathrm{A}$ are constant, that is,

$$v_1(t_1) = v_1(t_2), \qquad v_2(t_1) = v_2(t_2), \tag{7.13.9}$$

$$w_1(t_1) = w_1(t_2), \qquad w_2(t_1) = w_2(t_2). \tag{7.13.10}$$

In the direction $\hat{\imath}_{\mathrm{A}}$ it follows from (7.13.6) that

$$m_1u_1(t_1) + m_2u_2(t_1) = m_1u_1(t_2) + m_2u_2(t_2). \tag{7.13.11}$$

Next, define

$$u_{\mathrm{c}} \triangleq u_1(t_{\mathrm{c}}) = u_2(t_{\mathrm{c}}), \tag{7.13.12}$$

which is the common speed of y_1 and y_2 in the direction $\hat{\imath}_{\mathrm{A}}$ at t_{c}. Therefore, by conservation of translational momentum, it follows from (7.13.11) that

$$m_1u_1(t_1) + m_2u_2(t_1) = m_1u_{\mathrm{c}} + m_2u_{\mathrm{c}} = m_1u_1(t_2) + m_2u_2(t_2). \tag{7.13.13}$$

Hence,

$$m_1u_1(t_2) - m_1u_{\mathrm{c}} = -[m_2u_2(t_2) - m_2u_{\mathrm{c}}] \tag{7.13.14}$$

and

$$m_1u_{\mathrm{c}} - m_1u_1(t_1) = -[m_2u_{\mathrm{c}} - m_2u_2(t_1)]. \tag{7.13.15}$$

Dividing the left- and right-hand sides of (7.13.14) and (7.13.15) yields

$$e \triangleq \frac{u_1(t_2) - u_{\mathrm{c}}}{u_{\mathrm{c}} - u_1(t_1)} = \frac{u_2(t_2) - u_{\mathrm{c}}}{u_{\mathrm{c}} - u_2(t_1)}, \tag{7.13.16}$$

where e is the *coefficient of restitution*.

Next, it follows from (7.13.16) that

$$u_{\mathrm{c}} = \frac{u_1(t_1)u_2(t_2) - u_1(t_2)u_2(t_1)}{u_1(t_1) + u_2(t_2) - u_1(t_2) - u_2(t_1)}. \tag{7.13.17}$$

Substituting this expression into (7.13.16) yields

$$e = \frac{u_1(t_2) - u_2(t_2)}{u_2(t_1) - u_1(t_1)}. \tag{7.13.18}$$

Hence, e is the ratio of the relative velocities of y_1 and y_2 in the direction $\hat{\imath}_{\mathrm{A}}$ before and after the collision. In particular, $e = 0$ corresponds to the case in which the particles are stuck together after the collision, whereas $e = 1$ captures the case in which the relative speed reverses its sign.

For the following result, let $T_{\mathcal{B}/w/\mathrm{A}}(t)$ be the kinetic energy of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$, which is given by

$$T_{\mathcal{B}/w/\mathrm{A}}(t) = \tfrac{1}{2}m_1[u_1^2(t) + v_1^2(t) + w_1^2(t)] + \tfrac{1}{2}m_2[u_2^2(t) + v_2^2(t) + w_2^2(t)]). \tag{7.13.19}$$

Fact 7.13.1. Let y_1 and y_2 be small spheres whose masses are m_1 and m_2, respectively, let w be an unforced particle, and let $\mathrm{F_A}$ be an inertial frame. Assume that a collision between y_1 and y_2 occurs at time t_{c}, assume that the reaction forces are parallel to $\hat{\imath}_{\mathrm{A}}$, let $u_1(t)$ and $u_2(t)$ be the signed speeds of y_1 and y_2 in the direction $\hat{\imath}_{\mathrm{A}}$, that is, $u_1(t) = \hat{\imath}'_{\mathrm{A}}\vec{v}_{y_1/w/\mathrm{A}}(t)$ and $u_2(t) = \hat{\imath}'_{\mathrm{A}}\vec{v}_{y_2/w/\mathrm{A}}(t)$, and let $t_1 < t_{\mathrm{c}} < t_2$. Then,

$$u_1(t_2) = \frac{m_1 - em_2}{m_1 + m_2}u_1(t_1) + \frac{(1+e)m_2}{m_1 + m_2}u_2(t_1), \tag{7.13.20}$$

$$u_2(t_2) = \frac{(1+e)m_1}{m_1 + m_2}u_1(t_1) + \frac{m_2 - em_1}{m_1 + m_2}u_2(t_1). \tag{7.13.21}$$

Furthermore,

$$T_{\mathcal{B}/w/\mathrm{A}}(t_2) = T_{\mathcal{B}/w/\mathrm{A}}(t_1) - (1-e^2)\frac{m_1 m_2}{m_1+m_2}[u_1(t_1) - u_2(t_1)]^2. \tag{7.13.22}$$

Finally,

$$0 \le e \le 1. \tag{7.13.23}$$

Proof. Using (7.13.11) and (7.13.18) yields (7.13.20) and (7.13.21), which in turn imply (7.13.22). To prove (7.13.23), we assume that y_1 and y_2 move along a line that is parallel to $\hat{\imath}_\mathrm{A}$. Consider the case where $\hat{\imath}'_\mathrm{A}\vec{r}_{y_1/y_2}(t_1) > 0$, $u_2(t_1) > 0$, and $u_1(t_1) < u_2(t_1)$. With these relative locations and signed speeds, a collision occurs at some time $t_\mathrm{c} > t_1$. At time t_c, $\hat{\imath}'_\mathrm{A}\vec{r}_{y_1/y_2}(t_\mathrm{c}) = 0$ and $u_1(t_\mathrm{c}) = u_2(t_\mathrm{c})$. Furthermore, the reaction force $\vec{f}_\mathrm{R}$ applied to y_1 is given by $\vec{f}_\mathrm{R} = f_\mathrm{R}\hat{\imath}_\mathrm{A}$, where $f_\mathrm{R} > 0$, whereas the reaction force $-\vec{f}_\mathrm{R}$ applied to y_2 is given by $-\vec{f}_\mathrm{R} = -f_\mathrm{R}\hat{\imath}_\mathrm{A}$, where $-f_\mathrm{R} < 0$. Consequently, at time t_2 after the collision, it follows that $\hat{\imath}'_\mathrm{A}\vec{r}_{y_1/y_2}(t_2) > 0$ and $u_2(t_2) \le u_1(t_2)$, where equality holds if and only if the particles stick together. Since $u_1(t_2) - u_2(t_2) \ge 0$ and $u_2(t_1) - u_1(t_1) > 0$, it follows that $e \ge 0$. An analogous argument applies in the case where $\hat{\imath}'_\mathrm{A}\vec{r}_{y_2/y_1}(t_1) > 0$, $u_2(t_1) > 0$, and $u_2(t_1) < u_1(t_1)$. In both cases, $e \ge 0$. Using (7.13.22), conservation of energy implies that $e \le 1$.

The case where y_1 and y_2 are not necessarily moving along the same line is left to the reader. □

Note that the second term on the right-hand side of (7.13.22) represents the amount of energy dissipated in the collision. Since the occurrence of a collision implies that $u_1(t_1) \ne u_2(t_1)$, it follows that energy is conserved if and only if $e = 1$. Furthermore, the amount of dissipated energy is larger for smaller values of e.

7.14 Center of Percussion and Percussive Center of Rotation

It follows from (7.4.7) that a force applied to a rigid body produces an acceleration of the center of mass of the body. Furthermore, it follows from (7.8.13) that, if the line of force does not pass through the center of mass of the body, then the moment on the body relative to the center of mass is nonzero and produces an angular acceleration of the body. It is reasonable to expect that the combined translational acceleration of the center of mass and angular acceleration of the body induces a change in the velocity of every point on the body. If, however, an impulsive force is applied to a point P in the body, then it turns out that there exists a point R in the body that does not initially accelerate. The point P is the *center of percussion*, and the point R is the *percussive center of rotation.* This property is made precise by the following result. The notation "0^-" denotes the left limit toward zero, which can be viewed as the response immediately after an impulse at time $t = 0$.

Fact 7.14.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let $\mathrm{F_A}$ be an inertial frame, assume that $\vec{\omega}_{\mathrm{B/A}}(0) = 0$, let w be an unforced particle, let P be a point fixed in $\mathcal{B}$, let $\vec{f}(t) = f_0\delta(t)\hat{n}_\mathrm{P}$ denote an impulsive force applied to $\mathcal{B}$ at P, and assume that $\hat{n}_\mathrm{P}$ and $\vec{r}_{\mathrm{P/c}}$ are not parallel and $\hat{n}_\mathrm{P}$ is a principal axis of $\vec{\vec{J}}_{\mathcal{B}/\mathrm{c}}$. Then there exists a point R fixed in $\mathcal{B}$ such that $\vec{v}_{\mathrm{R}/w/\mathrm{A}}(0^-) = 0$. In particular, the location of one such point R is given by

$$\vec{r}_{\mathrm{R/c}} = \frac{1}{m|\vec{\vec{J}}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_\mathrm{P})|^2}[\vec{\vec{J}}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_\mathrm{P})]\times\hat{n}_\mathrm{P}. \tag{7.14.1}$$

Proof. It follows from (7.4.7) that

$$m\vec{v}_{\mathrm{c}/w/\mathrm{A}}(0^-) = f_0\hat{n}_{\mathrm{P}} \tag{7.14.2}$$

and, since $\vec{\omega}_{\mathrm{B/A}}(0) = 0$, it follows from (7.8.13) that

$$\vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}}(0^-) = f_0\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}}. \tag{7.14.3}$$

Using (7.14.2) and (7.14.3) yields

$$\begin{aligned}\vec{v}_{\mathrm{R}/w/\mathrm{A}}(0^-) &= \vec{v}_{\mathrm{R/c/A}}(0^-) + \vec{v}_{\mathrm{c}/w/\mathrm{A}}(0^-)\\ &= \vec{\omega}_{\mathrm{B/A}}(0^-) \times \vec{r}_{\mathrm{R/c}} + \vec{v}_{\mathrm{c}/w/\mathrm{A}}(0^-)\\ &= [f_0\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}})] \times \vec{r}_{\mathrm{R/c}} + \vec{v}_{\mathrm{c}/w/\mathrm{A}}(0^-)\\ &= f_0(\vec{a} \times \vec{r}_{\mathrm{R/c}} + \frac{1}{m}\hat{n}_{\mathrm{P}}),\end{aligned} \tag{7.14.4}$$

where $\vec{a} \triangleq \vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}})$. Since, by assumption, $\hat{n}_{\mathrm{P}}$ and $\vec{r}_{\mathrm{P/c}}$ are not parallel, it follows that $\vec{a}$ is not zero. In addition, since, by assumption, $\hat{n}_{\mathrm{P}}$ is a principal axis of $\vec{J}_{\mathcal{B}/\mathrm{c}}$, it follows that there exists $\gamma > 0$ such that $\vec{J}_{\mathcal{B}/\mathrm{c}}\hat{n}_{\mathrm{P}} = \gamma\hat{n}_{\mathrm{P}}$, and thus $\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}\hat{n}_{\mathrm{P}} = (1/\gamma)\hat{n}_{\mathrm{P}}$. Therefore,

$$\hat{n}'_{\mathrm{P}}\vec{a} = \hat{n}'_{\mathrm{P}}\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}}) = (1/\gamma)\hat{n}'_{\mathrm{P}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}}) = 0. \tag{7.14.5}$$

Since $\vec{a}$ and $\hat{n}_{\mathrm{P}}$ are mutually orthogonal, it follows that $\vec{a} \times \hat{n}_{\mathrm{P}}$ is not zero. Hence, let R be the point fixed in $\mathcal{B}$ such that

$$\vec{r}_{\mathrm{R/c}} = \alpha\vec{a} \times \hat{n}_{\mathrm{P}}, \tag{7.14.6}$$

where $\alpha \triangleq 1/(m|\vec{a}|^2)$. It thus follows from (7.14.5) that

$$\begin{aligned}\vec{v}_{\mathrm{R}/w/\mathrm{A}}(0^-) &= f_0(\vec{a} \times \vec{r}_{\mathrm{R/c}} + \frac{1}{m}\hat{n}_{\mathrm{P}})\\ &= f_0[\vec{a} \times (\alpha\vec{a} \times \hat{n}_{\mathrm{P}}) + \frac{1}{m}\hat{n}_{\mathrm{P}}]\\ &= f_0(-\alpha|\vec{a}|^2\hat{n}_{\mathrm{P}} + \frac{1}{m}\hat{n}_{\mathrm{P}})\\ &= 0.\end{aligned}$$

□

Example 7.14.2. Consider a free rigid body $\mathcal{B}$ consisting of particles y_1 and y_2 with masses m_1 and m_2, respectively, connected by a massless rigid link of length ℓ. The body frame is chosen such that $\vec{r}_{y_2/y_1} = \ell\hat{\imath}_{\mathrm{B}}$. The location of the center of mass is $\vec{r}_{\mathrm{c}/y_1} = \ell_{\mathrm{c}}\hat{\imath}_{\mathrm{B}}$, where $\ell_{\mathrm{c}} \triangleq \frac{m_2\ell}{m_1+m_2}$, and the physical inertia matrix is given by

$$\vec{J}_{\mathcal{B}/\mathrm{c}} = \varepsilon\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + \beta(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}), \tag{7.14.7}$$

where ε is a small positive number that allows inversion of $\vec{J}_{\mathcal{B}/\mathrm{c}}$ and

$$\beta \triangleq m_1\ell_{\mathrm{c}}^2 + m_2(\ell - \ell_{\mathrm{c}})^2 = \frac{m_1m_2\ell^2}{m_1 + m_2}. \tag{7.14.8}$$

Hence,

$$\vec{J}_{\mathcal{B}/c}^{-1} = \frac{1}{\varepsilon}\hat{\imath}_B\hat{\imath}_B' + \frac{1}{\beta}(\hat{\jmath}_B\hat{\jmath}_B' + \hat{k}_B\hat{k}_B'). \tag{7.14.9}$$

The impulsive force applied to $\mathcal{B}$ at the point P is given by $\vec{f}(t) = f_0\delta(t)\hat{n}_P$, where $\hat{n}_P = \hat{\jmath}_B$ and $\vec{r}_{P/y_1} = \ell_P\hat{\imath}_B$. It thus follows that

$$\begin{aligned}\vec{J}_{\mathcal{B}/c}^{-1}(\vec{r}_{P/c}\times\hat{n}_P) &= [\frac{1}{\varepsilon}\hat{\jmath}_B\hat{\jmath}_B' + \frac{1}{\beta}(\hat{\jmath}_B\hat{\jmath}_B' + \hat{k}_B\hat{k}_B')][(\ell_P - \ell_c)\hat{\imath}_B \times \hat{\jmath}_B]\\ &= \frac{\ell_P - \ell_c}{\beta}\hat{k}_B.\end{aligned} \tag{7.14.10}$$

Therefore, it follows from (7.14.1) that

$$\begin{aligned}\vec{r}_{R/c} &= \frac{1}{(m_1+m_2)|\vec{J}_{\mathcal{B}/c}^{-1}(\vec{r}_{P/c}\times\hat{n}_P)|^2}[\vec{J}_{\mathcal{B}/c}^{-1}(\vec{r}_{P/c}\times\hat{n}_P)]\times\hat{n}_P\\ &= -\frac{\beta^2}{(m_1+m_2)(\ell_P-\ell_c)^2}\frac{\ell_P-\ell_c}{\beta}\hat{\imath}_B\\ &= \frac{m_1m_2\ell^2}{(m_1+m_2)^2(\ell_c-\ell_P)}\hat{\imath}_B.\end{aligned} \tag{7.14.11}$$

To confirm this result, note that

$$\begin{aligned}\vec{v}_{R/w/A}(0^-) &= f_0([\vec{J}_{\mathcal{B}/c}^{-1}(\vec{r}_{P/c}\times\hat{n}_P)]\times\vec{r}_{R/c} + \frac{1}{m_1+m_2}\hat{n}_P)\\ &= f_0\left(\frac{(m_1+m_2)(\ell_P-\ell_c)}{m_1m_2\ell^2}\hat{k}_B \times \frac{m_1m_2\ell^2}{(m_1+m_2)^2(\ell_c-\ell_P)}\hat{\imath}_B + \frac{1}{m_1+m_2}\hat{\jmath}_B\right)\\ &= 0.\end{aligned} \tag{7.14.12}$$

As a special case, letting $P = y_2$ yields $\vec{r}_{R/c} = -\ell_c\hat{\imath}_B$, and thus the percussive center of rotation is y_1. ◇

Example 7.14.3. Consider a rigid body $\mathcal{B}$ consisting of a thin uniform bar of length ℓ and mass m. Let y_1 and y_2 denote the endpoints of the bar, and let the body frame be chosen such that $\vec{r}_{y_2/y_1} = \ell\hat{\imath}_B$. The location of the center of mass is $\vec{r}_{c/y_1} = \frac{\ell}{2}\hat{\imath}_B$, and the physical inertia matrix is given by

$$\vec{J}_{\mathcal{B}/c} = \varepsilon\hat{\imath}_B\hat{\imath}_B' + \beta(\hat{\jmath}_B\hat{\jmath}_B' + \hat{k}_B\hat{k}_B'), \tag{7.14.13}$$

where ε is a small positive number that allows inversion of $\vec{J}_{\mathcal{B}/c}$ and

$$\beta \triangleq \frac{1}{12}m\ell^2. \tag{7.14.14}$$

Hence,

$$\vec{J}_{\mathcal{B}/c}^{-1} = \frac{1}{\varepsilon}\hat{\imath}_B\hat{\imath}_B' + \frac{1}{\beta}(\hat{\jmath}_B\hat{\jmath}_B' + \hat{k}_B\hat{k}_B'). \tag{7.14.15}$$

The impulsive force applied to $\mathcal{B}$ at the point P is given by $\vec{f}(t) = f_0\delta(t)\hat{n}_P$, where $\hat{n}_P = \hat{\jmath}_B$ and

$\vec{r}_{\mathrm{P}/y_1} = \ell_{\mathrm{P}}\hat{\imath}_{\mathrm{B}}$. It thus follows that

$$\vec{J}_{\mathcal{B}/\mathrm{c}}^{-1}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}}) = [\frac{1}{\varepsilon}\hat{\jmath}_{\mathrm{B}}\hat{\jmath}_{\mathrm{B}}' + \frac{1}{\beta}(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}_{\mathrm{B}}' + \hat{k}_{\mathrm{B}}\hat{k}_{\mathrm{B}}')][(\ell_{\mathrm{P}} - \ell/2)\hat{\imath}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{B}}]$$
$$= \frac{12(\ell_{\mathrm{P}} - \ell/2)}{m\ell^2}\hat{k}_{\mathrm{B}}. \tag{7.14.16}$$

Therefore, it follows from (7.14.1) that

$$\begin{aligned}
\vec{r}_{\mathrm{R/c}} &= \frac{1}{m|\vec{J}_{\mathcal{B}/\mathrm{c}}^{-1}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}})|^2}[\vec{J}_{\mathcal{B}/\mathrm{c}}^{-1}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}})] \times \hat{n}_{\mathrm{P}} \\
&= -\frac{m^2\ell^4}{144m(\ell_{\mathrm{P}} - \ell/2)^2}\frac{12(\ell_{\mathrm{P}} - \ell/2)}{m\ell^2}\hat{\imath}_{\mathrm{B}} \\
&= \frac{\ell^2}{12(\ell/2 - \ell_{\mathrm{P}})}\hat{\imath}_{\mathrm{B}}.
\end{aligned} \tag{7.14.17}$$

To confirm this result, note that

$$\begin{aligned}
\vec{v}_{\mathrm{R}/w/\mathrm{A}}(0^-) &= f_0([\vec{J}_{\mathcal{B}/\mathrm{c}}^{-1}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}})] \times \vec{r}_{\mathrm{R/c}} + \frac{1}{m_1 + m_2}\hat{n}_{\mathrm{P}}) \\
&= f_0\left(\frac{12(\ell_{\mathrm{P}} - \ell_{\mathrm{c}})}{m\ell^2}\hat{k}_{\mathrm{B}} \times \frac{\ell^2}{12(\ell_{\mathrm{c}} - \ell_{\mathrm{P}})}\hat{\imath}_{\mathrm{B}} + \frac{1}{m}\hat{\jmath}_{\mathrm{B}}\right) \\
&= 0.
\end{aligned} \tag{7.14.18}$$

As a special case, letting $\ell_{\mathrm{P}} = \frac{2}{3}\ell$ yields $\vec{r}_{\mathrm{R/c}} = -\frac{1}{2}\ell\hat{\imath}_{\mathrm{B}}$, and thus the percussive center of rotation is y_1. ◇

7.15 Examples

Example 7.15.1. A simple pendulum consists of a particle y with mass m and a rigid massless link of length ℓ connected to an inertially nonrotating massive rigid body at the point w. Determine the reaction force on the particle y, the reaction force on the inertially nonrotating massive rigid body at w, and derive the equations of motion in terms of θ.

Solution: Assume that $\mathrm{F_A}$ is an inertial frame, and let $\mathrm{F_B}$ be a body-fixed frame. These frames are related by $\mathrm{F_A} \xrightarrow[1]{\theta} \mathrm{F_B}$,

$$\begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix},$$

and $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}\hat{\imath}_{\mathrm{A}} = \dot{\theta}\hat{\imath}_{\mathrm{B}}$. The position of y relative to w is given by $\vec{r}_{y/w} = \ell\hat{k}_{\mathrm{B}}$. Therefore,

$$\vec{v}_{y/w/\mathrm{A}} = \ell\overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}} = \ell\vec{\omega}_{\mathrm{B/A}} \times \hat{k}_{\mathrm{B}} = \ell\dot{\theta}\hat{\imath}_{\mathrm{B}} \times \hat{k}_{\mathrm{B}} = -\ell\dot{\theta}\hat{\jmath}_{\mathrm{B}}.$$

Furthermore,

$$\begin{aligned}
\vec{a}_{y/w/\mathrm{A}} &= -\ell\ddot{\theta}\hat{\jmath}_{\mathrm{B}} - \ell\dot{\theta}\overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}} \\
&= -\ell\ddot{\theta}\hat{\jmath}_{\mathrm{B}} - \ell\dot{\theta}^2\hat{\imath}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{B}}
\end{aligned}$$

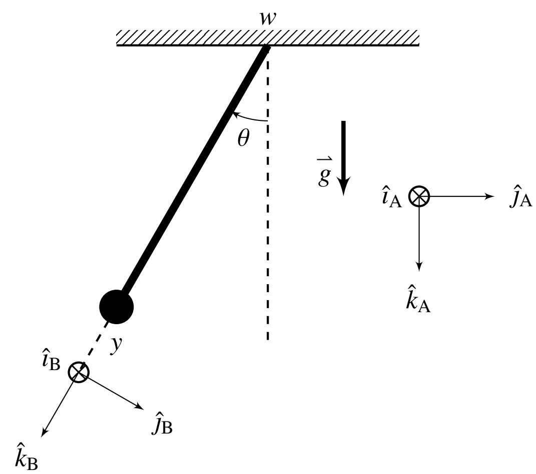


Figure 7.15.1: Simple pendulum for Example 7.15.1.

$$= -\ell\ddot{\theta}\hat{\jmath}_{\rm B} - \ell\dot{\theta}^2\hat{k}_{\rm B}.$$

The reaction force applied to y due to the rigid massless link is given by $\vec{f}_{\rm R} = f_{\rm R}\hat{k}_{\rm B}$, and thus the total force on y is given by

$$\begin{aligned}\vec{f}_y &= m\vec{g} + \vec{f}_{\rm R}\\ &= mg\hat{k}_{\rm A} + f_{\rm R}\hat{k}_{\rm B}\\ &= mg[(\sin\theta)\hat{\jmath}_{\rm B} + (\cos\theta)\hat{k}_{\rm B}] + f_{\rm R}\hat{k}_{\rm B}\\ &= mg(\sin\theta)\hat{\jmath}_{\rm B} + (mg\cos\theta + f_{\rm R})\hat{k}_{\rm B}.\end{aligned}$$

Next, it follows from Newton's second law $m\vec{a}_{y/w/{\rm A}} = \vec{f}_y$ that

$$-\ell\ddot{\theta}\hat{\jmath}_{\rm B} - \ell\dot{\theta}^2\hat{k}_{\rm B} = mg(\sin\theta)\hat{\jmath}_{\rm B} + (mg\cos\theta + f_{\rm R})\hat{k}_{\rm B},$$

and thus

$$-\ell\ddot{\theta} = mg\sin\theta$$

and

$$-m\ell\dot{\theta}^2 = mg\cos\theta + f_{\rm R}.$$

Therefore,

$$\ell\ddot{\theta} + mg\sin\theta = 0$$

and

$$f_{\rm R} = -m(\ell\dot{\theta}^2 + g\cos\theta).$$

The reaction force on the link and thus on w, is thus

$$-\vec{f}_{\rm R} = m(\ell\dot{\theta}^2 + g\cos\theta)\hat{k}_{\rm B}$$

$$= -m(\ell\dot{\theta}^2 + g\cos\theta)(\sin\theta)\hat{\jmath}_A + m(\ell\dot{\theta}^2 + g\cos\theta)(\cos\theta)\hat{k}_A. \qquad \diamond$$

Example 7.15.2. A physical pendulum consists of a body $\mathcal{B}$ connected to an inertially nonrotating massive rigid body by means of a frictionless pin joint at the point w. The center of mass of $\mathcal{B}$ is the point c, and the distance from w to c is ℓ. The mass of $\mathcal{B}$ is m, and the component of the moment of inertia of $\mathcal{B}$ relative to c along an axis perpendicular to the page is J_0. The angle between the line passing through w and c and the vertical direction is θ.

i) Use Newton-Euler dynamics to derive the equations of motion for $\mathcal{B}$ in terms of θ. Also, determine the reaction force on $\mathcal{B}$ at w. Do this in two different ways, namely, by applying Newton's second law for rotation relative to w and relative to c.

ii) Specialize the equations of motion and the reaction force to the case where $\mathcal{B}$ is a thin bar of length ℓ_0 and mass m.

iii) Specialize the equations of motion and reaction force to the case of a simple pendulum, that is, where $\mathcal{B}$ consists of a rigid massless link of length ℓ_1 with a particle of mass m on its end.

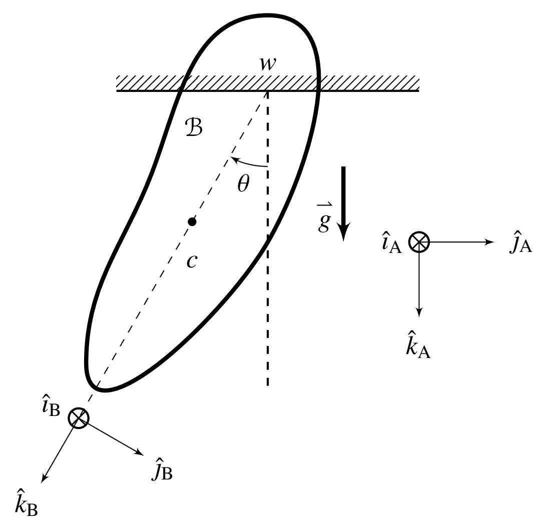


Figure 7.15.2: Physical pendulum for Example 7.15.2

Solution: *i*) Assume that F_A is an inertial frame, and let F_B be a body-fixed frame. These frames are related by $F_A \xrightarrow[1]{\theta} F_B$,

$$\begin{bmatrix} \hat{\imath}_B \\ \hat{\jmath}_B \\ \hat{k}_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\imath}_A \\ \hat{\jmath}_A \\ \hat{k}_A \end{bmatrix},$$

and $\vec{\omega}_{B/A} = \dot{\theta}\hat{\imath}_A = \dot{\theta}\hat{\imath}_B$. The position of c relative to w is given by $\vec{r}_{c/w} = \ell\hat{k}_B$. Therefore,

$$\vec{v}_{c/w/A} = \overset{A\bullet}{\vec{r}}_{c/w} = \ell\,\overset{A\bullet}{\hat{k}}_B = \ell\vec{\omega}_{B/A} \times \hat{k}_B = \ell\dot{\theta}\hat{\imath}_B \times \hat{k}_B = -\ell\dot{\theta}\hat{\jmath}_B,$$

and thus

$$\vec{a}_{c/w/A} = \ell\ddot{\theta}\hat{\jmath}_B - \ell\dot{\theta}\overset{A\bullet}{\hat{\jmath}_B} = -\ell\ddot{\theta}\hat{\jmath}_B - \ell\dot{\theta}^2\hat{\imath}_B \times \hat{\jmath}_B = -\ell\ddot{\theta}\hat{\jmath}_B - \ell\dot{\theta}^2\hat{k}_B.$$

The reaction force $\vec{f}_R$ on $\mathcal{B}$ at w is given by

$$\vec{f}_R = f_1\hat{\jmath}_B + f_2\hat{k}_B,$$

and thus the total force on $\mathcal{B}$ is given by

$$\vec{f}_{\mathcal{B}} = \vec{f}_R + m\vec{g} = (f_1 + mg\sin\theta)\hat{\jmath}_B + (f_2 + mg\cos\theta)\hat{k}_B.$$

Next, it follows from Newton's second law $m\vec{a}_{y/w/A} = \vec{f}_y$ that

$$-\ell\ddot{\theta}\hat{\jmath}_B - \ell\dot{\theta}^2\hat{k}_B = (f_1 + mg\sin\theta)\hat{\jmath}_B + (f_2 + mg\cos\theta)\hat{k}_B,$$

and thus

$$-m\ell\ddot{\theta} = f_1 + mg\sin\theta,$$
$$-m\ell\dot{\theta}^2 = f_2 + mg\cos\theta.$$

Therefore,

$$f_1 = -m(\ell\ddot{\theta} + g\sin\theta),$$
$$f_2 = -m(\ell\dot{\theta}^2 + g\cos\theta).$$

Next, since $\mathcal{B}$ lies in the $\hat{\jmath}_B$-$\hat{k}_A$ plane, it follows that $\vec{J}_{\mathcal{B}/w} = (J_0 + m\ell^2)\hat{\imath}_A\hat{\imath}'_A$. Hence, since $\vec{\omega}_{B/A} = \dot{\theta}\hat{\imath}_A$, it follows that $\vec{\omega}_{B/A} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{B/A} = 0$. Therefore, Newton's second law for rotation relative to w implies that

$$\vec{J}_{\mathcal{B}/w}\overset{B\bullet}{\vec{\omega}_{B/A}} = \vec{M}_{\mathcal{B}/w}.$$

The moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \vec{r}_{c/w} \times m\vec{g} + \vec{r}_{w/w} \times \vec{f}_R = \ell\hat{k}_B \times mg[(\sin\theta)\hat{\jmath}_B + (\cos\theta)\hat{k}_B] = -\ell mg(\sin\theta)\hat{\imath}_B.$$

Therefore, since $\overset{B\bullet}{\vec{\omega}_{B/A}} = \ddot{\theta}\hat{\imath}_B$, it follows that

$$(J_0 + m\ell^2)\ddot{\theta} = \hat{\imath}'_A\vec{J}_{\mathcal{B}/w}\overset{B\bullet}{\vec{\omega}_{B/A}} = \hat{\imath}'_A[-\ell mg(\sin\theta)\hat{\imath}_B] = -\ell mg\sin\theta.$$

Hence,

$$\ddot{\theta} + \frac{\ell mg}{J_0 + m\ell^2}\sin\theta = 0.$$

Using this expression for $\ddot{\theta}$ yields

$$f_1 = -m\left(\frac{-\ell^2 mg}{J_0 + m\ell^2}\sin\theta + g\sin\theta\right) = \frac{mgJ_0\sin\theta}{J_0 + m\ell^2}.$$

Alternatively, applying Newton's second law for rotation relative to c implies that

$$\vec{J}_{\mathcal{B}/\mathrm{c}} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \vec{M}_{\mathcal{B}/\mathrm{c}}.$$

Using

$$\vec{J}_{\mathcal{B}/\mathrm{c}} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = J_0\ddot{\theta}\hat{\imath}_{\mathrm{A}},$$

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \vec{r}_{w/\mathrm{c}} \times \vec{f}_{\mathrm{R}} = -\ell\hat{k}_{\mathrm{B}} \times (f_1\hat{\jmath}_{\mathrm{B}} + f_2\hat{k}_{\mathrm{B}}) = \ell f_1\hat{\imath}_{\mathrm{B}} = -m\ell(\ell\ddot{\theta} + g\sin\theta),$$

it follows that

$$J_0\ddot{\theta} = -m\ell(\ell\ddot{\theta} + g\sin\theta),$$

that is,

$$\ddot{\theta} + \frac{\ell mg}{J_0 + m\ell^2}\sin\theta = 0.$$

ii) Now assume that $\mathcal{B}$ is a thin bar of length ℓ_0. It thus follows that $\ell_0 = 2\ell$, and thus $J_0 = \frac{1}{12}m\ell_0^2 = \frac{1}{3}m\ell^2$, and thus $J_0 + m\ell^2 = \frac{1}{3}m\ell_0^2 = \frac{4}{3}m\ell^2$. Therefore, θ satisfies

$$\ddot{\theta} + \frac{3g}{2\ell_0}\sin\theta = 0.$$

In addition, the components of the reaction force are given by

$$f_1 = -\tfrac{1}{4}mg\sin\theta,$$
$$f_2 = -mg(\tfrac{1}{2}\ell\dot{\theta}^2 + g\cos\theta).$$

iii) Now assume that $\mathcal{B}$ consists of a rigid massless link of length ℓ_1 with a particle of mass m on its end. In this case, it follows that $J_0 = 0$ and thus $J_0 + m\ell^2 = m\ell_1^2$. Therefore, $\ddot{\theta}$ satisfies

$$\ddot{\theta} + \frac{g}{\ell_0}\sin\theta = 0.$$

In addition, the components of the reaction force are given by

$$f_1 = -m(\ell_1\ddot{\theta} + g\sin\theta) = -m\ell_1\left(\ddot{\theta} + \frac{g}{\ell_1}\sin\theta\right) = 0,$$
$$f_2 = -m(\ell_1\dot{\theta}^2 + g\cos\theta).$$

Note that these expressions agree with Example 7.15.1. ◇

Example 7.15.3. Consider a long rigid wire that rotates around a pin joint at the point w connected to an inertially nonrotating massive body at the rate $\omega = \dot{\theta}$ relative to an inertial frame. A bead whose mass is m slides without friction along the wire. The distance from w to the bead is x, where $x(0) > 0$ and $\dot{x}(0) = 0$. The moment of inertia of the wire around the axis of rotation is I.

i) Assume that $\omega > 0$ is constant. Derive the equation of motion for the bead and determine the reaction force between the bead and the wire.

ii) Assume that the wire has nonzero initial angular rate $\dot{\theta}(0) > 0$, the pivot joint at w is frictionless, and no external moments are applied to the wire. Consequently, as the bead moves, the rotation rate $\dot{\theta}$ is not constant. Determine the equations of motion of the body $\mathcal{B}$ consisting of the bead and the wire, and use conservation of angular momentum with respect to an inertial frame to determine a constant of the motion.

iii) Use the equations from *ii*) to show that the bead speeds up and the wire slows down, in particular, show that, as $t \to \infty$, $\dot\theta(t) \to 0$, $x(t) \to \infty$, and $\dot x(t) \to v_\infty$. In addition, determine an expression for the terminal velocity v_∞. Finally, show that $\theta(t)$ converges and find its limiting value. (Note: $\dot\theta \to 0$ does not imply that θ converges.)

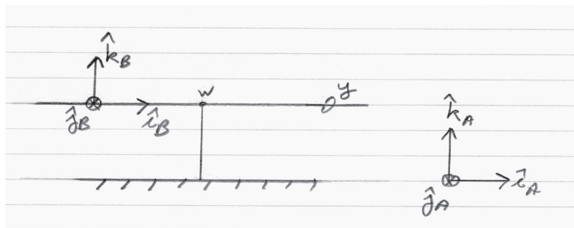


Figure 7.15.3: Bead on a rotating wire for Example 7.15.3.

Solution: *i*) Assume that $\mathrm{F_A}$ is an inertial frame, and let $\mathrm{F_B}$ be a body-fixed frame. These frames are related by $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$,

$$\begin{bmatrix} \hat\imath_\mathrm{B} \\ \hat\jmath_\mathrm{B} \\ \hat k_\mathrm{B} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat\imath_\mathrm{A} \\ \hat\jmath_\mathrm{A} \\ \hat k_\mathrm{A} \end{bmatrix},$$

and $\vec\omega_{\mathrm{B/A}} = \omega\hat k_\mathrm{A} = \omega\hat k_\mathrm{B}$, where $\omega \triangleq \dot\theta$. The position of y relative to w is given by $\vec r_{y/w} = x\hat\imath_\mathrm{B}$. Therefore,

$$\vec v_{y/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec r}_{y/w} = \overset{\mathrm{B}\bullet}{\vec r}_{y/w} + \vec\omega_{\mathrm{B/A}} \times \vec r_{y/w} = \dot x\hat\imath_\mathrm{B} + \omega\hat k_\mathrm{B} \times x\hat\imath_\mathrm{B} = \dot x\hat\imath_\mathrm{B} + \omega x\hat\jmath_\mathrm{B}.$$

Furthermore,

$$\begin{aligned} \vec a_{y/w/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec v}_{y/w/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\vec v}_{y/w/\mathrm{A}} + \vec\omega_{\mathrm{B/A}} \times \vec v_{y/w/\mathrm{A}} \\ &= \ddot x\hat\imath_\mathrm{B} + \omega\dot x\hat\jmath_\mathrm{B} + \omega\hat k_\mathrm{B} \times (\dot x\hat\imath_\mathrm{B} + \omega x\hat\jmath_\mathrm{B} \\ &= \ddot x\hat\imath_\mathrm{B} + \omega\dot x\hat\jmath_\mathrm{B} + \omega\dot x\hat\jmath_\mathrm{B} - \omega^2 x\hat\imath_\mathrm{B} \\ &= (\ddot x - \omega^2 x)\hat\imath_\mathrm{B} + 2\omega\dot x\hat\jmath_\mathrm{B}. \end{aligned}$$

Note that $\omega^2 x\hat\imath_\mathrm{B}$ is the centripetal acceleration and $2\omega\dot x\hat\jmath_\mathrm{B}$ is the Coriolis acceleration.

Next, since the bead slides without friction along the wire, the reaction force $\vec f_\mathrm{R}$ on the bead is in the $\hat\jmath_\mathrm{B}$ direction. Hence,

$$\vec f_\mathrm{R} = f_\mathrm{R}\hat\jmath_\mathrm{B}.$$

Now, it follows from Newton's second law $m\vec a_{y/w/\mathrm{A}} = \vec f_\mathrm{R}$ that

$$m(\ddot x - \omega^2 x)\hat\imath_\mathrm{B} + 2m\omega\dot x\hat\jmath_\mathrm{B} = f_\mathrm{R}\hat\jmath_\mathrm{B}.$$

Therefore,

$$f_{\rm R} = 2m\omega\dot{x},$$
$$\ddot{x} = \omega^2 x.$$

The solution of this differential equation is given by

$$x(t) = \tfrac{1}{2}(e^{\omega t} + e^{-\omega t})x(0),$$

and thus

$$\dot{x}(t) = \tfrac{1}{2}\omega(e^{\omega t} - e^{-\omega t})x(0).$$

Note that, since $x(0) > 0$, it follows that, for all $t \ge 0$, $x(t) > 0$, and thus $f_{\rm R} > 0$, which is consistent with the assumption that $\omega > 0$. Furthermore, note that $x(t) \to \infty$ as $t \to \infty$.

ii) We no longer assume that $\omega = \dot{\theta}$ is constant. Following the same steps as in the case of constant ω yields

$$f_{\rm R} = 2m\dot{\theta}\dot{x}, \tag{7.15.1}$$
$$\ddot{x} = \dot{\theta}^2 x. \tag{7.15.2}$$

To determine $\omega(t)$, note that the moment of inertia of the wire relative to w is given by

$$J_{{\rm wire}/w|{\rm B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}.$$

It thus follows from Euler's equation relative to w resolved in $\rm F_B$ that

$$\begin{aligned}\begin{bmatrix} 0 \\ 0 \\ J\ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ J\dot{\theta} \end{bmatrix} &= [\vec{r}_{y/w} \times (-\vec{f}_{\rm R})]\Big|_{\rm B} \\ &= \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -f_{\rm R} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -xf_{\rm R} \end{bmatrix}.\end{aligned}$$

Therefore,

$$J\ddot{\theta} = -xm(\ddot{\theta}x + 2\dot{\theta}\dot{x}),$$

and thus

$$(J\ddot{\theta} + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} = 0.$$

Next, since the external moment applied to the bead and wire is zero, angular momentum is conserved. To determine this constant of the motion, note that

$$\begin{aligned}\vec{H}_{y/w/{\rm A}} &= \vec{r}_{{\rm bead}/w} \times m\vec{v}_{y/w/{\rm A}} = x\hat{\imath}_{\rm B} \times m \overbrace{x\hat{\imath}_{\rm B}}^{{\rm A}\bullet} \\ &= x\hat{\imath}_{\rm B} \times (\dot{x}\hat{\imath}_{\rm B} + x\hat{\imath}_{\rm B}) = mx\hat{\imath}_{\rm B} \times x(\vec{\omega}_{\rm B/A} \times \hat{\imath}_{\rm B}) \\ &= mx^2\hat{\imath}_{\rm B} \times \dot{\theta}(\hat{k}_{\rm B} \times \hat{\imath}_{\rm B}) = m\dot{\theta}x^2\hat{\imath}_{\rm B} \times \hat{\jmath}_{\rm B} = m\dot{\theta}x^2\hat{k}_{\rm A}\end{aligned}$$

and

$$\vec{H}_{{\rm wire}/w/{\rm A}} = \vec{J}_{{\rm wire}/w}\vec{\omega}_{\rm A/B} = J\dot{\theta}\hat{k}_{\rm A}.$$

Therefore,

$$\overset{\rightharpoonup}{H}_{\mathcal{B}/w/\mathrm{A}} = (J + mx^2)\dot{\theta}\hat{k}_{\mathrm{A}}.$$

It thus follows from Newton's second law for rotation relative to w that

$$\overset{\mathrm{A}\bullet}{\overset{\rightharpoonup}{H}}_{\mathcal{B}/w/\mathrm{A}} = \overset{\rightharpoonup}{M}_{\mathcal{B}/w} = 0.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}[(J + mx^2)\dot{\theta}] = 0.$$

Thus, $(J + mx^2)\dot{\theta}$ is a constant of the motion, that is, there exists c_0 such that, for all $t \geq 0$,

$$(J + mx^2)\dot{\theta} = c_0.$$

Since, by assumption, $\dot{\theta}(0) > 0$, it follows that $c_0 > 0$.

iii) It follows from *ii*) that, for all $t \geq 0$,

$$\dot{x}(t) = \int_0^t \ddot{x}(s)\,\mathrm{d}s = \int_0^t \dot{\theta}^2(s)x(s)\,\mathrm{d}s. \tag{7.15.3}$$

By assumption, $x(0) > 0$. Suppose that there exists $\varepsilon > 0$ such that, for all $t \in [0, \varepsilon)$, $x(t) > 0$ and such that $x(\varepsilon) = 0$. It thus follows from (7.15.3) that, for all $t \in [0, \varepsilon)$, $\dot{x}(t) > 0$. Therefore,

$$x(\varepsilon) = x(0) + \int_0^\varepsilon \dot{x}(s)\,\mathrm{d}s > 0,$$

which is a contradiction. Hence, $x(t) > 0$ for all $t \geq 0$. Consequently, (7.15.3) implies that, for all $t \geq 0$, $\dot{x}(t) > 0$. Furthermore, it follows from (7.15.2) that, for all $t \geq 0$, $\ddot{x}(t) > 0$. Hence, $\dot{x}(t)$ is positive and increasing on $[0, \infty)$, and it follows from

$$x(t) = x(0) + \int_0^t \dot{x}(s)\,\mathrm{d}s$$

that $x(t) \to \infty$ as $t \to \infty$. Hence,

$$\lim_{t\to\infty} \dot{\theta}(t) = \lim_{t\to\infty} \frac{c_0}{J + mx^2(t)} = 0.$$

The fact that $\lim_{t\to\infty} \dot{\theta}(t) = 0$ is not sufficient to conclude that the angle of the wire converges. To analyze the convergence of θ, we note that the total energy $E(t)$ of the system is conserved, where

$$E(t) = \tfrac{1}{2}[J + mx^2(t)]\dot{\theta}^2(t) + \tfrac{1}{2}m\dot{x}^2(t).$$

In fact, note that

$$\begin{aligned}\dot{E}(t) &= mx\dot{x}\dot{\theta}^2 + (J + mx^2)\dot{\theta}\ddot{\theta} + m\dot{x}\ddot{x} = 2mx\dot{x}\dot{\theta}^2 + (J + mx^2)\dot{\theta}\ddot{\theta}\\ &= 2mx\dot{x}\dot{\theta}^2 + (J + mx^2)\dot{\theta}\frac{-2mx\dot{x}\dot{\theta}}{J + mx^2} = 0.\end{aligned}$$

Therefore, for all $t \geq 0$, it follows that

$$\tfrac{1}{2}c_0\dot{\theta}(t) + \tfrac{1}{2}m\dot{x}^2(t) = E(0),$$

and thus, for all $t \geq 0$,

$$\dot{x}(t) = \sqrt{\frac{2E(0) - c_0\dot{\theta}(t)}{m}}.$$

Consequently, the terminal velocity v_∞ of the bead is given by

$$v_\infty = \lim_{t\to\infty} \dot{x}(t) = \sqrt{\frac{2E(0)}{m}}.$$

Finally, to determine whether $\lim_{t\to\infty} \dot{\theta}$ is finite, note that

$$\theta(t) = \int_0^t \frac{c_0}{J + mx^2(s)}\,\mathrm{d}s,$$

which implies that $\theta(t)$ is increasing on $[0, \infty)$. Now, let ε satisfy $0 < \varepsilon v_\infty$. Since $v(t) \triangleq \dot{x}(t)$ is increasing and $v(t) \to \infty$ as $t \to \infty$, it follows that there exists $T > 0$ such that, for all $t > T$, $v_\infty - \varepsilon < v(t) < v_\infty$. Therefore, since, for all $s \geq T$, $x(s) \geq x(T) + (v_\infty - \varepsilon)(s - T)$, it follows that, for all $t \geq T$,

$$\begin{aligned}
\theta(t) &= \theta(T) + \int_T^\infty \frac{c_0}{J + mx^2(s)}\,\mathrm{d}s \\
&\leq \theta(T) + \int_T^\infty \frac{c_0}{J + m[x(T) + (v_\infty - \varepsilon)\sigma]^2}\,\mathrm{d}s \\
&= \theta(T) + \frac{c_0}{m(v_\infty - \varepsilon)^2}\int_0^\infty \frac{1}{\sigma^2 + \frac{2x(T)}{v_\infty - \varepsilon}\sigma + \frac{J + mx^2(T)}{m(v_\infty - \varepsilon)^2}}\,\mathrm{d}\sigma \\
&= \theta(T) + \frac{c_0}{\sqrt{mJ}(v_\infty - \varepsilon)}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{mx^2(T)}{J}}\right).
\end{aligned}$$

Since $\theta(t)$ is increasing on $[0, \infty)$, $\lim_{t\to\infty} \dot{\theta}(t)$ exists, and thus the wire comes to rest. ◇

Example 7.15.4. Consider a ball rolling along a beam subject to gravity. The beam rotates around a pivot joint at its center w due to the weight of the ball as well as an external torque. The the point w connected to an inertially nonrotating massive body

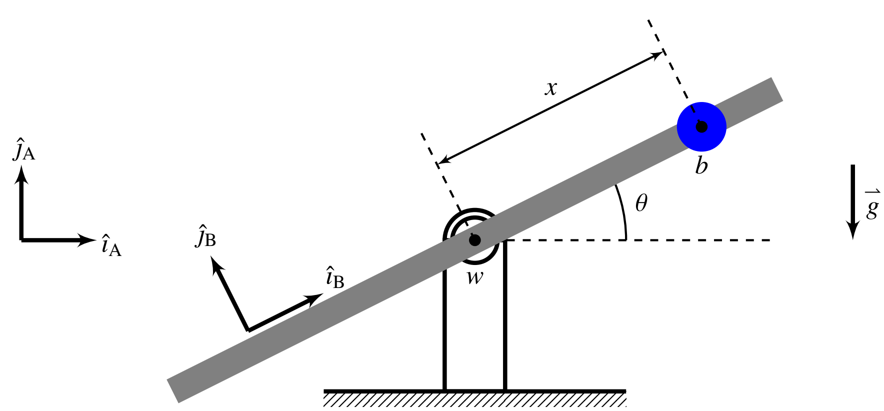


Figure 7.15.4: Ball and beam for Example 7.15.4.

Let $\mathrm{F_A}$ be fixed to the ground such that $\hat{\imath}_{\mathrm{A}}$ is horizontal and $\hat{\jmath}_{\mathrm{A}}$ points upward, and let $\mathrm{F_B}$ be fixed to beam such that $\hat{\imath}_{\mathrm{B}}$ and $\hat{\imath}_{\mathrm{A}}$ are parallel when the beam is horizontal. Then, $\mathrm{F_A}$ and $\mathrm{F_B}$ are related by $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$ so that

$$\begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix}$$

and $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}\hat{k}_{\mathrm{A}}$. Furthermore, since $\vec{r}_{b/w} = x\hat{\imath}_{\mathrm{B}}$, it follows that

$$\vec{v}_{b/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{b/w} = \dot{x}\hat{\imath}_{\mathrm{B}} + x\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} = \dot{x}\hat{\imath}_{\mathrm{B}} + x\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}} = \dot{x}\hat{\imath}_{\mathrm{B}} + x\dot{\theta}\hat{\jmath}_{\mathrm{B}},$$

$$\begin{aligned} \vec{a}_{b/w/\mathrm{A}} &= \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{b/w} \\ &= \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{b/w} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{r}}_{b/w} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{b/w} + \vec{\omega}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{b/w} \\ &= (\ddot{x} - \dot{\theta}^2 x)\hat{\imath}_{\mathrm{B}} + (2\dot{\theta}\dot{x} + \ddot{\theta}x)\hat{\jmath}_{\mathrm{B}}. \end{aligned}$$

The reaction force applied to the ball by the beam is given by $\vec{f}_{\mathrm{R}} = f_{\mathrm{R}}\hat{\jmath}_{\mathrm{B}}$, and thus total force on the ball is given by

$$\vec{f}_b = \vec{f}_{\mathrm{R}} + m\vec{g} = f_{\mathrm{R}}\hat{\jmath}_{\mathrm{B}} - mg\hat{\jmath}_{\mathrm{A}} = -mg\sin\theta\hat{\imath}_{\mathrm{B}} + (f_{\mathrm{R}} - mg\cos\theta)\hat{\jmath}_{\mathrm{B}}.$$

It thus follows from Newton's second law $m\vec{a}_{b/w/\mathrm{A}} = \vec{f}_b$ that

$$m(\ddot{x} - \dot{\theta}^2 x)\hat{\imath}_{\mathrm{B}} + m(2\dot{\theta}\dot{x} + \ddot{\theta}x)\hat{\jmath}_{\mathrm{B}} = -mg\sin\theta\hat{\imath}_{\mathrm{B}} + (f_{\mathrm{R}} - mg\cos\theta)\hat{\jmath}_{\mathrm{B}}.$$

Hence,

$$\begin{gathered} \ddot{x} = \dot{\theta}^2 x - g\sin\theta, \\ f_{\mathrm{R}} = mg\cos\theta + 2m\dot{\theta}\dot{x} + m\ddot{\theta}x. \end{gathered}$$

Next, note that $\vec{J}_{\mathcal{B}/w} = J_{xx}\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + J_{yy}\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} + J_{zz}\hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}$. Therefore,

$$\begin{gathered} \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}} = 0, \\ \vec{J}_{\mathcal{B}/w}\overset{B\bullet}{\vec{\omega}}_{\mathrm{B/A}} = J_{zz}\ddot{\theta}\hat{k}_{\mathrm{B}}. \end{gathered}$$

The moment applied to the beam relative to c is given by

$$\vec{M}_{\mathcal{B}/w} = \vec{r}_{b/w} \times (-\vec{f}_{\mathrm{R}}) + \vec{M}_{\mathrm{ext}} = (-xf_{\mathrm{R}} + \tau)\hat{k}_{\mathrm{B}}.$$

Newton's second law for rotation relative to w implies

$$\vec{J}_{\mathcal{B}/w}\overset{B\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \vec{M}_{\mathcal{B}/w}.$$

Therefore,

$$J_{zz}\ddot{\theta}\hat{k}_{\mathrm{B}} = (-xf_{\mathrm{R}} + \tau)\hat{k}_{\mathrm{B}},$$

and thus

$$(J_{zz} + mx^2)\ddot{\theta} = -x(mg\cos\theta + 2m\dot{\theta}\dot{x}) + \tau. \qquad \diamond$$

7.16 Theoretical Problems

Problem 7.16.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be rigid bodies with masses m_1 and m_2, respectively, let $\vec{f}$ be a force applied to $\mathcal{B}_1$, and assume that $\mathcal{B}_2$ is in contact with $\mathcal{B}_1$ in such a way that both bodies move along a straight line. Determine the magnitude of the reaction force between $\mathcal{B}_1$ and $\mathcal{B}_2$.

Problem 7.16.2. Let $\mathrm{F_A}$ be an inertial frame, let y_1 and y_2 be particles with masses m_1 and m_2, respectively, and assume that the force $\vec{f}_1$ is applied to y_1 and the force $\vec{f}_2$ is applied to y_2. Show that

$$\frac{m_1 m_2}{m_1+m_2}\vec{a}_{y_1/y_2/\mathrm{A}} = \frac{m_2}{m_1+m_2}\vec{f}_1 - \frac{m_1}{m_1+m_2}\vec{f}_2.$$

In particular, show that, if $\vec{f}_1 = \vec{f}$ and $\vec{f}_2 = -\vec{f}$, then

$$\frac{m_1 m_2}{m_1+m_2}\vec{a}_{y_1/y_2/\mathrm{A}} = \vec{f}.$$

Now, specialize the result to the cases where the force is due to a spring, dashpot, or inerter connecting y_1 and y_2. In each case, provide a differential equation whose solution is the distance between the particles. In addition, in each case, use the solution of the differential equation to describe the qualitative behavior of the distance.

Problem 7.16.3. A body consists of particles y_1 and y_2 with masses m_1 and m_2, respectively, connected by a massless link of length ℓ. A force $\vec{f}$ is applied to the point p, which is either along the link or colocated with one of the particles. The force $\vec{f}$ and all motion is confined to the plane spanned by $\hat{\imath}_\mathrm{A}$ and $\hat{\jmath}_\mathrm{A}$, where $\mathrm{F_A}$ is an inertial frame. Determine the reaction forces $\vec{f}_{\mathrm{R1}}$ and $\vec{f}_{\mathrm{R2}}$ on y_1 and y_2, respectively, in terms of the angular velocity and angular acceleration of the body relative to $\mathrm{F_A}$.

Solution: Let $\mathrm{F_B}$ be a body-fixed frame, where the direction of $\hat{\imath}_\mathrm{B}$ is along the link from y_1 to y_2, and let $\vec{f} = f_1\hat{\imath}_\mathrm{B} + f_2\hat{\jmath}_\mathrm{B}$. Let c denote the center of mass of the body, and let $\vec{\omega}_{\mathrm{B/A}} = \omega\hat{k}_\mathrm{B}$. Then,

$$\dot{\omega} = \frac{\vec{r}_{p/\mathrm{c}}{}' \hat{\imath}_\mathrm{B} f_2}{J_{zz}}, \quad J_{zz} = \frac{m_1 m_2 \ell^2}{m_1+m_2},$$

$$\vec{f}_{\mathrm{R1}} = \frac{m_1 m_2 \ell}{m_1+m_2}(\omega^2\hat{\imath}_\mathrm{B} - \dot{\omega}\hat{\jmath}_\mathrm{B}) + \frac{m_1}{m_1+m_2}\vec{f}, \quad \vec{f}_{\mathrm{R2}} = \frac{m_1 m_2 \ell}{m_1+m_2}(-\omega^2\hat{\imath}_\mathrm{B} + \dot{\omega}\hat{\jmath}_\mathrm{B}) + \frac{m_2}{m_1+m_2}\vec{f}.$$

Problem 7.16.4. Let $\mathrm{F_A}$ be an inertial frame, let y_1, y_2, and y_3 be particles with masses m_1, m_2, and m_3, respectively, and assume that the forces $\vec{f}_{12}$ and $\vec{f}_{13}$ are applied to y_1, the forces $-\vec{f}_{12}$ and $\vec{f}_{23}$ are applied to y_2, and the forces $-\vec{f}_{13}$ and $-\vec{f}_{23}$ are applied to y_3. Show that

$$\frac{m_1 m_3}{m_1+m_3}\vec{a}_{y_1/y_3/\mathrm{A}} = \vec{f}_{13} + \frac{1}{m_1+m_3}(m_1\vec{f}_{23} + m_3\vec{f}_{12}),$$

$$\frac{m_2 m_3}{m_2+m_3}\vec{a}_{y_2/y_3/\mathrm{A}} = \vec{f}_{23} + \frac{1}{m_2+m_3}(m_2\vec{f}_{13} - m_3\vec{f}_{12}).$$

Now, specialize the result to the cases in which the forces are due to springs, dashpots, or inerters connecting y_1, y_2, and y_3. In each case, provide a differential equation whose solution is the distance between the particles. Finally, describe the qualitative behavior of the solution in each case.

Problem 7.16.5. Let $\mathrm{F_A}$ be an inertial frame, let y and z be unforced particles, let x be a particle, and let $\vec{f}$ be a force applied to x. Show that

$$\vec{a}_{x/z/\mathrm{A}} = \vec{a}_{x/y/\mathrm{A}}.$$

Finally, explain why the equation

$$\vec{v}_{x/z/\mathrm{A}} = \vec{v}_{x/y/\mathrm{A}}.$$

is not true in general.

Problem 7.16.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be inertial frames, and let w and y be unforced particles. Show that

$$\overset{\mathrm{B}\bullet}{\underset{}{\overset{\mathrm{A}\bullet}{\vec{r}}}}_{y/w} = \overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{r}}}_{y/w} = 0.$$

Problem 7.16.7. Consider a rigid body spinning around a principal axis relative to its center of mass without any applied moments. Use Euler's equation to show that the body spins indefinitely around the principal axis.

7.17 Applied Problems

Problem 7.17.1. In Figure 7.17.1, a ball under the influence of gravity rolls without slipping down a slanted surface of a moving cart. The angle between the slanted surface and the ground is θ, and the distance between a and b is x. The mass of the ball is m, the radius of the ball is r, and the mass of the cart is M. The cart is mounted on frictionless, massless wheels, which allow the cart to translate on the ground, which is an inertially nonrotating massive body. Derive the equations of motion for the ball and the cart, and determine the normal and tangential components of the reaction force between the ball and the slanted surface as well as the vertical reaction force between the slanted surface and the ground.

Problem 7.17.2. The rotating platform in Figure 7.17.2 is connected to the ground by a pin joint at the point a. The ground can be viewed as an inertially nonrotating massive body with body-fixed inertial frame $\mathrm{F_A}$. The body $\mathcal{B}$ consists of a massless bar of length $2R$ with center at the point b and with identical small spheres of mass m mounted on each end. The distance from a to b is L. A motor spins the bar relative to the platform at the rate $\omega_1 > 0$, and the platform spins at the rate $\omega_2 > 0$ relative to the ground. Neither ω_1 nor ω_2 is necessarily constant, and the spin directions are shown in the figure.

i) Determine the total force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ resolved in a platform-fixed frame.

ii) Determine the angular momentum $\vec{H}_{\mathcal{B}/b/\mathrm{A}}$ of $\mathcal{B}$ relative to b with respect to $\mathrm{F_A}$ resolved in a platform-fixed frame.

iii) Determine the moment $\vec{M}_{\mathcal{B}/b}$ on $\mathcal{B}$ relative to b resolved in a platform-fixed frame.

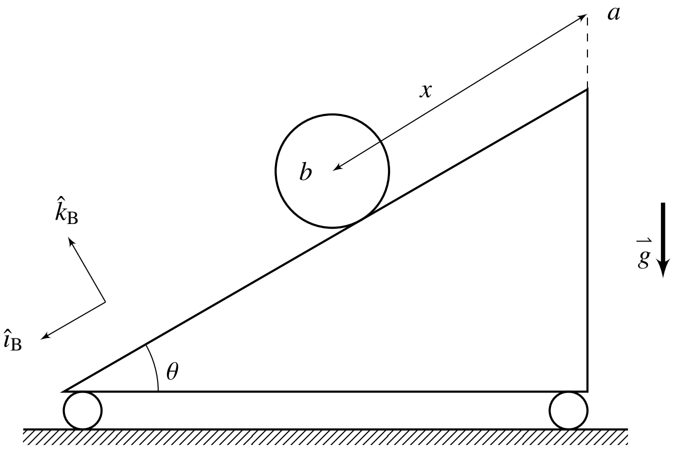


Figure 7.17.1: Problem 7.17.1. Ball rolling down a slanted surface of a moving cart for Problem 7.17.1.

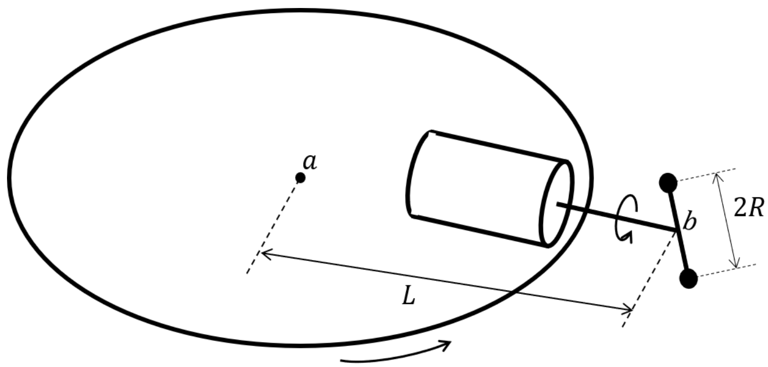


Figure 7.17.2: Rotating platform with a rotating bar for Problem 7.17.2

Problem 7.17.3. The uniform disk and uniform shaft shown in Figure 7.17.3 are welded together in such a way that the angle between the shaft and the line perpendicular to the disk and passing through its center is θ. The distance from the left end of the shaft to the center of the disk is d. The mass, length, and radius of the shaft are m_1, l_1, and r_1, respectively, while the mass, thickness, and radius of the disk are m_2, l_2, and r_2, respectively. Resolve the physical inertia matrix of the assembly relative to its center of mass in a frame F_A whose axis $\hat{\imath}_A$ is aligned with the longitudinal axis of the shaft and whose axis $\hat{\jmath}_A$ is such that the line perpendicular to the disk and passing through its center lies in the $\hat{\imath}_A$-$\hat{\jmath}_A$ plane.

Problem 7.17.4. The vertical rotating shaft in Figure 7.17.4 rotates at the constant rate $\Omega \geq 0$ in the direction shown. The cable, which is horizontal, supports a uniform bar of length l and mass m. The angle between the bar and the vertical direction is θ. Assume that the bearings that support the shaft are mounted on an inertially nonrotating massive body. Gravity acts in the direction shown. Determine the tension in the cable, the reaction force on the bar at point a, and the reaction torque on the vertical shaft at point a. (Hint: Consider the case $\Omega = 0$ first, and note that every point on the shaft is colocated with an unforced particle.)

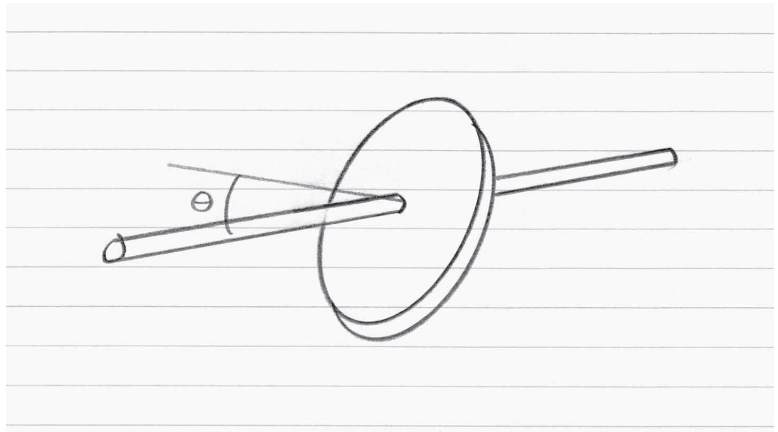


Figure 7.17.3: Disk welded to a shaft for Problem 7.17.3

Problem 7.17.5. The massless rotating shaft in Figure 7.17.5 is welded to a cylinder whose mass is m so that the center of mass of the cylinder lies on the shaft and the angle between the shaft and centerline of the cylinder is θ. The ends of the shaft are prevented from moving transversely by bearings, which are mounted on an inertially nonrotating massive body. No gravity is present, and a torque $\vec{\tau}$ is applied to the shaft. The length of the shaft is l, the radius and length of the cylinder are r and h, respectively, and the rotation rate of the assembly relative to the bearings is $\Omega > 0$, which is not necessarily constant. Determine the reaction forces applied by the bearings to the shaft and resolve these vectors in a frame that is attached to the shaft and one of whose axes is aligned with the shaft.

Problem 7.17.6. The vertical rotating shaft in Figure 7.17.6 is welded to a horizontal arm, which is connected by a frictionless pin joint to a slanted bar, where the angle between the slanted bar and the horizontal direction is θ. The tip of the slanted bar slides over the ground as the shaft rotates in the direction shown at the constant angular rate $\Omega > 0$. Assume that the ground is an inertially nonrotating massive body, and that gravity acts in the direction shown. The length of the horizontal arm is l_1, and the mass and length of the slanted bar are m and l_2, respectively. The coefficient of friction between the tip of the slanted bar and the ground is $\mu > 0$. Determine the reaction force acting on the slanted bar due to the horizontal arm, the reaction torque acting on the slanted bar due to the pin joint, the upward reaction force of the ground acting on the slanted bar, and the horizontal friction force acting on the slanted bar. Resolve these quantities in a frame fixed to the slanted bar, and express all of their components in terms of $\theta, \Omega, l_1, m, l_2, g, \mu$. (Hint: The friction force opposes the motion of the slanted bar; its magnitude is obtained by multiplying μ by the magnitude of the upward reaction force of the ground on the slanted bar.)

Problem 7.17.7. The wheel in Figure 7.17.7 is attached to a thin bar by means of a frictionless pin joint at the center of the wheel. The wheel rolls without slipping on the ground, which is an inertially nonrotating massive body. Gravity acts in the direction shown. At the time instant $t = 0$ shown, the disk rotates in the direction shown at the angular rate $\dot{\phi}(0) > 0$, and the angle θ between the arm and the horizontal direction has the value $\theta(0)$ and the rate $\dot{\theta}(0) > 0$. The radius and mass of the wheel are r and m_1, respectively, while the length and mass of the arm are l and m_2, respectively. At time $t = 0$, determine $\ddot{\phi}(0)$ and $\ddot{\theta}(0)$, as well as the horizontal and vertical components of the

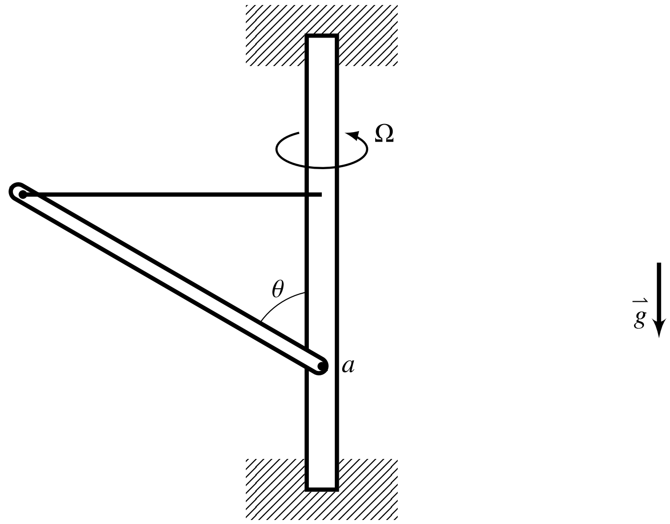


Figure 7.17.4: Rotating shaft with an attached bar and cable for Problem 7.17.4

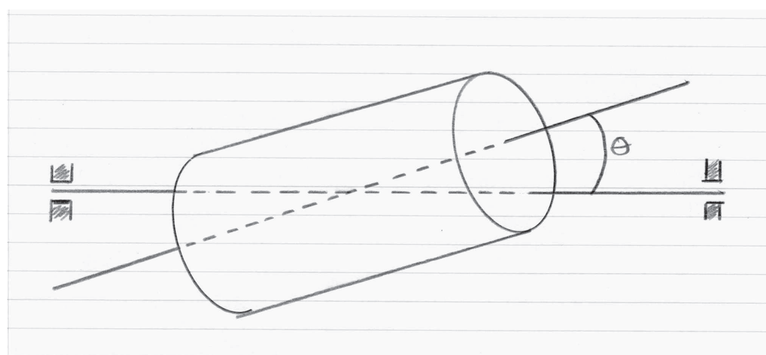


Figure 7.17.5: Rotating shaft with a rigidly attached cylinder for Problem 7.17.5

reaction force at the pin joint.

Problem 7.17.8. The simple pendulum on a cart shown in Figure 7.17.8 consists of a particle w with mass M that slides without friction on a horizontal surface on an inertially nonrotating massive rigid body. The particle w is connected by a rigid massless link of length ℓ to a particle y of mass m. Determine the reaction force on the particle y and derive the equations of motion of the cart and pendulum in terms of the directed angle θ and the distance x from w to the point a fixed in the inertially nonrotating massive rigid body. Finally, show that the equations of motion for the simple pendulum are recovered in the limit as $M \to \infty$.

7.18 Solutions to the Applied Problems

Solution to Problem 7.17.1.

$$\ddot{x} = \frac{5(m+M)g\sin\theta}{7M + (2 + 5\sin^2\theta)m}, \quad \ddot{y} = g\tan\theta - \frac{7}{5\cos\theta}\ddot{x},$$

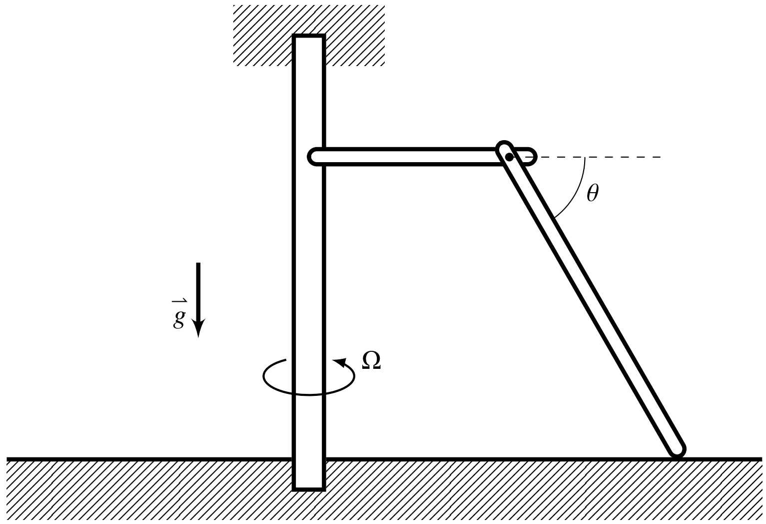


Figure 7.17.6: Rotating shaft with a slanted bar contacting the ground for Problem 7.17.6

$$f_{\rm T} = \frac{-2(m+M)mg\sin\theta}{7M+(2+5\sin^2\theta)m}, \quad f_{\rm N} = \frac{mg}{\cos\theta}\left(1 - \frac{7(m+M)\sin^2\theta}{7M+(2+5\sin^2\theta)m}\right),$$

where x is the distance from a to the center of the ball, and y is the distance from the vertical edge of the cart to an unforced particle on the ground.

Solution to Problem 7.17.2.

$$\vec{f}_{\mathcal{B}} = -2mL\dot{\omega}_2\hat{\imath}_{\rm B} - 2mL\omega_2^2\hat{\jmath}_{\rm B}, \quad \vec{H}_{\mathcal{B}/b/A} = 2mR^2[-\omega_2(\sin\theta)(\cos\theta)\hat{\imath}_{\rm B} + \omega_1\hat{\jmath}_{\rm B} + \omega_2(\sin^2\theta)\hat{k}_{\rm B}],$$

$$\vec{M}_{\mathcal{B}/b} = mR^2[-(\dot{\omega}_2\sin 2\theta + 4\omega_1\omega_2\cos^2\theta)\hat{\imath}_{\rm B} + (2\dot{\omega}_1 - \omega_2^2\sin 2\theta)\hat{\jmath}_{\rm B} + (2\dot{\omega}_2\sin^2\theta + 2\omega_1\omega_2\sin 2\theta)\hat{k}_{\rm B}],$$

where $\hat{\jmath}_{\rm B}$ is aligned with the motor axis and $\hat{k}_{\rm B}$ is pointing up. $\rm F_C$ is defined such that $\hat{k}_{\rm C}$ is aligned with the masses, and $\hat{\jmath}_{\rm C}$ is aligned with the motor axis.

Solution to Problem 7.17.3.

$$J_{\mathcal{B}_3/c_3|\rm A} = \begin{bmatrix} \alpha_1+\delta_1 & \delta_2 & 0 \\ \delta_3 & \beta_1 + m_1 l_4^2 + \delta_4 + m_2 l_5^2 & 0 \\ 0 & 0 & \beta_1 + m_1 l_4^2 + \beta_2 + m_2 l_5^2 \end{bmatrix},$$

$$m_3 = m_1 + m_2, \quad l_3 = \frac{l_1}{2} + \frac{m_2}{m_3}\left(d - \frac{l_1}{2}\right), \quad l_4 = \frac{m_2}{m_3}\left(\frac{l_1}{2} - d\right), \quad l_5 = d - \frac{l_1}{2} + \frac{m_2}{m_3}\left(\frac{l_1}{2} - d\right),$$

$$\alpha_1 = \frac{1}{2}m_1 r_1^2, \quad \alpha_2 = \frac{1}{2}m_2 r_2^2, \quad \beta_1 = \frac{1}{12}m_1(3r_1^2 + l_1^2), \quad \beta_2 = \frac{1}{12}m_2(3r_2^2 + l_2^2),$$

$$\delta_1 = \alpha_2\cos^2\theta + \beta_2\sin^2\theta, \quad \delta_2 = (-\alpha_2 + \beta_2)(\cos\theta)\sin\theta,$$

$$\delta_3 = (-\alpha_2 + \beta_2)(\cos\theta)\sin\theta, \quad \delta_4 = \alpha_2\sin^2\theta + \beta_2\cos^2\theta,$$

where $\mathcal{B}_3 = \mathcal{B}_1 \cup \mathcal{B}_2$ and c_3 is the center of mass of $\mathcal{B}_3$.

Solution to Problem 7.17.4.

$$T = \frac{1}{2}mg\tan\theta + \frac{1}{3}ml(\sin\theta)\Omega^2, \quad f_{ay} = -\frac{1}{2}mg\tan\theta + \frac{1}{6}ml(\sin\theta)\Omega^2, \quad f_{az} = mg.$$

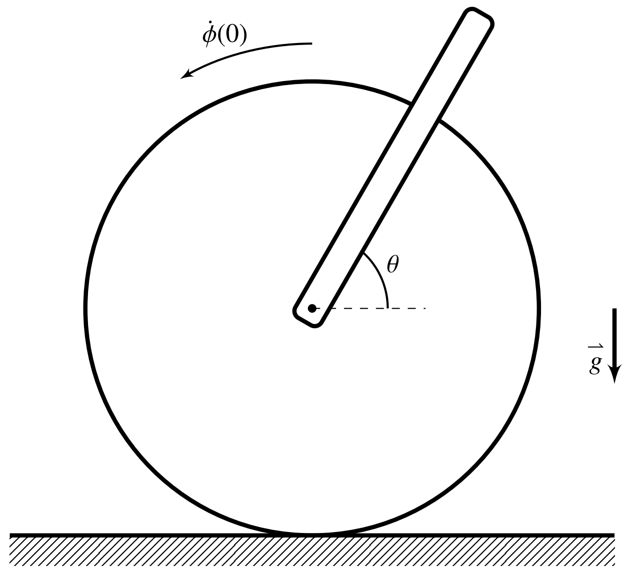


Figure 7.17.7: Rotating wheel with a pivoted bar for Problem 7.17.7

Solution to Problem 7.17.5.

$$\dot{\Omega} = \frac{12\tau}{m[6r^2 + (h^2 - 3r^2)\sin^2\theta]}, \quad f_{ax} = \frac{m\Omega^2 \sin 2\theta}{24L}(h^2 - 3r^2), \quad f_{ay} = \frac{-m\dot{\Omega}\sin 2\theta}{24L}(h^2 - 3r^2).$$

Solution to Problem 7.17.6.

$$f_{ay} = -\frac{1}{2}mg\sin\theta - \frac{1}{6}m\Omega^2[2l_2 + 6l_1\cos\theta + l_2\cos^2\theta + 3l_1(\tan\theta)\sin\theta],$$

$$f_{az} = -\frac{1}{6}m\Omega^2(\sin\theta)(3l_1 + l_2\cos\theta) + \frac{1}{2}mg\cos\theta, \quad f_b = \frac{1}{6}m[3g - \Omega^2(2l_2\sin\theta + 3l_1\tan\theta)],$$

$$f_{ax} = -\mu f_b, \quad f_f = \mu f_b, \quad M_{az} = \mu l_2 f_b.$$

f_{ax}, f_{ay}, f_{az} are the components of $\vec{f}_a$ resolved in $\mathrm{F_C}$, and $\mathrm{F_C}$ is defined such that $\hat{\jmath}_\mathrm{C}$ is aligned with the slanted bar pointing toward the ground, and $\hat{\imath}_\mathrm{C}$ is out of the page. M_{az} is a component of $\vec{M}_{\mathcal{B}/\mathrm{c}}$ resolved in $\mathrm{F_C}$, where $\mathcal{B}$ is the slanted bar, and c is its center of mass. The remaining components of $\vec{M}_{\mathcal{B}/\mathrm{c}}$ resolved in $\mathrm{F_C}$ are zero.

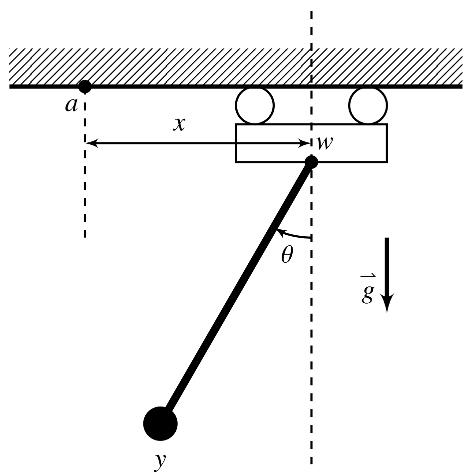


Figure 7.17.8: Problem 7.17.8. Simple pendulum attached to a cart for Problem 7.17.8.

Symbol	Definition
dm	Mass element of $\mathcal{B}$
$\vec{p}_{y/w/\mathrm{A}}$	Momentum of the particle y relative to w
$\vec{p}_{\mathcal{B}/x/\mathrm{A}}$	Momentum of the body $\mathcal{B}$ relative to w
$\vec{J}_{\mathcal{B}/w}$	Physical inertia matrix of body $\mathcal{B}$ relative to w
$\vec{H}_{\mathcal{B}/w/\mathrm{A}}$	Angular momentum of the body $\mathcal{B}$ relative to w

Table 7.1: Symbols for Chapter 7.

Chapter Eight
Kinetic and Potential Energy

8.1 Kinetic Energy of Particles and Bodies

Definition 8.1.1. Let let y be a particle with mass m, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *kinetic energy of y relative to w with respect to* $\mathrm{F_A}$ is defined by

$$T_{y/w/\mathrm{A}} \triangleq \tfrac{1}{2}m|\vec{v}_{y/w/\mathrm{A}}|^2. \tag{8.1.1}$$

Definition 8.1.2. Let $\mathcal{B}$ be a body composed of particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *kinetic energy* of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is defined by

$$T_{\mathcal{B}/w/\mathrm{A}} \triangleq \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/w/\mathrm{A}}|^2. \tag{8.1.2}$$

The follow result shows that the kinetic energy of a body is the sum of the kinetic energies of its component bodies.

Fact 8.1.3. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let $\mathcal{B}$ be the body consisting of $\mathcal{B}_1$ and $\mathcal{B}_2$, let w be a point, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = T_{\mathcal{B}_1/w/\mathrm{A}} + T_{\mathcal{B}_2/w/\mathrm{A}}. \tag{8.1.3}$$

The following result relates the kinetic energies of a body relative to two different points.

Fact 8.1.4. Let $\mathcal{B}$ be a body composed of particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w and z be points, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = T_{\mathcal{B}/z/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2. \tag{8.1.4}$$

Proof. Note that

$$\begin{aligned}
T_{\mathcal{B}/w/\mathrm{A}} &= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/w/\mathrm{A}}\vec{v}_{y_i/w/\mathrm{A}}\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i(\vec{v}_{y_i/z/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}})'(\vec{v}_{y_i/z/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}})\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{A}} + \sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{z/w/\mathrm{A}} + \tfrac{1}{2}m_{\mathcal{B}}\sum_{i=1}^{l}\vec{v}'_{z/w/\mathrm{A}}\vec{v}_{z/w/\mathrm{A}}
\end{aligned}$$

$$= T_{\mathcal{B}/z/\mathrm{A}} + m_{\mathcal{B}} \overset{\rightharpoonup}{v}'_{z/w/\mathrm{A}} \overset{\rightharpoonup}{v}_{\mathrm{c}/z/\mathrm{A}} + \tfrac{1}{2} m_{\mathcal{B}} |\overset{\rightharpoonup}{v}_{z/w/\mathrm{A}}|^2. \qquad \square$$

Choosing z to be the center of mass yields the following result, which shows that the kinetic energy of a body relative to an arbitrary point can be viewed as the kinetic energy of the body relative to its center of mass plus the kinetic energy of the mass of the body concentrated at the center of mass.

Fact 8.1.5. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w be a point, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \tfrac{1}{2} m_{\mathcal{B}} |\overset{\rightharpoonup}{v}_{\mathrm{c}/w/\mathrm{A}}|^2. \tag{8.1.5}$$

The following result expresses the kinetic energy $T_{\mathcal{B}/\mathrm{c}/\mathrm{A}}$ of $\mathcal{B}$ relative to its center of mass in terms of the relative velocities of its particles.

Fact 8.1.6. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = \tfrac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{l} \frac{m_i m_j}{m_{\mathcal{B}}} |\overset{\rightharpoonup}{v}_{y_j/y_i/\mathrm{A}}|^2. \tag{8.1.6}$$

Now assume that $\mathcal{B}$ is rigid and $\mathrm{F_A}$ is body fixed. Then,

$$T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = 0. \tag{8.1.7}$$

Finally, let w be a point. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = \tfrac{1}{2} m_{\mathcal{B}} |\overset{\rightharpoonup}{v}_{\mathrm{c}/w/\mathrm{A}}|^2. \tag{8.1.8}$$

Proof. Note that, for all $i = 1, \ldots, l$,

$$\begin{aligned}
\overset{\rightharpoonup}{v}_{\mathrm{c}/y_i/\mathrm{A}} &= \frac{1}{m_{\mathcal{B}}} \sum_{j=1}^{l} m_j \overset{\rightharpoonup}{v}_{y_j/y_i/\mathrm{A}} \\
&= \frac{1}{m_{\mathcal{B}}} \sum_{j=1}^{l} m_j (\overset{\rightharpoonup}{v}_{y_j/\mathrm{c}/\mathrm{A}} - \overset{\rightharpoonup}{v}_{y_i/\mathrm{c}/\mathrm{A}}).
\end{aligned}$$

Next, we note the identity

$$\sum_{i=1}^{l} m_i \left| \sum_{j=1}^{l} m_j (\overset{\rightharpoonup}{v}_{y_j/\mathrm{c}/\mathrm{A}} - \overset{\rightharpoonup}{v}_{y_i/\mathrm{c}/\mathrm{A}}) \right|^2 = \frac{m_{\mathcal{B}}}{2} \sum_{i,j=1}^{l} m_i m_j |\overset{\rightharpoonup}{v}_{y_j/\mathrm{c}/\mathrm{A}} - \overset{\rightharpoonup}{v}_{y_i/\mathrm{c}/\mathrm{A}}|^2.$$

Therefore,

$$\begin{aligned}
T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} &= \tfrac{1}{2} \sum_{i=1}^{l} m_i |\overset{\rightharpoonup}{v}_{y_i/\mathrm{c}/\mathrm{A}}|^2 \\
&= \tfrac{1}{2} \sum_{i=1}^{l} \frac{m_i}{m_{\mathcal{B}}^2} \left| \sum_{j=1}^{l} m_j (\overset{\rightharpoonup}{v}_{y_j/\mathrm{c}/\mathrm{A}} - \overset{\rightharpoonup}{v}_{y_i/\mathrm{c}/\mathrm{A}}) \right|^2
\end{aligned}$$

$$= \tfrac{1}{4}\sum_{i,j=1}^{l} \frac{m_i m_j}{m_{\mathcal{B}}} |\vec{v}_{y_j/\mathrm{c}/\mathrm{A}} - \vec{v}_{y_i/\mathrm{c}/\mathrm{A}}|^2$$

$$= \tfrac{1}{4}\sum_{i,j=1}^{l} \frac{m_i m_j}{m_{\mathcal{B}}} |\vec{v}_{y_j/y_i/\mathrm{A}}|^2$$

$$= \tfrac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{l} \frac{m_i m_j}{m_{\mathcal{B}}} |\vec{v}_{y_j/y_i/\mathrm{A}}|^2.$$

Now assume that $\mathcal{B}$ is rigid and $\mathrm{F_A}$ is body fixed. Then, for all $i, j \in \{1, \ldots, l\}$, it follows that $\vec{v}_{y_j/y_i/\mathrm{A}} = 0$, which implies (8.1.7). Finally, (8.1.8) follows from (8.1.7) and (8.1.5). □

The following result relates the kinetic energies of a body relative to the same point but with respect to two different frames.

Fact 8.1.7. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let z be a point, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$T_{\mathcal{B}/z/\mathrm{A}} + T_{\mathcal{B}/z/\mathrm{B}} = \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}} \vec{J}_{\mathcal{B}/z} \vec{\omega}_{\mathrm{B}/\mathrm{A}} + \sum_{i=1}^{l} m_i \vec{v}'_{y_i/z/\mathrm{A}} \vec{v}_{y_i/z/\mathrm{B}}. \tag{8.1.9}$$

If $\vec{\omega}_{\mathrm{B}/\mathrm{A}} = 0$, then

$$T_{\mathcal{B}/z/\mathrm{A}} = T_{\mathcal{B}/z/\mathrm{B}}. \tag{8.1.10}$$

Alternatively, if $\mathcal{B}$ is rigid, $\mathrm{F_B}$ is body fixed, and z is fixed in $\mathcal{B}$, then

$$T_{\mathcal{B}/z/\mathrm{A}} = \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}} \vec{J}_{\mathcal{B}/z} \vec{\omega}_{\mathrm{B}/\mathrm{A}}. \tag{8.1.11}$$

Proof. Note that

$$\begin{aligned}
T_{\mathcal{B}/z/\mathrm{A}} + T_{\mathcal{B}/z/\mathrm{B}} &= \tfrac{1}{2}\sum_{i=1}^{l} m_i \vec{v}'_{y_i/z/\mathrm{A}} \vec{v}_{y_i/z/\mathrm{A}} + \tfrac{1}{2}\sum_{i=1}^{l} m_i \vec{v}'_{y_i/z/\mathrm{B}} \vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i (\vec{v}_{y_i/z/\mathrm{A}} - \vec{v}_{y_i/z/\mathrm{B}})'(\vec{v}_{y_i/z/\mathrm{A}} - \vec{v}_{y_i/z/\mathrm{B}}) + \sum_{i=1}^{l} m_i \vec{v}'_{y_i/z/\mathrm{A}} \vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i (\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{y_i/z})'(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{y_i/z}) + \sum_{i=1}^{l} m_i \vec{v}'_{y_i/z/\mathrm{A}} \vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i \vec{\omega}'_{\mathrm{B}/\mathrm{A}} \vec{r}^{\,\times\prime}_{y_i/z} \vec{r}^{\,\times}_{y_i/z} \vec{\omega}_{\mathrm{B}/\mathrm{A}} + \sum_{i=1}^{l} m_i \vec{v}'_{y_i/z/\mathrm{A}} \vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}} \vec{J}_{\mathcal{B}/z} \vec{\omega}_{\mathrm{B}/\mathrm{A}} + \sum_{i=1}^{l} m_i \vec{v}'_{y_i/z/\mathrm{A}} \vec{v}_{y_i/z/\mathrm{B}}.
\end{aligned}$$

□

Choosing z to be c in Fact 8.1.7 and using Fact 8.1.6 yields the following result.

Fact 8.1.8. Let $\mathrm{F_A}$ be a frame, let $\mathcal{B}$ be a rigid body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $\mathrm{F_B}$ be a body-fixed frame, and let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$.

Then,

$$T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{B}/\mathrm{A}} = \tfrac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{l}\frac{m_i m_j}{m_{\mathcal{B}}}|\vec{v}_{y_j/y_i/\mathrm{A}}|^2. \tag{8.1.12}$$

The following result expresses the kinetic energy of a body in terms of its physical inertia matrix and center of mass.

Fact 8.1.9. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w and z be points, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2 + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/z/\mathrm{B}}|^2. \tag{8.1.13}$$

If, in addition, $\mathcal{B}$ is rigid, $\mathrm{F_B}$ is body fixed, and z is fixed in $\mathcal{B}$, then

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}} &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2 \\ &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{H}_{\mathcal{B}/z/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2. \end{aligned} \tag{8.1.14}$$

Proof. It follows from (8.1.4) that

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}} &= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2 \\ &= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{B}}\vec{v}_{y_i/z/\mathrm{B}} + \tfrac{1}{2}\sum_{i=1}^{l} m_i(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{y_i/z})'(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{y_i/z}) \\ &\quad + \vec{v}'_{z/w/\mathrm{A}}\sum_{i=1}^{l} m_i(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{y_i/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2 \\ &= \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/z/\mathrm{B}}|^2 + \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2. \qquad \square \end{aligned}$$

Choosing z to be colocated with w in Fact 8.1.9 yields the following result.

Fact 8.1.10. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let z be a point, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$T_{\mathcal{B}/z/\mathrm{A}} = \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/z/\mathrm{B}}|^2. \tag{8.1.15}$$

If, in addition, $\mathcal{B}$ is rigid, $\mathrm{F_B}$ is body fixed, and z is fixed in $\mathcal{B}$, then

$$\begin{aligned} T_{\mathcal{B}/z/\mathrm{A}} &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} \\ &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{H}_{\mathcal{B}/z/\mathrm{A}}. \end{aligned} \tag{8.1.16}$$

In particular,

$$T_{\mathcal{B}/c/A} = \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A}$$
$$= \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{H}_{\mathcal{B}/c/A}. \tag{8.1.17}$$

Alternatively, choosing z to be the center of mass in Fact 8.1.9 yields the following result.

Fact 8.1.11. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w be a point, and let F_A and F_B be frames. Then,

$$T_{\mathcal{B}/w/A} = \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2 + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/c/B}|^2. \tag{8.1.18}$$

Now assume that $\mathcal{B}$ is rigid and F_B is body fixed. Then,

$$T_{\mathcal{B}/w/A} = \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2$$
$$= \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{H}_{\mathcal{B}/c/A} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2. \tag{8.1.19}$$

If, in addition, F_A is body fixed, then

$$T_{\mathcal{B}/w/A} = \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2. \tag{8.1.20}$$

Note that (8.1.20) is identical to (8.1.8).

Finally, choosing w to be the center of mass in Fact 8.1.11 yields the following result.

Fact 8.1.12. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, and let F_A and F_B be frames. Then,

$$T_{\mathcal{B}/c/A} = \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/c/B}|^2. \tag{8.1.21}$$

Now assume that $\mathcal{B}$ is rigid and F_B is body fixed. Then,

$$T_{\mathcal{B}/c/A} = \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A}$$
$$= \tfrac{1}{2}\vec{\omega}'_{B/A}\vec{H}_{\mathcal{B}/c/A}. \tag{8.1.22}$$

If, in addition, F_A is body fixed, then

$$T_{\mathcal{B}/c/A} = 0. \tag{8.1.23}$$

Note that (8.1.23) is identical to (8.1.7).

It follows from Fact 8.1.12 that, if $\mathcal{B}$ is rigid and F_A is a body-fixed frame, then $T_{\mathcal{B}/c/A} = 0$, which is the second statement of Fact 8.1.6.

8.2 Work Done by Forces and Moments on a Body

Energy is a relative concept. Potential energy depends on position relative to a specified reference point, while kinetic energy depends on velocity relative to a reference point and with respect to a specified frame. The *work done* on a body by a force or moment is the energy transferred to or removed from the body due to the force or moment.

The following result is a law of physics, and thus is not proved. Since this result concerns the effect of a force on a particle, and since this effect is governed by Newton's second law, the reference point is taken to be an unforced particle.

Fact 8.2.1. Let y be a particle and let w be an unforced particle. Then, the work done on y relative to w by the force $\vec{f}_y$ applied to y as y moves along the path $\mathcal{C}_y$ is given by

$$W_{y/w}(\vec{f}_y, \mathcal{C}_y) = \int_{\mathcal{C}_y} \vec{f}_y \cdot \mathrm{d}\vec{r}_{y/w}. \tag{8.2.1}$$

Now, let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$. The work done on $\mathcal{B}$ relative to the point w by the force $\vec{f}_{y_i}$ applied to y_i as y_i moves along the path $\mathcal{C}_{y_i}$ for all $i = 1, \ldots, l$ is given by

$$W_{\mathcal{B}/w}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_{y_1}, \ldots, \mathcal{C}_{y_l}) = \sum_{i=1}^{l} W_{y_i/w}(\vec{f}_{y_i}, \mathcal{C}_{y_i}). \tag{8.2.2}$$

Notice that the mass of y plays no role. Therefore, we can consider a point y in place of a particle with the understanding that $W_{y/w}(\vec{f}_y, \mathcal{C}_y)$ denotes the energy transferred to y if y were a particle.

Let $\mathrm{F_A}$ be a frame. Then, we can rewrite (8.2.1) as

$$\begin{aligned} W_{y/w}(\vec{f}_y, \mathcal{C}_y) &= \int_0^{s_\mathrm{f}} \vec{f}_y(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\,\mathrm{d}s \\ &= \int_0^{s_\mathrm{f}} \vec{f}_y(s) \cdot \hat{e}_\mathrm{t}(s)\,\mathrm{d}s, \end{aligned} \tag{8.2.3}$$

where the path $\mathcal{C}$ is parameterized by the path length s in the interval $[0, s_\mathrm{f}]$ and, by (5.5.14), $\hat{e}_\mathrm{t}(s) = \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)$, which is the unit tangent vector to $\mathcal{C}_y$ at s.

The following result is the analogue of Fact 8.2.1 for the case of forces applied to a rigid body.

Fact 8.2.2. Let $\mathcal{B}$ be a rigid body with particles $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ denote the force applied to y_i, and let $\mathcal{C}_{y_i}$ denote the path of y_i. Furthermore, let y be a point fixed in $\mathcal{B}$, let $\mathcal{C}_y$ denote the path of y, let $\vec{f}_\mathcal{B}$ denote the total force on $\mathcal{B}$, and let w be an unforced particle, and let $\mathrm{F_A}$ be a frame. In addition, let the rotational path $\mathcal{C}_\mathcal{B}$ of $\mathcal{B}$ be given by $\vec{R}_{\mathrm{B/A}}(\alpha) = e^{\vec{\Theta}^{\times}_{\mathrm{B/A}}(\alpha)}$, where $\alpha \in [0, \alpha_\mathrm{f}]$. Then, the work done on $\mathcal{B}$ relative to the point w by the forces $\vec{f}_1, \ldots, \vec{f}_l$ is given by

$$W_{\mathcal{B}/w}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_{y_1}, \ldots, \mathcal{C}_{y_l}) = W_{y/w}(\vec{f}_\mathcal{B}, \mathcal{C}_y) + W_{\mathcal{B}/\mathrm{A}}(\vec{M}_\mathcal{B}, \mathcal{C}_\mathcal{B}), \tag{8.2.4}$$

where

$$W_{\mathcal{B}/\mathrm{A}}(\vec{M}_\mathcal{B}, \mathcal{C}_\mathcal{B}) \triangleq \int_{\mathcal{C}_\mathcal{B}} \vec{M}_\mathcal{B} \cdot G(\Theta_{\mathrm{B/A}})\,\mathrm{d}\Theta_{\mathrm{B/A}}, \tag{8.2.5}$$

$$\vec{M}_\mathcal{B} \triangleq \sum_{i=1}^{l} \vec{r}_{y_i/y} \times \vec{f}_{y_i}, \tag{8.2.6}$$

and

$$G(\vec{\Theta}_{\mathrm{B/A}}) \triangleq \frac{1}{|\vec{\Theta}_{\mathrm{B/A}}|^2}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{J} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right). \tag{8.2.7}$$

Proof. Let s denote the path length variable for the point y, for $i = 1, \ldots, l$, let s_i denote the path length variable for the point y_i. Therefore, for $i = 1, \ldots, l$, using the fact that $\vec{r}_{y_i/y}(\alpha) = \vec{R}_{\mathrm{B/A}}(\alpha)\vec{r}_{y_i/y}(0)$ as well as (4.9.8), we have

$$\begin{aligned}
\mathrm{d}\vec{r}_{y_i/y} &= \overset{\alpha\bullet}{\vec{r}}_{y_i/y}(\alpha)\mathrm{d}\alpha \\
&= \overset{\alpha\bullet}{\vec{R}}_{\mathrm{B/A}}(\alpha)\mathrm{d}\alpha\,\vec{r}_{y_i/y}(0) \\
&= \vec{\omega}^{\times}_{\mathrm{B/A}}(\alpha)\vec{R}_{\mathrm{B/A}}(\alpha)\mathrm{d}\alpha\,\vec{r}_{y_i/y}(0) \\
&= \vec{\omega}^{\times}_{\mathrm{B/A}}(\alpha)\vec{r}_{y_i/y}(\alpha)\mathrm{d}\alpha \\
&= \vec{\omega}_{\mathrm{B/A}}(\alpha) \times \vec{r}_{y_i/y}(\alpha)\mathrm{d}\alpha \\
&= -\vec{r}_{y_i/y}(\alpha) \times \vec{\omega}_{\mathrm{B/A}}(\alpha)\mathrm{d}\alpha \\
&= -\vec{r}_{y_i/y}(\alpha) \times \frac{1}{|\vec{\Theta}_{\mathrm{B/A}}|^2}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{J} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\overset{\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\,\mathrm{d}\alpha \\
&= -\vec{r}_{y_i/y} \times G(\vec{\Theta}_{\mathrm{B/A}})\,\mathrm{d}\vec{\Theta}_{\mathrm{B/A}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&W_{\mathcal{B}/w}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_{y_1}, \ldots, \mathcal{C}_{y_l}) \\
&\quad = \sum_{i=1}^{l} W_{y_i/w}(\vec{f}_{y_i}, \mathcal{C}_{y_i}) \\
&\quad = \sum_{i=1}^{l} \int_{\mathcal{C}_{y_i}} \vec{f}_{y_i} \cdot \mathrm{d}\vec{r}_{y_i/w} \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \mathrm{d}\vec{r}_{y_i/w}(s) \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y_i/w}(s)\,\mathrm{d}s \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot [\overset{\mathrm{A}s\bullet}{\vec{r}}_{y_i/y}(s) + \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)]\,\mathrm{d}s \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y_i/y}(s)\,\mathrm{d}s + \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\,\mathrm{d}s \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{s_i,y_i/y}(s_i(s))\,\mathrm{d}s + \int_0^{s_\mathrm{f}} \sum_{i=1}^{l} \vec{f}_{y_i}(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \overset{\mathrm{A}s_i\bullet}{\vec{r}}_{y_i/y}(s_i) \frac{\mathrm{d}s_i}{\mathrm{d}s} \mathrm{d}s + \int_0^{s_\mathrm{f}} \vec{f}_{\mathcal{B}}(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\, \mathrm{d}s \\
&= \sum_{i=1}^{l} \int_0^{s_{i,\mathrm{f}}} \vec{f}_{y_i}(s_i) \cdot \overset{\mathrm{A}s_i\bullet}{\vec{r}}_{y_i/y}(s_i)\, \mathrm{d}s_i + \int_0^{s_\mathrm{f}} \vec{f}_{\mathcal{B}}(s) \cdot \mathrm{d}\vec{r}_{y/w}(s) \\
&= \sum_{i=1}^{l} \int_{\mathcal{C}_{y_i}} \vec{f}_{y_i} \cdot \mathrm{d}\vec{r}_{y_i/y} + \int_0^{s_\mathrm{f}} \vec{f}_{\mathcal{B}}(s) \cdot \mathrm{d}\vec{r}_{y/w}(s) \\
&= -\int_{\mathcal{C}_{\mathcal{B}}} \sum_{i=1}^{l} \vec{f}_{y_i} \cdot (\vec{r}_{y_i/y} \times G(\vec{\Theta}_{\mathrm{B/A}})\, \mathrm{d}\vec{\Theta}_{\mathrm{B/A}}) + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= -\int_{\mathcal{C}_{\mathcal{B}}} \sum_{i=1}^{l} (\vec{f}_{y_i} \times \vec{r}_{y_i/y}) \cdot G(\vec{\Theta}_{\mathrm{B/A}})\, \mathrm{d}\vec{\Theta}_{\mathrm{B/A}} + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= \int_{\mathcal{C}_{\mathcal{B}}} \sum_{i=1}^{l} (\vec{r}_{y_i/y} \times \vec{f}_{y_i}) \cdot G(\vec{\Theta}_{\mathrm{B/A}})\, \mathrm{d}\vec{\Theta}_{\mathrm{B/A}} + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= \int_{\mathcal{C}_{\mathcal{B}}} \vec{M}_{\mathcal{B}} \cdot G(\vec{\Theta}_{\mathrm{B/A}})\, \mathrm{d}\vec{\Theta}_{\mathrm{B/A}} + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= W_{\mathcal{B}/\mathrm{A}}(\vec{M}_{\mathcal{B}}, \mathcal{C}_{\mathcal{B}}) + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y). \qquad \square
\end{aligned}
$$

8.3 Potential Energy of Particles and Bodies

The following result is fundamental.

Fact 8.3.1. Let y be a particle, let $\vec{f}_y$ be a force acting on y that depends only on the position of y, and let w be a point. Then, the following statements are equivalent:

i) For every path $\mathcal{C}$, $W_{y/w}(\vec{f}_y, \mathcal{C})$ depends on the initial and final endpoints z_0 and z_1, respectively, of $\mathcal{C}$ but is otherwise independent of $\mathcal{C}$.

ii) There exists a function $U_{y/w}$ that maps physical position vectors into real numbers such that

$$\vec{f}_y = -\vec{\partial} U_{y/w}(\vec{r}_{y/w}). \tag{8.3.1}$$

In this case,

$$W_{y/w}(\vec{f}_y, \mathcal{C}) = U_{y/w}(\vec{r}_{z_0/w}) - U_{y/w}(\vec{r}_{z_1/w}), \tag{8.3.2}$$

that is,

$$U_{y/w}(\vec{r}_{z_1/w}) - U_{y/w}(\vec{r}_{z_0/w}) = -\int_{\mathcal{C}} \vec{f}_y \cdot \mathrm{d}\vec{r}_{y/w}. \tag{8.3.3}$$

Proof. To prove *ii*) implies *i*), let $z(s)$ denote the path of y along $\mathcal{C}$. Then, note that

$$
\begin{aligned}
W_{y/w}(\vec{f}_y, \mathcal{C}) &= \int_{\mathcal{C}} \vec{f}_y \cdot \mathrm{d}\vec{r}_{y/w} \\
&= -\int_{\mathcal{C}} \vec{\partial} U_{y/w}(\vec{r}_{z/w}) \cdot \mathrm{d}\vec{r}_{y/w}
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{s_0}^{s_1} \vec{\partial} U_{y/w}(\vec{r}_{z(s)/w}) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{z(s)/w} \,\mathrm{d}s \\
&= -\int_{s_0}^{s_1} \partial U_{y/w|\mathrm{A}} \left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}} \right) \cdot \left. \overset{\mathrm{A}s\bullet}{\vec{r}}_{z(s)/w} \right|_{\mathrm{A}} \mathrm{d}s \\
&= -\int_{s_0}^{s_1} \partial U_{y/w|\mathrm{A}} \left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}} \right) \cdot \frac{\mathrm{d}}{\mathrm{d}s} \left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}} \right) \mathrm{d}s \\
&= -\int_{s_0}^{s_1} \frac{\mathrm{d}}{\mathrm{d}s} U_{y/w|\mathrm{A}} \left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}} \right) \mathrm{d}s \\
&= U_{y/w|\mathrm{A}} \left(\vec{r}_{z(s_0)/w}\Big|_{\mathrm{A}} \right) - U_{y/w|\mathrm{A}} \left(\vec{r}_{z(s_1)/w}\Big|_{\mathrm{A}} \right) \\
&= U_{y/w}(\vec{r}_{z(s_0)/w}) - U_{y/w}(\vec{r}_{z(s_1)/w}) \\
&= U_{y/w}(\vec{r}_{z_0/w}) - U_{y/w}(\vec{r}_{z_1/w}).
\end{aligned}
$$

To prove *i*) implies *ii*), suppose that y is located at z_0, and let $\mathcal{C}$ denote a path from z_0 to z_1. Since $W_{y/w}(\vec{f}_y, \mathcal{C})$ is independent of $\mathcal{C}$, we define

$$U_{y/w}(\vec{r}_{z_1/w}) \triangleq -W_{y/w}(\vec{f}_y, \mathcal{C}).$$

Next, let $\varepsilon > 0$, and let $z(s)$ for $s \in [0, \varepsilon]$ denote the path $\mathcal{C}_\varepsilon$ given by $\vec{r}_{z(s)/w} = s\hat{r}_{z_1/w}$. Hence $z(\varepsilon)$ denotes the point such that $\vec{r}_{z(\varepsilon)/z_0} = \varepsilon\hat{r}_{z_1/w}$. Note that $|\vec{r}_{z(\varepsilon)/w}| = \varepsilon$. We thus have

$$U_{y/w}(\vec{r}_{z(\varepsilon)/w}) = -\int_{\mathcal{C}_\varepsilon} \vec{f}_y \cdot \mathrm{d}\vec{r}_{z(s)/w} = -\int_0^\varepsilon \vec{f}_y(s)\,\mathrm{d}s \cdot \hat{r}_{z_1/w}.$$

□ Needs work

The force $\vec{f}_y$ acting on the particle y moving along the path $\mathcal{C}$ is a *potential force* if, for every point w, condition *i*) or, equivalently, condition *ii*), is satisfied. Furthermore, $U_{y/w}(\vec{r}_{z/w})$ is the *potential energy* of y located at z relative to w associated with the force $\vec{f}_y$. Note that $U_{y/w}(\vec{0}) = 0$.

Now consider a body $\mathcal{B}$ each of whose particles $y_1, \ldots, y_l$ is acted on by a potential force f_{y_i} associated with the potential energy $U_{y_i}(\vec{r}_{z_i/w})$, where z_{i0} and z_{i1} are the initial and final locations of y_i along $\mathcal{C}_i$, and w is a point. Then,

$$W_{\mathcal{B}}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_1, \ldots, \mathcal{C}_l, w) = \sum_{i=1}^{l} [U_{y_i/w}(\vec{r}_{z_{i0}/w}) - U_{y_i/w}(\vec{r}_{z_{i1}/w})]. \tag{8.3.4}$$

The moment $\vec{M}_{\mathcal{B}/w}$ acting on $\mathcal{B}$ relative to w is a *potential moment* if it arises from forces that are potential.

We first consider the gravitational potential energy in a uniform gravitational field. Let $\vec{g}$ denote the acceleration due to gravity.

Fact 8.3.2. Let y be a particle located at the point z, let m be the mass of y, let w be a point, let $\mathrm{F_A}$ be a frame, and let the acceleration due to gravity be given by $\vec{g}$. Then, the force acting on y due to gravity is given by $\vec{f}_y = -m\vec{g}$, and the potential energy of y relative to w is given by

$$U_{y/w}(\vec{r}_{z/w}) = -m\vec{r}_{z/w} \cdot \vec{g}. \tag{8.3.5}$$

Proof. Need proof. □

Let $\mathrm{F_A}$ be as in Fact 8.3.2, let $\vec{g} = -g\hat{k}_\mathrm{A}$, and let $\vec{r}\Big|_\mathrm{A} = [r_1 \;\; r_2 \;\; r_3]^\mathrm{T}$. Then,

$$U_{y/w|\mathrm{A}}(r) = mgr_3. \tag{8.3.6}$$

If, in addition, $\mathrm{F_B}$ is a frame, then

$$U_{y/w\mathrm{B}}(r) = mge_3^\mathrm{T}\mathcal{O}_{\mathrm{B/A}}r. \tag{8.3.7}$$

The gravitational potential energy of a body $\mathcal{B}$ due to a uniform gravitational field is defined to be the sum of the gravitational potential energy of each particle in $\mathcal{B}$.

Definition 8.3.3. Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ located at points $z_1, \ldots, z_l$, respectively. Then, the gravitational potential energy of $\mathcal{B}$ relative to w is defined by

$$U_{\mathcal{B}/w} \triangleq \sum_{i=1}^{l} U_{y_iw}(\vec{r}_{z_i/w}). \tag{8.3.8}$$

Fact 8.3.4. Let $\mathcal{B}$ be a body with total mass $m_\mathcal{B}$, let w be a point, let c denote the center of mass of $\mathcal{B}$, and let $\vec{g}$ be the acceleration due to gravity. Then, the gravitational potential energy of $\mathcal{B}$ relative to w is given by

$$U_{\mathcal{B}/w} = -m_\mathcal{B}\vec{r}_{\mathrm{c}/w} \cdot \vec{g}. \tag{8.3.9}$$

Proof. Note that

$$\begin{aligned} U_{\mathcal{B}/w} &= -\sum_{i=1}^{l}(m_i\vec{r}_{y_i/w} \cdot \vec{g}) \\ &= -\left(\sum_{i=1}^{l} m_i\vec{r}_{y_i/w}\right) \cdot \vec{g} \\ &= -m_\mathcal{B}\vec{r}_{\mathrm{c}/w} \cdot \vec{g}. \end{aligned}$$

□

Next we consider the potential energy due to a central gravitational field on the Earth with origin $\mathrm{O_E}$ at the center of the Earth. Let $\mu_\mathrm{E} = GM_\mathrm{E}$ denote the gravitational constant for the Earth, and

Fact 8.3.5. Let y be a particle with mass m, let $\mathrm{O_E}$ be the center of the Earth, let $\mathrm{F_E} = \mathrm{F_{sph}}$, and let the potential energy of y relative to $\mathrm{O_E}$ due to gravity be given by

$$U_{y/\mathrm{O_E}}(\vec{r}_{y/\mathrm{O_E}}) = -\frac{\mu_\mathrm{E}m}{\vec{r}_{y/\mathrm{O_E}} \cdot \hat{e}_\mathrm{u}}. \tag{8.3.10}$$

Then, the force $\vec{f}_\mathrm{grav}$ due to gravity is given by

$$\vec{f}_\mathrm{grav} = -mg\hat{e}_\mathrm{u},$$

where $g \triangleq \mu_\mathrm{E}/|\vec{r}_{y/\mathrm{O_E}}|^2$.

Proof. Recall that $\mathrm{F_{sph}} = [\hat{e}_\mathrm{u}\ \ \hat{e}_\mathrm{e}\ \ \hat{e}_\mathrm{n}]$. Writing

$$U_{y/\mathrm{O_E}|\mathrm{E}}(\vec{r}_{y/\mathrm{O_E}}) = -\frac{\mu_\mathrm{E} m}{r_1},$$

where $\vec{r}_{y/\mathrm{O_E}}\Big|_\mathrm{E} = [r_1\ \ r_1\ \ r_3]^\mathrm{T}$, it follows that

$$-U_{y/\mathrm{O_E}|\mathrm{E}}(\vec{r}_{y/\mathrm{O_E}}) = \begin{bmatrix} -\frac{\mu_\mathrm{E} m}{r_1^2} & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned}\vec{f}_\mathrm{grav}\Big|_\mathrm{E}^\mathrm{T} &= -\vec{\partial} U_{y/\mathrm{O_E}}(\vec{r}_{y/\mathrm{O_E}})\Big|_\mathrm{E} \\ &= -\partial U_{y/\mathrm{O_E}}(|\vec{r}_{y/\mathrm{O_E}}|e_1) \\ &= -\frac{\mu_\mathrm{E} m}{|\vec{r}_{y/\mathrm{O_E}}|^2} e_1^\mathrm{T} \\ &= -mg\ \hat{e}_\mathrm{u}|_\mathrm{E}^\mathrm{T}. \end{aligned}$$

□

Next, we consider the potential energy of a spring.

Fact 8.3.6. Let y be an inertia point, and assume that y is connected to the inertia point w by a spring whose stiffness is k and whose relaxed length is d. Then, the force acting on y due to the spring is given by

$$\vec{f}_y = -k(|\vec{r}_{y/w}| - d)\hat{r}_{y/w} = -\vec{\partial} U_{y/w|\mathrm{A}}(\vec{r}_{z/w}), \tag{8.3.11}$$

where the potential energy $U_{y/w}$ of y relative to w is given by

$$U_{y/w}(\vec{r}_{z/w}) = \tfrac{1}{2}k(|\vec{r}_{y/w}| - d)^2. \tag{8.3.12}$$

Proof. Let $\vec{r}_{z/w}\Big|_\mathrm{A} = \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}$. Then,

$$U_{y/w|\mathrm{A}}\left(\vec{r}_{z/w}\Big|_\mathrm{A}\right) = \tfrac{1}{2}k(\bar{x}^2 + \bar{y}^2 + \bar{z}^2 - 2d\sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2} + d^2). \tag{8.3.13}$$

Therefore,

$$\partial_{\bar{x}} U_{y/w|\mathrm{A}}\left(\vec{r}_{z/w}\Big|_\mathrm{A}\right) = \frac{k}{|\vec{r}_{y/w}|}(|\vec{r}_{y/w}| - d)\bar{x},$$

and thus

$$\begin{aligned}\partial^\mathrm{T} U_{y/w|\mathrm{A}}\left(\vec{r}_{z/w}\Big|_\mathrm{A}\right) &= \frac{k}{|\vec{r}_{y/w}|}(|\vec{r}_{y/w}| - d)\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \\ &= -k(|\vec{r}_{y/w}| - d)\ \hat{r}_{y/w}\Big|_\mathrm{A} \\ &= -\vec{f}_y\Big|_\mathrm{A}. \end{aligned}$$

Hence,

$$\vec{\partial} U_{y/w|\mathrm{A}}(\vec{r}_{z/w}) = k(|\vec{r}_{y/w}| - d)\hat{r}_{y/w}. \qquad \square$$

8.4 Conservation of Energy

The *total energy* of a body $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is defined by

$$E_{\mathcal{B}/w/\mathrm{A}} \triangleq T_{\mathcal{B}/w/\mathrm{A}} + U_{\mathcal{B}/w}. \tag{8.4.1}$$

Note that $E_{\mathcal{B}/w/\mathrm{A}}$ includes the kinetic energy of $\mathcal{B}$ as well as the potential energy associated with all potential forces acting on the particles in $\mathcal{B}$, such as gravity (which may be central or uniform) and springs. The body $\mathcal{B}$ is assumed to have finite mass. If $\mathcal{B}$ interacts with a massive body, then all reaction forces due to the massive body are viewed as internal forces, but the kinetic and potential energy of the massive body are not included. If $\mathcal{B}$ consists of particles $y_1, \ldots, y_l$, then

$$E_{\mathcal{B}/w/\mathrm{A}} = \sum_{i=1}^{l} [T_{y_i/w/\mathrm{A}} + U_{y_i/w}(\vec{r}_{y_i/w})]. \tag{8.4.2}$$

where, for $i = 1, \ldots, l$, $T_{y_i/w/\mathrm{A}} = \frac{1}{2} m_i |\vec{v}_{y_i/w/\mathrm{A}}|^2$.

The particles in a body can be subjected to both internal forces (that is, reaction forces) and external forces. Internal forces, such as sliding with friction, can lead to a decrease in energy, while external forces can lead to either a decrease or increase in energy. If the internal or external force $\vec{f}$ does not increase or decrease the total energy of the body, then $\vec{f}$ is a *conservative force*. The following result shows that all potential forces are conservative forces.

Fact 8.4.1. Let $\mathcal{B}$ be a body, let $\mathrm{F_A}$ be an inertial frame, let w be an unforced particle, and assume that all internal and external forces applied to $\mathcal{B}$ are potential forces. Then, the total energy of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is conserved.

Proof. For $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ denote the total force applied to y_i. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathcal{B}/w/\mathrm{A}} &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{l} [\tfrac{1}{2} m_i |\vec{v}_{y_i/w/\mathrm{A}}|^2 + U_{y_i/w}(\vec{r}_{y_i/w})] \\
&= \sum_{i=1}^{l} \left[\tfrac{1}{2} m_i \frac{\mathrm{d}}{\mathrm{d}t} (\vec{v}_{y_i/w/\mathrm{A}} \cdot \vec{v}_{y_i/w/\mathrm{A}}) + \partial U_{y_i/w}(\vec{r}_{y_i/w}) \cdot \vec{v}_{y_i/w/\mathrm{A}} \right] \\
&= \sum_{i=1}^{l} (m_i \vec{a}_{y_i/w/\mathrm{A}} \cdot \vec{v}_{y_i/w/\mathrm{A}} - \vec{f}_{y_i} \cdot \vec{v}_{y_i/w/\mathrm{A}}) \\
&= \sum_{i=1}^{l} [(m_i \vec{a}_{y_i/w/\mathrm{A}} - \vec{f}_{y_i}) \cdot \vec{v}_{y_i/w/\mathrm{A}}] \\
&= 0.
\end{aligned} \qquad \square$$

8.5 Theoretical Problems

Problem 8.5.1. Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ with masses $m_1, \ldots, m_l$, respectively, let c denote the center of mass of $\mathcal{B}$, and let w be an unforced particle. Assume that, for all distinct $i, j \in \{1, \ldots, l\}$, the particles y_i and y_j are connected by a dashpot with viscosity c_{ij}. (Note that $c_{ij} = c_{ji}$ and $c_{ii} = 0$.) Assume, in addition, that no external forces are applied to $\mathcal{B}$.

i) Show that, for all $i \in \{1, \ldots, l\}$, $\lim_{t\to\infty} \vec{v}_{y_i/w/\mathrm{A}}$ exists and, for all distinct $i, j \in \{1, \ldots, l\}$, $\lim_{t\to\infty} \vec{v}_{y_i/w/\mathrm{A}} = \lim_{t\to\infty} \vec{v}_{y_j/w/\mathrm{A}}$.

ii) Show that, for all distinct $i, j \in \{1, \ldots, l\}$, $\lim_{t\to\infty} \vec{v}_{y_i/y_j/\mathrm{A}} = 0$.

iii) Show that $\lim_{t\to\infty} T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = 0$.

iv) Show that $\lim_{t\to\infty} T_{\mathcal{B}/w/\mathrm{A}} = m_{\mathcal{B}} |\vec{v}_{\mathrm{c}/w/\mathrm{A}}|^2$.

Chapter Nine
Lagrangian Dynamics

9.1 Lagrangian Dynamics versus Newton-Euler Dynamics

The equations of motion for a body that consists of multiple rigid bodies can be derived by applying Newton-Euler dynamics individually to each rigid body. Rigid bodies can interact with each other through rolling, sliding (with or without friction), and pivoting (with or without friction). This interaction can occur through joints, which may be revolute (rotatable) or prismatic (extendable).

To apply Newton-Euler dynamics to each rigid body, we must determine the forces and moments acting on each rigid body. Since each rigid body can interact with all other rigid bodies (including massive bodies) through reaction forces and moments as determined by Newton's third law, the total force and moment on each rigid body must include the forces and moments due to the interaction with the remaining bodies as well as the forces and moments due to gravity.

A rigid body may also be subject to a constraint, such as a connection to another, possibly massive, rigid body by means of a revolute or prismatic joint. The constraint can be viewed as equivalent to forces and moments that prevent the interconnected rigid bodies from moving in ways that violate the constraint. The effect of constraints on the rigid bodies in a body is thus to introduce contact reaction forces and moments between the rigid bodies; these forces are known as *constraint forces and moments*. The massive body is unaffected by constraint forces and moments.

Unfortunately, the task of determining reaction forces and moments due to the interactions between rigid bodies can be difficult. In 1788, Lagrange discovered that the equations of motion for a body consisting of multiple rigid bodies can be obtained by performing operations on the kinetic and potential energies. Lagrangian dynamics circumvents the need to determine the contact forces and moments between particles and rigid bodies. If the contact forces and moments are of interest, then these can be determined by using Newton-Euler methods in conjunction with the equations derived from Lagrangian dynamics.

9.2 Generalized Coordinates

We consider a discrete or continuum body $\mathcal{B}$ that consists of particles that can translate as well as rigid bodies that can translate and rotate. The *configuration* of $\mathcal{B}$ refers to the spatial arrangement of the particles and rigid bodies in $\mathcal{B}$. The configuration of $\mathcal{B}$ can be modeled by using *generalized coordinates* q_i, each of which is either a position or an angle. More specifically, q_i is either a scalar position along a dimensionless unit vector or a signed angle around a dimensionless unit vector. The maximum number of generalized coordinates needed to determine the configuration of a body is the number of *degrees of freedom* of the body. Each particle that is not constrained is described by three coordinates, and thus has three degrees of freedom. Likewise, each rigid body that is not constrained can be modeled by six coordinates and thus has six degrees of freedom. Cylindrical and spherical coordinates are generalized coordinates. If $\mathcal{B}$ consists of l_1 particles and l_2 rigid bodies, then $\mathcal{B}$ can have up to $3l_1 + 6l_2$ degrees of freedom. If the particles and rigid bodies in $\mathcal{B}$ are constrained, for example, through revolute or prismatic joints, then the number of degrees of freedom is less than this number.

Ignoring constraints that may be present, the position of the particle y_i in the body $\mathcal{B}$ relative to the point w can be modeled by the three coordinates

$$\vec{r}_{y_i/w}\Big|_{\mathrm{A}} = \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{bmatrix}. \tag{9.2.1}$$

However, when constraints are considered, it is often convenient to model the configuration of the body by using r generalized coordinates

$$q \triangleq \begin{bmatrix} q_1 \\ \vdots \\ q_r \end{bmatrix} \in \mathbb{R}^r. \tag{9.2.2}$$

The coordinates $\bar{x}_i, \bar{y}_i, \bar{z}_i$ for y_i can thus be written as

$$\bar{x}_i = \bar{x}_i(q, t), \tag{9.2.3}$$

$$\bar{y}_i = \bar{y}_i(q, t), \tag{9.2.4}$$

$$\bar{z}_i = \bar{z}_i(q, t), \tag{9.2.5}$$

For example, the position of the tip of a pendulum connected by a massless arm to an inertially nonrotating massive rigid body by means of a revolute joint is constrained to lie on a circle centered at the joint, while a rigid body can be thought of as being composed of innumerable particles constrained to move in such a manner that the distance between each pair of particles is constant. It is, therefore, useful to consider a body of l particles with p independent physical constraints modeled by

$$\phi_j(\bar{x}_1, \bar{y}_1, \bar{z}_1, \ldots, \bar{x}_l, \bar{y}_l, \bar{z}_l) = 0, \ \ j = 1, \ldots, p. \tag{9.2.6}$$

This body possesses $r = 3l - p$ degrees of freedom since only r independent quantities need be known to determine the configuration. The configuration space of a pendulum is the circle on which the tip of the pendulum moves, while the configuration space of a rigid body—regardless of the number of particles comprising the body—is the set of all positions of its center of mass together with the set of all possible orientations.

Constraints such as (9.2.6), which constrain the positions but not the derivatives of position, are *holonomic*. When the constraints on a body are holonomic, it is often possible (but not always possible, see Chapter 9) to find independent, that is, unconstrained, generalized coordinates that describe the configuration of the body. The positions of all of the particles, when expressed in terms of these independent coordinates, automatically satisfy the physical constraints of the body. For instance, the generalized coordinate for the pendulum is its angular position. Likewise, the coordinates of the center of mass of a rigid body together with its three Euler angles provide six generalized coordinates for the rigid body regardless of the number of particles comprising the body. For the remainder of this chapter we assume that all bodies are holonomic with configurations that are described by independent, unconstrained generalized coordinates. This assumption is removed in Chapter 9.

9.3 Generalized Velocities and Kinetic Energy

The rate of change of the configuration of the body $\mathcal{B}$ induces the generalized coordinates to change with rates called *generalized velocities*, which are denoted by $\dot{q} = [\dot{q}_1 \ \cdots \ \dot{q}_r]^{\mathrm{T}}$. If $q_1, \ldots, q_r$ are independent, then so are $\dot{q}_1, \ldots, \dot{q}_r$. By applying the chain rule to (9.2.3)–(9.2.5), we obtain the components of the velocity of the ith particle relative to the point w and with respect to $\mathrm{F_A}$ as

$$\vec{v}_{y_i/w/\mathrm{A}}\Big|_{\mathrm{A}} = \begin{bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{w}_i \end{bmatrix}, \tag{9.3.1}$$

where

$$\bar{u}_i = \bar{u}_i(q, \dot{q}, t) = \sum_{j=1}^{r} \partial_{q_j}\bar{x}_i(q, t)\dot{q}_j + \partial_t\bar{x}_i(q, t), \tag{9.3.2}$$

$$\bar{v}_i = \bar{v}_i(q, \dot{q}, t) = \sum_{j=1}^{r} \partial_{q_j}\bar{y}_i(q, t)\dot{q}_j + \partial_t\bar{y}_i(q, t), \tag{9.3.3}$$

$$\bar{w}_i = \bar{w}_i(q, \dot{q}, t) = \sum_{j=1}^{r} \partial_{q_j}\bar{z}_i(q, t)\dot{q}_j + \partial_t\bar{z}_i(q, t). \tag{9.3.4}$$

Defining the gradient $\partial_q\bar{x}_i(q, t) \in \mathbb{R}^{1\times r}$ of $x_i(q, t)$ by

$$\partial_q\bar{x}_i(q, t) \triangleq \begin{bmatrix} \partial_{q_1}\bar{x}_i(q, t) & \cdots & \partial_{q_r}\bar{x}_i(q, t) \end{bmatrix}, \tag{9.3.5}$$

which is a row vector, and

$$\alpha_i(q, t) \triangleq \partial_t\bar{x}_i(q, t), \quad \beta_i(q, t) \triangleq \partial_t\bar{y}_i(q, t), \quad \gamma_i(q, t) \triangleq \partial_t\bar{z}_i(q, t), \tag{9.3.6}$$

we can rewrite (9.3.2)–(9.3.4) as

$$\bar{u}_i = \partial_q\bar{x}_i(q, t)\dot{q} + \alpha_i(q, t), \tag{9.3.7}$$

$$\bar{v}_i = \partial_q\bar{y}_i(q, t)\dot{q} + \beta_i(q, t), \tag{9.3.8}$$

$$\bar{w}_i = \partial_q\bar{z}_i(q, t)\dot{q} + \gamma_i(q, t). \tag{9.3.9}$$

Note that, if the dynamics of the body are time invariant, then $\alpha_i(q, t)$, $\beta_i(q, t)$, and $\gamma_i(q, t)$ are zero.

Next, letting m_i denote the mass of the ith particle of the body $\mathcal{B}$, we write the kinetic energy of $\mathcal{B}$ relative to the point w with respect to $\mathrm{F_A}$ in terms of its generalized coordinates and generalized velocities as

$$\begin{aligned}
T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}, t) &= \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/w/\mathrm{A}}|^2 \\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i[\bar{u}_i^2(q, \dot{q}, t) + \bar{v}_i^2(q, \dot{q}, t) + \bar{w}_i^2(q, \dot{q}, t)] \\
&= \tfrac{1}{2}\dot{q}^{\mathrm{T}}\sum_{i=1}^{l} m_i[\partial_q^{\mathrm{T}}\bar{x}_i(q, t)\partial_q\bar{x}_i(q, t) + \partial_q^{\mathrm{T}}\bar{y}_i(q, t)\partial_q\bar{y}_i(q, t) + \partial_q^{\mathrm{T}}\bar{z}_i(q, t)\partial_q\bar{z}_i(q, t)]\dot{q} \\
&\quad + \sum_{i=1}^{l} m_i[\alpha_i(q, t)\partial_q\bar{x}_i(q, t) + \beta_i(q, t)\partial_q\bar{y}_i(q, t) + \gamma_i(q, t)\partial_q\bar{z}_i(q, t)]\dot{q}
\end{aligned}$$

$$+\tfrac{1}{2}\sum_{i=1}^{l} m_i[\alpha_i^2(q,t)+\beta_i^2(q,t)+\gamma_i^2(q,t)]. \tag{9.3.10}$$

Hence

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) = \tfrac{1}{2}\dot q^{\mathrm{T}} M(q,t)\dot q + F(q,t)\dot q + G(q,t), \tag{9.3.11}$$

where the *mass matrix* $M(q,t) \in \mathbb{R}^{r\times r}$ is defined by

$$M(q,t) \triangleq \sum_{i=1}^{l} m_i[\partial_q^{\mathrm{T}}\bar x_i(q,t)\partial_q\bar x_i(q,t) + \partial_q^{\mathrm{T}}\bar y_i(q,t)\partial_q\bar y_i(q,t) + \partial_q^{\mathrm{T}}\bar z_i(q,t)\partial_q\bar z_i(q,t)] \tag{9.3.12}$$

and $F(q,t) \in \mathbb{R}^{1\times l}$ and $G(q,t) \in \mathbb{R}$ are defined by

$$F(q,t) \triangleq \sum_{i=1}^{l} m_i[\alpha_i(q,t)\partial_q\bar x_i(q,t) + \beta_i(q,t)\partial_q\bar y_i(q,t) + \gamma_i(q,t)\partial_q\bar z_i(q,t)] \tag{9.3.13}$$

and

$$G(q,t) \triangleq \tfrac{1}{2}\sum_{i=1}^{l} m_i[\alpha_i^2(q,t)+\beta_i^2(q,t)+\gamma_i^2(q,t)]. \tag{9.3.14}$$

Note that the (j,k) entry of $M(q,t)$ is given by

$$M_{jk}(q,t) = \sum_{i=1}^{l} m_i[\partial_{q_j}\bar x_i(q,t)\partial_{q_k}\bar x_i(q,t) + \partial_{q_j}\bar y_i(q,t)\partial_{q_k}\bar y_i(q,t) + \partial_{q_j}\bar z_i(q,t)\partial_{q_k}\bar z_i(q,t)]. \tag{9.3.15}$$

Finally, if the dynamics of the body are time invariant, then (9.3.11) becomes

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) = \tfrac{1}{2}\dot q^{\mathrm{T}} M(q,t)\dot q. \tag{9.3.16}$$

Next, suppose that $\mathcal{B}$ is a rigid body with mass m and let $\mathrm{F_B}$ be a body-fixed frame. Then, it follows from (8.1.19) that

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q) = \tfrac{1}{2}m\vec v'_{\mathrm{c}/w/\mathrm{A}}\vec v_{\mathrm{c}/w/\mathrm{A}} + \tfrac{1}{2}\vec\omega'_{\mathrm{B/A}}\vec J_{\mathcal{B}/\mathrm{c}}\vec\omega_{\mathrm{B/A}}. \tag{9.3.17}$$

In order to express $\vec\omega_{\mathrm{B/A}}$ as the derivative of components of q, we define $q \in \mathbb{R}^6$ by

$$q = \begin{bmatrix} r_{\mathrm{c}} \\ \Phi \\ \Theta \\ \Psi \end{bmatrix}, \tag{9.3.18}$$

where

$$r_{\mathrm{c}} \triangleq \vec r_{\mathrm{c}/w}\Big|_{\mathrm{A}} \tag{9.3.19}$$

and Ψ, Θ, Φ are (3,2,1) Euler angles that define the orientation of $\mathrm{F_B}$ relative to $\mathrm{F_A}$. It thus follows from (2.12.32) that $\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi)$. Consequently, $T_{\mathcal{B}/w/\mathrm{A}}$ can be written as

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q) &= \tfrac{1}{2}m\ \vec v_{\mathrm{c}/w/\mathrm{A}}\Big|_{\mathrm{A}}^{\mathrm{T}}\ \vec v_{\mathrm{c}/w/\mathrm{A}}\Big|_{\mathrm{A}} + \tfrac{1}{2}\ \vec\omega_{\mathrm{B/A}}\Big|_{\mathrm{B}}^{\mathrm{T}}\ \vec J_{\mathcal{B}/\mathrm{c}}\Big|_{\mathrm{B}}\ \vec\omega_{\mathrm{B/A}}\Big|_{\mathrm{B}} \\ &= \tfrac{1}{2}m\|v_{\mathrm{c}}\|^2 + \tfrac{1}{2}\begin{bmatrix}\dot\Phi\\ \dot\Theta\\ \dot\Psi\end{bmatrix}^{\mathrm{T}} S^{\mathrm{T}}(\Phi,\Theta)\ \vec J_{\mathcal{B}/\mathrm{c}}\Big|_{\mathrm{B}}\ S(\Phi,\Theta)\begin{bmatrix}\dot\Phi\\ \dot\Theta\\ \dot\Psi\end{bmatrix}, \end{aligned} \tag{9.3.20}$$

where

$$v_{\rm c} \stackrel{\triangle}{=} \dot{r}_{\rm c} = \left.\overset{\rm A\bullet}{\vec{r}}_{{\rm c}/w}\right|_{\rm A} = \left.\vec{v}_{{\rm c}/w/{\rm A}}\right|_{\rm A}, \tag{9.3.21}$$

and, from (4.10.7),

$$\left.\vec{\omega}_{\rm B/A}\right|_{\rm B} = S(\Phi,\Theta)\begin{bmatrix}\dot{\Phi}\\ \dot{\Theta}\\ \dot{\Psi}\end{bmatrix}, \tag{9.3.22}$$

where

$$S(\Phi,\Theta) \stackrel{\triangle}{=} \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\cos\Theta)\sin\Phi \\ 0 & -\sin\Phi & (\cos\Theta)\cos\Phi \end{bmatrix}. \tag{9.3.23}$$

Hence,

$$T_{\mathcal{B}/w/{\rm A}}(q,\dot{q}) = \tfrac{1}{2}\dot{q}^{\rm T}M(q)\dot{q}, \tag{9.3.24}$$

where $M(q) \in \mathbb{R}^{6\times 6}$ is defined by

$$M(q) \stackrel{\triangle}{=} \begin{bmatrix} mI_3 & 0 \\ 0 & S^{\rm T}(\Phi,\Theta)\left.\vec{J}_{\mathcal{B}/{\rm c}}\right|_{\rm B} S(\Phi,\Theta) \end{bmatrix}. \tag{9.3.25}$$

As shown by Problem 4.21.2, however, (9.3.22) is not able to represent all possible angular velocities due to gimbal lock. To address this issue, (4.9.11) can be used to express $\vec{\omega}_{\rm B/A}$ in terms of the eigenaxis angle vector and its derivative. In this case, define $q \in \mathbb{R}^6$ by

$$q = \begin{bmatrix} r_{\rm c} \\ \Theta_{\rm B/A} \end{bmatrix}, \tag{9.3.26}$$

where

$$r_{\rm c} \stackrel{\triangle}{=} \left.\vec{r}_{{\rm c}/w}\right|_{\rm A} \tag{9.3.27}$$

and, using (2.14.8)

$$\Theta_{\rm B/A} \stackrel{\triangle}{=} \left.\vec{\Theta}_{\rm B/A}\right|_{\rm B} = \left.\vec{\Theta}_{\rm B/A}\right|_{\rm A}. \tag{9.3.28}$$

Consequently, $T_{\mathcal{B}/w/{\rm A}}$ can be written as

$$\begin{aligned} T_{\mathcal{B}/w/{\rm A}}(q,\dot{q}) &= \tfrac{1}{2}m \left.\vec{v}_{{\rm c}/w/{\rm A}}\right|_{\rm A}^{\rm T} \left.\vec{v}_{{\rm c}/w/{\rm A}}\right|_{\rm A} + \tfrac{1}{2}\left.\vec{\omega}_{\rm B/A}\right|_{\rm B}^{\rm T} \left.\vec{J}_{\mathcal{B}/{\rm c}}\right|_{\rm B} \left.\vec{\omega}_{\rm B/A}\right|_{\rm B} \\ &= \tfrac{1}{2}m\|v_{\rm c}\|^2 + \tfrac{1}{2}\dot{\Theta}_{\rm B/A}^{\rm T}K^{\rm T}(\Theta_{\rm B/A})\left.\vec{J}_{\mathcal{B}/{\rm c}}\right|_{\rm B} K(\Theta_{\rm B/A})\dot{\Theta}_{\rm B/A}, \end{aligned} \tag{9.3.29}$$

where

$$v_{\rm c} \stackrel{\triangle}{=} \dot{r}_{\rm c} = \left.\vec{v}_{{\rm c}/w/{\rm A}}\right|_{\rm A} \tag{9.3.30}$$

and, from (4.9.11),

$$\left.\vec{\omega}_{\rm B/A}\right|_{\rm B} = G(\Theta_{\rm B/A})\dot{\Theta}_{\rm B/A} \tag{9.3.31}$$

where

$$K(\Theta_{\mathrm{B/A}}) \triangleq I_3 - \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^2}\Theta_{\mathrm{B/A}}^{\times} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^3}\Theta_{\mathrm{B/A}}^{\times 2} \tag{9.3.32}$$

and $\theta_{\mathrm{B/A}} \triangleq \|\Theta_{\mathrm{B/A}}\|$. Hence,

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) = \tfrac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q}, \tag{9.3.33}$$

where $M(q) \in \mathbb{R}^{6\times 6}$ is defined by

$$M(q) \triangleq \begin{bmatrix} mI_3 & 0 \\ 0 & G^{\mathrm{T}}(\Theta_{\mathrm{B/A}}) \ \vec{J}_{\mathcal{B}/\mathrm{c}}\Big|_{\mathrm{B}} \ G(\Theta_{\mathrm{B/A}}) \end{bmatrix}. \tag{9.3.34}$$

Alternatively, the angular velocity vector can be expressed in terms of Euler angles and their derivatives as well as Euler parameters and their derivatives.

The following result shows that the kinetic energy is a positive-definite quadratic form in the generalized velocities.

Fact 9.3.1. For all generalized coordinates q, the mass matrix $M(q)$ is positive definite.

Proof. Note that $M(q)$ is symmetric and positive semidefinite. To show that $M(q)$ is positive definite, suppose that $\dot{q}$ is nonzero. Then, there exists an inertia point y_i in $\mathcal{B}$ such that $\vec{v}_{y_i/w/\mathrm{A}}$ is nonzero. Consequently, $T_{\mathcal{B}/w/\mathrm{A}}$ is positive. Hence $M(q)$ is positive definite. □

Note that Fact 9.3.1 does not apply to (9.3.25) since $S(\Phi,\Theta)$ is singular when gimbal lock occurs.

9.4 Generalized Forces and Moments for Bodies with Forces

Each particle in a body $\mathcal{B}$ moves under the influence of *external forces*, which are external to the body in origin, as well as *internal forces*, which result from interactions between the particles and rigid bodies in $\mathcal{B}$. Internal forces can include reaction forces due to contact between particles and rigid bodies, as well as forces due to interconnections comprised of springs and dashpots. We assume in this section that all moments on $\mathcal{B}$ are expressed in terms of forces.

Assume that the body $\mathcal{B}$ has inertia points $y_1,\ldots,y_l$, let $q_1,\ldots,q_r$ be independent generalized coordinates for $\mathcal{B}$, let w be an unforced particle in (9.2.1) and (9.3.1), and let $\mathrm{F_A}$ be an inertial frame, as necessitated by the Lagrangian dynamics later in this chapter. Furthermore, for $i = 1,\ldots,l$, let $\vec{f}_i$ be the total internal and external force acting on y_i excluding conservative contact forces, and let

$$\vec{f}_i(t)\Big|_{\mathrm{A}} = \begin{bmatrix} f_{x_i}(t) \\ f_{y_i}(t) \\ f_{z_i}(t) \end{bmatrix}. \tag{9.4.1}$$

Then, for $j = 1,\ldots,r$, the *generalized force or moment* Q_j due to $\vec{f}_1,\ldots,\vec{f}_l$ is defined by

$$Q_j(q,\dot{q},t) \triangleq \sum_{i=1}^{l}[f_{\bar{x}_i}(t)\partial_{q_j}\bar{x}_i(q) + f_{\bar{y}_i}(t)\partial_{q_j}\bar{y}_i(q) + f_{\bar{z}_i}(t)\partial_{q_j}\bar{z}_i(q)], \tag{9.4.2}$$

which we can be rewritten as

$$Q_j(q,\dot q,t) = \sum_{i=1}^{l} \vec{f}_i\Big|_{\mathrm{A}}^{\mathrm{T}} \partial_{q_j}\left(\vec{r}_{y_i/w}\Big|_{\mathrm{A}}\right). \tag{9.4.3}$$

Equivalently, (9.4.2) can be written in terms of velocities as

$$Q_j(q,\dot q,t) = \sum_{i=1}^{l} \vec{f}_i\Big|_{\mathrm{A}}^{\mathrm{T}} \partial_{\dot q_j}\left(\vec{v}_{y_i/w/\mathrm{A}}\Big|_{\mathrm{A}}\right). \tag{9.4.4}$$

Note that, if q_j is a position, then Q_j is a force, whereas, if q_j is an angle, then Q_j is a moment. Furthermore, note that the ith term in the summation in (9.4.3) is a measure of the effect of the force $\vec{f}_i$ on the sensitivity of the position of the inertia point y_i to changes in the generalized coordinate q_j.

9.5 Generalized Forces and Moments for Bodies with Moments

If the body $\mathcal{B}$ consists of at least one rigid body, then moments may be specified rather than forces. In this case, each moment can be replaced by a pair of balanced forces, and (9.4.3) can be used to determine the resulting generalized forces and moments.

Alternatively, (9.4.3) can be replaced by an expression involving moments. In particular, suppose that $\mathcal{B}$ is a rigid body subject to exactly two nonzero forces that are not due to conservative contact. For convenience, assume that these forces are $\vec{f}$ and $-\vec{f}$ applied to inertia points y_1 and y_2, respectively. Then, the generalized force Q_j is given by

$$\begin{aligned}
Q_j(q,\dot q,t) &= \vec{f}\Big|_{\mathrm{A}}^{\mathrm{T}} \partial_{\dot q_j}\left(\vec{v}_{y_1/w/\mathrm{A}}\Big|_{\mathrm{A}}\right) - \vec{f}\Big|_{\mathrm{A}}^{\mathrm{T}} \partial_{\dot q_j}\left(\vec{v}_{y_2/w/\mathrm{A}}\Big|_{\mathrm{A}}\right)\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathrm{T}} \left[\partial_{\dot q_j}\left(\vec{v}_{y_1/w/\mathrm{A}}\Big|_{\mathrm{A}}\right) - \partial_{\dot q_j}\left(\vec{v}_{y_2/w/\mathrm{A}}\Big|_{\mathrm{A}}\right)\right]\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathrm{T}} \left[\partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} \times \vec{r}_{y_1/w}\Big|_{\mathrm{A}}\right) - \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} \times \vec{r}_{y_2/w}\Big|_{\mathrm{A}}\right)\right]\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathrm{T}} \left[\partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right) \times \left(\vec{r}_{y_1/w}\Big|_{\mathrm{A}} - \vec{r}_{y_2/w}\Big|_{\mathrm{A}}\right)\right]\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathrm{T}} \left[\partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right) \times \vec{r}_{y_1/y_2}\Big|_{\mathrm{A}}\right]\\
&= -\vec{f}\Big|_{\mathrm{A}}^{\mathrm{T}} \left[\vec{r}_{y_1/y_2}\Big|_{\mathrm{A}} \times \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right)\right]\\
&= -\left(\vec{f}\Big|_{\mathrm{A}} \times \vec{r}_{y_1/y_2}\Big|_{\mathrm{A}}\right)^{\mathrm{T}} \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right)\\
&= \left(\vec{r}_{y_1/y_2}\Big|_{\mathrm{A}} \times \vec{f}\Big|_{\mathrm{A}}\right)^{\mathrm{T}} \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right)\\
&= \vec{M}_k\Big|_{\mathrm{A}}^{\mathrm{T}} \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right).
\end{aligned} \tag{9.5.1}$$

More generally, suppose that the body $\mathcal{B}$ consists of m rigid bodies $\mathcal{B}_1,\ldots,\mathcal{B}_m$, and, for $k = 1,\ldots,m$, let $\mathrm{F}_{\mathrm{B}_k}$ be a body-fixed frame for $\mathcal{B}_k$ and let $\vec{M}_k$ denote the total internal and external torque applied to $\mathcal{B}_k$ excluding conservative contact torques. Then, for $j = 1,\ldots,r$, the generalized force or moment Q_j due to $\vec{M}_1, \ldots, \vec{M}_m$ is given in analogy with (9.4.4) by

$$Q_j(q, \dot{q}, t) = \sum_{k=1}^{m} \vec{M}_k\Big|_{\mathrm{A}}^{\mathrm{T}} \partial_{\dot{q}_j}\left(\vec{\omega}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}}\right). \tag{9.5.2}$$

As in the case of forces, if q_j is a position, then Q_j is a force, whereas, if q_j is an angle, then Q_j is a moment.

In order to apply (9.5.2), each angular velocity vector $\vec{\omega}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}}$ must be expressed in terms of the derivatives of the generalized coordinates. To demonstrate this approach in terms of the Cartesian coordinates of the centers of mass and eigenaxis angle vectors of the rigid bodies, we define $q \in \mathbb{R}^{9m}$ by

$$q = \begin{bmatrix} r_{\mathrm{c},1} \\ \vdots \\ r_{\mathrm{c},m} \\ \Theta_{\mathrm{B}_1/\mathrm{A}} \\ \vdots \\ \Theta_{\mathrm{B}_m/\mathrm{A}} \\ \dot{\Theta}_{\mathrm{B}_1/\mathrm{A}} \\ \vdots \\ \dot{\Theta}_{\mathrm{B}_m/\mathrm{A}} \end{bmatrix}, \tag{9.5.3}$$

where, using (2.14.8),

$$\Theta_{\mathrm{B}_k/\mathrm{A}} \triangleq \vec{\Theta}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}} = \vec{\Theta}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{B}}. \tag{9.5.4}$$

Next, it follows from (4.9.10) that

$$\vec{\omega}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}} = H(\Theta_{\mathrm{B}_k/\mathrm{A}})\dot{\Theta}_{\mathrm{B}_m/\mathrm{A}}, \tag{9.5.5}$$

where

$$H(\Theta_{\mathrm{B}_k/\mathrm{A}}) \triangleq I_3 + \frac{1-\cos\theta_{\mathrm{B}_k/\mathrm{A}}}{\theta^2_{\mathrm{B}_k/\mathrm{A}}}\Theta^{\times}_{\mathrm{B}_k/\mathrm{A}} + \frac{\theta_{\mathrm{B}_k/\mathrm{A}} - \sin\theta_{\mathrm{B}_k/\mathrm{A}}}{\theta^3_{\mathrm{B}_k/\mathrm{A}}}\Theta^{\times 2}_{\mathrm{B}_k/\mathrm{A}} \tag{9.5.6}$$

and $\theta_{\mathrm{B}_k/\mathrm{A}} \triangleq \|\Theta_{\mathrm{B}_k/\mathrm{A}}\|$. Now, using the fact that $\partial_{\dot{q}_j}\dot{\Theta}_{\mathrm{B}_k/\mathrm{A}} = \partial_{q_j}\Theta_{\mathrm{B}_k/\mathrm{A}}$, it follows from (9.5.2) and (9.5.5) that

$$Q_j(q, \dot{q}, t) = \sum_{k=1}^{m} \vec{M}_k\Big|_{\mathrm{A}}^{\mathrm{T}} H(\Theta_{\mathrm{B}_k/\mathrm{A}})\partial_{q_j}(\Theta_{\mathrm{B}_k/\mathrm{A}}), \tag{9.5.7}$$

If both forces and moments are present, then (9.4.3) and (9.5.7) can be combined to obtain

$$Q_j(q, \dot{q}, t) = \sum_{i=1}^{l} \vec{f}_i\Big|_{\mathrm{A}}^{\mathrm{T}} \partial_{q_j}\left(\vec{r}_{y_i/w}\Big|_{\mathrm{A}}\right) + \sum_{k=1}^{m} \vec{M}_k\Big|_{\mathrm{A}}^{\mathrm{T}} H(\Theta_{\mathrm{B}_k/\mathrm{A}})\partial_{q_j}(\Theta_{\mathrm{B}_k/\mathrm{A}}). \tag{9.5.8}$$

9.6 Lagrange's Equations: Kinetic Energy Form

The following result provides equations of motion for a body in terms of generalized coordinates. Generalized forces and moments that are due to conservative internal contact forces do not need to be considered, and thus these forces and moments are excluded.

Fact 9.6.1. Let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^{\mathrm{T}}$, let w be an unforced particle, and let Q denote all generalized forces and moments except those that arise from conservative contact, and let $\mathrm{F_A}$ be an inertial frame. Then, $q(t)$ satisfies

$$\mathrm{d}_t \partial_{\dot{q}}^{\mathrm{T}} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t) - \partial_q^{\mathrm{T}} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t) = Q(q(t), \dot{q}(t), t). \tag{9.6.1}$$

Proof. Prove this result. □

Equation (9.6.1) can be rewritten as

$$\mathrm{d}_t[M(q(t), t)\dot{q}(t) + F(q(t), t)] - \partial_q^{\mathrm{T}}[\dot{q}^{\mathrm{T}}(t)M(q(t), t)\dot{q}(t) + F(q(t), t)\dot{q}(t) + G(q(t), t)] = Q(q(t), \dot{q}(t), t). \tag{9.6.2}$$

In terms of components $j = 1, \ldots, r$, (9.6.1) can be rewritten as

$$\mathrm{d}_t \partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t) - \partial_{q_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t) = Q_j(q(t), \dot{q}(t), t). \tag{9.6.3}$$

Furthermore,

$$\partial_q^{\mathrm{T}} T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}, t) = \begin{bmatrix} \frac{1}{2}\dot{q}^{\mathrm{T}}\partial_{q_1}M(q, t)\dot{q} \\ \vdots \\ \frac{1}{2}\dot{q}^{\mathrm{T}}\partial_{q_r}M(q, t)\dot{q} \end{bmatrix} + \begin{bmatrix} \partial_{q_1}F(q, t)\dot{q} \\ \vdots \\ \partial_{q_r}F(q, t)\dot{q} \end{bmatrix} + \partial_q^{\mathrm{T}} G(q, t). \tag{9.6.4}$$

Equations (9.6.1) are *Lagrange's equations*. These equations consist of r second-order ordinary differential equations. Each solution, therefore, requires $2r$ initial values of the variables $q_1, \ldots, q_r, \dot{q}_1, \ldots, \dot{q}_r$. Thus the motion of a body depends on its initial configuration as well as the initial velocities of all of its constituent particles. The generalized coordinates and the corresponding generalized velocities at each instant together uniquely determine the subsequent motion of the body.

For the following identity, we suppress the arguments of $T_{\mathcal{B}/w/\mathrm{A}}$.

Fact 9.6.2. Let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^{\mathrm{T}}$, let w be a point, and let $\mathrm{F_A}$ be a frame. For all $j = 1, \ldots, r$, the kinetic energy $T_{\mathcal{B}/w/\mathrm{A}}$ satisfies

$$\mathrm{d}_t \partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}} = \partial_{\dot{q}_j}^2 T_{\mathcal{B}/w/\mathrm{A}} \ddot{q}_j + \mathrm{d}_t[\partial_{\dot{q}_j}^2 T_{\mathcal{B}/w/\mathrm{A}}]\dot{q}_j + \mathrm{d}_t[\,\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}})\big|_{\dot{q}_j=0}]. \tag{9.6.5}$$

Proof. For each $j \in \{1, \ldots, r\}$, we can write

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}, t) &= \tfrac{1}{2}\dot{q}^{\mathrm{T}} M(q, t)\dot{q} + F(q, t)\dot{q} + G(q, t) \\ &= \tfrac{1}{2}M_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)\dot{q}_j{}^2 + L_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)\dot{q}_j + K_j(\bar{q}_j, \dot{\bar{q}}_j, t), \end{aligned}$$

where $\bar{q}_j$ denotes q with q_j omitted, and M_j, L_j, and K_j are appropriate functions of the indicated arguments. We thus have

$$\begin{aligned} \mathrm{d}_t \partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}, t) &= \tfrac{1}{2}\mathrm{d}_t \partial_{\dot{q}_j}[M_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)\dot{q}_j^2] + \mathrm{d}_t \partial_{\dot{q}_j}[L_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)\dot{q}_j] + \mathrm{d}_t \partial_{\dot{q}_j}[K_j(\bar{q}_j, \dot{\bar{q}}_j, t)] \\ &= \mathrm{d}_t[M_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)\dot{q}_j] + \mathrm{d}_t L_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t) \\ &= M_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)\ddot{q}_j + \mathrm{d}_t[M_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)]\dot{q}_j + \mathrm{d}_t L_j(q_j, \bar{q}_j, \dot{\bar{q}}_j, t) \\ &= \partial_{\dot{q}_j}^2 T_{\mathcal{B}/w/\mathrm{A}}(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)\ddot{q}_j + \mathrm{d}_t[\partial_{\dot{q}_j}^2 T_{\mathcal{B}/w/\mathrm{A}}(q_j, \bar{q}_j, \dot{\bar{q}}_j, t)]\dot{q}_j \end{aligned}$$

$$+\, \mathrm{d}_t[\,\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q_j, \bar{q}_j, \dot{\bar{q}}_j, t))\big|_{\dot{q}_j=0}]. \qquad \square$$

Using Fact 9.6.2 we can rewrite Fact 9.6.1 as follows.

Fact 9.6.3. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^{\mathrm{T}}$, let w be an unforced particle, and let Q denote all generalized forces and moments except those that arise from conservative contact. Then, for all $j = 1, \ldots, r$, $q(t)$ satisfies

$$\partial^2_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}} \ddot{q}_j + \mathrm{d}_t[\partial^2_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}]\dot{q}_j + \mathrm{d}_t[\,\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}})\big|_{\dot{q}_j=0}] - \partial_{q_j} T_{\mathcal{B}/w/\mathrm{A}} = Q_j. \tag{9.6.6}$$

The following observation provides a constant of the motion in special cases.

Fact 9.6.4. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^{\mathrm{T}}$, let w be an unforced particle, and let Q denote all generalized forces and moments except those that arise from conservative contact. Furthermore, let $j \in \{1, \ldots, r\}$ and assume that $Q_j = 0$ and $T_{\mathcal{B}/w/\mathrm{A}}$ does not depend on q_j. Then,

$$\mathrm{d}_t \partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t) = 0. \tag{9.6.7}$$

That is, $\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t)$ is a constant of the motion.

If q_j is a position, then the constant $\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t)$ is a constant of momentum, whereas, if q_j is an angle, then the constant $\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t)$ is a constant of angular momentum

9.7 Derivation of Euler's Equation from Lagrange's Equations

For the case of a single rigid body, we now use Lagrange's equations to derive Euler's equation given by Fact 7.8.5. We do this for the case where the angular velocity is represented in terms of 3-2-1 Euler angles. To begin, we rewrite (4.10.6) as

$$\vec{\omega}_{\mathrm{D/A}} = f(\Theta, \dot{\Phi}, \dot{\Psi})\hat{\imath}_{\mathrm{D}} + g(\Phi, \Theta, \dot{\Theta}, \dot{\Psi})\hat{\jmath}_{\mathrm{D}} + h(\Phi, \Theta, \dot{\Theta}, \dot{\Psi})\hat{k}_{\mathrm{D}}, \tag{9.7.1}$$

where

$$f(\Theta, \dot{\Phi}, \dot{\Psi}) \triangleq -\dot{\Psi}(\sin\Theta) + \dot{\Phi}, \tag{9.7.2}$$

$$g(\Phi, \Theta, \dot{\Theta}, \dot{\Psi}) \triangleq \dot{\Psi}(\sin\Phi)(\cos\Theta) + \dot{\Theta}\cos\Phi, \tag{9.7.3}$$

$$h(\Phi, \Theta, \dot{\Theta}, \dot{\Psi}) \triangleq \dot{\Psi}(\cos\Phi)(\cos\Theta) - \dot{\Theta}(\sin\Phi). \tag{9.7.4}$$

Alternatively, we can rewrite $\vec{\omega}_{\mathrm{D/A}}$ as

$$\vec{\omega}_{\mathrm{D/A}} = \dot{\Phi}\vec{x} + \dot{\Theta}\vec{y}(\Phi) + \dot{\Psi}\vec{z}(\Phi, \Theta), \tag{9.7.5}$$

where

$$\vec{x} \triangleq \hat{\imath}_{\mathrm{D}}, \tag{9.7.6}$$

$$\vec{y}(\Phi) \triangleq (\cos\Phi)\hat{\jmath}_{\mathrm{D}} - (\sin\Phi)\hat{k}_{\mathrm{D}}, \tag{9.7.7}$$

$$\vec{z}(\Phi, \Theta) \triangleq -(\sin\Theta)\hat{\imath}_{\mathrm{D}} + (\sin\Phi)(\cos\Theta)\hat{\jmath}_{\mathrm{D}} + (\sin\Phi)(\cos\Theta)\hat{k}_{\mathrm{D}}. \tag{9.7.8}$$

By writing (9.7.5) as

$$\vec{\omega}_{\mathrm{D/A}} = \begin{bmatrix} \vec{x} & \vec{y}(\Phi) & \vec{z}(\Phi, \Theta) \end{bmatrix} \dot{\theta}, \tag{9.7.9}$$

where

$$\theta \triangleq \begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix}, \tag{9.7.10}$$

it follows that

$$\vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{D}} = S(\Phi, \Theta)\dot{\theta}, \tag{9.7.11}$$

where

$$\begin{aligned} S(\Phi,\Theta) &\triangleq \begin{bmatrix} \vec{x}\Big|_{\mathrm{D}} & \vec{y}(\Phi)\Big|_{\mathrm{D}} & \vec{z}(\Phi,\Theta)\Big|_{\mathrm{D}} \end{bmatrix} \\ &= \begin{bmatrix} S_1 & S_2(\Phi) & S_3(\Phi,\Theta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}. \end{aligned} \tag{9.7.12}$$

Next, we have

$$\begin{aligned} \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}} &= \dot{f}(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \dot{g}(\Phi,\Theta,\dot{\Theta},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \dot{h}(\Phi,\Theta,\dot{\Theta},\dot{\Psi})\hat{k}_{\mathrm{D}} \\ &= \nabla f\dot{\theta}\hat{\imath}_{\mathrm{D}} + \nabla g\dot{\theta}\hat{\jmath}_{\mathrm{D}} + \nabla h\dot{\theta}k_{\mathrm{D}} \\ &= \dot{\Phi}[\partial_\Phi f(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \partial_\Phi g(\Theta,\dot{\Phi},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \partial_\Phi h(\Theta,\dot{\Phi},\dot{\Psi})\hat{k}_{\mathrm{D}}] \\ &\quad + \dot{\Theta}[\partial_\Theta f(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \partial_\Theta g(\Theta,\dot{\Phi},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \partial_\Theta h(\Theta,\dot{\Phi},\dot{\Psi})\hat{k}_{\mathrm{D}}] \\ &\quad + \dot{\Psi}[\partial_\Psi f(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \partial_\Psi g(\Theta,\dot{\Phi},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \partial_\Psi h(\Theta,\dot{\Phi},\dot{\Psi})\hat{k}_{\mathrm{D}}] \\ &= \dot{\Phi}\partial_\Phi\vec{\omega}_{\mathrm{D/A}} + \dot{\Theta}\partial_\Theta\vec{\omega}_{\mathrm{D/A}} + \dot{\Psi}\partial_\Psi\vec{\omega}_{\mathrm{D/A}}, \end{aligned} \tag{9.7.13}$$

where

$$\partial_\Phi\vec{\omega}_{\mathrm{D/A}} = [\dot{\Psi}(\cos\Phi)(\cos\Theta) - \dot{\Theta}\sin\Phi]\hat{\jmath}_{\mathrm{D}} + [-\dot{\Psi}(\sin\Phi)\cos\Theta - \dot{\Theta}(\cos\Phi)]\hat{k}_{\mathrm{D}}, \tag{9.7.14}$$

$$\partial_\Theta\vec{\omega}_{\mathrm{D/A}} = -\dot{\Psi}(\cos\Theta)\hat{\imath}_{\mathrm{D}} - \dot{\Psi}(\sin\Phi)(\sin\Theta)\hat{\jmath}_{\mathrm{D}} - \dot{\Psi}(\cos\Phi)(\sin\Theta)\hat{k}_{\mathrm{D}}, \tag{9.7.15}$$

$$\partial_\Psi\vec{\omega}_{\mathrm{D/A}} = 0. \tag{9.7.16}$$

Furthermore, it follows from (9.7.5) that

$$\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{D/A}} = \dot{\Phi}\overset{\mathrm{A}\bullet}{\vec{x}} + \dot{\Theta}\overset{\mathrm{A}\bullet}{\vec{y}}(\Phi) + \dot{\Psi}\overset{\mathrm{A}\bullet}{\vec{z}}(\Phi,\Theta). \tag{9.7.17}$$

Since $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{D/A}} = \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}}$, it follows from (9.7.13) and (9.7.17) that

$$\partial_\Phi\vec{\omega}_{\mathrm{D/A}} = \overset{\mathrm{A}\bullet}{\vec{x}} = \overset{\mathrm{D}\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{D/A}} \times \vec{x}, \tag{9.7.18}$$

$$\partial_\Theta\vec{\omega}_{\mathrm{D/A}} = \overset{\mathrm{A}\bullet}{\vec{y}}(\Phi) = \overset{\mathrm{D}\bullet}{\vec{y}}(\Phi) + \vec{\omega}_{\mathrm{D/A}} \times \vec{y}(\Phi), \tag{9.7.19}$$

$$\partial_\Psi\vec{\omega}_{\mathrm{D/A}} = \overset{\mathrm{A}\bullet}{\vec{z}}(\Phi,\Theta) = \overset{\mathrm{D}\bullet}{\vec{z}}(\Phi,\Theta) + \vec{\omega}_{\mathrm{D/A}} \times \vec{z}(\Phi,\Theta). \tag{9.7.20}$$

Resolving (9.7.18)–(9.7.20) in $\mathrm{F_D}$ yields

$$\partial_\Phi S(\Phi,\Theta)\dot\theta = \dot S_1 + [S(\Phi,\Theta)\dot\theta]\times S_1, \tag{9.7.21}$$

$$\partial_\Theta S(\Phi,\Theta)\dot\theta = \dot S_2(\Phi) + [S(\Phi,\Theta)\dot\theta]\times S_2(\Phi), \tag{9.7.22}$$

$$\partial_\Psi S(\Phi,\Theta)\dot\theta = \dot S_3(\Phi,\Theta) + [S(\Phi,\Theta)\dot\theta]\times S_3(\Phi,\Theta), \tag{9.7.23}$$

where

$$\partial_\Phi S(\Phi,\Theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \\ 0 & -\cos\Phi & -(\sin\Phi)\cos\Theta \end{bmatrix}, \tag{9.7.24}$$

$$\partial_\Theta S(\Phi,\Theta) = \begin{bmatrix} 0 & 0 & -\cos\Phi \\ 0 & 0 & -(\sin\Phi)\sin\Theta \\ 0 & 0 & -(\cos\Phi)\sin\Theta \end{bmatrix}, \tag{9.7.25}$$

$$\partial_\Psi S(\Phi,\Theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.7.26}$$

Note that (9.7.21)–(9.7.23) can be written as

$$\begin{bmatrix} \partial_\Phi S(\Phi,\Theta)\dot\theta & \partial_\Theta S(\Phi,\Theta)\dot\theta & \partial_\Psi S(\Phi,\Theta)\dot\theta \end{bmatrix} = \dot S(\Phi,\Theta) + [S(\Phi,\Theta)\dot\theta]^\times S(\Phi,\Theta). \tag{9.7.27}$$

Now, let $\mathcal{B}$ be a rigid body, let $\vec M$ denote the moment applied to $\mathcal{B}$, let $\mathrm{F_A}$ be an inertial frame, let w be an unforced particle, and let $\mathrm{F_D}$ be a body-fixed frame. Furthermore, let Ψ, Θ, and Φ denote 3-2-1 Euler angles so that, by (2.12.28), $\vec R_{\mathrm{B/A}} = \vec R_{\hat\imath_{\mathrm C}}(\Phi)\vec R_{\hat\jmath_{\mathrm B}}(\Theta)\vec R_{\hat k_{\mathrm A}}(\Psi)$, where $\mathrm{F_B} \triangleq \vec R_{\hat k_{\mathrm A}}(\Psi)\mathrm{F_A}$, $\mathrm{F_C} \triangleq \vec R_{\hat\jmath_{\mathrm B}}(\Theta)\mathrm{F_B}$ and $\mathrm{F_D} \triangleq \vec R_{\hat\imath_{\mathrm C}}(\Phi)\mathrm{F_C}$. Next, define the generalized coordinates $q_1 \triangleq \Phi$, $q_2 \triangleq \Theta$, and $q_3 \triangleq \Psi$ so that $q = \theta$. It thus follows from (9.5.2) that the generalized moment $Q_1(q,\dot q)$ is given by

$$\begin{aligned} Q_1(q,\dot q) &= \vec M\Big|_{\mathrm A}^{\mathrm T} \partial_\Phi\left(\vec\omega_{\mathrm{D/A}}\Big|_{\mathrm A}\right) \\ &= \left(\mathcal{O}_{\mathrm{A/D}}\,\vec M\Big|_{\mathrm D}\right)^{\mathrm T} \partial_{\dot\Phi}\left(\mathcal{O}_{\mathrm{A/D}}\,\vec\omega_{\mathrm{D/A}}\Big|_{\mathrm D}\right) \\ &= \vec M\Big|_{\mathrm D}^{\mathrm T} \partial_{\dot\Phi}\left(\vec\omega_{\mathrm{D/A}}\Big|_{\mathrm D}\right) \\ &= M^{\mathrm T}\partial_{\dot\Phi}(\omega_{\mathrm{D/A}}) \\ &= M^{\mathrm T}\partial_{\dot\Phi}[S(\Phi,\Theta)\dot\theta] \\ &= M^{\mathrm T}\partial_{\dot\Phi}[\dot\Phi S_1 + \dot\Theta S_2(\Phi) + \dot\Psi S_3(\Phi,\Theta)] \\ &= M^{\mathrm T} S_1 \\ &= S_1^{\mathrm T} M. \end{aligned} \tag{9.7.28}$$

where $M \triangleq \vec M\Big|_{\mathrm D}$ and $\omega_{\mathrm{D/A}} \triangleq \vec\omega_{\mathrm{D/A}}\Big|_{\mathrm D}$. Therefore,

$$Q(q,\dot q) = \begin{bmatrix} Q_1(q,\dot q) \\ Q_2(q,\dot q) \\ Q_3(q,\dot q) \end{bmatrix} = \begin{bmatrix} S_1^{\mathrm T} M \\ S_2^{\mathrm T}(\Phi) M \\ S_3^{\mathrm T}(\Phi,\Theta) M \end{bmatrix} = \begin{bmatrix} S_1^{\mathrm T} \\ S_2^{\mathrm T}(\Phi) \\ S_3^{\mathrm T}(\Phi,\Theta) \end{bmatrix} M = S^{\mathrm T}(\Phi,\Theta) M. \tag{9.7.29}$$

Next, define $J \triangleq \vec{J}_{\mathcal{B}/c}\Big|_{\mathrm{D}}$. Then, it follows from (8.1.19) that the kinetic energy of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is given by

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}} &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{D/A}}\vec{J}_{\mathcal{B}/c}\vec{\omega}_{\mathrm{D/A}} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/\mathrm{A}}|^2 \\ &= \tfrac{1}{2}\dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)JS(\Phi,\Theta)\dot{\theta} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/\mathrm{A}}|^2. \end{aligned} \tag{9.7.30}$$

Therefore, using (9.7.11) it follows that

$$\begin{aligned} \mathrm{d}_t\partial_{\dot{q}}^{\mathrm{T}}T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) &= \mathrm{d}_t\partial_{\dot{q}}^{\mathrm{T}}[\tfrac{1}{2}\dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)JS(\Phi,\Theta)\dot{\theta}] \\ &= \mathrm{d}_t[S^{\mathrm{T}}(\Phi,\Theta)JS(\Phi,\Theta)\dot{\theta}] \\ &= \mathrm{d}_t[S^{\mathrm{T}}(\Phi,\Theta)J\omega_{\mathrm{D/A}}] \\ &= S^{\mathrm{T}}(\Phi,\Theta)J\dot{\omega}_{\mathrm{D/A}} + \dot{S}^{\mathrm{T}}(\Phi,\Theta)J\omega_{\mathrm{D/A}}. \end{aligned} \tag{9.7.31}$$

Furthermore, using (9.7.27) we obtain

$$\begin{aligned} \partial_q^{\mathrm{T}}T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) &= \partial_q^{\mathrm{T}}[\tfrac{1}{2}\dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)JS(\Phi,\Theta)\dot{\theta}] \\ &= \begin{bmatrix} \partial_\Phi[\tfrac{1}{2}\dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)JS(\Phi,\Theta)\dot{\theta}] \\ \partial_\Theta[\tfrac{1}{2}\dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)JS(\Phi,\Theta)\dot{\theta}] \\ \partial_\Psi[\tfrac{1}{2}\dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)JS(\Phi,\Theta)\dot{\theta}] \end{bmatrix} \\ &= \begin{bmatrix} \dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)J\partial_\Phi S(\Phi,\Theta)\dot{\theta} \\ \dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)J\partial_\Theta S(\Phi,\Theta)\dot{\theta} \\ \dot{\theta}^{\mathrm{T}}S^{\mathrm{T}}(\Phi,\Theta)J\partial_\Psi S(\Phi,\Theta)\dot{\theta} \end{bmatrix} \\ &= \left(\omega_{\mathrm{D/A}}^{\mathrm{T}}J\begin{bmatrix} \partial_\Phi S(\Phi,\Theta)\dot{\theta} & \partial_\Theta S(\Phi,\Theta)\dot{\theta} & \partial_\Psi S(\Phi,\Theta)\dot{\theta} \end{bmatrix}\right)^{\mathrm{T}} \\ &= \left(\omega_{\mathrm{D/A}}^{\mathrm{T}}J[\dot{S}(\Phi,\Theta) + \omega_{\mathrm{D/A}}^{\times}S(\Phi,\Theta)]\right)^{\mathrm{T}} \\ &= \dot{S}^{\mathrm{T}}(\Phi,\Theta)J\omega_{\mathrm{D/A}} - S^{\mathrm{T}}(\Phi,\Theta)(\omega_{\mathrm{D/A}}\times J\omega_{\mathrm{D/A}}). \end{aligned} \tag{9.7.32}$$

Consequently, using (9.7.29), (9.7.31), and (9.7.32), it follows from Fact 9.6.1 that

$$S^{\mathrm{T}}(\Phi,\Theta)J\dot{\omega}_{\mathrm{D/A}} + S^{\mathrm{T}}(\Phi,\Theta)(\omega_{\mathrm{D/A}}\times J\omega_{\mathrm{D/A}}) = S^{\mathrm{T}}(\Phi,\Theta)M. \tag{9.7.33}$$

Therefore, assuming that $\det S(\Phi,\Theta) = \cos\Theta \neq 0$ so that $S(\Phi,\Theta)$ is nonsingular, it follows that

$$J\dot{\omega}_{\mathrm{D/A}} + \omega_{\mathrm{D/A}}\times J\omega_{\mathrm{D/A}} = M, \tag{9.7.34}$$

which is Euler's equation given by Fact 7.8.5.

Alternatively, we can parameterize the angular velocity in terms of the Euler parameters

$$q_{\mathrm{D/A}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \eta_{\mathrm{D/A}} \\ \varepsilon_{\mathrm{D/A}} \end{bmatrix} \triangleq \begin{bmatrix} \cos\tfrac{1}{2}\theta_{\mathrm{D/A}} \\ (\sin\tfrac{1}{2}\theta_{\mathrm{D/A}})n_{\mathrm{D/A}} \end{bmatrix}. \tag{9.7.35}$$

It follows from (4.11.12) that

$$\begin{aligned} \omega_{\mathrm{D/A}} &= 2(\eta_{\mathrm{D/A}}\dot{\varepsilon}_{\mathrm{D/A}} - \dot{\eta}_{\mathrm{D/A}}\varepsilon_{\mathrm{D/A}} - \varepsilon_{\mathrm{D/A}}\times\dot{\varepsilon}_{\mathrm{D/A}}) \\ &= \mathcal{Q}(q_{\mathrm{D/A}})\dot{q}_{\mathrm{D/A}}, \end{aligned} \tag{9.7.36}$$

where

$$\mathcal{Q}(q_{\mathrm{D/A}}) \triangleq 2\begin{bmatrix} -q_2 & q_1 & q_4 & -q_3 \\ -q_3 & -q_4 & q_1 & q_2 \\ -q_4 & q_3 & -q_2 & q_1 \end{bmatrix}. \tag{9.7.37}$$

Since $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$, we can write

$$q_1 = \sqrt{1 - q_2^2 - q_3^2 - q_4^2} \tag{9.7.38}$$

and define the vector of generalized coordinates

$$q \triangleq \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}. \tag{9.7.39}$$

Therefore,

$$q_{\mathrm{D/A}} \triangleq \begin{bmatrix} \sqrt{1 - q_2^2 - q_3^2 - q_4^2} \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{9.7.40}$$

and, assuming that $q_1 \neq 0$,

$$\dot{q}_{\mathrm{D/A}} = \begin{bmatrix} \dfrac{-(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}. \tag{9.7.41}$$

Furthermore,

$$\begin{aligned} \omega_{\mathrm{D/A}} &= 2\begin{bmatrix} \dfrac{q_2(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} + q_1\dot{q}_2 + q_4\dot{q}_3 - q_3\dot{q}_4 \\ \dfrac{q_3(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} - q_4\dot{q}_2 + q_1\dot{q}_3 + q_2\dot{q}_4 \\ \dfrac{q_4(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} + q_3\dot{q}_2 - q_2\dot{q}_3 + q_1\dot{q}_4 \end{bmatrix} \\ &= 2\begin{bmatrix} \left(q_1 + \dfrac{q_2^2}{q_1}\right)\dot{q}_2 + \left(q_4 + \dfrac{q_2q_3}{q_1}\right)\dot{q}_3 + \left(-q_3 + \dfrac{q_2q_4}{q_1}\right)\dot{q}_4 \\ \left(-q_4 + \dfrac{q_2q_3}{q_1}\right)\dot{q}_2 + \left(q_1 + \dfrac{q_3^2}{q_1}\right)\dot{q}_3 + \left(q_2 + \dfrac{q_3q_4}{q_1}\right)\dot{q}_4 \\ \left(q_3 + \dfrac{q_2q_4}{q_1}\right)\dot{q}_2 + \left(-q_2 + \dfrac{q_3q_4}{q_1}\right)\dot{q}_3 + \left(q_1 + \dfrac{q_4^2}{q_1}\right)\dot{q}_4 \end{bmatrix} \\ &= S(q)\dot{q}, \end{aligned} \tag{9.7.42}$$

where

$$S(q) \triangleq 2\begin{bmatrix} q_1 + \dfrac{q_2^2}{q_1} & q_4 + \dfrac{q_2q_3}{q_1} & -q_3 + \dfrac{q_2q_4}{q_1} \\ -q_4 + \dfrac{q_2q_3}{q_1} & q_1 + \dfrac{q_3^2}{q_1} & q_2 + \dfrac{q_3q_4}{q_1} \\ q_3 + \dfrac{q_2q_4}{q_1} & -q_2 + \dfrac{q_3q_4}{q_1} & q_1 + \dfrac{q_4^2}{q_1} \end{bmatrix}. \tag{9.7.43}$$

For convenience, we write $S(q) = [S_1(q)\ \ S_2(q)\ \ S_3(q)]$.

Next, it follows from (9.5.2) that, for $i = 2, 3, 4$, the generalized moment $Q_i(q, \dot{q})$ is given by

$$\begin{aligned} Q_i(q, \dot{q}) &= M^{\mathrm{T}}\partial_{\dot{q}_i}(\omega_{\mathrm{D/A}}) \\ &= M^{\mathrm{T}}\partial_{\dot{q}_i}[S(q)\dot{q}] \\ &= M^{\mathrm{T}}\partial_{\dot{q}_i}[\dot{q}_2S_1(q) + \dot{q}_3S_2(q) + \dot{q}_4S_3(q)] \\ &= M^{\mathrm{T}}S_i(q) \\ &= S_i^{\mathrm{T}}(q)M. \end{aligned} \tag{9.7.44}$$

Therefore,

$$Q(q, \dot{q}) = S^{\mathrm{T}}(q)M. \tag{9.7.45}$$

Next, since

$$T_{\mathcal{B}/w/\mathrm{A}} = \tfrac{1}{2}\dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)JS(q)\dot{q} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{\mathrm{c}/w/\mathrm{A}}|^2, \tag{9.7.46}$$

it follows that

$$\begin{aligned} \mathrm{d}_t\partial_{\dot{q}}^{\mathrm{T}}T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}) &= \mathrm{d}_t\partial_{\dot{q}}^{\mathrm{T}}[\tfrac{1}{2}\dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)JS(q)\dot{q}] \\ &= \mathrm{d}_t[S^{\mathrm{T}}(q)J\omega_{\mathrm{D/A}}] \\ &= S^{\mathrm{T}}(q)J\dot{\omega}_{\mathrm{D/A}} + \dot{S}^{\mathrm{T}}(q)J\omega_{\mathrm{D/A}}. \end{aligned} \tag{9.7.47}$$

Next, a lengthy calculation shows that

$$\begin{bmatrix} \partial_{q_2}[S(q)\dot{q}] & \partial_{q_3}[S(q)\dot{q}] & \partial_{q_4}[S(q)\dot{q}] \end{bmatrix} = \dot{S}(q) + [S(q)\dot{q}]^{\times}S(q). \tag{9.7.48}$$

Therefore,

$$\begin{aligned} \partial_q^{\mathrm{T}}T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}) &= \partial_q^{\mathrm{T}}[\tfrac{1}{2}\dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)JS(q)\dot{q}] \\ &= \begin{bmatrix} \partial_{q_2}[\tfrac{1}{2}\dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)JS(q)\dot{q}] \\ \partial_{q_3}[\tfrac{1}{2}\dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)JS(q)\dot{q}] \\ \partial_{q_4}[\tfrac{1}{2}\dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)JS(q)\dot{q}] \end{bmatrix} \\ &= \begin{bmatrix} \dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)J\partial_{q_2}[S(q)\dot{q}] \\ \dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)J\partial_{q_3}[S(q)\dot{q}] \\ \dot{q}^{\mathrm{T}}S^{\mathrm{T}}(q)J\partial_{q_4}[S(q)\dot{q}] \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \omega_{\mathrm{D/A}}^{\mathrm{T}} J \partial_{q_2}[S(q)\dot{q}] \\ \omega_{\mathrm{D/A}}^{\mathrm{T}} J \partial_{q_3}[S(q)\dot{q}] \\ \omega_{\mathrm{D/A}}^{\mathrm{T}} J \partial_{q_4}[S(q)\dot{q}] \end{bmatrix}$$

$$= \left(\omega_{\mathrm{D/A}}^{\mathrm{T}} J \begin{bmatrix} \partial_{q_2}[S(q)\dot{q}] & \partial_{q_3}[S(q)\dot{q}] & \partial_{q_4}[S(q)\dot{q}] \end{bmatrix}\right)^{\mathrm{T}}$$

$$= \left(\omega_{\mathrm{D/A}}^{\mathrm{T}} J[\dot{S}(q) + \omega_{\mathrm{D/A}}^{\times} S(q)]\right)^{\mathrm{T}}$$

$$= \dot{S}^{\mathrm{T}}(q) J \omega_{\mathrm{D/A}} - S^{\mathrm{T}}(q)(\omega_{\mathrm{D/A}} \times J\omega_{\mathrm{D/A}}). \tag{9.7.49}$$

Consequently, using (9.7.45), (9.7.47), and (9.7.49), it follows from Fact 9.6.1 that

$$S^{\mathrm{T}}(q) J\dot{\omega}_{\mathrm{D/A}} + S^{\mathrm{T}}(q)(\omega_{\mathrm{D/A}} \times J\omega_{\mathrm{D/A}}) = S^{\mathrm{T}}(q) M. \tag{9.7.50}$$

Therefore, since by assumption $\det S(q) = 8/q_1 \neq 0$ and thus $S(q)$ is nonsingular, it follows that

$$J\dot{\omega}_{\mathrm{D/A}} + \omega_{\mathrm{D/A}} \times J\omega_{\mathrm{D/A}} = M, \tag{9.7.51}$$

which is Euler's equation given by Fact 7.8.5.

9.8 Lagrange's Equations: Potential Energy Form

Potential energy gives rise to generalized forces and moments, and it is convenient to distinguish generalized forces and moments arising from potential energy from the remaining generalized forces and moments. We thus write

$$Q(q, \dot{q}, t) = Q_{\mathrm{p}}(q) + Q_{\mathrm{np}}(q, \dot{q}, t), \tag{9.8.1}$$

where Q_{p} denotes the generalized forces and moments arising from potential energy and Q_{np} denotes the remaining generalized forces and moments. To determine the generalized forces and moments arising from potential energy, we write $U_{\mathcal{B}}(q)$ to denote the potential energy of $\mathcal{B}$ in terms of generalized coordinates. The following result is analogous to Fact 8.3.1.

Fact 9.8.1. Let $\mathcal{B}$ be a body, let $U_{\mathcal{B}}(q)$ denote the potential energy of $\mathcal{B}$ in terms of the generalized coordinates q. Then, the generalized force Q associated with $U_{\mathcal{B}}(q)$ is given by

$$Q_{\mathrm{p}}(q) = -\partial_q^{\mathrm{T}} U_{\mathcal{B}}(q). \tag{9.8.2}$$

Proof. Need proof. □

A generalized force or moment associated with the potential energy is called a *generalized potential force or moment*. The potential energy and associated generalized potential force for a collection of springs is given by the following result.

Fact 9.8.2. Consider a collection of s springs such that, for each $i = 1, \ldots, s$, the ith spring has stiffness $k_i > 0$, relaxed length $d_i \geq 0$, and connects inertia points y_i and w_i, respectively. Furthermore, assume that the force of the ith spring is given by

$$\vec{f}_{y_i/w_i} = -k_i(|\vec{r}_{y_i/w_i}| - d_i)\hat{r}_{y_i/w_i}. \tag{9.8.3}$$

Furthermore, let $q \in \mathbb{R}^r$ denote generalized coordinates, and assume that

$$U(q) \triangleq \frac{1}{2}\sum_{i=1}^{s} k_i(|\vec{r}_{y_i/w_i}| - d_i)^2 \tag{9.8.4}$$

can be written as

$$U(q) = \frac{1}{2}(q-d)^{\mathrm{T}}K(q-d), \tag{9.8.5}$$

where $K \in \mathbb{R}^{r\times r}$ is a symmetric matrix and $d \triangleq [d_1 \quad \cdots \quad d_s]^{\mathrm{T}}$. Then, K is positive semidefinite, and the corresponding generalized force is given by

$$Q_{\mathrm{p}}(q) = -K(q-d). \tag{9.8.6}$$

The following result considers rotational springs.

Fact 9.8.3. Consider a collection of s rotational springs such that, for each $i = 1,\dots,s$, let $\mathcal{B}_i$ and $\hat{\mathcal{B}}_i$ be rigid bodies that are connected by a pin joint at a point fixed in both bodies. Let $\hat{z}$ be a unit dimensionless vector that is parallel with the pin joint, and assume that the ith rotational spring has rotational stiffness $\kappa_i > 0$, and applies torques that are parallel with $\hat{z}$. Let $\hat{x}_{i1}$ and $\hat{x}_{i2}$ be unit dimensionless vectors that are fixed in $\mathcal{B}_i$ and $\hat{\mathcal{B}}_i$, respectively, and that are orthogonal to $\hat{z}$. Assume that the rotational spring is relaxed when $\hat{x}_{i1}$ and $\hat{x}_{i2}$ are parallel and that the torque of the ith rotational spring is given by

$$\vec{M}_{\mathcal{B}_i/\hat{\mathcal{B}}_i} = -\kappa_i\theta_{\hat{x}_{i1}/\hat{x}_{i2}/\hat{z}_i}\hat{z}_i. \tag{9.8.7}$$

Furthermore, let $q \in \mathbb{R}^r$ denote generalized coordinates, and assume that

$$U(q) \triangleq \frac{1}{2}\sum_{i=1}^{s} \kappa_i\theta^2_{\hat{x}_1/\hat{x}_2/\hat{z}} \tag{9.8.8}$$

can be written as

$$U(q) = \frac{1}{2}q^{\mathrm{T}}\mathcal{K}q, \tag{9.8.9}$$

where $\mathcal{K} \in \mathbb{R}^{r\times r}$ is a symmetric matrix. Then, $\mathcal{K}$ is positive semidefinite, and the corresponding generalized force is given by

$$Q_{\mathrm{p}}(q) = -\mathcal{K}q. \tag{9.8.10}$$

The following result reconsiders Fact 8.4.1 in terms of generalized forces and moments. This result uses Lagrange's equations to obtain a constant of the motion, namely, the total energy.

Fact 9.8.4. Let $\mathcal{B}$ be a body, let w be an unforced particle, let $\mathrm{F_A}$ be an inertial frame, and assume that all generalized forces and moments are potential forces and moments. Then, the total energy of $\mathcal{B}$ is conserved.

Proof. First we consider the simpler case where $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of q. In this case, it follows from (9.6.6) that

$$M\ddot{q}(t) = Q_{\mathrm{p}}(q(t)).$$

Omitting the argument t for convenience and using (9.8.2) yields

$$\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) = \mathrm{d}_t T_{\mathcal{B}/w/\mathrm{A}}(\dot{q}) + \mathrm{d}_t U_{\mathcal{B}}(q)$$

$$
\begin{aligned}
&= \dot{q}^{\mathrm{T}} M \ddot{q} + \partial_q U_{\mathcal{B}}(q)\dot{q} \\
&= \dot{q}^{\mathrm{T}} Q_{\mathrm{p}}(q) - Q_{\mathrm{p}}^{\mathrm{T}}(q)\dot{q} \\
&= 0.
\end{aligned}
$$

Next, we consider the case where $T_{\mathcal{B}/w/\mathrm{A}}$ may depend on q. In this case, it follows from (9.6.1) that

$$
\begin{aligned}
\dot{q}^{\mathrm{T}} Q(q) &= \tfrac{1}{2}\dot{q}^{\mathrm{T}} \mathrm{d}_t \partial_{\dot{q}}^{\mathrm{T}}[\dot{q}^{\mathrm{T}} M(q)\dot{q}] - \tfrac{1}{2}\dot{q}^{\mathrm{T}}\partial_q^{\mathrm{T}}[\dot{q}^{\mathrm{T}} M(q)\dot{q}] \\
&= \dot{q}^{\mathrm{T}} \mathrm{d}_t[M(q)\dot{q}] - \tfrac{1}{2}\dot{q}^{\mathrm{T}}\partial_q^{\mathrm{T}}\left[\sum_{j,k=1}^{r} M_{j,k}(q)\dot{q}_j\dot{q}_k\right] \\
&= \dot{q}^{\mathrm{T}} \mathrm{d}_t \begin{bmatrix} \sum_{j=1}^{r} M_{1,j}(q)\dot{q}_j \\ \vdots \\ \sum_{j=1}^{r} M_{r,j}(q)\dot{q}_j \end{bmatrix} - \tfrac{1}{2}\dot{q}^{\mathrm{T}} \begin{bmatrix} \sum_{j,k}^{r} \partial_{q_1} M_{j,k}(q)\dot{q}_j\dot{q}_k \\ \vdots \\ \sum_{j,k}^{r} \partial_{q_r} M_{j,k}(q)\dot{q}_j\dot{q}_k \end{bmatrix} \\
&= \dot{q}^{\mathrm{T}} \begin{bmatrix} \sum_{j=1}^{r}\sum_{k=1}^{r} \partial_{q_k} M_{1,j}(q)\dot{q}_k\dot{q}_j + M_{1,j}(q)\ddot{q}_j \\ \vdots \\ \sum_{j=1}^{r}\sum_{k=1}^{r} \partial_{q_k} M_{r,j}(q)\dot{q}_k\dot{q}_j + M_{r,j}(q)\ddot{q}_j \end{bmatrix} - \tfrac{1}{2}\dot{q}^{\mathrm{T}} \begin{bmatrix} \sum_{j,k}^{r} \partial_{q_1} M_{j,k}(q)\dot{q}_j\dot{q}_k \\ \vdots \\ \sum_{j,k}^{r} \partial_{q_r} M_{j,k}(q)\dot{q}_j\dot{q}_k \end{bmatrix} \\
&= \sum_{i,j,k=1}^{r} \dot{q}_i \partial_{q_k} M_{i,j}(q)\dot{q}_k\dot{q}_j + \sum_{i,j=1}^{r} M_{i,j}(q)\dot{q}_i\ddot{q}_j - \tfrac{1}{2}\sum_{i,j,k=1}^{r} \dot{q}_i \partial_{q_k} M_{i,j}(q)\dot{q}_k\dot{q}_j \\
&= \tfrac{1}{2}\sum_{i,j,k=1}^{r} \partial_{q_k} M_{i,j}(q)\dot{q}_i\dot{q}_j\dot{q}_k + \sum_{i,j=1}^{r} M_{i,j}(q)\dot{q}_i\ddot{q}_j.
\end{aligned}
\tag{9.8.11}
$$

Therefore, using (9.8.11) it follows that

$$
\begin{aligned}
\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) &= \tfrac{1}{2}\mathrm{d}_t[\dot{q}^{\mathrm{T}} M(q)\dot{q}] + \mathrm{d}_t U_{\mathcal{B}}(q) \\
&= \tfrac{1}{2}\mathrm{d}_t \sum_{i,j=1}^{r} M_{i,j}\dot{q}_i\dot{q}_j + \partial_q U_{\mathcal{B}}(q)\dot{q} \\
&= \tfrac{1}{2}\sum_{i,j=1}^{r}\left[\sum_{k=1}^{r} \partial_{q_k} M_{i,j}\dot{q}_k\dot{q}_i\dot{q}_j + M_{i,j}\ddot{q}_i\dot{q}_j + M_{i,j}\dot{q}_i\ddot{q}_j\right] - Q_{\mathrm{p}}(q)^{\mathrm{T}}\dot{q} \\
&= \tfrac{1}{2}\sum_{i,j,k=1}^{r} \partial_{q_k} M_{i,j}\dot{q}_i\dot{q}_j\dot{q}_k + \sum_{i,j=1}^{r} M_{i,j}\dot{q}_i\ddot{q}_j - \dot{q}^{\mathrm{T}} Q_{\mathrm{p}}(q) \\
&= 0.
\end{aligned}
$$

□

The *Lagrangian* of a system with potential $U_{\mathcal{B}}(q)$ and kinetic energy $T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q})$ is the function

$$L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) \triangleq T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) - U_{\mathcal{B}}(q). \tag{9.8.12}$$

Lagrange's equations for a system with potential force are given by the following result. This result is a specialization of Fact 9.6.1.

Fact 9.8.5. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^{\mathrm{T}}$, let w be an unforced particle, define $L_{\mathcal{B}/w/\mathrm{A}}$ by (9.8.12), where $U_{\mathcal{B}}$ includes all internal and external potential forces and moments, and let Q_{np} denote all generalized forces and moments except those that arise from $U_{\mathcal{B}}$ or conservative contact. Then, $q(t)$ satisfies

$$\mathrm{d}_t\partial_{\dot q}^{\mathrm{T}}T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot q(t),t)-\partial_q^{\mathrm{T}}L_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot q(t),t)=Q_{\mathrm{np}}(q(t),\dot q(t),t). \tag{9.8.13}$$

Proof. Show algebra to go from (9.6.1) to (9.8.13). □

If $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of q, then (9.8.13) can be written as

$$M\ddot q(t)+\partial_q^{\mathrm{T}}U_{\mathcal{B}}(q(t))=Q_{\mathrm{np}}(q(t),\dot q(t),t). \tag{9.8.14}$$

9.9 Lagrange's Equations: Rayleigh Dissipation Function Form

In some cases, it is convenient to decompose the nonpotential force Q_{np} in terms of a function R called a *Rayleigh dissipation function*. We thus write

$$Q_{\mathrm{np}}(q,\dot q,t)=Q_{\mathrm{R}}(q,\dot q,t)+Q_{\mathrm{npnR}}(q,\dot q,t), \tag{9.9.1}$$

where Q_{R} arises from a Rayleigh dissipation function and Q_{npnR} are generalized forces and moments that do not arise from a Rayleigh dissipation function. The force or moment Q_{R} arising from a Rayleigh dissipation function $R=R(q,\dot q)$ has the form

$$Q_{\mathrm{R}}(q,\dot q)=-\partial_{\dot q}^{\mathrm{T}}R(q,\dot q). \tag{9.9.2}$$

It is usually the case that $\partial_{\dot q}R(q,0)=0$ for all q, which implies that no energy dissipation occurs when the generalized velocities are zero.

The Rayleigh dissipation function for a collection of dashpots is given by the following result.

Fact 9.9.1. Consider a collection of s dashpots such that, for each $i=1,\ldots,s$, the ith dashpot has viscosity $c_i>0$ and connects inertia points y_i and w_i. Furthermore, for the ith dashpot assume that the force is given by

$$\vec{f}_{y_i/w_i}=-c_i(\mathrm{d}_t|\vec{r}_{y_i/w_i}|)\hat{r}_{y_i/w_i}. \tag{9.9.3}$$

Furthermore, let $q\in\mathbb{R}^r$ denote generalized coordinates, and assume that

$$R(q,\dot q)\triangleq\tfrac{1}{2}\sum_{i=1}^{s}c_i(\mathrm{d}_t|\vec{r}_{y_i/w_i}|)^2 \tag{9.9.4}$$

can be written as

$$R(q,\dot q)=\tfrac{1}{2}\dot q^{\mathrm{T}}C(q)\dot q, \tag{9.9.5}$$

where, for all $q\in\mathbb{R}^r$, $C(q)\in\mathbb{F}^{r\times r}$ is a symmetric matrix. Then, for all $q\in\mathbb{R}^r$, $C(q)$ is positive semidefinite and the corresponding generalized force is given by

$$Q_{\mathrm{R}}(q,\dot q)=-C(q)\dot q. \tag{9.9.6}$$

The following result presents Lagrange's equations for a body with Lagrangian $L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q)$, Rayleigh dissipation function $R(q,\dot q)$, and generalized forces and moments $Q_{\mathrm{npnR}}(q,\dot q)$ that do not arise from the Rayleigh dissipation function, a potential, or conservative contact.

Fact 9.9.2. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q=[q_1\ \cdots\ q_r]^{\mathrm{T}}$, let w be an unforced particle, define $L_{\mathcal{B}/w/\mathrm{A}}$ by (9.8.12), where $U_{\mathcal{B}}$ includes all potential generalized forces and moments, let R be a Rayleigh dissipation function, and let Q_{npnR}

denote all generalized forces and moments except those that arise from R, $U_{\mathcal{B}}$, or conservative contact. Then, q satisfies

$$\mathrm{d}_t\partial_{\dot{q}}^{\mathrm{T}}T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t)) - \partial_q^{\mathrm{T}}L_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t)) + \partial_{\dot{q}}^{\mathrm{T}}R(q(t),\dot{q}(t)) = Q_{\mathrm{npnR}}(q(t),\dot{q}(t),t). \tag{9.9.7}$$

The following result shows that the system dissipates energy if $q(t)$ and $\dot{q}(t)$ are such that $[\partial_{\dot{q}}R(q(t),\dot{q}(t))]\dot{q}(t) > 0$.

Fact 9.9.3. Let $\mathcal{B}$ be a body, let w be an unforced particle, let $\mathrm{F_A}$ be an inertial frame, and assume that all generalized forces or moments that are neither conservative reaction forces or moments nor potential forces or moments are given by the Rayleigh dissipation function R. Then

$$\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t)) = -[\partial_{\dot{q}}R(q(t),\dot{q}(t))]\dot{q}(t). \tag{9.9.8}$$

Proof. First we consider the simpler case where $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of q. In this case, it follows from (9.9.7) that

$$M\ddot{q}(t) = Q_{\mathrm{p}}(q(t)) - \partial_{\dot{q}}^{\mathrm{T}}R(q,\dot{q}).$$

Omitting the argument t for convenience, it follows from (9.8.14) that

$$\begin{aligned}\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}} &= \dot{q}^{\mathrm{T}}M\ddot{q} + \partial_q U_{\mathcal{B}}(q)\dot{q}\\ &= \dot{q}^{\mathrm{T}}[-\partial_q U_{\mathcal{B}}(q) + Q_{\mathrm{R}}(q,\dot{q})] + \partial_q U_{\mathcal{B}}(q)\dot{q}\\ &= Q_{\mathrm{R}}^{\mathrm{T}}(q,\dot{q})\dot{q}\\ &= -[\partial_{\dot{q}}R(q,\dot{q})]\dot{q}.\end{aligned}$$

Finally, for the case where $T_{\mathcal{B}/w/\mathrm{A}}$ depends on q, see the proof of Fact 9.8.4. □

9.10 Examples

Example 9.10.1. Derive the equations of motion for the damped SDOF oscillator.

Solution: A convenient generalized coordinate for such an oscillator is the extension q of the spring, while the corresponding generalized velocity is the instantaneous rate of extension $\dot{q}$ of the spring. The kinetic energy of the oscillator is $T_{\mathcal{B}/w/\mathrm{A}}(\dot{q}) = \frac{1}{2}m\dot{q}^2$, while the potential energy is $U_{\mathcal{B}}(q) = \frac{1}{2}kq^2$. The Rayleigh dissipation function for a dashpot with viscosity c is $R(q,\dot{q}) = \frac{1}{2}c\dot{q}^2$. The remaining nonconservative generalized force is the external applied force f. Lagrange's equations (9.9.7) then yield

$$m\ddot{q} + c\dot{q} + kq = f. \tag{9.10.1}$$

Example 9.10.2. Derive the equations of motion for the bead on the wire considered in Example 7.15.3.

Solution: First we consider the case where $\omega = \dot{\theta}$ is constant, and thus $q_1 = x$ is the only generalized coordinate. The kinetic energy of the body relative to w with respect to $\mathrm{F_A}$ is given by

$$\begin{aligned}T_{\mathcal{B}/w/\mathrm{A}} &= T_{\mathrm{wire}/w/\mathrm{A}} + T_{y/w/\mathrm{A}}\\ &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B/A}}\vec{J}_{\mathrm{wire}/w}\vec{\omega}_{\mathrm{B/A}} + \tfrac{1}{2}m|v_{y/w/\mathrm{A}}|^2\end{aligned}$$

$$= \tfrac{1}{2}\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} + \tfrac{1}{2}m(\dot{x}^2 + x^2\omega^2)$$
$$= \tfrac{1}{2}(I + mx^2)\omega^2 + \tfrac{1}{2}m\dot{x}^2.$$

It thus follows from Lagrange's equations (9.6.1) that

$$\mathrm{d}_t\partial_{\dot{x}}T_{\mathcal{B}/w/\mathrm{A}} - \partial_x T_{\mathcal{B}/w/\mathrm{A}} = 0$$

that

$$\mathrm{d}_t(m\dot{x}) - m\omega^2 x = 0$$

that is,

$$\ddot{x} = \omega^2 x.$$

To determine the reaction force between the bead and the wire, Newton's second law implies that

$$\begin{aligned} \vec{f}_{\mathrm{R}} &= m\vec{a}_{y/w/\mathrm{A}} \\ &= (\ddot{x} - \omega^2 x)\hat{\imath}_{\mathrm{B}} + 2\omega\dot{x}\hat{\jmath}_{\mathrm{B}} \\ &= 2\omega\dot{x}\hat{\jmath}_{\mathrm{B}}. \end{aligned}$$

Therefore, the force applied to the bead is transverse to the wire. Although the bead moves away from w, the component of the force along the wire is zero. Now, suppose that a stopper is placed on the wire preventing the bead from moving further along the wire. Then, the reaction force on the stopper due to the bead is the centrifugal reaction force, whereas the reaction force on the bead due to the stopper is the centripetal reaction force.

Next, we consider the case where $\dot{\theta}$ is not necessarily constant, and thus the generalized coordinates are $q_1 = x$ and $q_2 = \theta$. The kinetic energy thus has the form

$$T_{\mathcal{B}/w/\mathrm{A}} = \tfrac{1}{2}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & I + mx^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}.$$

For q_1 it follows from Lagrange's equations that

$$\ddot{x} = \dot{\theta}^2 x.$$

Since $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of $q_2 = \theta$, it follows from Lagrange's equations that

$$\mathrm{d}_t\partial_{\dot{\theta}}T_{\mathcal{B}/w/\mathrm{A}} = 0,$$

and thus

$$\mathrm{d}_t[(I + mx^2)\dot{\theta}] = 0,$$

which shows that the component of momentum $(I+mx^2)\dot{\theta}$ is a constant of the motion. Consequently,

$$(I + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} = 0.$$

To determine the reaction force between the bead and the wire, Newton's second law implies that

$$\begin{aligned} \vec{f}_{\mathrm{R}} &= m\vec{a}_{y/w/\mathrm{A}} \\ &= (\ddot{x} - \dot{\theta}^2 x)\hat{\imath}_{\mathrm{B}} + (\ddot{\theta}x + 2\dot{\theta}\dot{x})\hat{\jmath}_{\mathrm{B}} \\ &= (\ddot{\theta}x + 2\dot{\theta}\dot{x})\hat{\jmath}_{\mathrm{B}}. \end{aligned}$$

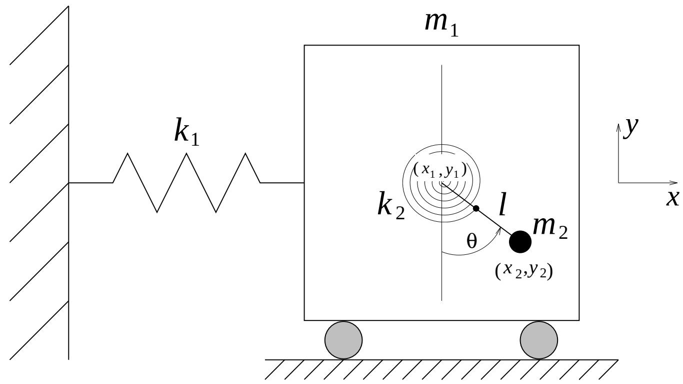


Figure 9.10.1: Eccentric rotating mass. This body has translational and rotational degrees of freedom

As in the case where $\dot{\theta}$ is constant, the reaction force on the bead due to the wire is transverse to the wire.

Example 9.10.3. Consider the eccentric rotating mass m_2 mounted on a cart of mass m_1 at the end of a rigid massless rod of length l as shown in Figure 9.10.1. The mass m_1 is constrained to move along a line under the action of an extensional spring with stiffness k_1. The mass m_2 is acted on by a torsional spring with torsional stiffness k_2, but is otherwise free to rotate. Derive the equation of motion for this body.

Solution: Choose $\mathrm{F_A}$ such that $\hat{\imath}_\mathrm{A}$ is aligned with the motion of the cart. Let x_1, y_1 and x_2, y_2 denote the coordinates of the masses m_1 and m_2, respectively. Then, x_1, y_1, x_2, y_2 are generalized coordinates for the body. However, these coordinates are not independent since they satisfy the constraints

$$y_1 = c_0, \tag{9.10.2}$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2, \tag{9.10.3}$$

where c_0 is a constant. It is possible to choose a coordinates such that the constant c in (9.10.2) is zero. With this choice, the constraints (9.10.2) and (9.10.3) have the form (9.2.6) with

$$\phi_1(x_1, y_1, x_2, y_2) = y_1 - c_0, \tag{9.10.4}$$

$$\phi_2(x_1, y_1, x_2, y_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2. \tag{9.10.5}$$

This body is thus holonomic. Since there are four generalized coordinates and two independent constraints, this body has two degrees of freedom. We thus look for two independent generalized coordinates. Notice that the mass m_1 moves in a straight line, while the mass m_2 moves in a circle. Thus the configuration space of this body is the Cartesian product of a line and a circle, in other words, the surface of a cylinder. Consequently, we choose the position of m_1 (with respect to a reference point) and the angle θ of m_2 as generalized coordinates. Accordingly, let q_1 be the extension of the spring k_1 and let $q_2 = \theta$.

In terms of these generalized coordinates, (9.2.3) and (9.2.4) have the form

$$x_1 = q_1 + c_1, \tag{9.10.6}$$
$$y_1 = 0, \tag{9.10.7}$$
$$x_2 = q_1 + l \sin q_2 + c_2, \tag{9.10.8}$$
$$y_2 = -l \cos q_2, \tag{9.10.9}$$

where c_1 and c_2 are constants. The generalized velocities corresponding to the chosen generalized coordinates are the rate $\dot{q}_1$ of deformation of the spring k_1 and the angular rate $\dot{q}_2$ of the are connected to m_2. Equations (9.3.2) and (9.3.3) become

$$u_1 = \dot{q}_1, \tag{9.10.10}$$
$$v_1 = 0, \tag{9.10.11}$$
$$u_2 = \dot{q}_1 + l\dot{q}_2 \cos q_2, \tag{9.10.12}$$
$$v_2 = l\dot{q}_2 \sin q_2. \tag{9.10.13}$$

Using (9.10.10)-(9.10.13) in (9.3.11), we obtain the kinetic energy of the body as

$$T_{\mathcal{B}/w/\mathrm{A}}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \tfrac{1}{2}(m_1 + m_2)\dot{q}_1^2 + m_2 l \dot{q}_1 \dot{q}_2 \cos q_2 + \tfrac{1}{2} m_2 l^2 \dot{q}_2^2. \tag{9.10.14}$$

The elastic potential energy of the springs is

$$U_{\mathcal{B}}(q_1, q_2) = \tfrac{1}{2}k_1 q_1^2 + \tfrac{1}{2}k_2 q_2^2.$$

Forming the Lagrangian as in (9.8.12) and expanding Lagrange's equations (9.8.13) yields the equations of motion

$$(m_1 + m_2)\ddot{q}_1 + m_2 l \ddot{q}_2 \cos q_2 - m_2 l \dot{q}_2^2 \sin q_2 + k_1 q_1 = 0,$$

$$m_2 l \ddot{q}_1 \cos q_2 + m_2 l^2 \ddot{q}_2 + k_2 q_2 = 0. \qquad \diamond$$

9.11 Lagrangian Dynamics with Constraints

Use Lagrange multipliers to address constraints on generalized coordinates.

Consider a linkage consisting of 3 links, with the ends pinned to a base. This mechanical system cannot be modeled by independent generalized coordinates. The sum of the internal angles is 360 degrees.

9.12 Lagrangian Dynamics for Nonholonomic Systems

Consider systems with constraints on the generalized velocities that cannot be integrated to yield constraints on generalized coordinates.

9.13 Hamiltonian Dynamics

Consider a particle y with mass m and generalized coordinates $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The total energy of y relative to an unforced particle w with respect to an inertial frame $\mathrm{F_A}$ is given by

$$\begin{aligned} \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) &= \tfrac{1}{2}m(\dot x^2+\dot y^2+\dot z^2)+V(q,t) \\ &= T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)+V(q,t) \\ &= 2T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)-L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t). \end{aligned} \tag{9.13.1}$$

In terms of the components of the momentum $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} m\dot x \\ m\dot y \\ m\dot z \end{bmatrix} = m\dot q$, it follows that

$$\begin{aligned} \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) &= \sum_{i=1}^{3} p_i\dot q_i - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) \\ &= p^{\mathrm{T}}\dot q - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t). \end{aligned} \tag{9.13.2}$$

Note that $p = \partial_{\dot q}T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)$.

More generally, let $\mathcal{B}$ be a body with particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let w be an unforced particle, and let $\mathrm{F_A}$ be a frame. As in the case of a single particle, define the *conjugate momentum* $p \in \mathbb{R}^l$ by

$$p \triangleq \partial_{\dot q}^{\mathrm{T}}T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t), \tag{9.13.3}$$

and, in analogy with (9.13.2), define the *Hamiltonian* $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)$ of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ by

$$H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) \triangleq p^{\mathrm{T}}\dot q - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t). \tag{9.13.4}$$

Next, recall from (9.3.11) that $T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)$ is given by

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) = \tfrac{1}{2}\dot q^{\mathrm{T}}M(q,t)\dot q + F(q,t)\dot q + G(q,t), \tag{9.13.5}$$

where $M(q,t)$, $F(q,t)$, and $G(q,t)$ are defined by (9.3.12), (9.3.13), and (9.3.14), respectively. It thus follows from (9.13.3) and (9.13.5) that

$$p = M(q,t)\dot q + F^{\mathrm{T}}(q,t). \tag{9.13.6}$$

Now, substituting (9.13.6) into (9.13.4) and using (9.13.5) yields

$$\begin{aligned} H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= [M(q,t)\dot q + F^{\mathrm{T}}(q,t)]^{\mathrm{T}}\dot q - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) \\ &= \dot q^{\mathrm{T}}M(q,t)\dot q + F(q,t)\dot q - T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) + V(q,t) \\ &= \tfrac{1}{2}\dot q^{\mathrm{T}}M(q,t)\dot q - G(q,t) + V(q,t) \\ &= \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) - F(q,t)\dot q - 2G(q,t). \end{aligned} \tag{9.13.7}$$

Therefore, if $F(q,t) = 0$ and $G(q,t) = 0$, then $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) = \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)$, as in the case of a single particle.

Next, to express $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)$ in terms of p rather than $\dot q$, note that it follows from (9.13.6) that

$$\dot q = M^{-1}(q,t)[p - F^{\mathrm{T}}(q,t)]. \tag{9.13.8}$$

Therefore, substituting (9.13.8) into (9.13.7) yields

$$\begin{aligned} H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= \tfrac{1}{2}[p - F^{\mathrm{T}}(q,t)]^{\mathrm{T}} M^{-1}(q,t)[p - F^{\mathrm{T}}(q,t)] - G(q,t) + V(q,t) \\ &= \tfrac{1}{2} p^{\mathrm{T}} M^{-1}(q,t)p - F(q,t)M^{-1}(q,t)p - \tfrac{1}{2}F(q,t)M^{-1}(q,t)F^{\mathrm{T}}(q,t) - G(q,t) + V(q,t). \end{aligned} \tag{9.13.9}$$

Next, note that

$$\mathrm{d}_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) = \partial_p H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)\dot{p} + \partial_q H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)\dot{q} + \partial_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t). \tag{9.13.10}$$

On the other hand, it follows from (9.13.4) and $p = \partial_{\dot{q}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)$ that

$$\begin{aligned} \mathrm{d}_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= \dot{p}^{\mathrm{T}}\dot{q} + p^{\mathrm{T}}\ddot{q} - [\partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} + \partial_{\dot{q}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\ddot{q} + \partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)] \\ &= \dot{p}^{\mathrm{T}}\dot{q} + p^{\mathrm{T}}\ddot{q} - [\partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} + p^{\mathrm{T}}\ddot{q} + \partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)] \\ &= \dot{p}^{\mathrm{T}}\dot{q} - \partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} - \partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t). \end{aligned} \tag{9.13.11}$$

Comparing (9.13.10) and (9.13.11), it follows that

$$\dot{q} = \partial_p^{\mathrm{T}} H_{\mathcal{B}/w/\mathrm{A}}(q,p,t), \tag{9.13.12}$$

$$\partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = -\partial_q H_{\mathcal{B}/w/\mathrm{A}}(q,p,t), \tag{9.13.13}$$

and

$$\partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = -\partial_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t). \tag{9.13.14}$$

Finally, using (9.13.3), the fact that $\partial_{\dot{q}}^{\mathrm{T}} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = \partial_{\dot{q}}^{\mathrm{T}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)$, and (9.13.13), it follows from (9.8.13) that

$$\dot{p} = -\partial_q^{\mathrm{T}} H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) + Q_{\mathrm{np}}. \tag{9.13.15}$$

Equations (9.13.12) and (9.13.15) are *Hamilton's equations*.

If $\partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = 0$, then it follows from (9.8.13) that

$$\begin{aligned} \mathrm{d}_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= \dot{p}^{\mathrm{T}}\dot{q} - \partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} \\ &= \mathrm{d}_t \partial_{\dot{q}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} - \partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} \\ &= Q_{\mathrm{np}}^{\mathrm{T}}\dot{q}. \end{aligned} \tag{9.13.16}$$

Therefore, if $Q_{\mathrm{np}} = 0$, then $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)$ is constant.

9.14 GAK Dynamics

Include Gibbs-Appel-Kane dynamics.

9.15 Theoretical Problems

Problem 9.15.1. Let $q \in \mathbb{R}^r$ denote generalized coordinates for a body $\mathcal{B}$, and let $B \in \mathbb{R}^{r\times r}$ be a positive-semidefinite matrix. For a collection of inerters with generalized force

$$Q_{\mathrm{inert}}(\ddot{q}) = -B\ddot{q},$$

define the *inertance function*

$$I(\ddot{q}) = \tfrac{1}{2}\ddot{q}^{\mathrm{T}} B\ddot{q}.$$

Show that the kinetic energy of the body with the inerters is given by

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) = \tfrac{1}{2}\dot{q}^{\mathrm{T}}[M(q)+B]\dot{q},$$

where $\frac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q}$ is the kinetic energy of $\mathcal{B}$ without the inerters.

9.16 Applied Problems

Problem 9.16.1. Two particles, three springs, and one dashpot are interconnected as shown in Figure 9.16.1. The masses are constrained to move along straight, frictionless tracks that are mutually orthogonal, as shown. No gravity is present. Springs k_1 and k_2 are connected to the point labeled a. The tracks and the point a are embedded in an inertially nonrotating massive body. The particles are interconnected by a spring with stiffness k and a dashpot with viscosity c. The relaxed lengths of the springs with stiffnesses k_1, k_2, and k are, respectively, r_1, r_2, and r. Derive the equations of motion, and then specialize these equations to the case where m_2 is fixed at a.

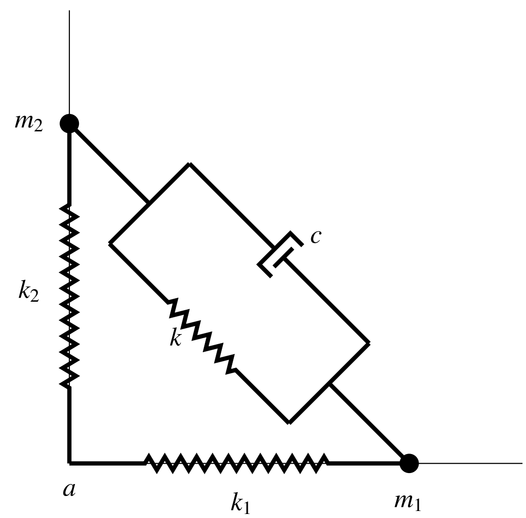


Figure 9.16.1: Two-particle body with springs and dashpot for Problem 9.16.1

Problem 9.16.2. In Figure 9.16.2, a rectangular rigid body whose mass is M and a thin bar whose mass is m and length is l move without friction (due to mounting on small wheels) over the surface of an inertially nonrotating massive body. The upper end of the thin bar is attached to a pin that moves without friction along a vertical track on the left edge of the rectangular rigid body. The pin at the end of the bar is attached to a spring with stiffness k and relaxed length r. The rectangular rigid body is connected to the right wall of the massive body by a dashpot whose viscosity is c. Finally, a force $\vec{f}$ is applied as shown to the lower left corner of the rectangular body. Derive the equations of motion.

Problem 9.16.3. The triangular cart in Figure 9.16.3 with mass m_1 is connected to a massive nonrotating body by means of a spring with stiffness k_1. The angle between the slanted surface of the cart and the horizontal direction is θ. The particle y, whose mass is m_2, slides without friction along the slanted surface attached to a spring with stiffness k_2. The relaxed length of both springs is zero, and gravity acts in the direction shown. Derive the equations of motion.

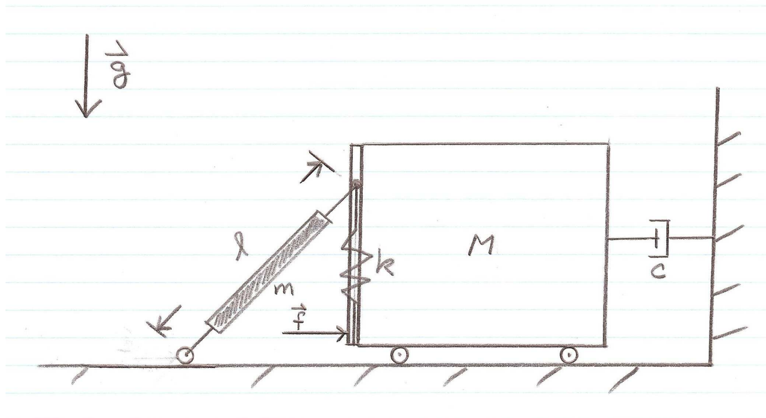


Figure 9.16.2: Rectangular rigid body and thin bar for Problem 9.16.2

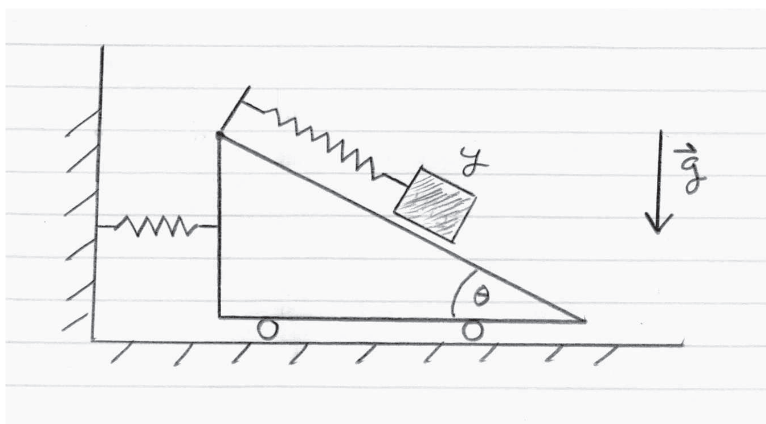


Figure 9.16.3: Triangular cart with particle and springs for Problem 9.16.3

Problem 9.16.4. The body in Figure 9.16.4 is a rigid bar that has two springs and two particles that slide without friction along the bar. The bar is attached by a frictionless pin joint to an inertially nonrotating massive body. Both particles y_1 and y_2 have mass m, and both springs have stiffness k. The relaxed length of each spring is l, and gravity acts in the direction shown. The angle θ between the bar and the direction shown is a prescribed function of time, and thus the inertia of the rigid bar is irrelevant. Derive the equations of motion.

Problem 9.16.5. The spherical pendulum in Figure 9.16.5 consists of a particle y attached by a rope to the top of a rigid bar. The length l of the rope is a prescribed function of time. The bar is attached to an inertially nonrotating massive body. Gravity acts in the vertical direction. Derive the equations of motion for this body in terms of the angle θ around the bar and the angle ϕ between the rope and the bar.

Problem 9.16.6. The two-link pendulum in Figure 9.16.6 is connected to a massive nonrotating body by means of a pin joint at the point a. The first link, whose ends are points a and b, has mass m_1 and length l_1. The second link, which is connected to the first link at point b by means of a pin joint, has mass m_2 and length l_2. The force $\vec{f}$ is applied to the tip of the second link at point c in a direction that is perpendicular to the second link. Derive the equations of motion for this body.

Problem 9.16.7. The rotating disk $\mathcal{D}$ in Figure 9.16.7 is subjected to a moment $\vec{\Gamma}$. The particle y slides without friction along a linear slot on the platform and is attached to a pair of identical springs, which are relaxed when y is at point b. The center of the disk is the point a, which is attached by a frictionless pin joint to a massive nonrotating body, and the moment of inertia of the disk around an axis that is perpendicular to the disk and relative to the point a is $J_{\mathcal{D}/a}$. The path of y is orthogonal to the line segment connecting a and b. The mass of the particle is m, the distance from a to b is h, and the stiffness of each spring is k. Derive the equations of motion.

Problem 9.16.8. Consider the two-bar linkage shown in Figure 9.16.8 consisting of two thin bars, one spring, and one particle. a, b, c, d, e are points. There are pin joints at a, b, c. The thin bar $\mathcal{B}_1$ between a and b has length l and mass m_1. The thin bar $\mathcal{B}_2$ between b and c has length l and mass m_2. The center of mass of $\mathcal{B}_1$ is at d, while the center of mass of $\mathcal{B}_2$ is at e. The spring connects points d and e. The relaxed length of the spring is $l/2$, and its stiffness is k. Because the bars $\mathcal{B}_1$ and $\mathcal{B}_2$ have equal length, there are four angles labeled θ. The particle y located at c is attached to the end of $\mathcal{B}_2$ and is mounted on a slider that allows it to move horizontally. The mass of y is m_3. All motion is frictionless. Uniform gravity acts in the direction shown, which is perpendicular to the line passing through a and c. The force $\vec{f}$ is applied to $\mathcal{B}_2$ at point e. The direction of $\vec{f}$ is the same as the direction of gravity. The moment $\vec{M}$ is a torque applied to $\mathcal{B}_1$ at the point a. Derive the equation of motion for this body in terms of θ.

Problem 9.16.9. Use Lagrangian dynamics to derive the equations of motion for the wheel with bar considered in Problem 7.17.7 in the form of a coupled pair of second-order differential equations involving ϕ and θ. Use these equations to determine the reaction forces at the pin joint.

Problem 9.16.10. Use Lagrangian dynamics to derive the equations of motion for the physical pendulum considered in Example 7.15.2. Use these equations to determine the reaction forces at

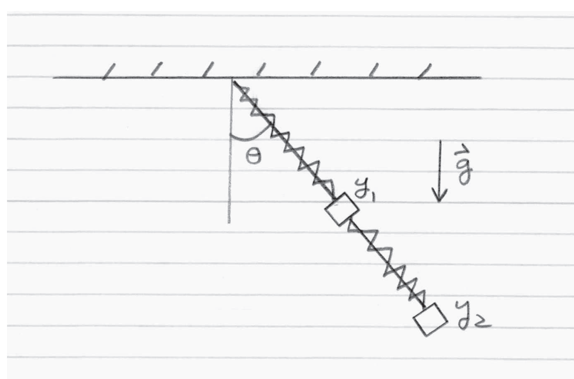


Figure 9.16.4: Rotating bar with two particles and two springs for Problem 9.16.4

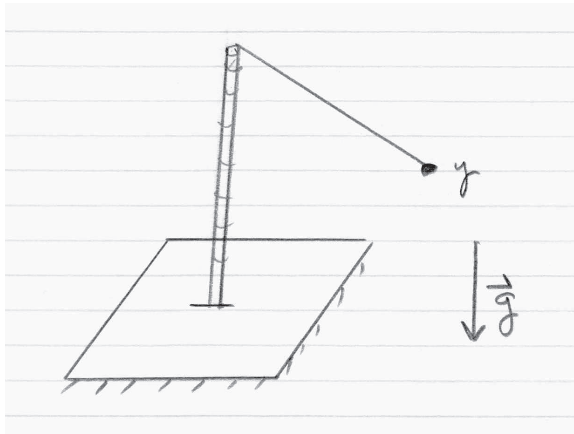


Figure 9.16.5: Spherical pendulum with variable length for Problem 9.16.5

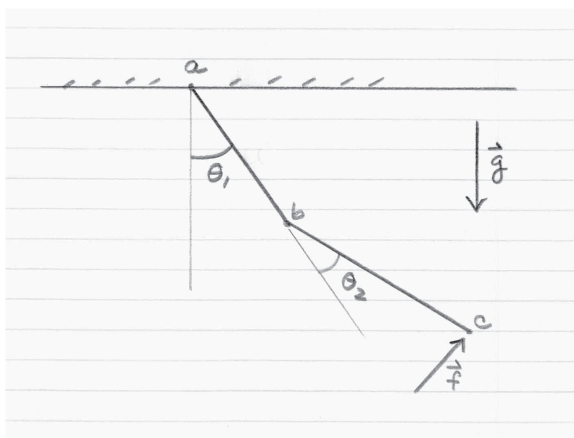


Figure 9.16.6: Two-link pendulum for Problem 9.16.6

the pin joint.

Problem 9.16.11. Use Lagrangian dynamics to derive the equations of motion for the ball rolling down the inclined plane considered in Problem 7.17.1. Use these equations to determine the normal and tangential components of the reaction force between the ball and the inclined plane as well as the vertical reaction force between the inclined plane and the ground.

Problem 9.16.12. Consider the offset rotating pendulum shown in Figure 9.16.9. The vertical shaft rotates at the rate $\omega(t)$, where $\omega(t)$ is a differentiable function of time and where $\omega(t) > 0$ corresponds to rotation in the direction shown. The horizontal arm is welded to the vertical shaft at point b and has length R. The pendulum is connected by a hinge joint to the point a on the horizontal arm. The pendulum consists of a massless link of length ℓ with a particle y of mass m attached to its tip. As shown, the pendulum swings in and out of the page. The inertial frame $\mathrm{F_A}$ is attached to the ground, the frame $\mathrm{F_B}$ is attached to the horizontal arm, and the frame $\mathrm{F_C}$ is attached to the massless

Figure 9.16.7: Rotating disk with translating particle for Problem 9.16.7

link. The angle between the massless link and the vertical direction is θ, where $\theta = 0$ corresponds to the downward direction and where $\theta > 0$ denotes clockwise rotation of the pendulum viewed from b to a. Gravity $\vec{g} = g\hat{k}_A$ is in the direction $\hat{k}_A$. Note: No information concerning the mass properties of the shaft or the arm is needed to do this problem.

i) Use Lagrangian dynamics to obtain a differential equation that describes the motion of the pendulum in terms of $\ddot{\theta}$, θ, ω, $\dot{\omega}$, R, ℓ, and g.

ii) Use Newton's second law to determine the components of the reaction force $\vec{f}_R = f_1\hat{\imath}_C + f_2\hat{\jmath}_C + f_3\hat{k}_C$ on the particle y resolved in F_C.

iii) Using your solution to 2), apply Euler's equation to the massless link (not including the particle y) to obtain additional information concerning f_1. Use this information to obtain a differential equation that describes the motion of the pendulum.

iv) Determine the reaction force and reaction torque on the horizontal arm at the point a.

9.17 Solutions to the Applied Problems

Solution to Problem 9.16.1.

$$m_1\ddot{q}_1 + \frac{c}{l^2}(q_1^2\dot{q}_1 + q_1q_2\dot{q}_2) + k_1(q_1 - r_1) + \frac{k}{l}(l - r)q_1 = 0,$$
$$m_2\ddot{q}_2 + \frac{c}{l^2}(q_2^2\dot{q}_2 + q_1q_2\dot{q}_1) + k_2(q_2 - r_2) + \frac{k}{l}(l - r)q_2 = 0,$$

where q_1 is the distance from mass m_1 to a, and q_2 is the distance from mass m_2 to a.

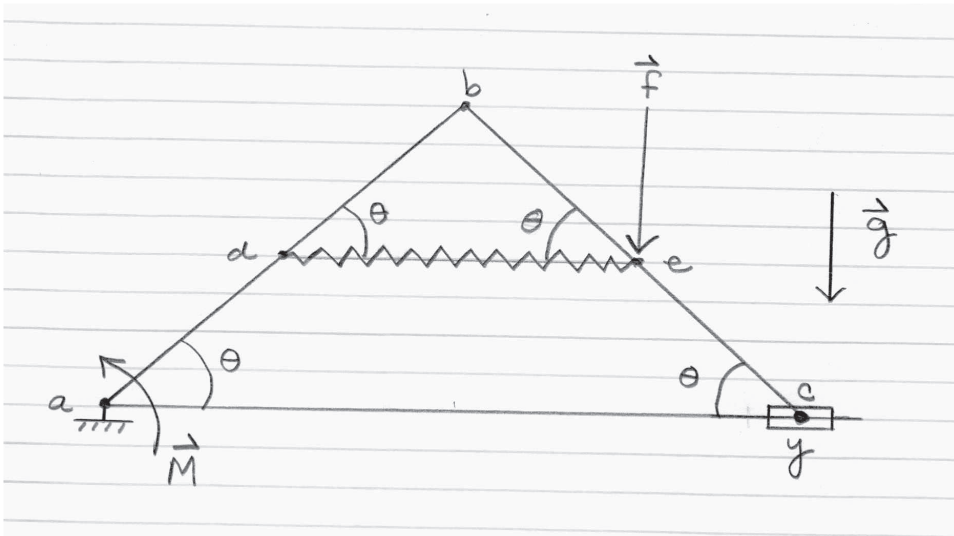


Figure 9.16.8: Two-bar linkage with spring for Problem 9.16.8

Solution to Problem 9.16.2.

$$(m+M)\ddot{x}+c\dot{x}-\tfrac{1}{2}ml[(\cos\theta)\dot{\theta}^2+(\sin\theta)\ddot{\theta}]=f,$$
$$\tfrac{1}{3}ml\ddot{\theta}-\tfrac{1}{2}m(\sin\theta)\ddot{x}+\tfrac{1}{2}mg\cos\theta+k(\cos\theta)(l\sin\theta-r)=0,$$

where x is the distance from the lower left corner of the cart to the wall, and θ is the angle between the inclined link and the floor.

Solution to Problem 9.16.3.

$$(m_1+m_2)\ddot{q}_1+m_2\cos\theta\ddot{q}_2+k_1q_1=0,$$
$$m_2[\ddot{q}_2+(\cos\theta)\ddot{q}_1]+k_2q_2=m_2g\sin\theta,$$

where q_1 is the distance between the vertical edge of the inclined plane and the wall, and q_2 is the distance between m_2 and the vertical edge of the inclined plane along the inclined plane.

Solution to Problem 9.16.4.

$$2\ddot{q}_1+\ddot{q}_2-(2q_1+q_2)\dot{\theta}^2+\frac{k}{m}(q_1-l)=2g\cos\theta,$$
$$\ddot{q}_1+\ddot{q}_2-(q_1+q_2)\dot{\theta}^2+\frac{k}{m}(q_2-l)=g\cos\theta,$$

where q_1 is the distance between the pin joint and first mass, and q_2 is the distance between the first and second mass.

Solution to Problem 9.16.5.

$$l^2\ddot{\phi}+2l\dot{l}\dot{\phi}-l^2\dot{\theta}^2(\sin\phi)\cos\phi+gl\sin\phi=0,$$
$$l(\sin\phi)\ddot{\theta}+[2\dot{l}\sin\phi+2l(\cos\phi)\dot{\phi}]\dot{\theta}=0,$$

where θ is the angle around the bar, and ϕ is the angle between the rope and the bar.

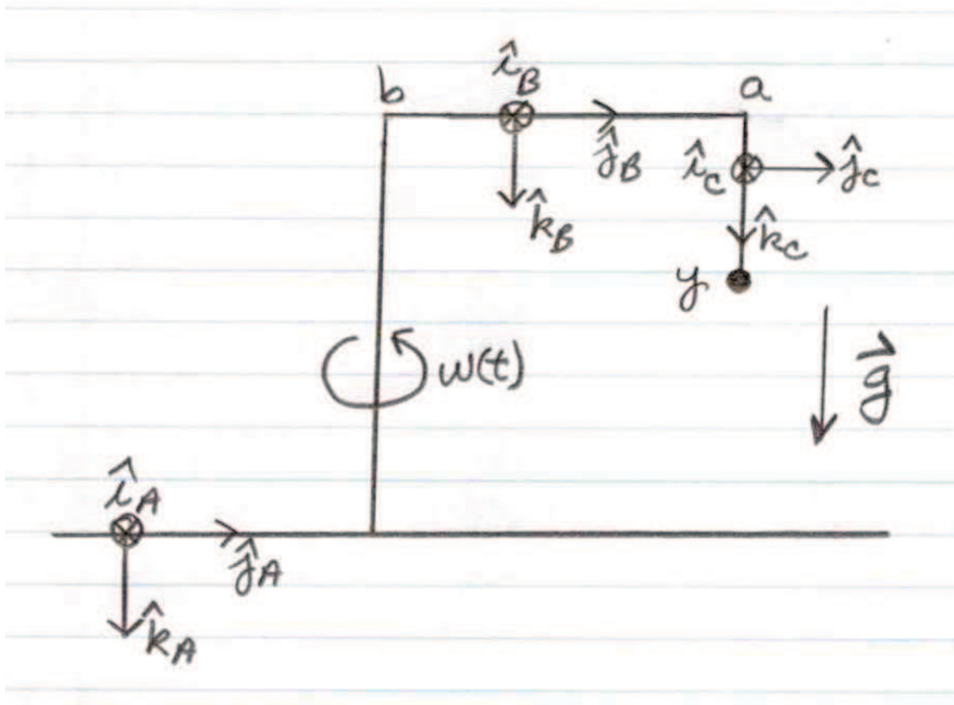


Figure 9.16.9: Rotating Pendulum for Problem 9.16.12

Solution to Problem 9.16.6.

$$\beta\ddot{\theta}_1 + \alpha\ddot{\theta}_2 - \tfrac{1}{2}m_2 l_1 l_2(\sin\theta_2)\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2) + (\tfrac{1}{2}m_1 + m_2)gl_1\sin\theta_1 + \gamma = f(\cos\theta_2 l_1 + l_2),$$
$$\alpha\ddot{\theta}_1 + J_2\ddot{\theta}_2 + \tfrac{1}{2}m_2 l_1 l_2 \sin\theta_2\dot{\theta}_1^2 + \gamma = fl_2,$$

where

$$\alpha = J_2 + \tfrac{1}{2}m_2 l_1 l_2\cos\theta_2, \quad \beta = J_1 + J_2 + m_2 l_1^2 + m_2 l_1 l_2\cos\theta_2, \quad \gamma = \tfrac{1}{2}m_2 g l_2\sin(\theta_1 + \theta_2).$$

Note that J_1 is the moment of inertia of bar 1 with respect to a around an axis coming out of the page, and J_2 is the moment of inertia of bar 2 with respect to b around an axis coming out of the page.

Solution to Problem 9.16.7.

$$(J_\mathrm{D} + mh^2 + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} - mh\ddot{x} = \tau, \quad \ddot{x} - h\ddot{\theta} - x\dot{\theta}^2 + \frac{2k}{m}x = 0,$$

where x is the distance between y and b, θ is the angle between $\vec{r}_{b/a}$ and $\hat{\imath}_\mathrm{A}$, and F_A is defined such that $\hat{\imath}_\mathrm{A}$ and $\hat{\jmath}_\mathrm{A}$ are in the plane of the disk, and $\hat{k}_\mathrm{A}$ is perpendicular to the disk.

Solution to Problem 9.16.8.

$$\tfrac{1}{3}(m_1 + m_2)l^2\ddot{\theta} + (2m_2 + 4m_3)l^2\sin\theta(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2) + kl^2\sin\theta(\tfrac{1}{2} - \cos\theta)$$
$$+ \tfrac{1}{2}(m_1 + m_2)gl\cos\theta = M - \tfrac{1}{2}fl\cos\theta.$$

Note that y can move only horizontally.

Chapter Ten
Aircraft Kinematics

10.1 Frames Used in Aircraft Kinematics

An aircraft is modeled as a rigid body that has six degrees of freedom, specifically, three translational degrees of freedom (translation in three orthogonal directions) and three rotational degrees of freedom (rotation about three orthogonal axes). The names of these degrees of freedom are given in Table 10.1.

Translational	Rotational
X: Range	Φ: Roll
Y: Drift	Θ: Pitch
Z: Plunge	Ψ: Yaw

Longitudinal	Lateral
X: Range	Φ: Roll
Θ: Pitch	Y: Drift
Z: Plunge	Ψ: Yaw

Table 10.1: Aircraft degrees of freedom using the 3-2-1 yaw-pitch-roll Euler-angle rotation sequence. On the left, the each degree of freedom is classified as translational or rotational, whereas, on the right, each degree of freedom is classified as longitudinal or lateral.

Since a single degree of freedom is modeled by one second-order differential equation and thus two first-order differential equations, the dynamics of a six-degree-of-freedom rigid body such as an aircraft are described by 6 second-order differential equations and thus 12 first-order differential equations.

To describe aircraft flight kinematics we consider eight frames, namely, the Earth frame, four intermediate Earth frames, the aircraft frame, the stability frame, and the wind frame. For simplicity, we assume throughout this book that the atmosphere is stagnant, and thus no ambient wind is present.

10.2 Earth Frame $\mathrm{F_E}$

The Earth frame is assumed to be an inertial frame. The origin $\mathrm{O_E}$ of the Earth frame is an arbitrary convenient point on the Earth. As shown in Figure 10.2.1, $\hat{\imath}_\mathrm{E}$ and $\hat{\jmath}_\mathrm{E}$ are horizontal, while $\hat{k}_\mathrm{E}$ points downward, that is, toward the center of the Earth. For convenience, we henceforth assume that the Earth is flat.

The acceleration due to gravity is the physical acceleration vector

$$\vec{g} = g\hat{k}_\mathrm{E}, \tag{10.2.1}$$

where $g = 9.8\ \mathrm{m/s^2} \approx 32.2\ \mathrm{ft/s^2}$. For a falling particle x that is unaffected by atmospheric forces,

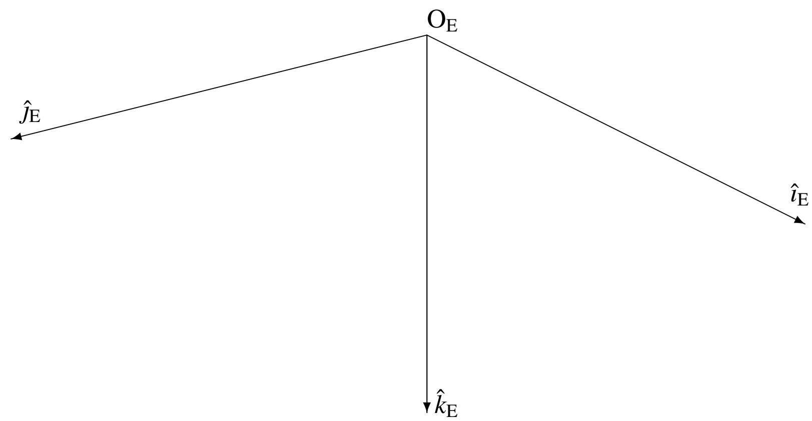


Figure 10.2.1: The Earth frame $\mathrm{F_E}$.

the acceleration of x relative to $\mathrm{O_E}$ with respect to $\mathrm{F_E}$ is given by

$$\vec{a}_{x/\mathrm{O_E}/\mathrm{E}} = \vec{g}. \tag{10.2.2}$$

10.3 Intermediate Earth Frames and Aircraft Frame $\mathrm{F_{AC}}$

The aircraft frame is fixed to the aircraft, and its origin is the center of mass c of the aircraft. We assume that the effect of gravity is uniform over the aircraft, and thus the center of mass of the aircraft coincides with the center of gravity of the aircraft. The aircraft frame origin c, along with the axes $\hat{\imath}_{\mathrm{AC}}$ and $\hat{k}_{\mathrm{AC}}$, are assumed to lie in the aircraft plane of symmetry, where the right wing lies on one side of the plane of symmetry and the left wing lies on the other side. The direction of $\hat{\imath}_{\mathrm{AC}}$ is typically, but not necessarily, through the nose of the aircraft, as shown in Figure 10.3.1.

The aircraft frame is related to the Earth frame by a 3-2-1 Euler-angle rotation sequence applied to the Earth frame. These operations yield two intermediate frames, namely, $\mathrm{F_{E'}}$ defined by the yaw rotation

$$\mathrm{F_{E'}} = \vec{R}_{\hat{k}_\mathrm{E}}(\Psi)\mathrm{F_E}, \tag{10.3.1}$$

$\mathrm{F_{E''}}$ defined by the pitch rotation

$$\mathrm{F_{E''}} = \vec{R}_{\hat{\jmath}_{\mathrm{E'}}}(\Theta)\mathrm{F_{E'}}, \tag{10.3.2}$$

and $\mathrm{F_{AC}}$ is achieved by the roll rotation

$$\mathrm{F_{AC}} = \vec{R}_{\hat{\imath}_{\mathrm{E''}}}(\Phi)\mathrm{F_{E''}}. \tag{10.3.3}$$

Therefore,

$$\mathrm{F_{AC}} = \vec{R}_{\hat{\imath}_{\mathrm{E''}}}(\Phi)\vec{R}_{\hat{\jmath}_{\mathrm{E'}}}(\Theta)\vec{R}_{\hat{k}_\mathrm{E}}(\Psi)\mathrm{F_E}, \tag{10.3.4}$$

and we can write

$$\mathrm{F_E} \xrightarrow[3]{\Psi} \mathrm{F_{E'}} \xrightarrow[2]{\Theta} \mathrm{F_{E''}} \xrightarrow[1]{\Phi} \mathrm{F_{AC}}. \tag{10.3.5}$$

Note that yaw rotation rotates $\mathrm{F_E}$ about the axis $\hat{k}_\mathrm{E}$ of $\mathrm{F_E}$, the pitch rotation rotates $\mathrm{F_{E'}}$ about the axis $\hat{\jmath}'_\mathrm{E}$ of $\mathrm{F_{E'}}$, and the roll rotation rotates $\mathrm{F_{E''}}$ about the axis $\hat{\imath}''_\mathrm{E}$ of $\mathrm{F_{E''}}$. The angles Ψ, Θ, and Φ are the signed angles given by

$$\Psi = \theta_{\hat{\imath}_{\mathrm{E}'}/\hat{\imath}_\mathrm{E}/\hat{k}_\mathrm{E}} = \theta_{\hat{\jmath}_{\mathrm{E}'}/\hat{\jmath}_\mathrm{E}/\hat{k}_\mathrm{E}}, \tag{10.3.6}$$

$$\Theta = \theta_{\hat{\imath}_{\mathrm{E}''}/\hat{\imath}_{\mathrm{E}'}/\hat{\jmath}_{\mathrm{E}'}} = \theta_{\hat{k}_{\mathrm{E}''}/\hat{k}_{\mathrm{E}'}/\hat{\jmath}_{\mathrm{E}'}}, \tag{10.3.7}$$

$$\Phi = \theta_{\hat{\jmath}_\mathrm{AC}/\hat{\jmath}_{\mathrm{E}''}/\hat{\imath}_{\mathrm{E}''}} = \theta_{\hat{k}_\mathrm{AC}/\hat{k}_{\mathrm{E}''}/\hat{\imath}_{\mathrm{E}''}}. \tag{10.3.8}$$

The transformation from $\mathrm{F_E}$ to $\mathrm{F_{AC}}$ thus involves two intermediate frames, namely, $\mathrm{F_{E'}}$ and $\mathrm{F_{E''}}$

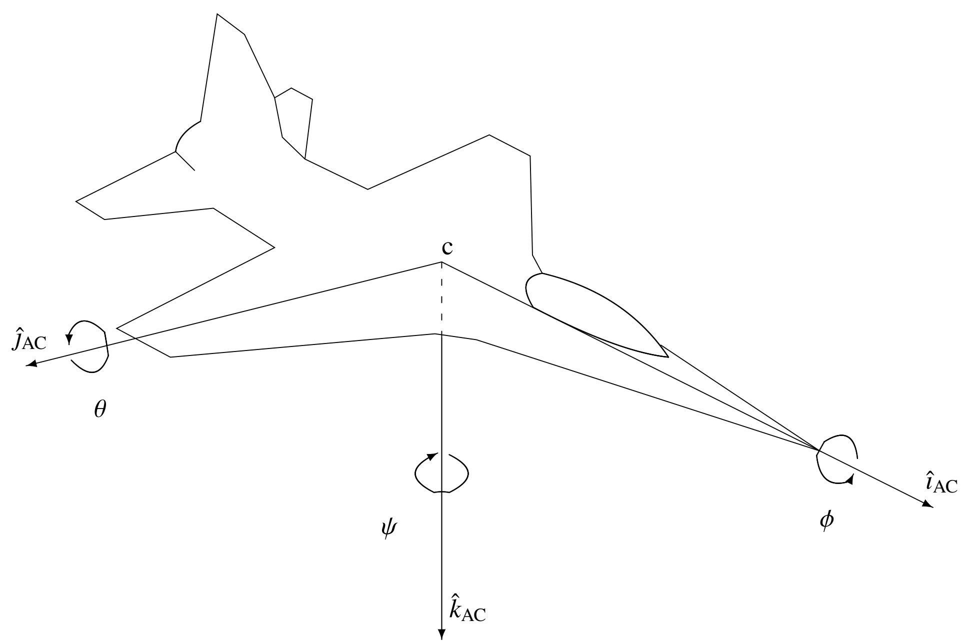


Figure 10.3.1: The aircraft frame $\mathrm{F_{AC}}$.

The orientation matrices corresponding to the physical rotation matrices $\vec{R}_{\hat{k}_\mathrm{E}}(\Psi)$, $\vec{R}_{\hat{\jmath}_{\mathrm{E}'}}(\Theta)$, and $\vec{R}_{\hat{\imath}_{\mathrm{E}''}}(\Phi)$, are given, respectively, by

$$\mathcal{O}_{\mathrm{E'/E}} = \mathcal{O}_3(\Psi) = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{10.3.9}$$

$$\mathcal{O}_{\mathrm{E''/E'}} = \mathcal{O}_2(\Theta) = \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix}, \tag{10.3.10}$$

$$\mathcal{O}_{\mathrm{AC/E''}} = \mathcal{O}_1(\Phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix}. \tag{10.3.11}$$

The vectrices of the four frames are related by

$$\begin{bmatrix} \hat{\imath}_{\mathrm{E'}} \\ \hat{\jmath}_{\mathrm{E'}} \\ \hat{k}_{\mathrm{E'}} \end{bmatrix} = \mathcal{O}_3(\Psi) \begin{bmatrix} \hat{\imath}_{\mathrm{E}} \\ \hat{\jmath}_{\mathrm{E}} \\ \hat{k}_{\mathrm{E}} \end{bmatrix}, \tag{10.3.12}$$

$$\begin{bmatrix} \hat{\imath}_{\mathrm{E''}} \\ \hat{\jmath}_{\mathrm{E''}} \\ \hat{k}_{\mathrm{E''}} \end{bmatrix} = \mathcal{O}_2(\Theta) \begin{bmatrix} \hat{\imath}_{\mathrm{E'}} \\ \hat{\jmath}_{\mathrm{E'}} \\ \hat{k}_{\mathrm{E'}} \end{bmatrix}, \tag{10.3.13}$$

$$\begin{bmatrix} \hat{\imath}_{\mathrm{AC}} \\ \hat{\jmath}_{\mathrm{AC}} \\ \hat{k}_{\mathrm{AC}} \end{bmatrix} = \mathcal{O}_1(\Phi) \begin{bmatrix} \hat{\imath}_{\mathrm{E''}} \\ \hat{\jmath}_{\mathrm{E''}} \\ \hat{k}_{\mathrm{E''}} \end{bmatrix}. \tag{10.3.14}$$

Combining (10.3.12), (10.3.13), and (10.3.14) yields

$$\begin{aligned} \begin{bmatrix} \hat{\imath}_{\mathrm{AC}} \\ \hat{\jmath}_{\mathrm{AC}} \\ \hat{k}_{\mathrm{AC}} \end{bmatrix} &= \mathcal{O}_{\mathrm{AC/E''}} \begin{bmatrix} \hat{\imath}_{\mathrm{E''}} \\ \hat{\jmath}_{\mathrm{E''}} \\ \hat{k}_{\mathrm{E''}} \end{bmatrix} \\ &= \mathcal{O}_{\mathrm{AC/E''}} \mathcal{O}_{\mathrm{E''/E'}} \begin{bmatrix} \hat{\imath}_{\mathrm{E'}} \\ \hat{\jmath}_{\mathrm{E'}} \\ \hat{k}_{\mathrm{E'}} \end{bmatrix} \\ &= \mathcal{O}_{\mathrm{AC/E''}} \mathcal{O}_{\mathrm{E''/E'}} \mathcal{O}_{\mathrm{E'/E}} \begin{bmatrix} \hat{\imath}_{\mathrm{E}} \\ \hat{\jmath}_{\mathrm{E}} \\ \hat{k}_{\mathrm{E}} \end{bmatrix} \\ &= \mathcal{O}_{\mathrm{AC/E}} \begin{bmatrix} \hat{\imath}_{\mathrm{E}} \\ \hat{\jmath}_{\mathrm{E}} \\ \hat{k}_{\mathrm{E}} \end{bmatrix}, \end{aligned} \tag{10.3.15}$$

where

$$\begin{aligned} \mathcal{O}_{\mathrm{AC/E}} &= \mathcal{O}_{\mathrm{AC/E''}} \mathcal{O}_{\mathrm{E''/E'}} \mathcal{O}_{\mathrm{E'/E}} = \mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (\cos\Theta)\cos\Psi & (\cos\Theta)\sin\Psi & -\sin\Theta \\ (\sin\Phi)(\sin\Theta)\cos\Psi - (\cos\Phi)\sin\Psi & (\sin\Phi)(\sin\Theta)\sin\Psi + (\cos\Phi)\cos\Psi & (\sin\Phi)\cos\Theta \\ (\cos\Phi)(\sin\Theta)\cos\Psi + (\sin\Phi)\sin\Psi & (\cos\Phi)(\sin\Theta)\sin\Psi - (\sin\Phi)\cos\Psi & (\cos\Phi)\cos\Theta \end{bmatrix}. \end{aligned} \tag{10.3.16}$$

Therefore,

$$\begin{aligned} \mathcal{O}_{\mathrm{E/AC}} &= \mathcal{O}_{\mathrm{E/E'}} \mathcal{O}_{\mathrm{E'/E''}} \mathcal{O}_{\mathrm{E''/AC}} = \mathcal{O}_3(-\Psi)\mathcal{O}_2(-\Theta)\mathcal{O}_1(-\Phi) \\ &= \begin{bmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} (\cos\Theta)\cos\Psi & (\sin\Phi)(\sin\Theta)\cos\Psi - (\cos\Phi)\sin\Psi & (\cos\Phi)(\sin\Theta)\cos\Psi + (\sin\Phi)\sin\Psi \\ (\cos\Theta)\sin\Psi & (\sin\Phi)(\sin\Theta)\sin\Psi + (\cos\Phi)\cos\Psi & (\cos\Phi)(\sin\Theta)\sin\Psi - (\sin\Phi)\cos\Psi \\ -\sin\Theta & (\sin\Phi)\cos\Theta & (\cos\Phi)\cos\Theta \end{bmatrix}. \quad (10.3.17)$$

10.4 Stability Frame $\mathrm{F_S}$

Let

$$\vec{r}_{\mathrm{AC/E}} \triangleq \vec{r}_{\mathrm{c/O_E}} \quad (10.4.1)$$

define the location of the aircraft center of mass relative to the origin of $\mathrm{F_E}$, and define the aircraft velocity vector $\vec{V}_{\mathrm{AC}}$ by

$$\vec{V}_{\mathrm{AC}} \triangleq \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{AC}}. \quad (10.4.2)$$

Next, define the projected velocity vector $\vec{V}_{\mathrm{AC,proj}}$, which is the projection of $\vec{V}_{\mathrm{AC}}$ onto the plane spanned by $\hat{\imath}_{\mathrm{AC}}$ and $\hat{k}_{\mathrm{AC}}$, that is,

$$\vec{V}_{\mathrm{AC,proj}} \triangleq \vec{P}_{\hat{\imath}_{\mathrm{AC}},\hat{k}_{\mathrm{AC}}} \vec{V}_{\mathrm{AC}}. \quad (10.4.3)$$

Since the plane spanned by $\hat{\imath}_{\mathrm{AC}}$ and $\hat{k}_{\mathrm{AC}}$ is the plane of symmetry of the aircraft, it follows that $\vec{V}_{\mathrm{AC,proj}}$ is independent of the direction of $\hat{\imath}_{\mathrm{AC}}$ relative to the nose of the aircraft.

Next, as shown in Figure 10.4.1 and assuming that $\vec{V}_{\mathrm{AC,proj}}$ is nonzero, define $\hat{\imath}_{\mathrm{S}}$ to be the unit vector aligned along $\vec{V}_{\mathrm{AC,proj}}$, that is,

$$\hat{\imath}_{\mathrm{S}} \triangleq \hat{V}_{\mathrm{AC,proj}}. \quad (10.4.4)$$

The *angle of attack* $\alpha \in (-\pi, \pi]$ is the signed angle from $\hat{\imath}_{\mathrm{S}}$ to $\hat{\imath}_{\mathrm{AC}}$ about $\hat{\jmath}_{\mathrm{AC}}$, that is,

$$\alpha \triangleq \theta_{\hat{\imath}_{\mathrm{AC}}/\hat{\imath}_{\mathrm{S}}/\hat{\jmath}_{\mathrm{AC}}} = \theta_{\hat{\imath}_{\mathrm{AC}}/\vec{V}_{\mathrm{AC,proj}}/\hat{\jmath}_{\mathrm{AC}}} = \theta_{\hat{k}_{\mathrm{AC}}/\hat{k}_{\mathrm{S}}/\hat{\jmath}_{\mathrm{AC}}}. \quad (10.4.5)$$

The *stability frame* $\mathrm{F_S}$ is defined by rotating the aircraft frame by the angle $-\alpha$ about $\hat{\jmath}_{\mathrm{AC}}$, that is,

$$\mathrm{F_S} \triangleq \vec{R}_{\hat{\jmath}_{\mathrm{AC}}}(-\alpha)\mathrm{F_{AC}}, \quad (10.4.6)$$

and thus

$$\vec{R}_{\mathrm{S/AC}} = \vec{R}_{\hat{\jmath}_{\mathrm{AC}}}(-\alpha), \quad (10.4.7)$$

$$\begin{bmatrix} \hat{\imath}_{\mathrm{S}} \\ \hat{\jmath}_{\mathrm{S}} \\ \hat{k}_{\mathrm{S}} \end{bmatrix} = \mathcal{O}_{\mathrm{S/AC}} \begin{bmatrix} \hat{\imath}_{\mathrm{AC}} \\ \hat{\jmath}_{\mathrm{AC}} \\ \hat{k}_{\mathrm{AC}} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\mathrm{AC}} \\ \hat{\jmath}_{\mathrm{AC}} \\ \hat{k}_{\mathrm{AC}} \end{bmatrix}. \quad (10.4.8)$$

Hence,

$$\mathrm{F_{AC}} = \vec{R}_{\hat{\jmath}_{\mathrm{AC}}}(\alpha)\mathrm{F_S}, \quad (10.4.9)$$

and thus

$$\vec{R}_{\mathrm{AC/S}} = \vec{R}_{\hat{\jmath}_{\mathrm{AC}}}(\alpha), \quad (10.4.10)$$

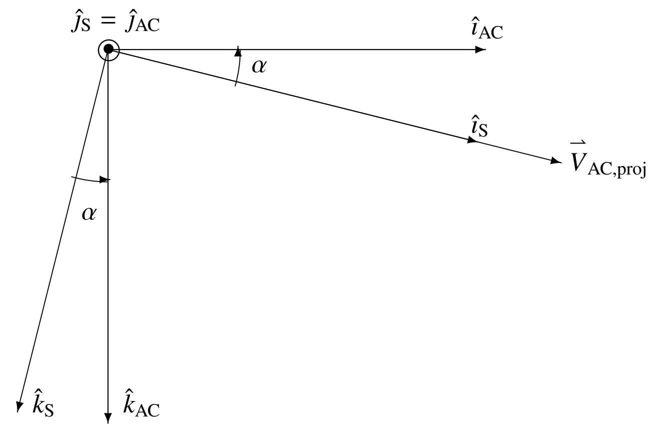


Figure 10.4.1: Definition of the angle of attack α, which is the angle of rotation about $\hat{\jmath}_S = \hat{\jmath}_{AC}$ from the stability frame F_S to the aircraft frame F_{AC}.

$$\begin{bmatrix} \hat{\imath}_{AC} \\ \hat{\jmath}_{AC} \\ \hat{k}_{AC} \end{bmatrix} = \mathcal{O}_{AC/S} \begin{bmatrix} \hat{\imath}_S \\ \hat{\jmath}_S \\ \hat{k}_S \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \hat{\imath}_S \\ \hat{\jmath}_S \\ \hat{k}_S \end{bmatrix}. \tag{10.4.11}$$

Note that

$$\mathcal{O}_{AC/S} = \mathcal{O}_2(\alpha) = \mathcal{O}_2^{T}(-\alpha) = \mathcal{O}_{S/AC}^{T}. \tag{10.4.12}$$

The case where $\vec{V}_{AC,proj}$ is zero must be handled separately since α is not defined by (10.4.5). In this case, we define α to be zero and $F_S \triangleq F_{AC}$. However, this definition causes discontinuities, which are unavoidable.

The stability frame is a velocity-dependent frame since its axes depend on the aircraft velocity vector $\vec{V}_{AC}$.

10.5 Wind Frame F_W

Assuming that $\vec{V}_{AC}$ is nonzero, we define $\hat{\imath}_W$ to be the unit vector that points along $\vec{V}_{AC}$ as shown in Figure 10.5.1, that is,

$$\hat{\imath}_W \triangleq \hat{V}_{AC}. \tag{10.5.1}$$

The *slideslip angle* $\beta \in (-\pi, \pi]$ is the signed angle from $\hat{\imath}_S$ to $\hat{\imath}_W$ about $\hat{k}_S$, that is,

$$\beta \triangleq \theta_{\hat{\imath}_W/\hat{\imath}_S/\hat{k}_S} = \theta_{\hat{\jmath}_W/\hat{\jmath}_S/\hat{k}_S}. \tag{10.5.2}$$

Therefore, the relative wind is in the pilot's right ear when β is positive.

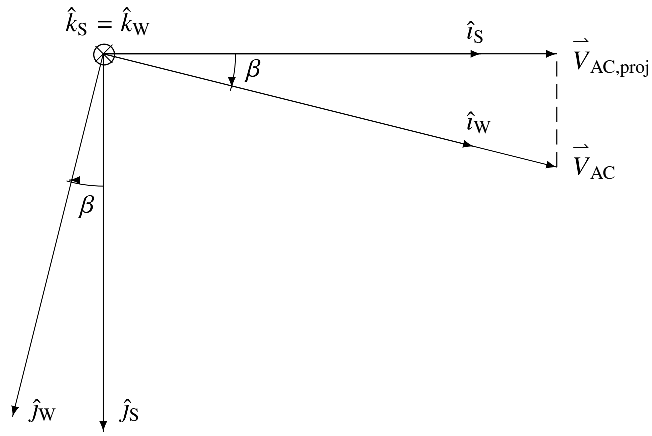


Figure 10.5.1: Definition of the sideslip angle β, which is the angle of rotation about $\hat{k}_S = \hat{k}_W$ from the stability frame F_S to the wind frame F_W.

The *wind frame* F_W is defined by rotating the stability frame by the angle β about $\hat{k}_S = \hat{k}_W$, that is,

$$F_W \triangleq \vec{R}_{\hat{k}_S}(\beta)F_S, \tag{10.5.3}$$

and thus

$$\vec{R}_{W/S} = \vec{R}_{\hat{k}_{AC}}(\beta), \tag{10.5.4}$$

$$\begin{bmatrix} \hat{\imath}_W \\ \hat{\jmath}_W \\ \hat{k}_W \end{bmatrix} = \mathcal{O}_{W/S} \begin{bmatrix} \hat{\imath}_S \\ \hat{\jmath}_S \\ \hat{k}_S \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\imath}_S \\ \hat{\jmath}_S \\ \hat{k}_S \end{bmatrix}. \tag{10.5.5}$$

Hence,

$$F_S = \vec{R}_{\hat{k}_S}(-\beta)F_W, \tag{10.5.6}$$

and thus

$$\vec{R}_{S/W} = \vec{R}_{\hat{k}_S}(-\beta), \tag{10.5.7}$$

$$\begin{bmatrix} \hat{\imath}_S \\ \hat{\jmath}_S \\ \hat{k}_S \end{bmatrix} = \mathcal{O}_{S/W} \begin{bmatrix} \hat{\imath}_W \\ \hat{\jmath}_W \\ \hat{k}_W \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\imath}_W \\ \hat{\jmath}_W \\ \hat{k}_W \end{bmatrix}. \tag{10.5.8}$$

Furthermore,

$$\mathcal{O}_{W/S} = \mathcal{O}_3(\beta) = \mathcal{O}_3^{T}(-\beta) = \mathcal{O}_{S/W}^{T}. \tag{10.5.9}$$

Consequently, the Earth frame is transformed to the wind frame by the 3-2-1-2-3 rotation sequence Ψ-Θ-Φ-$(-\alpha)$-β given by

$$\mathrm{F_E} \xrightarrow[3]{\Psi} \mathrm{F_{E'}} \xrightarrow[2]{\Theta} \mathrm{F_{E''}} \xrightarrow[1]{\Phi} \mathrm{F_{AC}} \xrightarrow[2]{-\alpha} \mathrm{F_S} \xrightarrow[3]{\beta} \mathrm{F_W}. \tag{10.5.10}$$

The wind frame is a velocity-dependent frame since its axes depend on the velocity of the aircraft.

10.6 Aircraft Velocity Vector

We resolve $\vec{V}_{\rm AC}$ in the aircraft frame as

$$\vec{V}_{\rm AC} = U\hat{\imath}_{\rm AC} + V\hat{\jmath}_{\rm AC} + W\hat{k}_{\rm AC},$$

and thus

$$\vec{V}_{\rm AC}\Big|_{\rm AC} = \begin{bmatrix} U \\ V \\ W \end{bmatrix}. \tag{10.6.1}$$

where U, V, and W are the range rate, drift rate, and plunge rate, respectively. It thus follows from (10.4.3) that

$$\vec{V}_{\rm AC,proj} = (\hat{\imath}_{\rm AC}\hat{\imath}'_{\rm AC} + \hat{k}_{\rm AC}\hat{k}'_{\rm AC})\vec{V}_{\rm AC} = U\hat{\imath}_{\rm AC} + W\hat{k}_{\rm AC}. \tag{10.6.2}$$

Hence,

$$\vec{V}_{\rm AC} = \vec{V}_{\rm AC,proj} + V\hat{\jmath}_{\rm AC}. \tag{10.6.3}$$

Next, define

$$\overline{U} \triangleq |\vec{V}_{\rm AC,proj}| = \sqrt{U^2 + W^2}. \tag{10.6.4}$$

Since $\hat{V}_{\rm AC,proj} = \hat{\imath}_{\rm S}$, it follows that

$$\vec{V}_{\rm AC,proj} = \overline{U}\hat{\imath}_{\rm S} = \overline{U}\hat{V}_{\rm AC,proj}. \tag{10.6.5}$$

Next, note that

$$\begin{aligned}
\hat{k}_{\rm S} \cdot \vec{V}_{\rm AC} &= \hat{k}_{\rm S} \cdot (\vec{V}_{\rm AC,proj} + V\hat{\jmath}_{\rm AC}) \\
&= \hat{k}_{\rm S} \cdot (\overline{U}\hat{\imath}_{\rm S} + V\hat{\jmath}_{\rm AC}) \\
&= V\hat{k}_{\rm S} \cdot \hat{\jmath}_{\rm AC} \\
&= V[-(\sin\alpha)\hat{\imath}_{\rm AC} + (\cos\alpha)\hat{k}_{\rm AC}] \cdot \hat{\jmath}_{\rm AC} \\
&= 0.
\end{aligned} \tag{10.6.6}$$

We thus have

$$\vec{V}_{\rm AC} = \overline{U}\hat{\imath}_{\rm S} + \overline{V}\hat{\jmath}_{\rm S},$$

where

$$\overline{V} \triangleq \hat{\jmath}_{\rm S} \cdot \vec{V}_{\rm AC} = \hat{\jmath}_{\rm AC} \cdot \vec{V}_{\rm AC} = V. \tag{10.6.7}$$

Therefore,

$$\vec{V}_{\mathrm{AC}}\Big|_{\mathrm{S}} = \begin{bmatrix} \overline{U} \\ \overline{V} \\ 0 \end{bmatrix}. \tag{10.6.8}$$

Next, we have

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathcal{O}_{\mathrm{AC/S}} \begin{bmatrix} \overline{U} \\ \overline{V} \\ 0 \end{bmatrix} = \mathcal{O}_2(\alpha) \begin{bmatrix} \overline{U} \\ \overline{V} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \overline{U} \\ \overline{V} \\ 0 \end{bmatrix}, \tag{10.6.9}$$

and thus

$$U = (\cos\alpha)\overline{U}, \tag{10.6.10}$$

$$V = \overline{V}, \tag{10.6.11}$$

$$W = (\sin\alpha)\overline{U}. \tag{10.6.12}$$

Hence,

$$\sin\alpha = \frac{W}{\sqrt{U^2 + W^2}}, \tag{10.6.13}$$

$$\cos\alpha = \frac{U}{\sqrt{U^2 + W^2}}, \tag{10.6.14}$$

$$\tan\alpha = \frac{W}{U}. \tag{10.6.15}$$

The reverse of (10.6.9) is

$$\begin{bmatrix} \overline{U} \\ \overline{V} \\ 0 \end{bmatrix} = \mathcal{O}_{\mathrm{S/AC}} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathcal{O}_2(-\alpha) \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}. \tag{10.6.16}$$

Therefore,

$$\overline{U} = (\cos\alpha)U + (\sin\alpha)W. \tag{10.6.17}$$

Finally, it follows from (10.4.5) and (2.3.8) that

$$\alpha = \theta_{\hat{\imath}_{\mathrm{AC}}/\vec{V}_{\mathrm{AC,proj}}/\hat{\jmath}_{\mathrm{AC}}} = -\theta_{\vec{V}_{\mathrm{AC,proj}}/\hat{\imath}_{\mathrm{AC}}/\hat{\jmath}_{\mathrm{AC}}} = -\mathrm{atan2}(-W, U) = \mathrm{atan2}(W, U), \tag{10.6.18}$$

which implies (10.6.15).

Now, consider the wind frame $\mathrm{F_W}$, and (10.5.1) implies that

$$\vec{V}_{\mathrm{AC}} = V_{\mathrm{AC}}\hat{\imath}_{\mathrm{W}} = V_{\mathrm{AC}}\hat{V}_{\mathrm{AC}}, \tag{10.6.19}$$

where the *airspeed* $V_{\mathrm{AC}} \geq 0$ is given by

$$V_{\mathrm{AC}} \triangleq |\vec{V}_{\mathrm{AC}}| = \sqrt{\overline{U}^2 + \overline{V}^2} = \sqrt{U^2 + V^2 + W^2}. \tag{10.6.20}$$

From (10.5.10) it follows that

$$\vec{V}_{\mathrm{AC}}\Big|_{\mathrm{W}} = \mathcal{O}_{\mathrm{W/S}}\, \vec{V}_{\mathrm{AC}}\Big|_{\mathrm{S}} = \mathcal{O}_3(\beta)\, \vec{V}_{\mathrm{AC}}\Big|_{\mathrm{S}},$$

and thus

$$\begin{bmatrix} \overline{U} \\ \overline{V} \\ 0 \end{bmatrix} = \vec{V}_{\rm AC}\Big|_{\rm S} = \mathcal{O}_{\rm S/W}\, \vec{V}_{\rm AC}\Big|_{\rm W} = \mathcal{O}_3(-\beta)\, \vec{V}_{\rm AC}\Big|_{\rm W} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{\rm AC} \\ 0 \\ 0 \end{bmatrix}. \tag{10.6.21}$$

Therefore,

$$\overline{U} = (\cos\beta)V_{\rm AC}, \tag{10.6.22}$$

$$\overline{V} = V = (\sin\beta)V_{\rm AC}, \tag{10.6.23}$$

which implies

$$\sin\beta = \frac{V}{V_{\rm AC}} = \frac{V}{\sqrt{\overline{U}^2 + \overline{V}^2}} = \frac{V}{\sqrt{U^2 + V^2 + W^2}}, \tag{10.6.24}$$

$$\cos\beta = \frac{\overline{U}}{V_{\rm AC}} = \frac{\overline{U}}{\sqrt{\overline{U}^2 + \overline{V}^2}} = \frac{\sqrt{U^2 + W^2}}{\sqrt{U^2 + V^2 + W^2}}, \tag{10.6.25}$$

$$\tan\beta = \frac{V}{\overline{U}} = \frac{V}{\sqrt{U^2 + W^2}}. \tag{10.6.26}$$

Finally, it follows from (10.6.26) that

$$\beta = \theta_{\hat\imath_{\rm W}/\hat\imath_{\rm S}/\hat k_{\rm S}} = {\rm atan2}(V, \overline{U}) = {\rm atan2}(V, \sqrt{U^2 + W^2}). \tag{10.6.27}$$

Combining (10.6.9) and (10.6.21) yields

$$\begin{aligned}
\begin{bmatrix} U \\ V \\ W \end{bmatrix} &= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{\rm AC} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (\cos\alpha)\cos\beta & -(\cos\alpha)\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ (\sin\alpha)\cos\beta & -(\sin\alpha)\sin\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} V_{\rm AC} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (\cos\alpha)(\cos\beta)V_{\rm AC} \\ (\sin\beta)V_{\rm AC} \\ (\sin\alpha)(\cos\beta)V_{\rm AC} \end{bmatrix}.
\end{aligned} \tag{10.6.28}$$

10.7 Range, Drift, Plunge, and Altitude

The position of the aircraft relative to the origin of the Earth frame is given by

$$\vec{r}_{\rm AC} = \vec{r}_{\rm c/O_E} = X\hat\imath_{\rm E} + Y\hat\jmath_{\rm E} + Z\hat k_{\rm E}, \tag{10.7.1}$$

that is,

$$\vec{r}_{\rm AC}\Big|_{\rm E} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{10.7.2}$$

where X, Y, and Z are the range, drift, and plunge, respectively. The altitude H is defined by $H = -Z$. Hence, the aircraft velocity vector $\vec{V}_{\rm AC}$ is given by

$$\vec{V}_{\rm AC} = \overset{\rm E\bullet}{\vec{r}}_{\rm AC} = \dot{X}\hat{\imath}_{\rm E} + \dot{Y}\hat{\jmath}_{\rm E} + \dot{Z}\hat{k}_{\rm E}. \tag{10.7.3}$$

Therefore,

$$\vec{V}_{\rm AC}\Big|_{\rm E} = \mathcal{O}_{\rm E/AC}\,\vec{V}_{\rm AC}\Big|_{\rm AC}, \tag{10.7.4}$$

it follows from (10.6.1) that

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \mathcal{O}_{\rm E/AC}\begin{bmatrix} U \\ V \\ W \end{bmatrix}, \tag{10.7.5}$$

where $\dot{X}$, $\dot{Y}$, and $\dot{Z}$ are the range rate, drift rate, and plunge rate, respectively.

10.8 Heading Angle, Flight-Path Angle, and Bank Angle

The *heading angle* η is defined to be the signed angle from $\hat{\imath}_{\rm E}$ to the projection of $\vec{V}_{\rm AC}$ onto the horizontal plane about $\hat{k}_{\rm E}$. This rotation defines the frame $\rm F_F \triangleq \vec{R}_{F/E}F_E$. to $\hat{\imath}_{\rm W}$ about $\hat{\jmath}_{\rm F}$. This rotation defines the frame $\rm F_G \triangleq \vec{R}_{G/F}F_F$. The 3-2-1-3-2 Euler-angle rotation sequence η-γ-μ-$(-\beta)$-α from $\rm F_E$ to $\rm F_W$ is thus given by

$$\rm F_E \xrightarrow[3]{\eta} F_F \xrightarrow[2]{\gamma} F_G \xrightarrow[1]{\mu} F_W \xrightarrow[3]{-\beta} F_S \xrightarrow[2]{\alpha} F_{AC}, \tag{10.8.1}$$

where the signed angle μ is the *bank angle*. The first two rotations align $\hat{\imath}_{\rm E}$ with $\hat{\imath}_{\rm W}$, while the third rotation rotates $\rm F_G$ about $\hat{\imath}_{\rm G} = \hat{\imath}_{\rm W} = \hat{V}_{\rm AC}$ by the bank angle μ in order to arrive at $\rm F_W$. The η-γ-μ-$(-\beta)$-α Euler-angle rotation sequence from $\rm F_E$ to $\rm F_W$ can be contrasted with the 3-2-1-2-3 rotation sequence Ψ-Θ-Φ-$(-\alpha)$-β given by (10.5.10), where

$$\rm F_E \xrightarrow[3]{\Psi} F_{E'} \xrightarrow[2]{\Theta} F_{E''} \xrightarrow[1]{\Phi} F_{AC} \xrightarrow[2]{-\alpha} F_S \xrightarrow[3]{\beta} F_W. \tag{10.8.2}$$

Together, (10.8.1) and (10.8.2) comprise eight rotation angles and eight frames. Merging these sequences yields

$$\rm F_E \xrightarrow[3]{\Psi} F_{E'} \xrightarrow[2]{\Theta} F_{E''} \xrightarrow[1]{\Phi} F_{AC} \xrightarrow[2]{-\alpha} F_S \xrightarrow[3]{\beta} F_W \xrightarrow[1]{-\mu} F_G \xrightarrow[2]{-\gamma} F_F \xrightarrow[3]{-\eta} F_E. \tag{10.8.3}$$

Comparing (10.8.1) and (10.8.2), it follows that

$$\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi) = \mathcal{O}_1(\mu)\mathcal{O}_2(\gamma)\mathcal{O}_3(\eta). \tag{10.8.4}$$

Equivalently,

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi-\eta) = I. \tag{10.8.5}$$

By setting certain angles in (10.8.5) to zero, rotations about the same axis become adjacent and thus can be combined into a single rotation, thereby reducing the number of factors. The simplest choice is to set $\Phi \equiv 0$ (wings level flight) in which case (10.8.5) can be written as

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(\Theta-\alpha)\mathcal{O}_3(\Psi-\eta) = I. \tag{10.8.6}$$

Since (10.8.6) involves a product of five Euler rotation matrices, setting one of these to zero yields a product of four Euler rotation matrices, which is amenable to either Fact 2.13.11, Fact 2.13.13, or Fact 2.13.15. This can be done in five different ways. In fact, setting either $\Theta \equiv \alpha$ or $\Psi \equiv \eta$ yields a product of three Euler rotation matrices. These two cases are considered by the next two results.

Fact 10.8.1. Assume that $\Phi \equiv 0$ and $\Theta \equiv \alpha$. Then, $\Psi, \beta, \eta, \gamma, \mu$ satisfy (10.8.5) if and only if either $\gamma \equiv \mu \equiv \beta + \Psi - \eta \equiv 0$ or $\gamma \equiv \mu \equiv \beta + \Psi - \eta \equiv \pi$.

Proof. The result follows by applying Proposition 3 to

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta + \Psi - \eta) = I. \qquad \square$$

Fact 10.8.2. Assume that $\Phi \equiv 0$ and $\Psi \equiv \eta$. Then, $\Theta, \alpha, \beta, \gamma, \mu$ satisfy (10.8.5) if and only if either $\mu \equiv \beta \equiv \Theta - \alpha - \gamma \equiv 0$ or $\mu \equiv \beta \equiv \Theta - \alpha - \gamma \equiv \pi$.

Proof. The result follows by applying Proposition 3 to

$$\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(\Theta - \alpha - \gamma) = I. \qquad \square$$

Combining Fact 10.8.1 and Fact 10.8.2 yields the following result.

Fact 10.8.3. Assume that $\Phi \equiv 0$, $\Theta \equiv \alpha$, and $\Psi \equiv \eta$. Then, β, γ, μ satisfy (10.8.5) if and only if either $\beta \equiv \gamma \equiv \mu \equiv 0$ or $\beta \equiv \gamma \equiv \mu \equiv \pi$.

The next result is a partial converse of Fact 10.8.3.

Fact 10.8.4. Assume that either $\beta \equiv \gamma \equiv \mu \equiv 0$ or $\beta \equiv \gamma \equiv \mu \equiv \pi$. Then, $\Psi, \Theta, \Phi, \alpha, \eta$ satisfy (10.8.5) if and only if *i*) $\Phi \equiv \Psi - \eta \equiv 0$ and $\alpha \equiv \Theta$, *ii*) $\Phi \equiv \Psi - \eta \equiv \pi$ and $-\alpha \equiv \Theta + \pi$, *iii*) $\alpha \equiv \Theta \equiv \pi/2$ and $\Phi \equiv \eta - \Psi$, or *iv*) $\alpha \equiv \Theta \equiv -\pi/2$ and $\Phi \equiv \Psi - \eta$.

Proof. The result follows by applying Fact 2.13.15 to

$$\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi - \eta) = I. \qquad \square$$

The following result considers wings-level, zero-sideslip flight, which is considered in deriving the equations of linearized flight.

Fact 10.8.5. Assume that $\Phi \equiv \beta \equiv 0$. Then, $\Psi, \Theta, \alpha, \eta, \gamma, \mu$ satisfy (10.8.5) if and only if *i*) $\mu \equiv \Psi - \eta \equiv 0$ and $\gamma \equiv \Theta - \alpha$, *ii*) $-\mu \equiv \Psi - \eta \equiv \pi$ and $-\gamma \equiv \Theta - \alpha + \pi$, *iii*) $\gamma \equiv \Theta - \alpha \equiv \pi/2$ and $\mu \equiv \Psi - \eta$, or *iv*) $\gamma \equiv \Theta - \alpha \equiv -\pi/2$ and $-\mu \equiv \Psi - \eta$.

Proof. The result follows by applying Proposition 5 to

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_2(\Theta - \alpha)\mathcal{O}_3(\Psi - \eta) = I. \qquad \square$$

Note that, in case *i*), the bank angle is zero, the yaw angle is equal to the heading angle, and $\Theta = \alpha + \gamma$. The condition $\Theta = \alpha + \gamma$ is illustrated in Figure 10.8.1 and Figure 10.8.2.

The next three results follow from Fact 2.13.13.

Fact 10.8.6. Assume that $\Phi = \mu = 0$. Then, $\Psi, \Theta, \alpha, \beta, \gamma, \eta$ satisfy

$$\mathcal{O}_2(-\gamma)\mathcal{O}_3(\beta)\mathcal{O}_2(\Theta - \alpha)\mathcal{O}_3(\Psi - \eta) = I, \tag{10.8.7}$$

if and only if one of the following conditions is satisfied:

i) $\beta = 0$, $\Psi = \eta$, and $\Theta = \alpha + \gamma$.

ii) $\gamma = 0$, $\Theta = \alpha$, and $\Psi = \eta - \beta$.

iii) $\beta = \pi$, $\Psi = \eta + \pi$, and $\Theta = \alpha - \gamma$.

iv) $\gamma = \pi$, $\Theta = \alpha + \pi$, and $\Psi = \eta + \beta$.

Fact 10.8.7. Assume that $\Theta = \alpha = \gamma = 0$. Then, $\Psi, \Phi, \beta, \eta, \mu$ satisfy

$$\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_1(\Phi)\mathcal{O}_3(\Psi - \eta) = I, \tag{10.8.8}$$

if and only if one of the following conditions is satisfied:

i) $\beta = 0$, $\Psi = \eta$, and $\Phi = \mu$.

ii) $\Phi = \mu = 0$ and $\Psi = \eta - \beta$.

iii) $\beta = \pi$, $\Psi = \eta + \pi$, and $\Phi = -\mu$.

iv) $\Phi = \mu = \pi$ and $\Psi = \eta + \beta$.

Fact 10.8.8. Assume that $\beta = 0$ and $\Psi = \eta$. Then, $\Theta, \Phi, \alpha, \gamma, \mu$ satisfy

$$\mathcal{O}_1(-\mu)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta - \gamma) = I, \tag{10.8.9}$$

if and only if one of the following conditions is satisfied:

i) $\alpha = 0$, $\Theta = \gamma$, and $\Phi = \mu$.

ii) $\Phi = \mu = 0$ and $\Theta = \alpha + \gamma$.

iii) $\alpha = \pi$, $\Theta = \gamma + \pi$, and $\Phi = -\mu$.

iv) $\Phi = \mu = \pi$ and $\Theta = \gamma - \alpha$.

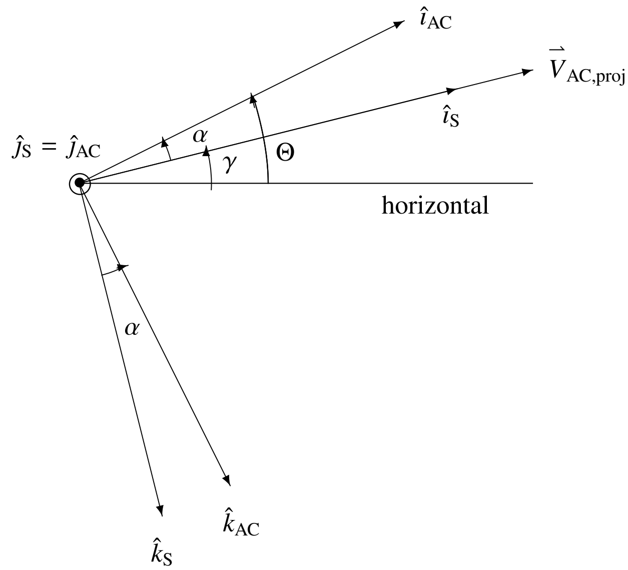


Figure 10.8.1: Flight path angle γ and pitch angle Θ. In this configuration, the angle of attack α, which is the angle from $\vec{V}_{\rm AC,proj}$ to $\hat{\imath}_{\rm AC}$, is positive. Note that $\Theta = \alpha + \gamma$.

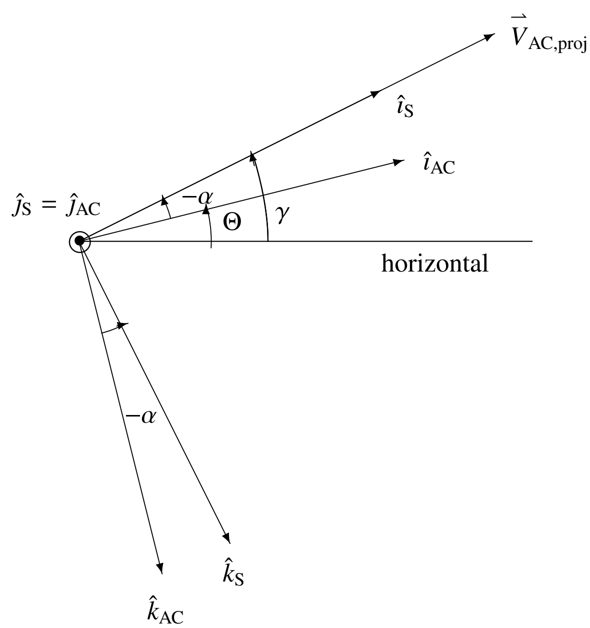


Figure 10.8.2: Flight path angle γ and pitch angle Θ. In this configuration, the angle of attack α is negative, and thus $-\alpha$, which is the angle from $\hat{\imath}_{\rm AC}$ to $\vec{V}_{\rm AC,proj}$, is positive. Note that $\Theta = \alpha + \gamma$.

10.9 Angular Velocity

The angular velocity of the aircraft relative to the Earth frame is given by $\vec{\omega}_{\mathrm{AC/E}}$. Resolving this vector in the aircraft frame, we define the notation

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} \triangleq \omega_{\mathrm{AC/E}} \triangleq \vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}}. \tag{10.9.1}$$

The angular velocity vector $\vec{\omega}_{\mathrm{AC/E}}$ can be related to the derivatives of the Euler angles. For 3-2-1 (yaw-pitch-roll) Euler angles Ψ, Θ, Φ (see (10.3.5)), it follows that have

$$\vec{\omega}_{\mathrm{AC/E}} = \vec{\omega}_{\mathrm{AC/E''}} + \vec{\omega}_{\mathrm{E''/E'}} + \vec{\omega}_{\mathrm{E'/E}} \tag{10.9.2}$$

$$= \dot{\Phi}\hat{\imath}_{\mathrm{AC}} + \dot{\Theta}\hat{\jmath}_{\mathrm{E''}} + \dot{\Psi}\hat{k}_{\mathrm{E'}}. \tag{10.9.3}$$

Since $\hat{\imath}_{\mathrm{AC}} = \hat{\imath}_{\mathrm{E''}}$, $\hat{\jmath}_{\mathrm{E''}} = \hat{\jmath}_{\mathrm{E'}}$, and $\hat{k}_{\mathrm{E'}} = \hat{k}_{\mathrm{E}}$, resolving $\vec{\omega}_{\mathrm{AC/E}}$ in $\mathrm{F_{AC}}$ yields

$$\vec{\omega}_{\mathrm{AC/E}} = \dot{\Phi}\hat{\imath}_{\mathrm{AC}} + \dot{\Theta}\hat{\jmath}_{\mathrm{E''}} + \dot{\Psi}\hat{k}_{\mathrm{E'}} \tag{10.9.4}$$

$$= \dot{\Phi}\hat{\imath}_{\mathrm{AC}} + \dot{\Theta}[(\cos\Phi)\hat{\jmath}_{\mathrm{AC}} - (\sin\Phi)\hat{k}_{\mathrm{AC}}] + \dot{\Psi}[(\cos\Theta)\hat{k}_{\mathrm{E''}} - (\sin\Theta)\hat{\imath}_{\mathrm{E''}}] \tag{10.9.5}$$

$$= \dot{\Phi}\hat{\imath}_{\mathrm{AC}} + \dot{\Theta}(\cos\Phi)\hat{\jmath}_{\mathrm{AC}} - \dot{\Theta}(\sin\Phi)\hat{k}_{\mathrm{AC}} + \dot{\Psi}(\cos\Theta)[(\cos\Phi)\hat{k}_{\mathrm{AC}} + (\sin\Phi)\hat{\jmath}_{\mathrm{AC}}] - \dot{\Psi}(\sin\Theta)\hat{\imath}_{\mathrm{AC}} \tag{10.9.6}$$

$$= [-\dot{\Psi}(\sin\Theta) + \dot{\Phi}]\hat{\imath}_{\mathrm{AC}} + [\dot{\Psi}(\sin\Phi)\cos\Theta + \dot{\Theta}\cos\Phi]\hat{\jmath}_{\mathrm{AC}} + [\dot{\Psi}(\cos\Phi)\cos\Theta - \dot{\Theta}(\sin\Phi)]\hat{k}_{\mathrm{AC}}. \tag{10.9.7}$$

It thus follows that

$$\begin{aligned} \omega_{\mathrm{AC/E}} &= (\vec{\omega}_{\mathrm{AC/E''}} + \vec{\omega}_{\mathrm{E''/E'}} + \vec{\omega}_{\mathrm{E'/E}})\Big|_{\mathrm{AC}} \\ &= (\dot{\Phi}\hat{\imath}_{\mathrm{AC}} + \dot{\Theta}\hat{\jmath}_{\mathrm{E''}} + \dot{\Psi}\hat{k}_{\mathrm{E'}})\Big|_{\mathrm{AC}} \\ &= \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\cos\Theta)\sin\Phi \\ 0 & -\sin\Phi & (\cos\Theta)\cos\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}. \end{aligned} \tag{10.9.8}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\cos\Theta)\sin\Phi \\ 0 & -\sin\Phi & (\cos\Theta)\cos\Phi \end{bmatrix}^{-1} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \\ &= \begin{bmatrix} 1 & (\sin\Phi)\tan\Theta & (\cos\Phi)\tan\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & (\sin\Phi)\sec\Theta & (\cos\Phi)\sec\Theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \end{aligned} \tag{10.9.9}$$

which can be integrated to provide the 3-2-1 sequence of Euler angles from $\mathrm{F_E}$ to $\mathrm{F_{AC}}$. Note that the inverse can fail to exist for $\Theta = \pm\frac{\pi}{2}$ due to gimbal lock.

For the 3-2-1-3-2 Euler-angle rotation sequence η-γ-μ-$(-\beta)$-α defined in (10.8.1), it follows that have

$$\vec{\omega}_{\mathrm{AC/E}} = \vec{\omega}_{\mathrm{AC/S}} + \vec{\omega}_{\mathrm{S/W}} + \vec{\omega}_{\mathrm{W/G}} + \vec{\omega}_{\mathrm{G/F}} + \vec{\omega}_{\mathrm{F/E}} \tag{10.9.10}$$

$$= \dot{\alpha}\hat{\jmath}_{\mathrm{AC}} - \dot{\beta}\hat{k}_{\mathrm{S}} + \dot{\mu}\hat{\imath}_{\mathrm{W}} + \dot{\gamma}\hat{\jmath}_{\mathrm{G}} + \dot{\eta}\hat{k}_{\mathrm{F}}. \tag{10.9.11}$$

Since $\hat{\jmath}_{\rm AC} = \hat{\jmath}_{\rm S}$, $\hat{k}_{\rm S} = \hat{k}_{\rm W}$, $\hat{\imath}_{\rm W} = \hat{\imath}_{\rm G}$, $\hat{\jmath}_{\rm G} = \hat{\jmath}_{\rm F}$, and $\hat{k}_{\rm F} = \hat{k}_{\rm E}$, resolving $\vec{\omega}_{\rm AC/E}$ in $\rm F_{AC}$ yields

$$\omega_{\rm AC/E} = \dot{\alpha}\ \hat{\jmath}_{\rm AC}|_{\rm AC} - \dot{\beta}\ \hat{k}_{\rm S}\big|_{\rm AC} + \dot{\mu}\ \hat{\imath}_{\rm W}|_{\rm AC} + \dot{\gamma}\ \hat{\jmath}_{\rm G}|_{\rm AC} + \dot{\eta}\ \hat{k}_{\rm F}\big|_{\rm AC}, \tag{10.9.12}$$

where

$$\hat{k}_{\rm S}\big|_{\rm AC} = \begin{bmatrix} -\sin\alpha \\ 0 \\ \cos\alpha \end{bmatrix}, \tag{10.9.13}$$

$$\hat{\imath}_{\rm W}|_{\rm AC} = \begin{bmatrix} (\cos\alpha)\cos\beta \\ \sin\beta \\ (\sin\alpha)\cos\beta \end{bmatrix}, \tag{10.9.14}$$

$$\hat{\jmath}_{\rm G}|_{\rm AC} = \begin{bmatrix} (\sin\alpha)\sin\mu - (\cos\alpha)(\sin\beta)\cos\mu \\ (\cos\beta)\cos\mu \\ -(\cos\alpha)\sin\mu - (\sin\alpha)(\sin\beta)\cos\mu \end{bmatrix}, \tag{10.9.15}$$

$$\hat{k}_{\rm F}\big|_{\rm AC} = \begin{bmatrix} -[(\sin\alpha)\cos\mu + (\cos\alpha)(\sin\beta)\sin\mu]\cos\gamma - (\cos\alpha)(\cos\beta)\sin\gamma \\ (\cos\beta)(\sin\mu)\cos\gamma - (\sin\beta)\sin\gamma \\ [(\cos\alpha)\cos\mu - (\sin\alpha)(\sin\beta)\sin\mu]\cos\gamma - (\sin\alpha)(\cos\beta)\sin\gamma \end{bmatrix}. \tag{10.9.16}$$

It thus follows from (10.9.12) that

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\rm W}|_{\rm AC} & \hat{\jmath}_{\rm G}|_{\rm AC} & \hat{k}_{\rm F}\big|_{\rm AC} & \hat{\jmath}_{\rm AC}|_{\rm AC} & -\hat{k}_{\rm S}\big|_{\rm AC} \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\eta} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix}, \tag{10.9.17}$$

which can be written as

$$\begin{bmatrix} P - \dot{\beta}\sin\alpha \\ Q - \dot{\alpha} \\ R + \dot{\beta}\cos\alpha \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\rm W}|_{\rm AC} & \hat{\jmath}_{\rm G}|_{\rm AC} & \hat{k}_{\rm F}\big|_{\rm AC} \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\eta} \end{bmatrix}. \tag{10.9.18}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} \hat{\imath}_{\rm W}|_{\rm AC} & \hat{\jmath}_{\rm G}|_{\rm AC} & \hat{k}_{\rm F}\big|_{\rm AC} \end{bmatrix}^{-1} \begin{bmatrix} P - \dot{\beta}\sin\alpha \\ Q - \dot{\alpha} \\ R + \dot{\beta}\cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} P - \dot{\beta}\sin\alpha \\ Q - \dot{\alpha} \\ R + \dot{\beta}\cos\alpha \end{bmatrix}, \end{aligned} \tag{10.9.19}$$

where

$$a = \frac{1}{\cos\gamma}\begin{bmatrix} -(\sin\alpha)(\cos\mu)\sin\gamma + (\cos\alpha)(\cos\beta)\cos\gamma - (\cos\alpha)(\sin\beta)(\sin\mu)\sin\gamma \\ [(\sin\alpha)\sin\mu - (\cos\alpha)(\sin\beta)\cos\mu]\cos\gamma \\ -(\sin\alpha)\cos\mu - (\cos\alpha)(\sin\beta)\sin\mu \end{bmatrix}, \tag{10.9.20}$$

$$b = \frac{1}{\cos\gamma}\begin{bmatrix} (\sin\beta)\cos\gamma + (\cos\beta)(\sin\mu)\sin\gamma \\ (\cos\beta)(\cos\mu)\cos\gamma \\ (\cos\beta)\sin\mu \end{bmatrix}, \tag{10.9.21}$$

$$c = \frac{1}{\cos\gamma}\begin{bmatrix} (\cos\alpha)(\cos\mu)\sin\gamma + (\sin\alpha)(\cos\beta)\cos\gamma - (\sin\alpha)(\sin\beta)(\sin\mu)\sin\gamma \\ [-(\cos\alpha)\sin\mu - (\sin\alpha)(\sin\beta)\cos\mu]\cos\gamma \\ (\cos\alpha)\cos\mu - (\sin\alpha)(\sin\beta)\sin\mu \end{bmatrix}. \tag{10.9.22}$$

Note that, $\det\begin{bmatrix} \hat{\imath}_{\mathrm{W}}|_{\mathrm{AC}} & \hat{\jmath}_{\mathrm{G}}|_{\mathrm{AC}} & \hat{k}_{\mathrm{F}}\big|_{\mathrm{AC}} \end{bmatrix} = \cos\gamma$, which implies that the inverse in (10.9.19) does not exist if and only if $\gamma = \pm\frac{\pi}{2}$.

Alternatively, to circumvent gimbal lock, the attitude of the aircraft can be obtained from Poisson's equation given by

$$\dot{\mathcal{O}}_{\mathrm{AC/E}} = -\omega^{\times}_{\mathrm{AC/E}}\mathcal{O}_{\mathrm{AC/E}}. \tag{10.9.23}$$

10.10 Frame Derivatives

Let $\vec{r}$ be a position vector. Then, the transport theorem given by Fact 4.4.1 implies that

$$\overset{\mathrm{E}\bullet}{\vec{r}} = \overset{\mathrm{AC}\bullet}{\vec{r}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{r}.$$

Next, note that

$$\overset{\mathrm{E}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} = \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{\omega}_{\mathrm{AC/E}} = \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}},$$

that is, the angular acceleration of the aircraft relative to the Earth frame is the same as the angular acceleration of the aircraft relative to the aircraft frame. Now consider the linear acceleration

$$\begin{aligned}
\overset{\mathrm{E}\bullet\bullet}{\vec{r}} &= \overset{\mathrm{E}\bullet}{\overset{\mathrm{AC}\bullet}{\vec{r}}} + \overbrace{\vec{\omega}_{\mathrm{AC/E}} \times \vec{r}}^{\mathrm{E}\bullet} \\
&= \overset{\mathrm{AC}\bullet\bullet}{\vec{r}} + \vec{\alpha}_{\mathrm{AC/E/E}} \\
&= \overset{\mathrm{AC}\bullet\bullet}{\vec{r}} + \vec{\omega}_{\mathrm{AC/E}} \times \overset{\mathrm{AC}\bullet}{\vec{r}} + \overset{\mathrm{E}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} \times \vec{r} + \vec{\omega}_{\mathrm{AC/E}} \times \left(\overset{\mathrm{AC}\bullet}{\vec{r}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{r} \right) \\
&= \overset{\mathrm{AC}\bullet\bullet}{\vec{r}} + \underbrace{2\vec{\omega}_{\mathrm{AC/E}} \times \overset{\mathrm{AC}\bullet}{\vec{r}}}_{a_{\mathrm{Cor}}} + \underbrace{\overset{\mathrm{E}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} \times \vec{r}}_{a_{\mathrm{ang}}} + \underbrace{\vec{\omega}_{\mathrm{AC/E}} \times \left(\vec{\omega}_{\mathrm{AC/E}} \times \vec{r}\right)}_{a_{\mathrm{cent}}},
\end{aligned} \tag{10.10.1}$$

where a_{Cor}, a_{ang}, and a_{cent} are the Coriolis acceleration, angular-acceleration acceleration, and centripetal acceleration, respectively.

The cross product of the angular velocity vector

$$\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \tag{10.10.2}$$

and the position vector

$$\vec{r}|_{\mathrm{AC}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{10.10.3}$$

is given by

$$\vec{\omega}_{\mathrm{AC/E}} \times \vec{r} = \det\begin{bmatrix} \hat{\imath}_{\mathrm{AC}} & \hat{\jmath}_{\mathrm{AC}} & \hat{k}_{\mathrm{AC}} \\ P & Q & R \\ r_1 & r_2 & r_3 \end{bmatrix} = (Qr_3 - Rr_2)\hat{\imath}_{\mathrm{AC}} - (Pr_3 - Rr_1)\hat{\jmath}_{\mathrm{AC}} + (Pr_2 - Qr_1)\hat{k}_{\mathrm{AC}}.$$

Since

$$\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}}^{\times} = \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix},$$

it follows that

$$(\vec{\omega}_{\mathrm{AC}} \times \vec{r})\Big|_{\mathrm{AC}} = \vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} \times \vec{r}\Big|_{\mathrm{AC}} = \vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}}^{\times} \vec{r}\Big|_{\mathrm{AC}} = \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{10.10.4}$$

10.11 Problems

Problem 10.11.1. Resolve the velocity vector

$$\vec{V}_{\mathrm{AC}} = 72 \text{ m/s } \hat{\imath}_{\mathrm{AC}} - 6.3 \text{ m/s } \hat{\jmath}_{\mathrm{AC}} - 43 \text{ m/s } \hat{k}_{\mathrm{AC}}$$

in the stability and wind frames.

Problem 10.11.2. An aircraft is flying with an angle of attack of 14° and no sideslip. The airspeed $|\vec{V}_{\mathrm{AC}}|$ is 94 m/s. Resolve $\vec{V}_{\mathrm{AC}}$ in the aircraft frame and in the stability frame.

Problem 10.11.3. An aircraft is flying with an angle of attack of 10° and a sideslip angle of −19°. The airspeed $|\vec{V}_{\mathrm{AC}}|$ is 80 m/s. Resolve $\vec{V}_{\mathrm{AC}}$ in the aircraft, stability, and wind frames.

Problem 10.11.4. Resolve the gravity vector $\vec{g}$ symbolically in the aircraft and stability frames. Then, set $\Psi = 22°$, $\Theta = 5°$, $\Phi = -24°$, and $\alpha = -17°$, and compute the components of $\vec{g}$ in the aircraft and stability frames.

Problem 10.11.5. Consider the position vector

$$\vec{r} = 6.2 \text{ m } \hat{\imath}_{\mathrm{E}} - 14.1 \text{ m } \hat{\jmath}_{\mathrm{E}} + 65.2 \text{ m } \hat{k}_{\mathrm{E}},$$

and assume that the orientation of the aircraft frame relative to the Earth frame is given by the yaw, pitch, and roll Euler angles $\Psi = 62°$, $\Theta = 7°$, $\Phi = -12°$. Then, resolve $\vec{r}$ in the aircraft frame. Check your solution by comparing the magnitude of $\vec{r}$ computed from $\vec{r}$ resolved in both frames.

Problem 10.11.6. An aircraft is flying with velocity $\vec{V}_{\mathrm{AC}}$, which is constant with respect to the Earth frame. Its angle of attack is −13°, sideslip angle is 24°, and airspeed $|\vec{V}_{\mathrm{AC}}|$ is 85 m/s. The aircraft then rolls about $\hat{\imath}_{\mathrm{AC}}$ by +23° while the velocity vector remains fixed with respect to the Earth frame. Resolve the velocity vector $\vec{V}_{\mathrm{AC}}$ in the aircraft frame after the roll is complete, and determine the new angle of attack and sideslip.

Problem 10.11.7. Write a Matlab program (and include your code) that implements the transformation (4.10.7) from 3-2-1 Euler-angle rates to angular velocity components. Next, derive and confirm the reverse transformation

$$\dot{\Phi} = P + Q(\sin\Phi)\tan\Theta + R(\cos\Phi)\tan\Theta,$$
$$\dot{\Theta} = Q\cos\Phi - R\sin\Phi,$$

$$\dot{\Psi} = (Q \sin \Phi + R \cos \Phi) \sec \Theta,$$

and write a Matlab program that implements it. Finally, let

$$(\Phi, \Theta, \Psi) = (-52^\circ, -29^\circ, 14^\circ),$$

$$(\dot{\Phi}, \dot{\Theta}, \dot{\Psi}) = (16 \text{ deg/sec}, -7 \text{ deg/sec}, 19 \text{ deg/sec}),$$

and compute (P, Q, R). Then, use the computed values of (P, Q, R) in the reverse transformation and compute $(\dot{\Phi}, \dot{\Theta}, \dot{\Psi})$. Verify that you recover the original values of $(\dot{\Phi}, \dot{\Theta}, \dot{\Psi})$.

Problem 10.11.8. Consider an aircraft flying at constant speed and zero roll angle in a horizontal circle of radius R and with center at the point $\mathrm{O_E}$. The left wing of the aircraft is pointing toward the center of the circle. The magnitude of the angular velocity vector $\vec{\omega}_{\mathrm{AC/E}}$ is ω. Use the double transport theorem to determine the acceleration $\overset{\mathrm{E}\bullet\bullet}{\vec{r}}_{\mathrm{AC}}$ in terms of $\hat{r}_{\mathrm{AC}}$. Resolve $\overset{\mathrm{E}\bullet\bullet}{\vec{r}}_{\mathrm{AC}}$ in $\mathrm{F_{AC}}$.

Problem 10.11.9. Let $\hat{e}_{\mathrm{rw}}$ denote a unit vector pointing from the fuselage of an aircraft along the right wing of the aircraft. Show that the stability and wind frames are given by

$$\begin{bmatrix} \hat{\imath}_{\mathrm{S}} \\ \hat{\jmath}_{\mathrm{S}} \\ \hat{k}_{\mathrm{S}} \end{bmatrix} = \begin{bmatrix} \hat{\jmath}_{\mathrm{S}} \times \hat{k}_{\mathrm{S}} \\ \hat{e}_{\mathrm{rw}} \\ \frac{\vec{V}_{\mathrm{AC}} \times \hat{e}_{\mathrm{rw}}}{|\vec{V}_{\mathrm{AC}} \times \hat{e}_{\mathrm{rw}}|} \end{bmatrix}, \quad \begin{bmatrix} \hat{\imath}_{\mathrm{W}} \\ \hat{\jmath}_{\mathrm{W}} \\ \hat{k}_{\mathrm{W}} \end{bmatrix} = \begin{bmatrix} \vec{V}_{\mathrm{AC}} \\ \hat{k}_{\mathrm{W}} \times \hat{\imath}_{\mathrm{W}} \\ \frac{\vec{V}_{\mathrm{AC}} \times \hat{e}_{\mathrm{rw}}}{|\vec{V}_{\mathrm{AC}} \times \hat{e}_{\mathrm{rw}}|} \end{bmatrix}.$$

(Remark: These expressions show that the stability and wind frames can be constructed from the aircraft velocity vector and one additional vector, namely, the body-fixed vector $\hat{e}_{\mathrm{rw}}$. Therefore, the stability and wind frames are velocity-dependent frames. Note that the LVLH frame for spacecraft is also a velocity-dependent frame, where the additional vector points from the spacecraft to the center of the Earth.)

Problem 10.11.10. Compare the $(2, 3)$ entries of $\mathcal{O}_{\mathrm{AC/E}}$ and $\mathcal{O}_{\mathrm{AC/W}}\mathcal{O}_{\mathrm{W/E}}$ to show that

$$\sin \mu_{\mathrm{a}} = \frac{(\sin \Phi) \cos \Theta + (\sin \beta_{\mathrm{a}}) \sin \gamma_{\mathrm{a}}}{(\cos \beta_{\mathrm{a}}) \cos \gamma_{\mathrm{a}}}.$$

Symbol	Definition
$\mathrm{F_E}$	Earth Frame
$\hat{\imath}_{\mathrm{E}}, \hat{\jmath}_{\mathrm{E}}, \hat{k}_{\mathrm{E}}$	Earth frame axes
$\mathrm{F_{AC}}$	Aircraft frame
$\hat{\imath}_{\mathrm{AC}}, \hat{\jmath}_{\mathrm{AC}}, \hat{k}_{\mathrm{AC}}$	Aircraft frame axes
$\mathrm{O_E}$	Origin of $\mathrm{F_E}$
c	Aircraft center of mass and origin of $\mathrm{F_{AC}}$
X, Y, Z	Components of $\vec{V}_{\mathrm{AC}}$ resolved in $\mathrm{F_{AC}}$
$\vec{r}_{\mathrm{AC/E}}$	$\vec{r}_{\mathrm{c/O_E}}$
$\mathrm{F_S}$	Stability frame
$\hat{\imath}_{\mathrm{S}}, \hat{\jmath}_{\mathrm{S}}, \hat{k}_{\mathrm{S}}$	Stability frame axes
$\mathrm{F_W}$	Wind frame
$\hat{\imath}_{\mathrm{W}}, \hat{\jmath}_{\mathrm{W}}, \hat{k}_{\mathrm{W}}$	Wind frame axes
U, V, W	Components of $\vec{V}_{\mathrm{AC}}$ resolved in $\mathrm{F_{AC}}$
$\overline{U}, \overline{V}, \overline{W}$	Components of $\vec{V}_{\mathrm{AC}}$ resolved in $\mathrm{F_S}$

Table 10.2: Symbols for Chapter 10, part 1.

Symbol	Definition
α	Angle of attack angle from $\hat{\imath}_{\mathrm{S}}$ to $\hat{\imath}_{\mathrm{AC}}$
β	Sideslip angle from $\hat{\imath}_{\mathrm{S}}$ to $\hat{\imath}_{\mathrm{W}}$
γ	Flight path angle from the horizontal in $\mathrm{F_E}$ to $\hat{\imath}_{\mathrm{S}}$
η	Heading angle
μ	Bank angle
Ψ	Yaw angle from $\hat{\imath}_{\mathrm{E}}$ to $\hat{\imath}_{\mathrm{AC}}$
Θ	Pitch angle from the horizontal in $\mathrm{F_E}$ to $\hat{\imath}_{\mathrm{AC}}$
Φ	Roll angle about $\hat{\imath}_{\mathrm{AC}}$
$\vec{\omega}_{\mathrm{AC/E}}$	Angular velocity of $\mathrm{F_{AC}}$ relative to $\mathrm{F_E}$
P, Q, R	Components of $\vec{\omega}_{\mathrm{AC/E}}$ resolved in $\mathrm{F_{AC}}$

Table 10.3: Symbols for Chapter 10, part 2.

Chapter Eleven
Aircraft Dynamics

11.1 Aerodynamic Forces

The total aerodynamic force $\vec{F}_{\rm A}$ on the center of mass c of the aircraft is given as the sum of force vectors

$$\vec{F}_{\rm A} = \vec{D} + \vec{E} + \vec{L}, \tag{11.1.1}$$

where

$$\vec{D} \triangleq -D\hat{\imath}_{\rm W}, \tag{11.1.2}$$

$$\vec{E} \triangleq -E\hat{\jmath}_{\rm W}, \tag{11.1.3}$$

$$\vec{L} \triangleq -L\hat{k}_{\rm W} \tag{11.1.4}$$

are the drag, side drag, and lift vectors, respectively. Note that D is a positive number, E may be positive or negative, and L is positive if the angle of attack is nonnegative. Hence,

$$\vec{F}_{\rm A} = -D\hat{\imath}_{\rm W} - E\hat{\jmath}_{\rm W} - L\hat{k}_{\rm W}, \tag{11.1.5}$$

and thus

$$\vec{F}_{\rm A}\Big|_{\rm W} = \begin{bmatrix} -D \\ -E \\ -L \end{bmatrix}. \tag{11.1.6}$$

It follows from (10.6.19) that the velocity vector can be written as $\vec{V}_{\rm AC} = V_{\rm AC}\hat{\imath}_{\rm W}$, and thus the direction of the drag vector $\vec{D}$ is opposite to the aircraft velocity. Therefore, $\vec{D}$ points in the direction of the wind relative to the aircraft. Furthermore, the lift force $\vec{L}$ and the side drag $\vec{E}$ are orthogonal to $\vec{V}_{\rm AC}$. The contribution of the control surfaces to $\vec{D}$, $\vec{E}$, and $\vec{L}$ and thus to the net aerodynamic force changes as the deflections of the control surfaces change.

In the stability frame, we write $\vec{F}_{\rm A}$ as

$$\vec{F}_{\rm A} = \overline{F}_{{\rm A}_x}\hat{\imath}_{\rm S} + \overline{F}_{{\rm A}_y}\hat{\jmath}_{\rm S} + \overline{F}_{{\rm A}_z}\hat{k}_{\rm S}, \tag{11.1.7}$$

and thus

$$\vec{F}_{\rm A}\Big|_{\rm S} = \begin{bmatrix} \overline{F}_{{\rm A}_x} \\ \overline{F}_{{\rm A}_y} \\ \overline{F}_{{\rm A}_z} \end{bmatrix}. \tag{11.1.8}$$

The overbar denotes a component of a vector resolved in the stability frame. Alternatively, in the aircraft frame we can write $\vec{F}_{\rm A}$ as

$$\vec{F}_{\rm A} = F_{{\rm A}_x}\hat{\imath}_{\rm AC} + F_{{\rm A}_y}\hat{\jmath}_{\rm AC} + F_{{\rm A}_z}\hat{k}_{\rm AC}, \tag{11.1.9}$$

and thus

$$\vec{F}_{\rm A}\Big|_{\rm AC} = \begin{bmatrix} F_{{\rm A}_x} \\ F_{{\rm A}_y} \\ F_{{\rm A}_z} \end{bmatrix}. \tag{11.1.10}$$

Next, since $\hat{k}_{\rm S} = \hat{k}_{\rm W}$, (11.1.7) can be written as

$$\vec{F}_{\rm A} = \overline{F}_{{\rm A}_x}\hat{\imath}_{\rm S} + \overline{F}_{{\rm A}_y}\hat{\jmath}_{\rm S} + \overline{F}_{{\rm A}_z}\hat{k}_{\rm W}. \tag{11.1.11}$$

Therefore, it follows from (11.1.5) and (11.1.11) that

$$L = -\overline{F}_{{\rm A}_z}. \tag{11.1.12}$$

Consequently,

$$-D\hat{\imath}_{\rm W} - E\hat{\jmath}_{\rm W} = \overline{F}_{{\rm A}_x}\hat{\imath}_{\rm S} + \overline{F}_{{\rm A}_y}\hat{\jmath}_{\rm S}, \tag{11.1.13}$$

and thus

$$D^2 + E^2 = \overline{F}^2_{{\rm A}_x} + \overline{F}^2_{{\rm A}_y}. \tag{11.1.14}$$

Next, since $\hat{\jmath}_{\rm AC} = \hat{\jmath}_{\rm S}$, (11.1.9) can be written as

$$\vec{F}_{\rm A} = F_{{\rm A}_x}\hat{\imath}_{\rm AC} + F_{{\rm A}_y}\hat{\jmath}_{\rm S} + F_{{\rm A}_z}\hat{k}_{\rm AC}. \tag{11.1.15}$$

Therefore, it follows from (11.1.7) and (11.1.15) that

$$F_{{\rm A}_y} = \overline{F}_{{\rm A}_y}. \tag{11.1.16}$$

Consequently,

$$F_{{\rm A}_x}\hat{\imath}_{\rm AC} + F_{{\rm A}_z}\hat{k}_{\rm AC} = \overline{F}_{{\rm A}_x}\hat{\imath}_{\rm S} + \overline{F}_{{\rm A}_z}\hat{k}_{\rm S}, \tag{11.1.17}$$

and thus

$$F^2_{{\rm A}_x} + F^2_{{\rm A}_z} = \overline{F}^2_{{\rm A}_x} + \overline{F}^2_{{\rm A}_z}. \tag{11.1.18}$$

Next, we have

$$\begin{aligned}
\vec{F}_{\rm A}\Big|_{\rm S} = \begin{bmatrix} \overline{F}_{{\rm A}_x} \\ \overline{F}_{{\rm A}_y} \\ \overline{F}_{{\rm A}_z} \end{bmatrix} &= \mathcal{O}_{\rm S/W}\, \vec{F}_{\rm A}\Big|_{\rm W} \\
&= \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -D \\ -E \\ -L \end{bmatrix} \\
&= \begin{bmatrix} -(\cos\beta)D + (\sin\beta)E \\ -(\sin\beta)D - (\cos\beta)E \\ -L \end{bmatrix}.
\end{aligned} \tag{11.1.19}$$

In terms of the aircraft frame, $\vec{F}_{\rm A}$ is given by

$$\vec{F}_{\rm A}\Big|_{\rm AC} = \begin{bmatrix} F_{{\rm A}_x} \\ F_{{\rm A}_y} \\ F_{{\rm A}_z} \end{bmatrix} = \mathcal{O}_{\rm AC/S}\, \vec{F}_{\rm A}\Big|_{\rm S}$$
$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \overline{F}_{{\rm A}_x} \\ \overline{F}_{{\rm A}_y} \\ \overline{F}_{{\rm A}_z} \end{bmatrix}$$
$$= \begin{bmatrix} (\cos\alpha)\overline{F}_{{\rm A}_x} - (\sin\alpha)\overline{F}_{{\rm A}_z} \\ \overline{F}_{{\rm A}_y} \\ (\sin\alpha)\overline{F}_{{\rm A}_x} + (\cos\alpha)\overline{F}_{{\rm A}_z} \end{bmatrix}. \tag{11.1.20}$$

Alternatively, using (11.1.19) it follows that

$$\vec{F}_{\rm A}\Big|_{\rm AC} = \begin{bmatrix} F_{{\rm A}_x} \\ F_{{\rm A}_y} \\ F_{{\rm A}_z} \end{bmatrix} = \mathcal{O}_{\rm AC/S}\, \vec{F}_{\rm A}\Big|_{\rm S} = \mathcal{O}_{\rm AC/S} \begin{bmatrix} \overline{F}_{{\rm A}_x} \\ \overline{F}_{{\rm A}_y} \\ \overline{F}_{{\rm A}_z} \end{bmatrix}$$
$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} -D\cos\beta + E\sin\beta \\ -(\sin\beta)D - (\cos\beta)E \\ -L \end{bmatrix}$$
$$= \begin{bmatrix} -(\cos\alpha)(\cos\beta)D + (\cos\alpha)(\sin\beta)E + (\sin\alpha)L \\ -(\sin\beta)D - (\cos\beta)E \\ -(\sin\alpha)(\cos\beta)D + (\sin\alpha)(\sin\beta)E - (\cos\alpha)L \end{bmatrix}. \tag{11.1.21}$$

Finally, if E is negligible, then it follows from (11.1.14) that

$$D = \sqrt{\overline{F}_{{\rm A}_x}^2 + \overline{F}_{{\rm A}_y}^2}, \tag{11.1.22}$$

$$\vec{F}_{\rm A}\Big|_{\rm S} = \begin{bmatrix} -(\cos\beta)D \\ -(\sin\beta)D \\ -L \end{bmatrix}, \tag{11.1.23}$$

and from (11.1.21) that

$$\vec{F}_{\rm A}\Big|_{\rm AC} = \begin{bmatrix} -(\cos\alpha)(\cos\beta)D + (\sin\alpha)L \\ -(\sin\beta)D \\ -(\sin\alpha)(\cos\beta)D - (\cos\alpha)L \end{bmatrix}. \tag{11.1.24}$$

11.2 Translational Momentum Equations

Recall that

$$\vec{r}_{\rm AC} \triangleq \vec{r}_{\rm c/O_E} \tag{11.2.1}$$

denotes the location of the center of mass of the aircraft relative to the origin of the Earth. In addition, the velocity $\vec{V}_{\rm AC}$ of the aircraft relative to the point a with respect to the Earth frame is

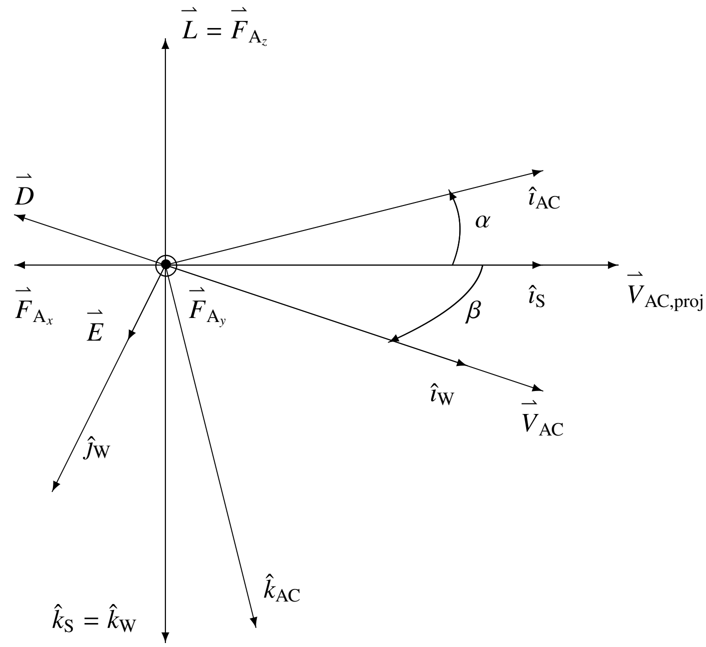


Figure 11.1.1: Aerodynamic forces. The vectors $\hat{\imath}_{\mathrm{W}}$ and $\hat{\jmath}_{\mathrm{W}}$ point obliquely out of the page, while $\vec{D}$ points obliquely into the page.

given by

$$\vec{V}_{\mathrm{AC}} \triangleq \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{AC}} . \tag{11.2.2}$$

To apply Newton's second law, we assume that $\mathrm{O_E}$ is an unforced particle and $\mathrm{F_E}$ is an inertial frame. The acceleration of the aircraft center of mass relative to $\mathrm{O_E}$ is thus given by

$$\vec{a}_{\mathrm{c/O_E/E}} = \overset{\mathrm{E}\bullet}{\vec{v}}_{\mathrm{c/O_E/E}} = \overset{\mathrm{E}\bullet}{\vec{V}}_{\mathrm{AC}} . \tag{11.2.3}$$

It thus follows from Newton's second law that

$$m\,\overset{\mathrm{E}\bullet}{\vec{V}}_{\mathrm{AC}} = m\vec{g} + \vec{F}_{\mathrm{A}} + \vec{F}_{\mathrm{T}}, \tag{11.2.4}$$

where $m\vec{g}$ is the weight of the aircraft and $\vec{F}_{\mathrm{T}}$ is the engine thrust force. Using the transport theorem (4.4) to introduce the derivative with respect to the aircraft frame $\mathrm{F_{AC}}$ into (11.2.4) yields

$$m(\overset{\mathrm{AC}\bullet}{\vec{V}}_{\mathrm{AC}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{V}_{\mathrm{AC}}) = m\vec{g} + \vec{F}_{\mathrm{A}} + \vec{F}_{\mathrm{T}}. \tag{11.2.5}$$

Resolving (11.2.5) in the aircraft frame yields

$$m\left(\overset{\mathrm{AC}\bullet}{\vec{V}}_{\mathrm{AC}}\Big|_{\mathrm{AC}} + (\vec{\omega}_{\mathrm{AC/E}} \times \vec{V}_{\mathrm{AC}})\Big|_{\mathrm{AC}} \right) = m\,\vec{g}\Big|_{\mathrm{AC}} + \vec{F}_{\mathrm{A}}\Big|_{\mathrm{AC}} + \vec{F}_{\mathrm{T}}\Big|_{\mathrm{AC}} . \tag{11.2.6}$$

Note that

$$\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}. \tag{11.2.7}$$

Resolving the gravity vector in the Earth frame as $\vec{g} = g\hat{k}_{\mathrm{E}}$ yields

$$\vec{g}\Big|_{\mathrm{E}} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}. \tag{11.2.8}$$

Using (10.3.16) to transform to the aircraft frame yields

$$\vec{g}\Big|_{\mathrm{AC}} = \mathcal{O}_{\mathrm{AC/E}}\, \vec{g}\Big|_{\mathrm{E}} = \begin{bmatrix} -(\sin\Theta)g \\ (\sin\Phi)(\cos\Theta)g \\ (\cos\Phi)(\cos\Theta)g \end{bmatrix}. \tag{11.2.9}$$

Hence, we define

$$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \triangleq \begin{bmatrix} -(\sin\Theta)g \\ (\sin\Phi)(\cos\Theta)g \\ (\cos\Phi)(\cos\Theta)g \end{bmatrix}. \tag{11.2.10}$$

Notice that the yaw angle Ψ does not appear in $\vec{g}\Big|_{\mathrm{AC}}$ due to the fact that the $\hat{k}_{\mathrm{E}}$-axis rotation of the Earth frame does not change the direction of gravity relative to the aircraft.

For the thrust force we express

$$\vec{F}_{\mathrm{T}}\Big|_{\mathrm{AC}} = \begin{bmatrix} F_{\mathrm{T}_x} \\ F_{\mathrm{T}_y} \\ F_{\mathrm{T}_z} \end{bmatrix} = \begin{bmatrix} \cos\Phi_{\mathrm{T}} & 0 & \sin\Phi_{\mathrm{T}} \\ 0 & 1 & 0 \\ -\sin\Phi_{\mathrm{T}} & 0 & \cos\Phi_{\mathrm{T}} \end{bmatrix} \begin{bmatrix} F_{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (\cos\Phi_{\mathrm{T}})F_{\mathrm{T}} \\ 0 \\ -(\sin\Phi_{\mathrm{T}})F_{\mathrm{T}} \end{bmatrix}, \tag{11.2.11}$$

where $F_{\mathrm{T}} = |\vec{F}_{\mathrm{T}}|$ is the engine force magnitude and Φ_{T} is the angle from $\hat{\imath}_{\mathrm{AC}}$ to the engine force direction, as shown in Figure 11.2.2. We assume that the component of the engine thrust in the direction $\hat{\jmath}_{\mathrm{AC}}$ is zero.

Now, since

$$\overset{\mathrm{AC}\bullet}{\vec{V}}_{\mathrm{AC}} = \overset{\mathrm{AC}\bullet}{\overbrace{\left(U\hat{\imath}_{\mathrm{AC}} + V\hat{\jmath}_{\mathrm{AC}} + W\hat{k}_{\mathrm{AC}}\right)}} = \dot{U}\hat{\imath}_{\mathrm{AC}} + \dot{V}\hat{\jmath}_{\mathrm{AC}} + \dot{W}\hat{k}_{\mathrm{AC}},$$

we have

$$\overset{\mathrm{AC}\bullet}{\vec{V}}_{\mathrm{AC}}\Bigg|_{\mathrm{AC}} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix}. \tag{11.2.12}$$

Substituting (11.2.7), (11.2.10), (11.2.11), and (11.2.12) into (11.2.6) we obtain the range-rate, drift-rate, and plunge-rate equations

$$m(\dot{U} - VR + WQ) = mg_x + F_{\mathrm{A}_x} + F_{\mathrm{T}_x}, \tag{11.2.13}$$

$$m(\dot{V} + UR - WP) = mg_y + F_{\mathrm{A}_y}, \tag{11.2.14}$$

$$m(\dot{W} - UQ + VP) = mg_z + F_{\mathrm{A}_z} + F_{\mathrm{T}_z}, \tag{11.2.15}$$

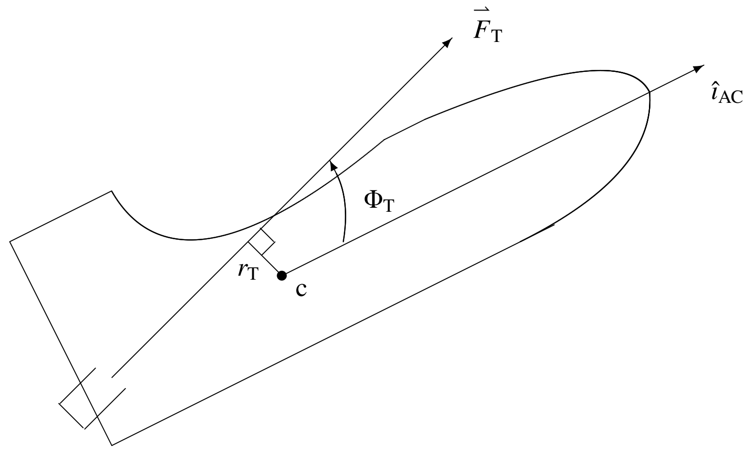


Figure 11.2.2: Thrust force and engine geometry.

or, equivalently, using (11.1.21),

$$m(\dot{U} - VR + WQ) = -(\sin\Theta)mg - (\cos\beta)(\cos\alpha)D + (\sin\alpha)L + (\cos\Phi_{\rm T})F_{\rm T}, \tag{11.2.16}$$

$$m(\dot{V} + UR - WP) = (\sin\Phi)(\cos\Theta)mg - (\sin\beta)D, \tag{11.2.17}$$

$$m(\dot{W} - UQ + VP) = (\cos\Phi)(\cos\Theta)mg - (\cos\beta)(\sin\alpha)D - (\cos\alpha)L - (\sin\Phi_{\rm T})F_{\rm T}. \tag{11.2.18}$$

11.3 Rotational Momentum Equations

Let

$$\vec{H}_{\rm AC/E} \triangleq \vec{H}_{\rm AC/c/E} = \vec{J}_{\rm AC/c}\vec{\omega}_{\rm AC/E} \tag{11.3.1}$$

denote the angular momentum of the aircraft relative to the center of mass c of the aircraft with respect to $\rm F_E$, where the physical inertia matrix of the aircraft relative to the center of mass is given by

$$\vec{J}_{\rm AC/c} = \int_{\rm AC} |\vec{r}_{{\rm d}m/{\rm c}}|^2 \vec{I} - \vec{r}_{{\rm d}m/{\rm c}}\vec{r}\,'_{{\rm d}m/{\rm c}}\,{\rm d}m. \tag{11.3.2}$$

Using (7.8.4) we have

$$\vec{H}_{\rm AC/E}\Big|_{\rm AC} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \vec{J}_{\rm AC/c}\Big|_{\rm AC} \vec{\omega}_{\rm AC/E}\Big|_{\rm AC} = \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{yy} & -J_{yz} \\ -J_{xz} & -J_{yz} & J_{zz} \end{bmatrix}\begin{bmatrix} P \\ Q \\ R \end{bmatrix}. \tag{11.3.3}$$

Assuming that $\hat{\imath}_{\rm AC}$-$\hat{k}_{\rm AC}$ is a plane of symmetry of the aircraft, it follows that

$$J_{xy} = \int_{\rm AC} xy\,{\rm d}m = 0, \tag{11.3.4}$$

$$J_{yz} = \int_{\rm AC} yz\,{\rm d}m = 0. \tag{11.3.5}$$

Consequently,

$$\vec{J}_{\mathrm{AC/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} J_{xx} & 0 & -J_{xz} \\ 0 & J_{yy} & 0 \\ -J_{xz} & 0 & J_{zz} \end{bmatrix}, \tag{11.3.6}$$

and thus

$$H_x = J_{xx}P - J_{xz}R, \tag{11.3.7}$$

$$H_y = J_{yy}Q, \tag{11.3.8}$$

$$H_z = J_{zz}R - J_{xz}P. \tag{11.3.9}$$

Next, Euler's equation (7.8.13) for the aircraft is given by

$$\vec{J}_{\mathrm{AC/c}} \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} + \vec{\omega}_{\mathrm{AC/E}} \times \vec{J}_{\mathrm{AC/c}}\vec{\omega}_{\mathrm{AC/E}} = \vec{M}_{\mathrm{AC/c}}, \tag{11.3.10}$$

where

$$\vec{M}_{\mathrm{AC/c}} \triangleq \vec{M}_{\mathrm{A/c}} + \vec{M}_{\mathrm{T/c}} \tag{11.3.11}$$

is the total moment acting on the aircraft relative to c, and $\vec{M}_{\mathrm{A/c}}$ and $\vec{M}_{\mathrm{T/c}}$ are the aerodynamic and thrust moments relative to c, respectively. The aerodynamic moment is produced by the air flow over both the fixed and variable (that is, control) surfaces of the aircraft.

In the aircraft frame, we have

$$\vec{M}_{\mathrm{AC/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} L_{\mathrm{AC}} \\ M_{\mathrm{AC}} \\ N_{\mathrm{AC}} \end{bmatrix} = \vec{M}_{\mathrm{A/c}}\Big|_{\mathrm{AC}} + \vec{M}_{\mathrm{T/c}}\Big|_{\mathrm{AC}}, \tag{11.3.12}$$

where

$$\vec{M}_{\mathrm{A/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} L_{\mathrm{A}} \\ M_{\mathrm{A}} \\ N_{\mathrm{A}} \end{bmatrix}, \quad \vec{M}_{\mathrm{T/c}}\Big|_{\mathrm{AC}} = \begin{bmatrix} L_{\mathrm{T}} \\ M_{\mathrm{T}} \\ N_{\mathrm{T}} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} L_{\mathrm{AC}} \\ M_{\mathrm{AC}} \\ N_{\mathrm{AC}} \end{bmatrix} = \begin{bmatrix} L_{\mathrm{A}} \\ M_{\mathrm{A}} \\ N_{\mathrm{A}} \end{bmatrix} + \begin{bmatrix} L_{\mathrm{T}} \\ M_{\mathrm{T}} \\ N_{\mathrm{T}} \end{bmatrix}. \tag{11.3.13}$$

Now resolving Euler's equation (11.3.10) in the aircraft frame yields

$$\left(\vec{J}_{\mathrm{AC/c}} \overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC}}\right)\Big|_{\mathrm{AC}} + \left(\vec{\omega}_{\mathrm{AC/E}} \times \vec{J}_{\mathrm{AC/c}}\vec{\omega}_{\mathrm{AC/E}}\right)\Big|_{\mathrm{AC}} = \vec{M}_{\mathrm{AC/c}}\Big|_{\mathrm{AC}}, \tag{11.3.14}$$

which yields

$$\begin{bmatrix} J_{xx}\dot{P} - J_{xz}\dot{R} \\ J_{yy}\dot{Q} \\ J_{zz}\dot{R} - J_{xz}\dot{P} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}\begin{bmatrix} J_{xx}P - J_{xz}R \\ J_{yy}Q \\ J_{zz}R - J_{xz}P \end{bmatrix} = \begin{bmatrix} L_{\mathrm{AC}} \\ M_{\mathrm{AC}} \\ N_{\mathrm{AC}} \end{bmatrix}. \tag{11.3.15}$$

Equation (11.3.15) can now be written component-wise as

$$J_{xx}\dot{P} + (J_{zz} - J_{yy})QR - J_{xz}(\dot{R} + PQ) = L_{\mathrm{AC}}, \tag{11.3.16}$$

$$J_{yy}\dot{Q} + (J_{xx} - J_{zz})PR + J_{xz}(P^2 - R^2) = M_{\rm AC}, \tag{11.3.17}$$

$$J_{zz}\dot{R} + (J_{yy} - J_{xx})PQ + J_{xz}(QR - \dot{P}) = N_{\rm AC}. \tag{11.3.18}$$

Each equation (11.3.16), (11.3.17), and (11.3.18) has three terms, each representing a physical effect. The first term is the angular acceleration, the next term is the gyroscopic precession, and the last term is the inertia coupling. For the $\hat{\imath}_{\rm AC}$-axis angular-velocity component,

$$\underbrace{J_{xx}\dot{P}}_{\text{angular acceleration}} + \underbrace{(J_{zz} - J_{yy})QR}_{\text{gyroscopic precession}} - \underbrace{J_{xz}(\dot{R} + PQ)}_{\text{inertial coupling}} = \underbrace{L_{\rm AC}}_{\text{roll moment}}. \tag{11.3.19}$$

11.4 Summary of the Aircraft Equations of Motion

Translational kinematics (see (10.7.5))

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \mathcal{O}_{\rm E/AC} \begin{bmatrix} U \\ V \\ W \end{bmatrix}, \tag{11.4.1}$$

where $\mathcal{O}_{\rm E/AC}$ is given by (10.3.17).

Rotational kinematics (see (10.9.8))

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\cos\Theta)\sin\Phi \\ 0 & -\sin\Phi & (\cos\Theta)\cos\Phi \end{bmatrix} \begin{bmatrix} \dot{\Psi} \\ \dot{\Theta} \\ \dot{\Phi} \end{bmatrix}. \tag{11.4.2}$$

Translational momentum (see (11.2.16), (11.2.17), and (11.2.18))

$$m(\dot{U} - VR + WQ) = -(\sin\Theta)mg - (\cos\beta)(\cos\alpha)D + (\sin\alpha)L + (\cos\Phi_{\rm T})F_{\rm T}, \tag{11.4.3}$$

$$m(\dot{V} + UR - WP) = (\sin\Phi)(\cos\Theta)mg - (\sin\beta)D, \tag{11.4.4}$$

$$m(\dot{W} - UQ + VP) = (\cos\Phi)(\cos\Theta)mg - (\cos\beta)(\sin\alpha)D - (\cos\alpha)L - (\sin\Phi_{\rm T})F_{\rm T}. \tag{11.4.5}$$

Rotational momentum (see (11.3.16), (11.3.17), and (11.3.18))

$$J_{xx}\dot{P} + (J_{zz} - J_{yy})QR - J_{xz}(\dot{R} + PQ) = L_{\rm AC} = L_{\rm A} + L_{\rm T}, \tag{11.4.6}$$

$$J_{yy}\dot{Q} + (J_{xx} - J_{zz})PR + J_{xz}(P^2 - R^2) = M_{\rm AC} = M_{\rm A} + M_{\rm T}, \tag{11.4.7}$$

$$J_{zz}\dot{R} + (J_{yy} - J_{xx})PQ + J_{xz}(QR - \dot{P}) = N_{\rm AC} = N_{\rm A} + N_{\rm T}. \tag{11.4.8}$$

11.5 Aircraft Equations of Motion in State Space Form

Consider the continuous-time state-space model $\dot{x} = f(x, u)$, where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^p$ is the input. Let

$$x = \begin{bmatrix} X & Y & Z & U & V & W & \Phi & \Theta & \Psi & P & Q & R \end{bmatrix}^{\rm T}, \tag{11.5.1}$$

$$u = \begin{bmatrix} D & L & F_{\rm T} & L_{\rm AC} & M_{\rm AC} & N_{\rm AC} \end{bmatrix}^{\rm T}. \tag{11.5.2}$$

The aircraft equations of motion (11.4.1)–(11.4.8) in the state-space form are given by

$$\begin{aligned}\dot{X} = (\cos\Theta)(\cos\Psi)U &+ [(\sin\Phi)(\sin\Theta)\cos\Psi - (\cos\Phi)\sin\Psi]\, V \\ &+ [(\cos\Phi)(\sin\Theta)\cos\Psi + (\sin\Phi)\sin\Psi]\, W, \end{aligned} \tag{11.5.3}$$

$$\dot{Y} = (\cos\Theta)(\sin\Psi)U + [(\sin\Phi)(\sin\Theta)\sin\Psi + (\cos\Phi)\cos\Psi]\,V + [(\cos\Phi)(\sin\Theta)\sin\Psi - (\sin\Phi)\cos\Psi]\,W, \tag{11.5.4}$$

$$\dot{Z} = -(\sin\Theta)U + (\sin\Phi)(\cos\Theta)V + (\cos\Phi)(\cos\Theta)W, \tag{11.5.5}$$

$$\dot{U} = VR - WQ - (\sin\Theta)g - (\cos\beta)(\cos\alpha)\frac{D}{m} + (\sin\alpha)\frac{L}{m} + (\cos\Phi_{\mathrm{T}})\frac{F_{\mathrm{T}}}{m}, \tag{11.5.6}$$

$$\dot{V} = -UR + WP + (\sin\Phi)(\cos\Theta)g - (\sin\beta)\frac{D}{m}, \tag{11.5.7}$$

$$\dot{W} = UQ - VP + (\cos\Phi)(\cos\Theta)g - (\cos\beta)(\sin\alpha)\frac{D}{m} - (\cos\alpha)\frac{L}{m} - (\sin\Phi_{\mathrm{T}})\frac{F_{\mathrm{T}}}{m}, \tag{11.5.8}$$

$$\dot{\Phi} = P + (\sin\Phi)(\tan\Theta)Q + (\cos\Phi)(\tan\Theta)R, \tag{11.5.9}$$

$$\dot{\Theta} = (\cos\Phi)Q - (\sin\Phi)R, \tag{11.5.10}$$

$$\dot{\Psi} = (\sin\Phi)(\sec\Theta)Q + (\cos\Phi)(\sec\Theta)R, \tag{11.5.11}$$

$$\dot{P} = \frac{1}{J_{xx}J_{zz} - J_{xz}^2}\left[(J_{yy}J_{zz} - J_{zz}^2 - J_{xz}^2)QR + J_{xz}(J_{xx} - J_{yy} + J_{zz})PQ + J_{zz}L_{\mathrm{AC}} + J_{xz}N_{\mathrm{AC}}\right], \tag{11.5.12}$$

$$\dot{Q} = \frac{1}{J_{yy}}\left[(J_{zz} - J_{xx})PR + J_{xz}(R^2 - P^2) + M_{\mathrm{AC}}\right], \tag{11.5.13}$$

$$\dot{R} = \frac{1}{J_{xx}J_{zz} - J_{xz}^2}\left[(-J_{xx}J_{yy} + J_{xx}^2 + J_{xz}^2)PQ + J_{xz}(-J_{xx} + J_{yy} - J_{zz})QR + J_{xz}L_{\mathrm{AC}} + J_{xx}N_{\mathrm{AC}}\right], \tag{11.5.14}$$

where

$$\alpha = \operatorname{atan2}(W, U), \tag{11.5.15}$$

$$\beta = \operatorname{atan2}(V, \sqrt{U^2 + W^2}). \tag{11.5.16}$$

11.6 Problems

Problem 11.6.1. Show that the moment on the rigid body $\mathcal{B}$ relative to the point w due to the force $\vec{f}$ applied to the particle x in $\mathcal{B}$ does not change if the force $\vec{f}$ is applied instead to another particle y in $\mathcal{B}$ located along the line that is parallel to $\vec{f}$ and that passes through x.

Problem 11.6.2. Consider Figure 11.2.2. Determine $\vec{M}_{\mathrm{T}}\Big|_{\mathrm{AC}}$ in terms of $F_{\mathrm{T}} = |\vec{F}_{\mathrm{T}}|$, Φ_{T}, and r_{T}. (Hint: Use Problem 11.6.1.)

Problem 11.6.3. Assuming the symmetry of a typical aircraft, use the discrete formulas given by (6.2.6)–(6.2.8) to show that four of the entries of the physical inertia matrix resolved in the aircraft frame are zero.

Symbol	Definition
$\vec{F}_{\mathrm{A}}$	Aerodynamic force vector
$\vec{D}$	Drag force
$\vec{E}$	Side force
$\vec{L}$	Lift force
D	Drag
E	Side drag
L	Lift
$\overline{F}_{\mathrm{A}_x}, \overline{F}_{\mathrm{A}_y}, \overline{F}_{\mathrm{A}_z}$	Force components in $\mathrm{F_S}$
$F_{\mathrm{A}_x}, F_{\mathrm{A}_y}, F_{\mathrm{A}_z}$	Force components in $\mathrm{F_{AC}}$
$\vec{F}_{\mathrm{T}}$	Engine thrust force vector
F_{T}	Magnitude of the thrust force vector
Φ_{T}	Angle between the thrust force and $\hat{\imath}_{\mathrm{AC}}$
$\vec{M}_{\mathrm{AC}}$	Total moment vector
$\overline{L}_{\mathrm{AC}}$	Total roll moment in $\mathrm{F_S}$
$\overline{M}_{\mathrm{AC}}$	Total pitch moment in $\mathrm{F_S}$
$\overline{N}_{\mathrm{AC}}$	Total yaw moment in $\mathrm{F_S}$
L_{AC}	Total roll moment in $\mathrm{F_{AC}}$
M_{AC}	Total pitch moment in $\mathrm{F_{AC}}$
N_{AC}	Total yaw moment in $\mathrm{F_{AC}}$

Table 11.1: Symbols for Chapter 11, part 1.

Symbol	Definition
$\vec{M}_{\mathrm{A}}$	Aerodynamic moment vector
$\overline{L}_{\mathrm{A}}$	Aerodynamic roll moment in $\mathrm{F_S}$
$\overline{M}_{\mathrm{A}}$	Aerodynamic pitch moment in $\mathrm{F_S}$
$\overline{N}_{\mathrm{A}}$	Aerodynamic yaw moment in $\mathrm{F_S}$
L_{A}	Aerodynamic roll moment in $\mathrm{F_{AC}}$
M_{A}	Aerodynamic pitch moment in $\mathrm{F_{AC}}$
N_{A}	Aerodynamic yaw moment in $\mathrm{F_{AC}}$
$\vec{M}_{\mathrm{T}}$	Thrust moment vector
$\overline{L}_{\mathrm{T}}$	Thrust roll moment in $\mathrm{F_S}$
$\overline{M}_{\mathrm{T}}$	Thrust pitch moment in $\mathrm{F_S}$
$\overline{N}_{\mathrm{T}}$	Thrust yaw moment in $\mathrm{F_S}$
L_{T}	Thrust roll moment in $\mathrm{F_{AC}}$
M_{T}	Thrust pitch moment in $\mathrm{F_{AC}}$
N_{T}	Thrust yaw moment in $\mathrm{F_{AC}}$
$\vec{J}_{\mathrm{AC/c}}$	Aircraft physical inertia matrix relative to c

Table 11.2: Symbols for Chapter 11, part 2.

Chapter Twelve
Steady Flight and Linearization

The goal of linearization is to approximate the nonlinear aircraft equations of motion with linear equations to facilitate the analysis of flight characteristics in the vicinity of steady flight. The linearized equations are easier to analyze than the original nonlinear equations and involve stability derivatives that can be estimated using computational fluid dynamics (CFD) codes or measurements obtained from experiments in a wind tunnel.

12.1 Steady Flight

Recall that

$$\vec{V}_{\mathrm{AC}} = \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{AC}}. \tag{12.1.1}$$

The conditions for steady flight are

$$\overset{\mathrm{AC}\bullet}{\vec{V}}_{\mathrm{AC}} = 0, \tag{12.1.2}$$

$$\overset{\mathrm{AC}\bullet}{\vec{\omega}}_{\mathrm{AC/E}} = 0. \tag{12.1.3}$$

These conditions state that the velocity vector $\vec{V}_{\mathrm{AC}}$ is constant with respect to the aircraft frame F_{AC}, and the angular velocity vector $\vec{\omega}_{\mathrm{AC/E}}$ is constant with respect to the aircraft frame F_{AC} and the Earth frame F_{E}. Under these conditions, we denote $\vec{V}_{\mathrm{AC}}$ and $\vec{\omega}_{\mathrm{AC/E}}$ by $\vec{V}_{\mathrm{AC}_0}$ and $\vec{\omega}_{\mathrm{AC/E}_0}$, respectively, where

$$\vec{V}_{\mathrm{AC}_0}\Big|_{\mathrm{AC}} = \begin{bmatrix} U_0 \\ V_0 \\ W_0 \end{bmatrix}, \quad \vec{\omega}_{\mathrm{AC/E}_0}\Big|_{\mathrm{AC}} = \begin{bmatrix} P_0 \\ Q_0 \\ R_0 \end{bmatrix}. \tag{12.1.4}$$

Therefore, $U_0, V_0, W_0, P_0, Q_0, R_0$ are constant.

In the case of zero ambient wind, the following steady flight regimes are defined, although not all of these regimes are feasible for many aircraft.

- Hovering flight. The aircraft maintains a constant location relative to the Earth. Therefore, $\vec{V}_{\mathrm{AC}} = 0$.
- Straight-line flight. The aircraft flies in a straight line relative to the Earth with its wings level, that is, with $\Phi_0 = 0$. Therefore, its translational velocity relative to the Earth is nonzero and constant with respect to both the aircraft and Earth frames, and its angular velocity vector

relative to the Earth is zero. Straight-line flight may be either climbing ($\gamma > 0$), horizontal ($\gamma = 0$), or descending ($\gamma < 0$). In all of these cases, $\overset{\mathrm{E}\bullet}{\vec{V}}_{\mathrm{AC}} = 0$, $\vec{V}_{\mathrm{AC}}$ is nonzero, and $\vec{\omega}_{\mathrm{AC/E}} = 0$.

- Bullet flight. The aircraft flies in a straight line relative to the Earth while rotating relative to the Earth around its velocity vector. Therefore, its translational velocity relative to the Earth is constant with respect to the Earth and is parallel to its angular velocity vector relative to the Earth. Hence, $\overset{\mathrm{E}\bullet}{\vec{V}}_{\mathrm{AC}} = 0$, and $\vec{V}_{\mathrm{AC}}$ and $\vec{\omega}_{\mathrm{AC/E}}$ are nonzero and parallel.
- Circular flight. The aircraft flies in a circle relative to the Earth. The circle may be either horizontal or tilted relative to the Earth, while the aircraft attitude along the circle may be either banked or level. Therefore, its translational velocity relative to the Earth is not constant and is perpendicular at all time instants to its angular velocity vector relative to the Earth. In all of these cases, $\vec{V}_{\mathrm{AC}}$ and $\vec{\omega}_{\mathrm{AC/E}}$ are nonzero and mutually orthogonal.
- Helical flight. The aircraft flies in a helix relative to the Earth. Therefore, its translational velocity vector is not constant with respect to the Earth and is neither parallel nor orthogonal to its angular velocity vector relative to the Earth. In this case, $\vec{V}_{\mathrm{AC}}$ and $\vec{\omega}_{\mathrm{AC/E}}$ are nonzero and neither parallel nor mutually orthogonal. A special case of helical flight is a barrel roll, where the wheels of the aircraft can be viewed as rolling on the inside of a cylinder.

In all steady flight regimes, V_{AC}, γ, τ, α, and β are constant. However, the Euler angles Ψ, Θ, and Φ and the altitude H may vary with time.

Steady flight where $\overset{\mathrm{AC}\bullet}{\vec{g}} = 0$, that is, the gravity vector is constant with respect to $\mathrm{F_{AC}}$, is called *trim flight*. Equivalently, Φ and Θ are constant. Since $\overset{\mathrm{AC}\bullet}{\vec{V}}_{\mathrm{AC}} = 0$, an equivalent condition is Φ and γ are constant. Therefore, steady flight is trim flight if and only if steady flight is either 1) hovering flight, 2) straight-line flight, 3) circular flight in a horizontal plane, or 4) helical flight around a vertical axis. In trim flight, the total force on the aircraft is constant with respect to $\mathrm{F_{AC}}$ (zero in hovering and straight-line flight), and the total moment on the aircraft is zero. The force and moment conditions are equivalent to the conditions that the thrust is constant and the control surfaces are set to constant *trim angles*.

12.2 Taylor Series and Linearization

The Taylor series expansion at a of an infinitely differentiable function f is expressed as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \tag{12.2.1}$$

In a more compact form, we have

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i. \tag{12.2.2}$$

Example 12.2.1. Using the Taylor expansion, we have

$$\sin(\Phi_0 + \phi) \approx \sin\Phi_0 + (\cos\Phi_0)\phi, \tag{12.2.3}$$

$$\cos(\Phi_0 + \phi) \approx \cos\Phi_0 - (\sin\Phi_0)\phi. \tag{12.2.4}$$

Hence, for $\Phi_0 = 0$, we have the small-angle approximations

$$\sin\phi \approx \phi, \tag{12.2.5}$$

$$\cos\phi \approx 1. \tag{12.2.6}$$

Using (12.2.5) and (12.2.6) and using trigonometric identities, it follows that

$$\begin{aligned}\sin(\Phi_0 + \phi) &= (\sin\Phi_0)\cos\phi + (\cos\Phi_0)\sin\phi \\ &\approx \sin\Phi_0 + (\cos\Phi_0)\phi,\end{aligned} \tag{12.2.7}$$

$$\begin{aligned}\cos(\Phi_0 + \phi) &= (\cos\Phi_0)\cos\phi - (\sin\Phi_0)\sin\phi \\ &\approx \cos\Phi_0 - (\sin\Phi_0)\phi,\end{aligned} \tag{12.2.8}$$

which agree with (12.2.3) and (12.2.4). ◇

For a multivariable function, we write

$$\begin{aligned} f(x,y) = f(a,b) &+ \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b) \\ &+ \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)}(x-a)^2 + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)}(x-a)(y-b) \\ &+ \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{(a,b)}(y-b)^2 + \cdots . \end{aligned} \tag{12.2.9}$$

Truncating the series (12.2.9) to the first power in x and y yields the approximation

$$f(x,y) \approx f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b). \tag{12.2.10}$$

The following alternative procedure is often convenient when linearizing the aircraft equations of motion:

i) Replace each variable with the sum of its steady value and a perturbation. For instance, $U = U_0 + u$, where U_0 is the steady value and u is the perturbation.

ii) Ignore products of perturbation variables.

iii) Subtract the steady equation from the equation obtained in *i*) to cancel all terms that involve only the steady values.

12.3 Linearization of the Aircraft Kinematics and Dynamics at Straight-Line, Horizontal, Wings-Level, Zero-Sideslip Steady Flight

We linearize the aircraft equations of motion at the steady flight condition where the aircraft flies in a horizontal straight line with wings level, constant angle of attack, and zero sideslip angle. Let Ψ_0, Θ_0, and $\Phi_0 = 0$ denote, respectively, the steady values of the 3-2-1 yaw, pitch, and roll Euler angles from F_E to F_{AC}, let α_0 and $\beta_0 = 0$ denote the steady values of the angle of attack and sideslip angle, and let $\eta_0 = \Psi_0$, $\gamma_0 = 0$, and $\mu_0 = 0$ denote, respectively, the steady values of the 3-2-1 heading, flight-path, and bank Euler angles from F_E to F_W. Fact 10.8.6, Fact 10.8.7, and Fact

10.8.8 provide special cases of these angles. For straight-line, horizontal, wings-level, steady flight, we consider the case where

$$\Psi_0 = \eta_0, \tag{12.3.1}$$

$$\Theta_0 = \alpha_0, \tag{12.3.2}$$

$$\Phi_0 = \mu_0 = 0, \tag{12.3.3}$$

$$\beta_0 = 0, \tag{12.3.4}$$

$$\gamma_0 = 0. \tag{12.3.5}$$

Under these conditions, $\Psi_0, \Theta_0, \Phi_0, \alpha_0, \beta_0, \eta_0, \gamma_0, \mu_0$ satisfy *i*) of Fact 10.8.6, *ii*) of Fact 10.8.6, and *ii*) of Fact 10.8.8. In the next section we redefine $\mathrm{F_{AC}}$ so that $\Theta_0 = 0$ and thus $\alpha_0 = 0$. With this redefinition, $\Psi_0, \Theta_0, \Phi_0, \alpha_0, \beta_0, \eta_0, \gamma_0, \mu_0$ also satisfy *i*) of Fact 10.8.7, *ii*) of Fact 10.8.7, and *i*) of Fact 10.8.8.

Steady flight implies that U_0, V_0, and W_0 are constant. Since $\beta_0 = 0$, it follows from (10.6.23) that $V_0 = 0$, while (10.6.15) implies that

$$W_0 = (\tan\alpha_0)U_0. \tag{12.3.6}$$

Therefore,

$$U_0 = \text{constant}, \quad V_0 = 0, \quad W_0 = (\tan\alpha_0)U_0 = \text{constant}. \tag{12.3.7}$$

Finally, (10.9.8) implies that

$$P_0 = Q_0 = R_0 = 0. \tag{12.3.8}$$

To introduce the perturbed aircraft velocities, we write

$$\vec{V}_{\mathrm{AC}}\Big|_{\mathrm{AC}} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} U_0 + u \\ V_0 + v \\ W_0 + w \end{bmatrix} = \begin{bmatrix} U_0 + u \\ v \\ W_0 + w \end{bmatrix}. \tag{12.3.9}$$

Linearizing (10.6.15) yields

$$\delta\alpha = \frac{-(\cos^2\alpha_0)W_0}{U_0^2}u + \frac{\cos^2\alpha_0}{U_0}w. \tag{12.3.10}$$

Next, note that

$$\frac{U_0^2}{U_0^2 + W_0^2} = \frac{1}{1 + (\frac{W_0}{U_0})^2} = \frac{1}{1 + (\frac{\sin\alpha_0}{\cos\alpha_0})^2} = \cos^2\alpha_0. \tag{12.3.11}$$

It follows from (12.3.11) that (12.3.10) can also be written as

$$\delta\alpha = \frac{-W_0}{U_0^2 + W_0^2}u + \frac{U_0}{U_0^2 + W_0^2}w. \tag{12.3.12}$$

Hence, if follows from (12.3.10) and (12.3.11) that

$$w = \frac{W_0}{U_0}u + (\sec^2\alpha_0)U_0\delta\alpha = \frac{W_0}{U_0}u + \frac{U_0^2 + W_0^2}{U_0}\delta\alpha. \tag{12.3.13}$$

Likewise, linearizing (10.6.24) yields

$$\delta\beta = \frac{1}{\sqrt{U_0^2 + W_0^2}}v. \tag{12.3.14}$$

Hence,

$$v = \frac{U_0}{\cos\alpha_0}\delta\beta = \sqrt{U_0^2 + W_0^2}\,\delta\beta. \tag{12.3.15}$$

Using (10.3.17) and (10.7.5), it follows that the steady range, drift, and plunge X_0, Y_0, and Z_0 satisfy

$$\dot{X}_0 = (\cos\Theta_0)(\cos\Psi_0)U_0 + (\sin\Theta_0)(\cos\Psi_0)W_0, \tag{12.3.16}$$

$$\dot{Y}_0 = (\cos\Theta_0)(\sin\Psi_0)U_0 + (\sin\Theta_0)(\sin\Psi_0)W_0, \tag{12.3.17}$$

$$\dot{Z}_0 = -(\sin\Theta_0)U_0 + (\cos\Theta_0)W_0. \tag{12.3.18}$$

In terms of altitude, (12.3.18) implies

$$\dot{H}_0 = (\sin\Theta_0)U_0 - (\cos\Theta_0)W_0. \tag{12.3.19}$$

The perturbations x, y, and z of $X = X_0 + x$, $Y = Y_0 + y$, and $Z = Z_0 + z$ satisfy

$$\begin{aligned}\dot{x} = {}& (\cos\Theta_0)(\cos\Psi_0)u - (\sin\Psi_0)v + (\sin\Theta_0)(\cos\Psi_0)w + (W_0\sin\Psi_0)\phi \\ & + (W_0\cos\Theta_0 - U_0\sin\Theta_0)(\cos\Psi_0)\theta - (U_0\cos\Theta_0 + W_0\sin\Theta_0)(\sin\Psi_0)\psi, \end{aligned} \tag{12.3.20}$$

$$\begin{aligned}\dot{y} = {}& (\cos\Theta_0)(\sin\Psi_0)u + (\cos\Psi_0)v + (\sin\Theta_0)(\sin\Psi_0)w - (W_0\cos\Psi_0)\phi \\ & + (W_0\cos\Theta_0 - U_0\sin\Theta_0)(\sin\Psi_0)\theta + (U_0\cos\Theta_0 + W_0\sin\Theta_0)(\cos\Psi_0)\psi, \end{aligned} \tag{12.3.21}$$

$$\dot{z} = -(\sin\Theta_0)u + (\cos\Theta_0)w - (U_0\cos\Theta_0 + W_0\sin\Theta_0)\theta. \tag{12.3.22}$$

The altitude perturbation h of $H = H_0 + h$ thus satisfies

$$\dot{h} = (\sin\Theta_0)u - (\cos\Theta_0)w + (U_0\cos\Theta_0 + W_0\sin\Theta_0)\theta. \tag{12.3.23}$$

Note that (12.3.20)–(12.3.23) are expressed in terms of u, v, w, ϕ, θ, and ψ. Alternatively, (12.3.20)–(12.3.23) can be expressed in terms of u, θ, ϕ, ψ, $\delta\alpha$, and $\delta\beta$. In particular, using (12.3.2), (12.3.13) and (12.3.15), (12.3.20)–(12.3.23) can be rewritten as

$$\begin{aligned}\dot{x} = {}& \frac{\cos\Psi_0}{\cos\Theta_0}u + (W_0\sin\Psi_0)\phi + (W_0\cos\Theta_0 - U_0\sin\Theta_0)(\cos\Psi_0)\theta \\ & - (U_0\cos\Theta_0 + W_0\sin\Theta_0)(\sin\Psi_0)\psi + \frac{W_0\cos\Psi_0}{\cos\Theta_0}\delta\alpha - \frac{U_0\sin\Psi_0}{\cos\Theta_0}\delta\beta, \end{aligned} \tag{12.3.24}$$

$$\begin{aligned}\dot{y} = {}& \frac{\sin\Psi_0}{\cos\Theta_0}u - (W_0\cos\Psi_0)\phi + (W_0\cos\Theta_0 - U_0\sin\Theta_0)(\sin\Psi_0)\theta \\ & + (U_0\cos\Theta_0 + W_0\sin\Theta_0)(\cos\Psi_0)\psi + \frac{W_0\sin\Psi_0}{\cos\Theta_0}\delta\alpha + \frac{U_0\cos\Psi_0}{\cos\Theta_0}\delta\beta, \end{aligned} \tag{12.3.25}$$

$$\dot{z} = -(U_0\cos\Theta_0 + W_0\sin\Theta_0)\theta + \frac{U_0}{\cos\Theta_0}\delta\alpha. \tag{12.3.26}$$

The altitude perturbation h of $H = H_0 + h$ thus satisfies

$$\dot{h} = (U_0\cos\Theta_0 + W_0\sin\Theta_0)\theta - \frac{U_0}{\cos\Theta_0}\delta\alpha. \tag{12.3.27}$$

Next, substituting $\Psi = \Psi_0 + \psi$, $\Theta = \Theta_0 + \theta$, $\Phi = \Phi_0 + \phi = \phi$, $\alpha = \alpha_0 + \delta\alpha = \Theta_0 + \delta\alpha$, $\beta = \beta_0 + \delta\beta = \delta\beta$, $\eta = \eta_0 + \delta\eta = \Psi_0 + \delta\eta$, $\gamma = \gamma_0 + \delta\gamma = \delta\gamma$, and $\mu = \mu_0 + \delta\mu = \delta\mu$, in (10.8.5) yields

$$\mathcal{O}_2(-\delta\gamma)\mathcal{O}_1(-\delta\mu)\mathcal{O}_3(\delta\beta)\mathcal{O}_2(-\Theta_0 - \delta\alpha)\mathcal{O}_1(\phi)\mathcal{O}_2(\Theta_0 + \theta)\mathcal{O}_3(\psi - \delta\eta) = I. \tag{12.3.28}$$

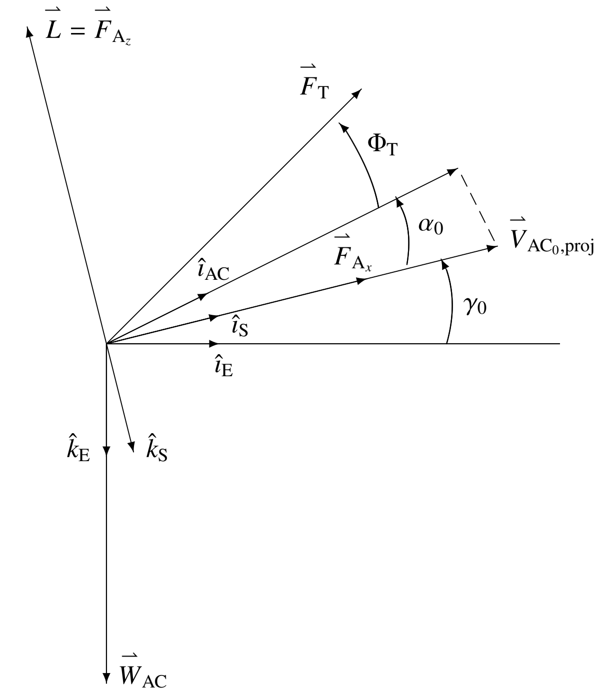


Figure 12.3.1: Steady flight definitions. All vectors shown lie in the aircraft plane of symmetry.

Using the Euler orientation matrices (2.12.14)–(2.12.16) with the trigonometric approximations (12.2.3)–(12.2.4) and neglecting the products of perturbation variables on the left side of (12.3.28) yields

$$\begin{bmatrix} 1 & -(\sin\Theta_0)\phi + \psi + \delta\beta - \delta\eta & -\theta + \delta\alpha + \delta\gamma \\ (\sin\Theta_0)\phi - \psi - \delta\beta + \delta\eta & 1 & -(\cos\Theta_0)\phi + \delta\mu \\ \theta - \delta\alpha - \delta\gamma & (\cos\Theta_0)\phi - \delta\mu & 1 \end{bmatrix} = I, \tag{12.3.29}$$

which yields the linearized relations

$$\theta = \delta\alpha + \delta\gamma, \tag{12.3.30}$$

$$\psi = \frac{-(\cos\Theta_0)\delta\beta + (\sin\Theta_0)\delta\mu + (\cos\Theta_0)\delta\eta}{\cos\Theta_0}, \tag{12.3.31}$$

$$\phi = \frac{\delta\mu}{\cos\Theta_0}. \tag{12.3.32}$$

Next, consider the perturbed angular-velocity components

$$\vec{\omega}_{AC/E}\Big|_{AC} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_0 + p \\ Q_0 + q \\ R_0 + r \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \tag{12.3.33}$$

and the perturbed 3-2-1 Euler angles

$$\begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix} = \begin{bmatrix} \Phi_0 + \phi \\ \Theta_0 + \theta \\ \Psi_0 + \psi \end{bmatrix} = \begin{bmatrix} \phi \\ \Theta_0 + \theta \\ \Psi_0 + \psi \end{bmatrix}. \tag{12.3.34}$$

Since Ψ_0 and Θ_0 are constant, linearizing (10.9.8) yields

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\Theta_0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\Theta_0 \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}, \tag{12.3.35}$$

that is,

$$p = \dot\phi - (\sin\Theta_0)\dot\psi, \tag{12.3.36}$$

$$q = \dot\theta, \tag{12.3.37}$$

$$r = (\cos\Theta_0)\dot\psi. \tag{12.3.38}$$

Therefore,

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\Theta_0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\Theta_0 \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & \tan\Theta_0 \\ 0 & 1 & 0 \\ 0 & 0 & \sec\Theta_0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \tag{12.3.39}$$

that is,

$$\dot\phi = p + (\tan\Theta_0)r, \tag{12.3.40}$$

$$\dot\theta = q, \tag{12.3.41}$$

$$\dot\psi = (\sec\Theta_0)r. \tag{12.3.42}$$

It thus follows from (12.3.30) and (12.3.37) that

$$q = \delta\dot\alpha + \delta\dot\gamma. \tag{12.3.43}$$

Furthermore, it follows from (12.3.31) and (12.3.38) that

$$r = -(\cos\Theta_0)\delta\dot\beta + (\sin\Theta_0)\delta\dot\mu + (\cos\Theta_0)\delta\dot\eta. \tag{12.3.44}$$

Next, consider the range equation (11.2.13) resolved in $\mathrm{F_{AC}}$ given by

$$m(\dot U - VR + WQ) = -(\sin\Theta)mg + F_{\mathrm{A}_x} + F_{\mathrm{T}_x}. \tag{12.3.45}$$

For straight-line steady flight, (12.3.45) becomes

$$m(\dot U_0 - V_0R_0 + W_0Q_0) = -(\sin\Theta_0)mg + F_{\mathrm{A}_{x0}} + F_{\mathrm{T}_{x0}}. \tag{12.3.46}$$

Since $\dot U_0 = Q_0 = R_0 = 0$, it follows from (12.3.46) that

$$-(\sin\Theta_0)mg + F_{\mathrm{A}_{x0}} + F_{\mathrm{T}_{x0}} = 0. \tag{12.3.47}$$

Now, replace each variable with the sum of its steady value and a perturbation to obtain

$$m[\dot U_0 + \dot u - (V_0 + v)(R_0 + r) + (W_0 + w)(Q_0 + q)] = -\sin(\Theta_0 + \theta)mg + F_{\mathrm{A}_{x0}} + f_{\mathrm{A}_x} + F_{\mathrm{T}_{x0}} + f_{\mathrm{T}_x}. \tag{12.3.48}$$

Using the trigonometric approximation (12.2.7) and neglecting the products of perturbation variables yields

$$\begin{aligned} m(\dot U_0 + \dot u - V_0r - R_0v - V_0R_0 + W_0Q_0 + W_0q + Q_0w) \\ = -[\sin\Theta_0 + (\cos\Theta_0)\theta]mg + F_{\mathrm{A}_{x0}} + f_{\mathrm{A}_x} + F_{\mathrm{T}_{x0}} + f_{\mathrm{T}_x}. \end{aligned} \tag{12.3.49}$$

Now subtracting the steady equation (12.3.47) from (12.3.49) and using the fact that $P_0 = Q_0 = R_0 = 0$ yields the linearized range-rate equation

$$m\dot{u} = -mW_0 q - (\cos\Theta_0)mg\theta + f_{\mathrm{A}_x} + f_{\mathrm{T}_x}, \tag{12.3.50}$$

which is linear in terms of the perturbation variables u, q, and θ.

Similarly, the linearized drift-rate and plunge-rate equations (11.2.14) and (11.2.15) are given by

$$m\dot{v} = -mU_0 r + mW_0 p + (\cos\Theta_0)mg\phi + f_{\mathrm{A}_y}, \tag{12.3.51}$$

$$m\dot{w} = mU_0 q - (\sin\Theta_0)mg\theta + f_{\mathrm{A}_z} + f_{\mathrm{T}_z}. \tag{12.3.52}$$

Using (12.3.13), (12.3.50), and (12.3.52) it follows that $\delta\alpha$ satisfies

$$m(\sec^2\alpha_0)\delta\dot{\alpha} = m\left(\frac{W_0^2}{U_0} + U_0\right)q + \left((\cos\Theta_0)\frac{W_0}{U_0} - \sin\Theta_0\right)mg\theta + f_{\mathrm{A}_z} + f_{\mathrm{T}_z} - \frac{W_0}{U_0}(f_{\mathrm{A}_x} + f_{\mathrm{T}_x}). \tag{12.3.53}$$

Finally, it follows from (11.3.16), (11.3.17), and (11.3.18) that the linearized roll-rate, pitch-rate, and yaw-rate equations are given by

$$J_{xx}\dot{p} - J_{xz}\dot{r} = l_{\mathrm{A}} + l_{\mathrm{T}}, \tag{12.3.54}$$

$$J_{yy}\dot{q} = m_{\mathrm{A}} + m_{\mathrm{T}}, \tag{12.3.55}$$

$$J_{zz}\dot{r} - J_{xz}\dot{p} = n_{\mathrm{A}} + n_{\mathrm{T}}. \tag{12.3.56}$$

12.4 Linearized Kinematics and Dynamics in the Case $\Theta_0 = 0$

In this section, we continue to consider the case where the aircraft flies in a horizontal straight line with wings level, constant angle of attack, and zero sideslip angle. In addition, we choose the aircraft frame $\mathrm{F_{AC}}$ such that $\hat{\imath}_{\mathrm{AC}} = \hat{V}_{\mathrm{AC}_0}$. Since $\beta_0 = 0$, it follows that $\hat{V}_{\mathrm{AC}_0,\mathrm{proj}} = \hat{V}_{\mathrm{AC}_0}$, and thus (10.4.5) implies that

$$\alpha_0 = \theta_{\hat{\imath}_{\mathrm{AC}}/\hat{V}_{\mathrm{AC}_0}/\hat{\jmath}_{\mathrm{AC}}} = 0. \tag{12.4.1}$$

Hence, (12.3.2) implies that $\Theta_0 = 0$, and (12.3.6) implies that $W_0 = 0$. Therefore, for this choice of $\mathrm{F_{AC}}$, $\Theta_0 = 0$. The benefit of working with this choice of $\mathrm{F_{AC}}$ instead of $\mathrm{F_{AC}}$ is the fact that many expressions are simplified in the case where $\Theta_0 = 0$. It is important to keep in mind that the frame $\mathrm{F_{AC}}$ is body-fixed but depends on the steady flight condition.

With this choice of $\mathrm{F_{AC}}$, it follows from (12.3.16)–(12.3.18) that

$$\dot{X}_0 = U_0, \tag{12.4.2}$$

$$\dot{Y}_0 = 0, \tag{12.4.3}$$

$$\dot{Z}_0 = 0. \tag{12.4.4}$$

Furthermore, it follows from (12.3.20)–(12.3.22) and (12.3.30) that the perturbations x, y, z of X_0, Y_0, Z_0 satisfy

$$\dot{x} = u, \tag{12.4.5}$$

$$\dot{y} = v + U_0\psi, \tag{12.4.6}$$

$$\dot{z} = U_0\delta\alpha - U_0\theta = -U_0\delta\gamma. \tag{12.4.7}$$

The altitude perturbation $h = -z$ of $H = H_0 + h$ thus satisfies

$$\dot{h} = U_0(\theta - \delta\alpha) = U_0\delta\gamma. \tag{12.4.8}$$

Next, (12.3.10) and (12.3.14) become

$$\delta\alpha = \frac{1}{U_0}w, \tag{12.4.9}$$

$$\delta\beta = \frac{1}{U_0}v. \tag{12.4.10}$$

It thus follows from (12.3.31), (12.4.6), and (12.4.10) that

$$\dot{y} = U_0(\delta\beta + \psi) = U_0\delta\eta. \tag{12.4.11}$$

Next, (12.3.30), (12.3.31), and (12.3.32) become

$$\theta = \delta\alpha + \delta\gamma, \tag{12.4.12}$$

$$\psi = -\delta\beta + \delta\eta, \tag{12.4.13}$$

$$\phi = \delta\mu. \tag{12.4.14}$$

Furthermore, it follows from (12.3.40)–(12.3.42) that

$$\dot{\phi} = p, \tag{12.4.15}$$

$$\dot{\theta} = q, \tag{12.4.16}$$

$$\dot{\psi} = r. \tag{12.4.17}$$

Hence, the perturbations of the angular-velocity components coincide with the perturbations of the Euler-angle rates. We thus refer to p, q, and r as the roll-rate, pitch-rate, and yaw-rate perturbations, respectively. In addition, the linearized longitudinal equations (12.3.50), (12.3.51), and (12.3.52) become

$$m\dot{u} = -mg\theta + f_{\mathrm{A}_x} + f_{\mathrm{T}_x}, \tag{12.4.18}$$

$$m\dot{v} = -mU_0r + mg\phi + f_{\mathrm{A}_y}, \tag{12.4.19}$$

$$m\dot{w} = mU_0q + f_{\mathrm{A}_z} + f_{\mathrm{T}_z}. \tag{12.4.20}$$

Using (12.4.10), equation (12.4.19) becomes

$$mU_0\delta\dot{\beta} = -mU_0r + mg\phi + f_{\mathrm{A}_y}. \tag{12.4.21}$$

Finally, using (12.4.9), equation (12.4.20) becomes

$$mU_0\delta\dot{\alpha} = mU_0q + f_{\mathrm{A}_z} + f_{\mathrm{T}_z}. \tag{12.4.22}$$

12.5 Summary of the Aircraft Equations of Motion Linearized at Straight-Line, Horizontal, Wings-Level, Zero-Sideslip Steady Flight

Assume $\Psi_0 = 0$.

Translational kinematics

$$\dot{x} = (\cos\Theta_0)u + (\sin\Theta_0)w + (W_0\cos\Theta_0 - U_0\sin\Theta_0)\theta, \tag{12.5.1}$$

$$\dot{y} = v - W_0\phi + (U_0\cos\Theta_0 + W_0\sin\Theta_0)\psi, \tag{12.5.2}$$

$$\dot{z} = -(\sin\Theta_0)u + (\cos\Theta_0)w - (U_0\cos\Theta_0 + W_0\sin\Theta_0)\theta. \tag{12.5.3}$$

Rotational kinematics

$$p = \dot{\phi} - (\sin\Theta_0)\dot{\psi}, \tag{12.5.4}$$
$$q = \dot{\theta}, \tag{12.5.5}$$
$$r = (\cos\Theta_0)\dot{\psi}. \tag{12.5.6}$$

Translational momentum

$$m\dot{u} = -mW_0 q - (\cos\Theta_0)mg\theta + f_{\mathrm{A}_x} + f_{\mathrm{T}_x}, \tag{12.5.7}$$
$$m\dot{v} = -mU_0 r + mW_0 p + (\cos\Theta_0)mg\phi + f_{\mathrm{A}_y}, \tag{12.5.8}$$
$$m\dot{w} = mU_0 q - (\sin\Theta_0)mg\theta + f_{\mathrm{A}_z} + f_{\mathrm{T}_z}. \tag{12.5.9}$$

Rotational momentum

$$J_{xx}\dot{p} - J_{xz}\dot{r} = l_{\mathrm{A}} + l_{\mathrm{T}}, \tag{12.5.10}$$
$$J_{yy}\dot{q} = m_{\mathrm{A}} + m_{\mathrm{T}}, \tag{12.5.11}$$
$$J_{zz}\dot{r} - J_{xz}\dot{p} = n_{\mathrm{A}} + n_{\mathrm{T}}. \tag{12.5.12}$$

12.6 Linearized Aircraft Equations of Motion in State Space Form

Consider the state-space model $\dot{x} = Ax + Bu$, where the state $x \in \mathbb{R}^{12}$ and the input $u \in \mathbb{R}^{11}$ are defined by

$$x \triangleq \begin{bmatrix} x & y & z & u & v & w & \phi & \theta & \psi & p & q & r \end{bmatrix}^{\mathrm{T}}, \tag{12.6.1}$$
$$u \triangleq \begin{bmatrix} f_{\mathrm{A}_x} & f_{\mathrm{A}_y} & f_{\mathrm{A}_z} & l_{\mathrm{A}} & m_{\mathrm{A}} & n_{\mathrm{A}} & f_{\mathrm{T}_x} & f_{\mathrm{T}_z} & l_{\mathrm{T}} & m_{\mathrm{T}} & n_{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{12.6.2}$$

The matrices $A \in \mathbb{R}^{12\times 12}$ and $B \in \mathbb{R}^{12\times 11}$ corresponding to the linearized kinematics and dynamics (12.5.1)–(12.5.12) are given by

$$A = \begin{bmatrix}
0 & 0 & 0 & \cos\Theta_0 & 0 & \sin\Theta_0 & 0 & W_0(\cos\Theta_0) - U_0\sin\Theta_0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & -W_0 & 0 & U_0(\cos\Theta_0) + W_0\sin\Theta_0 & 0 & 0 & 0\\
0 & 0 & 0 & -\sin\Theta_0 & 0 & \cos\Theta_0 & 0 & -U_0(\cos\Theta_0) - W_0\sin\Theta_0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\cos\Theta_0)g & 0 & 0 & -W_0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & (\cos\Theta_0)g & 0 & 0 & W_0 & 0 & -U_0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\Theta_0 & 0 & 0 & U_0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tan\Theta_0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sec\Theta_0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{12.6.3}$$

■

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{m} & 0 & 0 & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{m} & 0 & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & 0 & \frac{J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xz}}{J_{xx}J_{zz}-J_{xz}^2} \\
0 & 0 & 0 & 0 & \frac{1}{J_{yy}} & 0 & 0 & 0 & 0 & \frac{1}{J_{yy}} & 0 \\
0 & 0 & 0 & \frac{J_{xz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xx}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & 0 & \frac{J_{xz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xx}}{J_{xx}J_{zz}-J_{xz}^2}
\end{bmatrix}. \tag{12.6.4}$$

12.7 Problems

Problem 12.7.1. A stunt plane is flying in steady circular flight, where the circular flight path is contained in a vertical plane. The steady sideslip angle, steady angle of attack, and steady roll angles are zero. At the lowest point on the circle the pilot's head is closer to the ground than the pilot's feet are. The plane completes one revolution in 83 sec, and the radius of the circle is 4,100 ft. Resolve $\vec{V}_{\mathrm{AC}}$ and $\vec{\omega}_{\mathrm{AC/E}}$ in the aircraft frame. Draw a diagram that illustrates your solution.

Problem 12.7.2. The acceleration due to gravity on the surface of the Earth is 9.8 m/sec^2, and the radius of the Earth is approximately 6.3×10^6 m. Use this data and Newton's law of universal gravitation to compute $\mu_{\mathrm{E}} = Gm_{\mathrm{E}}$, where G is the universal gravitational constant and m_{E} is the mass of the Earth.

Problem 12.7.3. Let $\mathrm{F_E}$ denote a star-oriented frame with origin $\mathrm{O_E}$ at the center of the Earth, and let $\mathrm{F_M}$ denote a body-fixed Moon frame with origin $\mathrm{O_M}$ at the center of the Moon. Next, define the position vector of the Moon relative to the Earth by

$$\vec{r}_{\mathrm{M/E}} \triangleq \vec{r}_{\mathrm{O_M/O_E}}$$

and the velocity vector of the Moon relative to the center of the Earth with respect to the star frame by

$$\vec{V}_{\mathrm{M}} \triangleq \vec{v}_{\mathrm{O_M/O_E/E}} = \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{M/E}} = \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{O_M/O_E}}.$$

Draw a diagram that shows the velocity vector $\vec{V}_{\mathrm{M}}$ and the angular velocity vector $\vec{\omega}_{\mathrm{M/E}}$. Explain why the Moon is in steady flight around the Earth, that is,

$$\overset{\mathrm{M}\bullet}{\vec{V}}_{\mathrm{M}} = 0,$$

$$\overset{\mathrm{M}\bullet}{\vec{\omega}}_{\mathrm{M/E}} = \overset{\mathrm{E}\bullet}{\vec{\omega}}_{\mathrm{M/E}} = 0.$$

Furthermore, by attaching another frame to the invisible arm linking the Earth to the Moon, show that

$$\overset{\mathrm{M}\bullet}{\vec{r}}_{\mathrm{M/E}} = 0.$$

Furthermore, using the fact that the centripetal acceleration of the Moon is equal to the acceleration of the Moon as given by Newton's law of universal gravitation, determine the distance from the Earth to the Moon by deriving and using the relation

$$|\vec{r}_{\mathrm{M/E}}| = \sqrt[3]{\frac{\mu_{\mathrm{E}}}{|\vec{\omega}_{\mathrm{M/E}}|^2}}.$$

(Hints: You will need the value of $\mu_{\mathrm{E}} = Gm_{\mathrm{E}}$ from the previous problem as well as the facts that the period of the Moon's orbit around the Earth is 28 days, the Moon's orbit around the Earth is a circle, and the same "side" of the Moon is always facing the Earth. Note that

$$\vec{V}_{\mathrm{M}} = \vec{\omega}_{\mathrm{M/E}} \times \vec{r}_{\mathrm{M/E}}$$

and that the centripetal acceleration of the Moon is

$$\vec{\alpha}_{\mathrm{cent}} = \vec{\omega}_{\mathrm{M/E}} \times \vec{V}_{\mathrm{M}}.$$

Next, show that the magnitude of $\vec{\alpha}_{\mathrm{cent}}$ is equal to the magnitude of the acceleration $\vec{g}$ due to Earth's gravity at the Moon's location, which is given by

$$|\vec{g}| = \frac{\mu_{\mathrm{E}}}{|\vec{r}_{\mathrm{M/E}}|^2}.)$$

(Remark: This exercise and Problem 12.7.3 show that knowing only the acceleration due to gravity on the surface of the Earth, the period of the Moon around the Earth, and the radius of the Earth, it is possible to determine the distance from the Earth to the Moon.)

Problem 12.7.4. Derive (12.3.10) and (12.3.14).

Problem 12.7.5. Derive (12.3.35).

Problem 12.7.6. Derive the linearized sway and plunge equations (12.3.51) and (12.3.52).

Problem 12.7.7. Derive the linearized roll, pitch, and yaw equations (12.3.54)–(12.3.56).

Problem 12.7.8. Consider the rotational kinematics equation

$$\dot{\Phi} = P + Q(\sin\Phi)\tan\Theta + R(\cos\Phi)\tan\Theta.$$

Linearize this equation near the steady (not necessarily zero) values $(P_0, Q_0, R_0, \Theta_0, \Phi_0)$. (Hint: Note that $(\mathrm{d}/\mathrm{d}\Theta)\tan\Theta = \sec^2\Theta$.)

Symbol	Definition
f_{A_x}	Frontal aerodynamic force perturbation in F_{AC}
f_{A_y}	Side aerodynamic force perturbation in F_{AC}
f_{A_z}	Downward aerodynamic force perturbation in F_{AC}
f_{T_x}	Thrust force perturbation in F_{AC} along $\hat{\imath}_{AC}$
f_{T_z}	Thrust force perturbation in F_{AC} along $\hat{k}_{AC}$
l_{AC}	Total roll-moment perturbation on the aircraft
m_{AC}	Total pitch-moment perturbation on the aircraft
n_{AC}	Total yaw-moment perturbation on the aircraft
l_A	Aerodynamic roll-moment perturbation in F_{AC}
m_A	Aerodynamic pitch-moment perturbation in F_{AC}
n_A	Aerodynamic yaw-moment perturbation in F_{AC}
l_T	Thrust roll-moment perturbation in F_{AC}
m_T	Thrust pitch-moment perturbation in F_{AC}
n_T	Thrust yaw-moment perturbation in F_{AC}

Table 12.1: Symbols for Chapter 12.

Chapter Thirteen
Static Stability and Stability Derivatives

Static stability means that the initial motion of the aircraft after a perturbation from steady flight is such that the magnitude of the perturbation decreases. Static stability is determined by the sign of the *stability derivatives*, which are the partial derivatives of the forces and moments with respect to perturbations from steady flight.

We use the following conventions for the signs of the control-surface deflections. For the rudder deflection, $\delta r > 0$ means trailing edge is left. This deflection causes negative yaw. For the elevator deflection, $\delta e > 0$ means trailing edge down. This deflection causes negative pitch. For the aileron deflection, $\delta a > 0$ means right aileron up. This deflection causes positive roll. The ailerons always move in opposite directions.

The analysis of aerodynamic forces and moments is greatly facilitated by choosing F_{AC} such that $\Theta_0 = 0$, and thus $W_0 = 0$ and $\alpha_0 = 0$. We make this assumption in all discussions of aerodynamic effects.

13.1 Force Coefficients

Let

$$\begin{aligned} S &\triangleq \text{wing area}, \\ b &\triangleq \text{wing tip-to-tip distance}, \\ \overline{c} &\triangleq \text{wing mean chord}, \\ \rho &\triangleq \text{air density}, \\ V_{\mathrm{AC}} &\triangleq \text{aircraft speed}, \\ p_{\mathrm{d}} &\triangleq \text{dynamic pressure} = \tfrac{1}{2}\rho V_{\mathrm{AC}}^2. \end{aligned}$$

These data are used to nondimensionalize physical variables.

The *drag coefficient* is defined by

$$C_D \triangleq \frac{D}{p_{\mathrm{d}} S}, \tag{13.1.1}$$

the *side drag coefficient* is defined by

$$C_E \triangleq \frac{E}{p_{\mathrm{d}} S}, \tag{13.1.2}$$

and the *lift coefficient* is defined by

$$C_L \triangleq \frac{L}{p_{\mathrm{d}} S}. \tag{13.1.3}$$

For the force components in the aircraft frame, the *aerodynamic force coefficients* are defined by

$$C_x \triangleq \frac{F_{\mathrm{A}_x}}{p_{\mathrm{d}}S}, \tag{13.1.4}$$

$$C_y \triangleq \frac{F_{\mathrm{A}_y}}{p_{\mathrm{d}}S}, \tag{13.1.5}$$

$$C_z \triangleq \frac{F_{\mathrm{A}_z}}{p_{\mathrm{d}}S}. \tag{13.1.6}$$

Likewise, we define the thrust coefficients

$$C_{\mathrm{T}x} \triangleq \frac{F_{\mathrm{T}_x}}{p_{\mathrm{d}}S}, \tag{13.1.7}$$

$$C_{\mathrm{T}y} \triangleq \frac{F_{\mathrm{T}_y}}{p_{\mathrm{d}}S}, \tag{13.1.8}$$

$$C_{\mathrm{T}z} \triangleq \frac{F_{\mathrm{T}_z}}{p_{\mathrm{d}}S}. \tag{13.1.9}$$

13.2 Steady Force Coefficients

We assume horizontal straight-line steady flight. The steady velocity vector $\vec{V}_{\mathrm{AC}_0}$ resolved in F_{AC} is given by

$$\vec{V}_{\mathrm{AC}_0}\Big|_{\mathrm{AC}} = \begin{bmatrix} U_0 \\ 0 \\ 0 \end{bmatrix}. \tag{13.2.1}$$

Therefore, the steady dynamic pressure is given by

$$p_{\mathrm{d}_0} = \tfrac{1}{2}\rho U_0^2. \tag{13.2.2}$$

The steady drag, side drag, and lift coefficients are defined by

$$C_{D_0} \triangleq \frac{D_0}{p_{\mathrm{d}_0}S}, \tag{13.2.3}$$

$$C_{E_0} \triangleq \frac{E_0}{p_{\mathrm{d}_0}S}, \tag{13.2.4}$$

$$C_{L_0} \triangleq \frac{L_0}{p_{\mathrm{d}_0}S}. \tag{13.2.5}$$

It follows from (13.1.4)–(13.1.6) that

$$C_{x_0} = \frac{F_{\mathrm{A}_{x_0}}}{p_{\mathrm{d}_0}S}, \tag{13.2.6}$$

$$C_{y_0} = \frac{F_{\mathrm{A}_{y_0}}}{p_{\mathrm{d}_0}S}, \tag{13.2.7}$$

$$C_{z_0} = \frac{F_{\mathrm{A}_{z_0}}}{p_{\mathrm{d}_0}S}. \tag{13.2.8}$$

For a symmetric aircraft, it follows that

$$F_{\mathrm{A}_{y0}} = -E_0 = 0, \tag{13.2.9}$$

and thus

$$C_{y_0} = 0. \tag{13.2.10}$$

Furthermore, it follows from (11.1.21) that

$$C_x = -(\cos\alpha)(\cos\beta)C_D + (\cos\alpha)(\sin\beta)C_E + (\sin\alpha)C_L, \tag{13.2.11}$$

and thus

$$C_{x_0} = -C_{D_0}, \tag{13.2.12}$$

which is consistent with the fact that, during steady flight, $\hat{\imath}_{\mathrm{AC}} = \hat{\imath}_{\mathrm{W}}$.

The steady thrust coefficients are given by

$$C_{\mathrm{T}x_0} \triangleq \frac{F_{\mathrm{T}_{x0}}}{p_{\mathrm{d}_0}S}, \tag{13.2.13}$$

$$C_{\mathrm{T}y_0} \triangleq \frac{F_{\mathrm{T}_{y0}}}{p_{\mathrm{d}_0}S}, \tag{13.2.14}$$

$$C_{\mathrm{T}z_0} \triangleq \frac{F_{\mathrm{T}_{z0}}}{p_{\mathrm{d}_0}S}. \tag{13.2.15}$$

13.3 Linearization of Forces

13.3.1 Lift Coefficient

Expanding (13.1.3) at steady flight yields

$$\begin{aligned} C_L(u, q, r, \delta\alpha, \delta\dot{\alpha}, \delta\beta, \delta\dot{\beta}, \delta e) \approx C_{L_0} + \frac{1}{U_0}C_{L_{u_0}}u + \frac{\bar{c}}{2U_0}C_{L_{q_0}}q + \frac{b}{2U_0}C_{L_{r_0}}r \\ + C_{L_{\alpha_0}}\delta\alpha + \frac{\bar{c}}{2U_0}C_{L_{\dot{\alpha}_0}}\delta\dot{\alpha} + C_{L_{\beta_0}}\delta\beta + \frac{b}{2U_0}C_{L_{\dot{\beta}_0}}\delta\dot{\beta} + C_{L_{\delta e_0}}\delta e, \end{aligned} \tag{13.3.1}$$

where C_{L_0} is the steady lift coefficient. It is usually the case that $C_{L_{\alpha_0}} > 0$ and $C_{L_{\delta e_0}} > 0$. The dynamic stability derivatives $C_{L_{u_0}}$, $C_{L_{q_0}}$, and $C_{L_{r_0}}$ are nondimensionalized using the conventions

$$C_{L_{u_0}} \triangleq \left.\frac{\partial C_L}{\partial\left(\frac{u}{U_0}\right)}\right|_0, \tag{13.3.2}$$

$$C_{L_{q_0}} \triangleq \left.\frac{\partial C_L}{\partial\left(\frac{\bar{c}q}{2U_0}\right)}\right|_0, \tag{13.3.3}$$

$$C_{L_{r_0}} \triangleq \left.\frac{\partial C_L}{\partial\left(\frac{br}{2U_0}\right)}\right|_0, \tag{13.3.4}$$

$$C_{L_{\alpha_0}} \triangleq \left.\frac{\partial C_L}{\partial \delta\alpha}\right|_0, \tag{13.3.5}$$

$$C_{L_{\dot{\alpha}_0}} \triangleq \left.\frac{\partial C_L}{\partial\left(\frac{\bar{c}\delta\dot{\alpha}}{2U_0}\right)}\right|_0, \tag{13.3.6}$$

$$C_{L_{\beta_0}} \triangleq \left.\frac{\partial C_L}{\partial \delta\beta}\right|_0, \tag{13.3.7}$$

$$C_{L_{\dot{\beta}_0}} \triangleq \left.\frac{\partial C_L}{\partial\left(\frac{b\delta\dot{\beta}}{2U_0}\right)}\right|_0. \tag{13.3.8}$$

13.3.2 Drag Coefficient

Ignoring the dependence on the elevator δe, we write the drag polar as the parabolic function

$$C_D = C_{D_{\rm par}} + \frac{C_L^2}{\pi e \mathrm{AR}} = C_{D_{\rm par}} + KC_L^2, \tag{13.3.9}$$

where $\mathrm{AR} \triangleq b^2/S$ is the aspect ratio, e is the Oswald efficiency factor, and $K \triangleq 1/(\pi e\mathrm{AR})$. Typically, $e \approx 0.8$, while e always satisfies $e < 1$, which accounts for nonelliptical lift distribution. $C_{D_{\rm par}}$ is the parasitic drag coefficient. Hence,

$$C_{D_0} \triangleq C_{D_0} = C_{D_{\rm par}} + KC_{L_0}^2 \geq C_{D_{\rm par}}, \tag{13.3.10}$$

where C_{L_0} is the lift coefficient in steady flight, and $C_{L_0} > 0$ by assumption.

Expanding C_D, we have

$$\begin{aligned} C_D(u, q, r, \delta\alpha, \delta\dot{\alpha}, \delta\beta, \delta\dot{\beta}, \delta e) \approx{}& C_{D_0} + \frac{1}{U_0}C_{D_{u_0}}u + \frac{\bar{c}}{2U_0}C_{D_{q_0}}q + \frac{b}{2U_0}C_{D_{r_0}}r \\ &+ C_{D_{\alpha_0}}\delta\alpha + C_{D_{\dot{\alpha}_0}}\delta\dot{\alpha} + C_{D_{\beta_0}}\delta\beta + C_{D_{\dot{\beta}_0}}\delta\dot{\beta} + C_{D_{\delta e_0}}\delta e. \end{aligned} \tag{13.3.11}$$

Differentiating (13.3.9) with respect to α yields

$$C_{D_{\alpha_0}} = \left.\frac{\partial}{\partial\alpha}[C_{D_{\rm par}} + KC_L^2]\right|_0 = 2KC_{L_0}C_{L_{\alpha_0}}, \tag{13.3.12}$$

and similarly for $u, q, r, \delta\dot{\alpha}, \delta\beta, \delta\dot{\beta}, \delta e$. We thus have

$$\begin{aligned} C_D(u, q, r, \delta\alpha, \delta\dot{\alpha}, \delta\beta, \delta\dot{\beta}, \delta e) \approx{}& C_{D_0} + \frac{2}{U_0}KC_{L_0}C_{L_{u_0}}u + \frac{\bar{c}}{U_0}KC_{L_0}C_{L_{q_0}}q + \frac{b}{U_0}KC_{L_0}C_{L_{r_0}}r \\ &+ 2KC_{L_0}C_{L_{\alpha_0}}\delta\alpha + 2KC_{L_0}C_{L_{\dot{\alpha}_0}}\delta\dot{\alpha} \\ &+ 2KC_{L_0}C_{L_{\beta_0}}\delta\beta + 2KC_{L_0}C_{L_{\dot{\beta}_0}}\delta\dot{\beta} + 2KC_{L_0}C_{L_{\delta e_0}}\delta e. \end{aligned} \tag{13.3.13}$$

Note that the steady value of δe is defined to be zero, although the trim angle of the elevator depends on the flight condition.

13.3.3 Force Coefficients in $\mathrm{F_{AC}}$

We now look at perturbations of the steady flight forces. The goal is to express the perturbed aerodynamic forces in terms of the stability derivatives. We represent the force perturbations in the aircraft frame. Perturbing $\vec{V}_{\mathrm{AC}}$ yields

$$\vec{V}_{\mathrm{AC}}\Big|_{\mathrm{AC}} = \begin{bmatrix} U_0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} U_0 + u \\ v \\ w \end{bmatrix}. \tag{13.3.14}$$

For the perturbed angular velocity $\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}}$, we have

$$\vec{\omega}_{\mathrm{AC/E}}\Big|_{\mathrm{AC}} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{13.3.15}$$

The perturbed aerodynamic forces resolved in the aircraft frame are

$$\vec{F}_{\mathrm{A}}\Big|_{\mathrm{AC}} = \begin{bmatrix} F_{\mathrm{A}_x} \\ F_{\mathrm{A}_y} \\ F_{\mathrm{A}_z} \end{bmatrix} = \begin{bmatrix} F_{\mathrm{A}_{x0}} + f_{\mathrm{A}_x} \\ F_{\mathrm{A}_{y0}} + f_{\mathrm{A}_y} \\ F_{\mathrm{A}_{z0}} + f_{\mathrm{A}_z} \end{bmatrix}, \tag{13.3.16}$$

where f_{A_x}, f_{A_y}, and f_{A_z} are perturbations to the components of the aerodynamic force resolved in the aircraft frame. We can express f_{A_x} and f_{A_z} as functions of u, $\delta\alpha$, $\delta\dot{\alpha}$, and δe, and f_{A_y} as a function of $v, p, \delta\beta$, $\delta\dot{\beta}$, δa, and δr. We do not consider the perturbations v and w since these perturbations are captured by $\delta\beta$ and $\delta\alpha$, respectively, as shown by (12.3.14) and (12.3.10), respectively.

We thus have the linear approximations

$$f_{\mathrm{A}_x}(u,q,\delta\alpha,\delta\dot{\alpha},\delta e) \approx \frac{\partial f_{\mathrm{A}_x}}{\partial u}\bigg|_0 u + \frac{\partial f_{\mathrm{A}_x}}{\partial q}\bigg|_0 q + \frac{\partial f_{\mathrm{A}_x}}{\partial \delta\alpha}\bigg|_0 \delta\alpha + \frac{\partial f_{\mathrm{A}_x}}{\partial \delta\dot{\alpha}}\bigg|_0 \delta\dot{\alpha} + \frac{\partial f_{\mathrm{A}_x}}{\partial \delta e}\bigg|_0 \delta e, \tag{13.3.17}$$

$$f_{\mathrm{A}_y}(p,r,\delta\beta,\delta\dot{\beta},\delta a,\delta r) \approx \frac{\partial f_{\mathrm{A}_y}}{\partial p}\bigg|_0 p + \frac{\partial f_{\mathrm{A}_y}}{\partial r}\bigg|_0 r + \frac{\partial f_{\mathrm{A}_y}}{\partial \delta\beta}\bigg|_0 \delta\beta + \frac{\partial f_{\mathrm{A}_y}}{\partial \delta\dot{\beta}}\bigg|_0 \delta\dot{\beta} + \frac{\partial f_{\mathrm{A}_y}}{\partial \delta a}\bigg|_0 \delta a + \frac{\partial f_{\mathrm{A}_y}}{\partial \delta r}\bigg|_0 \delta r, \tag{13.3.18}$$

$$f_{\mathrm{A}_z}(u,q,\delta\alpha,\delta\dot{\alpha},\delta e) \approx \frac{\partial f_{\mathrm{A}_z}}{\partial u}\bigg|_0 u + \frac{\partial f_{\mathrm{A}_z}}{\partial q}\bigg|_0 q + \frac{\partial f_{\mathrm{A}_z}}{\partial \delta\alpha}\bigg|_0 \delta\alpha + \frac{\partial f_{\mathrm{A}_z}}{\partial \delta\dot{\alpha}}\bigg|_0 \delta\dot{\alpha} + \frac{\partial f_{\mathrm{A}_z}}{\partial \delta e}\bigg|_0 \delta e. \tag{13.3.19}$$

To obtain nondimensional partial derivatives of dimensionless coefficients, we use $\frac{u}{U_0}$, $\frac{bp}{2U_0}$, $\frac{\bar{c}q}{2U_0}$, $\frac{br}{2U_0}$, $\frac{\bar{c}\delta\dot{\alpha}}{2U_0}$, and $\frac{b\delta\dot{\beta}}{2U_0}$. For example, (13.3.17) becomes

$$f_{\mathrm{A}_x}(u,q,\delta\alpha,\delta\dot{\alpha},\delta e) = \frac{\partial f_{\mathrm{A}_x}}{\partial\left(\frac{u}{U_0}\right)}\Bigg|_0 \frac{1}{U_0}u + \frac{\partial f_{\mathrm{A}_x}}{\partial\left(\frac{\bar{c}q}{2U_0}\right)}\Bigg|_0 \frac{\bar{c}}{2U_0}q + \frac{\partial f_{\mathrm{A}_x}}{\partial \delta\alpha}\bigg|_0 \delta\alpha + \frac{\partial f_{\mathrm{A}_x}}{\partial\left(\frac{\bar{c}\delta\dot{\alpha}}{2U_0}\right)}\Bigg|_0 \frac{\bar{c}}{2U_0}\delta\dot{\alpha} + \frac{\partial f_{\mathrm{A}_x}}{\partial \delta e}\bigg|_0 \delta e, \tag{13.3.20}$$

and likewise for (13.3.18) and (13.3.19).

13.3.4 Linearization of F_{A_x}

We write

$$F_{\mathrm{A}_x} = p_{\mathrm{d}} S C_x \approx F_{\mathrm{A}_{x_0}} + f_{\mathrm{A}_x}. \tag{13.3.21}$$

Note that

$$p_{\mathrm{d}} \triangleq \frac{1}{2}\rho V_{\mathrm{AC}}^2 = \frac{1}{2}\rho[(U_0 + u)^2 + v^2 + w^2]. \tag{13.3.22}$$

Using (13.3.21), the first partial in (13.3.20) is given by

$$\left.\frac{\partial f_{\mathrm{A}_x}}{\partial\left(\frac{u}{U_0}\right)}\right|_0 = \left.\frac{\partial(p_{\mathrm{d}} S C_x)}{\partial\left(\frac{u}{U_0}\right)}\right|_0 = p_{\mathrm{d}_0} S \left.\frac{\partial C_x}{\partial\left(\frac{u}{U_0}\right)}\right|_0 + \left.\frac{\partial p_{\mathrm{d}}}{\partial\left(\frac{u}{U_0}\right)}\right|_0 S C_{x_0}. \tag{13.3.23}$$

Now, we define

$$C_{x_{u_0}} \triangleq \left.\frac{\partial C_x}{\partial\left(\frac{u}{U_0}\right)}\right|_0, \tag{13.3.24}$$

$$p_{\mathrm{d}_{u_0}} \triangleq \left.\frac{\partial p_{\mathrm{d}}}{\partial\left(\frac{u}{U_0}\right)}\right|_0 \tag{13.3.25}$$

so that

$$\left.\frac{\partial f_{\mathrm{A}_x}}{\partial\left(\frac{u}{U_0}\right)}\right|_0 = p_{\mathrm{d}_0} S C_{x_{u_0}} + p_{\mathrm{d}_{u_0}} S C_{x_0}. \tag{13.3.26}$$

Differentiating (13.2.11) with respect to u/U_0 yields

$$C_{x_{u_0}} = -C_{D_{u_0}}, \tag{13.3.27}$$

where $C_{D_{u_0}}$ is the *speed damping derivative*.

Next, to determine $p_{\mathrm{d}_{u_0}}$, we use (12.3.13) to express w in (13.3.22) in terms of u and $\delta\alpha$. Therefore,

$$\left.\frac{\partial p_{\mathrm{d}}}{\partial u}\right|_0 = \rho(U_0 + u) \approx \rho U_0, \tag{13.3.28}$$

and thus, from (13.3.25),

$$p_{\mathrm{d}_{u_0}} = \rho U_0^2 = 2 p_{\mathrm{d}_0}. \tag{13.3.29}$$

Therefore,

$$\left.\frac{\partial f_{\mathrm{A}_x}}{\partial\left(\frac{u}{U_0}\right)}\right|_0 = -p_{\mathrm{d}_0} S (2C_{D_0} + C_{D_{u_0}}). \tag{13.3.30}$$

Next, since $p_{\mathrm{d}_{q_0}} = 0$, it follows that

$$\left.\frac{\partial f_{\mathrm{A}_x}}{\partial\left(\frac{\bar{c}q}{2U_0}\right)}\right|_0 = -p_{\mathrm{d}_0} S C_{D_{q_0}}. \tag{13.3.31}$$

Finally, since $p_{\mathrm{d}_{\alpha_0}} = 0$ it follows that

$$\left.\frac{\partial f_{\mathrm{A}_x}}{\partial \delta\alpha}\right|_0 = p_{\mathrm{d}_0} S (C_{L_0} - C_{D_{\alpha_0}}). \tag{13.3.32}$$

Substituting (13.3.30), (13.3.31), and (13.3.32), into (13.3.20) yields

$$\begin{aligned} f_{\mathrm{A}_x}(u, q, \delta\alpha, \delta\dot{\alpha}, \delta e) \approx &-\frac{p_{\mathrm{d}_0} S}{U_0}(2C_{D_0} + C_{D_{u_0}})u - \frac{p_{\mathrm{d}_0} S \bar{c}}{2U_0} C_{D_{q_0}} q \\ &+ p_{\mathrm{d}_0} S (C_{L_0} - C_{D_{\alpha_0}})\delta\alpha - \frac{p_{\mathrm{d}_0} S \bar{c}}{2U_0} C_{D_{\dot{\alpha}_0}} \delta\dot{\alpha} - p_{\mathrm{d}_0} S C_{D_{\delta e_0}} \delta e. \end{aligned} \tag{13.3.33}$$

13.3.5 Linearization of F_{A_y}

From (13.1.5) we have

$$F_{\mathrm{A}_y} = p_{\mathrm{d}} S C_y \approx F_{\mathrm{A}_{y_0}} + f_{A_y}. \tag{13.3.34}$$

It thus follows from (11.1.21) that

$$C_y = -(\sin\beta)C_D - (\cos\beta)C_E, \tag{13.3.35}$$

and thus

$$C_{y_0} = -C_{E_0} = 0. \tag{13.3.36}$$

Using (13.3.34) and (13.3.35) yields

$$\left.\frac{\partial f_{\mathrm{A}_y}}{\partial\left(\frac{bp}{2U_0}\right)}\right|_0 = -p_{\mathrm{d}_0} S C_{E_{p_0}}, \tag{13.3.37}$$

$$\left.\frac{\partial f_{\mathrm{A}_y}}{\partial\left(\frac{br}{2U_0}\right)}\right|_0 = -p_{\mathrm{d}_0} S C_{E_{r_0}}, \tag{13.3.38}$$

$$\left.\frac{\partial f_{\mathrm{A}_y}}{\partial \delta\beta}\right|_0 = -p_{\mathrm{d}_0} S (C_{D_0} + C_{E_{\beta_0}}), \tag{13.3.39}$$

$$\left.\frac{\partial f_{\mathrm{A}_y}}{\partial\left(\frac{b\delta\dot{\beta}}{2U_0}\right)}\right|_0 = -p_{\mathrm{d}_0} S C_{E_{\dot{\beta}_0}}, \tag{13.3.40}$$

$$\left.\frac{\partial f_{\mathrm{A}_y}}{\partial \delta a}\right|_0 = -p_{\mathrm{d}_0} S C_{E_{\delta a_0}}, \tag{13.3.41}$$

$$\left.\frac{\partial f_{\mathrm{A}_y}}{\partial \delta r}\right|_0 = -p_{\mathrm{d}_0} S C_{E_{\delta r_0}}. \tag{13.3.42}$$

Substituting (13.3.37)–(13.3.42) into (13.3.18) yields

$$f_{\mathrm{A}_y}(p, r, \delta\beta, \delta\dot{\beta}, \delta a, \delta r) \approx -\frac{p_{\mathrm{d}_0} S b}{2U_0} C_{E_{p_0}} p - \frac{p_{\mathrm{d}_0} S b}{2U_0} C_{E_{r_0}} r - p_{\mathrm{d}_0} S (C_{D_0} + C_{E_{\beta_0}})\delta\beta - \frac{p_{\mathrm{d}_0} S b}{2U_0} C_{E_{\dot{\beta}_0}} \delta\dot{\beta}$$

$$- p_{\mathrm{d}_0} S C_{E_{\delta a_0}} \delta a - p_{\mathrm{d}_0} S C_{E_{\delta r_0}} \delta r. \tag{13.3.43}$$

The condition $C_{E_{\beta_0}} < 0$ implies static stability since sideslip to the right induces a force to the left. On the other hand, if $\delta r > 0$, then the left rudder causes a right force, and thus $C_{E_{\delta r_0}} > 0$.

13.3.6 Linearization of F_{A_z}

From (13.1.6) we have

$$F_{\mathrm{A}_z} = p_{\mathrm{d}} S C_z \approx F_{\mathrm{A}_{z_0}} + f_{A_z}. \tag{13.3.44}$$

Note that it follows from (11.1.21) that

$$C_z = -(\sin\alpha)(\cos\beta) C_D + (\sin\alpha)(\sin\beta) C_E - (\cos\alpha) C_L, \tag{13.3.45}$$

and thus

$$C_{z_0} = -C_{L_0}. \tag{13.3.46}$$

Then,

$$\left. \frac{\partial f_{\mathrm{A}_z}}{\partial\left(\frac{u}{U_0}\right)} \right|_0 = -p_{\mathrm{d}_0} S \left(C_{L_{u_0}} + 2C_{L_0} \right). \tag{13.3.47}$$

In addition,

$$\left. \frac{\partial f_{\mathrm{A}_z}}{\partial\left(\frac{\bar{c}q}{2U_0}\right)} \right|_0 = -p_{\mathrm{d}_0} S C_{L_{q_0}}, \tag{13.3.48}$$

where

$$C_{L_{q_0}} \triangleq \left. \frac{\partial C_L}{\partial\left(\frac{\bar{c}q}{2U_0}\right)} \right|_0. \tag{13.3.49}$$

Furthermore,

$$\left. \frac{\partial f_{\mathrm{A}_z}}{\partial \delta\alpha} \right|_0 = -p_{\mathrm{d}_0} S \left(C_{L_{\alpha_0}} + C_{D_0} \right), \tag{13.3.50}$$

$$C_{L_{\dot\alpha_0}} \triangleq \left. \frac{\partial C_L}{\partial\left(\frac{\bar{c}\delta\dot\alpha}{2U_0}\right)} \right|_0. \tag{13.3.51}$$

Substituting into (13.3.19), we have

$$\begin{aligned} f_{\mathrm{A}_z}(u, q, \delta\alpha, \delta\dot\alpha, \delta e) \approx & -\frac{p_{\mathrm{d}_0} S}{U_0} \left(2C_{L_0} + C_{L_{u_0}} \right) u - \frac{p_{\mathrm{d}_0} S \bar{c}}{2U_0} C_{L_{q_0}} q - p_{\mathrm{d}_0} S (C_{L_{\alpha_0}} + C_{D_0}) \delta\alpha \\ & - \frac{p_{\mathrm{d}_0} S \bar{c}}{2U_0} C_{L_{\dot\alpha_0}} \delta\dot\alpha - p_{\mathrm{d}_0} S C_{L_{\delta e_0}} \delta e. \end{aligned} \tag{13.3.52}$$

13.3.7 Linearization of Thrust Coefficients

The perturbed thrust forces resolved in the aircraft frame are given by

$$\vec{F}_{\mathrm{T}}\Big|_{\mathrm{AC}} = \begin{bmatrix} F_{\mathrm{T}_x} \\ F_{\mathrm{T}_y} \\ F_{\mathrm{T}_z} \end{bmatrix} = \begin{bmatrix} F_{\mathrm{T}_{x0}} + f_{\mathrm{T}_x} \\ F_{\mathrm{T}_{y0}} + f_{\mathrm{T}_y} \\ F_{\mathrm{T}_{z0}} + f_{\mathrm{T}_z} \end{bmatrix}, \tag{13.3.53}$$

where f_{T_x}, f_{T_y}, and f_{T_z} are perturbations of the components of the thrust force resolved in the aircraft frame.

Expanding the thrust-force perturbations yields

$$f_{\mathrm{T}_x}(u, q, \delta\alpha, \delta\dot{\alpha}, \delta e) \approx \frac{\partial f_{\mathrm{T}_x}}{\partial u}\bigg|_0 u + \frac{\partial f_{\mathrm{T}_x}}{\partial q}\bigg|_0 q + \frac{\partial f_{\mathrm{T}_x}}{\partial \delta\alpha}\bigg|_0 \delta\alpha + \frac{\partial f_{\mathrm{T}_x}}{\partial \delta\dot{\alpha}}\bigg|_0 \delta\dot{\alpha} + \frac{\partial f_{\mathrm{T}_x}}{\partial \delta e}\bigg|_0 \delta e, \tag{13.3.54}$$

$$f_{\mathrm{T}_y}(p, r, \delta\beta, \delta\dot{\beta}, \delta a, \delta r) \approx \frac{\partial f_{\mathrm{T}_y}}{\partial p}\bigg|_0 p + \frac{\partial f_{\mathrm{T}_y}}{\partial r}\bigg|_0 r + \frac{\partial f_{\mathrm{T}_y}}{\partial \delta\beta}\bigg|_0 \delta\beta + \frac{\partial f_{\mathrm{T}_y}}{\partial \delta\dot{\beta}}\bigg|_0 \delta\dot{\beta} + \frac{\partial f_{\mathrm{T}_y}}{\partial \delta a}\bigg|_0 \delta a + \frac{\partial f_{\mathrm{T}_y}}{\partial \delta r}\bigg|_0 \delta r, \tag{13.3.55}$$

$$f_{\mathrm{T}_z}(u, q, \delta\alpha, \delta e) \approx \frac{\partial f_{\mathrm{T}_z}}{\partial u}\bigg|_0 u + \frac{\partial f_{\mathrm{T}_z}}{\partial q}\bigg|_0 q + \frac{\partial f_{\mathrm{T}_z}}{\partial \delta\alpha}\bigg|_0 \delta\alpha + \frac{\partial f_{\mathrm{T}_z}}{\partial \delta\dot{\alpha}}\bigg|_0 \delta\dot{\alpha} + \frac{\partial f_{\mathrm{T}_z}}{\partial \delta e}\bigg|_0 \delta e. \tag{13.3.56}$$

In addition, we define the thrust force derivatives

$$C_{\mathrm{T}x_{u0}} \triangleq \frac{\partial C_{\mathrm{T}x}}{\partial\left(\frac{u}{U_0}\right)}\Bigg|_0, \tag{13.3.57}$$

$$C_{\mathrm{T}z_{u0}} \triangleq \frac{\partial C_{\mathrm{T}z}}{\partial\left(\frac{u}{U_0}\right)}\Bigg|_0. \tag{13.3.58}$$

13.4 Moment Coefficients

For the moment components in the aircraft frame, the *aerodynamic moment coefficients* are defined by

$$C_l \triangleq \frac{L_{\mathrm{A}}}{p_{\mathrm{d}} S b}, \tag{13.4.1}$$

$$C_m \triangleq \frac{M_{\mathrm{A}}}{p_{\mathrm{d}} S \bar{c}}, \tag{13.4.2}$$

$$C_n \triangleq \frac{N_{\mathrm{A}}}{p_{\mathrm{d}} S b}. \tag{13.4.3}$$

Likewise, the *thrust moment coefficients* are defined by

$$C_{\mathrm{T}l} \triangleq \frac{L_{\mathrm{T}}}{p_{\mathrm{d}} S b}, \tag{13.4.4}$$

$$C_{\mathrm{T}m} \triangleq \frac{M_{\mathrm{T}}}{p_{\mathrm{d}} S \bar{c}}, \tag{13.4.5}$$

$$C_{\mathrm{T}n} \triangleq \frac{N_{\mathrm{T}}}{p_{\mathrm{d}} S b}. \tag{13.4.6}$$

13.5 Linearization of Moments

13.5.1 Linearization of the Roll Moment

We now express the perturbed aerodynamic moments in terms of the stability derivatives. For the roll moment we have

$$L_{\mathrm{A}} = p_{\mathrm{d}} S b C_l \approx L_{\mathrm{A}_0} + l_{\mathrm{A}}, \tag{13.5.1}$$

and thus

$$L_{\mathrm{A}_0} = p_{\mathrm{d}_0} S b C_{l_0}. \tag{13.5.2}$$

Therefore,

$$\begin{aligned} l_{\mathrm{A}}(p, r, \delta\beta, \delta\dot{\beta}, \delta a, \delta r) \approx{} & \frac{p_{\mathrm{d}_0} S b^2}{2U_0} C_{l_{p_0}} p + \frac{p_{\mathrm{d}_0} S b^2}{2U_0} C_{l_{r_0}} r + p_{\mathrm{d}_0} S b C_{l_{\beta_0}} \delta\beta + \frac{p_{\mathrm{d}_0} S b^2}{2U_0} C_{l_{\dot{\beta}_0}} \delta\dot{\beta} \\ & + p_{\mathrm{d}_0} S b C_{l_{\delta a_0}} \delta a + p_{\mathrm{d}_0} S b C_{l_{\delta r_0}} \delta r. \end{aligned} \tag{13.5.3}$$

13.5.1.1 $C_{l_{p_0}}$

As the aircraft rolls, the induced drag creates a moment.

13.5.1.2 $C_{l_{r_0}}$

As the aircraft yaws, the wing tip that is moving faster due to the rotation around $\hat{k}_{\mathrm{AC}}$ creates greater lift than the wing tip that is moving more slowly. This difference in lift creates a roll moment.

13.5.1.3 $C_{l_{\beta_0}}$

Suppose the aircraft velocity $\vec{V}_{\mathrm{AC}}$ is perturbed by $\Delta\vec{V}_{\mathrm{AC}}$ with $\delta\beta > 0$. We need $L_{\mathrm{A}} < 0$ and hence $C_{l_{\beta_0}} < 0$ in order to roll away from the sideslip perturbation and thus reduce the sideslip perturbation in accordance with static stability. Note that $\delta\alpha$ increases as $\delta\beta$ decreases.

$C_{l_{\beta_0}}$ is the stability derivative corresponding to the roll moment L_{A} caused by sideslip. It is affected by the fuselage, the wing dihedral, the wing position on the fuselage, and its sweep angle.

13.5.1.4 Effect of Wing Dihedral on $C_{l_{\beta_0}}$

We first consider the wing dihedral with the aircraft sideslipping to the right, as shown in Figure 13.5.1.

Note that, if $\mathit{\Gamma} > 0$, the wing is *dihedral*, otherwise it is *anhedral*. In Figure 13.5.2, V_{n_1} is the normal component of V_{air} due to V, and V_{n_2} is the normal component of V_{air} due to W. The normal

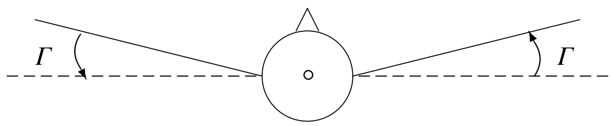


Figure 13.5.1: Dihedral wings with angle Γ. The aircraft is pointing out of the page.

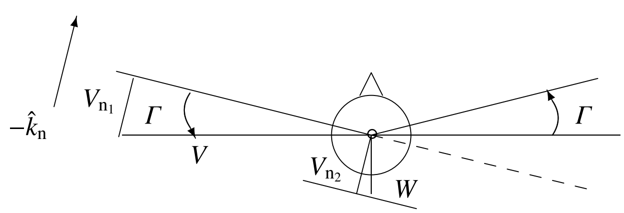


Figure 13.5.2: Wing normal velocity (front view). The aircraft is sideslipping to its right.

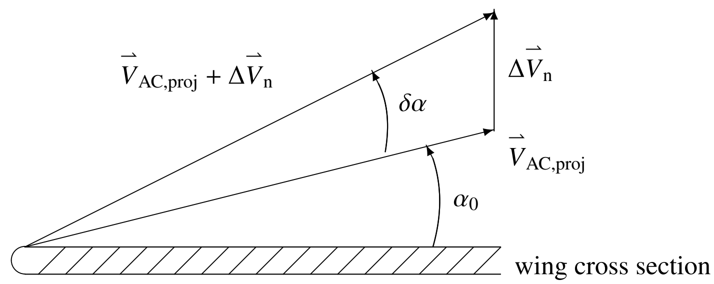


Figure 13.5.3: Perturbed angle of attack.

velocity $\vec{V}_{\text{n}}$ of $\vec{V}_{\text{air}}$ on the right wing is then

$$\vec{V}_{\text{n}} = (V_{\text{n}_1} + V_{\text{n}_2})(-\hat{k}_{\text{n}}), \tag{13.5.4}$$

and the change ΔV_{n} in normal velocity of $\vec{V}_{\text{air}}$ on the right wing due to dihedral is

$$\Delta V_{\text{n}} = (V_{\text{n}_1} + V_{\text{n}_2} - W)(-\hat{k}_{\text{n}}). \tag{13.5.5}$$

Note that

$$\cos\Gamma = \frac{V_{\text{n}_2}}{W} \tag{13.5.6}$$

and

$$\sin\Gamma = \frac{V_{\text{n}_1}}{V}. \tag{13.5.7}$$

Substituting (13.5.6) and (13.5.7) into (13.5.5), we have

$$\Delta V_{\text{n}} = (\cos\Gamma)W + (\sin\Gamma)V - W \approx W + V\Gamma - W = V\Gamma. \tag{13.5.8}$$

From Figure 13.5.4, we have

$$V = (\tan\beta)U \approx \delta\beta U. \tag{13.5.9}$$

Hence,

$$\tan\delta\alpha = \frac{\Delta V_{\text{n}}}{U} \approx \frac{\Gamma V}{U} \approx \frac{\delta\beta U\Gamma}{U} = \delta\beta\Gamma. \tag{13.5.10}$$

For small perturbations, we have,

$$\delta\alpha \approx \delta\beta\Gamma. \tag{13.5.11}$$

Thus, α_0 is perturbed by $\delta\alpha$, which effectively increases the angle of attack of the right wing due to sideslipping to the right, as shown in Figure 13.5.4. A similar analysis shows that, on the left wing, the angle of attack is decreased by $\delta\alpha$. This effect results in a negative roll moment. Hence $C_{l_{\beta_0}} < 0$, which implies static stability.

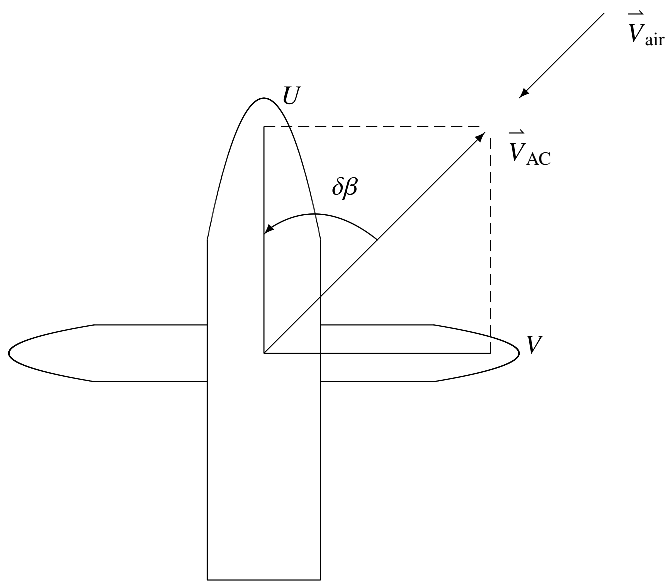


Figure 13.5.4: Effect of sideslip and dihedral.

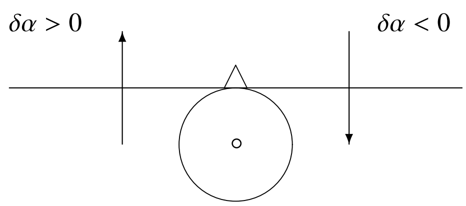


Figure 13.5.5: High wing (front view). The aircraft is sideslipping to its right. The flow around the fuselage perpendicular to the wing produces a negative roll moment.

13.5.1.5 Effect of Wing Position on $C_{l_{\beta_0}}$

Consider first a high wing as in Figure 13.5.5. The cross flow field velocity due to sideslip to the right is equal to $\delta\beta U$. This sideslip causes the aircraft to roll to the left, with a roll moment $L_{\text{A}} < 0$. Hence, since positive sideslip induces a negative roll moment, it follows that $C_{l_{\beta_0}} < 0$. Hence, high wing has the same effect as wing dihedral.

Consider now a low wing as in Figure 13.5.6, and a positive sideslipping to the right, with, again, a flow field velocity $\delta\beta U$. In this case, $C_{l_{\beta_0}} > 0$, and a positive sideslip induces a positive roll moment, while a negative sideslip causes a negative roll moment. Wing dihedral is sometimes used to counteract this effect.

13.5.1.6 Effect of Wing Sweep on $C_{l_{\beta_0}}$

As shown in Figure 13.5.7, the sweep angle of the wings affects the aircraft response to sideslip. A sideslip to the right causes increased lift by the right wing, and thus, causes a negative roll moment.

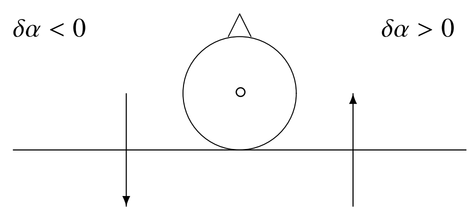


Figure 13.5.6: Low wing (front view). The aircraft is sideslipping to its right. The flow around the fuselage perpendicular to the wing produces a positive roll moment.

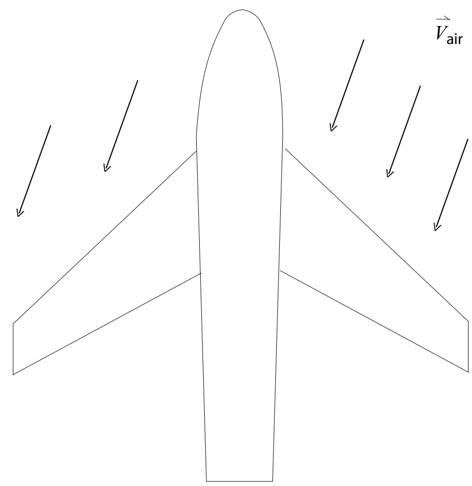


Figure 13.5.7: $C_{l_{\beta_0}}$ is negative due to wing sweep since for $\delta\beta > 0$ the lift of the right wing is greater than the lift of the left wing.

Finally, note that $C_{l_{\beta_0}}$ is also affected by the horizontal and vertical tails.

13.5.1.7 $C_{l_{\delta a_0}}$

The control derivative $C_{l_{\delta a_0}}$ is the *aileron-roll derivative*. Since right aileron up is positive, it follows that $C_{l_{\delta a_0}}$ is positive.

13.5.1.8 $C_{l_{\delta r_0}}$

The control derivative $C_{l_{\delta r_0}}$ is the *adverse rudder-roll derivative*.

13.5.2 Linearization of the Pitch Moment

For the pitch moment we have

$$M_{\rm A} = p_{\rm d} S \bar{c} C_m \approx M_{\rm A_0} + m_{\rm A}, \tag{13.5.12}$$

and thus

$$M_{\mathrm{A}_0} = p_{\mathrm{d}_0} S \bar{c} C_{m_0}. \tag{13.5.13}$$

Therefore,

$$m_{\mathrm{A}}(u, q, \delta\alpha, \delta\dot{\alpha}, \delta e) \approx \frac{p_{\mathrm{d}_0} S \bar{c}}{U_0}(2C_{m_0} + C_{m_{u_0}})u + \frac{p_{\mathrm{d}_0} S \bar{c}^2}{2U_0} C_{m_{q_0}} q$$
$$+ p_{\mathrm{d}_0} S \bar{c} C_{m_{\alpha_0}} \delta\alpha + \frac{p_{\mathrm{d}_0} S \bar{c}^2}{2U_0} C_{m_{\dot{\alpha}_0}} \delta\dot{\alpha} + p_{\mathrm{d}_0} S \bar{c} C_{m_{\delta e_0}} \delta e. \tag{13.5.14}$$

13.5.2.1 $C_{m_{\alpha_0}}$

For pitch static stability, we perturb $\alpha_0 = 0$ to $\alpha = \alpha_0 + \delta\alpha = \delta\alpha$, to obtain

$$C_m(\delta\alpha) \approx C_{m_0} + C_{m_{\alpha_0}} \delta\alpha, \tag{13.5.15}$$

where $C_{m_0} \triangleq C_m(0)$ and the pitch moment is

$$M_{\mathrm{A}} = M_{\mathrm{A}_0} + m_{\mathrm{A}}, \tag{13.5.16}$$

with

$$M_{\mathrm{A}_0} = C_{m_0} p_{\mathrm{d}} S \bar{c}. \tag{13.5.17}$$

The perturbed moment is

$$M_{\mathrm{A}} = C_m(\delta\alpha) p_{\mathrm{d}} S \bar{c} = C_{m_0} p_{\mathrm{d}} S \bar{c} + \Delta C_m p_{\mathrm{d}} S \bar{c}, \tag{13.5.18}$$

where

$$\Delta C_m \triangleq C_m(\delta\alpha) - C_m(0) \approx C_{m_{\alpha_0}} \delta\alpha. \tag{13.5.19}$$

Hence, the change in moment is

$$m_{\mathrm{A}} = \Delta C_m p_{\mathrm{d}} S \bar{c} \approx C_{m_{\alpha_0}} p_{\mathrm{d}} S \bar{c} \delta\alpha. \tag{13.5.20}$$

If $\delta\alpha > 0$, that is, the nose goes up, then the change in moment is negative. Thus,

$$\Delta C_m < 0, \tag{13.5.21}$$

and hence

$$C_{m_{\alpha_0}} \approx \frac{\Delta C_m}{\delta\alpha} < 0. \tag{13.5.22}$$

Next, we link C_{m_α} to the *center of mass* c. First, referring to Figure 13.5.8, we note that ac is the *aerodynamic center*, which, for a wing alone, is usually fixed at the location $x_{\mathrm{ac}} = c/4$. Furthermore, the moment on the aircraft

$$\vec{M}_{\mathrm{ac}} = M_{\mathrm{ac}} \hat{\jmath}_{\mathrm{S}} \tag{13.5.23}$$

relative to ac is independent of α and is generally nonzero. The point cp is the *center of pressure*, where x_{cp} moves as α changes. The center of pressure has the property that the moment on the aircraft relative to cp is zero for all α.

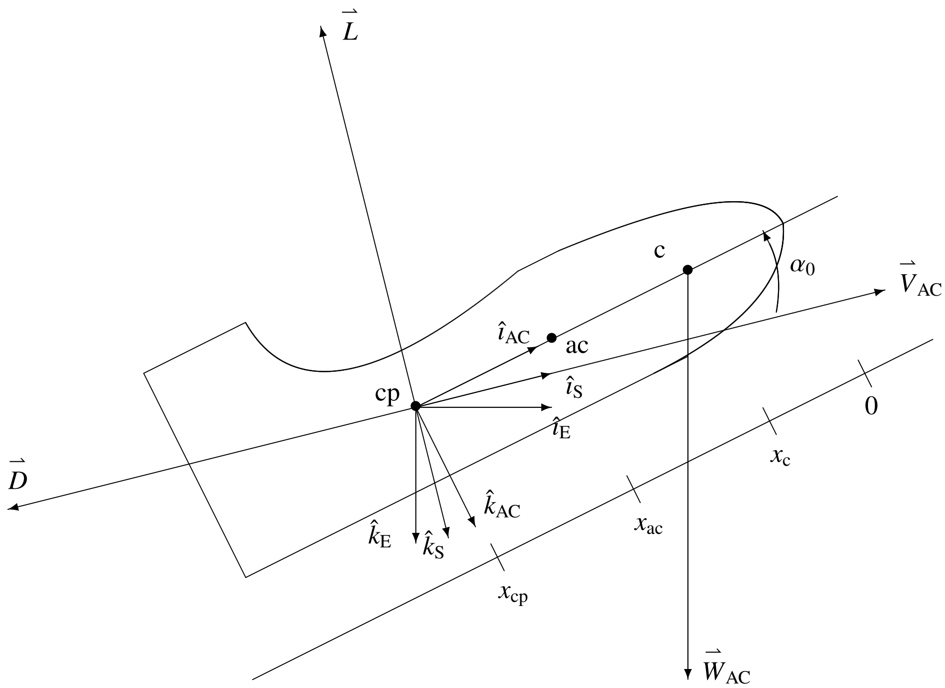


Figure 13.5.8: Pitching moment analysis assuming zero sideslip.

We compute the moment $\vec{M}_c$ on the aircraft relative to the center of mass,

$$\begin{aligned}\vec{M}_c &= \underbrace{(x_c - x_{cp})}_{<0}\hat{\imath}_{AC} \times \vec{L}\\ &= \underbrace{(x_c - x_{ac})\hat{\imath}_{AC} \times \vec{L}}_{\text{aerodynamic moment}} + \underbrace{(x_{ac} - x_{cp})\hat{\imath}_{AC} \times \vec{L}}_{\text{aerodynamic moment}}\\ &= (x_c - x_{ac})L(\cos\alpha)\hat{\jmath}_S + \vec{M}_{ac}\\ &= [(x_c - x_{ac})L + M_{ac}](\cos\alpha)\hat{\jmath}_S\\ &\cong [\underbrace{(x_c - x_{ac})}_{<0}L + \underbrace{M_{ac}}_{<0}]\hat{\jmath}_S.\end{aligned} \tag{13.5.24}$$

Writing

$$\vec{M}_c = M_c\hat{\jmath}_S, \tag{13.5.25}$$

we have

$$\frac{M_c}{p_d S\bar{c}} = [(x_c - x_{ac})L + M_{ac}]\frac{1}{p_d S\bar{c}}$$

or

$$\frac{M_c}{p_d S\bar{c}} = \frac{x_c - x_{ac}}{\bar{c}}\frac{L}{p_d S} + \frac{M_{ac}}{p_d S\bar{c}}. \tag{13.5.26}$$

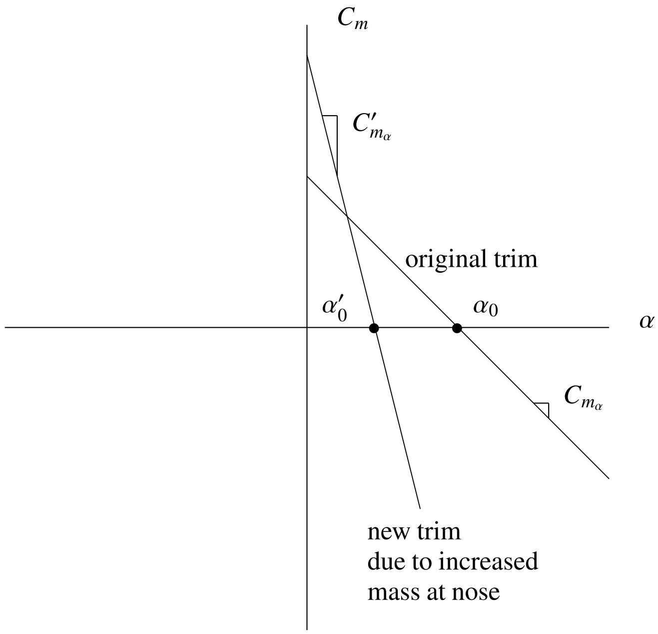


Figure 13.5.9: Change in pitching moment coefficient due to a change in trim.

Therefore, defining

$$C_{m_c} \triangleq \frac{M_c}{p_d S \bar{c}}, \tag{13.5.27}$$

we have

$$C_{m_c} = \frac{x_c - x_{ac}}{\bar{c}} C_L + C_{m_{ac}}, \tag{13.5.28}$$

where the pitch moment coefficient $C_{m_{ac}}$ is independent of α. Since $C_{m_\alpha} = C_{m_c}$, we have

$$C_{m_\alpha} = \frac{x_c - x_{ac}}{\bar{c}} C_{L_\alpha} + 0. \tag{13.5.29}$$

At the steady flight conditions, (13.5.29) becomes

$$C_{m_{\alpha_0}} = \frac{x_c - x_{ac}}{\bar{c}} C_{L_{\alpha_0}}. \tag{13.5.30}$$

Since $x_c - x_{ac} < 0$ we have $C_{m_{\alpha_0}} < 0$ with static stability as shown. On the other hand, if $x_c - x_{ac} > 0$, then $C_{m_{\alpha_0}} > 0$ and the aircraft is statically unstable.

Now suppose that additional mass is added at the nose. Then, from Figure 13.5.9, we can see that $C'_{m_\alpha} < C_{m_\alpha} < 0$. Hence, C_{m_α} becomes more negative as x_c decreases.

13.5.2.2 $C_{m_{u_0}}$

$C_{m_{u_0}}$ is the *Mach tuck derivative*. Suppose that $C_{m_{u_0}}$ is positive. Therefore, an increase in speed (that is, a positive speed perturbation) leads to an increase in the pitch moment (that is, a positive pitch-moment perturbation), which causes the aircraft to pitch up. Since $C_{D_{\alpha_0}} > 0$, the increase in speed leads to an increase in drag. The additional drag, however, counteracts the increase in speed.

The speed perturbation thus has a tendency to decrease, which shows that $C_{m_{u_0}} > 0$ corresponds to static stability. However, in many aircraft, $C_{m_{u_0}}$ is negative, and thus the aircraft is statically unstable. In particular, as u increases, $x_{\rm ac}$ shifts toward the rear of the plane. It follows from (13.5.29) that C_{m_α} decreases, which means that, for $\alpha > 0$, the pitch moment decreases and the nose goes down. The drag thus decreases, which leads to an increase in speed.

13.5.2.3 $C_{m_{q_0}}$

$C_{m_{q_0}}$ is the *pitch-damping derivative*. Values $C_{m_{q_0}} < 0$ corresponds to static stability.

13.5.2.4 $C_{m_{\delta e_0}}$

The control derivative $C_{m_{\delta e_0}}$ is the *elevator pitch derivative*. For a rear-mounted elevator, $C_{m_{\delta e_0}}$ is negative.

13.5.3 Linearization of the Yaw Moment

For the yaw moment we have

$$N_{\rm A} = p_{\rm d} S b C_n \approx N_{{\rm A}_0} + n_{\rm A}, \tag{13.5.31}$$

and thus

$$N_{{\rm A}_0} = p_{{\rm d}_0} S b C_{n_0}. \tag{13.5.32}$$

Therefore,

$$\begin{aligned} n_{\rm A}(p, r, \delta\beta, \delta\dot\beta, \delta r, \delta a) \approx{}& \frac{p_{{\rm d}_0} S b^2}{2U_0} C_{n_{p_0}} p + \frac{p_{{\rm d}_0} S b^2}{2U_0} C_{n_{r_0}} r + p_{{\rm d}_0} S b C_{n_{\beta_0}} \delta\beta \\ &+ \frac{p_{{\rm d}_0} S b^2}{2U_0} C_{n_{\dot\beta_0}} \delta\dot\beta + p_{{\rm d}_0} S b C_{n_{\delta r_0}} \delta r + p_{{\rm d}_0} S b C_{n_{\delta a_0}} \delta a. \end{aligned} \tag{13.5.33}$$

For a symmetric airfoil, $C_{n_0} = 0$.

13.5.3.1 $C_{n_{\beta_0}}$

$C_{n_{\beta_0}}$ is the *static directional stability derivative*, also called the *weathervane stability derivative*. $C_{n_{\beta_0}} > 0$ implies static stability in yaw due to sideslip since sideslip to the right causes a yaw moment to the right. Therefore, yawing into sideslip so that $\delta\beta$ decreases is statically stable. In contrast, rolling away from sideslip so that $\delta\beta$ decreases implies static stability.

The rudder control derivative $C_{n_{\delta r_0}}$ is negative. The term $C_{n_{\delta a_0}}$ is the *adverse aileron-yaw derivative*, which is negative.

13.5.4 Linearization of the Thrust Moments

We linearize the thrust roll moment as

$$L_{\rm T} = p_{\rm d} S b C_{{\rm T}l} \approx L_{{\rm T}_0} + l_{\rm T}, \tag{13.5.34}$$

where

$$l_{\rm T}(\delta\beta) \approx \left.\frac{\partial l_{\rm T}}{\partial \delta\beta}\right|_0 \delta\beta. \tag{13.5.35}$$

Therefore,

$$l_{\rm T}(\delta\beta) \approx p_{\rm d_0} S b C_{{\rm T}l_{\beta_0}} \delta\beta, \tag{13.5.36}$$

where

$$C_{{\rm T}l_{\beta_0}} \triangleq \left.\frac{\partial C_{{\rm T}l}}{\partial \delta\beta}\right|_0. \tag{13.5.37}$$

We linearize the thrust pitch moment as

$$M_{\rm T} = p_{\rm d} S \bar{c} C_{{\rm T}m} \approx M_{{\rm T}_0} + m_{\rm T}, \tag{13.5.38}$$

where

$$m_{\rm T}(u, \delta\alpha) \approx \left.\frac{\partial m_{\rm T}}{\partial u}\right|_0 u + \left.\frac{\partial m_{\rm T}}{\partial \delta\alpha}\right|_0 \delta\alpha. \tag{13.5.39}$$

Therefore,

$$m_{\rm T}(u, \delta\alpha) \approx \frac{p_{\rm d_0} S \bar{c}}{U_0}(2C_{{\rm T}m_0} + C_{{\rm T}m_{u_0}})u + p_{\rm d_0} S \bar{c} C_{{\rm T}m_{\alpha_0}} \delta\alpha, \tag{13.5.40}$$

where

$$C_{{\rm T}m_{u_0}} \triangleq \left.\frac{\partial C_{{\rm T}m}}{\partial \left(\frac{u}{U_0}\right)}\right|_0, \tag{13.5.41}$$

$$C_{{\rm T}m_{\alpha_0}} \triangleq \left.\frac{\partial C_{{\rm T}m}}{\partial \delta\alpha}\right|_0. \tag{13.5.42}$$

We linearize the thrust yaw moment as

$$N_{\rm T} = p_{\rm d} S b C_{{\rm T}n} \approx N_{{\rm T}_0} + n_{\rm T}, \tag{13.5.43}$$

where

$$n_{\rm T}(\delta\beta) \approx \left.\frac{\partial n_{\rm T}}{\partial \delta\beta}\right|_0 \delta\beta. \tag{13.5.44}$$

Therefore,

$$n_{\rm T}(\delta\beta) \approx p_{\rm d_0} S b C_{{\rm T}n_{\beta_0}} \delta\beta, \tag{13.5.45}$$

where

$$C_{{\rm T}n_{\beta_0}} \triangleq \left.\frac{\partial C_{{\rm T}n}}{\partial \delta\beta}\right|_0. \tag{13.5.46}$$

13.6 Adverse Control Derivatives

13.6.1 Adverse Aileron-Yaw

As shown in Figure 13.6.10, since the left wing has increased lift due to aileron down, it also has increased drag. Hence, the aircraft turns right but sideslips to the left, which explains the “adverse” terminology. To cancel this moment, we can use the rudder δr. From the yaw moment expression (13.5.33), we have

$$C_{n_{\delta r_0}} \delta r + C_{n_{\delta a_0}} \delta a = 0, \tag{13.6.1}$$

that is,

$$\delta r = -\frac{C_{n_{\delta a_0}}}{C_{n_{\delta r_0}}} \delta a. \tag{13.6.2}$$

Note that $C_{n_{\delta a_0}}$ and $C_{n_{\delta r_0}}$ do not change sign, whereas δa and δr change sign as these control surfaces move. The adverse aileron-yaw moment is due to $C_{n_{\delta a_0}} < 0$.

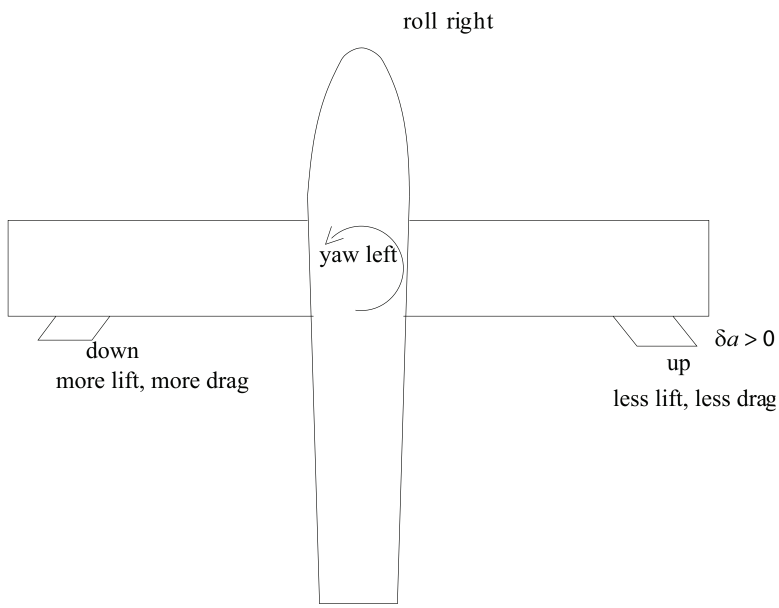


Figure 13.6.10: Adverse aileron-yaw.

13.6.2 Adverse Rudder-Roll

To cancel the adverse rudder-roll moment, we use the aileron. It follows from the roll moment expression (13.5.3) that

$$C_{l_{\delta r_0}} \delta r + C_{l_{\delta a_0}} \delta a = 0. \tag{13.6.3}$$

Solving for the aileron deflection δa yields

$$\delta a = -\frac{C_{l_{\delta r_0}}}{C_{l_{\delta a_0}}} \delta r. \tag{13.6.4}$$

Note that $C_{l_{\delta a_0}}$ does not change sign, but δa and δr change sign as the control surfaces move. In addition, the sign of $C_{l_{\delta r_0}}$ depends on the angle of attack. To see this, let $\vec{L}_{\delta r}$ denote the lift due to the rudder. As shown in Figure 13.6.11 and Figure 13.6.12 for the case of left rudder, $\vec{L}_{\delta r}$ is aligned with $\hat{\jmath}_{\rm AC}$ and applied to the aircraft at the point c′ on the vertical tail. The component of the roll moment along $\hat{\imath}_{\rm AC}$ due to $\vec{L}_{\delta r}$ relative to c is thus given by

$$L_{\delta r} = (\vec{r}_{\rm c'/c} \times \vec{L}_{\delta r}) \cdot \hat{\imath}_{\rm AC}, \tag{13.6.5}$$

where $\vec{L}_{\delta r} = \lambda\hat{\jmath}_{\rm AC}$ and $\vec{r}_{\rm c'/c} = \mu_x\hat{\imath}_{\rm AC} + \mu_z\hat{k}_{\rm AC}$. Note that $\lambda > 0$ and $\mu_x < 0$. However, the sign of μ_z depends on the angle of attack. In particular, if the angle of attack is low, then Figure 13.6.11 shows that μ_z is negative, whereas, if the angle of attack is high, then Figure 13.6.12 shows that μ_z is positive. Note that, in both cases, the axis $\hat{\imath}_{\rm AC}$ is defined to be aligned with $\vec{V}_{\rm AC_0}$ in steady flight.

To determine whether the rudder-roll effect is adverse or proverse, note that

$$\begin{aligned} L_{\delta r} &= \vec{r}_{\rm c'/c} \cdot (\vec{L}_{\delta r} \times \hat{\imath}_{\rm AC}) \\ &= \vec{r}_{\rm c'/c} \cdot (\lambda\hat{\jmath}_{\rm AC} \times \hat{\imath}_{\rm AC}) \\ &= -\lambda(\mu_x\hat{\imath}_{\rm AC} + \mu_z\hat{k}_{\rm AC}) \cdot \hat{k}_{\rm AC} \\ &= -\lambda\mu_z. \end{aligned} \tag{13.6.6}$$

If the angle of attack is low, and thus μ_z is negative, it follows from (13.6.6) that $L_{\delta r}$ is positive, and thus $\vec{L}_{\delta r}$ is aligned with $\hat{\jmath}_{\rm AC}$. Hence, $C_{l_{\delta r_0}}$ is positive. Therefore, in a left turn, that is, when δr is positive, the aircraft has a tendency to roll right. Therefore, the rudder-roll effect is adverse. Alternatively, if the angle of attack is high, and thus μ_z is positive, it follows from (13.6.6) that $L_{\delta r}$ is negative, and thus $\vec{L}_{\delta r}$ is aligned with $-\hat{\jmath}_{\rm AC}$. Hence $C_{l_{\delta r_0}}$ is negative. Therefore, in a left turn, that is, when δr is positive, the aircraft has a tendency to roll left. Therefore, the rudder-roll effect is proverse.

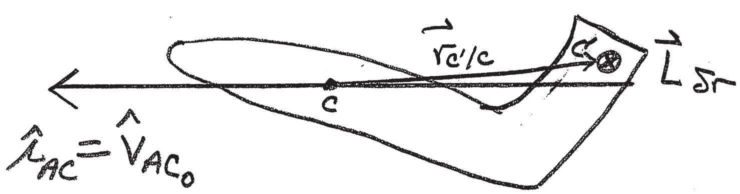


Figure 13.6.11: Adverse rudder-roll. The case of left rudder ($\delta r > 0$) for low angle of attack is shown here. The lift $\vec{L}_{\delta r}$ due to the rudder is aligned with $\hat{\jmath}_{\rm AC}$ and is applied to the point c′ on the vertical tail. The aircraft center of mass is denoted by c. The roll due to rudder is adverse since the component μ_z of $\vec{r}_{\rm c/c'}$ along $\hat{k}_{\rm AC}$ is negative.

		u	q	$\delta\alpha$	$\delta\dot{\alpha}$	δe
Range	C_D	> 0	< 0	< 0	< 0	> 0
Pitch	C_m	> 0	< 0	< 0	< 0	< 0
Plunge	C_L	> 0	> 0	> 0	> 0	> 0

Table 13.1: Signs of longitudinal stability derivatives for static stability, as well as the signs of the control derivatives.

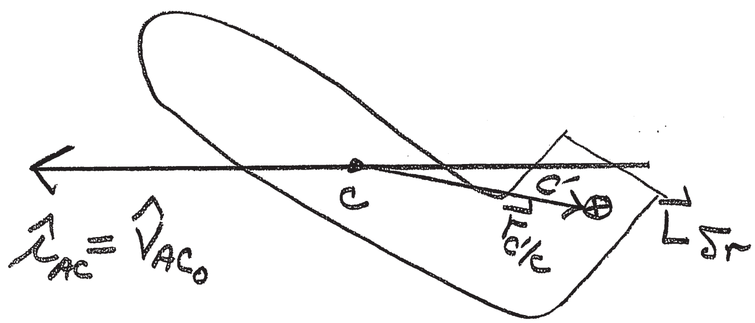


Figure 13.6.12: Proverse rudder-roll. The case of left rudder ($\delta r > 0$) for high angle of attack is shown here. The lift $\vec{L}_{\delta r}$ due to the rudder is aligned with $\hat{\jmath}_{AC}$ and is applied to the point c′ on the vertical tail. The aircraft center of mass is denoted by c. The roll due to rudder is proverse since the component μ_z of $\vec{r}_{c/c'}$ along $\hat{k}_{AC}$ is positive.

		p	r	$\delta\beta$	$\delta\dot{\beta}$	δa	δr
Roll	C_l	< 0	< 0	< 0	< 0	> 0	$\pm$
Drift	C_E	< 0	> 0	< 0	< 0	< 0	> 0
Yaw	C_n	< 0	< 0	> 0	> 0	< 0	< 0

Table 13.2: Signs of lateral stability derivatives for static stability, as well as the signs of the control derivatives.

13.7 Problems

Problem 13.7.1. Consider the expression (13.6.2), which shows how to set the rudder to cancel adverse aileron-yaw. Then, draw two diagrams (one for right aileron up and one for left aileron up) and check the signs of all terms in the equation to confirm that they are all correct.

Problem 13.7.2. Consider the expression (13.6.4), which shows how to set the ailerons to cancel adverse rudder-roll. Then, draw two diagrams (one for rudder right and one for rudder left) and check the signs of all terms in the equation to confirm that they are all correct.

Problem 13.7.3. Derive (13.3.27).

$C_{\mathrm{L}}(u,q,r,\delta\alpha,\delta\dot{\alpha},\delta e)$	$C_{L_0} + \frac{1}{U_0}C_{L_{u_0}}u + \frac{\bar{c}}{2U_0}C_{L_{q_0}}q + \frac{b}{2U_0}C_{L_{r_0}}r$ $+ C_{L_{\alpha_0}}\delta\alpha + \frac{\bar{c}}{2U_0}C_{L_{\dot{\alpha}_0}}\delta\dot{\alpha} + C_{L_{\delta e_0}}\delta e$
C_{L_0}	$\frac{L_0}{p_{\mathrm{d}_0}S}$
$C_{L_{u_0}}$	$\left.\frac{\partial C_L}{\partial\left(\frac{u}{U_0}\right)}\right\|_0$
$C_{L_{q_0}}$	$\left.\frac{\partial C_L}{\partial\left(\frac{\bar{c}q}{2U_0}\right)}\right\|_0$
$C_{L_{r_0}}$	$\left.\frac{\partial C_L}{\partial\left(\frac{br}{2U_0}\right)}\right\|_0$
$C_{L_{\alpha_0}}$	$\left.\frac{\partial C_L}{\partial\delta\alpha}\right\|_0$
$C_{L_{\dot{\alpha}_0}}$	$\left.\frac{\partial C_L}{\partial\left(\frac{\bar{c}\delta\dot{\alpha}}{2U_0}\right)}\right\|_0$
$C_{L_{\delta e_0}}$	$\left.\frac{\partial C_L}{\partial\delta e}\right\|_0$
$C_{\mathrm{D}}(u,q,r,\delta\alpha,\delta\dot{\alpha},\delta e)$	$C_{D_0} + \frac{1}{U_0}C_{D_{u_0}}u + \frac{\bar{c}}{2U_0}C_{D_{q_0}}q + \frac{b}{2U_0}C_{D_{r_0}}r$ $+ C_{D_{\alpha_0}}\delta\alpha + C_{D_{\dot{\alpha}_0}}\delta\dot{\alpha} + C_{D_{\delta e_0}}\delta e$
C_{D_0}	$\frac{D_0}{p_{\mathrm{d}_0}S}$
$C_{D_{u_0}}$	$2KC_{L_0}C_{L_{u_0}}$
$C_{D_{q_0}}$	$2KC_{L_0}C_{L_{q_0}}$
$C_{D_{r_0}}$	$2KC_{L_0}C_{L_{r_0}}$
$C_{D_{\alpha_0}}$	$2KC_{L_0}C_{L_{\alpha_0}}$
$C_{D_{\dot{\alpha}_0}}$	$2KC_{L_0}C_{L_{\dot{\alpha}_0}}$
$C_{D_{\delta e_0}}$	$2KC_{L_0}C_{L_{\delta e_0}}$

Table 13.3: Lift and drag stability derivatives. These stability derivatives model the aerodynamic forces applied to the aircraft due to perturbations away from steady longitudinal flight.

Chapter Fourteen
Linearized Dynamics and Flight Modes

14.1 Linearized Longitudinal Equations of Motion

We now incorporate the stability derivatives within the linearized longitudinal equations of motion (12.4.18), (12.4.22), and (12.3.55) for the range-rate, angle-of-attack, and pitch-rate perturbations, respectively. This yields the linearized longitudinal equations of motion

$$\begin{aligned}\dot{u} = {} & \underbrace{-\frac{p_{\mathrm{d}_0} S}{m U_0}\left(2C_{D_0} + C_{D_{u_0}}\right)}_{X_{u_0}} u + \underbrace{\frac{p_{\mathrm{d}_0} S}{m U_0}(2C_{\mathrm{T}x_0} + C_{\mathrm{T}x_{u_0}})}_{X_{\mathrm{T}_{u_0}}} u + \underbrace{\frac{p_{\mathrm{d}_0} S}{m}\left(C_{L_0} - C_{D_{\alpha_0}}\right)}_{X_{\alpha_0}} \delta\alpha \\ & + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}}{2 m U_0} C_{D_{q_0}}}_{X_{q_0}} q + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}}{2 m U_0} C_{D_{\dot{\alpha}_0}}}_{X_{\dot{\alpha}_0}} \delta\dot{\alpha} - g\theta + \underbrace{\frac{p_{\mathrm{d}_0} S}{m} C_{D_{\delta e_0}}}_{X_{\delta e_0}} \delta e,\end{aligned} \tag{14.1.1}$$

$$\begin{aligned}U_0 \delta\dot{\alpha} = {} & \underbrace{-\frac{p_{\mathrm{d}_0} S}{m U_0}(2C_{L_0} + C_{L_{u_0}})}_{Z_{u_0}} u \underbrace{- \frac{p_{\mathrm{d}_0} S}{m U_0}(2C_{\mathrm{T}z_0} + C_{\mathrm{T}z_{u_0}})}_{Z_{\mathrm{T}_{u_0}}} u \underbrace{- \frac{p_{\mathrm{d}_0} S}{m}(C_{L_{\alpha_0}} + C_{D_0})}_{Z_{\alpha_0}} \delta\alpha \\ & + U_0 q \underbrace{- \frac{p_{\mathrm{d}_0} S \overline{c}}{2 m U_0} C_{L_{q_0}}}_{Z_{q_0}} q \underbrace{- \frac{p_{\mathrm{d}_0} S \overline{c}}{2 m U_0} C_{L_{\delta\dot{\alpha}_0}}}_{Z_{\dot{\alpha}_0}} \delta\dot{\alpha} \underbrace{- \frac{p_{\mathrm{d}_0} S}{m} C_{L_{\delta e_0}}}_{Z_{\delta e_0}} \delta e,\end{aligned} \tag{14.1.2}$$

$$\begin{aligned}\dot{q} = {} & \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}}{J_{yy} U_0}(2C_{m_0} + C_{m_{u_0}})}_{M_{u_0}} u + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}}{J_{yy} U_0}(2C_{\mathrm{T}m_0} + C_{\mathrm{T}m_{u_0}})}_{M_{\mathrm{T}_{u_0}}} u + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}}{J_{yy}} C_{m_{\alpha_0}}}_{M_{\alpha_0}} \delta\alpha \\ & + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}}{J_{yy}} C_{\mathrm{T}m_{\alpha_0}}}_{M_{\mathrm{T}_{\alpha_0}}} \delta\alpha + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}^2}{2 J_{yy} U_0} C_{m_{q_0}}}_{M_{q_0}} q + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}^2}{2 J_{yy} U_0} C_{m_{\delta\dot{\alpha}_0}}}_{M_{\dot{\alpha}_0}} \delta\dot{\alpha} + \underbrace{\frac{p_{\mathrm{d}_0} S \overline{c}}{J_{yy}} C_{m_{\delta e_0}}}_{M_{\delta e_0}} \delta e.\end{aligned} \tag{14.1.3}$$

By introducing the longitudinal stability parameters, which are defined in Table 14.1, (14.1.1)–(14.1.3) can be written as

$$\dot{u} - X_{\dot{\alpha}_0} \delta\dot{\alpha} = \left(X_{u_0} + X_{\mathrm{T}_{u_0}}\right) u + X_{\alpha_0} \delta\alpha + X_{q_0} q - g\theta + X_{\delta e_0} \delta e, \tag{14.1.4}$$

$$(U_0 - Z_{\dot{\alpha}_0}) \delta\dot{\alpha} = (Z_{u_0} + Z_{\mathrm{T}_{u_0}}) u + Z_{\alpha_0} \delta\alpha + (U_0 + Z_{q_0}) q + Z_{\delta e_0} \delta e, \tag{14.1.5}$$

$$\dot{q} - M_{\dot{\alpha}_0} \delta\dot{\alpha} = \left(M_{u_0} + M_{\mathrm{T}_{u_0}}\right) u + \left(M_{\alpha_0} + M_{\mathrm{T}_{\alpha_0}}\right) \delta\alpha + M_{q_0} q + M_{\delta e_0} \delta e. \tag{14.1.6}$$

Stability Parameter	Definition	Value	Units
X_{u_0}	$-\frac{p_{d_0} S}{m U_0}(2C_{D_0} + C_{D_{u_0}})$	−0.0074	1/sec
X_{q_0}	$\frac{p_{d_0} S \bar{c}}{2m U_0} C_{D_{q_0}}$	0.0000	1/sec
X_{α_0}	$\frac{p_{d_0} S}{m}(C_{L_0} - C_{D_{\alpha_0}})$	8.9782	ft/sec^2-rad
$X_{\dot\alpha_0}$	$\frac{p_{d_0} S \bar{c}}{m U_0} C_{D_{\dot\alpha_0}}$	0.0000	1/sec
$X_{\delta e_0}$	$\frac{p_{d_0} S}{m} C_{D_{\delta e_0}}$	0.0000	ft/sec^2-rad
$X_{\mathrm{T}u_0}$	$\frac{p_{d_0} S}{m U_0}(2C_{\mathrm{T}x_0} + C_{\mathrm{T}x_{u_0}})$	0.0000	1/sec
Z_{u_0}	$-\frac{p_{d_0} S}{m U_0}(2C_{L_0} + C_{L_{u_0}})$	−0.1390	1/sec
Z_{q_0}	$-\frac{p_{d_0} S \bar{c}}{2m U_0} C_{L_{q_0}}$	−1.8598	ft/sec-rad
Z_{α_0}	$-\frac{p_{d_0} S}{m}(C_{L_{\alpha_0}} + C_{D_0})$	−445.7224	ft/sec^2-rad
$Z_{\dot\alpha_0}$	$-\frac{p_{d_0} S \bar{c}}{2m U_0} C_{L_{\dot\alpha_0}}$	−0.8705	ft/sec-rad
$Z_{\delta e_0}$	$-\frac{p_{d_0} S}{m} C_{L_{\delta e_0}}$	−42.1968	ft/sec^2-rad
$Z_{\mathrm{T}u_0}$	$-\frac{p_{d_0} S}{m U_0}(2C_{\mathrm{T}z_0} + C_{\mathrm{T}z_{u_0}})$	0.0000	1/sec
M_{u_0}	$\frac{p_{d_0} S \bar{c}}{J_{yy} U_0}(2C_{m_0} + C_{m_{u_0}})$	0.0011	rad/ft-sec
M_{q_0}	$\frac{p_{d_0} S \bar{c}^2}{2J_{yy} U_0} C_{m_{q_0}}$	−0.9397	1/sec
M_{α_0}	$\frac{p_{d_0} S \bar{c}}{J_{yy}} C_{m_{\alpha_0}}$	−7.4416	1/sec^2
$M_{\dot\alpha_0}$	$\frac{p_{d_0} S \bar{c}^2}{2J_{yy} U_0} C_{m_{\dot\alpha_0}}$	−0.4062	1/sec
$M_{\delta e_0}$	$\frac{p_{d_0} S \bar{c}}{J_{yy}} C_{m_{\delta e_0}}$	−17.6737	1/sec^2
$M_{\mathrm{T}u_0}$	$\frac{p_{d_0} S \bar{c}}{J_{yy} U_0}(2C_{\mathrm{T}m_0} + C_{\mathrm{T}m_{u_0}})$	−0.0002	1/ft-sec
$M_{\mathrm{T}\alpha_0}$	$\frac{p_{d_0} S \bar{c}}{J_{yy}} C_{\mathrm{T}m_{\alpha_0}}$	0.0000	1/sec^2

Table 14.1: Longitudinal stability parameters. These data for a business jet with air speed $U_0 = 400$ kt are given in [10, p. 330].

Combining (14.1.4), (14.1.5), (14.1.6), and (12.4.16) for u, $\delta\alpha$, q, and θ yields

$$\Upsilon \begin{bmatrix} \dot u \\ \delta\dot\alpha \\ \dot q \\ \dot\theta \end{bmatrix} = \begin{bmatrix} X_{u_0} + X_{\mathrm{T}u_0} & X_{\alpha_0} & X_{q_0} & -g \\ Z_{u_0} + Z_{\mathrm{T}u_0} & Z_{\alpha_0} & U_0 + Z_{q_0} & 0 \\ M_{u_0} + M_{\mathrm{T}u_0} & M_{\alpha_0} + M_{\mathrm{T}\alpha_0} & M_{q_0} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \\ M_{\delta e_0} \\ 0 \end{bmatrix} \delta e, \qquad (14.1.7)$$

where

$$\Upsilon = \begin{bmatrix} 1 & -X_{\dot{\alpha}_0} & 0 & 0 \\ 0 & U_0 - Z_{\dot{\alpha}_0} & 0 & 0 \\ 0 & -M_{\dot{\alpha}_0} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14.1.8}$$

Assuming that $U_0 - Z_{\dot{\alpha}_0}$ is not zero, it follows that

$$\begin{bmatrix} \dot{u} \\ \delta\dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \Upsilon^{-1} \begin{bmatrix} X_{u_0} + X_{\mathrm{T}u_0} & X_{\alpha_0} & X_{q_0} & -g \\ Z_{u_0} + Z_{\mathrm{T}u_0} & Z_{\alpha_0} & U_0 + Z_{q_0} & 0 \\ M_{u_0} + M_{\mathrm{T}u_0} & M_{\alpha_0} + M_{\mathrm{T}\alpha_0} & M_{q_0} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \end{bmatrix} + \Upsilon^{-1} \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \\ M_{\delta e_0} \\ 0 \end{bmatrix} \delta e, \tag{14.1.9}$$

where

$$\Upsilon^{-1} = \begin{bmatrix} 1 & \frac{X_{\dot{\alpha}_0}}{U_0 - Z_{\dot{\alpha}_0}} & 0 & 0 \\ 0 & \frac{1}{U_0 - Z_{\dot{\alpha}_0}} & 0 & 0 \\ 0 & \frac{M_{\dot{\alpha}_0}}{U_0 - Z_{\dot{\alpha}_0}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14.1.10}$$

Consequently, (14.1.9) can be written as

$$\dot{x} = Ax + B\delta e, \tag{14.1.11}$$

where

$$x \triangleq \begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \end{bmatrix}, \tag{14.1.12}$$

$$A \triangleq \begin{bmatrix} X_{u_0} + X_{\mathrm{T}u_0} + \frac{X_{\dot{\alpha}_0}(Z_{u_0} + Z_{\mathrm{T}u_0})}{U_0 - Z_{\dot{\alpha}_0}} & X_{\alpha_0} + \frac{X_{\dot{\alpha}_0} Z_{\alpha_0}}{U_0 - Z_{\dot{\alpha}_0}} & X_{q_0} + \frac{X_{\dot{\alpha}_0}(U_0 + Z_{q_0})}{U_0 - Z_{\dot{\alpha}_0}} & -g \\ \frac{Z_{u_0} + Z_{\mathrm{T}u_0}}{U_0 - Z_{\dot{\alpha}_0}} & \frac{Z_{\alpha_0}}{U_0 - Z_{\dot{\alpha}_0}} & \frac{U_0 + Z_{q_0}}{U_0 - Z_{\dot{\alpha}_0}} & 0 \\ M_{u_0} + M_{\mathrm{T}u_0} + \frac{M_{\dot{\alpha}_0}(Z_{u_0} + Z_{\mathrm{T}u_0})}{(U_0 - Z_{\dot{\alpha}_0})^2} & M_{\alpha_0} + M_{\mathrm{T}\alpha_0} + \frac{M_{\dot{\alpha}_0} Z_{\alpha_0}}{(U_0 - Z_{\dot{\alpha}_0})^2} & M_{q_0} + \frac{M_{\dot{\alpha}_0}(U_0 + Z_{q_0})}{(U_0 - Z_{\dot{\alpha}_0})^2} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{14.1.13}$$

$$B \triangleq \begin{bmatrix} X_{\delta e_0} + \frac{X_{\dot{\alpha}_0} Z_{\delta e_0}}{U_0 - Z_{\dot{\alpha}_0}} \\ \frac{Z_{\delta e_0}}{U_0 - Z_{\dot{\alpha}_0}} \\ M_{\delta e_0} + \frac{M_{\dot{\alpha}_0} Z_{\delta e_0}}{U_0 - Z_{\dot{\alpha}_0}} \\ 0 \end{bmatrix}. \tag{14.1.14}$$

Combining (14.1.11)–(14.1.14) with the range- and altitude-perturbation equations (12.4.5) and

(12.4.8) yields the linearized longitudinal equations of motion given by

$$\begin{bmatrix} \dot{u} \\ \delta\dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{x} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} X_{u_0} + X_{\mathrm{T}u_0} + \frac{X_{\dot{\alpha}_0}(Z_{u_0}+Z_{\mathrm{T}u_0})}{U_0-Z_{\dot{\alpha}_0}} & X_{\alpha_0} + \frac{X_{\dot{\alpha}_0}Z_{\alpha_0}}{U_0-Z_{\dot{\alpha}_0}} & X_{q_0} + \frac{X_{\dot{\alpha}_0}(U_0+Z_{q_0})}{U_0-Z_{\dot{\alpha}_0}} & -g & 0 & 0 \\ \frac{Z_{u_0}+Z_{\mathrm{T}u_0}}{U_0-Z_{\dot{\alpha}_0}} & \frac{Z_{\alpha_0}}{U_0-Z_{\dot{\alpha}_0}} & \frac{U_0+Z_{q_0}}{U_0-Z_{\dot{\alpha}_0}} & 0 & 0 & 0 \\ M_{u_0} + M_{\mathrm{T}u_0} + \frac{M_{\dot{\alpha}_0}(Z_{u_0}+Z_{\mathrm{T}u_0})}{(U_0-Z_{\dot{\alpha}_0})^2} & M_{\alpha_0} + M_{\mathrm{T}\alpha_0} + \frac{M_{\dot{\alpha}_0}Z_{\alpha_0}}{(U_0-Z_{\dot{\alpha}_0})^2} & M_{q_0} + \frac{M_{\dot{\alpha}_0}(U_0+Z_{q_0})}{(U_0-Z_{\dot{\alpha}_0})^2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -U_0 & 0 & U_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \\ x \\ h \end{bmatrix}$$

$$+ \begin{bmatrix} X_{\delta e_0} + \frac{X_{\dot{\alpha}_0}Z_{\delta e_0}}{U_0-Z_{\dot{\alpha}_0}} \\ \frac{Z_{\delta e_0}}{U_0-Z_{\dot{\alpha}_0}} \\ M_{\delta e_0} + \frac{M_{\dot{\alpha}_0}Z_{\delta e_0}}{U_0-Z_{\dot{\alpha}_0}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \delta e. \tag{14.1.15}$$

In the special case where $X_{\dot{\alpha}_0} = 0$, $Z_{\dot{\alpha}_0} = 0$, and $M_{\dot{\alpha}_0} = 0$, (14.1.15) becomes

$$\begin{bmatrix} \dot{u} \\ \delta\dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{x} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} X_{u_0} + X_{\mathrm{T}u_0} & X_{\alpha_0} & X_{q_0} & -g & 0 & 0 \\ \frac{Z_{u_0}+Z_{\mathrm{T}u_0}}{U_0} & \frac{Z_{\alpha_0}}{U_0} & \frac{U_0+Z_{q_0}}{U_0} & 0 & 0 & 0 \\ M_{u_0} + M_{\mathrm{T}u_0} & M_{\alpha_0} + M_{\mathrm{T}\alpha_0} & M_{q_0} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -U_0 & 0 & U_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \\ x \\ h \end{bmatrix} + \begin{bmatrix} X_{\delta e_0} \\ \frac{Z_{\delta e_0}}{U_0} \\ M_{\delta e_0} \\ 0 \\ 0 \\ 0 \end{bmatrix} \delta e. \tag{14.1.16}$$

14.2 Transfer Functions for Longitudinal Motion

Assuming zero initial conditions, taking the Laplace transform of the linearized longitudinal equations (14.1.4), (14.1.5), (14.1.6), and using $\hat{q}(s) = s\hat{\theta}(s)$ from (12.4.16), we obtain

$$s\hat{u}(s) = \left(X_{u_0} + X_{\mathrm{T}_{u_0}}\right)\hat{u}(s) + X_{\alpha_0}\delta\hat{\alpha}(s) + X_{q_0}s\hat{\theta}(s) + X_{\dot{\alpha}_0}s\delta\hat{\alpha}(s) - g\hat{\theta}(s) + X_{\delta e_0}\delta\hat{e}(s), \tag{14.2.1}$$

$$U_0 s\delta\hat{\alpha}(s) = (Z_{u_0} + Z_{\mathrm{T}u_0})\hat{u}(s) + Z_{\alpha_0}\delta\hat{\alpha}(s) + (U_0 + Z_{q_0})s\hat{\theta}(s) + Z_{\dot{\alpha}_0}s\delta\hat{\alpha}(s) + Z_{\delta e_0}\delta\hat{e}(s), \tag{14.2.2}$$

$$s^2\hat{\theta}(s) = (M_{u_0} + M_{\mathrm{T}u_0})\,\hat{u}(s) + (M_{\alpha_0} + M_{\mathrm{T}\alpha_0})\,\delta\hat{\alpha}(s) + M_{q_0}s\hat{\theta}(s) + M_{\dot{\alpha}_0}s\delta\hat{\alpha}(s) + M_{\delta e_0}\delta\hat{e}(s). \tag{14.2.3}$$

These equations can be written in matrix form as

$$\begin{bmatrix} s - (X_{u_0} + X_{\mathrm{T}u_0}) & -X_{\dot{\alpha}_0}s - X_{\alpha_0} & -X_{q_0}s + g \\ -(Z_{u_0} + Z_{\mathrm{T}u_0}) & (U_0 - Z_{\dot{\alpha}_0})s - Z_{\alpha_0} & -(U_0 + Z_{q_0})s \\ -(M_{u_0} + M_{\mathrm{T}u_0}) & -M_{\dot{\alpha}_0}s - (M_{\alpha_0} + M_{\mathrm{T}\alpha_0}) & s^2 - M_{q_0}s \end{bmatrix} \begin{bmatrix} \hat{u}(s) \\ \delta\hat{\alpha}(s) \\ \hat{\theta}(s) \end{bmatrix} = \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \\ M_{\delta e_0} \end{bmatrix} \delta\hat{e}(s). \tag{14.2.4}$$

Inverting the 3×3 matrix coefficient in (14.2.4) yields the transfer functions

$$G_{u/\delta e}(s) = \frac{\hat{u}(s)}{\delta\hat{e}(s)} = \frac{A_u s^3 + B_u s^2 + C_u s + D_u}{s^4 + Es^3 + Fs^2 + Gs + H}, \tag{14.2.5}$$

$$G_{\delta\alpha/\delta e}(s) = \frac{\delta\hat{\alpha}(s)}{\delta\hat{e}(s)} = \frac{A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha}{s^4 + Es^3 + Fs^2 + Gs + H}, \tag{14.2.6}$$

$$G_{\theta/\delta e}(s) = \frac{\hat{q}(s)}{\delta\hat{e}(s)} = \frac{B_\theta s^3 + C_\theta s^2 + D_\theta s}{s^4 + Es^3 + Fs^2 + Gs + H}, \tag{14.2.7}$$

$$G_{q/\delta e}(s) = \frac{\hat{\theta}(s)}{\delta\hat{e}(s)} = \frac{B_\theta s^2 + C_\theta s + D_\theta}{s^4 + Es^3 + Fs^2 + Gs + H}. \tag{14.2.8}$$

The coefficients of the transfer functions (14.2.5)–(14.2.8) are functions of the stability parameters. In particular, since the leading numerator coefficient is CB, it follows that

$$A_u = X_{\delta e_0} + \frac{X_{\dot{\alpha}_0} Z_{\delta e_0}}{U_0 - Z_{\dot{\alpha}_0}}, \quad A_\alpha = \frac{Z_{\delta e_0}}{U_0 - Z_{\dot{\alpha}_0}}, \quad B_\theta = M_{\delta e_0} + \frac{M_{\dot{\alpha}_0} Z_{\delta e_0}}{U_0 - Z_{\dot{\alpha}_0}}. \tag{14.2.9}$$

Note that the numerator in (14.2.8) is quadratic rather than cubic, which is due to the fact that, for the output θ, $CB = 0$. For details, see Section 15.14.

Next, it follows from (12.4.5) that the range perturbation x satisfies

$$\dot{x} = u, \tag{14.2.10}$$

and thus

$$\hat{x}(s) = \frac{1}{s}\hat{u}(s). \tag{14.2.11}$$

It thus follows from (14.2.6) and (14.2.11) that

$$G_{x/\delta e}(s) = \frac{A_u s^3 + B_u s^2 + C_u s + D_u}{s(s^4 + Es^3 + Fs^2 + Gs + H)}. \tag{14.2.12}$$

For an elevator impulse $\delta e(t) = \delta\overline{e}\delta(t)$, it follows that the asymptotic range perturbation is

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sG_{x/\delta e}(s)\delta\overline{e} = \frac{D_u}{H}\delta\overline{e}. \tag{14.2.13}$$

Next, it follows from (12.4.8) that the altitude perturbation h satisfies

$$\dot{h} = U_0\delta\gamma = U_0(\theta - \delta\alpha), \tag{14.2.14}$$

and thus

$$\hat{h}(s) = \frac{U_0}{s}\delta\hat{\gamma}(s) = \frac{U_0}{s}[\hat{\theta}(s) - \delta\hat{\alpha}(s)]. \tag{14.2.15}$$

Hence,

$$\begin{aligned} G_{h/\delta e}(s) &= \frac{U_0}{s}\frac{\hat{\delta}\gamma(s)}{\delta\hat{e}(s)} = \frac{U_0}{s}\left(\frac{\hat{\theta}(s)}{\delta\hat{e}(s)} - \frac{\delta\hat{\alpha}(s)}{\delta\hat{e}(s)}\right) = \frac{U_0}{s}[G_{\theta/\delta e}(s) - G_{\delta\alpha/\delta e}(s)] \\ &= \frac{-U_0 A_\alpha s^3 + U_0(B_\theta - B_\alpha)s^2 + U_0(C_\theta - C_\alpha)s + U_0(D_\theta - D_\alpha)}{s(s^4 + Es^3 + Fs^2 + Gs + H)}. \end{aligned} \tag{14.2.16}$$

For an elevator impulse $\delta e(t) = \overline{\delta e}\delta(t)$, it thus follows that the asymptotic altitude perturbation is given by

$$\lim_{t\to\infty} h(t) = \lim_{s\to 0} s\hat{h}(s) = \frac{U_0(D_\theta - D_\alpha)}{H}\overline{\delta e}. \tag{14.2.17}$$

The characteristic polynomial $p(s) = s^4 + Es^3 + Fs^2 + Gs + H$ of A given by (14.1.13) can be factored as

$$p(s) = (s^2 + 2\zeta_{\rm ph}\omega_{\rm n,ph}s + \omega_{\rm n,ph}^2)(s^2 + 2\zeta_{\rm sp}\omega_{\rm n,sp}s + \omega_{\rm n,sp}^2). \tag{14.2.18}$$

The roots of p, which are the eigenvalues of A, depend on the flight condition, the mass distribution, and the airplane geometry. The eigenvalues and corresponding eigenvectors of A given by (14.1.13) define eigensolutions that represent flight modes. These flight modes are the phugoid mode and the short period mode.

14.3 Linearized Lateral Equations of Motion

We now incorporate the stability derivatives within the lateral equations of motion (12.3.56), (12.4.21), and (12.3.54) for yaw rate, sideslip, and roll rate, respectively. This yields the linearized lateral equations of motion

$$\begin{aligned}\dot{r} - \frac{J_{xz}}{J_{zz}}\dot{p} = {} & \underbrace{\frac{p_{\rm d_0}Sb^2}{2J_{zz}U_0}C_{n_{r_0}}}_{N_{r_0}} r + \left(\underbrace{\frac{p_{\rm d_0}Sb}{J_{zz}}C_{n_{\beta_0}}}_{N_{\beta_0}} + \underbrace{\frac{p_{\rm d_0}Sb}{J_{zz}}C_{{\rm T}n_{\beta_0}}}_{N_{{\rm T}\beta_0}}\right)\delta\beta + \underbrace{\frac{p_{\rm d_0}Sb^2}{2J_{zz}U_0}C_{n_{p_0}}}_{N_{p_0}} p \\ & + \underbrace{\frac{p_{\rm d_0}Sb^2}{2J_{zz}U_0}C_{n_{\dot\beta_0}}}_{N_{\dot\beta_0}} \delta\dot\beta + \underbrace{\frac{p_{\rm d_0}Sb}{J_{zz}}C_{n_{\delta a_0}}}_{N_{\delta a_0}} \delta a + \underbrace{\frac{p_{\rm d_0}Sb}{J_{zz}}C_{n_{\delta r_0}}}_{N_{\delta r_0}} \delta r,\end{aligned} \tag{14.3.1}$$

$$\begin{aligned}U_0\delta\dot\beta = {} & -\underbrace{\frac{p_{\rm d_0}Sb}{2mU_0}C_{E_{r_0}}}_{-Y_{r_0}} r - U_0 r - \underbrace{\frac{p_{\rm d_0}S}{m}(C_{D_0} + C_{E_{\beta_0}})}_{-Y_{\beta_0}}\delta\beta - \underbrace{\frac{p_{\rm d_0}Sb}{2mU_0}C_{E_{p_0}}}_{-Y_{p_0}} p - \underbrace{\frac{p_{\rm d_0}Sb}{2mU_0}C_{E_{\dot\beta_0}}}_{-Y_{\dot\beta_0}} \delta\dot\beta \\ & + g\phi - \underbrace{\frac{p_{\rm d_0}S}{m}C_{E_{\delta a_0}}}_{-Y_{\delta a_0}} \delta a - \underbrace{\frac{p_{\rm d_0}S}{m}C_{E_{\delta r_0}}}_{-Y_{\delta r_0}} \delta r,\end{aligned} \tag{14.3.2}$$

$$\begin{aligned}\dot{p} - \frac{J_{xz}}{J_{xx}}\dot{r} = {} & \underbrace{\frac{p_{\rm d_0}Sb^2}{2J_{xx}U_0}C_{l_{r_0}}}_{L_{r_0}} r + \left(\underbrace{\frac{p_{\rm d_0}Sb}{J_{xx}}C_{l_{\beta_0}}}_{L_{\beta_0}} + \underbrace{\frac{p_{\rm d_0}Sb}{J_{xx}}C_{{\rm T}l_{\beta_0}}}_{L_{{\rm T}\beta_0}}\right)\delta\beta + \underbrace{\frac{p_{\rm d_0}Sb^2}{2J_{xx}U_0}C_{l_{p_0}}}_{L_{p_0}} p \\ & + \underbrace{\frac{p_{\rm d_0}Sb^2}{2J_{xx}U_0}C_{l_{\dot\beta_0}}}_{L_{\dot\beta_0}} \delta\dot\beta + \underbrace{\frac{p_{\rm d_0}Sb}{J_{xx}}C_{l_{\delta a_0}}}_{L_{\delta a_0}} \delta a + \underbrace{\frac{p_{\rm d_0}Sb}{J_{xx}}C_{l_{\delta r_0}}}_{L_{\delta r_0}} \delta r.\end{aligned} \tag{14.3.3}$$

By introducing the lateral stability parameters, which are defined in Table 14.2, (14.3.1)–(14.3.3) can be written as

$$\dot r - \frac{J_{xz}}{J_{zz}}\dot p = N_{r_0} r + (N_{\beta_0} + N_{\mathrm{T}\beta_0})\delta\beta + N_{p_0} p + N_{\dot\beta_0}\delta\dot\beta + N_{\delta a_0}\delta a + N_{\delta r_0}\delta r, \tag{14.3.4}$$

$$U_0\delta\dot\beta = (Y_{r_0} - U_0) r + Y_{\beta_0}\delta\beta + Y_{p_0} p + Y_{\dot\beta_0}\delta\dot\beta + g\phi + Y_{\delta a_0}\delta a + Y_{\delta r_0}\delta r, \tag{14.3.5}$$

$$\dot p - \frac{J_{xz}}{J_{xx}}\dot r = L_{r_0} r + (L_{\beta_0} + L_{\mathrm{T}\beta_0})\delta\beta + L_{p_0} p + L_{\dot\beta_0}\delta\dot\beta + L_{\delta a_0}\delta a + L_{\delta r_0}\delta r. \tag{14.3.6}$$

Combining (14.3.4)–(14.3.6) and (12.4.15) for r, $\delta\beta$, p, and ϕ yields

$$\Gamma\begin{bmatrix}\dot r\\ \delta\dot\beta\\ \dot p\\ \dot\phi\end{bmatrix} = \begin{bmatrix} N_{r_0} & N_{\beta_0}+N_{\mathrm{T}\beta_0} & N_{p_0} & 0\\ Y_{r_0}-U_0 & Y_{\beta_0} & Y_{p_0} & g\\ L_{r_0} & L_{\beta_0}+L_{\mathrm{T}\beta_0} & L_{p_0} & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix} r\\ \delta\beta\\ p\\ \phi\end{bmatrix} + \Gamma\begin{bmatrix} N_{\delta a_0} & N_{\delta r_0}\\ Y_{\delta a_0} & Y_{\delta r_0}\\ L_{\delta a_0} & L_{\delta r_0}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta a\\ \delta r\end{bmatrix}, \tag{14.3.7}$$

where

$$\Gamma \triangleq \begin{bmatrix} 1 & -N_{\dot\beta_0} & -\frac{J_{xz}}{J_{zz}} & 0\\ 0 & U_0 - Y_{\dot\beta_0} & 0 & 0\\ -\frac{J_{xz}}{J_{xx}} & -L_{\dot\beta_0} & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{14.3.8}$$

Assuming that $U_0 - Y_{\dot\beta_0}$ is not zero, it follows that

$$\begin{bmatrix}\dot r\\ \delta\dot\beta\\ \dot p\\ \dot\phi\end{bmatrix} = \Gamma^{-1}\begin{bmatrix} N_{r_0} & N_{\beta_0}+N_{\mathrm{T}\beta_0} & N_{p_0} & 0\\ Y_{r_0}-U_0 & Y_{\beta_0} & Y_{p_0} & g\\ L_{r_0} & L_{\beta_0}+L_{\mathrm{T}\beta_0} & L_{p_0} & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix} r\\ \delta\beta\\ p\\ \phi\end{bmatrix} + \Gamma^{-1}\begin{bmatrix} N_{\delta a_0} & N_{\delta r_0}\\ Y_{\delta a_0} & Y_{\delta r_0}\\ L_{\delta a_0} & L_{\delta r_0}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta a\\ \delta r\end{bmatrix}, \tag{14.3.9}$$

where

$$\Gamma^{-1} = \begin{bmatrix} \frac{J_{xx}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & \frac{J_{xx}(J_{zz}N_{\dot\beta_0}+J_{xz}L_{\dot\beta_0})}{(U_0-Y_{\dot\beta_0})(J_{xx}J_{zz}-J_{xz}^2)} & \frac{J_{xz}J_{xx}}{J_{xx}J_{zz}-J_{xz}^2} & 0\\ 0 & \frac{1}{U_0-Y_{\dot\beta_0}} & 0 & 0\\ \frac{J_{xz}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & \frac{J_{zz}(J_{xx}L_{\dot\beta_0}+J_{xz}N_{\dot\beta_0})}{(U_0-Y_{\dot\beta_0})(J_{xx}J_{zz}-J_{xz}^2)} & \frac{J_{xx}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{14.3.10}$$

In the special case where $Y_{\dot\beta_0} = 0$, $L_{\dot\beta_0} = 0$, and $N_{\dot\beta_0} = 0$, (14.3.9) becomes

$$\begin{bmatrix}\dot r\\ \delta\dot\beta\\ \dot p\\ \dot\phi\end{bmatrix} = \Gamma_0^{-1}\begin{bmatrix} N_{r_0} & N_{\beta_0}+N_{\mathrm{T}\beta_0} & N_{p_0} & 0\\ \frac{Y_{r_0}-U_0}{U_0} & \frac{Y_{\beta_0}}{U_0} & \frac{Y_{p_0}}{U_0} & \frac{g}{U_0}\\ L_{r_0} & L_{\beta_0}+L_{\mathrm{T}\beta_0} & L_{p_0} & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix} r\\ \delta\beta\\ p\\ \phi\end{bmatrix} + \Gamma_0^{-1}\begin{bmatrix} N_{\delta a_0} & N_{\delta r_0}\\ \frac{Y_{\delta a_0}}{U_0} & \frac{Y_{\delta r_0}}{U_0}\\ L_{\delta a_0} & L_{\delta r_0}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta a\\ \delta r\end{bmatrix}, \tag{14.3.11}$$

where

$$\Gamma_0 \triangleq \begin{bmatrix} 1 & 0 & -\frac{J_{xz}}{J_{zz}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{J_{xz}}{J_{xx}} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_0^{-1} = \begin{bmatrix} \frac{J_{xx}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xz}J_{xx}}{J_{xx}J_{zz}-J_{xz}^2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{J_{xz}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xx}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14.3.12}$$

With the aileron and rudder deflections δa and δr as inputs, (14.3.9) can be written as the state space equation

$$\dot{x} = Ax + B\begin{bmatrix} \delta a \\ \delta r \end{bmatrix}, \tag{14.3.13}$$

where

$$x \triangleq \begin{bmatrix} r \\ \delta\beta \\ p \\ \phi \end{bmatrix}, \quad A \triangleq \Gamma^{-1}\begin{bmatrix} N_{r_0} & N_{\beta_0} + N_{\mathrm{T}\beta_0} & N_{p_0} & 0 \\ Y_{r_0} - U_0 & Y_{\beta_0} & Y_{p_0} & g \\ L_{r_0} & L_{\beta_0} + L_{\mathrm{T}\beta_0} & L_{p_0} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B \triangleq \Gamma^{-1}\begin{bmatrix} N_{\delta a_0} & N_{\delta r_0} \\ Y_{\delta a_0} & Y_{\delta r_0} \\ L_{\delta a_0} & L_{\delta r_0} \\ 0 & 0 \end{bmatrix}. \tag{14.3.14}$$

Writing Γ^{-1} as

$$\Gamma^{-1} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ 0 & \gamma_{22} & 0 & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{11} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14.3.15}$$

it follows that

$$B = \begin{bmatrix} \gamma_{11}N_{\delta a_0} + \gamma_{12}Y_{\delta a_0} + \gamma_{13}L_{\delta a_0} & \gamma_{11}N_{\delta r_0} + \gamma_{12}Y_{\delta r_0} + \gamma_{13}L_{\delta r_0} \\ \gamma_{22}Y_{\delta a_0} & \gamma_{22}Y_{\delta r_0} \\ \gamma_{31}N_{\delta a_0} + \gamma_{32}Y_{\delta a_0} + \gamma_{11}L_{\delta a_0} & \gamma_{31}N_{\delta r_0} + \gamma_{32}Y_{\delta r_0} + \gamma_{11}L_{\delta r_0} \\ 0 & 0 \end{bmatrix}. \tag{14.3.16}$$

Combining (14.3.13) with the yaw- and drift-perturbation equations (12.4.17) and (12.4.6) yields the linearized lateral equations of motion given by

$$\begin{bmatrix} \dot{r} \\ \delta\dot{\beta} \\ \dot{p} \\ \dot{\phi} \\ \dot{\psi} \\ \dot{y} \end{bmatrix} = \tilde{\Gamma}^{-1}\begin{bmatrix} N_{r_0} & N_{\beta_0} + N_{\mathrm{T}\beta_0} & N_{p_0} & 0 & 0 & 0 \\ Y_{r_0} - U_0 & Y_{\beta_0} & Y_{p_0} & g & 0 & 0 \\ L_{r_0} & L_{\beta_0} + L_{\mathrm{T}\beta_0} & L_{p_0} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_0 & 0 & 0 & U_0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \delta\beta \\ p \\ \phi \\ \psi \\ y \end{bmatrix} + \tilde{\Gamma}^{-1}\begin{bmatrix} N_{\delta a_0} & N_{\delta r_0} \\ Y_{\delta a_0} & Y_{\delta r_0} \\ L_{\delta a_0} & L_{\delta r_0} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta a \\ \delta r \end{bmatrix}, \tag{14.3.17}$$

where

$$\tilde{\Gamma} \triangleq \begin{bmatrix} \Gamma & 0 \\ 0 & I_2 \end{bmatrix}, \quad \tilde{\Gamma}^{-1} = \begin{bmatrix} \Gamma^{-1} & 0 \\ 0 & I_2 \end{bmatrix}. \tag{14.3.18}$$

Stability Parameter	Definition	Value	Units
Y_{β_0}	$-\frac{p_{d_0}S}{m}(C_{D_0} + C_{E_{\beta_0}})$	–55.4022	ft/sec^2-rad
Y_{p_0}	$-\frac{p_{d_0}Sb}{2mU_0}C_{E_{p_0}}$	0.0000	ft/sec-rad
Y_{r_0}	$-\frac{p_{d_0}Sb}{2mU_0}C_{E_{r_0}}$	0.7689	ft/sec-rad
$Y_{\dot{\beta}_0}$	$-\frac{p_{d_0}Sb}{2mU_0}C_{E_{\dot{\beta}_0}}$	0.0000	ft/sec-rad
$Y_{\delta a_0}$	$-\frac{p_{d_0}S}{m}C_{E_{\delta a_0}}$	0.0000	ft/sec^2-rad
$Y_{\delta r_0}$	$-\frac{p_{d_0}S}{m}C_{E_{\delta r_0}}$	10.4733	ft/sec^2-rad
L_{β_0}	$\frac{p_{d_0}Sb}{J_{xx}}C_{l_{\beta_0}}$	–4.1845	1/sec^2
L_{p_0}	$\frac{p_{d_0}Sb^2}{2J_{xx}U_0}C_{l_{p_0}}$	–0.4365	1/sec
L_{r_0}	$\frac{p_{d_0}Sb^2}{2J_{xx}U_0}C_{l_{r_0}}$	0.1571	1/sec
$L_{\dot{\beta}_0}$	$\frac{p_{d_0}Sb^2}{2J_{xx}U_0}C_{l_{\dot{\beta}_0}}$	0.0000	1/sec
$L_{\delta a_0}$	$\frac{p_{d_0}Sb}{J_{xx}}C_{l_{\delta a_0}}$	6.7714	1/sec^2
$L_{\delta r_0}$	$\frac{p_{d_0}Sb}{J_{xx}}C_{l_{\delta r_0}}$	0.6543	1/sec^2
$L_{\mathrm{T}\beta_0}$	$\frac{p_{d_0}Sb}{J_{xx}}C_{\mathrm{T}l_{\beta_0}}$	0.0000	1/sec^2
N_{β_0}	$\frac{p_{d_0}Sb}{J_{zz}}C_{n_{\beta_0}}$	2.8643	1/sec^2
N_{p_0}	$\frac{p_{d_0}Sb^2}{2J_{zz}U_0}C_{n_{p_0}}$	0.0046	1/sec
N_{r_0}	$\frac{p_{d_0}Sb^2}{2J_{zz}U_0}C_{n_{r_0}}$	–0.1148	1/sec
$N_{\dot{\beta}_0}$	$\frac{p_{d_0}Sb^2}{2J_{zz}U_0}C_{n_{\dot{\beta}_0}}$	0.0000	1/sec
$N_{\delta a_0}$	$\frac{p_{d_0}Sb}{J_{zz}}C_{n_{\delta a_0}}$	–0.3879	1/sec^2
$N_{\delta r_0}$	$\frac{p_{d_0}Sb}{J_{zz}}C_{n_{\delta r_0}}$	–1.6847	1/sec^2
$N_{\mathrm{T}\beta_0}$	$\frac{p_{d_0}Sb}{J_{zz}}C_{\mathrm{T}n_{\beta_0}}$	0.0000	1/sec^2
J_{xx}		27915	slg-ft^2
J_{zz}		47085	slg-ft^2
J_{xz}		450	slg-ft^2

Table 14.2: Lateral stability parameters. These data for a business jet with air speed $U_0 = 400$ kt are given in [10, p. 330].

14.4 Transfer Functions for Lateral Motion

Assuming zero initial conditions, taking the Laplace transform of the linearized lateral equations (14.3.4)–(14.3.6) and using $s\hat{\phi}(s) = \hat{p}(s)$ from (12.4.15) and $s\hat{\psi}(s) = \hat{r}(s)$ from (12.4.17) yields

$$s\hat{r}(s) - \frac{J_{xz}}{J_{zz}} s^2\hat{\phi}(s) = N_{r_0}\hat{r}(s) + (N_{\beta_0} + N_{\mathrm{T}\beta_0})\delta\hat{\beta}(s) + N_{p_0}s\hat{\phi}(s) + N_{\dot{\beta}_0}s\delta\hat{\beta}(s) + N_{\delta a_0}\delta\hat{a}(s) + N_{\delta r_0}\delta\hat{r}(s), \tag{14.4.1}$$

$$U_0 s\delta\hat{\beta}(s) = (Y_{r_0} - U_0)\hat{r}(s) + Y_{\beta_0}\delta\hat{\beta}(s) + Y_{p_0}s\hat{\phi}(s) + Y_{\dot{\beta}_0}s\delta\hat{\beta}(s) + g\hat{\phi}(s) + Y_{\delta a_0}\delta\hat{a}(s) + Y_{\delta r_0}\delta\hat{r}(s), \tag{14.4.2}$$

$$s^2\hat{\phi}(s) - \frac{J_{xz}}{J_{xx}} s\hat{r}(s) = L_{r_0}\hat{r}(s) + (L_{\beta_0} + L_{\mathrm{T}\beta_0})\delta\hat{\beta}(s) + L_{p_0}s\hat{\phi}(s) + L_{\dot{\beta}_0}s\delta\hat{\beta}(s) + L_{\delta a_0}\delta\hat{a}(s) + L_{\delta r_0}\delta\hat{r}(s). \tag{14.4.3}$$

These equations can be written in matrix form as

$$\begin{bmatrix} s - N_{r_0} & -N_{\dot{\beta}_0}s - (N_{\beta_0} + N_{\mathrm{T}\beta_0}) & -(\frac{J_{xz}}{J_{zz}}s^2 + N_{p_0}s) \\ U_0 - Y_{r_0} & (U_0 - Y_{\dot{\beta}_0})s - Y_{\beta_0} & -(Y_{p_0}s + g) \\ -(\frac{J_{xz}}{J_{xx}}s + L_{r_0}) & -L_{\dot{\beta}_0}s - (L_{\beta_0} + L_{\mathrm{T}\beta_0}) & s^2 - L_{p_0}s \end{bmatrix} \begin{bmatrix} \hat{r}(s) \\ \delta\hat{\beta}(s) \\ \hat{\phi}(s) \end{bmatrix} = \begin{bmatrix} Y_{\delta a_0} & Y_{\delta r_0} \\ L_{\delta a_0} & L_{\delta r_0} \\ N_{\delta a_0} & N_{\delta r_0} \end{bmatrix} \begin{bmatrix} \delta\hat{a}(s) \\ \delta\hat{r}(s) \end{bmatrix}. \tag{14.4.4}$$

Inverting the 3×3 matrix coefficient in (14.4.4) and considering aileron deflection yields

$$G_{r/\delta a}(s) = \frac{\hat{r}(s)}{\delta\hat{a}(s)} = \frac{A_{r,\delta a}s^3 + B_{r,\delta a}s^2 + C_{r,\delta a}s + D_{r,\delta a}}{s^4 + E's^3 + F's^2 + G's + H'}, \tag{14.4.5}$$

$$G_{\delta\beta/\delta a}(s) = \frac{\delta\hat{\beta}(s)}{\delta\hat{a}(s)} = \frac{A_{\beta,\delta a}s^3 + B_{\beta,\delta a}s^2 + C_{\beta,\delta a}s + D_{\beta,\delta a}}{s^4 + E's^3 + F's^2 + G's + H'}, \tag{14.4.6}$$

$$G_{p/\delta a}(s) = \frac{\hat{p}(s)}{\delta\hat{a}(s)} = \frac{B_{\phi,\delta a}s^3 + C_{\phi,\delta a}s^2 + D_{\phi,\delta a}s}{s^4 + E's^3 + F's^2 + G's + H'}, \tag{14.4.7}$$

$$G_{\phi/\delta a}(s) = \frac{\hat{\phi}(s)}{\delta\hat{a}(s)} = \frac{B_{\phi,\delta a}s^2 + C_{\phi,\delta a}s + D_{\phi,\delta a}}{s^4 + E's^3 + F's^2 + G's + H'}. \tag{14.4.8}$$

Likewise, for rudder deflection, (14.4.4) yields

$$G_{r/\delta r}(s) = \frac{\hat{r}(s)}{\delta\hat{r}(s)} = \frac{A_{r,\delta r}s^3 + B_{r,\delta r}s^2 + C_{r,\delta r}s + D_{r,\delta r}}{s^4 + E's^3 + F's^2 + G's + H'}, \tag{14.4.9}$$

$$G_{\delta\beta/\delta r}(s) = \frac{\delta\hat{\beta}(s)}{\delta\hat{r}(s)} = \frac{A_{\beta,\delta r}s^3 + B_{\beta,\delta r}s^2 + C_{\beta,\delta r}s + D_{\beta,\delta r}}{s^4 + E's^3 + F's^2 + G's + H'}, \tag{14.4.10}$$

$$G_{p/\delta r}(s) = \frac{\hat{p}(s)}{\delta\hat{r}(s)} = \frac{B_{\phi,\delta r}s^3 + C_{\phi,\delta r}s^2 + D_{\phi,\delta r}s}{s^4 + E's^3 + F's^2 + G's + H'}, \tag{14.4.11}$$

$$G_{\phi/\delta r}(s) = \frac{\hat{\phi}(s)}{\delta\hat{r}(s)} = \frac{B_{\phi,\delta r}s^2 + C_{\phi,\delta r}s + D_{\phi,\delta r}}{s^4 + E's^3 + F's^2 + G's + H'}. \tag{14.4.12}$$

The coefficients of the transfer functions (14.4.5)–(14.4.12) are functions of the stability parameters. In particular, since the leading numerator coefficient is CB, it follows using the notation (14.3.15), that

$$A_{r,\delta a} = \gamma_{11}N_{\delta a_0} + \gamma_{12}Y_{\delta a_0} + \gamma_{13}L_{\delta a_0}, \quad A_{r,\delta r} = \gamma_{11}N_{\delta r_0} + \gamma_{12}Y_{\delta r_0} + \gamma_{13}L_{\delta r_0} \tag{14.4.13}$$

$$A_{\beta,\delta a} = \gamma_{22}Y_{\delta a_0}, \quad A_{\beta,\delta r} = \gamma_{22}Y_{\delta r_0}, \tag{14.4.14}$$

$$B_{\phi,\delta a} = \gamma_{31} N_{\delta a_0} + \gamma_{32} Y_{\delta a_0} + \gamma_{11} L_{\delta a_0}, \quad B_{\phi,\delta r} = \gamma_{31} N_{\delta r_0} + \gamma_{32} Y_{\delta r_0} + \gamma_{11} L_{\delta r_0}. \tag{14.4.15}$$

Note that the numerators in (14.4.8) and (14.4.12) are quadratic rather than cubic, which is due to the fact that, for the output ϕ, $CB = 0$. For details, see Section 15.14.

Next, it follows from (12.4.17) that the yaw perturbation ψ satisfies

$$\dot{\psi} = r, \tag{14.4.16}$$

and thus

$$\hat{\psi}(s) = \frac{1}{s}\hat{r}(s). \tag{14.4.17}$$

For aileron deflection, it follows from (14.4.5) and (14.4.17) that

$$G_{\psi/\delta a}(s) = \frac{\hat{\psi}(s)}{\delta\hat{a}(s)} = \frac{A_{r,\delta a}s^3 + B_{r,\delta a}s^2 + C_{r,\delta a}s + D_{r,\delta a}}{s(s^4 + E's^3 + F's^2 + G's + H')}. \tag{14.4.18}$$

For an aileron impulse $\delta a(t) = \delta\overline{a}\delta(t)$, it thus follows that the asymptotic yaw perturbation is

$$\lim_{t\to\infty} \psi(t) = \lim_{s\to 0} s\hat{\psi}(s) = \frac{D_{r,\delta a}}{H'}\delta\overline{a}. \tag{14.4.19}$$

Likewise, for rudder deflection, it follows from (14.4.9) and (14.4.17) that

$$G_{\psi/\delta r}(s) = \frac{\hat{\psi}(s)}{\delta\hat{r}(s)} = \frac{A_{r,\delta r}s^3 + B_{r,\delta r}s^2 + C_{r,\delta r}s + D_{r,\delta r}}{s(s^4 + E's^3 + F's^2 + G's + H')}. \tag{14.4.20}$$

For a rudder impulse $\delta r(t) = \delta\overline{r}\delta(t)$, it thus follows that the asymptotic yaw perturbation is

$$\lim_{t\to\infty} \psi(t) = \lim_{s\to 0} s\hat{\psi}(s) = \frac{D_{r,\delta r}}{H'}\delta\overline{r}. \tag{14.4.21}$$

Next, it follows from (12.4.11) that the drift perturbation y satisfies

$$\dot{y} = U_0\delta\eta = U_0(\psi + \delta\beta). \tag{14.4.22}$$

and thus

$$\hat{y}(s) = \frac{U_0}{s}\delta\hat{\eta}(s) = \frac{U_0}{s}[\hat{\psi}(s) + \delta\hat{\beta}(s)]. \tag{14.4.23}$$

Hence, for aileron deflection,

$$\begin{aligned} G_{y/\delta a}(s) &= \frac{U_0}{s}\frac{\hat{\delta}\eta(s)}{\delta\hat{a}(s)} = \frac{U_0}{s}\left(\frac{\hat{\psi}(s)}{\delta\hat{a}(s)} + \frac{\delta\hat{\beta}(s)}{\delta\hat{a}(s)}\right) = \frac{U_0}{s}[G_{\psi/\delta a}(s) + G_{\delta\beta/\delta a}(s)] \\ &= \frac{U_0}{s}\left[\frac{A_{r,\delta a}s^3 + B_{r,\delta a}s^2 + C_{r,\delta a}s + D_{r,\delta a}}{s(s^4 + E's^3 + F's^2 + G's + H')} + \frac{A_{\beta,\delta a}s^3 + B_{\beta,\delta a}s^2 + C_{\beta,\delta a}s + D_{\beta,\delta a}}{s^4 + E's^3 + F's^2 + G's + H'}\right] \\ &= \frac{U_0 A_{\beta,\delta a}s^4 + U_0(A_{r,\delta a} + B_{\beta,\delta a})s^3 + U_0(B_{r,\delta a} + C_{\beta,\delta a})s^2 + U_0(C_{r,\delta a} + D_{\beta,\delta a})s + U_0 D_{r,\delta a}}{s^2(s^4 + E's^3 + F's^2 + G's + H')}. \end{aligned} \tag{14.4.24}$$

For an aileron impulse $\delta a(t) = \delta\overline{a}\delta(t)$, it thus follows that the asymptotic drift perturbation is

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) = \operatorname{sign}\left(\frac{U_0 D_{r,\delta a}}{H'}\delta\overline{a}\right)\infty. \tag{14.4.25}$$

An analogous result holds for a rudder impulse.

The characteristic polynomial $p(s) = s^4 + E's^3 + F's^2 + G's + H'$ of A given by (14.3.14) can be factored as

$$p(s) = (s^2 + 2\zeta_{\rm Dr}\omega_{\rm n,Dr}s + \omega_{\rm n,Dr}^2)\left(s + \frac{1}{\tau_{\rm r}}\right)\left(s + \frac{1}{\tau_{\rm s}}\right), \tag{14.4.26}$$

where $\tau_{\rm r}$ and $\tau_{\rm s}$ are real numbers. The roots of p, which are the eigenvalues of A, depend on the flight condition, the mass distribution, and the airplane geometry. The eigenvalues and corresponding eigenvectors of A given by (14.3.14) define eigensolutions that represent flight modes. These flight modes are the Dutch roll mode, spiral mode, and roll mode.

14.5 Combined Linearized Longitudinal and Lateral Equations of Motion

By combining the linearized longitudinal equations of motion (14.1.15) with the linearized lateral equations of motion (14.3.17) as well as the linearized kinematic equations of motion (12.4.15)–(12.4.17), we obtain the complete linearized longitudinal and lateral equations

$$\begin{bmatrix} \dot{u} \\ \delta\dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{x} \\ \dot{h} \\ \delta\dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \\ \dot{y} \end{bmatrix} =$$

$$\overline{\Gamma}\begin{bmatrix}
X_{u_0} + X_{{\rm T}u_0} & X_{\alpha_0} & X_{q_0} & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_{u_0} + Z_{{\rm T}u_0} & Z_{\alpha_0} & U_0 + Z_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M_{u_0} + M_{{\rm T}u_0} & M_{\alpha_0} + M_{{\rm T}\alpha_0} & M_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -U_0 & 0 & U_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Y_{\beta_0} & Y_{p_0} & Y_{r_0} - U_0 & g & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & L_{\beta_0} + L_{{\rm T}\beta_0} & L_{p_0} & L_{r_0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & N_{\beta_0} + N_{{\rm T}\beta_0} & N_{p_0} & N_{r_0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & U_0 & 0 & 0 & 0 & U_0 & 0
\end{bmatrix}\begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \\ x \\ h \\ \delta\beta \\ p \\ r \\ \phi \\ \psi \\ y \end{bmatrix}$$

$$
+\ \overline{\Gamma}\begin{bmatrix} X_{\delta e_0} & 0 & 0 \\ Z_{\delta e_0} & 0 & 0 \\ M_{\delta e_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & Y_{\delta a_0} & Y_{\delta r_0} \\ 0 & L_{\delta a_0} & L_{\delta r_0} \\ 0 & N_{\delta a_0} & N_{\delta r_0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta e \\ \delta a \\ \delta r \end{bmatrix}, \tag{14.5.1}
$$

where

$$
\overline{\Gamma} \triangleq \begin{bmatrix} I_6 & 0 & 0 \\ 0 & \Gamma & 0 \\ 0 & 0 & I_2 \end{bmatrix}, \quad \overline{\Gamma}^{-1} = \begin{bmatrix} I_6 & 0 & 0 \\ 0 & \Gamma^{-1} & 0 \\ 0 & 0 & I_2 \end{bmatrix}, \tag{14.5.2}
$$

and Γ_0 and its inverse are defined by (14.3.12).

14.6 Eigenflight

Eigenflight is the motion of an aircraft near steady flight as given by an eigensolution of the linearized dynamics. Eigensolutions are also called *modal solutions*. For details see Section 15.3.

Consider the unforced linear time invariant system

$$
\dot{x}(t) = Ax(t) \tag{14.6.1}
$$

with initial condition

$$
x(0) = x_0,
$$

where $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$, and consider a solution to (14.6.1) of the form

$$
x(t) = \mathrm{Re}(e^{\lambda t}v), \tag{14.6.2}
$$

whose initial value is

$$
x_0 = \mathrm{Re}\, v. \tag{14.6.3}
$$

Substituting (14.6.2) into (14.6.1) yields the eigenvalue-eigenvector equation

$$
Av = \lambda v. \tag{14.6.4}
$$

Note that λ may be complex, in which case the associated eigenvector v may also be complex. Consequently, if $\lambda \in \mathbb{C}$ is an eigenvalue of A and $v \in \mathbb{C}^n$ is an associated eigenvector, then $x(t)$ given by (14.6.2) is a solution of (14.6.1) with the initial condition (14.6.3). The solution $x(t)$ is an eigensolution.

If λ represents underdamped motion with damping ratio ζ and undamped natural frequency $\omega_{\rm n}$, then we can write

$$\lambda = -\zeta\omega_{\rm n} + \jmath\omega_{\rm n}\sqrt{1-\zeta^2}. \tag{14.6.5}$$

In terms of the decay rate

$$\sigma \triangleq -\zeta\omega_{\rm n} \tag{14.6.6}$$

and the damped natural frequency

$$\omega_{\rm d} = \omega_{\rm n}\sqrt{1-\zeta^2}, \tag{14.6.7}$$

λ can be written as

$$\lambda = \sigma + \jmath\omega_{\rm d}. \tag{14.6.8}$$

Next, since v is complex, we can write

$$v = v_{\rm R} + \jmath v_{\rm I}, \tag{14.6.9}$$

where $v_{\rm R}$ and $v_{\rm I}$ are real vectors. Then, substituting λ and v into (14.6.2) yields

$$\begin{aligned} x(t) &= \operatorname{Re} e^{(\sigma+\jmath\omega_{\rm d})t}(v_{\rm R} + \jmath v_{\rm I}) \\ &= \operatorname{Re} e^{\sigma t}[\cos(\omega_{\rm d}t) + \jmath\sin(\omega_{\rm d}t)](v_{\rm R} + \jmath v_{\rm I}) \\ &= e^{\sigma t}[v_{\rm R}\cos(\omega_{\rm d}t) - v_{\rm I}\sin(\omega_{\rm d}t)]. \end{aligned} \tag{14.6.10}$$

Alternatively, we can represent each component of v in polar form by writing

$$v = \begin{bmatrix} r_1e^{\jmath\phi_1} \\ r_2e^{\jmath\phi_2} \\ r_3e^{\jmath\phi_3} \\ r_4e^{\jmath\phi_4} \end{bmatrix}. \tag{14.6.11}$$

The component-wise magnitude and angle of v are given by

$$|v| = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}, \quad \angle v = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}. \tag{14.6.12}$$

Substituting v into (14.6.2) yields

$$x(t) = \operatorname{Re} e^{(\sigma+\jmath\omega_{\rm d})t}\begin{bmatrix} r_1e^{\jmath\phi_1} \\ r_2e^{\jmath\phi_2} \\ r_3e^{\jmath\phi_3} \\ r_4e^{\jmath\phi_4} \end{bmatrix} = e^{\sigma t}\operatorname{Re}\begin{bmatrix} r_1e^{\jmath(\omega_{\rm d}t+\phi_1)} \\ r_2e^{\jmath(\omega_{\rm d}t+\phi_2)} \\ r_3e^{\jmath(\omega_{\rm d}t+\phi_3)} \\ r_4e^{\jmath(\omega_{\rm d}t+\phi_4)} \end{bmatrix} = e^{\sigma t}\begin{bmatrix} r_1\cos(\omega_{\rm d}t+\phi_1) \\ r_2\cos(\omega_{\rm d}t+\phi_2) \\ r_3\cos(\omega_{\rm d}t+\phi_3) \\ r_4\cos(\omega_{\rm d}t+\phi_4) \end{bmatrix}, \tag{14.6.13}$$

where $e^{\sigma t}$ is the decay envelope, and ϕ_i is the phase shift of the ith component. Note that

$$x(0) = \begin{bmatrix} r_1\cos\phi_1 \\ r_2\cos\phi_2 \\ r_3\cos\phi_3 \\ r_4\cos\phi_4 \end{bmatrix} = \operatorname{Re} v. \tag{14.6.14}$$

Hence the ith component of the initial value $x(0)$ is given by $r_i\cos\phi_i$.

14.7 Longitudinal Flight Modes

The roots of the longitudinal characteristic polynomial (14.2.18), which are typically underdamped, give rise to the longitudinal flight modes. In particular, the complex root

$$\lambda_{\mathrm{ph}} = -\zeta_{\mathrm{ph}}\omega_{\mathrm{n,ph}} + \jmath\omega_{\mathrm{n,ph}}\sqrt{1-\zeta_{\mathrm{ph}}^2} \tag{14.7.1}$$

corresponds to the *phugoid mode*. Furthermore, the complex root

$$\lambda_{\mathrm{sp}} = -\zeta_{\mathrm{sp}}\omega_{\mathrm{n,sp}} + \jmath\omega_{\mathrm{n,sp}}\sqrt{1-\zeta_{\mathrm{sp}}^2} \tag{14.7.2}$$

corresponds to the *short period mode*. The natural frequency $\omega_{\mathrm{n,ph}}$ of λ_{ph} is lower than the natural frequency $\omega_{\mathrm{n,sp}}$ of λ_{sp}.

14.7.1 Phugoid Mode

The phugoid mode is associated with low damping, that is, $\zeta_{\mathrm{ph}} << 1$, and a long period $T_{\mathrm{ph}} = \frac{2\pi}{\omega_{\mathrm{n,ph}}}$ of oscillation, typically 30 sec to 120 sec.

We now show, by means of a numerical example, that the phugoid mode is a roller-coaster-type oscillation, which trades kinetic and potential energy. In particular, we show that the perturbation states θ and u are oscillatory and $\delta\alpha \approx 0$, which implies that α is approximately constant.

Consider a 747-100 aircraft flying at 40,000 ft at Mach 0.8. The aircraft state is $\dot{x} = Ax$, where

$$x = \begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \end{bmatrix}, \quad A = \begin{bmatrix} -2.02e(-2) & 7.88 & -6.5e(-1) & -3.22e(+1) \\ -2.54e(-4) & -1.02 & 9.05e(-1) & 0 \\ 7.95e(-11) & -2.5 & -1.39 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{14.7.3}$$

where the units of x are given by

$$[x] = \begin{bmatrix} \mathrm{m/s} \\ \mathrm{rad} \\ \mathrm{rad/s} \\ \mathrm{rad} \end{bmatrix}. \tag{14.7.4}$$

Computing the eigenvalues of A we find

$$\lambda_{\mathrm{ph}} = -0.0087 \pm \jmath 0.074 \text{ rad/sec}$$

for the phugoid mode and

$$\lambda_{\mathrm{sp}} = -1.206 \pm \jmath 1.49 \text{ rad/sec}$$

for the short period mode, which are represented in Figure 14.7.1. Therefore, $\omega_{\mathrm{d,ph}} = 0.074$ rad/sec, $\omega_{\mathrm{n,ph}} = 0.0745$ rad/sec, $\zeta_{\mathrm{ph}} = 0.117$, $\omega_{\mathrm{d,sp}} = 1.49$ rad/sec, $\omega_{\mathrm{n,sp}} = 1.92$ rad/sec, and $\zeta_{\mathrm{sp}} = 0.628$. These results are summarized in Table 14.3

By normalizing the range velocity component to 1, the phugoid eigenvector is given by

$$v_{\mathrm{ph}} = \begin{bmatrix} 1 \text{ m/sec} \\ -9.6e(-5) - \jmath 5.0e(-7) \text{ rad} \\ 1.74e(-4) - \jmath 8.4e(-6) \text{ rad/sec} \\ -3.8e(-4) - \jmath 2.3e(-3) \text{ rad} \end{bmatrix}. \tag{14.7.5}$$

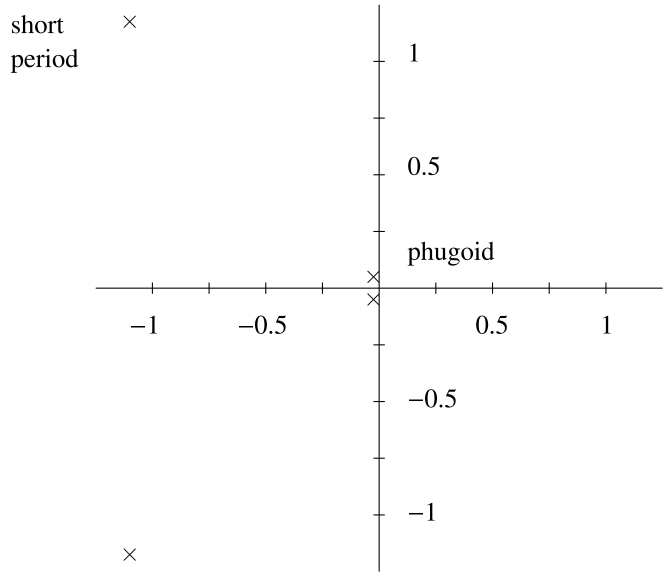


Figure 14.7.1: Phugoid and short period eigenvalues.

Mode	ω_n	ζ	$\zeta\omega_\mathrm{n}$	ω_d	$T = \frac{2\pi}{\omega_\mathrm{n}}$	$T_{1/2} = \frac{\ln 2}{\zeta\omega_\mathrm{n}}$
phugoid	0.0745 rad/sec	0.117	.0087	0.074	84.34 sec	79.5 sec
short period	1.92 rad/sec	0.628	1.21	1.49	3.27 sec	0.57 sec

Table 14.3: Longitudinal mode characteristics.

The component-wise magnitude of $v_{\rm ph}$ is

$$|v_{\rm ph}| = \begin{bmatrix} 1 \text{ m/sec} \\ 9.6e(-5) \text{ rad} \\ 1.74e(-4) \text{ rad/sec} \\ 2.33e(-3) \text{ rad} \end{bmatrix} \tag{14.7.6}$$

with phase

$$\angle v_{\rm ph} = \begin{bmatrix} 0^\circ \\ -179.7^\circ \\ -2.83^\circ \\ -99.48^\circ \end{bmatrix} = \begin{bmatrix} 0 \text{ rad} \\ -3.137 \text{ rad} \\ -0.0484 \text{ rad} \\ -1.736 \text{ rad} \end{bmatrix}. \tag{14.7.7}$$

Using (14.6.13) the eigensolution for the state $x(t)$ defined by (14.7.3) is written as

$$x(t) = e^{\sigma_{\rm ph}t}\,\mathrm{Re}\, e^{\jmath\omega_{\rm d,ph}t} \begin{bmatrix} r_1 e^{\jmath\phi_1} \\ r_2 e^{\jmath\phi_2} \\ r_3 e^{\jmath\phi_3} \\ r_4 e^{\jmath\phi_4} \end{bmatrix} = e^{\sigma_{\rm ph}t} \begin{bmatrix} r_1 \cos(\omega_{\rm d,ph}t + \phi_1) \\ r_2 \cos(\omega_{\rm d,ph}t + \phi_2) \\ r_3 \cos(\omega_{\rm d,ph}t + \phi_3) \\ r_4 \cos(\omega_{\rm d,ph}t + \phi_4) \end{bmatrix}. \tag{14.7.8}$$

It thus follows from (14.7.6) and (14.7.7) that

$$u(t) = e^{-0.0087t}\cos(\omega_{\rm d,ph}t) \text{ m/sec}, \tag{14.7.9}$$

$$\delta\alpha(t) = 9.6e(-5)e^{-0.0087t}\cos(\omega_{\rm d_{ph}}t - 3.136) \text{ rad}, \tag{14.7.10}$$

$$q(t) = 1.74e(-4)e^{-0.0087t}\cos(\omega_{\rm d,ph}t - 0.0484) \text{ rad/sec}, \tag{14.7.11}$$

$$\theta(t) = 2.33e(-3)e^{-0.0087t}\cos(\omega_{\rm d,ph}t - 1.736) \text{ rad}. \tag{14.7.12}$$

Note that (14.7.10) shows that $\delta\alpha$ is small compared to θ. Consequently, $\theta \approx \delta\gamma$.

To approximate the phugoid mode, we use the fact, based on the numerical results above, that $\delta\alpha \approx 0$. Therefore, ignoring $\delta\alpha$, it follows from the first two equations in (14.2.4) that

$$\begin{bmatrix} s - (X_{u_0} + X_{{\rm T}u_0}) & -X_{q_0}s + g \\ -(Z_{u_0} + Z_{{\rm T}u_0}) & -(U_0 + Z_{q_0})s \end{bmatrix} \begin{bmatrix} \hat{u}(s) \\ \hat{\theta}(s) \end{bmatrix} = \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \end{bmatrix} \delta\hat{e}(s). \tag{14.7.13}$$

The characteristic polynomial is thus given by

$$p(s) = s^2 - \left(X_{u_0} + X_{{\rm T}u_0} - \frac{(Z_{u_0} + Z_{{\rm T}u_0})X_{q_0}}{U_0 + Z_{q_0}}\right)s - \frac{(Z_{u_0} + Z_{{\rm T}u_0})g}{U_0 + Z_{q_0}}. \tag{14.7.14}$$

Assuming that $X_{q_0} = 0$, $Z_{q_0} = 0$, and $Z_{{\rm T}_{u_0}} = 0$, $p(s)$ becomes

$$p(s) = s^2 - (X_{u_0} + X_{{\rm T}u_0})\,s - \frac{Z_{u_0}g}{U_0}. \tag{14.7.15}$$

Using the expression for Z_{u_0} given in Table 14.1, it follows that

$$\omega_{\rm n,ph} \approx \sqrt{\frac{-Z_{u_0}g}{U_0}} = \sqrt{\frac{g p_{\rm d_0} S(2C_{L_0} + C_{L_{u_0}})}{mU_0^2}}. \tag{14.7.16}$$

Since $C_{L_{u_0}} << C_{L_0}$ and $C_{L_0} = \frac{mg}{p_{\rm d_0}S}$, we obtain

$$\omega_{\rm n,ph} \approx \sqrt{\frac{g p_{\rm d_0} S}{mU_0^2}\left(\frac{2mg}{p_{\rm d_0}S}\right)} = \sqrt{\frac{2g^2}{U_0^2}} = \sqrt{2}\frac{g}{U_0}. \tag{14.7.17}$$

For the phugoid damping it follows from (14.7.15) that

$$\zeta_{\rm ph} \approx \frac{-(X_{u_0} + X_{{\rm T}u_0})}{2\omega_{\rm n,ph}}. \tag{14.7.18}$$

Table 14.1 implies

$$X_{u_0} = \frac{-p_{{\rm d}_0} S\,(2C_{D_0} + C_{D_{u_0}})}{mU_0}, \tag{14.7.19}$$

$$X_{{\rm T}u_0} = \frac{p_{{\rm d}_0} S\,(2C_{{\rm T}x_0} + C_{{\rm T}x_{u_0}})}{mU_0}. \tag{14.7.20}$$

Letting $C_{{\rm T}x_{u_0}} = 0$ and $C_{{\rm T}x_0} = 0$, it follows from (14.7.18)–(14.7.20) that

$$\zeta_{\rm ph} = \frac{p_{{\rm d}_0} S\,(2C_{D_0} + C_{D_{u_0}})}{2mU_0\omega_{\rm n,ph}} = \frac{p_{{\rm d}_0} S\,(2C_{D_0} + C_{D_{u_0}})}{2\,\sqrt{2}mg} = \frac{(2C_{D_0} + C_{D_{u_0}})}{2\,\sqrt{2}}\frac{1}{C_{L_0}}. \tag{14.7.21}$$

At low speed, $C_{D_{u_0}} \approx 0$, and thus (14.7.21) becomes

$$\zeta_{\rm ph} = \frac{C_{D_0}}{\sqrt{2}C_{L_0}}, \tag{14.7.22}$$

which is proportional to $\frac{1}{L/D}$. Therefore, for high L/D, $\zeta_{\rm ph}$ is low.

14.7.2 Short Period Mode

The short period mode is associated with the higher frequency eigenvalue $\lambda_{\rm sp}$ in (14.2.18). The short period mode is underdamped with high $\zeta_{\rm sp}$ compared to $\zeta_{\rm ph}$, high frequency $\omega_{\rm n,sp}$, and period $T_{\rm sp} = \frac{2\pi}{\omega_{\rm n,sp}}$, Typically, $T_{\rm sp}$ is approximately 2 sec. Computational results show that the eigensolution characteristics are that u is constant while θ and $\delta\alpha$ oscillate.

To approximate the short period mode, set $u = 0$ in (14.2.4), which yields

$$\begin{bmatrix} (U_0 - Z_{\dot\alpha_0})\,s - Z_{\alpha_0} & -\left(U_0 + Z_{q_0}\right)s \\ -M_{\dot\alpha_0}s - (M_{\alpha_0} + M_{{\rm T}\alpha_0}) & s^2 - M_{q_0}s \end{bmatrix}\begin{bmatrix} \delta\hat\alpha(s) \\ \hat\theta(s) \end{bmatrix} = \begin{bmatrix} Z_{\delta e_0} \\ M_{\delta e_0} \end{bmatrix}\delta\hat e(s). \tag{14.7.23}$$

For convenience, we now assume that $Z_{\dot\alpha_0} = 0$, $Z_{q_0} = 0$, $M_{\dot\alpha_0} = 0$, and $M_{{\rm T}\alpha_0} = 0$. Then, (14.7.23) becomes

$$\begin{bmatrix} U_0s - Z_{\alpha_0} & -U_0s \\ -M_{\alpha_0} & s^2 - M_{q_0}s \end{bmatrix}\begin{bmatrix} \delta\hat\alpha(s) \\ \hat\theta(s) \end{bmatrix} = \begin{bmatrix} Z_{\delta e_0} \\ M_{\delta e_0} \end{bmatrix}\delta\hat e(s). \tag{14.7.24}$$

Then, the characteristic equation obtained from (14.7.24) is given by

$$(s^2 - M_{q_0}s)\,(U_0s - Z_{\alpha_0}) - U_0M_{\alpha_0}s = 0, \tag{14.7.25}$$

which can be written as

$$U_0s^3 - (M_{q_0}U_0 + Z_{\alpha_0})s^2 + \left(M_{q_0}Z_{\alpha_0} - M_{\alpha_0}U_0\right)s = 0. \tag{14.7.26}$$

Dividing (14.7.26) by s and U_0 yields the quadratic equation

$$p(s) = s^2 - \left(M_{q_0} + \frac{Z_{\alpha_0}}{U_0}\right)s + \left(\frac{M_{q_0}Z_{\alpha_0}}{U_0} - M_{\alpha_0}\right) = 0. \tag{14.7.27}$$

Hence, the transfer functions (14.2.6) and (14.2.8) can be written as

$$G_{\delta\alpha(s)/\delta e(s)} = \frac{Z_{\delta e_0} s + M_{\delta e_0} U_0 - M_{q_0} Z_{\delta e_0}}{U_0 p(s)}, \tag{14.7.28}$$

$$G_{\theta(s)/\delta e(s)} = \frac{U_0 M_{\delta e_0} s + M_{\alpha_0} Z_{\delta e_0} - Z_{\alpha_0} M_{\delta e_0}}{U_0 s p(s)}. \tag{14.7.29}$$

It follows from (14.7.27) that the short period undamped natural frequency and damping ratio are given by

$$\omega_{\mathrm{n,sp}} = \sqrt{\frac{M_{q_0} Z_{\alpha_0}}{U_0} - M_{\alpha_0}}, \tag{14.7.30}$$

$$\zeta_{\mathrm{sp}} = -\frac{M_{q_0} + \frac{Z_{\alpha_0}}{U_0}}{2\omega_{\mathrm{n,sp}}}. \tag{14.7.31}$$

Note that $M_{\alpha_0} < 0$ is required for static stability. For dynamic stability, the roots of the quadratic polynomial (14.7.27) must have negative real parts. This condition is satisfied if and only if the argument of the square root in (14.7.30) is positive, that is,

$$M_{\alpha_0} < \frac{M_{q_0} Z_{\alpha_0}}{U_0}, \tag{14.7.32}$$

$$M_{q_0} + \frac{Z_{\alpha_0}}{U_0} < 0. \tag{14.7.33}$$

Now, if the center of mass c of the aircraft is sufficiently forward of the center of pressure, then using the expression for M_{α_0} given by Table 14.1 $\omega_{\mathrm{n,sp}}$ becomes

$$\omega_{\mathrm{n,sp}} \approx \sqrt{-M_{\alpha_0}} = \sqrt{\frac{-C_{m_{\alpha_0}} p_{\mathrm{d}_0} S \bar{c}}{J_{yy}}}. \tag{14.7.34}$$

Note that $\omega_{\mathrm{n,sp}}$ increases with $-C_{m_{\alpha_0}}$ and p_{d_0}, and decreases with J_{yy}. We also note that the main component of ζ_{sp} is M_{q_0}, which is the pitch-damping derivative.

14.8 Lateral Flight Modes

Equation (14.4.26) yields the eigenvalues for the Dutch roll mode, which involves sideslip, roll, yaw oscillations, the roll mode, and the spiral mode. The spiral mode, which involves slow roll and a slow yaw, is stable for $C_{l_{\beta_0}} << 0$. The roll mode is unstable for large α and τ_{r} and is largely unaffected by δa.

14.8.1 Dutch Roll Mode

To be written.

14.8.2 Roll Mode

To be written.

14.8.3 Spiral Mode

To be written.

14.9 Problems

Problem 14.9.1. Show that the numerator of the transfer function in (14.2.8) from δe to θ is second order. Hint: Show $CB = 0$ and see Section 15.14.

Problem 14.9.2. Determine A_u, A_α, and B_θ in terms of the stability parameters. Hint: See Section 15.14.

Problem 14.9.3. Consider the transfer function (14.4.20) from rudder δr to yaw-angle perturbation ψ. Show that this transfer function is not asymptotically stable, and use Routh to derive conditions that guarantee that it is semistable (see Chapter 15). For an impulse rudder deflection, use (14.4.20) to determine the asymptotic yaw-angle perturbation, and use (14.4.10) to determine the asymptotic sideslip perturbation. Explain physically why a rudder impulse produces new steady yaw and heading angles, and contrast this behavior with pitch motion, where the pitch angle returns to its original value after an elevator impulse.

This type of stability is consistent with the fact that, unlike pitch motion, which is affected by the direction of gravity, the to return its heading to the original value after a sideslip or roll perturbation, thus leading to a new steady-state heading angle.

Problem 14.9.4. The linearized longitudinal dynamics of a business jet in straight, horizontal, wings-level flight have the characteristic polynomial

$$p(s) = s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068.$$

Check stability using the Routh criterion (see Chapter 15), compute the roots of p using Matlab, identify the short period and phugoid roots, determine the damping ratio and natural frequencies of the roots, and compute the time for each eigensolution to decay by 50%.

Problem 14.9.5. The linearized longitudinal dynamics of an F-104 fighter flying in straight, horizontal, wings-level flight are modeled by the 4th-order system $\dot{x} = Ax$, where $x = [u \;\; \delta\alpha \;\; \theta \;\; \dot{\theta}]^{\mathrm{T}}$ and the dynamics matrix is given by

$$A = \begin{bmatrix} -2.02e(-2) & 7.88 & -3.22e(+1) & -6.5e(-1) \\ -2.54e(-4) & -1.02 & 0 & 9.05e(-1) \\ 0 & 0 & 0 & 1 \\ 7.95e(-11) & -2.5 & 0 & -1.39 \end{bmatrix}.$$

Note the order of the states, which is slightly different from (14.7.3). The units of u are m/sec, the units of $\delta\alpha$ and θ are rad, and the units of $\dot{\theta}$ are rad/sec. Then, do the following:

i) Compute the phugoid and short period eigenvalues. Plot these poles as four ×'s in the complex plane.

ii) Compute the damping ratio, undamped natural frequency, damped natural frequency, undamped period, and time to 50% decay for each mode.

Problem 14.9.6. For the F-104 in Problem 14.9.5:

i) Compute the eigenvector v_{ph} for the phugoid mode, and normalize the u component to 1.

ii) Convert the phugoid eigenvector into its magnitude and angle components.

Problem 14.9.7. Simulate the phugoid eigenflight mode of the F-104 in Problem 14.9.5 using the eigensolution $x(t) = \text{Re}\, e^{\lambda_{\text{ph}} t} v_{\text{ph}}$, where v_{ph} is the phugoid eigenvector. Assume that the initial condition for θ is given by $\theta(0) = 0.5$ deg and assume that the steady speed is $U_0 = 500$ miles per hour. Then, use Matlab to plot all four states for three full oscillations of the damped motion as functions of time. Furthermore, compute the altitude perturbation and plot three full oscillations of the altitude as a function of range.

Problem 14.9.8. Let $q(t)$ and $\theta(t)$ be given by the eigensolution expressions (14.7.11) and (14.7.12), respectively. Show that $\dot{\theta}(t) = q(t)$.

Chapter Fifteen

Linear Dynamical Systems

15.1 Vectors and Matrices

A *mathematical vector* is a column of scalars

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

The transpose of x is the row vector

$$x^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{1\times n}.$$

The *dot product* of the vectors $x, y \in \mathbb{R}^n$ is given by

$$x^{\mathrm{T}} y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in \mathbb{R},$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n.$$

A matrix has the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \ddots & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} \in \mathbb{R}^{n\times m}.$$

Letting

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = Ax,$$

it follows that, for $i = 1, ..., n$,

$$y_i = \mathrm{row}_i(A)x = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,m} \end{bmatrix} x,$$

where $\mathrm{row}_i(A)$ is the ith row of A. Letting

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,l} \\ b_{21} & b_{2,2} & \cdots & b_{2,l} \\ \vdots & \ddots & \vdots & \\ b_{m,1} & b_{m,2} & \cdots & b_{m,l} \end{bmatrix} \in \mathbb{R}^{m\times l},$$

the product of A and B is given by

$$AB = A\begin{bmatrix} \mathrm{col}_1(B) & \cdots & \mathrm{col}_l(B) \end{bmatrix} = \begin{bmatrix} A\mathrm{col}_1(B) & \cdots & A\mathrm{col}_l(B) \end{bmatrix} \in \mathbb{R}^{n\times l},$$

where $\mathrm{col}_i(B)$ is the ith column of B. The matrix $A \in \mathbb{R}^{n\times n}$ is square. The $n \times n$ identity matrix is given by

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Suppose $A, B \in \mathbb{R}^{n\times n}$ and $AB = I$. Then $B = A^{-1}$, that is, B is the inverse of A. A has an inverse if and only if $\det A \neq 0$, that is, the determinant of A is nonzero. For $n = 2$ the determinant is given by

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc,$$

while, for $n = 3$, we have

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}.$$

The inverses of these matrices are given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}\begin{bmatrix} \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} & -\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} & \det\begin{bmatrix} d & e \\ g & h \end{bmatrix} \\ -\det\begin{bmatrix} b & c \\ h & i \end{bmatrix} & \det\begin{bmatrix} a & c \\ g & i \end{bmatrix} & -\det\begin{bmatrix} a & b \\ g & h \end{bmatrix} \\ \det\begin{bmatrix} b & c \\ e & f \end{bmatrix} & -\det\begin{bmatrix} a & c \\ d & f \end{bmatrix} & \det\begin{bmatrix} a & b \\ d & e \end{bmatrix} \end{bmatrix}^{\mathrm{T}}.$$

For $A \in \mathbb{R}^{n\times n}$, Table 15.1 presents several types of special matrices.

Name	Property	Example
Symmetric	$A^{\mathrm{T}} = A$	$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$
Skew Symmetric	$A^{\mathrm{T}} = -A$	$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$
Orthogonal	$A^{-1} = A^{\mathrm{T}}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Nilpotent	$A^2 = 0$	$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$
Idempotent	$A^2 = A$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
Nonsingular	$\det A \neq 0$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Table 15.1: Special matrices.

15.1.1 Existence and Uniqueness of Solutions to Linear Equations

Let $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$, and consider

$$Ax = b. \tag{15.1.1}$$

We wish to determine whether $x \in \mathbb{R}^n$ satisfying (15.1.1) exists, and, if so, whether the solution x is unique. It is helpful to consider the special case

$$Ax = 0. \tag{15.1.2}$$

Fact 15.1.1. $x = 0$ is a solution of (15.1.2).

Note that $A(\alpha x) = 0$ for all $\alpha \in \mathbb{R}$. Hence, (15.1.2) has either one solution or an infinite number of solutions.

Fact 15.1.2. $x = 0$ is the unique solution of (15.1.2) if and only if $\det A \neq 0$.

Fact 15.1.3. (15.1.1) has a unique solution if and only if $\det A \neq 0$. In this case, the unique solution is $x = A^{-1}b$.

15.2 Complex Numbers, Vectors and Matrices

A *complex number* $z \in \mathbb{C}$ is written as

$$z = x + \jmath y,$$

where $\jmath = \sqrt{-1}$ and x and y are real numbers.

Let $z_1 = x_1 + y_1 \jmath \in \mathbb{C}$ and $z_2 = x_2 + y_2 \jmath \in \mathbb{C}$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Definition 15.2.1. *Complex addition* is defined by

$$z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)\jmath.$$

Definition 15.2.2. *Complex multiplication* is defined by

$$\begin{aligned} z_1 z_2 &= (x_1 + \jmath y_1)(x_2 + y_2 \jmath) \\ &= x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1)\jmath. \end{aligned}$$

Definition 15.2.3. *Complex conjugation* is defined by

$$\bar{z_1} = x_1 - y_1 \jmath.$$

Definition 15.2.4. *Complex division* is defined by

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z_2}}{z_2 \bar{z_2}} = \frac{x_1 x_2 + y_1 y_2 + (x_2 y_1 - x_1 y_2)\jmath}{{x_2}^2 + {y_2}^2}.$$

Definition 15.2.5. The *magnitude* $|z|$ of the complex number z is defined as

$$|z_1| = \sqrt{z_1 \bar{z_1}} = \sqrt{{x_1}^2 + {y_1}^2}.$$

The complex number $z = x + y\jmath$ is written in *polar form* as

$$z = |z| e^{\jmath\theta} = |z|(\cos\theta + (\sin\theta)\jmath),$$

where $\theta = \operatorname{atan2}(y, x)$. Therefore, for all $z \neq 0$ and all $\alpha \in \mathbb{R}$,

$$z^\alpha = |z|^\alpha \left[(\cos\alpha\theta) + (\sin\alpha\theta)\jmath\right].$$

Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be real numbers. Then a *complex vector* is a vector of complex numbers. We write

$$z = \begin{bmatrix} x_1 + y_1 \jmath \\ x_2 + y_2 \jmath \\ \vdots \\ x_n + y_n \jmath \end{bmatrix} \in \mathbb{C}^n.$$

Furthermore, let $x_{1,1}, \ldots, x_{n,m}, y_{1,1}, \ldots, y_{n,m}$ be real numbers. Then a *complex matrix* is a matrix of complex numbers. We write

$$A = \begin{bmatrix} x_{1,1} + y_{1,1}\jmath & \cdots & x_{1,m} + y_{1,m}\jmath \\ \vdots & & \vdots \\ x_{n,1} + y_{n,1}\jmath & \cdots & x_{n,m} + y_{n,m}\jmath \end{bmatrix} \in \mathbb{C}^{n \times m}.$$

Define

$$\bar{z} \triangleq \begin{bmatrix} x_1 - y_1 J \\ x_2 - y_2 J \\ \vdots \\ x_n - y_n J \end{bmatrix} \in \mathbb{C}^n$$

and

$$\bar{A} \triangleq \begin{bmatrix} x_{1,1} - y_{1,1} J & \cdots & x_{1,m} - y_{1,m} J \\ \vdots & & \vdots \\ x_{n,1} - y_{n,1} J & \cdots & x_{n,m} - y_{n,m} J \end{bmatrix} \in \mathbb{C}^{n\times m}.$$

Vector and matrix addition are given by entry-wise addition.

Define the *conjugate transpose* of the complex vector $z \in \mathbb{C}^n$ by

$$z^* \triangleq \bar{z}^{\mathrm{T}},$$

as well as its *magnitude* by

$$|z| \triangleq \sqrt{\bar{z}^{\mathrm{T}} z} = \sqrt{z^* z}.$$

The *dot product* of the complex vectors $z_1, z_2 \in \mathbb{C}^n$ is given by

$$z_1{}^* z_2 = \bar{z_1}^{\mathrm{T}} z_2 \ \in \ \mathbb{C}.$$

The *conjugate transpose* of the matrix $A \in \mathbb{C}^{n\times m}$ is defined by

$$A^* \triangleq \bar{A}^{\mathrm{T}} = \bar{A^{\mathrm{T}}}.$$

15.3 Eigenvalues and Eigenvectors

Let $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$, assume that x is nonzero, and assume that

$$Ax = \lambda x. \tag{15.3.1}$$

Then λ is an *eigenvalue* of A, and x is an associated *eigenvector*. Note that an eigenvector must be nonzero by definition. Eigenvalues are related to modal frequencies, while eigenvectors are related to mode shapes.

Fact 15.3.1. (15.3.1) holds if and only if $(\lambda I - A)x = 0$.

Fact 15.3.2. (15.3.1) has a nonzero solution x if and only if $\det(\lambda I - A) = 0$.

Example 15.3.3. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then

$$(\lambda I - A)x = 0$$

implies that

$$\begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$(\lambda - 1)x_1 = 0 \quad \text{and} \quad (\lambda - 2)x_2 = 0. \tag{15.3.2}$$

Since $x \neq 0$, we must have either $x_1 \neq 0$ or $x_2 \neq 0$, and, correspondingly, either $\lambda = 1$ or $\lambda = 2$. If $\lambda_1 = 1$, then

$$x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \qquad \text{where } x_1 \neq 0, \tag{15.3.3}$$

while, if $\lambda_2 = 2$, then

$$x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \qquad \text{where } x_2 \neq 0. \tag{15.3.4}$$

Example 15.3.4. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$(\lambda I - A)x = 0$$

implies that

$$\begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$\lambda x_1 + x_2 = 0,$$
$$-x_1 + \lambda x_2 = 0,$$

which implies that $\lambda = \jmath$ or $\lambda = -\jmath$. If $\lambda = \jmath$, then

$$x = \begin{bmatrix} 1 \\ -\jmath \end{bmatrix},$$

while, if $\lambda = -\jmath$, then

$$x = \begin{bmatrix} 1 \\ \jmath \end{bmatrix}.$$

Definition 15.3.5. The *characteristic polynomial* of A is

$$p(s) \triangleq \det(sI - A). \tag{15.3.5}$$

Definition 15.3.6. The *characteristic equation* of A is

$$p(\lambda) = 0. \tag{15.3.6}$$

Note that s represents an arbitrary complex number in (15.3.5), whereas $s = \lambda$ in (15.3.6) denotes a root of p.

Fact 15.3.7. Let $A \in \mathbb{R}^{n\times n}$. Then the characteristic polynomial of A is an nth-order polynomial with real coefficients.

Fact 15.3.8. Let $A \in \mathbb{R}^{n\times n}$, and let $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A, that is, λ satisfies the characteristic equation of A.

It follows from Fact 15.3.7 and Fact 15.3.8 that a real $n \times n$ matrix has n eigenvalues, which are not necessarily distinct.

Fact 15.3.9. If A is diagonal, then the diagonal entries of A are the eigenvalues of A.

Fact 15.3.10. Let $A \in \mathbb{R}^{n\times n}$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then

$$\det A = \prod_{i=1}^{n} \lambda_i \tag{15.3.7}$$

and

$$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i. \tag{15.3.8}$$

Fact 15.3.11. Let $A \in \mathbb{R}^{n\times n}$, and let λ be an eigenvalue of A. Then λ^2 is an eigenvalue of A^2.

Proof. Note that

$$Av = \lambda v.$$

Therefore,

$$A^2 v = \lambda A v = \lambda^2 v. \qquad \square$$

Fact 15.3.12. Let $A \in \mathbb{R}^{n\times n}$, assume that A is nonsingular, and let λ be an eigenvalue of A. Then $1/\lambda$ is an eigenvalue of A^{-1}.

Proof. Note that

$$\lambda v = Av.$$

Therefore,

$$\lambda A^{-1} v = A^{-1} A v,$$

and thus

$$A^{-1} v = \frac{1}{\lambda} v. \qquad \square$$

Fact 15.3.13. Let $A \in \mathbb{R}^{n\times n}$, and assume that A is symmetric. Then every eigenvalue of A is real.

Fact 15.3.14. Let $A \in \mathbb{R}^{n\times n}$, and assume that A is skew symmetric. Then every eigenvalue of A is imaginary.

Definition 15.3.15. Let $A \in \mathbb{R}^{n\times n}$, assume that A is symmetric, and let $x \in \mathbb{R}^n$. Then $x^{\mathrm{T}}Ax$ is a *quadratic form*.

Definition 15.3.16. Let $A \in \mathbb{R}^{n\times n}$, and assume that A is symmetric. Then, A is *positive semidefinite* (PSD) if $x^{\mathrm{T}}Ax \geq 0$ for all $x \in \mathbb{R}^n$. Furthermore, A is *positive definite* (PD) if $x^{\mathrm{T}}Ax > 0$ for all nonzero $x \in \mathbb{R}^n$.

Fact 15.3.17. Let $A \in \mathbb{R}^{n\times n}$ and assume that A is symmetric. Then, A is PSD if and only if every eigenvalue of A is nonnegative.

Fact 15.3.18. Let $A \in \mathbb{R}^{n\times n}$ and assume that A is symmetric. Then, A is PD if and only if every eigenvalue of A is positive.

15.4 Single-Degree-of-Freedom Systems

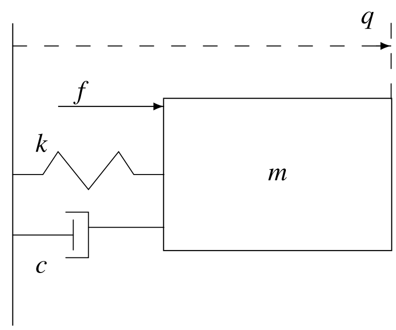


Figure 15.4.1: The damped oscillator

We study the dynamics of the spring-mass-damper system shown in Figure 15.4.1, where $q(t)$ is the position of the mass m. Note that $q(t) > 0$ if and only if the spring extends to the right from its relaxed configuration, whereas $q(t) < 0$ if and only if the spring compresses to the left from its relaxed configuration. Likewise, $\dot{q}(t) > 0$ if and only if the dashpot pulls to the left with force $-c\dot{q}(t)$, whereas $\dot{q}(t) < 0$ if and only if the dashpot pushes to the right with force $-c\dot{q}(t)$.

Applying Newton's second law (see Chapter 3) to the mass yields

$$m\ddot{q}(t) = f_{\mathrm{total}}(t) = f(t) - kq(t) - c\dot{q}(t).$$

Dividing through by m yields

$$\ddot{q}(t) + \frac{c}{m}\dot{q}(t) + \frac{k}{m}q(t) = \frac{1}{m}f(t).$$

Next define the *natural frequency*

$$\omega_{\mathrm{n}} \triangleq \sqrt{\frac{k}{m}}$$

and the *damping ratio*

$$\zeta \triangleq \frac{c}{2\sqrt{mk}}.$$

Then

$$\ddot{q}(t) + 2\zeta\omega_n\dot{q}(t) + \omega_n^2 q(t) = \frac{1}{m}f(t).$$

The following special cases are of interest.

Undamped Rigid Body (URB) ($k = 0$, $c = 0$)

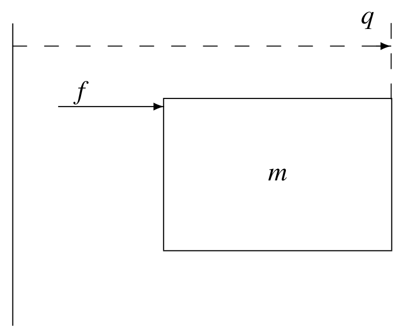


Figure 15.4.2: Undamped rigid body

$$\ddot{q}(t) = \frac{1}{m}f(t) \tag{15.4.1}$$

Damped Rigid Body (DRB) ($k = 0$)

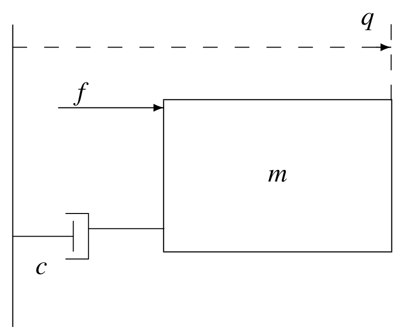


Figure 15.4.3: Damped rigid body

$$\ddot{q}(t) + \frac{c}{m}\dot{q}(t) = \frac{1}{m}f(t) \tag{15.4.2}$$

Undamped Oscillator (UO) ($c = 0$)

$$\ddot{q}(t) + \frac{k}{m}q(t) = \frac{1}{m}f(t). \tag{15.4.3}$$

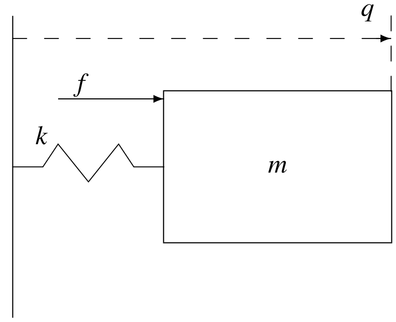


Figure 15.4.4: Undamped oscillator

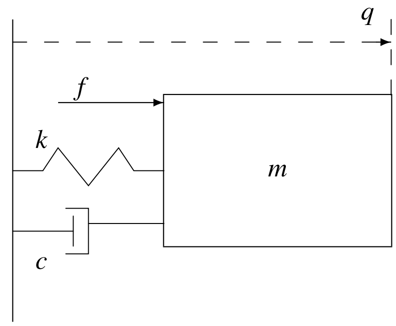


Figure 15.4.5: Damped oscillator

Damped Oscillator (DO)

$$\ddot{q}(t) + \frac{c}{m}\dot{q}(t) + \frac{k}{m}q(t) = \frac{f(t)}{m}. \tag{15.4.4}$$

15.5 Matrix Differential Equations

For $t \geq 0$ consider the scalar differential equation

$$\dot{x}(t) = ax(t), \qquad x(0) = x_0, \tag{15.5.1}$$

where a is a real number. Then

$$x(t) = e^{at}x_0$$

is a solution of (15.5.1), as can be verified by substitution. Since (15.5.1) is a linear differential equation, it has a unique solution.

Now consider the matrix differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{15.5.2}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Then

$$x(t) = e^{At}x_0 \tag{15.5.3}$$

is the unique solution, where e^{At} is the *matrix exponential* defined by the Taylor expansion

$$e^{At} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots .$$

To verify that (15.5.3) is a solution of (15.5.2) note that

$$\frac{\mathrm{d}(e^{At})}{\mathrm{d}t} = 0 + A + tA^2 + \frac{1}{2}t^2A^3 + \cdots = A\left(I + tA + \frac{1}{2}t^2A^2 + \cdots\right) = Ae^{At}.$$

Example 15.5.1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

From Example 15.3.3 we know that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. The matrix exponential is given by

$$\begin{aligned} e^{At} &= I + t\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \frac{1}{2}t^2\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + t + \frac{1}{2}t^2 + \cdots & 0 \\ 0 & 1 + 2t + \frac{1}{2}t^2(4) + \cdots \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}. \end{aligned}$$

Note that e^{At} involves $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

Example 15.5.2. Let

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

From Example 15.3.3 we know that $\lambda_1 = \jmath\omega$ and $\lambda_2 = -\jmath\omega$. The matrix exponential is given by

$$\begin{aligned} e^{At} &= I + \omega t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{1}{2}\omega^2t^2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{3!}\omega^3t^3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 - \frac{1}{2}\omega^2t^2 + \cdots & \omega t - \frac{1}{6}\omega^3t^3 + \cdots \\ -\omega t + \frac{1}{6}\omega^3t^3 - \cdots & 1 - \frac{1}{2}\omega^2t^2 + \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}. \end{aligned}$$

15.6 Eigensolutions

Fact 15.6.1. Let $A \in \mathbb{R}^{n\times n}$, let $\lambda \in \mathbb{C}$ be an eigenvalue of A, and let $v \in \mathbb{C}^n$ be an eigenvector associated with λ. Furthermore, let

$$\dot{x}(t) = Ax(t), \qquad x(0) = \operatorname{Re} v. \tag{15.6.1}$$

Then

$$x(t) = \operatorname{Re}(e^{\lambda t}v) \tag{15.6.2}$$

is a solution of (15.6.1).

Proof. Let $v = y + \jmath z$, where $y, z \in \mathbb{R}^n$. Then,

$$\begin{aligned} x(t) &= \mathrm{Re}(e^{\lambda t} v) \\ &= \mathrm{Re}[e^{(\sigma+\jmath\omega)t}(y + \jmath z)] \\ &= \mathrm{Re}[e^{\sigma t}(\cos\omega t + \jmath\sin\omega t)(y + \jmath z)] \\ &= e^{\sigma t}(y\cos\omega t - z\sin\omega t). \end{aligned}$$

Hence,

$$\dot{x}(t) = e^{\sigma t}[(\sigma\cos\omega t - \omega\sin\omega t)y - (\sigma\sin\omega t + \omega\cos\omega t)z].$$

Furthermore,

$$\begin{aligned} Ax(t) &= A\,\mathrm{Re}(e^{\lambda t} v) \\ &= \mathrm{Re}(e^{\lambda t} A v) \\ &= \mathrm{Re}(e^{\lambda t} \lambda v) \\ &= \mathrm{Re}(e^{\sigma t}(\cos\omega t + \jmath\sin\omega t)(\sigma + \jmath\omega)(y + \jmath z)) \\ &= e^{\sigma t}[(\sigma\cos\omega t - \omega\sin\omega t)y - (\sigma\sin\omega t + \omega\cos\omega t)z]. \end{aligned}$$

Therefore,

$$\dot{x}(t) = Ax(t), \quad x(0) = \mathrm{Re}\ v,$$

which shows that $x(t) = \mathrm{Re}[e^{\lambda t} v]$ is a solution of $\dot{x}(t) = Ax(t)$.

As an alternative proof, we do not decompose the complex variables λ and v into their real and complex parts. Then,

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}(e^{\lambda t} v) = \mathrm{Re}\left[\frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda t} v)\right] = \mathrm{Re}(e^{\lambda t}\lambda v) = \mathrm{Re}(e^{\lambda t} A v) = A\,\mathrm{Re}(e^{\lambda t} v) = Ax(t),$$

which, together with $x(0) = \mathrm{Re}(e^{\lambda 0} v) = \mathrm{Re}\ v$, shows that $x(t)$ satisfies (15.6.1). □

Definition 15.6.2. The solution (15.6.2) of (15.6.1) is an *eigensolution*.

15.7 State Space Form

Consider the nth-order differential equation

$$\frac{\mathrm{d}^n q(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} q(t)}{\mathrm{d}t^{n-1}} + \cdots + a_1\frac{\mathrm{d}q(t)}{\mathrm{d}t} + a_0 q(t) = b_0 u(t) \tag{15.7.1}$$

and define the state vector

$$x \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \triangleq \begin{bmatrix} q \\ \frac{\mathrm{d}q}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}^{n-2}q}{\mathrm{d}t^{n-2}} \\ \frac{\mathrm{d}^{n-1}q}{\mathrm{d}t^{n-1}} \end{bmatrix}. \tag{15.7.2}$$

In *state space form* the differential equation (15.7.1) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u(t). \tag{15.7.3}$$

Equation (15.7.3) is the *state equation*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{15.7.4}$$

while the *output equation* has the form

$$y(t) = Cx(t) + Du(t). \tag{15.7.5}$$

Example 15.7.1. Consider the differential equation for the damped oscillator

$$\ddot{q}(t) = -\frac{c}{m}\dot{q}(t) - \frac{k}{m}q(t) + \frac{f(t)}{m}.$$

Define the state variables $x_1(t) \triangleq q(t)$ and $x_2(t) \triangleq \dot{q}(t)$ and assume that the output $y(t)$ is given by the position $q(t)$. Then

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t), \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned} \tag{15.7.6}$$

15.8 Linear Systems with Forcing

Consider the state space system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{15.8.1}$$

$$y(t) = Cx(t) + Du(t). \tag{15.8.2}$$

Then, the solution $x(t)$ of (15.8.1) is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \tag{15.8.3}$$

and thus

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{free response}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t)}_{\text{forced response}}. \tag{15.8.4}$$

Define the *impulse response function*

$$H(t) \triangleq Ce^{At}B + \delta(t)D, \tag{15.8.5}$$

where $\delta(t)$ is the unit impulse function at $t = 0$ defined in the next section. Then it follows from the sifting property (15.9.2) that

$$y(t) = Ce^{At}x_0 + \underbrace{\int_0^t H(t-\tau)u(\tau)\,\mathrm{d}\tau}_{\text{convolution}}. \tag{15.8.6}$$

15.9 Standard Input Signals

Definition 15.9.1. The *unit impulse function* $\delta(t)$ at $t = 0$ has the property

$$\delta(t) \triangleq \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0, \end{cases}$$

and

$$\int_a^b \delta(t)\,\mathrm{d}t = 1, \qquad a \leq 0 < b. \tag{15.9.1}$$

Note that $\delta(t)$ is a right-sided impulse function (see Figure 15.9.7). It follows from (15.9.1) that the units of the unit impulse function are given by $[\delta(t)] = 1/\mathrm{sec}$. The unit impulse function has the *sifting property*

$$\int_a^b g(t)\delta(t-t_0)\,\mathrm{d}t = g(t_0), \qquad a \leq t_0 < b, \tag{15.9.2}$$

where $\delta(t-t_0)$ is the *delayed unit impulse function* at $t = t_0$.

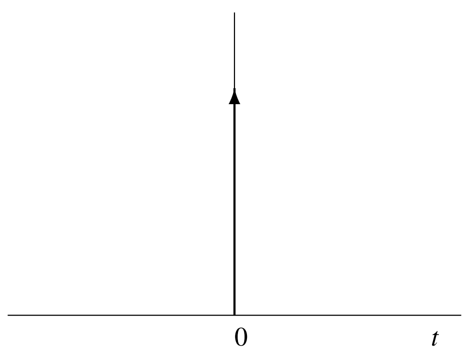


Figure 15.9.6: The unit impulse function $\delta(t)$

A force impulse at time t_0 has the form

$$f(t) = f_0\delta(t-t_0), \tag{15.9.3}$$

where the dimensions $[f_0]$ of f_0 are momentum.

An impulse at $t = 0$ imparts a nonzero initial velocity but zero change in position. The forced response with an impulse is thus equivalent to a particular free response.

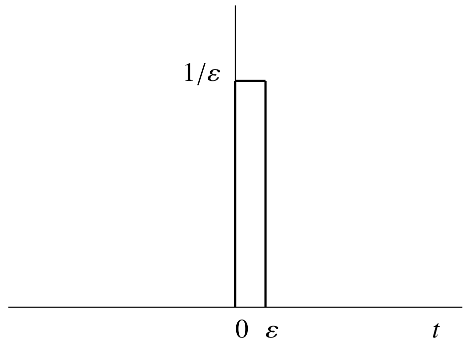


Figure 15.9.7: Approximate right-sided unit impulse function

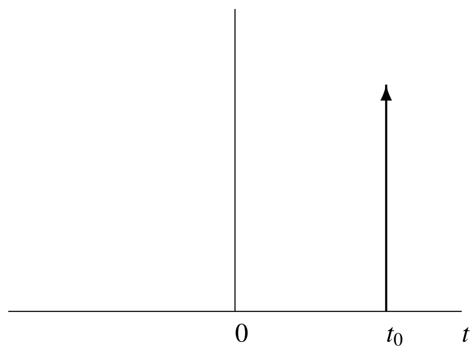


Figure 15.9.8: The delayed unit impulse function $\delta(t - t_0)$

Definition 15.9.2. The function $\mathbf{1}(t)$ is a *unit step function*, where

$$\mathbf{1}(t) \triangleq \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

The function $\mathbf{1}(t - t_0)$ is a *delayed unit step function*.

The unit step function is dimensionless. A force impulse at time t_0 has the form

$$f(t) = f_0 1(t - t_0), \tag{15.9.4}$$

where the dimensions $[f_0]$ of f_0 are force.

Definition 15.9.3. The function $t\mathbf{1}(t)$ is the *unit ramp*. The function $(t - t_0)\mathbf{1}(t - t_0)$ is a *delayed unit ramp*.

Definition 15.9.4. The function

$$f(t) = f_0 \sin(\omega t + \phi) \tag{15.9.5}$$

is a *sinusoid*.

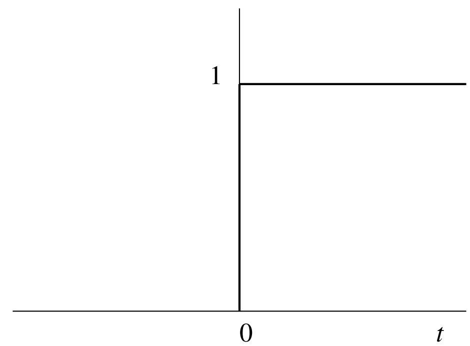


Figure 15.9.9: The unit step function $\mathbf{1}(t)$

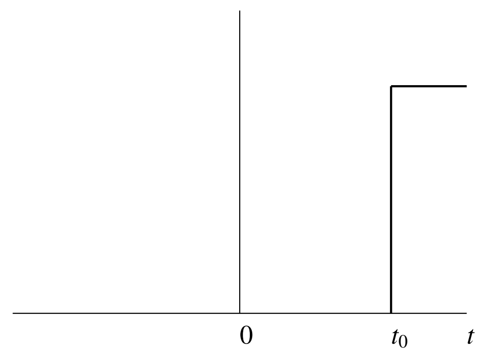


Figure 15.9.10: The delayed unit step function $\mathbf{1}(t-t_0)$

15.10 Laplace Transform

Definition 15.10.1. Given the function $q(t)$, the *Laplace transform* of $q(t)$ is defined by

$$\hat{q}(s) \triangleq \mathcal{L}\{q(t)\} \triangleq \int_0^\infty e^{-st} q(t)\,\mathrm{d}t. \tag{15.10.1}$$

Note that $[\hat{q}(s)] = \sec \times [q(t)]$ and $[s] = 1/\sec$. From (15.10.1) it follows that

$$\mathcal{L}\{\dot{q}(t)\} = \int_0^\infty e^{-st}\dot{q}(t)\,\mathrm{d}t = e^{-st}q(t)\Big|_{t=0}^{\infty} + s\int_0^\infty e^{-st}q(t)\,\mathrm{d}t = s\hat{q}(s) - q(0).$$

Similarly,

$$\mathcal{L}\{\ddot{q}(t)\} = s^2\hat{q}(s) - sq(0) - \dot{q}(0).$$

Note the formulas

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty e^{-st}\delta(t)\,\mathrm{d}t = e^0 = 1,$$

$$\mathcal{L}\{\mathbf{1}(t)\} = \int_0^\infty e^{-st}\mathbf{1}(t)\,\mathrm{d}t = \int_0^\infty e^{-st}\,\mathrm{d}t = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s},$$

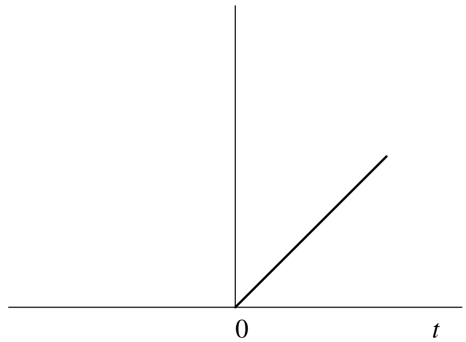


Figure 15.9.11: The unit ramp function

$$\mathcal{L}\{t\mathbf{1}(t)\} = \frac{1}{s^2}, \quad \mathcal{L}\{t^2\mathbf{1}(t)\} = \frac{2}{s^3},$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \quad \mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a},$$

$$\mathcal{L}\left\{\int_0^t q(\tau)\,\mathrm{d}\tau\right\} = \frac{1}{s}\mathcal{L}\{q(t)\},$$

$$\mathcal{L}\left\{\sin(\omega t)\right\} = \mathcal{L}\left\{\frac{1}{2\jmath}\left(e^{\jmath\omega t} - e^{-\jmath\omega t}\right)\right\} = \frac{1}{2\jmath}\left(\frac{1}{s-\jmath\omega} - \frac{1}{s+\jmath\omega}\right) = \frac{1}{2\jmath}\frac{2\jmath\omega}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2},$$

$$\mathcal{L}\left\{\cos(\omega t)\right\} = \mathcal{L}\left\{\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\sin(\omega t)\right\} = \frac{1}{\omega}s\frac{\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}.$$

Note that

$$\mathcal{L}\left\{\cos(0t)\right\} = \frac{s}{s^2+0^2} = \frac{1}{s} = \mathcal{L}\left\{\mathbf{1}(t)\right\}.$$

15.10.1 Time Delay

The Laplace transform of a time delay is given by

$$\mathcal{L}\{y(t-\tau)\} = e^{-\tau s}\hat{y}(s). \tag{15.10.2}$$

15.10.2 s-shift

An s-shift in the Laplace transform is represented in the time domain as

$$\mathcal{L}\{e^{-at}y(t)\} = \hat{y}(s+a). \tag{15.10.3}$$

Hence

$$\mathcal{L}^{-1}\{\hat{y}(s+a)\} = e^{-at}y(t) \tag{15.10.4}$$

and

$$\mathcal{L}^{-1}\{\hat{y}(s-a)\} = e^{at}y(t). \tag{15.10.5}$$

Example 15.10.2.

$$\mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s+a)^2+\omega^2}.$$

15.10.3 Time Multiplication

Time multiplication corresponds to s-differentiation. In particular,

$$\mathcal{L}\{ty(t)\} = -\hat{y}'(s), \tag{15.10.6}$$

where $\hat{y}'(s)$ is the derivative of $\hat{y}(s)$ with respect to s. Hence,

$$\mathcal{L}^{-1}\{\hat{y}'(s)\} = -ty(t).$$

Note the dual formula

$$\mathcal{L}\{\dot{y}(t)\} = s\hat{y}(s) - y(0). \tag{15.10.7}$$

15.11 Solving Differential Equations

Taking the Laplace transform of

$$\ddot{q}(t) + q(t) = 0$$

yields

$$\mathcal{L}\{\ddot{q}(t)\} + \mathcal{L}\{q(t)\} = 0.$$

Therefore,

$$s^2\hat{q}(s) - sq(0) - \dot{q}(0) + \hat{q}(s) = 0$$

and

$$\hat{q}(s) = \frac{sq(0)}{s^2+1} + \frac{\dot{q}(0)}{s^2+1}.$$

Hence,

$$q(t) = q(0)\cos(t) + \dot{q}(0)\sin(t).$$

Example 15.11.1. Consider the differential equation

$$\ddot{q}(t) = -\frac{c}{m}\dot{q}(t) - \frac{k}{m}q(t) + \frac{f(t)}{m}.$$

Taking the Laplace transform yields

$$\hat{q}(s) = \underbrace{\frac{mq(0)s + cq(0) + m\dot{q}(0)}{ms^2+cs+k}}_{\text{free response}} + \underbrace{\frac{1}{ms^2+cs+k}\hat{f}(s)}_{\text{forced response}}.$$

Note that the free response is a ratio of polynomials in s. Furthermore, the coefficient of $\hat{f}(s)$ is a ratio of polynomials in s. A ratio of polynomials in s is a *rational function*, while the rational function from the input to the output is a *transfer function*.

Definition 15.11.2. A *zero* of a rational function is a root of the numerator. A *pole* of a rational function is a root of the denominator.

Example 15.11.3. Consider the forced response of the damped rigid body

$$\ddot{q}(t) + 2\dot{q}(t) = u(t).$$

If $u(t) = \delta(t)$, then the position is given by

$$\hat{q}(s) = \frac{1}{s^2 + 2s}.$$

Define the velocity $v(t) = \dot{q}(t)$. Then,

$$\dot{v}(t) + 2v(t) = u(t).$$

Alternatively, if $u(t) = \mathbf{1}(t)$, then the velocity is given by

$$\hat{v}(s) = \frac{1}{s(s+2)}.$$

Therefore, the position impulse response is identical to the velocity response.

15.11.1 Partial Fractions

Partial fractions is a technique for finding the time-domain signal corresponding to a Laplace transform.

Example 15.11.4. Consider

$$\hat{y}(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}.$$

Use partial fractions to show

$$y(t) = \frac{8}{3}\mathbf{1}(t) - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}.$$

15.12 Initial Value and Initial Slope Theorems

The following result is the *initial value theorem*.

Fact 15.12.1.

$$y(0^+) \triangleq \lim_{t\to 0} y(t) = \lim_{s\to\infty} s\hat{y}(s).$$

The following result is the *initial slope theorem*.

Fact 15.12.2.

$$\dot{y}(0^+) \triangleq \lim_{t\to 0} \dot{y}(t) = \lim_{s\to\infty} s[s\hat{y}(s) - y(0)].$$

Example 15.12.3. If $y(t) = \mathbf{1}(t)$, then

$$y(0^+) = \lim_{s\to\infty} s\frac{1}{s} = 1.$$

If $y(t) = \cos(\omega t)$, then

$$y(0^+) = \lim_{s\to\infty} s\frac{s}{s^2+\omega^2} = 1.$$

Example 15.12.4. Consider the unit-slope ramp

$$y(t) = t.$$

Then,

$$\dot{y}(0^+) = \lim_{s\to\infty} s[s\hat{y}(s) - y(0)] = \lim_{s\to\infty} s(s\frac{1}{s^2} - 0) = 1.$$

15.13 Final Value Theorem

We say that $\lim_{t\to\infty} y(t)$ exists if it is a number. If either $\lim_{t\to\infty} y(t) = \infty$ or $\lim_{t\to\infty} y(t) = -\infty$, then $\lim_{t\to\infty} y(t)$ does not exist but is infinite. This convention is consistent with the fact that ∞ is not a number. The following cases can occur.

i) $y(t)$ remains bounded but $\lim_{t\to\infty} y(t)$ does not exist. For example, $y(t) = \sin(\omega t)$.

ii) $y(t)$ is not bounded (and thus $\lim_{t\to\infty} y(t)$ does not exist) and, in addition, neither $\lim_{t\to\infty} y(t) = \infty$ nor $\lim_{t\to\infty} y(t) = -\infty$. For example, $y(t) = e^t \sin t$.

iii) $y(t)$ is not bounded and $\lim_{t\to\infty} y(t)$ does not exist but is infinite. For example, $y(t) = e^t$, in which case, $\lim_{t\to\infty} y(t) = \infty$.

iv) $\lim_{t\to\infty} y(t)$ exists.

The following result is the *final value theorem*.

Fact 15.13.1. Assume that every pole of $\hat{y}(s)$ is either in the open left half plane or is zero. Then,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0^+} s\hat{y}(s). \tag{15.13.1}$$

Fact 15.13.1 implies that, if every pole of $\hat{y}(s)$ is either in the open left half plane or is zero, then the right-hand side of (15.13.1) gives the correct limit of $y(t)$ as $t \to \infty$. In this case we say that the final value theorem is *legal*. Note, however, that $\lim_{t\to\infty} y(t)$ exists if and only if $s = 0$ is not a repeated pole of $\hat{y}(s)$. Hence, the limit on the left hand side of (15.13.1) does not exist in the case where $\hat{y}(s)$ has a repeated pole at zero. Nevertheless, the use of (15.13.1) in this case is legal.

Example 15.13.2. If $y(t) = \mathbf{1}(t)$, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0^+} s\frac{1}{s} = 1.$$

If $y(t) = -t\mathbf{1}(t)$, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0^+} s\frac{-1}{s^2} = \lim_{s\to 0^+} \frac{-1}{s} = -\infty.$$

If $y(t) = e^{-t}$, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0^+} s\frac{1}{s+1} = 0.$$

If $y(t) = e^{t}$, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0^+} s\frac{1}{s-1} = 0,$$

which is incorrect.

Note that the limit from the right $s \to 0^+$ is needed in the second example to obtain the correct sign.

Example 15.13.3. Consider the damped oscillator with a real zero z given by

$$y(t) = \frac{s-z}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}.$$

Assume that the input $f(t)$ is the unit step $f(t) = f_0\mathbf{1}(t)$. From the final value theorem it follows that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{s-z}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}\frac{f_0}{s} = \frac{-f_0 z}{\omega_\mathrm{n}^2}.$$

Furthermore, from the initial value theorem it follows that

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} s\frac{s-z}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}\frac{f_0}{s} = 0,$$

while, from the initial slope theorem it follows that

$$\lim_{t\to 0} \dot y(t) = \lim_{s\to\infty} s^2\frac{s-z}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}\frac{f_0}{s} = f_0.$$

It can be seen that, if $z < 0$, then the values of the step response for small values of time have the same sign as the limiting value of the step response. However, if $z > 0$, then the values of the step response for small values of time have the opposite sign of the limiting value of the step response. The latter case is known as *initial undershoot*. The positive zero, which is called a *nonminimum phase zero*, is responsible for the initial undershoot.

15.14 Laplace Transforms of State Space Models

Consider

$$\dot x(t) = Ax(t) + Bu(t), \tag{15.14.1}$$

$$y(t) = Cx(t) + Du(t). \tag{15.14.2}$$

Then

$$s\hat x(s) - x(0) = A\hat x(s) + B\hat u(s), \tag{15.14.3}$$

$$\hat y(s) = C\hat x(s) + D\hat u(s). \tag{15.14.4}$$

Hence,

$$\hat y(s) = \underbrace{C(sI-A)^{-1}x(0)}_{\text{free response}} + \underbrace{\overbrace{\Big(C(sI-A)^{-1}B + D\Big)}^{G(s)}\hat u(s)}_{\text{forced response}}. \tag{15.14.5}$$

$G(s)$ thus has the *state space realization*

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

We write

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \frac{1}{s}C\left(I - \frac{1}{s}A\right)^{-1} B + D \\ &= \frac{1}{s}C\left(I + \frac{1}{s}A + \frac{1}{s^2}A^2 + \cdots\right)B + D \\ &= D + \frac{1}{s}CB + \frac{1}{s^2}CAB + \cdots \\ &= H_0 + \frac{1}{s}H_1 + \frac{1}{s^2}H_2 + \cdots, \end{aligned} \tag{15.14.6}$$

where H_0, H_1, and H_2 are *Markov parameters*. Note that the expansion (15.14.6) of $G(s)$ converges for all sufficiently large values of $|s|$.

Consider the third-order transfer function $G(s)$ of the form

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}. \tag{15.14.7}$$

Taking the limit of $G(s)$ we obtain

$$b_3 = \lim_{s\to\infty} G(s) = D. \tag{15.14.8}$$

In the case $D = 0$, $G(s)$ has the form

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}, \tag{15.14.9}$$

and thus

$$b_2 = \lim_{s\to\infty} sG(s) = CB. \tag{15.14.10}$$

Furthermore, if $D = 0$ and $CB = 0$, then $G(s)$ has the form

$$G(s) = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}, \tag{15.14.11}$$

and thus

$$b_1 = \lim_{s\to\infty} s^2 G(s) = CAB. \tag{15.14.12}$$

Finally, if $D = 0$, $CB = 0$, and $CAB = 0$, then $G(s)$ has the form

$$G(s) = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}, \tag{15.14.13}$$

and thus

$$b_0 = \lim_{s\to\infty} s^3 G(s) = CA^2 B. \tag{15.14.14}$$

Note that all leading zero coefficients of the numerator as well as the first nonzero coefficient of the numerator are Markov parameters. The remaining coefficients of the numerator are not Markov parameters. The same pattern holds if $G(s)$ is of arbitrary order.

Example 15.14.1. Consider the differential equation for the damped oscillator

$$\ddot{q}(t) = -\frac{c}{m}\dot{q}(t) - \frac{k}{m}q(t) + \frac{f(t)}{m}.$$

Define the state variables $x_1(t) \triangleq q(t)$ and $x_2(t) \triangleq \dot{q}(t)$ and assume that the output $y(t)$ is the velocity $\dot{q}(t)$. Then

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{s}{ms^2 + cs + k}.$$

Every pole of $G(s)$ is an eigenvalue of A. In addition, in most cases every eigenvalue of A is a pole of $G(s)$. As an exception, consider the DRB with velocity output $y(t) = v(t) = \dot{q}(t)$. Then the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = -\frac{c}{m}$. Also,

$$\hat{y}(s) = \hat{v}(s) = \frac{s}{s^2 + \frac{c}{m}s} = \frac{1}{s + \frac{c}{m}}. \tag{15.14.15}$$

The pole at 0 is canceled by the zero at 0 since the position state is *unobservable* by the velocity measurement.

15.15 Pole Locations and Response

Consider the transfer function for the damped oscillator with position output, which is given by

$$G(s) = \frac{1/m}{s^2 + 2\zeta\omega_\mathrm{n}s + \omega_\mathrm{n}^2}. \tag{15.15.1}$$

For $0 \le \zeta \le 1$, the poles of (15.15.1) are given by

$$\lambda_1, \lambda_2 = -\zeta\omega_\mathrm{n} \pm \sqrt{1-\zeta^2}\,\omega_\mathrm{n}\jmath. \tag{15.15.2}$$

If $\zeta = 0$, then the poles are imaginary and are given by $\lambda = \pm\omega_\mathrm{n}\jmath$, which is the *undamped* case. If $0 < \zeta < 1$, then the poles are complex, which is the *underdamped* case. Defining the *damped natural frequency*

$$\omega_\mathrm{d} = \sqrt{1-\zeta^2}\,\omega_\mathrm{n}, \tag{15.15.3}$$

(15.15.2) can be written as

$$\lambda_1, \lambda_2 = -\zeta\omega_\mathrm{n} \pm \omega_\mathrm{d}\jmath. \tag{15.15.4}$$

If $\zeta = 1$, then the poles are repeated and are given by $\lambda = -\zeta\omega_{\rm n}$, which is the *critically damped* case. In the undamped, underdamped, and critically damped cases we have

$$|\lambda| = \omega_{\rm n}. \tag{15.15.5}$$

Therefore, the distance from the pole to the origin determines the natural frequency. Furthermore, defining the angle θ by

$$\zeta = \sin\theta, \tag{15.15.6}$$

it follows that the angle that each pole subtends at the origin from the imaginary axis determines the damping.

If $\zeta \geq 1$, then the poles are real and are given by

$$\lambda_1, \lambda_2 = -\zeta\omega_{\rm n} \pm \sqrt{\zeta^2 - 1}\omega_{\rm n}.$$

If $\zeta > 1$, then the roots are real and distinct, which is the *overdamped* case. If ζ is large, then $\lambda_1 \approx -\frac{1}{2}\omega_{\rm n}/\zeta$ and $\lambda_2 \approx -2\zeta\omega_{\rm n}$. Thus λ_2 is the faster pole, and λ_1 is the slower pole.

Consider the case in which the input is the force impulse $f(t) = f_0\delta(t)$. Then, for the undamped case, the impulse response of (15.15.1) is given by

$$y(t) = \frac{f_0}{m\omega_{\rm n}}\sin\omega_{\rm n}t. \tag{15.15.7}$$

For the underdamped case, the impulse response of (15.15.1) is given by

$$y(t) = \frac{f_0}{m\omega_{\rm d}}e^{-\zeta\omega_{\rm n}t}\sin\omega_{\rm d}t. \tag{15.15.8}$$

For the critically damped case, the impulse response of (15.15.1) is given by

$$y(t) = \frac{f_0}{m}te^{-\omega_{\rm n}t}. \tag{15.15.9}$$

Finally, for the overdamped case, the impulse response of (15.15.1) is given by

$$y(t) = \frac{f_0}{m\sqrt{\zeta^2 - 1}\omega_{\rm n}}e^{-\zeta\omega_{\rm n}t}\sinh(\sqrt{\zeta^2 - 1}\omega_{\rm n}t). \tag{15.15.10}$$

In all cases, the real part of the root determines the rate of decay, and the imaginary part determines the frequency of oscillation.

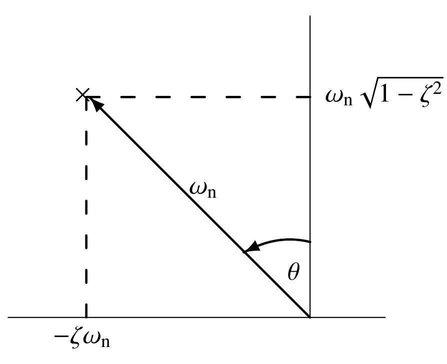


Figure 15.15.12: Relationships among pole location, damping, and natural frequency

Open Left Half Plane (OLHP)	Imaginary Axis (IA)	Open Right Half Plane (ORHP)
not repeated $ce^{-at}\sin(\omega t)$ repeated $cte^{-at}\sin(\omega t)$	not repeated $\sin(\omega t)$ repeated $t\sin(\omega t)$	not repeated $ce^{at}\sin(\omega t)$ repeated $cte^{at}\sin(\omega t)$
not repeated ce^{-at} repeated cte^{-at}	not repeated c repeated ct	not repeated ce^{at} repeated cte^{at}
not repeated $ce^{-at}\sin(\omega t)$ repeated $cte^{-at}\sin(\omega t)$	not repeated $\sin(\omega t)$ repeated $t\sin(\omega t)$	not repeated $ce^{at}\sin(\omega t)$ repeated $cte^{at}\sin(\omega t)$

Figure 15.15.13: This figure shows the type of response that arises from nonrepeated and repeated poles located in the open left half plane, imaginary axis, and open right half plane. The center column represents the imaginary axis, the center row represents the real axis, and the center square represents the origin.

The free responses for the various damping cases are summarized in Table **??**.

Property	Name	Free Response
$\zeta = 0$	undamped	sinusoid
$0 < \zeta < 1$	underdamped	decaying sinusoid
$\zeta = 1$	critically damped	decaying exponentials
$\zeta > 1$	overdamped	decaying exponentials

Table 15.2: Free responses for damping cases

15.16 Stability

Stability concerns the free response only, and thus can be determined by either the matrix A of a state space model or the poles of G. We assume that G is formed from the state space model without pole-zero cancellation. For example, pole-zero cancellation can occur for the damped rigid body with velocity output.

Equivalent conditions for G to be Unstable (US):

- $x(t) \to \infty$ as $t \to \infty$ for at least one $x(0)$.

- At least one entry of e^{At} is unbounded.
- Not Lyapunov stable.
- At least one pole of $G(s)$ is in the ORHP or is repeated on the imaginary axis.
- Example: Undamped rigid body.

Equivalent conditions for G to be Lyapunov Stable (LS):

- For all $x(0)$, $x(t)$ is bounded.
- Every entry of e^{At} is bounded.
- Each eigenvalue of A is in the OLHP or is not repeated on the imaginary axis.
- Each pole of $G(s)$ is in the OLHP or is not repeated on the imaginary axis.
- Example: Undamped oscillator.

Equivalent conditions for G to be Semistable (SS):

- For all $x(0)$, $\lim_{t\to\infty} x(t)$ exists.
- Each eigenvalue of A is in the OLHP or is not repeated at the origin.
- $\lim_{t\to\infty} e^{At}$ exists.
- Each pole of $G(s)$ is in the OLHP or is not repeated at the origin.
- Example: Damped rigid body.

Equivalent conditions for G to be Asymptotic Stable (AS):

- For all $x(0)$, $x(t) \to 0$ as $t \to \infty$.
- Each eigenvalue of A is in the OLHP.
- $\lim_{t\to\infty} e^{At} = 0$.
- Each pole of $G(s)$ is in the OLHP.
- Example: Damped oscillator.

The following result, which is illustrated by Figure 15.16.14, shows how the three types of stability are related.

Fact 15.16.1. AS $\Longrightarrow$ SS $\Longrightarrow$ LS

Definition 15.16.2. The transfer function $G(s)$ is *bounded-input, bounded-output (BIBO) stable* if, for every bounded input signal $u(t)$, the output $y(t)$ of $G(s)$ is bounded.

Example 15.16.3. Consider the undamped rigid body

$$\hat{v}(s) = \frac{1}{ms}\hat{f}(s)$$

with velocity output and force step input $f(t) = f_0\mathbf{1}(t)$. Then the velocity $v(t)$ is unbounded due to the constant forcing and thus constant acceleration. Mathematically, $v(t)$ is the integral of a step function. Physically, the velocity under nonzero constant forcing increases without bound. Hence, the transfer function

$$G(s) = \frac{1}{ms}$$

is not BIBO stable.

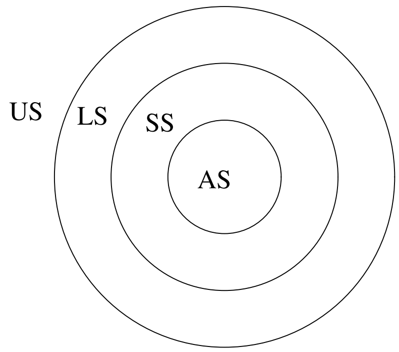


Figure 15.16.14: Stability Venn Diagram

Example 15.16.4. Consider the undamped oscillator with position output and harmonic force input $f(t) = \sin \omega_{\rm n} t$. The output $y(t)$ has repeated poles on the imaginary axis. The corresponding time-domain response, called *resonance*, is an oscillation of increasing amplitude with a linear envelope. Hence

$$G(s) = \frac{1}{s^2 + \omega_{\rm n}^2} \tag{15.16.1}$$

is not BIBO stable.

Fact 15.16.5. $G(s)$ is BIBO stable if and only if $G(s)$ is asymptotically stable.

15.17 Routh Test

Fact 15.17.1. Suppose that all of the roots of the polynomial

$$p(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

are in the open left half plane. Then $a_0, \ldots, a_{n-1}$ are positive.

Fact 15.17.2. Let $n = 2$. Then both roots of

$$p(s) = s^2 + a_1 s + a_0$$

are in the open left half plane if and only if a_0 and a_1 are positive.

Fact 15.17.3. Let $n = 3$. Then all three roots of

$$p(s) = s^3 + a_2 s^2 + a_1 s + a_0$$

are in the open left half plane if and only if a_0, a_1, a_2 are positive and

$$a_0 < a_1 a_2. \tag{15.17.1}$$

Fact 15.17.4. Let $n = 4$. Then all four roots of

$$p(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

are in the open left half plane if and only if a_0, a_1, a_2, a_3 are positive and

$$a_0 a_3^2 + a_1^2 < a_1 a_2 a_3. \tag{15.17.2}$$

The above results are special cases of the *Routh test*. Conditions can be derived for arbitrary values of n.

15.18 Matlab Operations

15.18.1 atan2

The Matlab command atan2(y, x) computes the angle $\theta \in (-\pi, \pi]$ of the complex number $x + \jmath y$.

Example 15.18.1. For $x = 1$ and $y = 1$,

$$\theta = \text{atan2}(y, x) = \text{atan2}(1, 1) = \frac{\pi}{4}.$$

For $x = -1$ and $y = -1$,

$$\theta = \text{atan2}(-1, -1) = -\frac{3\pi}{4}.$$

Note that, in both cases, $\tan\theta = 1$.

15.18.2 expm

The Matlab command expm(A) computes the exponential e^A of the matrix A.

15.18.3 rlocus

Consider the basic servo loop consisting of the transfer function $G(s) = N(s)/D(s)$ and the controller $K(s) = kN_K(s)/D_K(s)$, where $k \geq 0$ is a constant. Then, the *loop transfer function* is defined by

$$L(s) \triangleq G(s)K(s) = kL_0(s), \tag{15.18.1}$$

where

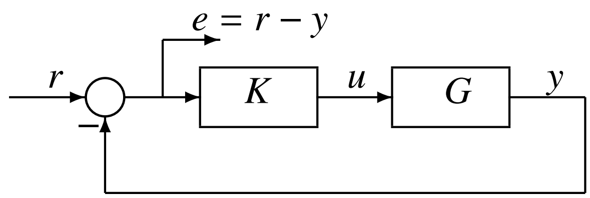
$$L_0(s) \triangleq \frac{N_0(s)}{D_0(s)}, \quad N_0(s) \triangleq N(s)N_K(s), \quad D_0(s) \triangleq D(s)D_K(s). \tag{15.18.2}$$

Let $u(t)$ and $y(t)$ be the input and output of the plant G, respectively, let $r(t)$ denote the *command*, and let $e(t) = r(t) - y(t)$ denote the *error*. Then, the error and command are related by

$$\hat{e}(s) = S(s)\hat{r}(s), \tag{15.18.3}$$

where the *sensitivity function* $S(s)$ is defined by

$$S(s) \triangleq \frac{1}{1 + L(s)}. \tag{15.18.4}$$



We use the notation

$$N_0(s) = (s - z_1) \cdots (s - z_m) = s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \tag{15.18.5}$$

and

$$D_0(s) = (s - p_1) \cdots (s - p_n) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \tag{15.18.6}$$

where $m < n$. Note that $z_1, \ldots, z_m$ and $p_1, \ldots, p_n$ are the zeros and poles of L_0, respectively. The numerator and denominator coefficients of L_0 are represented by the row vectors num and den, respectively, where

$$\text{num} = \begin{bmatrix} 1 & b_{m-1} \cdots & b_1 & b_0 \end{bmatrix}, \quad \text{den} = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}. \tag{15.18.7}$$

The MATLAB command rlocus(num,den) plots the root locus of L.

Note that N_0 and D_0 are assumed to be monic polynomials. If these polynomials are not monic, then they can be scaled to be monic, and the scaling is incorporated in the gain k. For the root locus analysis in this section, we assume that k is nonnegative. The case in which k is negative can also be considered but is less useful in practice.

The *root locus* is a sketch of the locations of the poles of $S(s)$ as k increases. To determine these locations, note that

$$S(s) = \frac{1}{1 + k\frac{N_0(s)}{D_0(s)}} = \frac{D_0(s)}{D_0(s) + kN_0(s)} = \frac{\tilde{N}(s)}{\tilde{D}(s)}. \tag{15.18.8}$$

For small values of k, the poles of $S(s)$ are approximately the roots of $D_0(s)$, that is, the poles of the loop transfer function; these poles are also called the *open-loop poles*. However, for large values of k, some of the poles of $S(s)$ are approximately roots of $N_0(s)$.

The following rules are useful for sketching the root locus, which is a plot of the locations of the closed-loop poles as k increases from zero to infinity.

Rule #1. Plot the poles and zeros of $L_0(s)$, using "×" to denote each pole, and "○" to denote each zero.

Rule #2. The root locus is conjugate symmetric.

Rule #3. Intervals on the real axis to the left of an odd number of real poles and zeros are subsets of the root locus, while the remaining intervals are not. Complex poles and zeros can be ignored when applying this rule.

Rule #4. Each zero attracts exactly one pole.

Rule #5. If a pole is repeated q times, then the departure angles of the q poles are all different and are given by

$$\phi_{\rm dep} = \frac{1}{q}\left[\sum_{i=1}^{m}\psi_i - \sum_{i=1}^{n-q}\phi_i \pm 180^\circ \pm 360^\circ\ell\right], \tag{15.18.9}$$

where $\sum_{i=1}^{m}\psi_i$ is the sum of the angles from all of the zeros to the departing poles, and $\sum_{i=1}^{n-q}\phi_i$ is the sum of the angles from the remaining (non-departing) poles to the departing poles. The signs $\pm$ and the integer ℓ can be chosen to obtain distinct angles between -180 deg and 180 deg. If the departing poles are real, then complex conjugate zeros can be ignored in the first sum and complex conjugate poles can be ignored in the second sum.

Rule #6. The $n-m$ excess poles approach infinity along $n-m$ asymptotes drawn from the center α, which is given by

$$\alpha \triangleq \frac{b_{m-1} - a_{n-1}}{n-m} = \frac{P-Z}{n-m}, \tag{15.18.10}$$

where P is the sum of all of the poles of $L_0(s)$ and Z is the sum of all of the zeros of $L_0(s)$.

Rule #7. If $n - m = 1$, then the asymptote is in the direction of the negative real axis. In this case, the location of the center is irrelevant. If $n - m = 2$, then the asymptotes point 90 and -90 degrees relative to the positive real axis. If $n - m = 3$, then the asymptotes point in directions that are 60 degrees, 180 degrees, and -60 degrees relative to the positive real axis. If $n-m = 4$, then the asymptotes point in directions that are 45 degrees, 135 degrees, and -135 degrees, and -45 degrees relative to the positive real axis. More generally, the number of asymptotes is $r = n - m$, and the angles between the asymptotes relative to the positive real axis are $(2i + 1)180/r$ degrees, where $i = 0, \ldots, r - 1$.

15.19 Dimensions and Units

15.19.1 Mass and Force

For pound force,

$$1 \text{ lb} = 1 \text{ slug-ft/sec}^2 = 4.4 \text{ N}, \tag{15.19.1}$$

where N denotes newton. On the surface of the Earth, 1 kg weighs 9.8 N, while 1 slug weighs 32.2 lb. Conversion between slugs and kilograms is

$$1 \text{ slug} = 14.59 \text{ kg}. \tag{15.19.2}$$

15.19.2 Force, Impulse, and Momentum

The impulse $\delta(t)$ has the units

$$[\delta(t)] = 1/\text{sec}. \tag{15.19.3}$$

For the force impulse $f(t) = f_0\delta t$, the coefficient f_0 has the units

$$[f_0] = [f(t)]/[\delta(t)] = \text{kg-m/sec}, \tag{15.19.4}$$

which are the units of momentum. Specific impulse is given by f_0/m, which has the units

$$[f_0/m] = \text{m/sec}, \tag{15.19.5}$$

that is, velocity.

15.20 Problems

Problem 15.20.1. Consider the 3×3 matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 4 & 3 & 5 \\ 7 & 2 & 6 \end{bmatrix}.$$

Manually compute the determinant and inverse of A. Then compute these quantities using Matlab to check your solution.

Problem 15.20.2. Use Matlab to compute the eigenvalues of the matrix A in Problem 15.20.1. Then use Matlab to show that the determinant is the product of the eigenvalues, and show that the trace (the sum of the diagonal entries) is the sum of the eigenvalues. Furthermore, use Matlab to compute the eigenvalues of A^2 and A^{-1}, and discuss how they are related to the eigenvalues of A.

Problem 15.20.3. Using the "randn" command in Matlab, form a random 4×2 matrix A. Then compute the 4×4 matrix AA^{T}. Using Matlab to compute the eigenvalues of the symmetric matrix AA^{T}, check whether this matrix is positive semidefinite. Then, show mathematically (by hand, not using Matlab) that $x^{\mathrm{T}}AA^{\mathrm{T}}x \geq 0$ for all vectors x. (Hint: Define $z = A^{\mathrm{T}}x$.)

Problem 15.20.4. Let A be an $n \times n$ matrix and let $p(s) = \det(sI - A)$ be the characteristic polynomial of A. The Cayley-Hamilton theorem states that $p(A) = 0$. Check this fact by obtaining the characteristic polynomial $p(s)$ for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix},$$

and then showing that $p(A) = 0$. Repeat these steps for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}.$$

Do all of these symbolic calculations by hand.

Problem 15.20.5. The eigenvalues of a matrix are the roots of its characteristic polynomial. Consider the 3×3 matrix in Problem 15.20.4 with $a_0 = -2$, $a_1 = 5$, and $a_2 = 3$, and use Matlab to show numerically that the roots of $p(s)$ are indeed the eigenvalues of A. Use roots(p) and eig(A) for your computations.

Problem 15.20.6. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal. Also, check whether the matrix

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

is orthogonal. Determine by hand (not by Matlab) the determinants and eigenvalues of these matrices. Finally, choose several values of θ and multiply the vector $[1 \;\; 1]^{\mathrm{T}}$ by these matrices. Discuss how the resulting vectors compare to the original vector in terms of their length and direction. You can use a calculator but do not use Matlab.

Problem 15.20.7. Use the dot product to compute the angle between the vectors $[3 \;\; 2]^{\mathrm{T}}$ and $[-2 \;\; 4]^{\mathrm{T}}$.

Problem 15.20.8. Solve the differential equation

$$\dot{x}(t) = ax(t) + b$$

analytically by evaluating the convolution integral given by (15.8.3). Under what conditions does $\lim_{t\to\infty} x(t)$ exist? In the case where the limit exists, determine the limiting value. How could you guess the limiting value without solving the equation?

Problem 15.20.9. Using the solution to Problem 15.20.8, write down the solution to the scalar ordinary differential equation

$$\dot{x}(t) = -2x(t) + 8.$$

This equation represents the step response of a linear system, where the constant 8 is the value of the step input. Plot the solutions for the initial conditions $x(0) = 5$ and $x(0) = -4$ in the same figure. Be sure to label all axes of your figure and give it an appropriate caption. Determine $\lim_{t\to\infty} x(t)$ from the plot and compare that numerical value with the analytical limit. Explain how the limiting value of $x(t)$ depends on the constants in the problem, namely, the coefficient of $x(t)$, the value of the input step, and the initial condition.

Problem 15.20.10. Consider $\dot{x} = Ax$, and let $\lambda = \sigma + \jmath\omega$ be a complex eigenvalue of A with associated complex eigenvector

$$v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \jmath \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Consider three different eigenvalues (of different matrices A) given by $\omega = 1$ and $\sigma = -1, 0, 1$. Then, for each value of σ, plot the corresponding eigensolutions

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathrm{Re}(e^{\lambda t} v)$$

in the x_1, x_2 plane, where the curve is parameterized by t. Use arrows to denote how the solution evolves as t increases, and explain how the properties of the eigensolution depend on σ.

Problem 15.20.11. Let $\lambda, \bar{\lambda} = -\zeta\omega_{\mathrm{n}} \pm \omega_{\mathrm{n}}\sqrt{1-\zeta^2}\,\jmath$ denote a complex conjugate pair of underdamped eigenvalues. Show that

$$\omega_{\mathrm{n}} = |\lambda|, \quad \zeta = -\frac{\lambda + \bar{\lambda}}{2|\lambda|}, \quad \omega_{\mathrm{d}} = \frac{\lambda - \bar{\lambda}}{2\jmath}.$$

Problem 15.20.12. Let $\lambda_1, \lambda_2 = -\zeta\omega_{\mathrm{n}} \pm \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1}$ denote a pair of overdamped eigenvalues. Show that

$$\omega_{\mathrm{n}} = \sqrt{\lambda_1\lambda_2}, \quad \zeta = -\frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1\lambda_2}}.$$

Problem 15.20.13. Use l'Hopital's rule to derive (15.15.9) from (15.15.8).

Problem 15.20.14. Consider the damped oscillator (DO) with acceleration output. Write this system in A, B, C, D form, where A is of size 2×2.

Problem 15.20.15. Use the initial value theorem to determine the initial value of each of the following functions: $y(t) = t$, $y(t) = t + 1$, $y(t) = \sin 2t$, and $y(t) = \cos 2t$.

Problem 15.20.16. Use the initial slope theorem to determine the initial slopes of the functions in the previous problem.

Problem 15.20.17. Use Laplace transforms to analytically determine the response of $\dot{v} + 2v = \sin 5t$ for an arbitrary initial condition $v(0)$. Then show that you can choose a special initial condition $v(0)$ so that the response is exactly harmonic, that is, there is no transient (non-harmonic) component of the solution. Finally, confirm your solution by using ODE45 to simulate the system with this special initial condition as well as another initial condition.

Problem 15.20.18. A motor with constant applied torque is modeled as the damped rigid body $J\ddot{\theta} + c\dot{\theta} = \tau_0 \mathbf{1}(t)$, where J is the load inertia, c is the viscous damping coefficient, and τ_0 is the moment. The initial angle $\theta(0)$ and initial angular rate $\dot{\theta}(0)$ are zero. Use Laplace transforms and the final value theorem to determine the terminal angular rate $\lim_{t\to\infty} \dot{\theta}(t)$. Also compute the same limit by using the time-domain solution obtained from Laplace transforms.

Problem 15.20.19. For

$$\hat{y}(s) = \frac{1}{s(s^2 + s + 1)},$$

use partial fractions to show that

$$y(t) = \mathbf{1}(t) - e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right).$$

Problem 15.20.20. Consider an object falling under the force of gravity. Ignore drag and model the motion as an undamped rigid body. Use Laplace transforms to express the position $q(t)$ as a function of t, $q(0)$, $\dot{q}(0)$, and g.

Problem 15.20.21. A body with mass M is falling under the force of gravity. Atmospheric drag is modeled by a dashpot coefficient D, so that the system is modeled as a damped rigid body. Assuming initial velocity v_0, use Laplace transforms to find the velocity $v(t)$ for $t > 0$. Then use the final value theorem to compute the terminal velocity $\lim_{t\to\infty} v(t)$.

Problem 15.20.22. Consider an iron sphere and a wooden sphere of the same size and smoothness. Does the iron sphere fall faster than the wooden sphere? Use the undamped rigid body to show that, if there is no drag, then the spheres fall at the same rate. Now consider the more realistic case in which drag is present. Using the damped rigid body, and assuming that the damping coefficient is the same for both spheres, determine whether or not the heavier body falls faster than the lighter body. (Hint: First consider the terminal velocities and perform some Matlab simulation.)

Problem 15.20.23. Consider the damped rigid body with ramp force input $u(t) = f_0 t$ and velocity output. Use Laplace transforms and partial fractions to determine the forced response for $t \geq 0$.

Problem 15.20.24. Consider the undamped rigid body in two cases, namely, 1) position output with unit impulse input, and 2) velocity output with unit step input. Show that the Laplace transforms of the forced outputs have the same form in both cases. Next, explain how the same expression correctly captures both outputs even though the outputs have different dimensions.

Problem 15.20.25. Consider the rigid body force-to-velocity transfer function

$$G(s) = \frac{1}{ms}$$

with harmonic input $f(t) = a \sin \omega t$. Use Laplace transforms to determine the forced velocity response $\dot{q}(t)$ for $t \geq 0$. For the forced response, assume that the initial conditions are zero so that the free response is zero.

Problem 15.20.26. Determine the forced response of $2\ddot{q} + 17q = 5f$ with output $y(t) = q(t)$ and input $f(t) = \sin 2t$. Check your solution by using either Matlab or Simulink to plot your analytical solution.

Problem 15.20.27. Consider the damped rigid body with position output, and write it in state space form $\dot{x} = Ax$. Then determine e^{At} by using the fact that the Laplace transform of the matrix exponential e^{At} is $(sI - A)^{-1}$ and taking the inverse Laplace transform of each entry of $(sI - A)^{-1}$. Use this result to determine the free response of the damped rigid body with initial position $q(0)$, initial velocity $\dot{q}(0)$, and output $y(t)$ given by the mass position. Finally, determine $\lim_{t\to\infty} e^{At}$ by using the expression you obtained for e^{At} as well as by applying the final value theorem to each separate entry of $(sI - A)^{-1}$. What kind of matrix is $\lim_{t\to\infty} e^{At}$?

Problem 15.20.28. An engineer has shown that the output response of a new airframe developed for a UAV application is given by

$$y(t) = 2e^{-6t}.$$

Use the initial value theorem to determine $\ddot{y}(0^+)$. (Note the two dots.) Check your solution by computing $\ddot{y}(t)$ and then setting $t = 0$. (Hint: Note that

$$\mathcal{L}\{\ddot{y}(t)\} = s^2\hat{y}(s) - sy(0) - \dot{y}(0).$$

You may use the time-domain expression for $y(t)$ to determine $y(0)$ and $\dot{y}(0)$.)

Problem 15.20.29. For each transfer function $G(s)$ below with input u and output y, determine whether the use of the final value theorem is legal, and, if so, use it to determine the limit of the output $y(t)$ as $t \to \infty$. Explain why or why not the use of the final value theorem is legal in each case.

i) $G(s) = \dfrac{-5}{s(s+7)^2}, \quad u(t) = 3e^{-2t}.$

ii) $G(s) = \dfrac{5}{s-3}, \quad u(t) = 7\mathbf{1}(t) - 3\delta(t - 4.2).$

iii) $G(s) = \dfrac{-4}{s^2 + 2.4s + .3}, \quad u(t) = 6t^2.$

Problem 15.20.30. For both systems described below (described by a transfer function and input) and without using a calculator or computer, determine whether the use of the final value theorem is legal, and, if so, use the final value theorem to determine the limit of the output $y(t)$ as $t \to \infty$. Explain why or why not the use of the final value theorem is legal in each case. Simulate the first system using the Matlab "impulse" command. Simulate the second system using Simulink with integrator blocks but not the transfer function block. (Hint: Use a pair of nested feedback loops each of which has an integrator in the forward path.) If $y(t)$ converges, verify that the value of $y(t)$ as t becomes very large agrees with the result of the final value theorem. Run your Simulink model from a Matlab script and plot $y(t)$ as a function of time from your Matlab script.

i) $G(s) = \dfrac{s^2 - 1}{s(3s^3 + 2s^2 + 4s + 3)}, \quad u(t) = 2\delta(t).$

ii) $G(s) = \dfrac{1}{s(s^2 + 3s + 1)}, \quad u(t) = -2e^{-t}\sin 2t.$

Problem 15.20.31. For the systems described below and without using a calculator or computer, determine whether the use of the final value theorem is legal, and, if so, use the final value theorem to determine the limit of the output $y(t)$ as $t \to \infty$. Explain why or why not the use of the final value theorem is legal in each case. Model the first system with Simulink and simulate the system. Simulate the second system using the Matlab "impulse" command. If $y(t)$ converges, verify that the value of $y(t)$ as t becomes very large agrees with the result of the final value theorem. Run your Simulink model from a Matlab script and plot $y(t)$ as a function of time from your Matlab script.

i) The free response of the damped rigid body with initial position $q(0)$, initial velocity $\dot{q}(0)$, and output $y(t)$ given by the mass position. Obtain the limit symbolically. For the Simulink model, use the numerical values $m = 3$ kg, $c = 4$ kg/s, $q(0) = 1$ m, and $\dot{q}(0) = -2$ m/s.

ii) The forced response of a system whose transfer function is

$$G(s) = \frac{4s^2 - 12s - 16}{2s^5 + 2s^4 + 4s^3 + 2s^2 + s}$$

with the impulsive input $u(t) = 3\delta(t - 1)$.

Problem 15.20.32. Without taking inverse Laplace transforms, use the initial and final value theorems as well as knowledge of the pole locations to sketch the step response of the transfer function

$$G(s) = \frac{4s - 3}{s^2 + 0.8s + 4}.$$

Be sure to qualitatively capture the direction of the step response for small positive time t as well as the asymptotic behavior for large time t. What features does your sketch illustrate? Model this system with Simulink and simulate the system. You may use the 'transfer function' block in your Simulink model. Run your Simulink model from a Matlab script and plot the step response from your Matlab script.

Problem 15.20.33. Flight testing of a new aircraft reveals that it has a pair of underdamped complex conjugate poles. Testing reveals that the time to 50% decay is T seconds, while analytical modeling shows that the imaginary parts of the poles are $\pm\omega_{\mathrm{d}}$, where $\omega_{\mathrm{d}} > 0$. Derive an expression for the damping ratio ζ in terms of T and ω_{d}. Finally, show that the expression for ζ satisfies $0 < \zeta < 1$. (Hint: $T = (\ln 2)/(\zeta\omega_{\mathrm{n}})$.)

Problem 15.20.34. Flight testing of a new aircraft reveals that it behaves like an overdamped oscillator. Analytical models are used to determine the value of ω_n. Measurements show that the time to 50% decay is T seconds. Derive an expression for the damping ratio ζ in terms of T and ω_n. Finally, show that the expression for ζ satisfies $\zeta \geq 1$. (Note: Consider only the slow eigenvalue of the overdamped oscillator.)

Problem 15.20.35. Consider the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}$$

for the undamped ($c = 0$) and damped ($c > 0$) oscillators. Let $k = 2.5$ kg/s^2 and $m = 7$ kg. Plot the locations of the eigenvalues as $\times$'s in the complex plane for a range of values of c. Choose a range that includes undamped, underdamped, critically damped, and overdamped cases. For each value of c, plot a "$\times$" in the complex plane.

Problem 15.20.36. Show analytically that the poles of the undamped, underdamped, and critically damped oscillators satisfy $|\lambda_1| = |\lambda_2| = \omega_n$. Furthermore, show that the poles of the overdamped oscillator satisfy $\lambda_1\lambda_2 = \omega_n^2$. Explain the meaning of these results in terms of the plot you made in the previous problem.

Problem 15.20.37. Consider the second-order state space (A, B, C, D) system with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0.$$

Determine the transfer function $G(s)$ for this system. Show that $G(s)$ is actually a first-order transfer function due to pole-zero cancelation. Finally, determine a first-order state space $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ system whose transfer function is the same G. (Remark: Given a transfer function G we are usually interested in a *realization* (A, B, C, D) of G of the lowest possible order, that is, where the size of A is as small as possible. For this first-order system you can easily construct this realization by looking at the form of the transfer function and setting $C = 1$.)

Problem 15.20.38. Consider the longitudinal dynamics of a commercial aircraft given by

$$\begin{aligned} \dot{\alpha}(t) &= -0.313\alpha(t) + 56.7q(t) + 0.232\delta e(t), \\ \dot{q}(t) &= -0.0139\alpha(t) - 0.426q(t), \\ \dot{\theta}(t) &= -0.5\theta(t) + 0.0203\delta e(t), \end{aligned}$$

where α is the angle of attack in rad, q is the pitch rate in rad/sec, θ is the pitch angle in rad, and δe is the elevator deflection angle in rad. Assume that $\alpha(0) = q(0) = \theta(0) = 0$. Let the input of the system be the elevator deflection angle δe, and the output of the system be the pitch angle θ. Derive equations for this system in state space and transfer function form. Model this system in Simulink using only integrator blocks, gain blocks, and summation blocks. Also model this system in Simulink using a state space block.

Force both Simulink models with a step input given by a 3-degree elevator deflection. Plot the resulting pitch angle and verify that the two models give exactly the same response.

Problem 15.20.39. Take the Simulink model you constructed without using the state space block in Problem 15.20.38, and then put that entire model inside a subsystem block. Set the input to the subsystem block to be the elevator input and set the output of the subsystem block to be the pitch angle. Now we want to use feedback control to make the aircraft maintain a pitch angle of 3 degrees. Set up proportional control whose input is the error between the true pitch angle and the desired pitch angle and whose output is the elevator deflection angle. Tune the proportional control to obtain the best possible response. Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same axes. For clarity, plot the true pitch angle using a solid blue line, and plot the desired pitch angle using a dashed red line. Are you able to follow the command using proportional control?

Now set up an integral controller whose input is the error between the true pitch angle and the desired pitch angle, and whose output is the elevator deflection angle. Add a gain block to the output of the integrator block and tune the gain to obtain the best possible result. Run your Simulink model from a MATLAB script and plot the true pitch angle and the desired pitch angle on the same axes. For clarity, plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using integral control? Note the relationship between the Laplace transform of a step command and the transfer function of an integrator.

Problem 15.20.40. Consider the same Simulink setup as in Problem 15.20.39. Now, we want to use feedback control to make the aircraft follow an oscillation in pitch angle given by $\sin t$. Set up integral control that takes the error between the true pitch angle and the desired pitch angle as input and outputs elevator deflection angle. Add a gain block to the output of the integrator block and tune the gain to obtain the best possible result. Change the relative tolerance to 10^{-10} by selecting "Simulation", then "Model Configuration Parameters." Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same axes. Plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using integral control?

Now set up the controller as the transfer function

$$G(s) = \frac{1}{s^2 + 1}. \tag{15.20.1}$$

Let the input to this transfer function be the error between the true pitch angle and the desired pitch angle and let its output be the elevator deflection angle. Add a gain block to the output of this transfer function and tune it to try and get the best result. Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same plot. Plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using this transfer function?

Next, set up the controller as the transfer function

$$G(s) = \frac{2}{s^2 + 4}. \tag{15.20.2}$$

Let this transfer function take the error between the true pitch angle and the desired pitch angle as input and let it output the elevator deflection angle. Add a gain block to the output of this transfer function and tune it to try and get the best result. Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same plot. Plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using this transfer function?

What can you infer from the last two problems in terms of the relationship between the command signal and the poles of the controller? Hint: Consider the Laplace transforms of $\sin t$ and $\sin 2t$?

Problem 15.20.41. Redo Problems 15.20.39 and 15.20.40, but instead of the dynamics of the aircraft, let the plant be given by the transfer function

$$G(s) = \frac{4s^2 + 4s + 1}{s^3 + s^2 + \pi s + 0.777}. \tag{15.20.3}$$

You may use the transfer function block in your Simulink model. Is your conclusion here the same as your conclusion in Problem 15.20.40?

Problem 15.20.42. Redo Problem 15.20.39 for different aircraft dynamics, given by

$$\begin{aligned}
\dot{\alpha}(t) &= -0.313\alpha(t) + 56.7q(t) + 0.232\delta e(t),\\
\dot{q}(t) &= -0.0139\alpha(t) - 0.426q(t) + 0.0203\delta e(t),\\
\dot{\theta}(t) &= 56.7q(t),
\end{aligned}$$

where α is the angle of attack in rad, q is the pitch rate in rad/sec, θ is the pitch angle in rad, and δe is the elevator deflection angle in rad. Assume that $\alpha(0) = q(0) = \theta(0) = 0$. Let the input of the system be the elevator deflection angle δe, and the output of the system be the pitch angle θ. You may use the state space block in your Simulink model. Is your conclusion here the same as your conclusion in Problem 15.20.39? Why or why not?

Problem 15.20.43. Consider the basic servo loop with a SISO plant G and SISO controller G_c chosen such that the closed-loop system is asymptotically stable. Assume that neither G nor G_c has a zero at zero. Then, do the following:

i) Case 1: Show that if G_c has at least one integrator, then the asymptotic error to a step command is zero (whether or not G has an integrator). Determine the asymptotic value of the control.

ii) Case 2: Show that if G has at least one integrator, then the asymptotic error to a step command is zero (whether or not G_c has an integrator). Determine the asymptotic value of the control.

iii) Case 3: Show that if G has at least one integrator but G_c has no integrators, then the asymptotic error to a step command in the presence of a nonzero constant disturbance is not zero. The disturbance is added to the control.

iv) Case 4: Show that if G_c has at least one integrator, then the asymptotic error to a step command in the presence of a nonzero constant disturbance is zero whether or not G has an integrator.

v) Illustrate the above four cases using Simulink with $G(s) = 1/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s)$, or vice versa. Plot the plant output, the command-following error, and the control input. Write an mfile to check stability using the 4th-order Routh conditions in the notes as well as root locus with the gain running from 0 to 1.

Problem 15.20.44. Consider the basic servo loop with a SISO plant G and SISO controller G_c chosen such that the closed-loop system is asymptotically stable. The command and disturbance are steps, which may or may not be zero. The disturbance is added to the control. Assume that the measurement y that is fed back is corrupted by an unknown nonzero constant bias, that is, $y = y_0 + b$, where y_0 is the plant output and b is the unknown constant bias. Since b is unknown, we are not able to use its value in the control law. The "false" error that the controller operates on is $e = r - y$. However, the *true error* is $e_{\text{true}} = r - y_0$. Then, do the following:

i) Case 1: Show that if G_c has at least one integrator, then the asymptotic error is zero (whether or not G has an integrator) but the asymptotic true error is not zero. Show that this statement holds for all values of the command and disturbance.

ii) Case 2: Assume that neither G nor G_c has an integrator, and assume that G has at least one zero at zero. Show that, if the command is zero, then the true error converges to zero for all values of the disturbance.

iii) Case 3: Assume that neither G nor G_c has an integrator, and assume that $L = GG_c$ has at least one zero at zero. Determine necessary and sufficient conditions on the command and disturbance such that the true error converges to zero. Use this result to show that, if the command is not zero and the disturbance is zero, then the true error does not converge to zero.

iv) Illustrate the above three cases using Simulink. For Case 1, let $G(s) = 1/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s)$. For Case 2, let $G(s) = s/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s + 2)$. For Case 3, let $G(s) = s/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s + 2)$.

v) Is it possible to follow a step command (in the sense of the true error) despite the presence of an unknown measurement bias?

Problem 15.20.45. Consider the second-order system described by the differential equation $\ddot{y} + \dot{y} - 3y = u$.

i) Assuming $y(0) = \dot{y}(0) = 0$, derive the transfer function of the system from $\hat{u}(s)$ to $\hat{y}(s)$.

ii) Simulate the system with the step input $u(t) = \mathbf{1}(t-1)$ for 100 sec. How does $y(t)$ evolve in time? Attach a plot of $y(t)$.

iii) Now include the proportional feedback control law $\hat{u}(s) = K_\mathrm{p}\hat{e}(s)$, and simulate the closed-loop system with the step command $r(t) = \mathbf{1}(t-1)$ for 20 sec. Choose K_p such that the maximum overshoot in $y(t)$ is less than 100%, and $|e(t)| = |r(t) - y(t)|$ is less than 0.2 for $t > 10$. Does the error converge to zero as t increases? Comparing your results from the previous part, what is the benefit of using proportional feedback compared to open loop? Attach a screenshot of your Simulink model, as well as plots of $e(t)$ and $y(t)$ showing that the design requirements are met.

iv) Using your choice of K_p from *iii*), now introduce the proportional-integral (PI) control law

$$\hat{u}(s) = \left(K_\mathrm{p} + \frac{K_\mathrm{I}}{s}\right)\hat{e}(s),$$

and simulate the closed-loop system with the command $r(t) = \mathbf{1}(t-1)$ for 50 sec. Adjust K_I so that the maximum overshoot in $y(t)$ is less than 100%, $|e(t)| < 0.2$ for $t > 10$, and $|e(t)| < 0.01$ for $t > 50$. Does the command-following error converge to zero as t increases? Comparing your results from the previous part, what is the benefit of integral control? Attach a screenshot of your Simulink model, as well as plots of $e(t)$ and $y(t)$ showing that the design requirements are met. Keep in mind that too much control might destabilize the closed-loop system. Keeping K_p the same, increase K_I and show that the closed-loop system becomes unstable.

v) Using K_p and K_I from *iv*), now consider the harmonic command $r(t) = \sin(2t)$, and simulate the closed-loop system for 50 sec. Attach a plot of $e(t)$. Does the PI control law drive the tracking error $e(t)$ to zero in this case?

vi) Replace the PI control law in *v*) with the feedback control law

$$\hat{u}(s) = \left(20 + \frac{5s}{s^2+4}\right)\hat{e}(s),$$

and simulate the closed-loop system with the harmonic reference input $r(t) = \sin(2t)$ for 50 seconds. Attach a plot of $e(t)$. Is the command-following error driven to zero in this case?

vii) Compare the Laplace transforms of the step input $r(t) = \mathbf{1}(t)$ and the harmonic input $r(t) = \sin(2t)$ with the Laplace transform of the PI controller, and the controller you used in *vi*). Compare the denominators of the input signals and the controller. Also, compare the Laplace transform of a step/ramp command and the denominator of the loop transfer function for a type I/II control system. What pattern do you see?

Problem 15.20.46. Consider the second-order transfer function

$$G(s) = \frac{1}{s^2+3s-2}$$

controlled by the PI controller

$$K(s) = K_{\mathrm{P}} + \frac{K_{\mathrm{I}}}{s}$$

in a servo loop. Determine the values of K_{P} and K_{I} that render the closed-loop system asymptotically stable by sketching the region in the $K_{\mathrm{P}}, K_{\mathrm{I}}$ plane of stabilizing gains.

Problem 15.20.47. Consider the type II control system with plant

$$G(s) = \frac{1}{s(s+1)}$$

and PI controller

$$K(s) = K_{\mathrm{P}} + \frac{K_{\mathrm{I}}}{s}$$

in a servo loop. First, determine the values of K_{P} and K_{I} that provide zero asymptotic error for a ramp command. Next, use Matlab or Simulink to simulate the closed-loop system, and choose K_{P} and K_{I} for good rise time and reasonable overshoot for a step command. Next, plot the error response for a ramp command, and verify numerically and analytically using the final value theorem that the ramp command error converges to zero.

Problem 15.20.48. Sketch the root loci for the following loop transfer functions by using the root locus rules and then check your sketches using Matlab's root locus function:

i) $L(s) = \frac{(s+1)(s+2)}{(s-1)(s-2)(s-3)}$.

ii) $L(s) = \frac{(s+2)^2}{(s+1)(s^2+1)}$.

Problem 15.20.49. At Mach 0.6, an experimental aircraft has unstable phugoid eigenvalues $0.04 \pm 0.12\jmath$ and stable short period eigenvalues $-4.3 \pm 5.7\jmath$. Wind tunnel testing reveals that the elevator-to-pitch transfer function has one zero located at -0.2. Using proportional control, sketch the root locus, determine the center and asymptotes, and discuss the stability of the closed-loop longitudinal dynamics for high values of k.

Problem 15.20.50. An experimental aircraft has unstable phugoid eigenvalues $0.04 \pm 0.12\jmath$ and stable short period eigenvalues $-4.3 \pm 5.7\jmath$. Wind tunnel testing reveals that, at Mach 0.6, the elevator-to-pitch transfer function has three real zeros located at $-0.3, -1.7, -8.4$. Sketch the root locus and discuss the stability of the closed-loop longitudinal dynamics using a proportional control. Indicate (but you do not need to compute) the point on the root locus at which all poles have at least $\sqrt{2}/2$ damping by marking the intersection of the root locus and the 0.707 damping-ratio line. (Note: Problem 16.8.5 explains why this minimum value of damping is desirable.)

Problem 15.20.51. Consider the linearized model of a Boeing 747 aircraft in straight and level flight given by

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du,$$

where

$$x = \begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix}, \quad u = \begin{bmatrix} \delta r \\ \delta a \end{bmatrix},$$

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.598 & -0.115 & -0.0318 & 0 \\ -3.05 & 0.388 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.00729 & 0 \\ -0.475 & 0.00775 \\ 0.153 & 0.143 \\ 0 & 0 \end{bmatrix}.$$

Determine the eigenvalues of A. What kind of stability does the aircraft have? Which poles correspond to the Dutch roll mode?

Problem 15.20.52. Consider the linearized aircraft model in Problem 15.20.51. Plot the zeros of the four SISO transfer functions from the inputs δr and δa to the outputs r and ϕ. Next, use Matlab or Simulink to plot the impulse response of all four transfer functions. Finally, apply root locus to each of the four transfer functions with proportional feedback.

Problem 15.20.53. The goal of this problem is to design a yaw-rate damper for the aircraft in Problem 15.20.49 in order to improve its response to disturbances. A yaw-rate damper is an essential part of an aircraft autopilot for improving handling quality. Note that "yaw-rate" refers to the $\hat{k}_{\mathrm{AC}}$-axis gyro perturbation $r = \dot{\psi}$. Design a yaw-rate damper using proportional feedback with the rudder as the control input, and determine the proportional feedback gain that yields the largest damping ratio for all closed-loop poles. Finally, impulse the rudder operating in closed loop and plot the response of all four states. Repeat the above steps for the ailerons. Which control input provides a better yaw-rate damper, rudder or ailerons?

Problem 15.20.54. Using full-state feedback ($C = I_4$) in Problem 15.20.49, design a controller using the "place" command in MATLAB. Plot the impulse response of the closed-loop system for all eight combinations of inputs and outputs, and compare these results to the results of Problem 15.20.51.

Problem 15.20.55. Determine all equilibria $\bar{x}$ and control offsets $\bar{u}$ that satisfy $A\bar{x} + B\bar{u} = 0$ for each of the following systems and give a physical explanation for each equilibrium (for example, constant position, zero velocity, constant force):

i) URB with position and velocity states.

ii) URB with just velocity state.

iii) DRB with position and velocity states.

iv) DRB with just velocity state.

v) DO with position and velocity states.

Problem 15.20.56. Consider state space models for URB, DRB, UO, and DO, where the state consists of the position and velocity of the mass. Then do the following:

i) Check the controllability of each model.

ii) Assume that the output is position. Check the observability of each model.

iii) Assume that the output is velocity. Check the observability of each model.

Problem 15.20.57. In this problem we study the effect of nonzero initial conditions. Consider the basic servo loop, where G and G_c are such that the closed-loop dynamics are asymptotically stable.

i) By using Laplace transforms that include the free response, show that, for each command r, the asymptotic response of the error is independent of the initial states of the plant and controller.

ii) Rerun Case 1 of Problem 6 of Homework 1 with Simulink, but now use state space blocks and set the initial conditions of the plant and controller to be nonzero. Are the conclusions for zero initial conditions still valid for nonzero initial conditions?

Problem 15.20.58. In this problem we study the effects of pole-zero cancellation, where the canceled pole and zero are either negative, zero, or positive. All cases are to be studied by using Simulink. Six Simulink models are needed to consider all cases.

i) Use state space models to form the cascade $G_1(s) = \frac{1}{s+1}$ followed by $G_2(s) = \frac{s+1}{(s+2)^2}$. Give both models nonzero initial conditions, force G_1 with a constant input, and add a constant disturbance to the output of G_1 to serve as the input to G_2. Describe the response of G_1 and G_2. How do the initial conditions and external inputs affect the outputs? What is the effect of reversing the order of G_1 and G_2?

ii) Same as part *i*) but now with $G_1(s) = \frac{1}{s}$ and $G_2(s) = \frac{s}{(s+2)^2}$. Consider the reverse case as well.

iii) Same as part *i*) but now with $G_1(s) = \frac{1}{s-1}$ and $G_2(s) = \frac{s-1}{(s+2)^2}$. Consider the reverse case as well.

Problem 15.20.59. With Simulink we cannot say that the pole-zero cancellation in Problem (15.20.58) *iii*) is exact since Simulink does not simulate the system exactly. Therefore, we now determine the effect of unstable pole-zero cancellation by exact analysis of an example. Consider both cases, that is, with G_1 preceding G_2 and vice versa. Assume that the external inputs to G_1 and G_2 are zero. Therefore, in the case where G_1 precedes G_2, it follows that G_1 has only a free response and G_2 has both a free response and a forced response due to the output of G_1. As an example, let

$G_1(s) = \frac{1}{s-1}$ and $G_2(s) = \frac{s-1}{(s+1)^2}$. It is helpful to note that

$$e^{\left[\begin{smallmatrix} 0 & 1 \\ -1 & -2 \end{smallmatrix}\right]t} = e^{-t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}.$$

Check this by showing that $e^{A0} = I_2$ and $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$. You may use symbolic integration to determine the response of these systems.

Chapter Sixteen
Frequency Response

16.1 Phase Shift and Time Shift

Consider harmonic signals with the same frequency. The harmonic signals in Figure 16.1.1 have different amplitudes but the same phase, whereas the harmonic signals in Figure 16.1.2 have the same amplitude but different phases. Our goal is to relate phase shift to time shift.

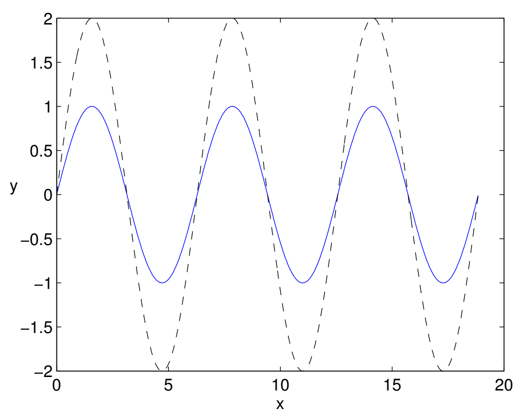


Figure 16.1.1: Sinusoids with the same phase shifts but different amplitudes.

To consider harmonic signals with the same amplitude but different phase shifts, consider

$$f(t) = \sin(\omega t), \tag{16.1.1}$$

$$g(t) = \sin(\omega t + \phi), \tag{16.1.2}$$

where $\phi \in [-\pi, \pi]$ is the phase shift. To solve for the time shift, we solve for the time delay Δt shown in Figure 16.1.3 from $g(t) = 0$. We obtain (see Figure 16.1.3)

$$\sin(\omega \Delta t + \phi) = 0 \tag{16.1.3}$$

and thus,

$$\Delta t = -\frac{\phi}{\omega}. \tag{16.1.4}$$

We thus have

$$g(t) = \sin(\omega t - \omega \Delta t) = \sin(\omega(t - \Delta t)). \tag{16.1.5}$$

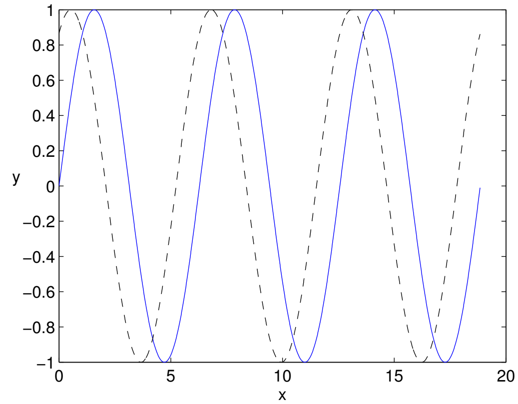


Figure 16.1.2: Sinusoids with the same amplitude but different phase shifts.

If $\phi > 0$, then $\Delta t < 0$, and thus $g(t)$ is advanced relative to $f(t)$. On the other hand, if $\phi < 0$, then $\Delta t > 0$, and thus $g(t)$ is delayed relative to $f(t)$.

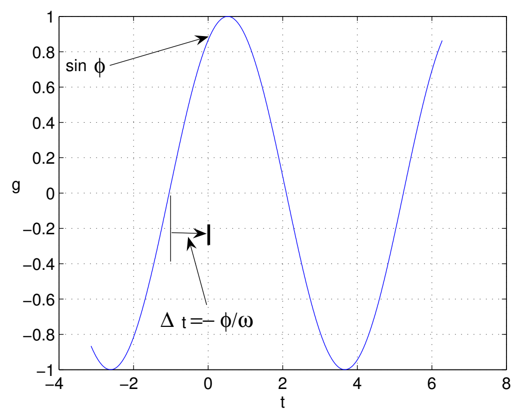


Figure 16.1.3: Phase shift as a time delay.

Now consider

$$f(t) = \sin(\omega t + \phi), \tag{16.1.6}$$

$$g(t) = \sin(\omega t + \hat{\phi}), \tag{16.1.7}$$

as shown in Figure 16.1.4. Now suppose that $0 < \hat{\phi} - \phi \leq 180$. In this case, we say that $g(t)$ *leads* $f(t)$ or $f(t)$ *lags* $g(t)$. Note that leading by more than 180° is equivalent to lagging by less than 180°.

We can express $f(t)$ as the phasor

$$f(t) = \operatorname{Im} e^{(\omega t + \phi)\jmath}, \tag{16.1.8}$$

which is shown in Figure 16.1.5. Leading and lagging are represented in Figure 16.1.6.

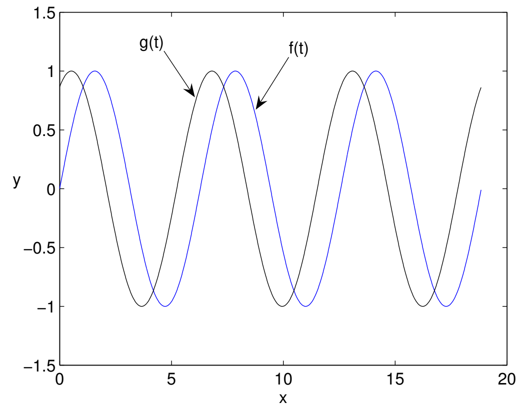


Figure 16.1.4: Sinusoids $g(t)$ and $f(t)$, where $g(t)$ leads $f(t)$.

16.2 Frequency Response Law for Linear Systems

Suppose that a harmonic input (such as a forcing) is applied to an asymptotically stable linear system. Then the output of the system, which consists of the sum of the free and forced response, approaches sinusoidal motion whose frequency is the same as the input frequency. The limiting sinusoidal motion is called the *harmonic steady-state response*. Note, however, that the response does not have a "limit" in the usual mathematical sense since it does not approach a constant value.

The harmonic steady-state response exists for semistable and Lyapunov stable systems as long as a pole of the Laplace transform of the input signal does not coincide with a pole of the transfer function.

The transient behavior of the system before its response reaches harmonic steady state depends on the poles and zeros of the system, the initial conditions of the internal states, and the sinusoidal input.

The ratio of the amplitude of the harmonic steady-state response to the amplitude of the harmonic input is equal to the magnitude of the transfer function evaluated at the input frequency ω, that is, $|G(\omega \jmath)|$, while the phase shift of the harmonic steady-state response relative to the phase of the input is given by the phase of the transfer function at the input frequency, that is, $\mathrm{atan2}(\mathrm{Im}\, G(\omega \jmath), \mathrm{Re}\, G(\omega \jmath))$. Note that the harmonic steady-state response leads the input if $0 < \angle G(\omega \jmath) < 180$, whereas the harmonic steady-state response lags the output if $-180 < \angle G(\omega \jmath) < 0$

For the input

$$u(t) = u_0 \sin(\omega t + \phi_0), \tag{16.2.1}$$

the harmonic steady-state output is given by

$$y_{\mathrm{hss}}(t) = M(\omega) u_0 \sin(\omega t + \phi_0 + \phi(\omega)), \tag{16.2.2}$$

where

$$M(\omega) \triangleq |G(\omega \jmath)| \tag{16.2.3}$$

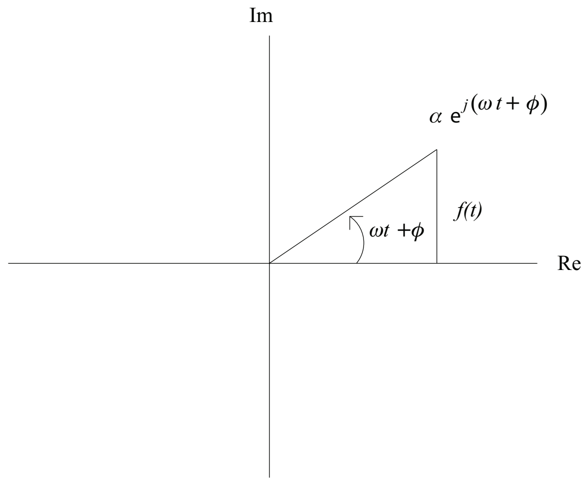


Figure 16.1.5: Phasor representation.

is the *magnitude* and

$$\phi(\omega) \triangleq \angle G(\omega \jmath) \tag{16.2.4}$$

is the *phase*. The magnitude and phase satisfy

$$M(\omega)e^{\jmath\phi(\omega)} = G(\omega \jmath). \tag{16.2.5}$$

16.3 Frequency Response Plots for Linear Systems Analysis

Plots of the magnitude $M(\omega)$ versus input frequency ω and of the phase $\phi(\omega)$ versus input frequency ω are called *Bode plots*. The plot of magnitude versus frequency is the magnitude plot, while the plot of phase versus frequency is the phase plot. It is convenient to express these plots using base-10 logarithmic scales for the magnitude and frequency. In particular, the magnitude is often scaled $20\log_{10}M(\omega)$, referred to as decibels (dB). Logarithmic scales are convenient for plotting the magnitude of a product of transfer functions since the logarithms of the magnitudes of the factors can be added.

The Matlab commands for creating the Bode plot are bode(A,B,C,D) and bode(num,den), which are described in Appendix C. The Bode plot can be constructed for all transfer functions $G(s)$ whether or not $G(s)$ is asymptotically stable, semistable, Lyapunov stable, or unstable. However, the magnitude and phase shift characteristics of $G(s)$ have a harmonic steady-state interpretation only when the harmonic steady-state response exists, as discussed in the previous subsection.

Properties of the Bode frequency response plots include:

- The magnitude $|G(\jmath\omega)|$ is finite if and only if $\jmath\omega$ is not a pole of G.
- The *DC gain* is the magnitude at zero frequency, that is, $|G(0)|$; the DC phase is either zero degrees or -180 degrees depending on whether $G(0)$ is positive or negative. In the Bode plot the DC gain is evident from the low frequency part of the magnitude plot.
- For a linear system with more poles than zeros, that is, $n > m$, the magnitude approaches zero

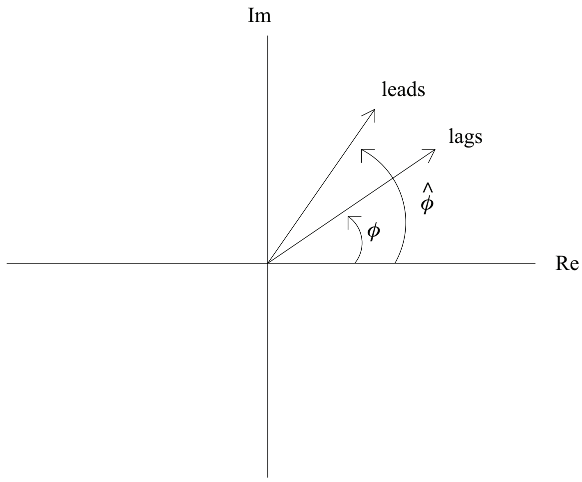


Figure 16.1.6: Phasor representation of leading and lagging sinusoids.

as the frequency tends to infinity; in the Bode plot, the magnitude plot tends to $-\infty$ dB as the frequency tends to infinity.

- Each local maximum at a nonzero frequency corresponds to a pair of complex eigenvalues with damping ratio less than $\sqrt{2}/2$, that is, an underdamped mode with sub-resonant damping.
- The total change in the phase from low frequency to high frequency is given by $(m-n)90^\circ$, where $d \triangleq n-m$ is the relative degree of G, that is, the excess of poles over zeros.
- The frequency $\omega_{\rm mco}$ at which the magnitude is unity (or, equivalently, zero dB) is the *magnitude crossover frequency*. The *phase margin* is defined as

$$\mathrm{PM} = 180^\circ - \angle G(\omega_{\rm mco}\jmath). \tag{16.3.1}$$

- The frequency $\omega_{\rm pco}$ at which the phase is 180 degrees is the *phase crossover frequency*. The *gain margin* is defined as

$$\mathrm{GM} = -20\log|G(\omega_{\rm pco}\jmath)|. \tag{16.3.2}$$

16.4 Pole at Zero

Consider

$$G(s) = \frac{1}{ms}.$$

With $s = \omega\jmath$, $G(s)$ becomes

$$G(\omega\jmath) = \frac{1}{m\omega\jmath} = \frac{-\jmath}{m\omega}.$$

Hence

$$|G(\omega\jmath)| = \frac{1}{m\omega}.$$

We plot 20log $|G(\omega\jmath)|$ versus log ω. We write

$$\begin{aligned} y &= 20\log|G(\omega\jmath)| \\ &= 20\log\left|\frac{1}{m\omega}\right| \\ &= -20\log|m\omega| \\ &= -20\log m - 20\log\omega \\ &= -20x + b, \end{aligned} \tag{16.4.1}$$

where $x \triangleq \log m$ and $b \triangleq -20\log\omega$.

Consider

$$G(\omega\jmath) = \frac{1}{m\omega\jmath} = \frac{-\jmath}{m\omega} = -\jmath\frac{1}{m\omega}. \tag{16.4.2}$$

The magnitude of $G(\omega\jmath)$ is

$$|G(\omega\jmath)| = \frac{1}{m\omega} \tag{16.4.3}$$

with phase

$$\angle G(\omega\jmath) = -90^\circ. \tag{16.4.4}$$

Figure 16.4.7 represents the phase of $G(\omega\jmath)$ on the unit circle, while Figure 16.4.8 represents $G(\omega\jmath)$ on the Bode phase plot.

The magnitude crossover frequency $\omega_{\rm mco}$ satisfies $|G(\omega_{\rm mco}\jmath)| = 1$ sec/kg. Therefore,

$$\frac{1}{m\omega_{\rm mco}} = 1\ \frac{\rm sec}{\rm kg{-}rad}, \tag{16.4.5}$$

and therefore

$$\omega_{\rm mco} = \frac{1}{m}\frac{\rm rad}{\rm sec}. \tag{16.4.6}$$

Note that the magnitude crossover frequency depends on the units chosen for time and mass, and thus is primarily useful for dimensionless transfer functions.

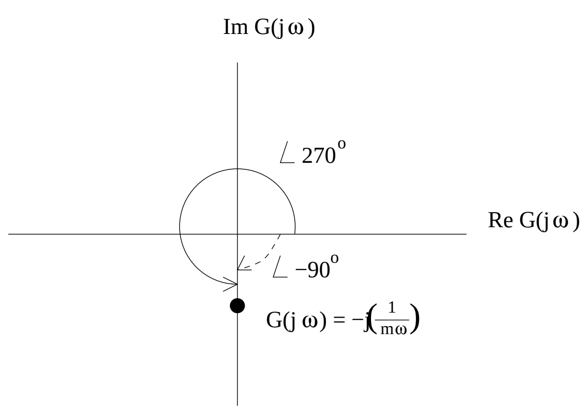


Figure 16.4.7: Phase representation on the unit circle.

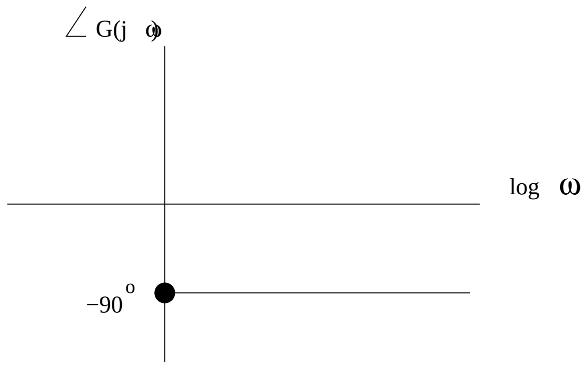


Figure 16.4.8: Phase representation in the Bode phase plot.

16.5 Real Poles

Consider

$$G(s) = \frac{\alpha}{s - p}. \tag{16.5.1}$$

Substituting $s = \omega \jmath$, we obtain

$$G(\omega \jmath) = \frac{\alpha}{\omega \jmath - p}. \tag{16.5.2}$$

Note that if $p < 0$, then G is asymptotically stable, whereas, if $p > 0$, then G is unstable. The magnitude of $G(\omega \jmath)$ is

$$|G(\omega \jmath)| = \frac{|\alpha|}{\sqrt{\omega^2 + p^2}}. \tag{16.5.3}$$

For $\omega << p$,

$$|G(\omega \jmath)| \approx \frac{|\alpha|}{|p|} \tag{16.5.4}$$

while, for $\omega >> p$,

$$|G(\omega \jmath)| \approx \frac{|\alpha|}{|\omega|}. \tag{16.5.5}$$

Now, for $\omega = |p|$, the magnitude of $G(\omega \jmath)$ becomes

$$|G(\jmath|p|)| = \frac{|\alpha|}{\sqrt{2p^2}} = \frac{\sqrt{2}}{2}\frac{|\alpha|}{|p|}. \tag{16.5.6}$$

Hence, ΔdB is found from

$$\begin{aligned}\Delta\text{dB} &= 20\log|G(\jmath|p|)| - 20\log|G(\jmath 0)| \\ &= 20\log\left(\frac{\sqrt{2}}{2}\frac{|\alpha|}{|p|}\right) - 20\log\left(\frac{|\alpha|}{|p|}\right) \\ &= 20\log\left(\frac{\sqrt{2}}{2}\right) = 20\log 2^{-1/2} = -10\log 2 = -3 \text{ dB}.\end{aligned} \tag{16.5.7}$$

Consider

$$z = a + \jmath b \tag{16.5.8}$$

as represented on Figure 16.5.9. To find the phase θ, we can write

$$\tan\theta = \frac{\operatorname{Im} z}{\operatorname{Re} z}. \tag{16.5.9}$$

Consequently,

$$\angle G(\omega\jmath) = \operatorname{atan2}(\operatorname{Im} G(\omega\jmath), \operatorname{Re} G(\omega\jmath)). \tag{16.5.10}$$

The function atan2(a,b) is discussed in Appendix C.

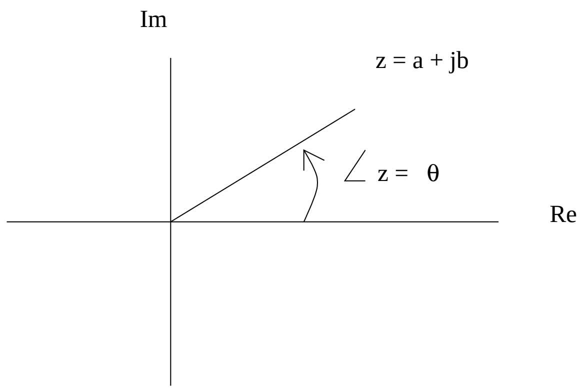


Figure 16.5.9: Phase representation in the complex plane.

16.6 Complex Poles

Consider

$$\hat{q}(s) = G(s)\hat{f}(s), \tag{16.6.1}$$

where

$$G(s) = \frac{1}{ms^2 + cs + k} = \frac{1/m}{s^2 + 2\zeta\omega_{\rm n}s + \omega_{\rm n}^2}. \tag{16.6.2}$$

Substituting $s = \omega\jmath$, yields

$$\begin{aligned} G(\omega\jmath) &= \frac{1/m}{-\omega^2 + 2\zeta\omega_{\rm n}(\omega\jmath) + \omega_{\rm n}^2} \\ &= \frac{1/m}{\omega_{\rm n}^2 - \omega^2 + (2\zeta\omega_{\rm n}\omega)\jmath} \\ &= \frac{(1/m)[\omega_{\rm n}^2 - \omega^2 - (2\zeta\omega_{\rm n}\omega)\jmath]}{(\omega_{\rm n}^2 - \omega^2)^2 + (2\zeta\omega_{\rm n}\omega)^2}. \end{aligned} \tag{16.6.3}$$

16.6.1 bode

The Matlab command for plotting the frequency response of a transfer function is given by bode(A,B,C,D), where A, B, C, D define the state equations for the linear system with the desired input variable and the desired output variable. The function can also be used in the form bode(num,den) where num and den are defined as in Subsection 15.18.3.

16.6.2 Asymptotic Phase Shift

It is useful to be able to estimate the phase shift of the transfer function at both asymptotically low and high frequency.

First, we consider the low-frequency limit. If $s = 0$ is neither a pole nor zero of G, then

$$\angle G(0) = \begin{cases} 0 \text{ deg}, & G(0) > 0, \\ 180 \text{ deg}, & G(0) < 0. \end{cases}$$

Next, if $s = 0$ is a zero of G and $G(s) = s^r \hat{G}(s)$, where s is not a zero of $\hat{G}$, then

$$\lim_{\omega \to 0^+} \angle G(\jmath\omega) = \begin{cases} 90r \text{ deg}, & \hat{G}(0) > 0, \\ -90r \text{ deg}, & \hat{G}(0) < 0. \end{cases}$$

Finally, if $s = 0$ is a pole of G and $G(s) = \hat{G}(s)/s^r$, where s is not a pole of $\hat{G}$, then

$$\lim_{\omega \to 0^+} \angle G(\jmath\omega) = \begin{cases} -90r \text{ deg}, & \hat{G}(0) > 0, \\ 90r \text{ deg}, & \hat{G}(0) < 0. \end{cases}$$

Next, we consider the high-frequency limit. In this case, we have

$$\lim_{\omega \to \infty} \angle G(\jmath\omega) = \lim_{\omega \to 0^+} \angle G(\jmath\omega) + (p_+ - p_- + z_- - z_+ + p_0 + z_0)90 \text{ deg},$$

where p_-, p_+, z_-, z_+ is the number of OLHP poles, ORHP poles, OLHP zeros, and ORHP zeros, respectively, and p_0, z_0 is the number of nonzero imaginary-axis poles and zeros, respectively.

16.7 Electrical Filter Example

A lowpass filter reduces noise at high frequency. From the impedance law, we have

$$V = IZ = ZI$$

or

$$I = \frac{V}{Z}.$$

Here $Z = Z(s)$ is a transfer function.

Example 16.7.1. Consider a circuit involve a resistor and capacitor in series with input voltage V_{in} across the both components and and output voltage V_{out} across the capacitor only. Then,

$$V_{\text{in}} - V_{\text{out}} = RI$$

and

$$V_{\text{out}} = \frac{I}{Cs} = \frac{V_{\text{in}} - V_{\text{out}}}{RCs}. \tag{16.7.1}$$

Solving for V_{out} in (16.7.1, we obtain

$$\left(1 + \frac{1}{RCs}\right) V_{\text{out}} = \frac{1}{RCs} V_{\text{in}}$$

or

$$V_{\text{out}} = \frac{1}{RCs + 1} V_{\text{in}} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} V_{\text{in}}.$$

Now, the current I is

$$I = \frac{Cs}{RCs + 1} V_{\text{in}}.$$

Hence, at high frequency,

$$I = \frac{C\omega\jmath}{RC\omega\jmath + 1} V_{\text{in}}(\omega\jmath) \approx \frac{1}{R} V_{\text{in}}(\omega\jmath).$$

16.8 Problems

Problem 16.8.1. Suppose that a sensor with first-order dynamics has a time constant of .031 sec and a DC gain of 9.2. (Hint: $G(s) = \frac{\alpha}{Ts+1}$.) Now suppose that the input to the sensor is 4.6 volts corrupted by (which means “added to”) 60-Hz electrical noise with amplitude 3.3 mV. Describe the sensor output after a large amount of time passes by giving the amplitude and phase of both of its harmonic components, that is, the DC component and the 60-Hz component. Do this two different ways. First, use Laplace transforms to determine the transient and harmonic steady-state components of the output. Next, use $G(\omega\jmath)$ to determine the magnitude and phase shift of the harmonic steady-state component of the output.

Problem 16.8.2. Suppose that a voltage amplifier with first-order dynamics $G(s) = \frac{\alpha}{Ts+1}$ has a time constant of $T = 0.037$ sec and DC gain of $\alpha = 43.1$. A sinusoidal input signal with amplitude of 2.8 volts yields, after an initial transient, a sinusoidal response with amplitude 76.2 volts. What were the frequencies of the input sinusoid and the output sinusoid?

Problem 16.8.3. A lag filter has a pole at -2, a zero at -6, and a DC gain of 10. At the frequency 4 rad/sec, what is the magnitude of the filter in dB and what is the phase of the filter in degrees? (Use just a calculator for this problem.)

Problem 16.8.4. Consider the transfer function

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_{\text{n}}s + \omega_{\text{n}}^2}$$

with $\omega_{\text{n}} = 1$ rad/sec. Use Matlab to plot the magnitude and phase Bode plots of this function for ω from .01 to 100 for $\zeta = .1, .3, .5, .7, .9$. Put all plots in the same figure.

Problem 16.8.5. Consider the transfer function

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_{\text{n}}s + \omega_{\text{n}}^2}.$$

Assume that the system is underdamped, that is, $0 < \zeta < 1$. Use calculus to determine the resonance frequency ω_r at which the magnitude $|G(j\omega)|$ is maximized, and determine $|G(j\omega_r)|$. Check whether your solution agrees with the figure from the previous problem. In addition, determine the range of values of ζ for which the magnitude of the transfer function is never greater than the value of its DC gain.

Problem 16.8.6. Sketch Bode magnitude and phase plots by hand for each of the following transfer functions. You can use Matlab to print an "empty" log-log grid for your sketch. Be sure that the range of ω is large enough to include all important features of your plots. Explain how each plot was constructed. Check your sketches by plotting with the Matlab Bode function.

i) $G(s) = \frac{s+2}{s+10}$ (lead).

ii) $G(s) = \frac{s+10}{s+2}$ (lag).

iii) $G(s) = \frac{s-2}{s+10}$.

iv) $G(s) = \frac{s-2}{s+2}$ (allpass).

v) $G(s) = \frac{s}{s+2}$ (washout).

Problem 16.8.7. For each of the following transfer functions, sketch the Bode magnitude and phase plots by hand. You can use Matlab to print an "empty" log-log grid for your sketch. Be sure that the range of ω is large enough to include all important features of your plots. Explain how each plot was constructed. Check your sketches by plotting with Matlab.

i) $G(s) = \frac{s^2}{(s+1)^2(s+10)^2}$ (rooftop).

ii) $G(s) = \frac{(s+1)(s+100)}{s(s+5)(s+10)}$.

Problem 16.8.8. Consider the asymptotically stable loop transfer function

$$L(s) = \frac{10}{s^2 + 2s + 1}.$$

Show that the closed-loop transfer function $S(s) = 1/(1 + L(s))$ is asymptotically stable, and use Matlab to plot the magnitude and phase Bode plots for S. Then, write a Matlab program to show numerically that

$$\int_0^\infty \ln |S(\omega j)|\, d\omega = 0.$$

Explain this result in terms of the "balance" between attenuation and amplification.

Problem 16.8.9. Consider the unstable loop transfer function

$$L(s) = \frac{4}{(s-1)(s+2)}.$$

Show that the closed-loop transfer function $S(s)$ is asymptotically stable, and use Matlab to plot the Bode plots for S. Then, write a Matlab program to show numerically that

$$\int_0^\infty \ln |S(\omega j)|\, d\omega = \pi.$$

Discuss the "balance" between attenuation and amplification.

Problem 16.8.10. Consider the loop transfer function

$$L(s) = \frac{2.5(s + 100)}{(s + 1)^2}.$$

Sketch the gain and phase Bode plots of $L(s)$. Use your plot to indicate the magnitude crossover frequency $\omega_{\rm mco}$, the phase crossover frequency $\omega_{\rm pco}$, the gain margin, and the phase margin.

Problem 16.8.11. Consider the damped rigid body plant

$$G(s) = \frac{1}{s(s + 1)}.$$

Then, do the following:

i) Assume $K(s) = 1$ so that $L(s) = G(s)$. Sketch the Nyquist plot and determine the gain and phase margins.

ii) Instead of unity feedback, consider the lead controller $G_{\rm c}(s) = k(s + 2)/(s + 20)$ so that $L(s) = G_{\rm c}(s)G(s)$. For $k = 1$, use Matlab to determine whether the lead provided by this lead controller increases the phase margin.

iii) Draw the root locus in terms of k.

iv) Choose k so that the complex conjugate poles have damping ratio $\zeta = \sqrt{2}/2$.

v) For the value of k that you chose, determine the asymptotic error for the unit ramp input $1/s^2$.

(Hint: You can solve the problem directly by equating the product of $(s - a)(s^2 + 2\zeta\omega_{\rm n}s + \omega_{\rm n}^2)$ with the cubic obtained from the closed loop transfer function with k as the unknown parameter. Then, you can get a cubic equation in a or $\omega_{\rm n}$.)

Problem 16.8.12. The lateral dynamics of an experimental aircraft are modeled by the transfer function $1/(\tau s + 1)$, where $\tau > 0$ is a time constant. For this transfer function a basic servo loop is closed with the integral controller $K_{\rm I}/s$.

i) Determine the values of $K_{\rm I}$ for which the closed-loop system is asymptotically stable.

ii) For which values of $K_{\rm I}$ is the asymptotic error to a unit-slope ramp command less than 0.025?

iii) For a value of $K_{\rm I}$ such that the closed-loop system is asymptotically stable, determine the amplitude of the harmonic steady-state component of the error due to the command $r(t) = r_0 \cos(\omega t)$.

Problem 16.8.13. At a given Mach number, the open-loop longitudinal dynamics of an experimental aircraft are given by $L(s) = 1/[s(s + 1)^2]$.

i) Sketch the Bode plot of L (magnitude and phase plots).

ii) Determine the phase crossover frequency $\omega_{\rm pco}$ and the gain margin in dB of the closed-loop system and illustrate them on the Bode plot. (Hint: $\log_{10} 2 \approx 0.3$.)

iii) Sketch the Nyquist plot of L.

iv) Indicate the phase crossover frequency $\omega_{\rm pco}$ and the gain margin on the Nyquist plot. Be sure that what you show on the Nyquist plot is consistent with the Bode plot.

Problem 16.8.14. At a given Mach number, the open-loop longitudinal dynamics of an unstable experimental aircraft are given by

$$L(s) = \frac{4(s+10)}{(s-1)(s-2)}.$$

i) Sketch the Bode plot of L (magnitude and phase plots).

ii) Sketch the Nyquist plot of L.

iii) Apply the Nyquist test to this system and use it to assess closed-loop stability.

Chapter Seventeen

Solutions to Chapter 15

Problem 15.20.1. Consider the 3×3 matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 4 & 3 & 5 \\ 7 & 2 & 6 \end{bmatrix}.$$

Manually compute the determinant and inverse of A. Then compute these quantities using Matlab to check your answer.

Solution 15.20.1.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 \\ 4 & 3 & 5 \\ 7 & 2 & 6 \end{bmatrix}$$

Then, its determinant is

$$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 3(3 \cdot 6 - 2 \cdot 5) - 0(4 \cdot 6 - 7 \cdot 5) + 2(4 \cdot 2 - 7 \cdot 3) \\ &= -2 \end{aligned}$$

where C_{ij} is the cofactor associated with its corresponding element. And its inverse is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} -4 & -2 & 3 \\ -5.5 & -2 & 3.5 \\ 6.5 & 3 & -4.5 \end{bmatrix}$$

Checking with Matlab,

```
>> A = [3 0 2; 4 3 5; 7 2 6]

A =

     3     0     2
     4     3     5
     7     2     6

>> detA = det(A)
```

```
detA =

    -2

>> invA = inv(A)

invA =

   -4.0000   -2.0000    3.0000
   -5.5000   -2.0000    3.5000
    6.5000    3.0000   -4.5000
```

Problem 15.20.2. Use Matlab to compute the eigenvalues of the matrix A in Problem 15.20.1. Then use Matlab to show that the determinant is the product of the eigenvalues, and show that the trace (the sum of the diagonal entries) is the sum of the eigenvalues. Furthermore, use Matlab to compute the eigenvalues of A^2 and A^{-1}, and discuss how they are related to the eigenvalues of A.

Solution 15.20.2.
The eigenvalues of A are

```
>> evalue_of_A = eig(A)
    evalue_of_A =  {9.8467, 2.2438, -0.0905}
```

The product of the eigenvalues is

```
>> prod(evalue_of_A)
    ans = -2.0000
```

which is equal to the determinant of A computed in the Solution 15.20.1. The sum of eigenvalues is

```
>> sum(evalue_of_A)
    ans = 12.0000
```

which is equal to $\mathrm{tr}A = 3 + 3 + 6 = 12$. The trace of A can also be computed using the following in MATLAB

```
>> trace(A)
    ans = 12.0000
```

The eigenvalues of A^2 and A^{-1} are respectively,

```
>> eig(A^2)
    ans =  {96.9570, 5.0349, 0.0082}

>> eig(inv(A))
    ans =   {-11.0472, 0.4457, 0.1016}
```

Each eigenvalue of A^2 is the square of each eigenvalue of A. Also, each eigenvalue of A^{-1} is the reciprocal of each eigenvalue of A. Checking with Matlab,

```
>> evalue_of_A
    ans = {10.6031, 0.1515, 1.2454}

>> evalue_of_A.^2
    ans = {96.9570, 5.0349, 0.0082}

>> 1./evalue_of_A
    ans = {-11.0472, 0.4457, 0.1016}
```

These relations can be generalized and analytically proven with the aid of Jordan canonical decomposition (or simply diagonalizaton if all of the eigenvalues are distinct like A above).

Problem 15.20.3. Using the "randn" command in Matlab, form a random 4×2 matrix A. Then compute the 4×4 matrix AA^{T}. Using Matlab to compute the eigenvalues of the symmetric matrix AA^{T}, check whether this matrix is positive semidefinite. Then, show mathematically (by hand, not using Matlab) that $x^{\mathrm{T}}AA^{\mathrm{T}}x \geq 0$ for all vectors x. (Hint: Define $z = A^{\mathrm{T}}x$.)

Solution 15.20.3.

```
>> A = rand(4,2)
A =

    0.9501    0.8913
    0.2311    0.7621
    0.6068    0.4565
    0.4860    0.0185

>> A*A'

ans =

    1.6972    0.8989    0.9834    0.4782
    0.8989    0.6342    0.4881    0.1264
    0.9834    0.4881    0.5766    0.3034
    0.4782    0.1264    0.3034    0.2365

>> eig(A*A')
    ans = {0.0000    0.0000    0.2632    2.8813}
```

All the eigenvalues of symmetric matrix AA^{T} are greater than or equal to zero, which implies that it is positive semidefinite. Finally,

$$x^{\mathrm{T}}AA^{\mathrm{T}}x = (x^{\mathrm{T}}A)(A^{\mathrm{T}}x) = (A^{\mathrm{T}}x)^{\mathrm{T}}(A^{\mathrm{T}}x) = z^{\mathrm{T}}z \geq 0$$

Note $z^{\mathrm{T}}z = \sum z_i^2$.
Since this inequality holds for all vectors x, it completes the proof.

Problem 15.20.4. Let A be an $n \times n$ matrix and let $p(s) = \det(sI - A)$ be the characteristic polynomial of A. The Cayley-Hamilton theorem states that $p(A) = 0$. Check this fact by obtaining the characteristic polynomial $p(s)$ for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix},$$

and then showing that $p(A) = 0$. Repeat these steps for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}.$$

Do all of these symbolic calculations by hand.

Solution 15.20.4.
The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$$

is

$$p(s) = |sI - A| = \begin{vmatrix} s & -1 \\ a_0 & s + a_1 \end{vmatrix} = s^2 + a_1 s + a_0.$$

To verify the Cayley-Hamilton theorem, we evaluate

$$p(A) = A^2 + a_1 A + a_0 I_2$$

where I_2 is the 2×2 identity matrix. This is evaluated to be

$$p(A) = \begin{bmatrix} -a_0 & -a_1 \\ a_0 a_1 & -a_0 + a_1^2 \end{bmatrix} + a_1 \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} + a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, A is seen to be a root of its characteristic polynomial.

For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix},$$

the characteristic polynomial is

$$p(s) = |sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_0 & a_1 & s + a_2 \end{vmatrix} = s^3 + a_2 s^2 + a_1 s + a_0.$$

We evaluate $\left[p(A) = A^3 + a_2 A^2 + a_1 A + a_0 I_3\right]$ where I_3 is the 3×3 identity matrix. Substituting for A, we obtain

$$\begin{aligned} p(A) = &\begin{bmatrix} -a_0 & -a_1 & -a_2 \\ a_0 a_2 & -a_0 + a_1 a_2 & -a_1 + a_2^2 \\ a_0 a_1 - a_0 a_2^2 & a_1^2 - a_2(-a_0 + a_1 a_2) & -a_0 + a_1 a_2 - a_2(-a_1 + a_2^2) \end{bmatrix} \\ &+ a_2 \begin{bmatrix} 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \\ a_0 a_2 & -a_0 + a_1 a_2 & -a_1 + a_2^2 \end{bmatrix} + a_1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \end{aligned}$$

$$+ a_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which verifies the Cayley-Hamilton theorem.

Problem 15.20.5. The eigenvalues of a matrix are the roots of its characteristic polynomial. Consider the 3×3 matrix in Problem 15.20.4 with $a_0 = -2$, $a_1 = 5$, and $a_2 = 3$, and use Matlab to show numerically that the roots of $p(s)$ are indeed the eigenvalues of A. Use roots(p) and eig(A) for your computations.

Solution 15.20.5.
Substituting for a_0, a_1, and a_2, we obtain

$$p(s) = s^3 + 3s^2 + 5s - 2 \text{ for } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & -3 \end{bmatrix}.$$

The roots of A are

```
>> roots([1 3 5 -2])

ans =

  -1.6641 + 1.8230i
  -1.6641 - 1.8230i
   0.3283
```

Also, the eigenvalues of A are given by

```
>> A = [ 0 1 0 ; 0 0 1 ; 2 -5 -3 ]
>> eig(A)

ans =

   0.3283
  -1.6641 + 1.8230i
  -1.6641 - 1.8230i
```

which numerically verifies that the roots of $p(s)$ are indeed the eigenvalues of A.

Problem 15.20.6. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal. Also, check whether the matrix

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

is orthogonal. Determine by hand (not by Matlab) the determinants and eigenvalues of these matrices. Finally, choose several values of θ and multiply the vector $[1 \ \ 1]^{\mathrm{T}}$ by these matrices. Discuss how the resulting vectors compare to the original vector in terms of their length and direction. You can use a calculator but do not use Matlab.

Solution 15.20.6.
For the first matrix,

$$\begin{aligned} AA^{\mathrm{T}} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

which verifies that A is orthogonal by definition. The determinant of A is

$$\det\mathrm{A} = \cos^2\theta + \sin^2\theta = 1$$

The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$. The roots of the polynomial are $\lambda = \cos\theta \pm \jmath \sin\theta$.
Similarly for the second matrix,

$$\begin{aligned} AA^{\mathrm{T}} &= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

which verifies that A is orthogonal by definition. Its determinant is

$$\det\mathrm{A} = -\cos^2\theta - \sin^2\theta = -1$$

The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$. The roots of the polynomial are $\lambda = \pm 1$.

Now let $\theta = 76^\circ = 1.3265$ rad. Then, for the first matrix,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2419 & 0.9703 \\ -0.9703 & 0.2419 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2122 \\ -0.7284 \end{bmatrix},$$

which shows that the given vector is rotated by 76° clockwise. Similarly for the second matrix,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2419 & 0.9703 \\ 0.9703 & -0.2419 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2122 \\ 0.7284 \end{bmatrix},$$

which shows that the original vector is rotated by 76° clockwise followed by mirroring about the x-axis while keeping the magnitude.

Problem 15.20.7. Use the dot product to compute the angle between the vectors $[3\ \ 2]^{\mathrm{T}}$ and $[-2\ \ 4]^{\mathrm{T}}$.

Solution 15.20.7.
We use the definition of dot product:

$$x \cdot y = |x||y| \cos\theta$$

Let $x = [3\ \ 2]^{\mathrm{T}}$ and $y = [-2\ 4]^{\mathrm{T}}$. Then, the angle between these vectors is

$$\cos\theta = \frac{x \cdot y}{|x||y|} = \frac{3 \cdot (-2) + 2 \cdot 4}{\sqrt{3^2 + 2^2}\,\sqrt{(-2)^2 + 4^2}} = 0.1240$$

$$\Rightarrow \theta = \arccos 0.1240 = 1.4464\text{rad} = 82.875\text{deg}$$

Problem 15.20.8. Solve the differential equation

$$\dot{x}(t) = ax(t) + b$$

analytically by evaluating the convolution integral expression given by (15.8.3). Under what conditions does $\lim_{t\to\infty} x(t)$ exist? When the limit exists, determine the limiting value. How could you guess the limiting value without solving the equation?

Solution 15.20.8.
Let $u = 1$, $A = a$, and $B = b$. Substituting these into the convolution integral expression yields,

$$\begin{aligned} x(t) &= e^{at}x_0 + \int_0^t e^{a(t-\tau)} b d\tau \\ &= e^{at}x_0 - \frac{b}{a} e^{a(t-\tau)}|_0^t \\ &= e^{at}x_0 - \frac{b}{a}\left(1 - e^{at}\right) \\ &= e^{at}\left(\frac{b}{a} + x_0\right) - \frac{b}{a}. \end{aligned}$$

The limiting value exist if and only if a is negative and the limiting value is $\lim_{t\to\infty} x(t) = -b/a$. Since the limiting value of $x(t)$ is the value of $x(t)$ when $\dot{x}(t) = 0$, it can be guessed from the differential equation that $\lim_{t\to\infty} x(t) = -b/a$.

Problem 15.20.9. Using the solution to Problem 15.20.8, write down the solution to the scalar ordinary differential equation

$$\dot{x}(t) = -2x(t) + 8.$$

This equation represents the step response of a linear system, where the constant 8 is the value of the step input. Plot the solutions for the initial conditions $x(0) = 5$ and $x(0) = -4$ in the same figure. Be sure to label all axes of your figure and give it an appropriate caption. Determine $\lim_{t\to\infty} x(t)$ from the plot and compare that numerical value with the analytical limit. Explain how the limiting value of $x(t)$ depends on the constants in the problem, namely, the coefficient of $x(t)$, the value of the input step, and the initial condition.

Solution 15.20.9.
Let $a = -2$ and $b = 8$. Substituting these into the solution (from Problem 15.20.8.) yields,

$$x(t) = e^{-2t}(-4 + x_0) + 4$$

The limiting value is, $\lim_{t\to\infty} x(t) = 4$.

For $x(0) = 5$ it yields,

$$x(t) = e^{-2t} + 4.$$

For $x(0) = -4$ it yields,

$$x(t) = -8e^{-2t} + 4.$$

The plot is shown in Figure A.0.1.

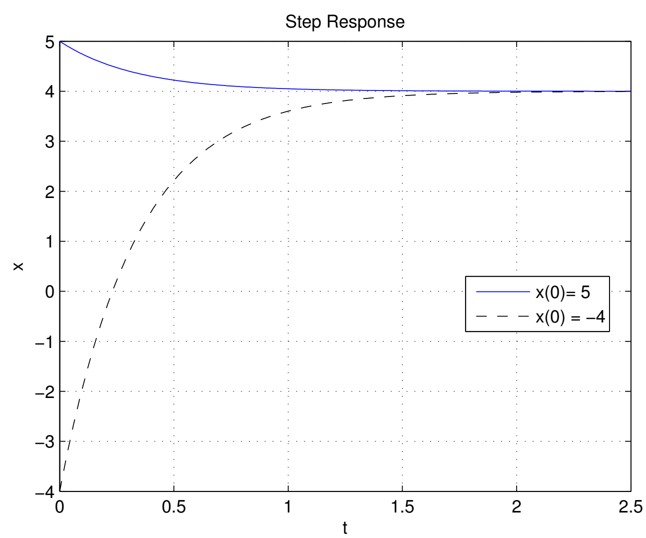


Figure 17.0.1: Problem 15.20.9 Step Response

The limiting value can be determined from Figure 1 as $\lim_{t\to\infty} x(t) = 4$. This value matches the analytical limit found earlier.

The M-file used to create the plot is given below.

```
<HWChapterLinSysProblem9.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem9 Soln

clear all; close all;

t = 0:0.01:2.5;

xa = exp(-3*t)+ 4; xb = -8*exp(-3*t)+ 4;

figure(1);
plot(t,xa,'b-',t,xb,'k--');
xlabel('t');
ylabel('x');
title('Step Response');
legend('x(0) = 5','x(0) = -4','location','best');
grid on;
```

Problem 15.20.10. Consider $\dot{x} = Ax$, and let $\lambda = \sigma + \jmath\omega$ be a complex eigenvalue of A with associated complex eigenvector

$$v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \jmath \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Consider three different eigenvalues (of different matrices A) given by $\omega = 1$ and $\sigma = -1, 0, 1$. Then, for each value of σ, plot the eigensolutions

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathrm{Re}(e^{\lambda t} v)$$

in the x_1, x_2 plane, where the curve is parameterized by t. Use arrows to denote how the solution evolves as t increases, and explain how the properties of the eigensolution depend on σ.

Solution 15.20.10.
We use the eigensolution

$$x(t) = \mathrm{Re}(e^{\lambda t} v)$$

Let $x(t) = [x_1(t)\ x_2(t)]^T$. Figures 17.0.2 – 17.0.6 show the time responses of x(t) for each eigenvalue. It is clear that the time response

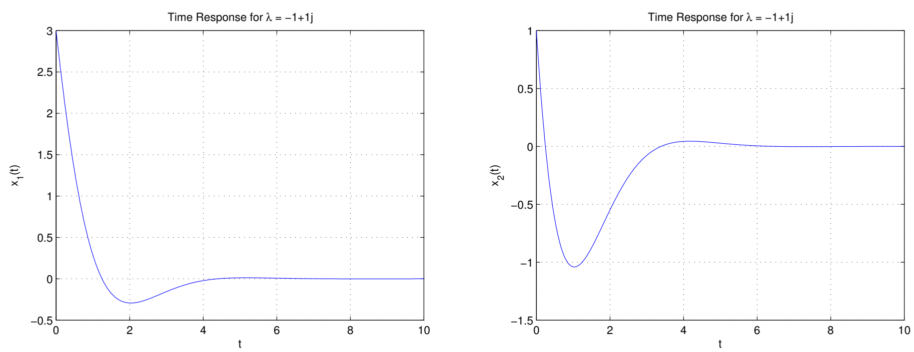


Figure 17.0.2: Problem 15.20.10. Time Responses of Eigensolution with $\lambda = -1 + 1\jmath$

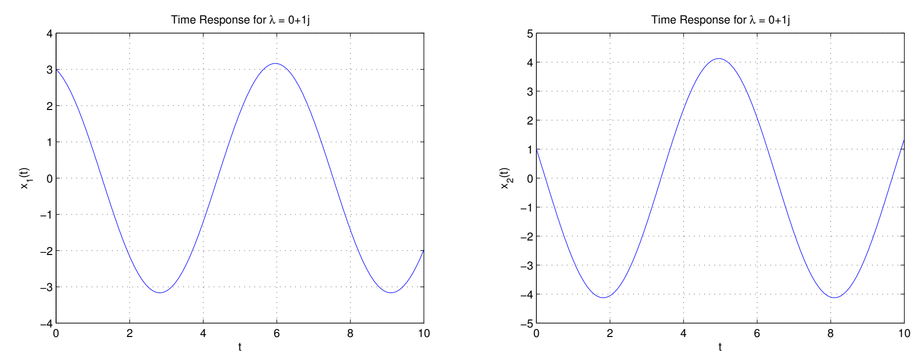


Figure 17.0.3: Problem 15.20.10. Time Responses of Eigensolution with $\lambda = 0 + 1\jmath$

- converges if $\mathrm{Re}\,\lambda < 0$

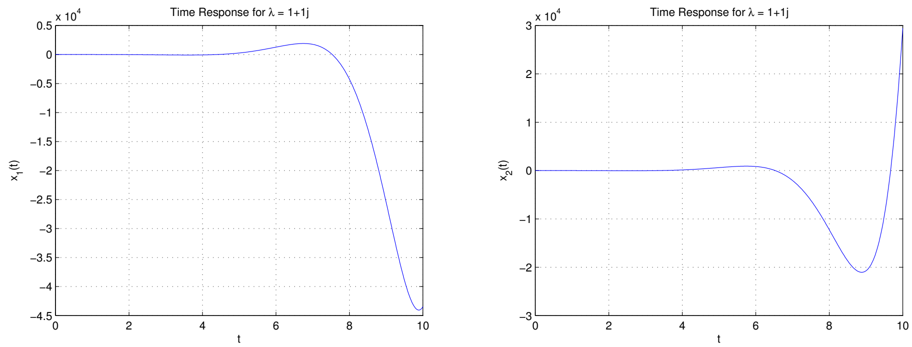


Figure 17.0.4: Problem 15.20.10. Time Responses of Eigensolution with $\lambda = 1 + 1\jmath$

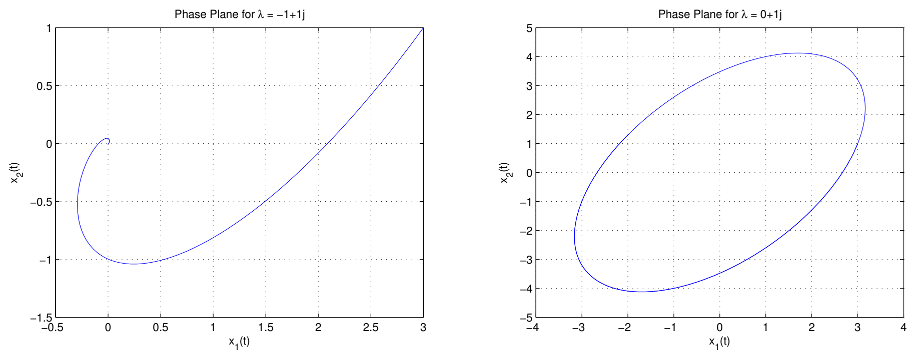


Figure 17.0.5: Problem 15.20.10. Phase Planes for Eigensolution

- oscillates if $\mathrm{Re}\,\lambda = 0$
- diverges if $\mathrm{Re}\,\lambda > 0$.

The M-file used to create the plots is given below

<HWChapterLinSysProblem10.m>

```
% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem10 Soln

clear all; home; close all;

v = [3 1]' + j*[ 1 4]'; lam1 = -1+j; lam2 = 0+j; lam3 = 1+j;

t=0:0.01:10; for ii=1:length(t)
    x1(:,ii) = real(exp(lam1*t(ii))*v);
    x2(:,ii) = real(exp(lam2*t(ii))*v);
    x3(:,ii) = real(exp(lam3*t(ii))*v);
end

figure(1); plot(t,x1(1,:),'-'); xlabel('t'); ylabel('x_1(t)'); grid
```

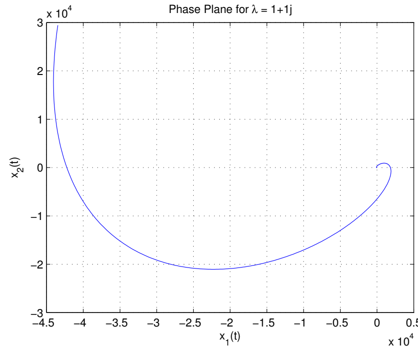


Figure 17.0.6: Problem 15.20.10. Phase Plane for Eigensolution

```
title('Time Response for \lambda = -1+1j')

figure(2); plot(t,x1(2,:),'-'); xlabel('t'); ylabel('x_2(t)'); grid
title('Time Response for \lambda = -1+1j')

figure(3); plot(t,x2(1,:),'-'); xlabel('t'); ylabel('x_1(t)'); grid
title('Time Response for \lambda = 0+1j')

figure(4); plot(t,x2(2,:),'-'); xlabel('t'); ylabel('x_2(t)'); grid
title('Time Response for \lambda = 0+1j')

figure(5); plot(t,x3(1,:),'-'); xlabel('t'); ylabel('x_1(t)'); grid
title('Time Response for \lambda = 1+1j')

figure(6); plot(t,x3(2,:),'-'); xlabel('t'); ylabel('x_2(t)'); grid
title('Time Response for \lambda = 1+1j')

figure(7); plot(x1(1,:),x1(2,:),'-'); xlabel('x_1(t)');
ylabel('x_2(t)'); grid on; title('Phase Plane for \lambda = -1+1j')

figure(8); plot(x2(1,:),x2(2,:),'-'); xlabel('x_1(t)');
ylabel('x_2(t)'); grid on; title('Phase Plane for \lambda = 0+1j')

figure(9); plot(x3(1,:),x3(2,:),'-'); xlabel('x_1(t)');
ylabel('x_2(t)'); grid on; title('Phase Plane for \lambda = 1+1j')
```

Problem 15.20.11. Let $\lambda, \bar{\lambda} = -\zeta\omega_{\mathrm{n}} \pm \jmath\omega_{\mathrm{n}}\sqrt{1-\zeta^2}$ denote a complex conjugate pair of underdamped eigenvalues. Show that

$$\omega_{\mathrm{n}} = |\lambda|, \quad \zeta = -\frac{\lambda + \bar{\lambda}}{2|\lambda|}, \quad \omega_{\mathrm{d}} = \frac{\lambda - \bar{\lambda}}{2\jmath}.$$

Solution 15.20.11.

Substituting λ into the right hand side we get,

$$\begin{aligned}
|\lambda| &= \sqrt{(-\zeta\omega_{\mathrm{n}})^2 + (\omega_{\mathrm{n}}\sqrt{1-\zeta^2})^2} \\
\Rightarrow |\lambda| &= \sqrt{(\zeta^2\omega_{\mathrm{n}}^2) + (\omega_{\mathrm{n}}^2(1-\zeta^2))} \\
\Rightarrow |\lambda| &= \omega_{\mathrm{n}}\sqrt{\zeta^2 + 1 - \zeta^2} \\
\Rightarrow |\lambda| &= \omega_{\mathrm{n}}
\end{aligned}$$

Next, substituting λ and $\bar{\lambda}$ into the right hand side we get,

$$\begin{aligned}
-\frac{\lambda + \bar{\lambda}}{2|\lambda|} &= -\frac{-2\zeta\omega_{\mathrm{n}}}{2\omega_{\mathrm{n}}} \\
\Rightarrow -\frac{\lambda + \bar{\lambda}}{2|\lambda|} &= \zeta
\end{aligned}$$

Once again, substituting λ and $\bar{\lambda}$ into the right hand side we get,

$$\begin{aligned}
\frac{\lambda - \bar{\lambda}}{2\jmath} &= \frac{2\jmath\omega_{\mathrm{n}}\sqrt{1-\zeta^2}}{2\jmath} \\
\Rightarrow \frac{\lambda - \bar{\lambda}}{2\jmath} &= \omega_{\mathrm{n}}\sqrt{1-\zeta^2} \\
\Rightarrow \frac{\lambda - \bar{\lambda}}{2\jmath} &= \omega_{\mathrm{d}}
\end{aligned}$$

Note that $\omega_{\mathrm{d}} \triangleq \omega_{\mathrm{n}}\sqrt{1-\zeta^2}$.

Problem 15.20.12. Let $\lambda_1, \lambda_2 = -\zeta\omega_\mathrm{n} \pm \omega_\mathrm{n}\sqrt{\zeta^2 - 1}$ denote a pair of overdamped eigenvalues. Show that

$$\omega_\mathrm{n} = \sqrt{\lambda_1\lambda_2} \tag{17.0.1}$$

and

$$\zeta = -\frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1\lambda_2}}. \tag{17.0.2}$$

Solution 15.20.12.
First, substitute λ_1 and λ_2 into the right side,

$$\begin{aligned}
&\sqrt{\lambda_1\lambda_2} = \sqrt{\lambda_1\lambda_2} \\
\Rightarrow\ &\sqrt{\lambda_1\lambda_2} = \sqrt{(-\zeta\omega_\mathrm{n} + \omega_\mathrm{n}\sqrt{\zeta^2 - 1})(-\zeta\omega_\mathrm{n} - \omega_\mathrm{n}\sqrt{\zeta^2 - 1})} \\
\Rightarrow\ &\sqrt{\lambda_1\lambda_2} = \sqrt{\zeta^2\omega_\mathrm{n}^2 - \omega_\mathrm{n}^2(\zeta^2 - 1)} \\
\Rightarrow\ &\sqrt{\lambda_1\lambda_2} = \omega_\mathrm{n}\sqrt{\zeta^2 - (\zeta^2 - 1)} \\
\Rightarrow\ &\sqrt{\lambda_1\lambda_2} = \omega_\mathrm{n}
\end{aligned}$$

Once again, substitute λ_1 and λ_2 into the right side and simplify the expression,

$$\begin{aligned}
&-\frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1\lambda_2}} = -\frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1\lambda_2}} \\
\Rightarrow\ &-\frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1\lambda_2}} = \frac{2\zeta\omega_\mathrm{n}}{2\omega_\mathrm{n}} \\
\Rightarrow\ &-\frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1\lambda_2}} = \zeta
\end{aligned}$$

Problem 15.20.13. Use l'Hopital's rule to derive (15.15.9) from (15.15.8).

Solution 15.20.13.
Knowing that $\omega_d = \sqrt{1-\zeta^2}\omega_n$, let

$$f(\zeta) = \frac{f_0}{m} e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\omega_n t), \tag{17.0.3}$$

$$g(\zeta) = \sqrt{1-\zeta^2}\omega_n. \tag{17.0.4}$$

Using l'Hopital's rule

$$\lim_{\zeta\to 1} \frac{f(\zeta)}{g(\zeta)} = \lim_{\zeta\to 1} \frac{f'(\zeta)}{g'(\zeta)} \tag{17.0.5}$$

$$= \lim_{\zeta\to 1} \frac{f_0}{m} \frac{-\omega_n t e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\omega_n t) + e^{-\zeta\omega_n t} \frac{-\zeta\omega_n t}{\sqrt{1-\zeta^2}} \cos(\sqrt{1-\zeta^2}\omega_n t)}{\frac{-\zeta\omega_n}{\sqrt{1-\zeta^2}}} \tag{17.0.6}$$

$$= \frac{f_0}{m} \lim_{\zeta\to 1} \frac{-\omega_n t e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\omega_n t)}{\frac{-\zeta\omega_n}{\sqrt{1-\zeta^2}}} + e^{-\zeta\omega_n t} t \cos(\sqrt{1-\zeta^2}\omega_n t) \tag{17.0.7}$$

$$= \frac{f_0}{m} t e^{-\zeta\omega_n t}. \tag{17.0.8}$$

Problem 15.20.14. Consider the damped oscillator (DO) with acceleration output. Write this system in A, B, C, D form, where A is of size 2×2.

Solution 15.20.14.
A damped oscillator is described by

$$m\ddot{q} + c\dot{q} + kq = F.$$

We need to write this system in the form,

$$\begin{aligned} \dot{x}(t) &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where u is the input and y is the output.

We first transform the 2nd order system into state space form.
Let $x = [x_1 \ x_2]^T = [q \ \dot{q}]^T$. Then,

$$\begin{aligned} \dot{x}_1 &= \dot{q} = x_2 \\ \dot{x}_2 &= \ddot{q} = -\frac{k}{m}q - \frac{c}{m}\dot{q} = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}F \end{aligned}$$

Re-writing into the matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F \Leftrightarrow \dot{x} = Ax + Bu$$

The system has an acceleration output, so

$$y = \ddot{q} = -\frac{k}{m}q - \frac{c}{m}\dot{q} = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}F$$

Re-writing into the matrix form,

$$y = \begin{bmatrix} -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{m}F \Leftrightarrow y = Cx + Du$$

So we have,

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\ C &= \begin{bmatrix} -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \\ D &= \begin{bmatrix} \frac{1}{m} \end{bmatrix} \end{aligned}$$

Problem 15.20.15. Use the initial value theorem to determine the initial value of each of the following functions: $y(t) = t$, $y(t) = t + 1$, $y(t) = \sin 2t$, and $y(t) = \cos 2t$.

Solution 15.20.15.
The initial value theorem states that

$$y(0^+) = \lim_{s\to\infty} s\hat{y}(s), \tag{17.0.9}$$

if the limit $\lim_{s\to\infty} s\hat{y}(s)$ exists, where $\hat{y}(s)$ is the Laplace transform of $y(t)$.

- $y(t) = t \Rightarrow \hat{y}(s) = \frac{1}{s^2}$. Hence, $y(0^+) = \lim_{s\to\infty} s\frac{1}{s^2} = 0$.
  - Setting $t = 0$ yields $y(0) = 0$, which confirms the initial value theorem.
- $y(t) = t + 1 \Rightarrow \hat{y}(s) = \frac{1}{s^2} + \frac{1}{s}$. Hence, $y(0^+) = \lim_{s\to\infty} s\left(\frac{1}{s^2} + \frac{1}{s}\right) = \lim_{s\to\infty}\left(\frac{1}{s} + 1\right) = 1$.
  - Setting $t = 0$ yields $y(0) = 1$.
- $y(t) = \sin 2t \Rightarrow \hat{y}(s) = \frac{2}{s^2+4}$. Hence, $y(0^+) = \lim_{s\to\infty} s\left(\frac{2}{s^2+4}\right) = \lim_{s\to\infty} \frac{2/s}{1+(4/s^2)} = 0$.
  - Setting $t = 0$ yields $y(0) = \sin 0 = 0$.
- $y(t) = \cos 2t \Rightarrow \hat{y}(s) = \frac{s}{s^2+4}$. Hence, $y(0^+) = \lim_{s\to\infty} s\left(\frac{s}{s^2+4}\right) = \lim_{s\to\infty} \frac{1}{1+(4/s^2)} = 1$.
- Setting $t = 0$ yields $y(0) = \cos 0 = 1$.

Problem 15.20.16. Use the initial value theorem to determine the initial slopes of the functions in the previous problem.

Solution 15.20.16.
From the initial value theorem, we know that

$$y'(0^+) = \lim_{s\to\infty} s(s\hat{y}(s) - y(0)), \tag{17.0.10}$$

if the limit $\lim_{s\to\infty} s^2\hat{y}(s)$ exists, where $\hat{y}(s)$ is the Laplace transform of $y(t)$.

- $y(t) = t \Rightarrow \hat{y}(s) = \frac{1}{s^2}$. Hence, $y'(0^+) = \lim_{s\to\infty} s^2\frac{1}{s^2} = 1$.
  - Since $y'(t) = 1$, setting $t = 0$ yields $y'(0) = 1$, which confirms the initial value theorem.
- $y(t) = t + 1 \Rightarrow \hat{y}(s) = \frac{1}{s^2} + \frac{1}{s}$. Hence, $y'(0^+) = \lim_{s\to\infty} s\left(s\left(\frac{1}{s^2} + \frac{1}{s}\right) - 1\right) = 1$.
  - Since $y'(t) = 1$, setting $t = 0$ yields $y'(0) = 1$.
- $y(t) = \sin 2t \Rightarrow \hat{y}(s) = \frac{2}{s^2+4}$. Hence, $y'(0^+) = \lim_{s\to\infty} s^2\left(\frac{2}{s^2+4}\right) = \lim_{s\to\infty} \frac{2}{1+(4/s^2)} = 2$.
  - Since $y'(t) = 2\cos 2t$, setting $t = 0$ yields $y'(0) = 2\cos 0 = 2$.
- $y(t) = \cos 2t \Rightarrow \hat{y}(s) = \frac{s}{s^2+4}$. Hence, $y'(0^+) = \lim_{s\to\infty} s\left(s\left(\frac{s}{s^2+4}\right) - 1\right) = 0$.
  - Since $y'(t) = 2\sin 2t$, setting $t = 0$ yields $y'(0) = 2\sin 0 = 0$.

Problem 15.20.17. Use Laplace transforms to analytically determine the response of $\dot{v} + 2v = \sin 5t$ for an arbitrary initial condition $v(0)$. Then show that you can choose a special initial condition $v(0)$ so that the response is exactly harmonic, that is, there is no transient (non-harmonic) component of the solution. Finally, confirm your answer by using ODE45 to simulate the system with this special initial condition as well as another initial condition.

Solution 15.20.17.
Taking into account the non-zero initial conditions, the Laplace transform of $\dot{v} + 2v = \sin 5t$ is given below

$$
\begin{aligned}
(s+2)\hat{v}(s) &= v(0) + \frac{5}{s^2+25} \\
\Rightarrow \hat{v}(s) &= \frac{v(0)}{s+2} + \frac{5}{(s+2)(s^2+25)}
\end{aligned}
$$

Using partial fraction decomposition, it yields

$$
\begin{aligned}
\hat{v}(s) &= \frac{v(0)}{s+2} + \frac{5}{(s+2)(s^2+25)} \\
\Rightarrow \hat{v}(s) &= \frac{v(0)}{s+2} + \frac{5}{29(s+2)} - \frac{5s-10}{29(s^2+25)} \\
\Rightarrow \hat{v}(s) &= \frac{v(0)}{s+2} + \frac{5}{29(s+2)} - \frac{5s}{29(s^2+25)} + \frac{10}{29(s^2+25)}
\end{aligned}
$$

The response is then obtained by taking the inverse Laplace transformation, which yields

$$\left[v(t) = \left(v(0) + \frac{5}{29} \right) e^{-2t} - \frac{5}{29}\cos 5t + \frac{2}{29}\sin 5t. \right]$$

If the initial condition is chosen to be $v(0) = -5/29$, then the exponential term is cancelled out, which yields

$$\left[v(t) = -\frac{5}{29}\cos 5t + \frac{2}{29}\sin 5t. \right]$$

Apparently, now the response is harmonic. Figure 17.0.7 compares the exact harmonic response with the response including a non-harmonic component.

The M-file used to create plots is given below

```
<HWChapterLinSysProblem17.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem17 Soln

function HWChapterLinSysProblem17

v01    = -5/29;   %yields only harmonic response
v02    = 1/5;     %choose something other than -5/29
tspan  = 0:0.1:5; %timespan for ode solution

% Solution using ODE45
[t1,v1] = ode45(@f,tspan,v01);
[t2,v2] = ode45(@f,tspan,v02);
```

```
% Plot
figure(1);clf;
hold on
plot(t1,v1,'b')
plot(t1,v2,'r--')
xlabel('Time (s)')
ylabel('v(t)')
legend('v(0) = -5/29','v(0) = 1/5')

end

function vdot = f(t,v)
vdot = -2*v + sin(5*t);
end
```

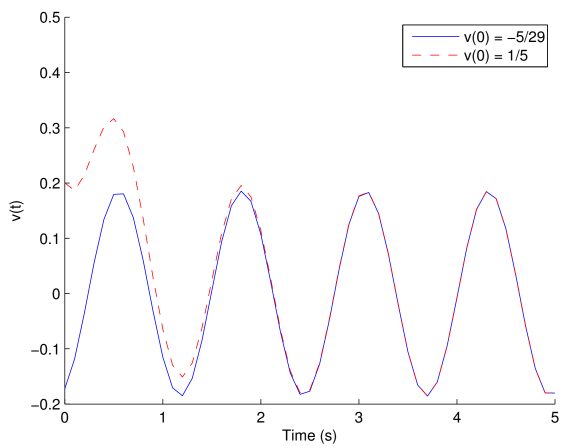


Figure 17.0.7: Problem 15.20.17. Response of the System

Problem 15.20.18. A motor with constant applied torque is modeled as the damped rigid body $J\ddot{\theta} + c\dot{\theta} = \tau_0 \mathbf{1}(t)$, where J is the load inertia, c is the viscous damping coefficient, and τ_0 is the moment. The initial angle $\theta(0)$ and initial angular rate $\dot{\theta}(0)$ are zero. Use Laplace transforms and the final value theorem to determine the terminal angular rate $\lim_{t\to\infty} \dot{\theta}(t)$. Also compute the same limit by using the time-domain solution obtained from Laplace transforms.

Solution 15.20.18.
Let $v(t) = \dot{\theta}(t)$. Then the given equation is expressed as

$$J\dot{v}(t) + cv = \tau_0 1(t)$$

Taking Laplace transforms on both sides of the equation with zero initial conditions, we get the forced response given by

$$\begin{aligned}
(Js + c)\hat{v}(s) &= \tau_0/s \\
\Rightarrow \hat{v}(s) &= \frac{\tau_0}{s(Js + c)} \\
\Rightarrow \hat{v}(s) &= \frac{\tau_0}{cs} - \frac{J\tau_0}{c(Js + c)}
\end{aligned}$$

The inverse transform then yields the time domain solution for the velocity

$$v(t) = \frac{\tau_0}{c} - \frac{\tau_0}{c} e^{-\frac{c}{J}t}.$$

Hence, the terminal velocity is given by

$$\lim_{t\to\infty} v(t) = \frac{\tau_0}{c}.$$

The final value theorem provides the same result:

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} s\hat{v}(s) = \frac{\tau_0}{c}$$

Also this result can be guessed from the given equation as follows: if there is a steady-state terminal velocity, then the acceleration $\dot{v} \to 0$ as $t \to \infty$. Thus,

$$\begin{aligned}
\lim_{t\to\infty}(J\dot{v}(t) + cv(t)) &= \lim_{t\to\infty} \tau_0 \\
\Rightarrow \lim_{t\to\infty} cv(t) &= \tau_0 \\
\Rightarrow \lim_{t\to\infty} v(t) &= \frac{\tau_0}{c}.
\end{aligned}$$

Problem 15.20.19. For

$$\hat{y}(s) = \frac{1}{s(s^2 + s + 1)},$$

use partial fractions to show that

$$y(t) = \mathbf{1}(t) - e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

Solution 15.20.19.
Using partial fractions, we can write

$$\frac{1}{s(s^2 + s + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + s + 1} = \frac{(A + B)s^2 + (A + C)s + A}{s(s^2 + s + 1)},$$

where we find $A = 1$ and $B = C = -1$. This gives

$$\hat{y}(s) = \frac{1}{s} + \frac{-(s + 1)}{s^2 + s + 1} = \frac{1}{s} + \frac{-(s + \frac{1}{2})}{(s + \frac{1}{2})^2 + \frac{3}{4}} + \frac{-\frac{1}{2}}{(s + \frac{1}{2})^2 + \frac{3}{4}}.$$

Then, taking the inverse Laplace of $\hat{y}(s)$ and using the s-shift rule, we obtain

$$\begin{aligned} y(t) &= \mathbf{1}(t) - e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2}\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ &= \mathbf{1}(t) - e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right). \end{aligned}$$

Problem 15.20.20. Consider an object falling under the force of gravity. Ignore drag and model the motion as an undamped rigid body. Use Laplace transforms to express the position $q(t)$ as a function of t, $q(0)$, $\dot{q}(0)$, and g.

Solution 15.20.20.
Let us start by defining

$$\ddot{q}(t) \triangleq g. \tag{17.0.11}$$

Taking the Laplace of this equation, we get

$$s^2 q(s) - sq(0) - \dot{q}(0) = \frac{g}{s} \tag{17.0.12}$$

$$q(s) = \frac{q(0)}{s} + \frac{\dot{q}(0)}{s^2} + \frac{g}{s^3}. \tag{17.0.13}$$

Now take the inverse Laplace to yield

$$q(t) = q(0) + \dot{q}(0)t + \frac{gt^2}{2}. \tag{17.0.14}$$

Problem 15.20.21. A body with mass M is falling under the force of gravity. Atmospheric drag is modeled by a dashpot coefficient D, so that the system is modeled as a damped rigid body. Assuming initial velocity v_0, use Laplace transforms to find the velocity $v(t)$ for $t > 0$. Then use the final value theorem to compute the terminal velocity $\lim_{t\to\infty} v(t)$.

Solution 15.20.21.
The damped rigid body is expressed as

$$\begin{aligned} M\ddot{q} + D\dot{q} &= Mg \\ \Rightarrow M\dot{v} + Dv &= Mg \\ v(0) &= v_0 \end{aligned}$$

Using Laplace

$$s\hat{v}(s) - v(0) + \frac{D}{M}\hat{v}(s) = \frac{g}{s}.$$

This gives

$$(s + \frac{D}{M})\hat{v}(s) = v_0 + \frac{g}{s}$$

or

$$\hat{v}(s) = \frac{v_0}{s + \frac{D}{M}} + \frac{g}{s(s + \frac{D}{M})}.$$

Now using partial fractions

$$\frac{g}{s(s + \frac{D}{M})} = \frac{a}{s} + \frac{b}{s + \frac{D}{M}} \quad \text{or} \quad g = as + \frac{D}{M}a + bs.$$

For $s = 0$,

$$g = \frac{aD}{M} \quad \Rightarrow \quad a = \frac{gM}{D}.$$

For $s = -\frac{D}{M}$,

$$g = -\frac{bD}{M} \quad \Rightarrow \quad b = -\frac{gM}{D}.$$

Taking the inverse Laplace of $\hat{v}(s)$, we obtain

$$\begin{aligned} v(t) &= v_0 e^{-\frac{D}{M}t} - \frac{gM}{D}e^{-\frac{D}{M}t} + \frac{gM}{D}\mathbf{1}(t) \\ &= \left(v_0 - \frac{gM}{D}\right)e^{-\frac{D}{M}t} + \frac{gM}{D}\mathbf{1}(t). \end{aligned}$$

Using the final value theorem, we obtain

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} s\hat{v}(s) = \frac{Mg}{D}.$$

Problem 15.20.22. Consider an iron sphere and a wooden sphere of the same size and smoothness. Does the iron sphere fall faster than the wooden sphere? Use the undamped rigid body to show that, if there is no drag, then the spheres fall at the same rate. Now consider the more realistic case in which drag is present. Using the damped rigid body, and assuming that the damping coefficient is the same for both spheres, determine whether or not the heavier body falls faster than the lighter body. (Hint: First consider the terminal velocities and perform some Matlab simulation.)

Solution 15.20.22. With no atmospheric drag, the equation of motion of a rigid body is,

$$\dot{v}(t) = g.$$

Since the mass cancels out, both objects fall equally.

With atmospheric drag,

$$\dot{v} + \frac{D}{M}v = g.$$

The analytical solution for $v(t)$ is (from Problem 15.20.21),

$$v(t) = \left(v_0 - \frac{gM}{D}\right)e^{-\frac{D}{M}t} + \frac{gM}{D}.$$

Assuming a zero initial velocity for simplicity, it yields,

$$v(t) = \frac{gM}{D}\left(1 - e^{-\frac{D}{M}t}\right).$$

Now, let v_1 and v_2 be the velocities of a wooden sphere and an iron sphere, respectively. Now, we prove that if $M_1 < M_2$, then for all $t \geq 0$, $v_1(t) \leq v_2(t)$.

First, we prove the following.

Fact 17.0.1. Given a differentiable function $f : [0, \infty) \to \mathcal{R}$, if $f(y) - f(x) \leq y - x, \forall y \geq x$, then $f'(x) \leq 1, \forall x$.

Proof.

$$\begin{aligned}
&f(y) - f(x) \leq y - x, \forall y \geq x \\
&\Rightarrow \frac{f(y) - f(x)}{y - x} \leq 1, \forall y \geq x \\
&\Rightarrow \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x} \leq 1, \\
&\Rightarrow f'(x) \leq 1, \forall x.
\end{aligned}$$

Fact 17.0.2. If $f(x) = xe^{-1/x}$, $x > 0$, then $f'(x) \leq 1$.

Proof. By power series expansion,

$$1 + \frac{1}{x} < e^{1/x} = 1 + \frac{1}{x} + \frac{1}{2!}\frac{1}{x^2} + \frac{1}{3!}\frac{1}{x^3} + \cdots.$$

So,

$$e^{-1/x}\left(1 + \frac{1}{x}\right) < 1.$$

Thus,

$$\begin{aligned} f'(x) &= e^{-1/x} + xe^{-1/x}\left(\frac{1}{x^2}\right) \\ &= e^{-1/x}\left(1+\frac{1}{x}\right) < 1. \end{aligned}$$

Using Fact 17.0.1 and 17.0.2, it is true that

$$\begin{aligned} & y \geq x \\ & \Rightarrow ye^{-1/y} - xe^{-1/x} \leq y - x. \end{aligned}$$

Now let $y = M_2/Dt$ and $x = M_1/Dt$ where $M_2 < M_1$. Then,

$$\begin{aligned} & \Rightarrow \frac{M_2}{Dt}e^{-Dt/M_2} - \frac{M_1}{Dt}e^{-Dt/M_1} < \frac{M_2}{Dt} - \frac{M_1}{Dt} \\ & \Rightarrow M_2 e^{-Dt/M_2} - M_1 e^{-Dt/M_1} < M_2 - M_1 \\ & \Rightarrow M_1\left(1 - e^{-Dt/M_1}\right) < M_2\left(1 - e^{-Dt/M_2}\right) \\ & \Rightarrow \frac{gM_1}{D}\left(1 - e^{-Dt/M_1}\right) < \frac{gM_2}{D}\left(1 - e^{-Dt/M_2}\right) \\ & \Rightarrow v_1(t) < v_2(t) \end{aligned}$$

Therefore, heavier objects fall faster than lighter objects in the presence of atmospheric drag.

Alternative Solution
For the terminal velocity,

$$\begin{aligned} \lim_{t\to\infty} v_i(t) &= \lim_{t\to\infty} \frac{gM_i}{D}\left(1 - e^{-Dt/M_i}\right) \\ &= \frac{gM_i}{D}, \text{ for } i = 1, 2 \end{aligned}$$

Thus, if $M_2 > M_1$ then

$$\begin{aligned} & \frac{gM_2}{D} > \frac{gM_1}{D} \\ \Rightarrow & v_2(t_\infty) > v_1(t_\infty) \end{aligned}$$

For the transient velocities, we show them numerically.

Note that the magnitude of the damping coefficient (D) determines the slope of the transient velocity, while the mass determines the settling time.

The M-file used to create the plot is given below.

```
<HWChapterLinSysProble22.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem22 Soln

clear all;clc;close all

D = 1;
g = 9.8;
```

```
M1 = 1;
M2 = 2;

t = 0:.1:10;

v1 = g*M1/D*(1-exp(-D*t/M1));
v2 = g*M2/D*(1-exp(-D*t/M2));

figure(1);clf
plot(t,v1)
hold on
plot(t,v2,'g--')
xlabel('Time (s)')
ylabel('v(t)')
title('Comparion of velocities')
legend('v_1(t)','v_2(t)')
```

Figure 17.0.8: Problem 15.20.22. Comparision of velocities.

Problem 15.20.23. Consider the damped rigid body with ramp force input $u(t) = f_0 t$ and velocity output. Use Laplace transforms and partial fractions to determine the forced response for $t \geq 0$.

Solution 15.20.23.
The equation of motion for a damped rigid body is given by,

$$m\ddot{q}(t) + c\dot{q}(t) = f(t)$$

Let $v(t) = \dot{q}(t)$. Then,

$$m\dot{v}(t) + cv(t) = f(t)$$

Taking the Laplace transform on both sides and rearranging,

$$\hat{v}(s) = \frac{1}{ms + c}\hat{f}(s)$$

Given, $f(t) = u(t) = f_0 t$, which implies that

$$\mathcal{L}\{f(t)\} = \hat{f}(s) = \frac{f_0}{s^2}$$

Hence,

$$\hat{v}(s) = \frac{1}{ms + c} \cdot \frac{f_0}{s^2} = \frac{f_0}{s^2(ms + c)}$$

Decomposition into partial fractions yields

$$\begin{aligned} \frac{1}{s^2(ms + c)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Z}{ms + c} \\ Z &= \frac{m^2}{c^2} \\ B &= \frac{1}{c} \\ A &= -\frac{m}{c^2} \end{aligned}$$

Thus,

$$\hat{v}(s) = f_0\left(-\frac{m}{c^2 s} + \frac{1}{cs^2} + \frac{m^2/c^2}{ms + c}\right)$$

Taking the inverse Laplace transform, yields

$$v(t) = f_0\left(-\frac{m}{c^2} + \frac{t}{c} + \frac{m}{c^2}\, e^{-\frac{c}{m}t}\right)$$

which is the velocity response of the damped rigid body for a ramp force input.

Problem 15.20.24. Consider the undamped rigid body in two cases, namely, 1) position output with unit impulse input, and 2) velocity output with unit step input. Show that the Laplace transforms of the outputs have the same form in both cases. Next, explain how the same expression correctly captures both outputs even though the outputs have different dimensions.

Solution 15.20.24.
The undamped rigid body is given by

$$\begin{aligned}\ddot{q}(t) &= \frac{1}{m} f(t) \\ s^2\hat{q}(s) - sq(0) - \dot{q}(0) &= \frac{1}{m}\hat{f}(s)\end{aligned}$$

1) position output with unit force impulse input $f(t) = f_0\delta(t)$

$$\begin{aligned}\hat{q}(s) &= \frac{1}{ms^2}\hat{f}(s) \\ &= \frac{1}{ms^2} \cdot f_0 \\ &= \frac{f_0}{ms^2}\end{aligned} \tag{17.0.15}$$

2) velocity output with unit force step input $f(t) = f_0 1(t)$

$$\begin{aligned}\hat{v}(s) = s\hat{q}(s) &= \frac{1}{ms}\hat{f}(s) \\ &= \frac{1}{ms} \cdot \frac{f_0}{s} \\ &= \frac{f_0}{ms^2}\end{aligned} \tag{17.0.16}$$

The same expression captures both outputs but the dimensions of the outputs are different.

Case 1): $[\hat{q}(s)]$ = m-sec

In this case $[f_0]$ is momentum.

$$[\frac{1}{ms^2}] \cdot [f_0] = (1/((\text{kg})(1/\text{sec}^2)) \cdot (\text{kg-m/sec}) = \text{m-sec}$$

Case 2): $[\hat{v}(s)]$ = m/sec · sec = m

In this case $[f_0]$ is force.

$$[\frac{1}{ms}] \cdot [f_0] = (1/((\text{kg})(1/\text{sec}^2)) \cdot (\text{kg-m/sec}^2) = \text{m}$$

Problem 15.20.25. Consider the rigid body force-to-velocity transfer function

$$G(s) = \frac{1}{ms}$$

with sinusoidal input $f(t) = a \sin \omega t$. Use Laplace transforms to determine the forced velocity response $\dot{q}(t)$ for $t \geq 0$. For the forced response, assume that the initial conditions are zero so that the free response is zero.

Solution 15.20.25.
Let $v(t) = \dot{q}(t)$. Then,

$$\begin{aligned} \hat{v}(s) &= G(s)\hat{f}(s) \\ &= \frac{1}{ms}\frac{a\omega}{s^2 + \omega^2} \\ &= \frac{a}{m\omega}(\frac{1}{s} - \frac{s}{s^2 + \omega^2}) \end{aligned}$$

Taking the inverse Laplace transform, yields

$$v(t) = \frac{a}{m\omega} - \frac{a}{m\omega}\cos(\omega(t)).$$

Problem 15.20.26. Determine the forced response of $2\ddot{q} + 17q = 5f$ with output $y(t) = q(t)$ and input $f(t) = \sin 2t$. Check your answer by using either Matlab or Simulink to plot your analytical solution.

Solution 15.20.26.

$$2\ddot{q}(t) + 17q(t) = 5f(t).$$

Taking the Laplace transform

$$\begin{aligned}\hat{q}(s) &= \frac{5\hat{f}(s)}{2s^2+17} \\ &= \frac{10}{(s^2+4)(2s^2+17)} \\ &= \frac{5}{9}\frac{2}{s^2+2^2} - \frac{10}{9}\sqrt{\frac{2}{17}}\frac{\sqrt{\frac{17}{2}}}{s^2+(\sqrt{\frac{17}{2}})^2}\end{aligned}$$

Taking the inverse Laplace transform

$$q(t) = \frac{5}{9}\sin(2t) - \frac{10}{9}\sqrt{\frac{2}{17}}\sin(\sqrt{\frac{17}{2}}t).$$

The M-file used to create the plot is given below.

```
<HWChapterLinSysProble26.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem26 Soln

clear all;clc;close all

t = 0:0.1:50;
q = 5/9*sin(2*t) - 10/9*sqrt(2/17)*sin(sqrt(17/2)*t);

figure(1);clf
plot(t,q)
xlabel('t')
ylabel('q(t)')
title('Matlab Result')
```

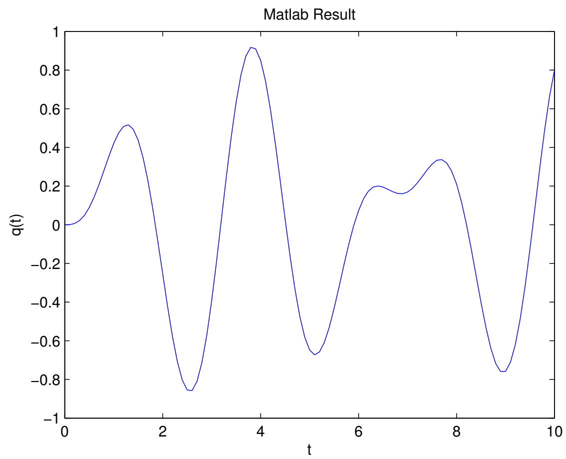


Figure 17.0.9: Problem 15.20.24. Analytical solution

Problem 15.20.27. Consider the damped rigid body with position output, and write it in state space form $\dot{x} = Ax$. Then determine e^{At} by using the fact that the Laplace transform of the matrix exponential e^{At} is $(sI - A)^{-1}$ and taking the inverse Laplace transform of each entry of $(sI - A)^{-1}$. Use this result to determine the free response of the damped rigid body with initial displacement $q(0)$, initial velocity $\dot{q}(0)$, and output $y(t)$ given by the mass position. Finally, determine $\lim_{t\to\infty} e^{At}$ by using the expression you obtained for e^{At} as well as by applying the final value theorem to each separate entry of $(sI - A)^{-1}$. What kind of matrix is $\lim_{t\to\infty} e^{At}$?

Solution 15.20.27.
A damped rigid body is described by,

$$m\ddot{q} + c\dot{q} = F.$$

Here we assume there is no input, so F = 0. We need to write this system in the form,

$$\dot{x} = Ax$$

We first transform the 2nd order system into state space form. Let $x = [x_1\ x_2]^T = [q\ \dot{q}]^T$. Then,

$$\begin{aligned}\dot{x}_1 &= \dot{q} = x_2\\ \dot{x}_2 &= \ddot{q} = -\frac{c}{m}\dot{q} = -\frac{c}{m}x_2\end{aligned}$$

Re-writing into the matrix form,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -\frac{c}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \Leftrightarrow \dot{x} = Ax$$

We are given that,

$$\begin{aligned}&\mathcal{L}\{e^{At}\} = (sI - A)^{-1}\\ \Rightarrow (sI - A)^{-1} &= \begin{bmatrix}s & -1\\ 0 & s+\frac{c}{m}\end{bmatrix}^{-1} = \frac{1}{s(s+\frac{c}{m})}\begin{bmatrix}s+\frac{c}{m} & 1\\ 0 & s\end{bmatrix} = \begin{bmatrix}\frac{1}{s} & \frac{1}{s(s+\frac{c}{m})}\\ 0 & \frac{1}{s+\frac{c}{m}}\end{bmatrix}\end{aligned}$$

FVT shows that,

$$\lim_{t\to\infty} e^{At} = \lim_{s\to 0}\{s(sI - A)^{-1}\} = \lim_{s\to 0} s \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+\frac{c}{m})} \\ 0 & \frac{1}{s+\frac{c}{m}} \end{bmatrix} = \lim_{s\to 0} \begin{bmatrix} 1 & \frac{1}{(s+\frac{c}{m})} \\ 0 & \frac{s}{s+\frac{c}{m}} \end{bmatrix}$$

$$\lim_{t\to\infty} e^{At} = \begin{bmatrix} 1 & \frac{m}{c} \\ 0 & 0 \end{bmatrix}$$

Converting to the time domain we have,

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} 1 & -\frac{m}{c} e^{-\frac{c}{m}t} + \frac{m}{c} \\ 0 & e^{-\frac{c}{m}t} \end{bmatrix}$$

$$\lim_{t\to\infty} e^{At} = \begin{bmatrix} 1 & \frac{m}{c} \\ 0 & 0 \end{bmatrix}$$

The free response of this system is described by $y(t) = Ce^{At}x(0)$. So we have,

$$\lim_{t\to\infty} y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{m}{c} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix} = q(0) + \frac{m}{c}\dot{q}(0)$$

The matrix $\lim_{t\to\infty} e^{At}$ is an idempotent matrix. An idempotent matrix holds the property $A = A^2$.

Problem 15.20.28. An engineer has shown that the output response of a new airframe developed for a UAV application is given by

$$y(t) = 2e^{-6t}.$$

Use the initial value to determine $\ddot{y}(0^+)$. (Note the two dots.) Check your answer by computing $\ddot{y}(t)$ and then setting $t = 0$. (Hint: Note that

$$\mathcal{L}\{\ddot{y}(t)\} = s^2\hat{y}(s) - sy(0) - \dot{y}(0).$$

You may use the time-domain expression for $y(t)$ to determine $y(0)$ and $\dot{y}(0)$.)

Solution 15.20.28.
First find the Laplace transform of $y(t)$,

$$\hat{y}(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{2e^{-6t}\} = \frac{2}{s+6}. \tag{17.0.17}$$

Let

$$z(t) = \ddot{y}(t), \tag{17.0.18}$$

then

$$\begin{aligned} \mathcal{L}\{z(t)\} &= \mathcal{L}\{\ddot{y}(t)\} \\ \Rightarrow \quad \hat{z}(s) &= s^2\hat{y}(s) - sy(0) - \dot{y}(0). \end{aligned} \tag{17.0.19}$$

Now rewrite the initial value theorem stated in the class in terms of $z(t)$ as below

$$z(0^+) = \lim_{s\to\infty} s\hat{z}(s). \tag{17.0.20}$$

Combining (17.0.19) and (17.0.20) we get

$$z(0^+) = \lim_{s\to\infty} s\left(s^2\hat{y}(s) - sy(0) - \dot{y}(0)\right). \tag{17.0.21}$$

Now using (17.0.17) we have

$$\begin{aligned} z(0^+) &= \lim_{s\to\infty} s\left(s^2\frac{2}{s+6} - s(2) - (-12)\right) \\ &= \lim_{s\to\infty}\left(\frac{72s}{s+6}\right) \\ &= 72. \end{aligned}$$

To check the answer, evaluate

$$\begin{aligned} \ddot{y}(t) &= 72e^{-6t}, \\ \Rightarrow \ddot{y}(0) &= 72. \end{aligned}$$

Problem 15.20.29. For each of transfer functions $G(s)$ below with input u and output y, determine whether the use of the final value theorem is legal, and, if so, use it to determine the limit of the output $y(t)$ as $t \to \infty$. Explain why or why not the use of the final value theorem is legal in each case.

i) $G(s) = \dfrac{-5}{s(s+7)^2}$, $\quad u(t) = 3e^{-2t}$.

ii) $G(s) = \dfrac{5}{s-3}$, $\quad u(t) = 7\mathbf{1}(t) - 3\delta(t-4.2)$.

iii) $G(s) = \dfrac{-4}{s^2+2.4s+.3}$, $\quad u(t) = 6t^2$.

Solution 15.20.29.
i)

$$\hat{u}(s) = \frac{3}{s+2}$$
$$\Rightarrow \hat{y}(s) = G(s)\hat{u}(s) = \frac{-15}{s(s+2)(s+7)^2}$$

It follows that $\hat{y}(s)$ has four poles, out of which, three are in the open left half plane (OLHP) and one at zero. Hence, the final value theorem is legal. Since the pole at the origin is not repeated, the limit is finite and is given by

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) = -15/98 \simeq -0.1531$$

ii)

$$\hat{u}(s) = \frac{7}{s} - 3e^{-4.2s}$$
$$\Rightarrow \hat{y}(s) = G(s)\hat{u}(s) = \frac{5}{s-3}\left(\frac{7}{s} - 3e^{-4.2s}\right)$$
$$\Rightarrow \hat{y}(s) = \frac{5(7-3se^{-4.2s})}{s(s-3)}$$

It follows that $\hat{y}(s)$ has a pole in the open right half plane (ORHP). Hence the final value theorem is not legal.

iii)

$$\hat{u}(s) = \frac{6\cdot 2}{s^3}$$
$$\Rightarrow \hat{y}(s) = G(s)\hat{u}(s) = \frac{-4}{s^2+2.4s+.3}\,\frac{12}{s^3}$$
$$\Rightarrow \hat{y}(s) = \frac{-48}{s^3(s^2+2.4s+.3)}$$

It follows that FVT is legal since the poles of $\hat{y}(s)$ are in the OLHP or at the origin. However, limit does not exist, but is infinite, since the pole at the origin is repeated.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) \to -\infty$$

Problem 15.20.30. For both systems described below (described by a transfer function and input) and without using a calculator or computer, determine whether the use of the final value theorem is legal, and, if so, use the final value theorem to determine the limit of the output $y(t)$ as $t \to \infty$. Explain why or why not the use of the final value theorem is legal in each case. Simulate the first system using the Matlab "impulse" command. Simulate the second system using Simulink with integrator blocks but not the transfer function block. (Hint: Use a pair of nested feedback loops each of which has an integrator in the forward path). If $y(t)$ converges, verify that the value of $y(t)$ as t becomes very large agrees with the result of the final value theorem. Run your Simulink model from a Matlab script and plot $y(t)$ as a function of time from your Matlab script.

$$i)\ \ G(s) = \frac{s^2 - 1}{s(3s^3 + 2s^2 + 4s + 3)}, \qquad u(t) = 2\delta(t).$$

$$ii)\ \ G(s) = \frac{1}{s(s^2 + 3s + 1)}, \qquad u(t) = -2e^{-t}\sin 2t.$$

Solution 15.20.30. *i*)

$$\begin{aligned}
&\mathcal{L}\{\delta(t)\} = 1 \\
&\Rightarrow \hat{u}(s) = 2 \\
&\Rightarrow \hat{y}(s) = G(s)\hat{u}(s) = \frac{2(s^2 - 1)}{s(3s^3 + 2s^2 + 4s + 6)} = \frac{\frac{2}{3}(s^2 - 1)}{s(s^3 + \frac{2}{3}s^2 + \frac{4}{3}s + \frac{6}{3})}
\end{aligned}$$

Using the Routh test for the cubic polynomial in the denominator, we have

$$a_2 a_1 - a_0 = -\frac{1}{9} < 0,$$

which implies that the cubic polynomial is not asymptotically stable. Therefore, $\hat{y}(s)$ has at least one nonzero pole in the CRHP. Hence the final value theorem is not legal. Use the MATLAB command

```
>> impulse([2 0 -2],[3 2 4 3 0],12)
```

to get the plot shown in Figure 17.0.10. As seen in the figure, the output diverges due to a pair of ORHP poles.

ii)

$$\begin{aligned}
\mathcal{L}\{-2e^{-t}\sin 2t\} = \hat{u}(s) &= \left(\frac{-2 \times 2}{(s+1)^2 + 2^2}\right) \\
\Rightarrow \hat{y}(s) = G(s)\hat{u}(s) &= \frac{-4}{s(s^2 + 2s + 5)(s^2 + 3s + 1)}
\end{aligned}$$

Using the Routh test for the second order polynomial in the denominator, which implies that all the poles of $s^2 + 2s + 5$ and $s^2 + 3s + 1$ are in OLHP, and thus are stable. Therefore, the final value theorem is legal. Since $s = 0$ is not a repeated pole, the limit exists.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) = -0.8$$

The M-file used to run the Simulink model (shown in Figure 17.0.11) and create the plot (Figure 17.0.12) is given below

```
<HWChapterLinSysProblem30.m>
```

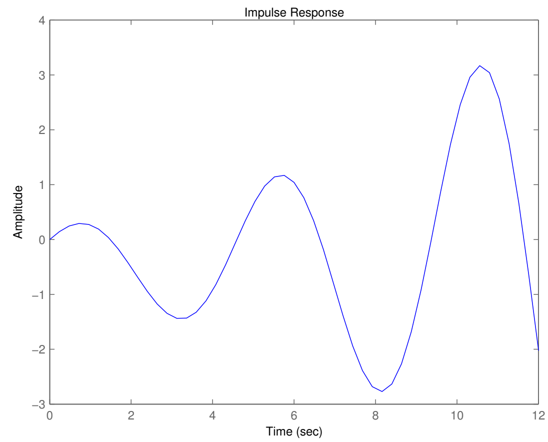


Figure 17.0.10: Problem 15.20.30 *i*). Impulse Response of the System $G(s) = \dfrac{s^2-1}{s(3s^3+2s^2+4s+5)}$.

```
% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem30 Soln

close all; clc; clear all;

k1 = 1/2; k2 = 2;

t = 0:0.1:100;
u = -2*exp(-t).*sin(2*t);

sim('Pb_30_Model',[t(1) t(end)])

figure(1);clf
plot(y.Time,y.Data)
grid on
ylabel('Response')
xlabel('Time (sec)')
ylim([-1.2 0.4])
```

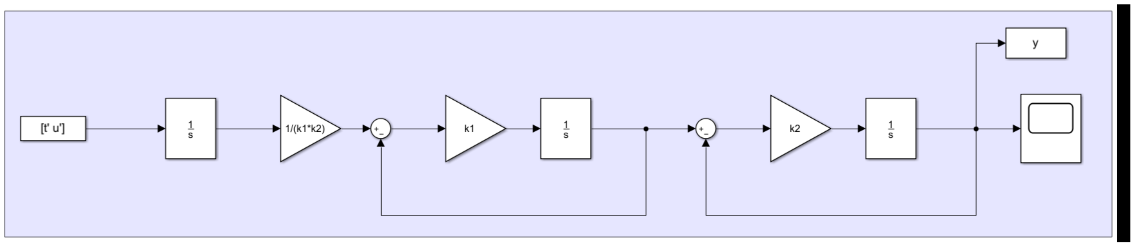


Figure 17.0.11: Problem 15.20.30 *ii*) Simulink Model 'Pb_30_Model'.

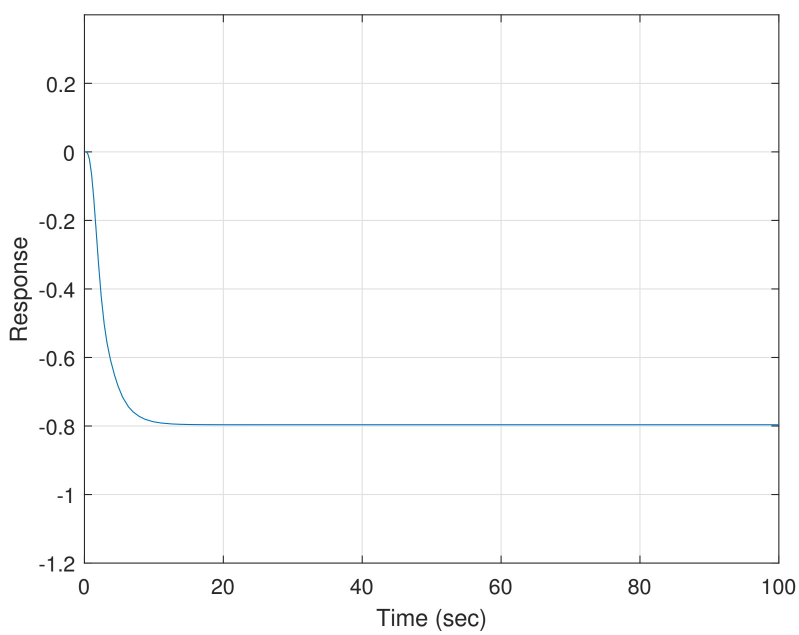


Figure 17.0.12: Problem 15.20.30 *ii*). Response of the System $G(s) = \dfrac{1}{s(2s^2+3s+1)}$ to the input $u(t) = -2e^{-t}\sin(2t)$..

Problem 15.20.31. For the systems described below and without using a calculator or computer, determine whether the use of the final value theorem is legal, and, if so, use the final value theorem to determine the limit of the output $y(t)$ as $t \to \infty$. Explain why or why not the use of the final value theorem is legal in each case. Model the first system with Simulink and simulate the system. Simulate the second system using the Matlab "impulse" command. If $y(t)$ converges verify that the value of $y(t)$ as t becomes very large agrees with the result of the final value theorem. Run your Simulink model from a Matlab script and plot $y(t)$ as a function of time from your Matlab script.

i) The free response of the damped rigid body with initial displacement $q(0)$, initial velocity $\dot{q}(0)$, and output $y(t)$ given by the mass position. Obtain the limit symbolically. For the Simulink model, use the numerical values $m = 3$ kg, $c = 4$ kg/s, $q(0) = 1$ m, and $\dot{q}(0) = -2$ m/s.

ii) The forced response of a system whose transfer function is

$$G(s) = \frac{4s^2 - 12s - 16}{2s^5 + 2s^4 + 4s^3 + 2s^2 + s}$$

with the impulsive input $u(t) = 3\delta(t-1)$.

Solution 15.20.31. *i*) We start from the equation of motion for the damped rigid body with no input:

$$m\ddot{q} + c\dot{q} = 0, \quad q(0), \quad \dot{q}(0)$$

$$\begin{aligned}
&\Rightarrow \mathcal{L}\{m\ddot{q} + c\dot{q}\} = 0 \\
&\Rightarrow m(s^2\hat{q} - sq(0) - \dot{q}(0)) + c(s\hat{q} - q(0)) = 0 \\
&\Rightarrow s(ms + c)\hat{q} = mq(0)s + m\dot{q}(0) + cq(0) \\
&\Rightarrow \hat{q} = \frac{mq(0)s + m\dot{q}(0) + cq(0)}{s(ms + c)}
\end{aligned}$$

Note that the poles s = 0, -c/m are located at the origin and in the OLHP, respectively. Thus, the final value theorem is legal. Hence,

$$q_\infty = \lim_{s\to 0} s\hat{q}(s) = \lim_{s\to 0} \frac{mq(0)s + m\dot{q}(0) + cq(0)}{ms + c} = \frac{m\dot{q}(0) + cq(0)}{c}$$

The M-file used to run the Simulink model (shown in Figure 17.0.13) and create the plot (Figure 17.0.14) is given below

```
<HWChapterLinSysProblem31.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem31 Soln

close all; clc; clear all;

% Mass Spring Damper System
m = 3; %kg
c = 4; %kg/sec

% System matrices
```

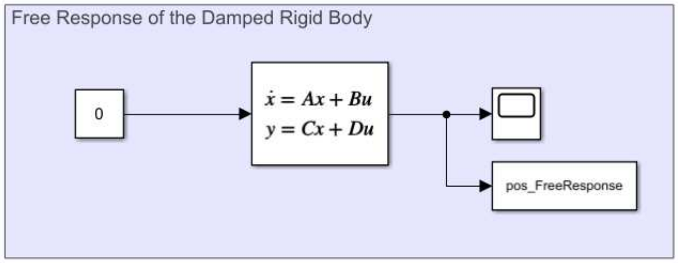


Figure 17.0.13: Problem 15.20.31 *ii*) Simulink Model 'Pb_31_Model'.

```
A = [0 1;
    0 -c/m];
B = [0;
    1/m];
C = [1 0];
D = 0;
% Initial condition
x0 = [1; %position
      -2];%velocity

% Running Simulink Model
sim('Pb_31_Model',[0 40]) %runs the model

% Plotting free response
figure(1);clf
plot(pos_FreeResponse.Time,pos_FreeResponse.Data)
grid on
ylabel('Position (m)')
xlabel('Time (sec)')
title('Free Response')
ylim([-0.8 1.2])
```

ii)

$$
\begin{aligned}
\mathcal{L}\{\delta(t-1)\} &= e^{-s}\\
\Rightarrow \hat{u}(s) &= 3e^{-s}\\
\Rightarrow \hat{y}(s) &= G(s)\hat{u}(s) = \frac{(4s^2-12s-16)(3e^{-s})}{2s^5+2s^4+4s^3+2s^2+s}\\
&= \frac{12(s^2-3s-4)e^{-s}}{2s(s^4+s^3+2s^2+s+0.5)}
\end{aligned}
$$

Using the Routh test for the quadratic polynomial in the denominator, we have $a_0 = \frac{1}{2} > 0$, $a_1 = 1 > 0$, $a_2 = 2 > 0$, $a_3 = 1 > 0$, $a_4 = 1 > 0$, and $a_0a_3^2 + a_1^2 = 1.5 < 2 = a_1a_2a_3$ which implies that the polynomial is asymptotically stable. Thus four poles of $\hat{y}(s)$ are located in the OLHP and one pole

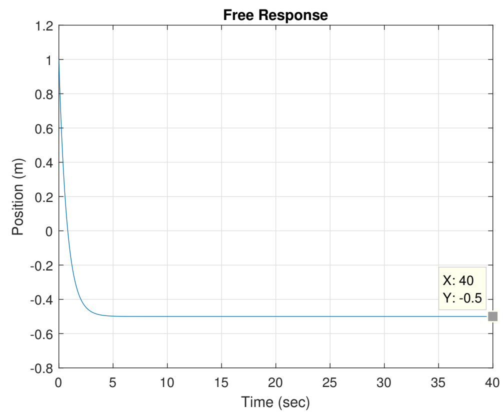


Figure 17.0.14: Problem 15.20.31 *ii*). Free response of the damped rigid body.

at the origin. Hence the final value theorem is legal and the limit is given below

$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) = -48$$

The MATLAB code

```
>> impulse(3*[4 -12 -16],[2 2 4 2 1 0],[1:.01:20])
```

gives the response plot as shown in Figure 17.0.15.

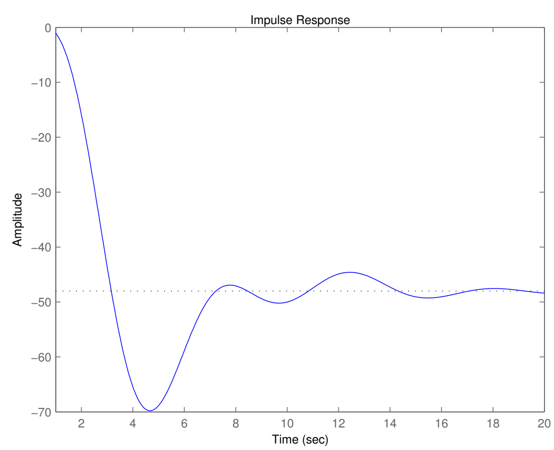


Figure 17.0.15: Problem 15.20.31. Response of the system $G(s) = \dfrac{4s^2 - 12s - 16}{2s^5 + 2s^4 + 4s^3 + 2s^2 + s}$ with the impulsive input $u(t) = 3\delta(t-1)$.

Problem 15.20.32. Without taking inverse Laplace transforms, use the initial and final value theorems as well as knowledge of the pole locations to sketch the step response of the transfer function

$$G(s) = \frac{4s - 3}{s^2 + 0.8s + 4}.$$

Be sure to qualitatively capture the direction of the step response for small positive time t as well as the settling behavior for large time t. What features does your sketch illustrate? Model this system with Simulink and simulate the system. You may use the 'transfer function' block in your Simulink model. Run your Simulink model from a Matlab script and plot the step response from your Matlab script.

Solution 15.20.32. Given input $u(t) = \mathbf{1}(t)$ and the transfer function $G(s) = \frac{4s-3}{s^2+.8s+4}$, we have the response,

$$\hat{y}(s) = G(s)\hat{u}(s) = \frac{4s - 3}{s(s^2 + .8s + 4)} \tag{17.0.22}$$

The Initial Value Theorem, the Initial Slope Theorem and the Final Value Theorem can be used to qualitatively plot the step response. The Initial Value Theorem states that

$$y(0^+) = \lim_{s\to\infty} s\hat{y}(s)$$

where $\hat{y}(s)$ is the Laplace transform of the response and $y(0^+)$ is the initial value of the response function. Applying the IVT to (17.0.22),

$$\begin{aligned} y(0^+) &= \lim_{s\to\infty} s\frac{4s - 3}{s(s^2 + .8s + 4)} \\ &= \lim_{s\to\infty} \frac{4s - 3}{(s^2 + .8s + 4)} = 0 \end{aligned}$$

The initial value of the response function is 0.

The Initial Slope Theorem states that

$$y'(0^+) = \lim_{s\to\infty} s(s\hat{y}(s) - y(0))$$

where $y'(0^+)$ is the initial slope of the response function. Applying the IST to (17.0.22),

$$\begin{aligned} y'(0^+) &= \lim_{s\to\infty} s^2\frac{4s - 3}{s(s^2 + .8s + 4)} \\ &= \lim_{s\to\infty} \frac{4 - \frac{3}{s}}{1 + \frac{0.8}{s} + \frac{4}{s^2}} = 4 \end{aligned}$$

The initial slope of the response function is positive, which means that the response function is positive initially.

Now apply the Final Value Theorem. The poles of $\hat{y}(s)$ are $0, -0.4 \pm \jmath 1.96$. Since all of the poles are either in the OLHP or at the origin, it is legal to use the FVT. Applying the FVT to (17.0.22) yields

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0^+} s\hat{y}(s) \\ &= \lim_{s\to 0^+} s\frac{4s - 3}{s(s^2 + .8s + 4)} = -\frac{3}{4} \end{aligned}$$

The response is negative for large time t.

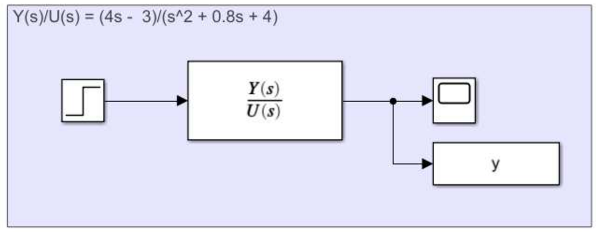


Figure 17.0.16: Problem 15.20.32. Simulink Model 'Pb_32_Model'.

Hence, initially the response is positive, but after some time the response reverses its direction and finally converges to a negative value. This phenomenon of starting in the "wrong" direction is because of the nonminimum-phase zero (real zero in the ORHP) in the transfer function $G(s)$. The response generated using Simulink is shown in Figure 17.0.17. Note that, because the poles have nonzero imaginary part, the response is oscillatory. The M-file used to run the Simulink model (shown in Figure 17.0.16) and create the plot (Figure 17.0.17) is given below

```
<HWChapterLinSysProblem32.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem32 Soln

close all; clc; clear all;

% Numerator and denominator of the transfer function
num = [4 -3];
den = [1 0.8 4];

% Running Simulink Model
sim('Pb_32_Model',[0 40]) %runs the model

% Plotting step response
figure(1);clf
plot(y.Time,y.Data)
grid on
ylabel('y(t)')
xlabel('Time (sec)')
ylim([-2 1.2])
```

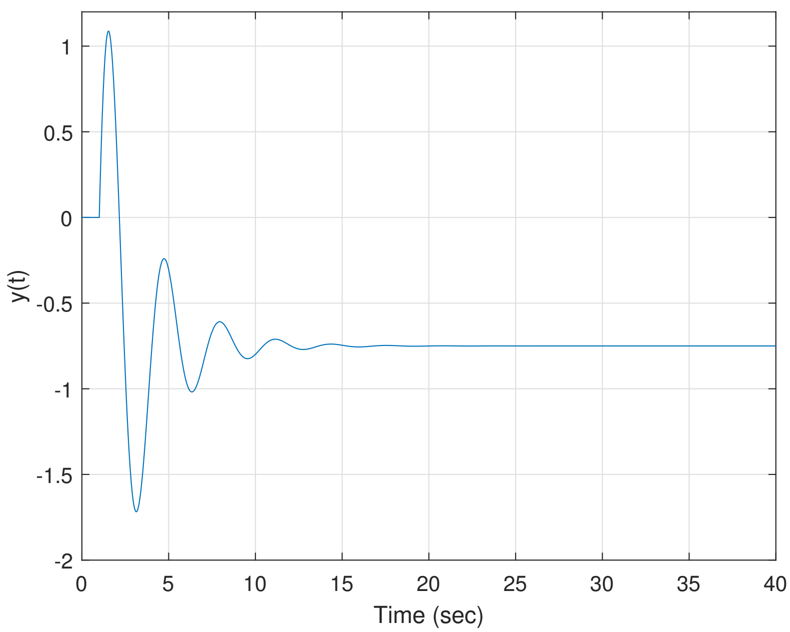


Figure 17.0.17: Problem 15.20.32. Step Response of the System $G(s) = \dfrac{4s - 3}{s(s^2 + 0.8s + 4)}$.

Problem 15.20.33.

Flight testing of a new aircraft reveals that it has a pair of underdamped complex conjugate poles. Testing reveals that the time to 50% decay is T seconds, while analytical modeling shows that the imaginary parts of the poles are $\pm\omega_{\rm d}$, where $\omega_{\rm d} > 0$. Derive an expression for the damping ratio ζ in terms of T and $\omega_{\rm d}$. Show that your expression for ζ satisfies $0 < \zeta < 1$. (Hint: $T = (\ln 2)/(\zeta\omega_{\rm n})$.)

Solution 15.20.33.

The underdamped complex conjugate poles are $-\zeta\omega_{\rm n} \pm \jmath\omega_{\rm n}\sqrt{1-\zeta^2}$, where $\zeta > 0$ and $\omega_{\rm n} > 0$. Given the imaginary part is $\pm\omega_{\rm d}$, where $\omega_{\rm d} > 0$, we have,

$$\omega_{\rm n}\sqrt{1-\zeta^2} = \omega_{\rm d}. \tag{17.0.23}$$

The response of the system will be a decaying sinusoid, where the amplitude is given by $e^{-\zeta\omega_{\rm n}t}$ (note that only the real part of the pole determines the amplitude of the response). At $t = 0$, the amplitude is 1. At $t = T$, the amplitude reduces by 50%. Hence, $\left[e^{-\zeta\omega_{\rm n}T} = \frac{1}{2},\right]$

$$\Longrightarrow \omega_{\rm n} = \frac{\ln 2}{\zeta T}. \tag{17.0.24}$$

From (17.0.23) and (17.0.24),

$$\begin{aligned}
& & \frac{\ln 2}{\zeta T}\sqrt{1-\zeta^2} &= \omega_{\rm d} \\
\Longleftrightarrow & & \sqrt{1-\zeta^2}\ln 2 &= \zeta\omega_{\rm d}T, \\
\Longleftrightarrow & & \sqrt{\frac{1}{\zeta^2}-1} &= \frac{\omega_{\rm d}T}{\ln 2}, \\
\Longleftrightarrow & & \frac{1}{\zeta^2} &= 1 + \left(\frac{\omega_{\rm d}T}{\ln 2}\right)^2, \\
\Longleftrightarrow & & \zeta &= \sqrt{\left[1/\left(1+\left(\tfrac{\omega_{\rm d}T}{\ln 2}\right)^2\right)\right]}.
\end{aligned}$$

Since, $\omega_{\rm d} > 0$, $T > 0$, and $\ln 2 > 0$, clearly $0 < \zeta < 1$.

Problem 15.20.34. Flight testing of a new aircraft reveals that it behaves like an overdamped oscillator. Analytical models are used to determine the value of ω_{n}. Measurements show that the time to 50% decay is T seconds. Derive an expression for the damping ratio ζ in terms of T and ω_{n}. Finally, show that your expression for ζ satisfies $\zeta \geq 1$. (Note: Consider only the slow eigenvalue of the overdamped oscillator.)

Solution 15.20.34.
The overdamped complex conjugate poles are $\lambda_1,\ \lambda_2 = -\zeta\omega_{\mathrm{n}} \pm \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1}$, where $\zeta > 0$ and $\omega_{\mathrm{n}} > 0$. Consider the slow eigenvalue λ_1 for evaluating the time take for 50% decay, then

$$
\begin{aligned}
& & e^{\lambda_1 T} &= \frac{1}{2} \\
&\Leftrightarrow & \lambda_1 &= -\frac{\ln 2}{T} \\
&\Leftrightarrow & -\zeta\omega_{\mathrm{n}} + \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1} &= -\frac{\ln 2}{T} \\
&\Leftrightarrow & \zeta\omega_{\mathrm{n}} - \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1} &= \frac{\ln 2}{T} \\
&\Leftrightarrow & \zeta - \sqrt{\zeta^2 - 1} &= \frac{\ln 2}{\omega_{\mathrm{n}} T} \\
&\Leftrightarrow & \zeta - \frac{\ln 2}{\omega_{\mathrm{n}} T} &= \sqrt{\zeta^2 - 1}.
\end{aligned}
\tag{17.0.25}
$$

Let $\frac{\ln 2}{\omega_{\mathrm{n}} T} = b$, a constant, then squaring equation (17.0.25) on both sides we obtain

$$
\begin{aligned}
& & \zeta^2 + b^2 - 2\zeta b &= \zeta^2 - 1 \\
&\Leftrightarrow & 2\zeta b &= b^2 + 1.
\end{aligned}
\tag{17.0.26}
$$

Hence,

$$
\zeta = \frac{1}{2}\left(b + \frac{1}{b}\right). \tag{17.0.27}
$$

Using λ_2 gives the same expression for ζ. Now, to prove that $\zeta \geq 1$ in (17.0.27), note that

$$
\begin{aligned}
& & (b-1)^2 &\geq 0 \\
&\Leftrightarrow & b^2 - 2b + 1 &\geq 0 \\
&\Leftrightarrow & b^2 + 1 &\geq 2b \\
&\Leftrightarrow & \frac{1}{2}\left(b + \frac{1}{b}\right) &\geq 1.
\end{aligned}
$$

Problem 15.20.35. Consider the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}$$

for the undamped ($c = 0$) and damped ($c > 0$) oscillators. Let $k = 2.5$ kg/s^2 and $m = 7$ m. Plot the locations of the eigenvalues as $\times$'s in the complex plane for a range of values of c. Choose a range that includes undamped, underdamped, critically damped, and overdamped cases. For each value of c, plot a "$\times$" in the complex plane.

Solution 15.20.35.
The characteristic equation of the given matrix A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ k/m & \lambda + c/m \end{bmatrix} = 0$$

$$\Rightarrow \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Hence the eigenvalues are

$$\lambda_1, \lambda_2 = \begin{cases} \pm \jmath \sqrt{\frac{k}{m}} & \text{if } c = 0 \quad \text{(undamped)} \\ \frac{-c \pm \jmath \sqrt{4mk - c^2}}{2m} & \text{if } c^2 < 4mk \quad \text{(underdamped)} \\ -\frac{c}{2m} & \text{if } c^2 = 4mk \quad \text{(critically damped)} \\ \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} & \text{if } c^2 > 4mk \quad \text{(overdamped)} \end{cases}$$

Figure 17.0.18 shows the loci of eigenvalues including all these cases when $k = 2.5\,\text{kg/s}^2$ and $m = 7\,\text{kg}$.

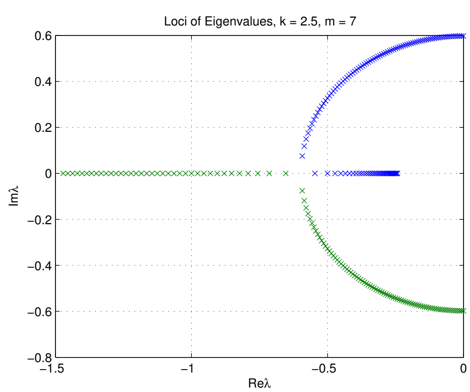


Figure 17.0.18: Problem 15.20.33. Loci of Eigenvalues $k = 2.5\,\text{kg/s}^2$ and $m = 7\,\text{kg}$.

Below is the Matlab code that was used to produce the figure.

```
<HWChapterLinSysProblem35.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem35 Soln
```

```
clear all; close all;

c = 0:0.1:12; k = 2.5; m = 7;

lambda1 = (-c + sqrt(c.^2 - 4*m*k))/(2*m); lambda2 = (-c - ...
    sqrt(c.^2 - 4*m*k))/(2*m);

figure(1);clf
plot(real(lambda1), imag(lambda1), 'x', ...
     real(lambda2), imag(lambda2), 'x')
xlabel('Re\lambda'); ylabel('Im\lambda');
title('Loci of Eigenvalues, k = 3, m = 6');
grid on;
```

Problem 15.20.36. Show analytically that the poles of the undamped, underdamped, and critically damped oscillators satisfy $|\lambda_1| = |\lambda_2| = \omega_{\mathrm{n}}$. Furthermore, show that the poles of the overdamped oscillator satisfy $\lambda_1\lambda_2 = \omega_{\mathrm{n}}^2$. Explain the meaning of these results in terms of the plot you made in the previous problem.

Solution 15.20.36.
For the undamped, underdamped, and critically damped oscillators, the eigenvalues of the oscillator can be expressed in terms of the damping ratio ζ and the natural frequency ω_n:[1]

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \jmath\omega_n \sqrt{1-\zeta^2}$$

Hence

$$|\lambda_1| = |\lambda_2| = |-\zeta\omega_n \pm \jmath\omega_n \sqrt{1-\zeta^2}| = \sqrt{(\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \omega_n$$

Note that, $|\lambda_1| = |\lambda_2| = \omega_n$ gives a circle in the complex plane. Similarly for the overdamped case, the eigenvalues are

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2-1}$$

from which

$$\lambda_1\lambda_2 \quad = (-\zeta\omega_n + \omega_n \sqrt{\zeta^2-1})(-\zeta\omega_n - \omega_n \sqrt{\zeta^2-1}) = \omega_n^2$$

[1] Note that the oscillator is undamped if $\zeta = 0$ and is critically damped if $\zeta = 1$.

Problem 15.20.37. Consider the second-order state space (A, B, C, D) system with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad D = 0.$$

Determine the transfer function $G(s)$ for this system. Show that $G(s)$ is actually a first-order transfer function due to pole-zero cancelation. Finally, determine a first-order state space $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ system whose transfer function is the same G. (Remark: Given a transfer function G we are usually interested in a *realization* (A, B, C, D) of G of the lowest possible order, that is, where the size of A is as small as possible. For this first-order system you can easily construct this realization by looking at the form of the transfer function and setting $C = 1$.)

Solution 15.20.37.
The transfer function can be evaluated using the relation,

$$G(s) = C(sI - A)^{-1}B + D.$$

Substituting the matrices we obtain,

$$G(s) = \frac{(s+1)}{(s+1)(s+2)} = \frac{1}{(s+2)}.$$

A first order state space system that has the same transfer function is given by

$$\begin{aligned} A &= -2; \qquad B = 1 \\ C &= 1; \qquad D = 0. \end{aligned}$$

Problem 15.20.38. Consider the longitudinal dynamics of a commercial aircraft given by

$$\dot{\alpha}(t) = -0.313\alpha(t) + 56.7q(t) + 0.232\delta e(t),$$
$$\dot{q}(t) = -0.0139\alpha(t) - 0.426q(t),$$
$$\dot{\theta}(t) = -0.5\theta(t) + 0.0203\delta e(t),$$

where α is the angle of attack in rad, q is the pitch rate in rad/sec, θ is the pitch angle in rad, and δe is the elevator deflection angle in rad. Assume that $\alpha(0) = q(0) = \theta(0) = 0$. Let the input of the system be the elevator deflection angle δe, and the output of the system be the pitch angle θ. Derive equations for this system in state space and transfer function form. Model this system in Simulink using only integrator blocks, gain blocks, and summation blocks. Also model this system in Simulink using a state space block.

Force both Simulink models with a step input given by a 3-degree elevator deflection. Plot the resulting pitch angle and verify that the two models give exactly the same response.

Solution 15.20.38.
The state space (A, B, C, D) system is given by

$$A = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.232 \\ 0 \\ 0.0203 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad D = 0.$$

The transfer function can be evaluated using the relation,

$$G(s) = C(sI - A)^{-1}B + D.$$

Substituting the matrices we obtain,

$$G(s) = \frac{0.0203}{(s + 0.5)}.$$

The M-file used to run the Simulink model (shown in Figure 17.0.19) and create the plot (Figure 17.0.20) is given below

```
<HWChapterLinSysProblem38.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem38 Soln

clear all; close all; clc

% State Space
A = [-0.313 56.7 0;   %alphadot
    -0.0139 -0.426 0; %qdot
    0 0 -0.5];        %thetadot
B = [0.232 0 0.0203]';
C = [0 0 1];
D = 0;

% Initial condition
alpha0 = 0; %For alphadot integrator block
q0     = 0; %For qdot integrator block
```

```
theta0 = 0; %For thetadot integrator block
x0     = [alpha0 q0 theta0]'; %For State Space Block

% Transfer function
s = tf('s');
G = C*inv(s*eye(3)-A)*B

% Input
delta_e = 3; %deg

% Simulation
sim('Pb_38_Model',[0 40])

% Plotting
figure(1);clf
hold on
plot(theta_model.Time,rad2deg(theta_model.Data))
plot(theta_StateSpace.Time,rad2deg(theta_StateSpace.Data),'r--')
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('From Model Based on Gains and Integrators','From State Space Model')
```

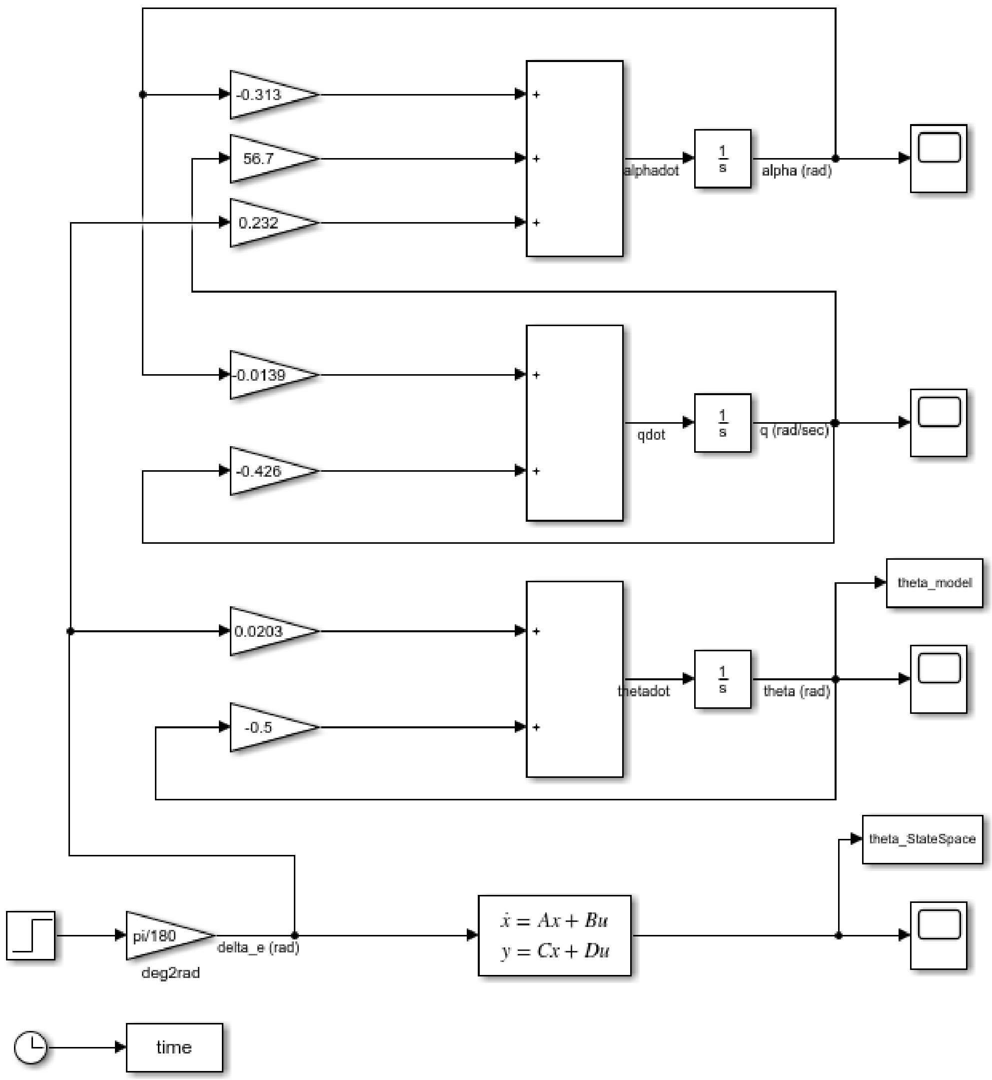


Figure 17.0.19: Problem 15.20.31 *ii*) Simulink Model 'Pb_38_Model'.

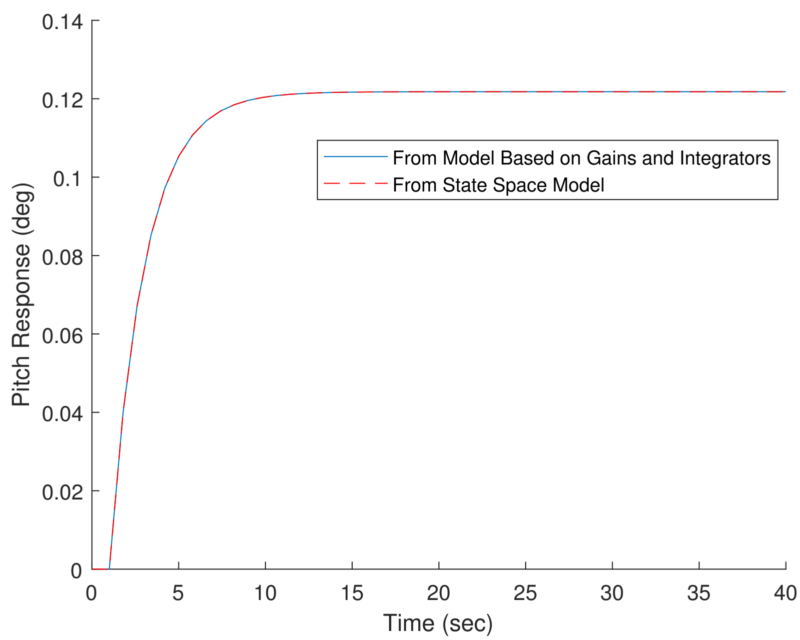


Figure 17.0.20: Problem 15.20.31. Comparison of the responses.

Problem 15.20.39. Take the Simulink model you constructed without using the state space block in Problem 15.20.38, and then put that entire model inside a subsystem block. Set the input to the subsystem block to be the elevator input and set the output of the subsystem block to be the pitch angle. Now we want to use feedback control to make the aircraft maintain a pitch angle of 3 degrees. Set up proportional control whose input is the error between the true pitch angle and the desired pitch angle and whose output is the elevator deflection angle. Tune the proportional control to obtain the best possible response. Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same axes. For clarity, plot the true pitch angle using a solid blue line, and plot the desired pitch angle using a dashed red line. Are you able to follow the command using proportional control?

Now set up an integral controller whose input is the error between the true pitch angle and the desired pitch angle, and whose output is the elevator deflection angle. Add a gain block to the output of the integrator block and tune the gain to obtain the best possible result. Run your Simulink model from a MATLAB script and plot the true pitch angle and the desired pitch angle on the same axes. For clarity, plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using integral control? Note the relationship between the Laplace transform of a step command and the transfer function of an integrator.

Solution 15.20.39.
For proportional control with gain K_p, the sensitivity function is given by

$$S(s) = \frac{1}{1+L} = \frac{s+0.5}{s+0.5+0.0203K_p}$$

The Laplace transform of the error signal $\hat{e}(s)$ for a step command $\hat{r}(s) = 0.0524/s$, is given by

$$\hat{e}(s) = S(s)\hat{r}(s) = \frac{s+0.5}{s+0.5+0.0203K_p}\frac{0.0524}{s}$$

Using the Routh test for first-order polynomial in the denominator, we have $a_0 = 0.5{+}0.0203K_p > 0$, which implies that the polynomial is asymptotically stable for all $K_p > -24.6$. Hence, we have one pole in the OLHP and one pole at the origin, and thus FVT is legal for all $K_p > -24.6$. FVT yields the following asymptotic error

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0^+} s\hat{e}(s)\\ &= \lim_{s\to 0^+} \frac{0.0524(s+0.5)}{s+0.5+0.0203K_p} = \frac{0.0262}{0.5+0.0203K_p}.\end{aligned}$$

It follows that the asymptotic error is non-zero. Also note that, increasing the gain K_p yields smaller error.

For integral control with gain K_i, the sensitivity function is given by

$$S(s) = \frac{1}{1+L} = \frac{s(s+0.5)}{s^2+0.5s+0.0203K_i}$$

The Laplace transform of the error signal $\hat{e}(s)$ for a step command $\hat{r}(s) = 0.0524/s$, is given by

$$\hat{e}(s) = S(s)\hat{r}(s) = \frac{0.0524(s+0.5)}{s^2+0.5s+0.0203K_i}$$

Using the Routh test for second-order polynomial in the denominator, we have $a_1 = 0.5 > 0$, $a_0 = 0.0203K_i > 0$, which implies that the polynomial is asymptotically stable for all $K_i > 0$.

Hence, FVT is legal for all $K_i > 0$. FVT yields the following asymptotic error

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0^+} s\hat{e}(s)\\ &= \lim_{s\to 0^+} \frac{0.0524s(s+0.5)}{s^2+0.5s+0.0203K_i} = 0.\end{aligned}$$

It follows that the asymptotic error is zero for any choice of $K_i > 0$, however, different choices of the gain K_i will yield different transient responses. Also, note that, the Laplace transform of a step command and the transfer function of an integrator have same poles.

The M-file used to run the Simulink models (shown in Figures 17.0.21 and 17.0.23) and create the plots (Figures 17.0.22 and 17.0.24) is given below

```
<HWChapterLinSysProblem39.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem39 Soln

clear all; clc

% Initial condition
alpha0 = 0;
q0     = 0;
theta0 = 0;

% Command
pitch_cmd = 3; %deg

% ---- Proportional Control -----

% Proportional Gain
Kp        = 100;

% Simulation
sim('Pb_39_ProportionalControl_Model',[0 40])

% Plotting
figure(1);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Proportional Control: Kp = 100')

% ---- Integral Control -----

% Integral Gain
Ki        = 4;
```

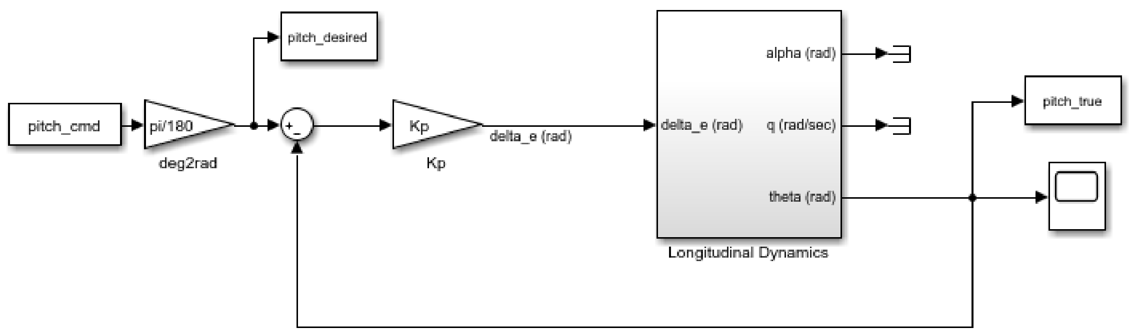


Figure 17.0.21: Problem 15.20.31 Simulink Model 'Pb_39_KpCntrl_Model'.

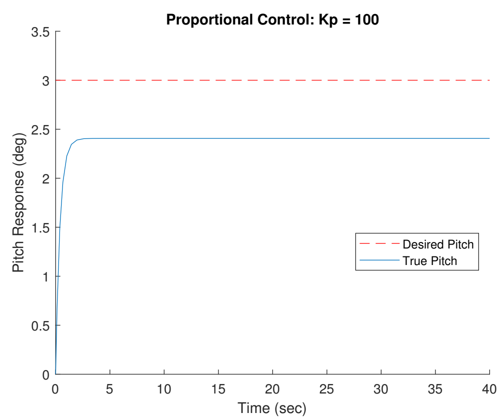


Figure 17.0.22: Problem 15.20.31. Response of the aircraft.

```
% Simulation
sim('Pb_39_IntegralControl_Model',[0 40])

% Plotting
figure(2);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Integral Control: Ki = 4')
```

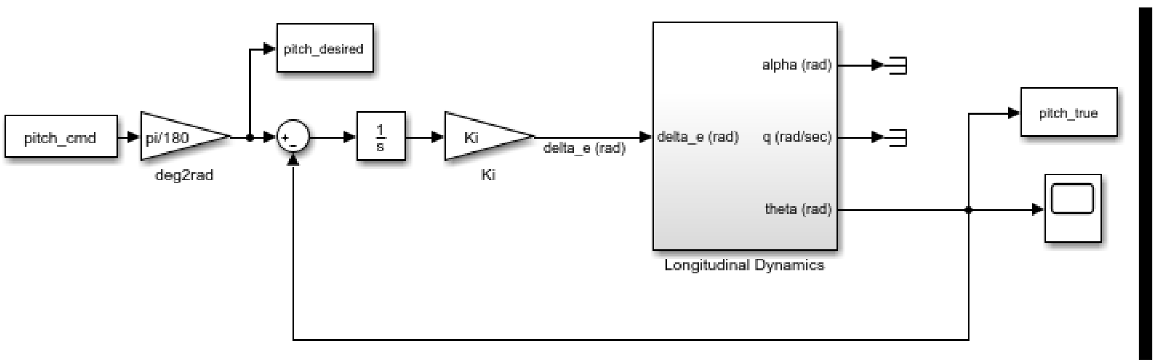


Figure 17.0.23: Problem 15.20.31 Simulink Model 'Pb_39_KiCntrl_Model'.

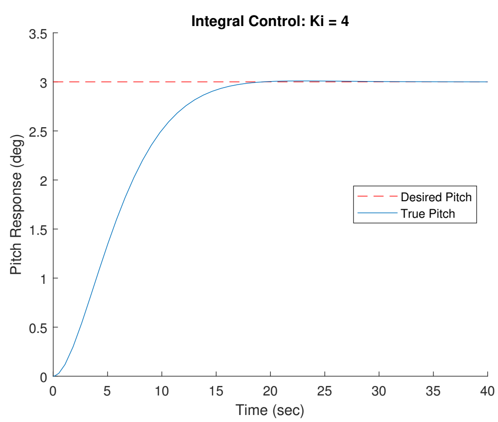


Figure 17.0.24: Problem 15.20.31. Response of the aircraft.

Problem 15.20.40. Consider the same Simulink setup as in Problem 15.20.39. Now, we want to use feedback control to make the aircraft follow an oscillation in pitch angle given by $\sin t$. Set up integral control that takes the error between the true pitch angle and the desired pitch angle as input and outputs elevator deflection angle. Add a gain block to the output of the integrator block and tune the gain to obtain the best possible result. Change the relative tolerance to 10^{-10} by selecting "Simulation", then "Model Configuration Parameters." Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same axes. Plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using integral control?

Now set up the controller as the transfer function

$$G(s) = \frac{1}{s^2 + 1}.$$

Let the input to this transfer function be the error between the true pitch angle and the desired pitch angle and let its output be the elevator deflection angle. Add a gain block to the output of this transfer function and tune it to try and get the best result. Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same plot. Plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using this transfer function?

Next, set up the controller as the transfer function

$$G(s) = \frac{2}{s^2 + 4}.$$

Let this transfer function take the error between the true pitch angle and the desired pitch angle as input and let it output the elevator deflection angle. Add a gain block to the output of this transfer function and tune it to try and get the best result. Run your Simulink model from a MATLAB script and plot both the true pitch angle and the desired pitch angle on the same plot. Plot the true pitch angle using a solid blue line and the desired pitch angle using a dashed red line. Are you able to follow the command using this transfer function?

What can you infer from the last two problems in terms of the relationship between the command signal and the poles of the controller? Hint: Consider the Laplace transforms of $\sin t$ and $\sin 2t$?

Solution 15.20.40.
For integral control with gain K_i, the sensitivity function is given by

$$S(s) = \frac{1}{1 + L} = \frac{s(s + 0.5)}{s^2 + 0.5s + 0.0203K_i}$$

The Laplace transform of the error signal $\hat{e}(s)$ for a harmonic command $\hat{r}(s) = 1/(s^2 + 1)$, is given by

$$\hat{e}(s) = S(s)\hat{r}(s) = \frac{s(s + 0.5)}{s^2 + 0.5s + 0.0203K_i}\frac{1}{s^2 + 1}$$

Note that $\hat{e}(s)$ has two poles on the imaginary axis, and hence FVT is not legal. Consequently, FVT can not be used to determine the gain K_i that gives a zero asymptotic error. However, K_i can be tuned to get the best possible response. Figure 17.0.26 shows the response of the aircraft for $K_i = 30$. Note that the error is nonzero.

For the controller $K/(s^2+1)$, the sensitivity function is given by

$$S(s) = \frac{1}{1+L} = \frac{(s^2+1)(s+0.5)}{s^3+0.5s^2+s+0.5+0.0203K}$$

The Laplace transform of the error signal $\hat{e}(s)$ for a harmonic command $\hat{r}(s) = 1/(s^2+1)$, is given by

$$\hat{e}(s) = S(s)\hat{r}(s) = \frac{s+0.5}{s^3+0.5s^2+s+0.5+0.0203K}$$

Using the Routh test for third-order polynomial in the denominator, we have $a_2 = 0.5 > 0$, $a_1 = 1 > 0$, $a_0 = 0.5+0.0203K > 0$ and $a_2a_1 = 0.5 > a_0 = 0.5+0.0203K$, which implies that the polynomial is asymptotically stable for all $-24.63 < K < 0$. Hence, FVT is legal for all $-24.63 < K < 0$. FVT yields the following asymptotic error

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0^+} s\hat{e}(s) \\ &= \lim_{s\to 0^+} \frac{s(s+0.5)}{s^3+0.5s^2+s+0.5+0.0203K} = 0.\end{aligned}$$

It follows that the asymptotic error is zero for any choice of $-24.63 < K < 0$, however, different choices of the gain K will yield different transient responses. Also, note that, the Laplace transform of the given harmonic command and the transfer function of the controller have same poles. Figure **??** shows the response of the aircraft for $K = -20$. Note that the error goes to zero asymptotically.

For the controller $2K/(s^2+4)$, the sensitivity function is given by

$$S(s) = \frac{1}{1+L} = \frac{(s^2+4)(s+0.5)}{s^3+0.5s^2+4s+2+0.0406K}$$

The Laplace transform of the error signal $\hat{e}(s)$ for a harmonic command $\hat{r}(s) = 1/(s^2+1)$, is given by

$$\hat{e}(s) = S(s)\hat{r}(s) = \frac{(s^2+4)(s+0.5)}{(s^3+0.5s^2+4s+2+0.0406K)}\frac{1}{(s^2+1)}$$

Note that $\hat{e}(s)$ has two poles on the imaginary axis, and hence FVT is not legal. Consequently, FVT can not be used to determine the gain K that gives a zero asymptotic error. However, K can be tuned to get the best possible response. Figure **??** shows the response of the aircraft for $K = -40$. Note that the error is nonzero.

The M-file used to run the Simulink models (shown in Figures 17.0.25, 17.0.27 and 17.0.29) and create the plots (Figures 17.0.26, 17.0.28 and 17.0.30) is given below

```
<HWChapterLinSysProblem40.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem40 Soln

clear all; clc

% Initial condition
alpha0 = 0;
q0     = 0;
theta0 = 0;
```

```
% ---- Integral Control -----

% Integral Gain
Ki        = 30;

% Simulation
sim('Pb_40_KiCntrl_Model',[0 40])

% Plotting
figure(1);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Integral Control')

% ---- Internal Model Control (Model with Cmd Freq) -----

% Gain
K        = -20;

% Simulation
sim('Pb_40_IMCntrlWithCmdFreq_Model',[0 40])

% Plotting
figure(2);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Controller: G(s) = -20/(s^2+1)')

% ---- Internal Model Control (Model without Cmd Freq) -----

% Gain
K        = -40;

% Simulation
sim('Pb_40_IMCntrlWithoutCmdFreq_Model',[0 40])

% Plotting
figure(3);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
```

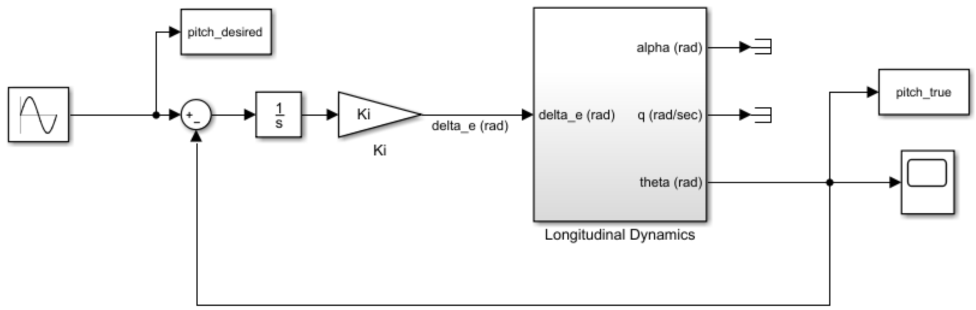


Figure 17.0.25: Problem 15.20.31 Simulink Model 'Pb_40_KiCntrl_Model'.

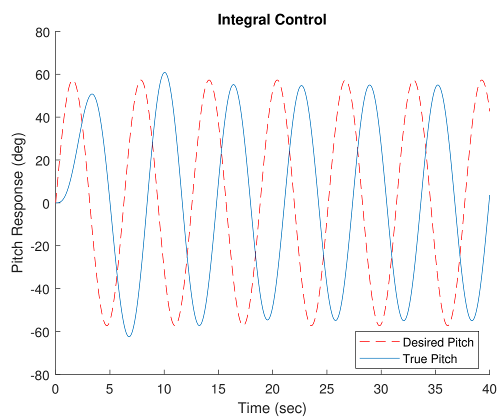


Figure 17.0.26: Problem 15.20.31. Response of the aircraft.

```
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Controller: G(s) = -2(40)/(s^2+4)')
```

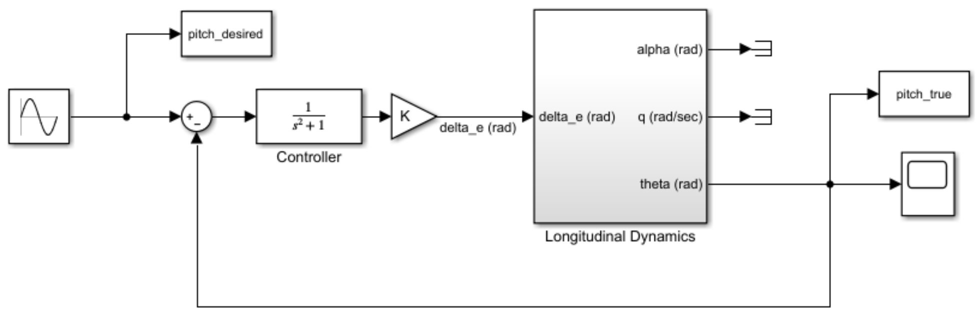


Figure 17.0.27: Problem 15.20.31 Simulink Model 'Pb_40_IMCntrlWithCmdFreq_Model'.

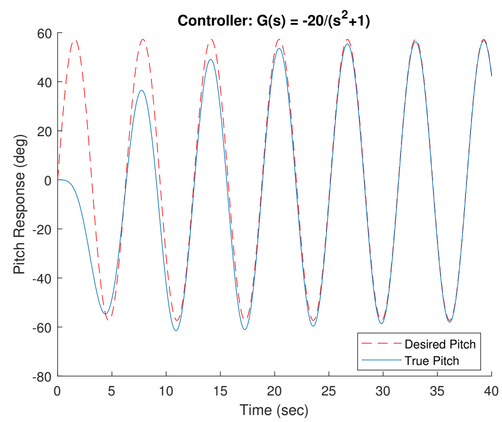


Figure 17.0.28: Problem 15.20.31. Response of the aircraft.

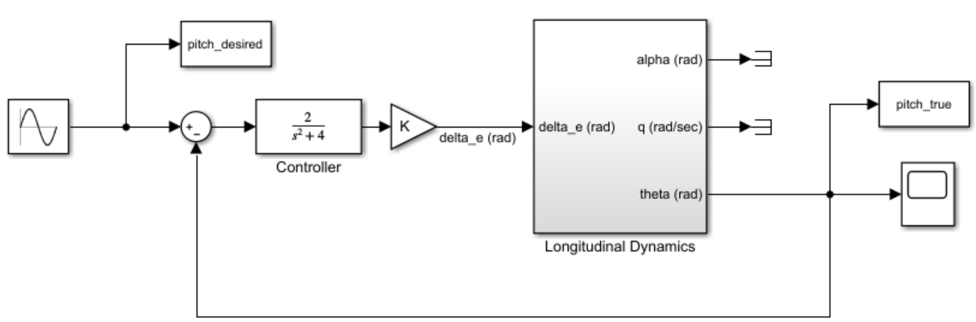


Figure 17.0.29: Problem 15.20.31 Simulink Model 'Pb_40_IMCntrlWithoutCmdFreq_Model'.

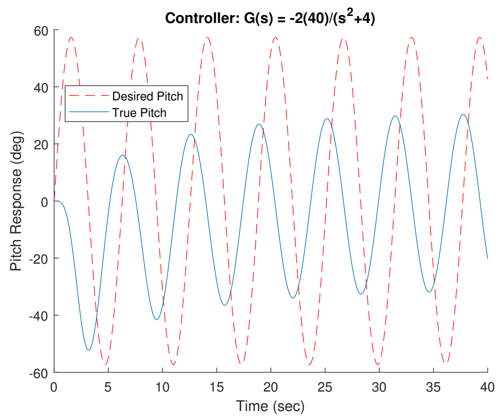


Figure 17.0.30: Problem 15.20.31. Response of the aircraft.

Problem 15.20.41. Redo Problems 15.20.39 and 15.20.40, but instead of the dynamics of the aircraft, let the plant be given by the transfer function

$$G(s) = \frac{4s^2 + 4s + 1}{s^3 + s^2 + \pi s + 0.777}.$$

You may use the transfer function block in your Simulink model. Is your conclusion here the same as your conclusion in Problem 15.20.40?

Solution 15.20.41.
A similar FVT analysis as in the solutions for Problems 15.20.39 and 15.20.40 can be done and reached to the aame conclusions.

The M-file used to run the Simulink models (shown in Figures 17.0.31, 17.0.33, 17.0.35, 17.0.37 and 17.0.39) and create the plots (Figures 17.0.32, 17.0.34, 17.0.36, 17.0.38 and 17.0.40) is given below

```
<HWChapterLinSysProblem41.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem41 Soln

clear all; clc

% ---------- Step Command ---------

% Command
pitch_cmd = 3; %deg

% ---- Proportional Control -----

% Proportional Gain
Kp        = 20;

% Simulation
sim('Pb_41_ProportionalControl_Model',[0 40])

% Plotting
figure(1);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Proportional Control: Kp = 20')


% ---- Integral Control -----
```

```
% Integral Gain
Ki       = 0.2;

% Simulation
sim('Pb_41_IntegralControl_Model',[0 40])

% Plotting
figure(2);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Integral Control: Ki = 0.2')

% ------------ Harmonic Command (sin t) --------

% Integral Gain
Ki       = 4;

% Simulation
sim('Pb_41_IntegralCntrl_HarmonicCmd_Model',[0 40])

% Plotting
figure(3);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Integral Control: Ki = 4')


% ---- Internal Model Control (Model with Cmd Freq) -----

% Gain
K       = 0.2;

% Simulation
sim('Pb_41_CntrllWithCmdFreq_HarmonicCmd_Model',[0 40])

% Plotting
figure(4);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
```

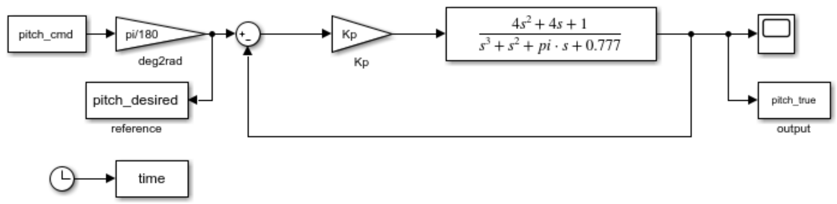


Figure 17.0.31: Problem 15.20.31 Simulink Model 'Pb_41_ProportionalControl_Model'.

```
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Controller: G(s) = 0.2/(s^2+1)')

% ---- Internal Model Control (Model without Cmd Freq) -----

% Gain
K        = -0.15;

% Simulation
sim('Pb_41_CntrllWithoutCmdFreq_HarmonicCmd_Model',[0 40])

% Plotting
figure(5);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Controller: G(s) = -2(0.15)/(s^2+4)')
```

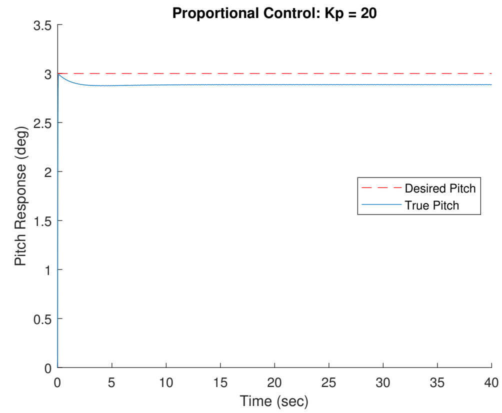


Figure 17.0.32: Problem 15.20.31. Response of the aircraft.

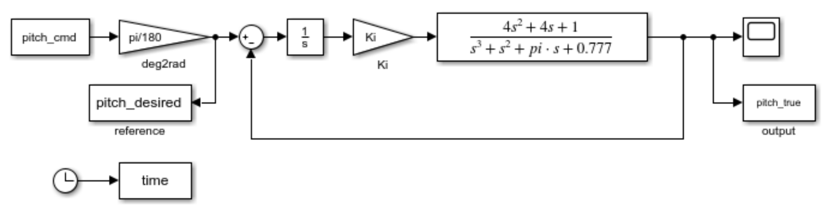


Figure 17.0.33: Problem 15.20.31 Simulink Model 'Pb_41_IntegralControl_Model'.

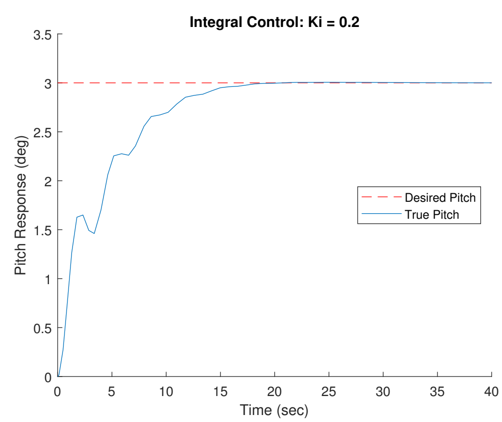


Figure 17.0.34: Problem 15.20.31. Response of the aircraft.

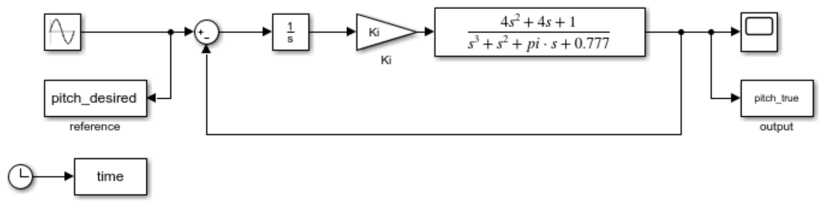


Figure 17.0.35: Problem 15.20.31 Simulink Model 'Pb_41_IntegralControl_HarmonicCmd_Model'.

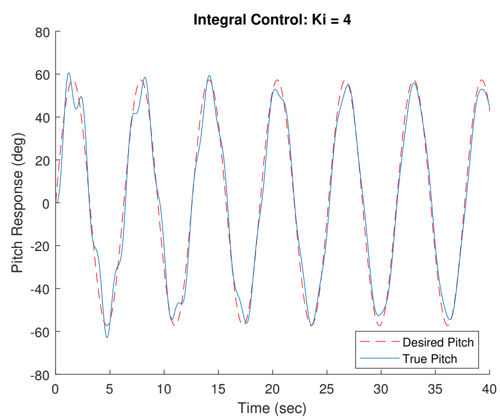


Figure 17.0.36: Problem 15.20.31. Response of the aircraft.

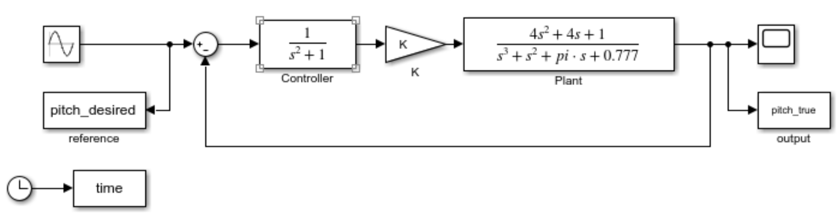


Figure 17.0.37: Problem 15.20.31 Simulink Model 'Pb_41_CntrllWithCmdFreq_HarmonicCmd_Model'. ∎

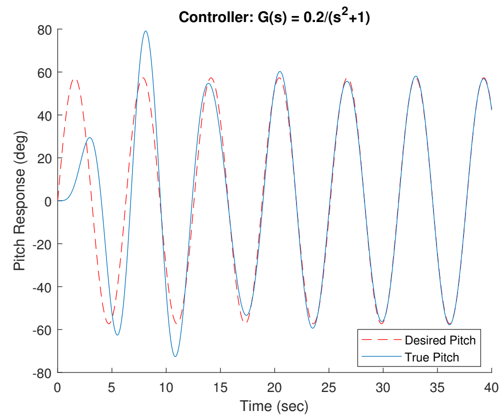


Figure 17.0.38: Problem 15.20.31. Response of the aircraft.

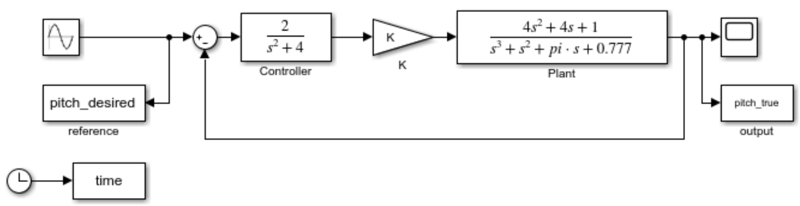


Figure 17.0.39: Problem 15.20.31 Simulink Model 'Pb_41_CntrllWithoutCmdFreq_HarmonicCmd_Model'. ■

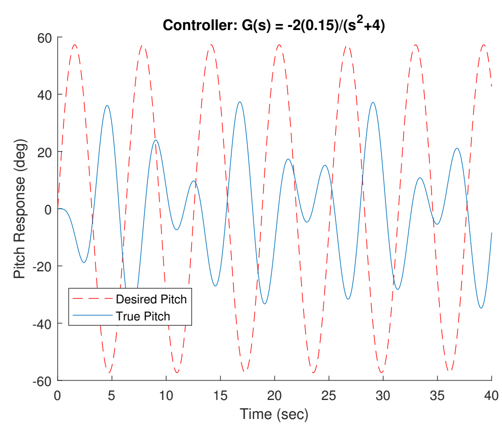


Figure 17.0.40: Problem 15.20.31. Response of the aircraft.

Problem 15.20.42. Redo Problem 15.20.39 for different aircraft dynamics, given by

$$\begin{aligned}\dot{\alpha}(t) &= -0.313\alpha(t) + 56.7q(t) + 0.232\delta e(t),\\ \dot{q}(t) &= -0.0139\alpha(t) - 0.426q(t) + 0.0203\delta e(t),\\ \dot{\theta}(t) &= 56.7q(t),\end{aligned}$$

where α is the angle of attack in rad, q is the pitch rate in rad/sec, θ is the pitch angle in rad, and δe is the elevator deflection angle in rad. Assume that $\alpha(0) = q(0) = \theta(0) = 0$. Let the input of the system be the elevator deflection angle δe, and the output of the system be the pitch angle θ. You may use the state space block in your Simulink model. Is your conclusion here the same as your conclusion in Problem 15.20.39? Why or why not?

Solution 15.20.42.
The state space (A, B, C, D) system is given by

$$A = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad D = 0.$$

The transfer function can be evaluated using the relation,

$$G(s) = C(sI - A)^{-1}B + D.$$

Substituting the matrices we obtain,

$$G(s) = \frac{1.151(s + 0.1541)}{s(s^2 + 0.739s + 0.9215)}.$$

For proportional control with gain K_p, the sensitivity function is given by

$$S(s) = \frac{1}{1 + L} = \frac{s(s^2 + 0.739s + 0.9215)}{s(s^2 + 0.739s + 0.9215) + 1.151K_p(s + 0.1541)}$$

The Laplace transform of the error signal $\hat{e}(s)$ for a step command $\hat{r}(s) = 0.0524/s$, is given by

$$\hat{e}(s) = S(s)\hat{r}(s) = \frac{0.0524(s^2 + 0.739s + 0.9215)}{s^3 + 0.739s^2 + (0.9215 + 1.151K_p)s + 0.1774K_p}$$

Using the Routh test for third-order polynomial in the denominator, we have $a_2 = 0.5 > 0$, $a_1 = 1 > 0$, $a_0 = 0.1774K_p > 0$ and $a_2a_1 = 0.739(0.9215 + 1.151K_p) > a_0 = 0.1774K_p$, which implies that the polynomial is asymptotically stable for all $K_p > 0$. Hence, FVT is legal for all $K_p > 0$. FVT yields the following asymptotic error

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0^+} s\hat{e}(s)\\ &= \lim_{s\to 0^+} \frac{0.0524s(s^2 + 0.739s + 0.9215)}{s^3 + 0.739s^2 + (0.9215 + 1.151K_p)s + 0.1774K_p} = 0.\end{aligned}$$

It follows that the asymptotic error is zero for any choice of $K_p > 0$, however, different choices of the gain K_p will yield different transient responses. Note that, this result is different from the result of Problem 15.20.39, where the proportional control yields a non-zero asymptotic error. Also, note that, sine the plant has a pole at 0, therefore, an integrator is not required in the controller for step command following.

A similar FVT analysis for the integral control with gain K_i yields zero asymptotic error.

The M-file used to run the Simulink models (shown in Figures 17.0.41 and 17.0.43) and create the plots (Figures 17.0.42 and 17.0.44) is given below

```
<HWChapterLinSysProblem42.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem42 Soln

clear all; close all; clc

% State Space
A = [-0.313 56.7 0;   %alphadot
    -0.0139 -0.426 0; %qdot
    0 56.7 0];        %thetadot
B = [0.232 0.0203 0]';
C = [0 0 1];
D = 0;

% Initial condition
alpha0 = 0; %For alphadot integrator block
q0     = 0; %For qdot integrator block
theta0 = 0; %For thetadot integrator block
x0     = [alpha0 q0 theta0]'; %For State Space Block

% ---------- Step Command ---------

% Command
pitch_cmd = 3; %deg

% ---- Proportional Control -----

% Proportional Gain
Kp       = 2;

% Simulation
sim('Pb_42_ProportionalControl_Model',[0 40])

% Plotting
figure(1);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Proportional Control: Kp = 2')


% ---- Integral Control -----
```

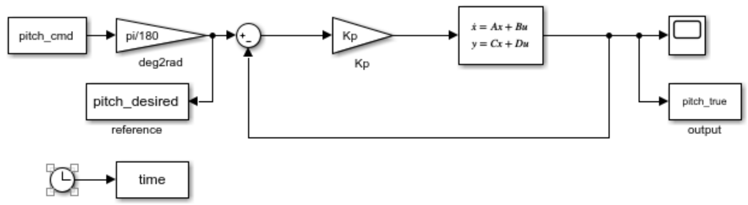


Figure 17.0.41: Problem 15.20.31 Simulink Model 'Pb_42_ProportionalControl_Model'.

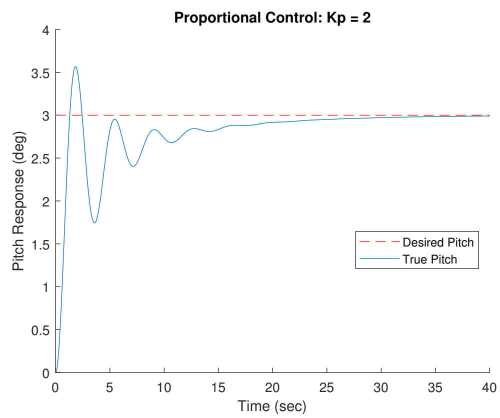


Figure 17.0.42: Problem 15.20.31. Response of the aircraft.

```
% Integral Gain
Ki         = 0.2;

% Simulation
sim('Pb_42_IntegralControl_Model',[0 40])

% Plotting
figure(2);clf
hold on
plot(time,rad2deg(pitch_desired),'r--')
plot(time,rad2deg(pitch_true))
xlabel('Time (sec)')
ylabel('Pitch Response (deg)')
legend('Desired Pitch','True Pitch','location','best')
title('Integral Control: Ki = 0.2')
```

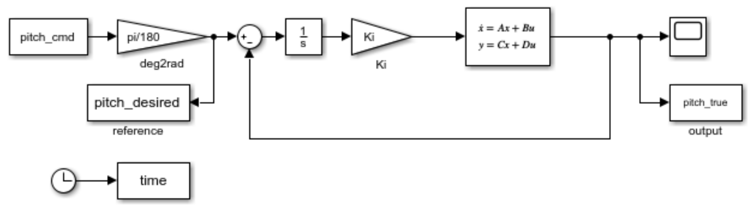


Figure 17.0.43: Problem 15.20.31 Simulink Model 'Pb_42_IntegralControl_Model'.

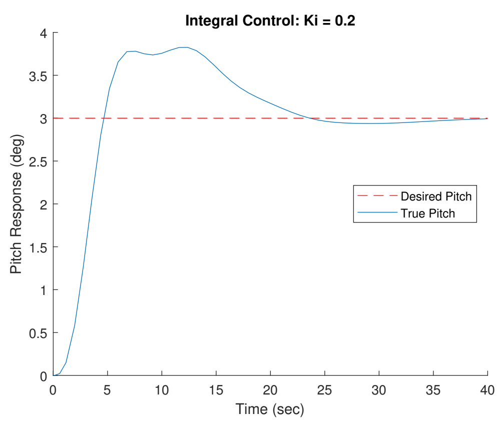


Figure 17.0.44: Problem 15.20.31. Response of the aircraft.

Problem 15.20.43. Consider the basic servo loop with a SISO plant G and SISO controller G_c chosen such that the closed-loop system is asymptotically stable. Assume that neither G nor G_c has a zero at zero. Then, do the following:

i) Case 1: Show that if G_c has at least one integrator, then the asymptotic error to a step command is zero (whether or not G has an integrator). Determine the asymptotic value of the control.

ii) Case 2: Show that if G has at least one integrator, then the asymptotic error to a step command is zero (whether or not G_c has an integrator). Determine the asymptotic value of the control.

iii) Case 3: Show that if G has at least one integrator but G_c has no integrators, then the asymptotic error to a step command in the presence of a nonzero constant disturbance is not zero. The disturbance is added to the control.

iv) Case 4: Show that if G_c has at least one integrator, then the asymptotic error to a step command in the presence of a nonzero constant disturbance is zero whether or not G has an integrator.

v) Illustrate the above four cases using Simulink with $G(s) = 1/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s)$, or vice versa. Plot the plant output, the command-following error, and the control input. Write an mfile to check stability using the 4th-order Routh conditions in the notes as well as root locus with the gain running from 0 to 1.

Solution 15.20.43.

Problem 15.20.44. Consider the basic servo loop with a SISO plant G and SISO controller G_c chosen such that the closed-loop system is asymptotically stable. The command and disturbance are steps, which may or may not be zero. The disturbance is added to the control. Assume that the measurement y that is fed back is corrupted by an unknown nonzero constant bias, that is, $y = y_0 + b$, where y_0 is the plant output and b is the unknown constant bias. Since b is unknown, we are not able to use its value in the control law. The "false" error that the controller operates on is $e = r - y$. However, the *true error* is $e_{\rm true} = r - y_0$. Then, do the following:

i) Case 1: Show that if G_c has at least one integrator, then the asymptotic error is zero (whether or not G has an integrator) but the asymptotic true error is not zero. Show that this statement holds for all values of the command and disturbance.

ii) Case 2: Assume that neither G nor G_c has an integrator, and assume that G has at least one zero at zero. Show that, if the command is zero, then the true error converges to zero for all values of the disturbance.

iii) Case 3: Assume that neither G nor G_c has an integrator, and assume that $L = GG_c$ has at least one zero at zero. Determine necessary and sufficient conditions on the command and disturbance such that the true error converges to zero. Use this result to show that, if the command is not zero and the disturbance is zero, then the true error does not converge to zero.

iv) Illustrate the above three cases using Simulink. For Case 1, let $G(s) = 1/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s)$. For Case 2, let $G(s) = s/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s + 2)$. For Case 3, let $G(s) = s/(s^2 + 2s + 1)$ and $G_c(s) = 1/(s^2 + 2s + 2)$.

v) Is it possible to follow a step command (in the sense of the true error) despite the presence of an unknown measurement bias?

Solution 15.20.44.

Problem 15.20.45. Consider the second-order system described by the differential equation $\ddot{y} + \dot{y} - 3y = u$.

i) Assuming $y(0) = \dot{y}(0) = 0$, derive the transfer function of the system from $\hat{u}(s)$ to $\hat{y}(s)$.

ii) Simulate the system with the step input $u(t) = \mathbf{1}(t-1)$ for 100 sec. How does $y(t)$ evolve in time? Attach a plot of $y(t)$.

iii) Now include the proportional feedback control law $\hat{u}(s) = K_\mathrm{p}\hat{e}(s)$, and simulate the closed-loop system with the step command $r(t) = \mathbf{1}(t-1)$ for 20 sec. Choose K_p such that the maximum overshoot in $y(t)$ is less than 100%, and $|e(t)| = |r(t) - y(t)|$ is less than 0.2 for $t > 10$. Does the error converge to zero as t increases? Comparing your results from the previous part, what is the benefit of using proportional feedback compared to open loop? Attach a screenshot of your Simulink model, as well as plots of $e(t)$ and $y(t)$ showing that the design requirements are met.

iv) Using your choice of K_p from *iii*), now introduce the proportional-integral (PI) control law

$$\hat{u}(s) = \left(K_\mathrm{p} + \frac{K_\mathrm{I}}{s}\right)\hat{e}(s),$$

and simulate the closed-loop system with the command $r(t) = \mathbf{1}(t-1)$ for 50 sec. Adjust K_I so that the maximum overshoot in $y(t)$ is less than 100%, $|e(t)| < 0.2$ for $t > 10$, and $|e(t)| < 0.01$ for $t > 50$. Does the command-following error converge to zero as t increases? Comparing your results from the previous part, what is the benefit of integral control? Attach a screenshot of your Simulink model, as well as plots of $e(t)$ and $y(t)$ showing that the design requirements are met. Keep in mind that too much control might destabilize the closed-loop system. Keeping K_p the same, increase K_I and show that the closed-loop system becomes unstable.

v) Using K_p and K_I from *iv*), now consider the harmonic command $r(t) = \sin(2t)$, and simulate the closed-loop system for 50 sec. Attach a plot of $e(t)$. Does the PI control law drive the tracking error $e(t)$ to zero in this case?

vi) Replace the PI control law in *v*) with the feedback control law

$$\hat{u}(s) = \left(20 + \frac{5s}{s^2+4}\right)\hat{e}(s),$$

and simulate the closed-loop system with the harmonic reference input $r(t) = \sin(2t)$ for 50 seconds. Attach a plot of $e(t)$. Is the command-following error driven to zero in this case?

vii) Compare the Laplace transforms of the step input $r(t) = \mathbf{1}(t)$ and the harmonic input $r(t) = \sin(2t)$ with the Laplace transform of the PI controller, and the controller you used in *vi*). Compare the denominators of the input signals and the controller. Also, compare the Laplace transform of a step/ramp command and the denominator of the loop transfer function for a type I/II control system. What pattern do you see?

Solution 15.20.45.

Problem 15.20.46. Consider the second-order transfer function

$$G(s) = \frac{1}{s^2 + 3s - 2}$$

controlled by the PI controller

$$K(s) = K_{\mathrm{P}} + \frac{K_{\mathrm{I}}}{s}$$

in a servo loop. Determine the values of K_{P} and K_{I} that render the closed-loop system stable by sketching the region in the $K_{\mathrm{P}}, K_{\mathrm{I}}$ plane of stabilizing gains.

Solution 15.20.46.
The closed-loop transfer function is,

$$\tilde{G}(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{K_P s + K_I}{s^3 + 3s^2 + (K_P - 2)s + K_I}.$$

Using the Routh test for the third order polynomial ($a_0 < a_2 a_1$), we determine the condition for the closed-loop system to be stable:

$$K_I < 3(K_P - 2) \quad , \quad K_P > 2 \quad , \quad K_I > 0$$

The shaded area in the figure below represents the region in $K_P K_I$-plane where the closed-loop system is stable.

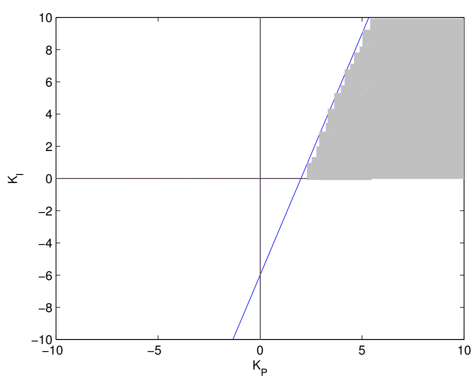


Figure 17.0.45: Problem 15.20.46 Stable Region in K_P, K_I-plane

Problem 15.20.47. Consider the type II control system with plant

$$G(s) = \frac{1}{s(s+1)}$$

and PI controller

$$K(s) = K_{\mathrm{P}} + \frac{K_{\mathrm{I}}}{s}$$

in a servo loop. First, determine which values of K_{P} and K_{I} provide zero steady-state error for a ramp command. Next, use Matlab or Simulink to simulate the closed-loop system, and choose K_{P} and K_{I} for good rise time and reasonable overshoot for a step command. Plot the error response for a ramp command, and verify numerically and analytically that the ramp command error converges to zero.

Solution 15.20.47.
The reference-to-output relation in the Laplace domain is,

$$Y(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} R(s) = \frac{K_P s + K_I}{s^3 + s^2 + K_P s + K_I} R(s)$$

from which the reference-to-error relation is,

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= \left(1 - \frac{G(s)K(s)}{1 + G(s)K(s)}\right) R(s) \\ &= \frac{1}{1 + G(s)K(s)} R(s) = S(s)R(s) \\ \Rightarrow E(s) &= \frac{s^2(s+1)}{s^3 + s^2 + K_P s + K_I} R(s) \end{aligned}$$

Substituting $R(s) = 1/s^2$ for the ramp input and using final value theorem yields,

$$e_\infty = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(s+1)}{s^3 + s^2 + K_P s + K_I} = 0$$

whatever values K_P and K_I might take as long as the denominator polynomial is asymptotically stable. Therefore using the Routh test for the stability of the 3rd order polynomial ($a_0 < a_2 a_1$), we have

$$0 < K_I < K_P$$

Figure 17.0.46 shows the output response $y(t)$ for a unit step command $r(t) = 1$ with

$$K_P = 15 \;\;,\;\; K_I = 1.$$

Here the transfer function is given by,

$$\tilde{G}(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{K_P s + K_I}{s^3 + s^2 + K_P s + K_I}$$

and the following diagram shows the relevant Simulink model.

Figure 17.0.47 shows the error response $e(t)$ for a unit ramp command $r(t) = t$ with $K_P = 5$, $K_I = 2$, which verifies that the ramp command error converges to zero. The diagram shows the relevant Simulink model.

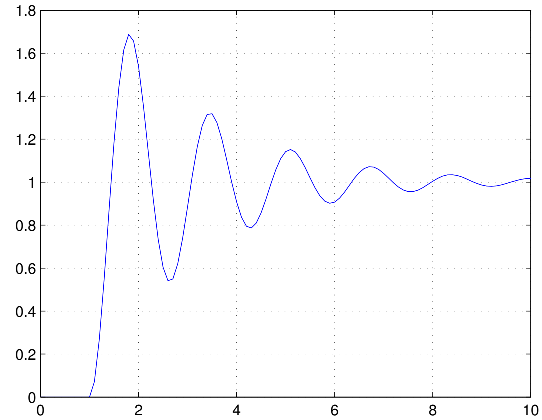
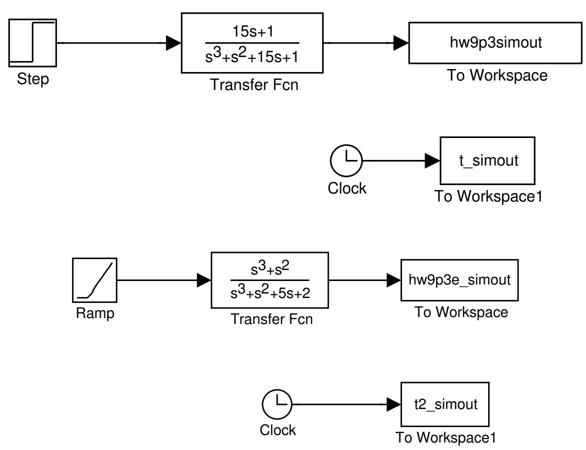


Figure 17.0.46: Problem 15.20.37. Output response for a unit step command with $K_P = 15$ $K_I = 1$



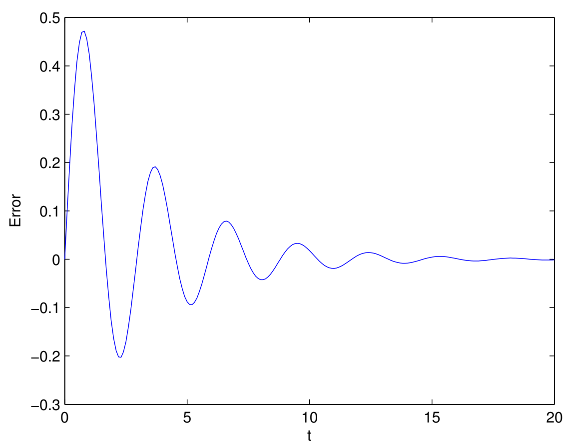


Figure 17.0.47: Problem 15.20.37. Error response for a unit ramp command with $K_P = 5$ $K_I = 2$

Problem 15.20.48. Sketch the root loci for the following loop transfer functions by using the root locus rules and then check your sketches using Matlab's root locus function:

i) $L(s) = \frac{(s+1)(s+2)}{(s-1)(s-2)(s-3)}$.

ii) $L(s) = \frac{(s+2)^2}{(s+1)(s^2+1)}$.

Solution 15.20.48.
Refer to Figure 17.0.48.

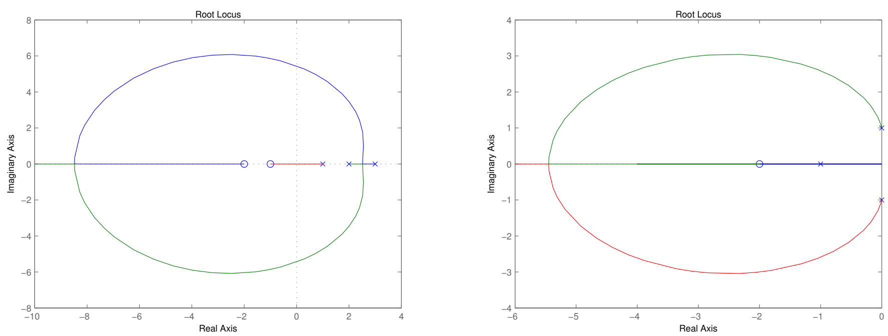


Figure 17.0.48: Problem 15.20.38. Root Locus for i) $L(s) = \frac{(s+1)(s+2)}{(s-1)(s-2)(s-3)}$ and ii) $L(s) = \frac{(s+2)^2}{(s+1)(s^2+1)}$

Problem 15.20.49. At Mach .6, an experimental aircraft has unstable phugoid eigenvalues $0.04 \pm 0.12 j$ and stable short period eigenvalues $-4.3 \pm 5.7 j$. Wind tunnel testing reveals that the elevator-to-pitch transfer function has one zero located at -0.2. For proportional control, sketch the root locus, determine the center and asymptotes, and discuss the stability of the closed-loop longitudinal dynamics for high values of k.

Solution 15.20.49.
From the given conditions, the open loop transfer function is,

$$G(s) = \frac{(s+0.2)}{[s-(0.04 \pm 0.12j)][s-(-4.3 \pm 5.7j)]}$$
$$\Rightarrow G(s) = \frac{(s+0.2)}{(s^2-0.08s+0.016)(s^2+8.6s+50.98)}$$

The root locus of the closed-loop transfer function in terms of the proportional gain K is given in Figure 17.0.49.

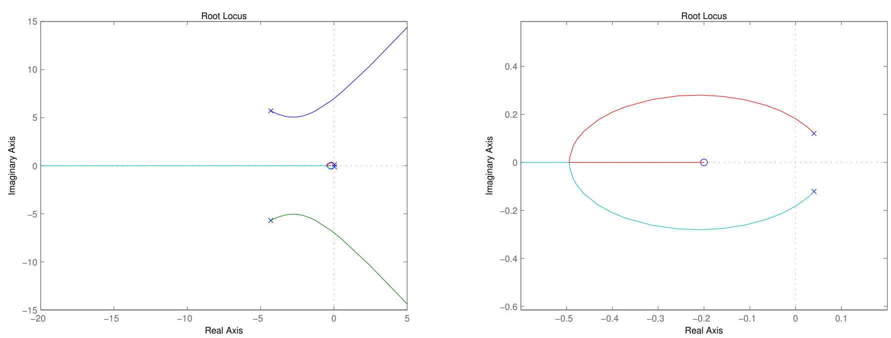


Figure 17.0.49: Problem 15.20.39. Root Locus for G(s) and its close-up around the short period mode

The center and asymptotes are determined as,

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{(0.04*2+(-4.3)*2)-(-0.2))}{4-1} = -2.7733$$
$$\phi = \frac{(2l+1)\pi}{n-m} = \frac{\pi}{3}, \pi, \frac{5\pi}{3} \qquad l = 0, 1, 2$$

In the root locus, as the gain k increases, the unstable phugoid modes becomes stable, whereas the stable short period mode becomes unstable. In order for both modes to be stable, the gain k should be selected in the approximate range of $4.25 < k < 417$.

Problem 15.20.50. An experimental aircraft has unstable short period eigenvalues $0.04 \pm 0.12 j$ and stable phugoid eigenvalues $-4.3 \pm 5.7 j$. Wind tunnel testing reveals that, at Mach .6, the elevator-to-pitch transfer function has three real zeros located at $-0.3, -1.7, -8.4$. Sketch the root locus and discuss the stability of the closed-loop longitudinal dynamics using a proportional control. Indicate (but you do not need to compute) the point on the root locus at which all poles have at least $\sqrt{2}/2$ damping by marking the intersection of the root locus and the 0.707 damping-ratio line. (Note: Problem 16.8.5 explains why this minimum value of damping is desirable.

Solution 15.20.50.
From the given condition, the open-loop transfer function is,

$$G(s) = \frac{(s+0.3)(s+1.7)(s+8.4)}{[s-(0.04 \pm 1.12j)][s-(-4.3 \pm 5.7j)]}$$
$$\Rightarrow G(s) = \frac{(s+0.3)(s+1.7)(s+8.4)}{(s^2 - 0.08s + 0.016)(s^2 + 8.6s + 50.98)}$$

The root locus of the closed-loop transfer function in terms of gain K is given below in the following figure. As shown in the figure, the stability characteristics of the short period mode is satisfactory

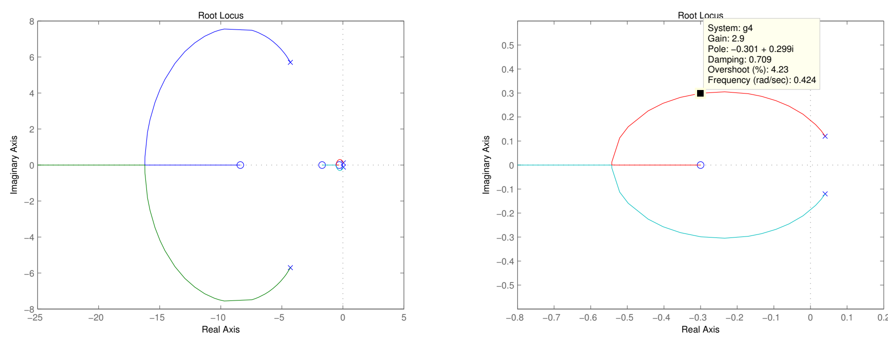


Figure 17.0.50: Problem 15.20.40. Root Locus for $G(s)$ and its close-up around the short period mode

over a large range of gain K. In order to enhance the stability characteristics of phugoid mode, choose $K \simeq 2.9$, which results in the following satisfactory characteristics:

$$s = -0.301 \pm 0.299j$$
$$\zeta_{sp} = 0.709$$
$$\omega_{n\,sp} = 0.424$$

Problem 15.20.51. Consider the linearized model of a Boeing 747 aircraft in straight and level flight given by

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du,$$

where

$$x = \begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix}, \quad u = \begin{bmatrix} \delta r \\ \delta a \end{bmatrix},$$

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.598 & -0.115 & -0.0318 & 0 \\ -3.05 & 0.388 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.00729 & 0 \\ -0.475 & 0.00775 \\ 0.153 & 0.143 \\ 0 & 0 \end{bmatrix}.$$

Determine the eigenvalues of A. What kind of stability does the aircraft have? Which poles correspond to the Dutch roll mode?

Solution 15.20.51.
The eigen values of A are

$$\lambda_{1,2} = -0.0329 \pm 0.9467j$$
$$\lambda_3 = -0.5627$$
$$\lambda_4 = -0.0073$$

Since all the eigen values are in OLHP, the aircraft is asymptotically stable in the lateral mode. $\lambda_{1,2}$ corresponds to the Dutch roll mode.

```
<HWChapterLinSysProblem51.m>

% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem51 Soln

clear all;

A = [-0.0558 -0.9968 0.0802 0.0415;
      0.598  -0.115  -0.0318  0;
      -3.05  0.388  -0.4650  0;
       0  0.0805  1  0];

B = [ 0.00729  0 ;
      -0.475  0.00775 ;
      0.153  0.143 ;
      0  0];

eig_val = eig(A)
```

Problem 15.20.52. Consider the linearized aircraft model in Problem 15.20.51. with outputs r and ϕ. Plot the zeros of all four SISO transfer functions. Next, use Matlab or Simulink to plot the impulse response of all four transfer functions. Finally, apply root locus to each of the four transfer functions with proportional feedback.

Solution 15.20.52.

The zeros, impulse-responses and root-loci of $G_{r/\delta r}$, $G_{r/\delta a}$, $G_{\phi/\delta r}$, and $G_{\phi/\delta a}$ are given by Figures 17.0.51, 17.0.52, 17.0.53, and 17.0.54, respectively.

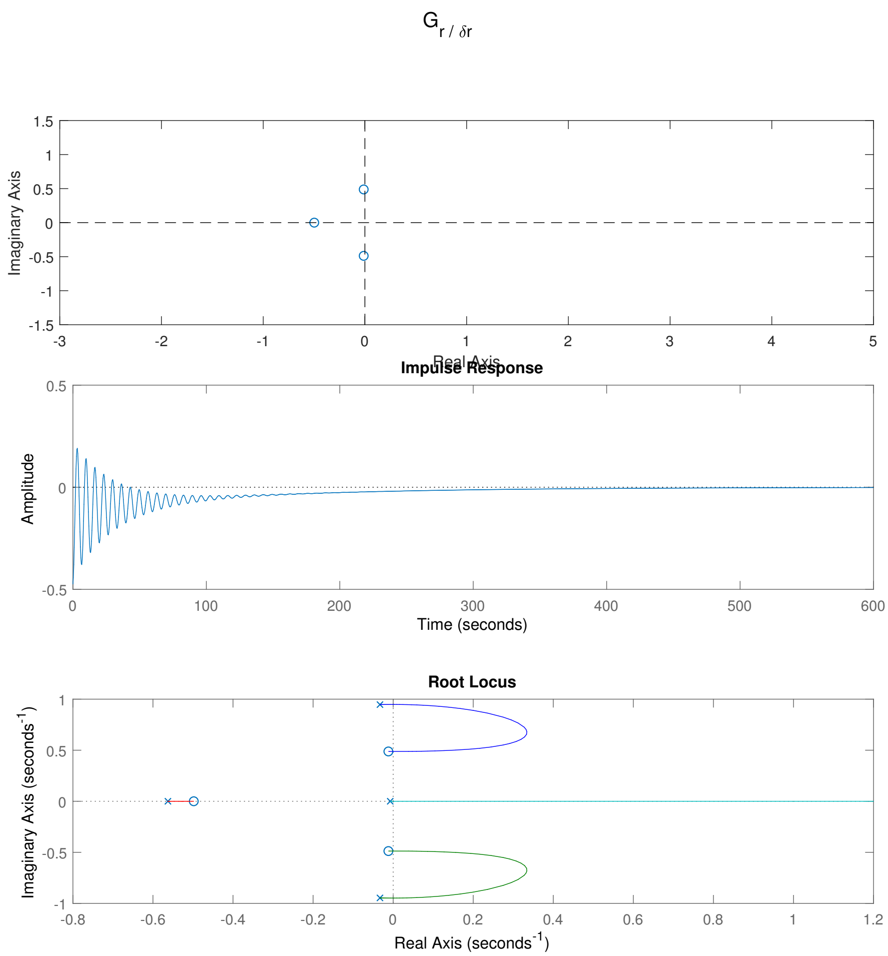


Figure 17.0.51: Problem 15.20.52. Zeros, impulse-response and root-locus of $G_{r/\delta r}$

.

<HWChapterLinSysProblem52.m>

```
% AE345: Flight Dynamics and Control
% Problem ChapterLinSysProblem52 Soln

clear all;

A = [-0.0558 -0.9968 0.0802 0.0415;
      0.598  -0.115  -0.0318  0;
      -3.05  0.388  -0.4650  0;
       0  0.0805  1  0];

B = [ 0.00729  0 ;
      -0.475  0.00775 ;
      0.153  0.143 ;
      0  0];

D = [0 0];

%%%%---------------- Output r
C_r = [0 1 0 0];
% Input rudder
[num_r_dr,den_r_dr] = ss2tf(A,B,C_r,D,1);
G_r_dr              = tf(num_r_dr,den_r_dr);
% Input aileron
[num_r_da,den_r_da] = ss2tf(A,B,C_r,D,2);
G_r_da              = tf(num_r_da,den_r_da);


%%%%---------------- Output phi
C_phi = [0 0 0 1];
% Input rudder
[num_phi_dr,den_phi_dr] = ss2tf(A,B,C_phi,D,1);
G_phi_dr                = tf(num_phi_dr,den_phi_dr);
% Input aileron
[num_phi_da,den_phi_da] = ss2tf(A,B,C_phi,D,2);
G_phi_da                = tf(num_phi_da,den_phi_da);


figure(1);clf;
subplot(3,1,1)
plot(real(roots(num_r_dr)),imag(roots(num_r_dr)),'o')
hold on
plot([0 0],[-1.5 1.5],'k--')
plot([-3 5],[0 0],'k--')
xlabel('Real Axis')
ylabel('Imaginary Axis')
subplot(3,1,2)
impulse(G_r_dr)
subplot(3,1,3)
rlocus(G_r_dr)
```

```
suptitle('G_{r / \deltar}')


figure(2);clf;
subplot(3,1,1)
plot(real(roots(num_r_da)),imag(roots(num_r_da)),'o')
hold on
plot([0 0],[-1.5 1.5],'k--')
plot([-3 5],[0 0],'k--')
xlabel('Real Axis')
ylabel('Imaginary Axis')
subplot(3,1,2)
impulse(G_r_da)
subplot(3,1,3)
rlocus(G_r_da)
suptitle('G_{r / \deltaa}')

figure(3);clf;
subplot(3,1,1)
plot(real(roots(num_phi_dr)),imag(roots(num_phi_dr)),'o')
hold on
plot([0 0],[-1.5 1.5],'k--')
plot([-3 5],[0 0],'k--')
xlabel('Real Axis')
ylabel('Imaginary Axis')
subplot(3,1,2)
impulse(G_phi_dr)
subplot(3,1,3)
rlocus(G_phi_dr)
suptitle('G_{\phi / \deltar}')

figure(4);clf;
subplot(3,1,1)
plot(real(roots(num_phi_da)),imag(roots(num_phi_da)),'o')
hold on
plot([0 0],[-1.5 1.5],'k--')
plot([-3 5],[0 0],'k--')
xlabel('Real Axis')
ylabel('Imaginary Axis')
subplot(3,1,2)
impulse(G_phi_da)
subplot(3,1,3)
rlocus(G_phi_da)
suptitle('G_{\phi / \deltaa}')
```

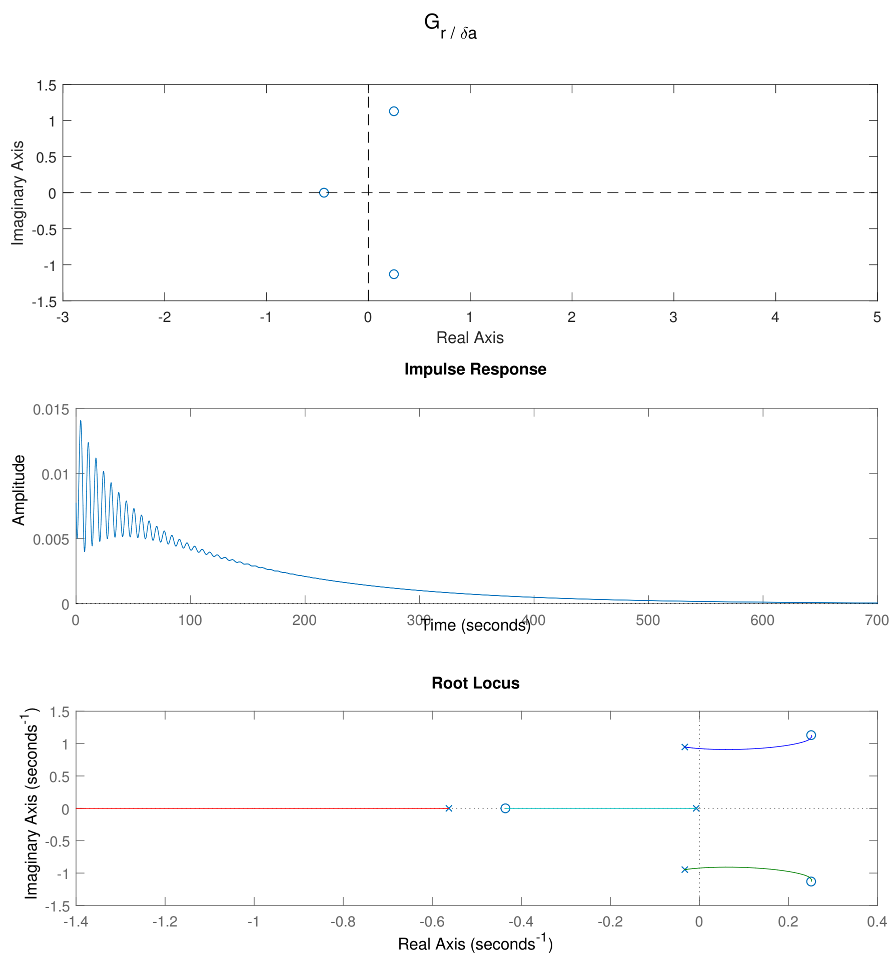


Figure 17.0.52: Problem 15.20.52. Zeros, impulse-response and root-locus of $G_{r/\delta a}$

.

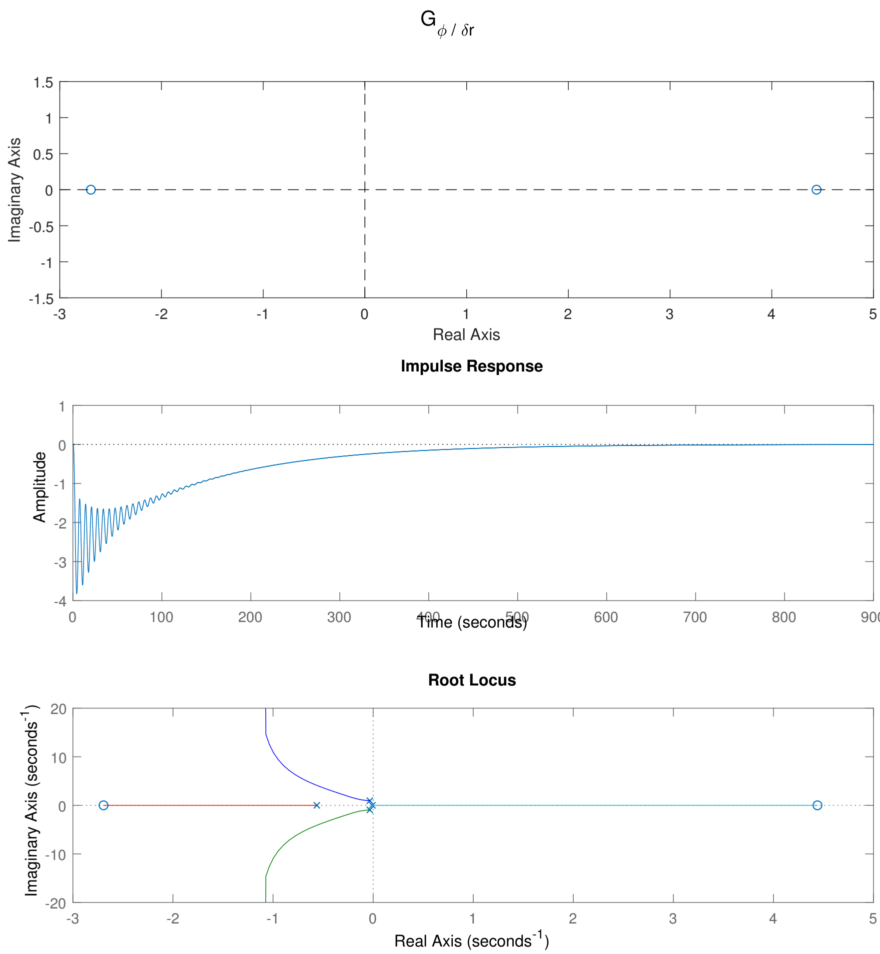


Figure 17.0.53: Problem 15.20.52. Zeros, impulse-response and root-locus of $G_{\phi/\delta r}$
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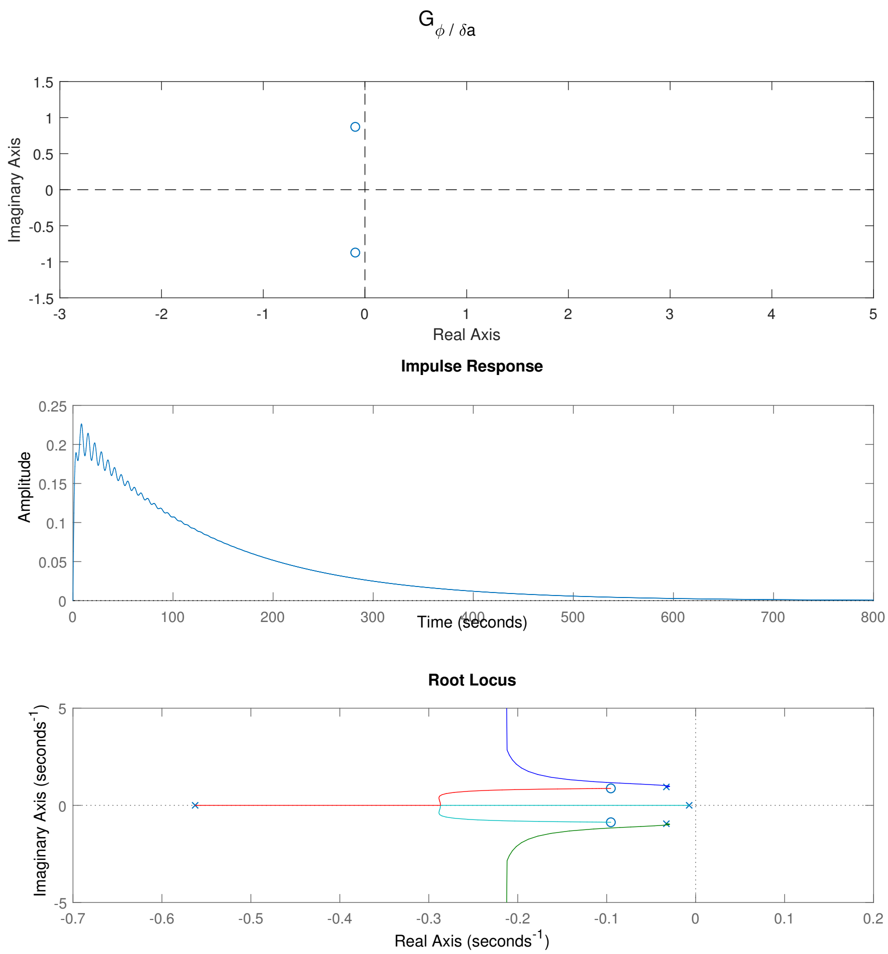

Figure 17.0.54: Problem 15.20.52. Zeros, impulse-response and root-locus of $G_{\phi/\delta a}$
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Chapter Eighteen

Solutions to Chapter 16

Problem 16.8.1. Suppose that a sensor with first-order dynamics has a time constant of .031 sec and a DC gain of 9.2. (Hint: $G(s) = \frac{\alpha}{Ts+1}$.) Now suppose that the input to the sensor is 4.6 volts corrupted by (which means "added to") 60-Hz electrical noise with amplitude 3.3 mV. Describe the sensor output after a large amount of time passes by giving the amplitude and phase of both of its harmonic components, that is, the DC component and the 60-Hz component. Do this two different ways. First use Laplace transform to determine the transient and harmonic steady-state components of the output. Next, use $G(\omega\jmath)$ to determine the magnitude and phase shift of the harmonic steady-state component of the output.

Solution 16.8.1.
The first-order sensor dynamics are given by,

$$G(s) = \frac{9.2}{0.031s + 1}$$

and the input is,

$$u(t) = 4.6\sin(0t + 90^\circ) + 3.3 \times 10^{-3}\sin(120\pi t)$$

The magnitude and phase of the harmonic components can be directly evaluated from the sensor transfer function and the input.
For the DC component, which corresponds to a zero frequency sinusoid:

$$\begin{aligned} 4.6|G(\jmath\omega)|_{\omega=0\pi\text{rad/sec}} &= 4.6\left|\frac{9.2}{0.031\jmath 0\pi + 1}\right| \\ &= 4.6(9.2) = 42.32 \\ \operatorname{Arg}G(\jmath\omega)|_{\omega=0\pi\text{rad/sec}} &= -\tan^{-1}(0.031 \times 0\pi) = 0^\circ \end{aligned}$$

Thus the amplitude and phase of this harmonic component are 42.32 and 0° respectively.

For the 60-Hz component:

$$\begin{aligned} 3.3 \times 10^{-3}|G(\jmath\omega)|_{\omega=120\pi\text{rad/sec}} &= 3.3 \times 10^{-3}\left|\frac{9.2}{0.031\jmath 120\pi + 1}\right| \\ &= 3.3 \times 10^{-3}(0.7844) = 2.6 \times 10^{-3} \\ \operatorname{Arg}G(\jmath\omega)|_{\omega=120\pi\text{rad/sec}} &= -\tan^{-1}(0.031 \times 120\pi) = -85.16^\circ \end{aligned}$$

Therefore, the amplitude and phase of this harmonic component are 2.6×10^{-3} and -85.15° respectively.

cf. Note that the magnitude and phase of the harmonic components can also be evaluated using

Laplace transforms to derive y(t). The output in the Laplace domain is given by,

$$U(s) = \frac{4.6}{s} + \frac{1.2441}{s^2 + (120\pi)^2}$$

$$Y(s) = G(s)U(s) = \frac{42.32}{s(0.031s + 1)} + \frac{11.4457}{(0.031s + 1)[s^2 + (120\pi)^2]}$$

$$\Rightarrow Y(s) = \frac{42.32}{s} - \frac{1.3118}{0.031s + 1} + \frac{-2.6 \times 10^{-3} s + 0.0832}{s^2 + (120\pi)^2}.$$

Calculating the inverse Laplace of the output, the DC component is:

$$f(t) = 42.32 \sin(0t + 90^\circ + 0^\circ)$$

and the 60-Hz component is:

$$\begin{aligned} h(t) &= h_1(t) + h_2(t) \\ &= -2.6 \times 10^{-3} \cos(120\pi t) + 2.2 \times 10^{-4} \sin(120\pi t) \\ &= 2.6 \times 10^{-3} \sin(120\pi t - 85.15^\circ). \end{aligned}$$

Each of the two harmonic components give the same amplitude and phase as calculated previously.

Problem 16.8.2. Suppose that a voltage amplifier with first-order dynamics $G(s) = \frac{\alpha}{Ts+1}$ has a time constant of $T = 0.037$ sec and DC gain of $\alpha = 43.1$. A sinusoidal input signal with amplitude of 2.8 volts yields, after an initial transient, a sinusoidal response with amplitude 76.2 volts. What were the frequencies of the input sinusoid and the output sinusoid?

Solution 16.8.2.
Recall that the input and output sinusoids have the same frequency for a linear system and that $|G(\jmath\omega)|$ represents the amplitude of the sinusoidal output due to the sinusoidal input with unit amplitude.

$$\begin{aligned} G(\jmath\omega) &= \frac{\alpha}{\jmath T\omega + 1} \\ \Rightarrow |G(\jmath\omega)| &= \frac{\alpha}{\sqrt{1 + T^2\omega^2}} \end{aligned}$$

¿From the given condition, we have,

$$\begin{aligned} \frac{1}{|G(\jmath\omega)|} &= \frac{2.8}{76.2} \\ \Rightarrow \frac{\alpha}{\sqrt{1 + T^2\omega^2}} &= \frac{76.2}{2.8} \\ \Rightarrow \omega &= \sqrt{\frac{(2.8\alpha/76.2)^2 - 1}{T^2}} = 33.19\text{rad/sec} \end{aligned}$$

Problem 16.8.3. A lag filter has a pole at -2, a zero at -6, and a DC gain of 10. At the frequency 4 rad/sec, what is the magnitude of the filter in dB and what is the phase angle of the filter in degrees? (Use just a calculator for this problem.)

Solution 16.8.3.
The transfer function for the lag filter is of the form,

$$G(s) = k\frac{s+6}{s+2},$$

which implies that,

$$G(\jmath\omega) = k\frac{\jmath\omega+6}{\jmath\omega+2}.$$

Given dc gain $G(0) = 10$, we have $k = \frac{10}{3}$. At $\omega = 4$ rad/sec,

$$G(4\jmath) = \frac{10}{3}\frac{4\jmath+6}{4\jmath+2}.$$

The magnitude and phase are given by,

$$\begin{aligned} 20\log_{10}|G(4\jmath)| &= 14.6, \\ \angle G(4\jmath) &= -29.75^\circ. \end{aligned}$$

Problem 16.8.4. Consider the transfer function

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with $\omega_n = 1$ rad/sec. Use Matlab to plot the magnitude and phase Bode plots of this function for ω from .01 to 100 for $\zeta = .1, .3, .5, .7, .9$. Put all plots in the same figure.

Solution 16.8.4.
Refer to Figure 18.0.1.

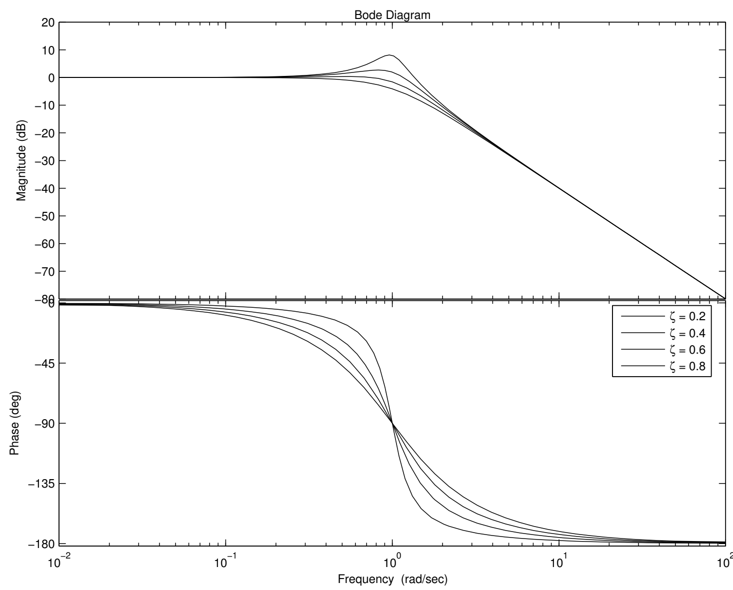

Figure 18.0.1: Problem 9. Bode Plot of $G(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$ ($\omega_n = 1$rad/sec , $\zeta = 0.1 - 0.9$)

Problem 16.8.5. Consider the transfer function

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2}.$$

Assume that the system is underdamped, that is, $0 < \zeta < 1$. Use calculus to determine the resonance frequency ω_{r} at which the magnitude $|G(j\omega)|$ is maximized, and determine $|G(j\omega_r)|$. Check whether your answer agrees with the figure from the previous problem. In addition, determine the range of values of ζ for which the magnitude of the transfer function is never greater than the value of its DC gain.

Solution 16.8.5.
The magnitude of the given transfer function is:

$$|G(j\omega)| = \left|\frac{1}{\omega_n^2 - \omega^2 + 2j\zeta\omega_n\omega}\right| = \frac{1}{\sqrt{\omega^4 + 2(2\zeta^2 - 1)\omega_n^2\omega^2 + \omega_n^4}}$$

From the extremum condition $d|G(j\omega)|/d\omega = 0$, it turns out that the resonance frequency is,

$$\begin{cases} |G(j\omega_r)| = \frac{1}{2\zeta\omega_n^2\sqrt{1-\zeta^2}} & \text{if} \quad \zeta < \frac{1}{\sqrt{2}} \simeq 0.707 \\ |G(j\omega_r)| = \frac{1}{\omega_n^2} & \text{if} \quad \zeta > \frac{1}{\sqrt{2}} \simeq 0.707 \end{cases}$$

Thus,

$$\begin{cases} \omega_r = \omega_n\sqrt{1 - 2\zeta^2} & \text{if} \quad \zeta < \frac{1}{\sqrt{2}} \simeq 0.707 \\ \omega_r = 0 & \text{if} \quad \zeta > \frac{1}{\sqrt{2}} \simeq 0.707 \end{cases}$$

In Problem 16.8.4, $\zeta = 0.1 - 0.7$ corresponds to the former and $\omega_r = \omega_n\sqrt{1 - 2\zeta^2} = 0.9592 - 0.5292$ whereas $\zeta = 0.9$ corresponds to the latter and $\omega_r = 0$. These results agree with the figure in Problem 16.8.4.

For $\zeta > 1/\sqrt{2}$ the gain of the transfer function is never greater than its DC gain (the gain at $\omega = 0$). $\zeta = 1/\sqrt{2}$ is called resonance damping.

Problem 16.8.6. Sketch Bode magnitude and phase plots AND the Nyquist plot by hand for each of the following transfer functions. You can use Matlab to print an "empty" log-log grid for your sketch. Be sure that the range of ω is large enough to include all important features of your plots. Explain how each plot was constructed. Check your sketches by plotting with the Matlab Bode and Nyquist functions. Question: What is strange about the Nyquist plots that Matlab draws?

i) $G(s) = \frac{s+2}{s+10}$ (lead).

ii) $G(s) = \frac{s+10}{s+2}$ (lag).

iii) $G(s) = \frac{s-2}{s+10}$.

iv) $G(s) = \frac{s-2}{s+2}$ (allpass).

v) $G(s) = \frac{s}{s+2}$ (washout).

Solution 16.8.6.
First sketch Bode plots for such basic forms as $G_1(s) = (s+n)$ and $G_2(s) = 1/(s+d)$ where n and d are constants. Noting that each $G(s)$ above can be expressed as $G(s) = G_1(s)G_2(s)$, simply use the additivity of magnitude (in log scale) and phase of complex variables. That is,

$$20\log_{10}|G(j\omega)| = 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)|$$
$$\text{Arg } G(j\omega) = \text{Arg } G_1(j\omega) + \text{Arg } G_2(j\omega)$$

In order to draw the Nyquist plots, first evaluate $|G(j\omega)|$ and Arg $G(j\omega)$ at such critical points as zero frequency, infinite frequency, crossover frequency, etc. Then use continuity to draw for $\omega \in [0,\infty]$ and employ symmetry about the ReG(jω)-axis to complete the plots for $\omega \in [-\infty,\infty]$. (Nyquist plots are symmetric about the ReG(jω)-axis.) Figures 6-11 are obtained from the Matlab commands 'bode' and 'nyquist'.

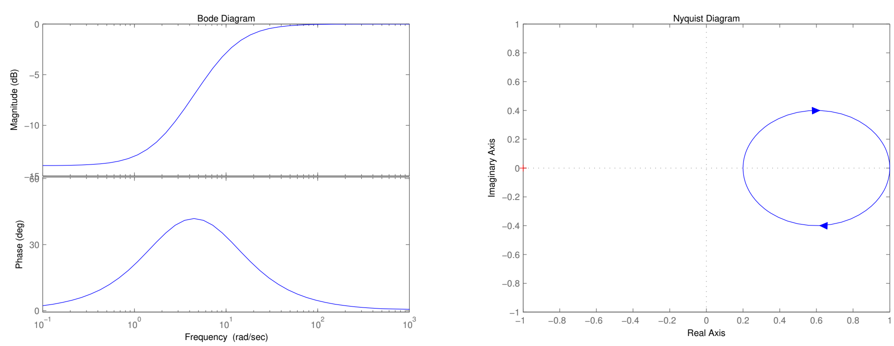

Figure 18.0.2: Problem 16.8.6(i). Bode and Nyquist Plots of $G(s) = \frac{s+2}{s+10}$

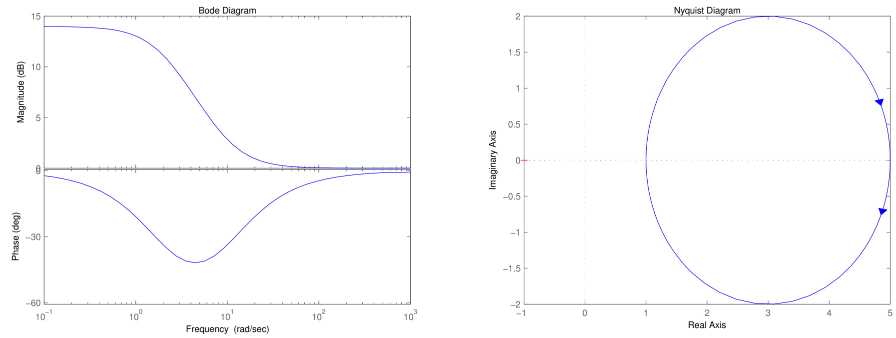

Figure 18.0.3: Problem 16.8.6(ii). Bode and Nyquist Plots of $G(s) = \frac{s+10}{s+2}$

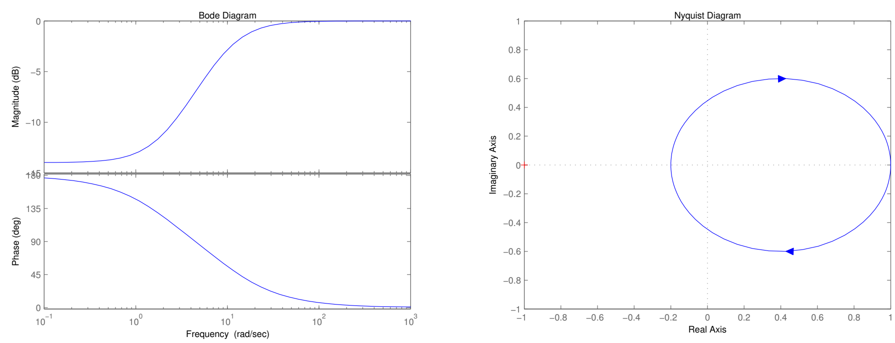

Figure 18.0.4: Problem 16.8.6(iii). Bode and Nyquist Plots of $G(s) = \frac{s-2}{s+10}$

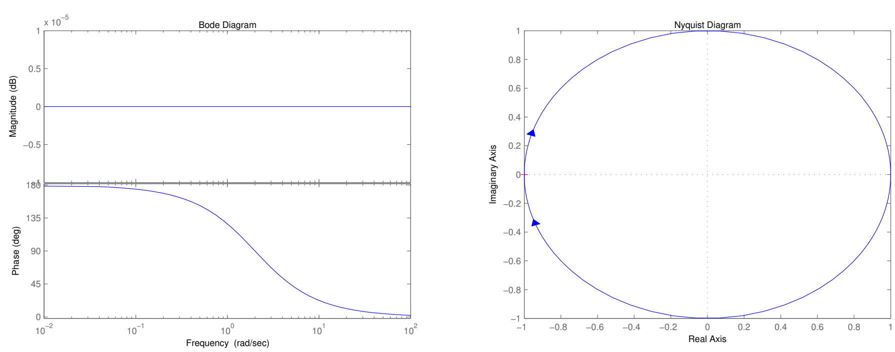

Figure 18.0.5: Problem 16.8.6(iv). Bode and Nyquist Plots of $G(s) = \frac{s-2}{s+2}$

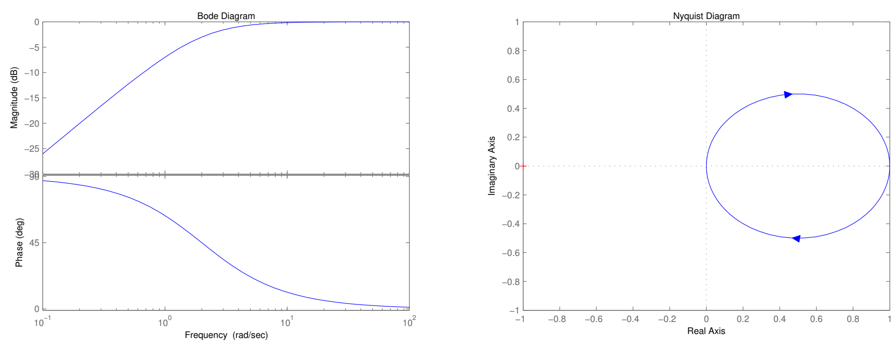

Figure 18.0.6: Problem 16.8.6(v). Bode and Nyquist Plots of $G(s) = \frac{s}{s+2}$

Problem 16.8.7. For each of the following transfer functions, sketch the Bode magnitude and phase plots by hand. You can use Matlab to print an "empty" log-log grid for your sketch. Be sure that the range of ω is large enough to include all important features of your plots. Explain how each plot was constructed. Check your sketches by plotting with Matlab.

i) $G(s) = \frac{s^2}{(s+1)^2(s+10)^2}$ (rooftop).

ii) $G(s) = \frac{(s+1)(s+100)}{s(s+5)(s+10)}$.

Solution 16.8.7.
For a brief description of how to construct bode plots, refer to the previous problem. Figure 16.8.7 is obtained from the Matlab command 'bode'.

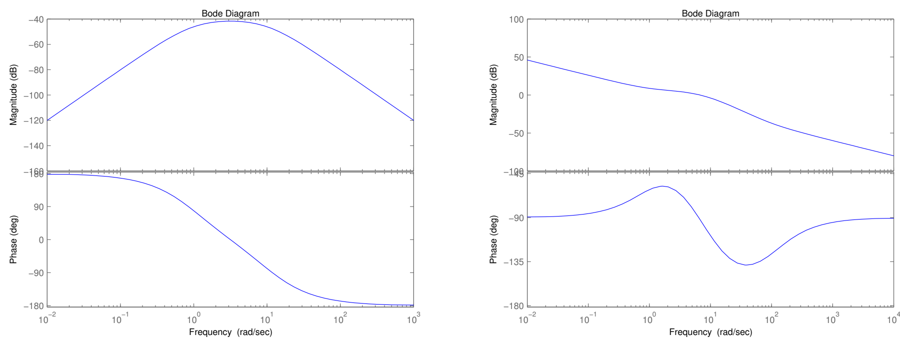

Figure 18.0.7: Problem 16.8.7. Bode Plots of $G(s) = \frac{s^2}{(s+1)^2(s+10)^2}$ and $G(s) = \frac{(s+1)(s+100)}{s(s+5)(s+10)}$

Problem 16.8.8. Consider the asymptotically stable loop transfer function

$$L(s) = \frac{-1}{s^2 + 2s + 3}.$$

Show that the closed-loop transfer function $S(s) = 1/(1 + L(s))$ is asymptotically stable, and use Matlab to plot the magnitude and phase Bode plots for S. Then, write a Matlab program to show numerically that

$$\int_0^\infty \ln|S(\jmath\omega)|\, d\omega = 0.$$

Explain this result in terms of the "balance" between attenuation and amplification.

Solution 16.8.8..
The closed-loop transfer function is,

$$S(s) = \frac{1}{1 + L(s)} = \frac{s^2 + 2s + 3}{s^2 + 2s + 2}.$$

Its poles are, $s = -1 \pm \jmath$, from which we see that $S(s)$ is asymptotically stable. Figure 18.0.8 shows the bode plot for $S(s)$. The following Matlab code numerically shows that $\int_0^\infty \ln|S(\jmath\omega)|\, d\omega =$

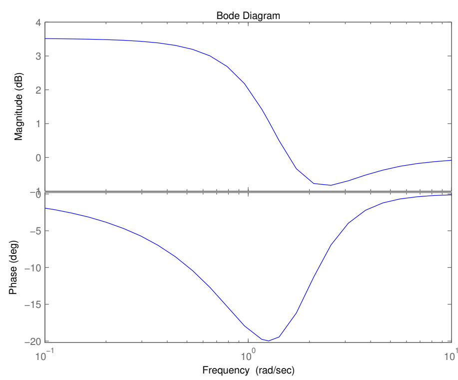

Figure 18.0.8: Problem 16.8.8. Bode Plot for $S(s) = \frac{s^2+2s+3}{s^2+2s+2}$

$0.0021 \simeq 0$. Attenuation is exactly balanced by amplification in this transfer function.

```
clear all;

w0 = 0; wf = 10^4; dw = 0.01; w = w0:dw:wf;

S = ((j*w).^2 + 2*j*w + 3)./((j*w).^2 + 2*j*w + 2);

Int_ = log(abs(S)); Int = sum(Int_)*dw
```

Problem 16.8.9. Consider the unstable loop transfer function

$$L(s) = \frac{4}{(s-1)(s+2)}.$$

Show that the closed-loop transfer function $S(s)$ is asymptotically stable, and use Matlab to plot the Bode plots for S. Then, write a Matlab program to show numerically that

$$\int_0^\infty \ln |S(j\omega)|\, d\omega = \pi.$$

Discuss the "balance" between attenuation and amplification.

Solution 16.8.9.
The closed-loop transfer function is,

$$S(s) = \frac{1}{1+L(s)} = \frac{s^2+s-2}{s^2+s+2}.$$

Its poles are $s = (-1 \pm \sqrt{7}j)/2$, from which we conclude that $S(s)$ is asymptotically stable. Figure 3 shows the bode plot for $S(s)$. As in Problem 1, it can be shown that, $\int_0^\infty \ln |S(j\omega)|\, d\omega = 3.1412 \simeq \pi$. This transfer function amplifies the input signal at every frequency.

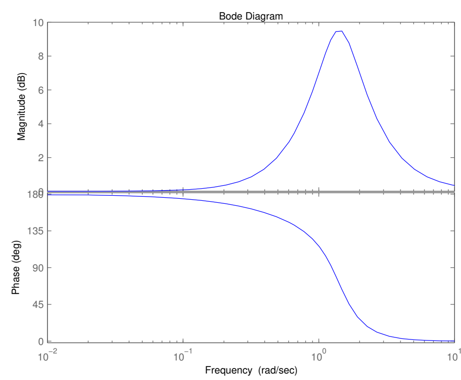

Figure 18.0.9: Problem 16.8.9. Bode Plot for $S(s) = \frac{s^2+s-2}{s^2+s+2}$

Problem 16.8.10. Consider the loop transfer function

$$L(s) = \frac{2.5(s + 100)}{(s + 1)^2}.$$

Sketch the gain and phase Bode plots of $L(s)$. Use your plot to indicate the magnitude crossover frequency ω_{mco}, the phase crossover frequency ω_{pco}, the gain margin, and the phase margin.

Solution 16.8.10.
The Bode plot and the loop transfer function is given in Figure 18.0.10.

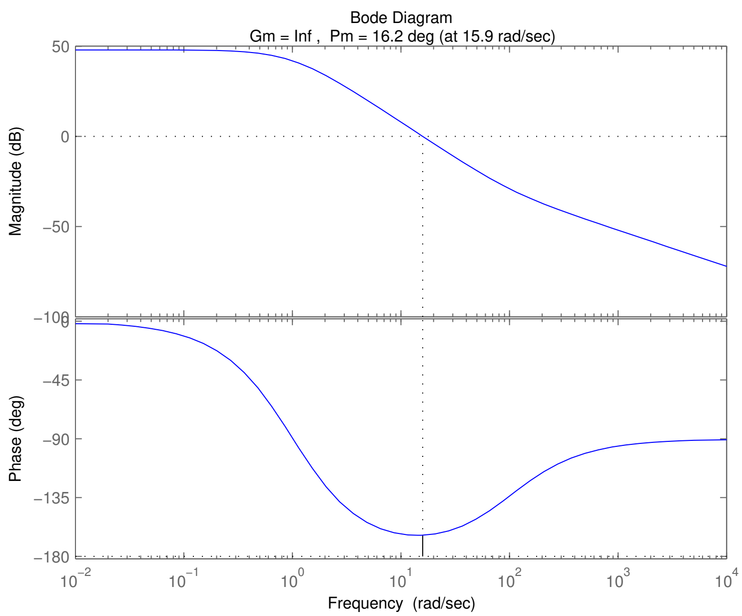

Figure 18.0.10: Problem 1, Bode Plot for $L(s) = \frac{2.5(s+100)}{(s+1)^2}$

From the graph, we see that,

$$\begin{aligned} &\text{GM} = \infty \\ &\text{PM} = 16.2 \text{ deg} \quad \text{at } \omega_{\text{mco}} = 15.9 \text{ rad/sec} \end{aligned}$$

Problem 16.8.11. Consider the damped rigid body plant

$$G(s) = \frac{1}{s(s+1)}.$$

i) Assume unity feedback so that $L(s) = G(s)$. Sketch the Nyquist plot and determine the gain and phase margins.

ii) Instead of unity feedback, consider the lead controller $G_c(s) = k(s+2)/(s+20)$ so that $L(s) = G_c(s)G(s)$. For $k = 1$, use Matlab to determine whether the lead provided by this lead controller increases the phase margin.

iii) Draw the root locus in terms of k.

iv) Choose k so that the complex conjugate poles have damping ratio $\zeta = \sqrt{2}/2$.

v) For the value of k that you chose, determine the steady state error for the unit ramp input $1/s^2$.

(Hint: You can solve the problem directly by equating the product of $(s-a)(s^2 + 2\zeta\omega_n s + \omega_n^2)$ with the cubic obtained from the closed loop transfer function with k as the unknown parameter. Then you can get a cubic equation in a or ω_n.)

Solution 16.8.11.

(i) Figure 18.0.11 shows the Nyquist plot of the given loop transfer function. Since it crosses the Re$G(j\omega)$-axis at the origin, its gain margin is infinite. Also it turns out to cross the unit circle centered at the origin when the frequency $\omega = 0.786$ rad/sec. Therefore the phase margin is PM = $\tan^{-1}$ [ImG(jω)/ReG(jω)] $\simeq$ 51.8 deg

(ii) Simply use the command 'margin' to obtain the gain and phase margin along with bode plot. Figure 18.0.12 shows the bode plot indicating phase margins before and after the lead compensator is used. The phase margin has been increased from PM = 51.8 deg to PM = 86.9 deg. (Also note that the margin (before the compensator is used) is consistent with the result in part (i).)

(iii) Figure shows the root locus for the closed-loop

(iv) The closed-loop transfer function is,

$$\tilde{G}(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)} = \frac{k(s+2)}{s^3 + 21s^2 + (k+20)s + 2k}$$

Balancing the cubic characteristic polynomial with the given factored form yields

$$\begin{aligned}
&s^3 + 21s^2 + (k+20)s + 2k = (s-a)(s^2 + 2\zeta\omega_n s + \omega_n^2) \\
\Rightarrow\ &s^3 + 21s^2 + (k+20)s + 2k = s^3 + (\sqrt{2}\omega_n - a)s^2 + (\omega_n^2 - \sqrt{2}a\omega_n)s - a\omega_n^2 \\
\Rightarrow\ &\begin{cases} 21 = \sqrt{2}\omega_n - a \\ k + 20 = \omega_n^2 - \sqrt{2}a\omega_n \\ 2k = -a\omega_n^2 \end{cases}
\end{aligned}$$

Rearranging for ω yields a cubic equation

$$\sqrt{2}\omega_n^3 - 23\omega_n^2 + 42\sqrt{2}\omega_n - 40 = 0,$$

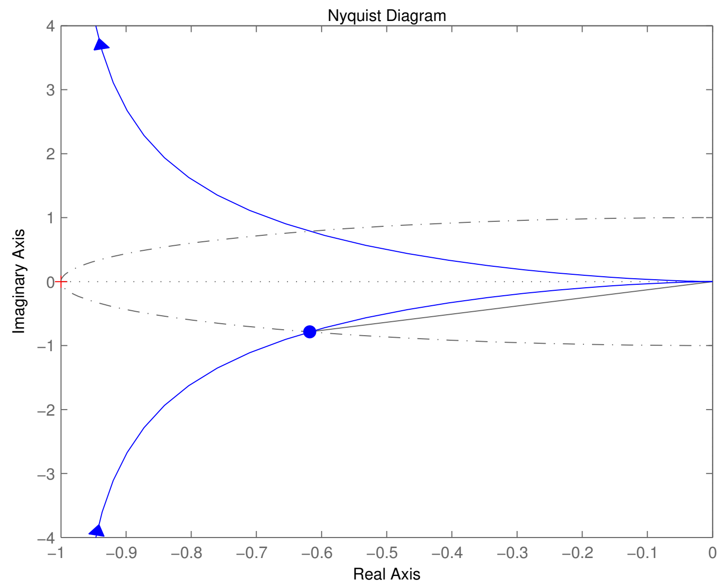

Figure 18.0.11: Problem 16.8.11(i). Nyquist plot for $L(s) = \frac{1}{s(s+1)}$

which results in

$$\begin{aligned} \omega_n &= \{13.2561, 1.8607, 1.1467\} \\ a &= \{-2.2531, -18.3686, -19.3783\} \\ k &= \{197.9617, 31.7973, 12.7410\} \end{aligned}$$

Now using Routh's test for stability, we have

$$a_0 - a_2 a_1 = 2k - 21(k+20) = -19k - 420 < 0 \ \text{ if k} > 0,$$

which implies that the closed-loop system is asymptotically stable if $k > 0$. Hence it follows that all three gains computed above yields the closed-loop stability. Finally introducing the three k's into the characteristic polynomial and solving for s results in

$$s = \begin{cases} -2.2531, -9.3734 \pm 9.3734 j \\ -18.3686, -1.3157 \pm 1.3157 j \\ -19.3783, -0.8109 \pm 0.8109 j \end{cases}$$

Note that the real and imaginary parts of all the complex roots have the same magnitude, which corresponds to the damping ratio of $\zeta = 1/\sqrt{2}$ and thus satisfies the given requirement.

(v) The reference-to-error transfer function is given by

$$\begin{aligned} E(s) &= R(s) - Y(s) = \frac{1}{1 + G(s)G_c(s)} R(s) \\ \Rightarrow E(s) &= \frac{s(s+1)(s+20)}{s^3 + 21s^2 + (k+20)s + 2k} R(s) \\ \Rightarrow e_\infty &= \lim_{s \to 0} sE(s) = \frac{10}{k} \end{aligned}$$

$$\Rightarrow e_\infty = \{0.0505, 0.3145, 0.7849\}$$

The highest gain $k = 197.9617$ yields the smallest steady state error $e_\infty = 0.0505$ for the ramp input. transfer function in terms of k.

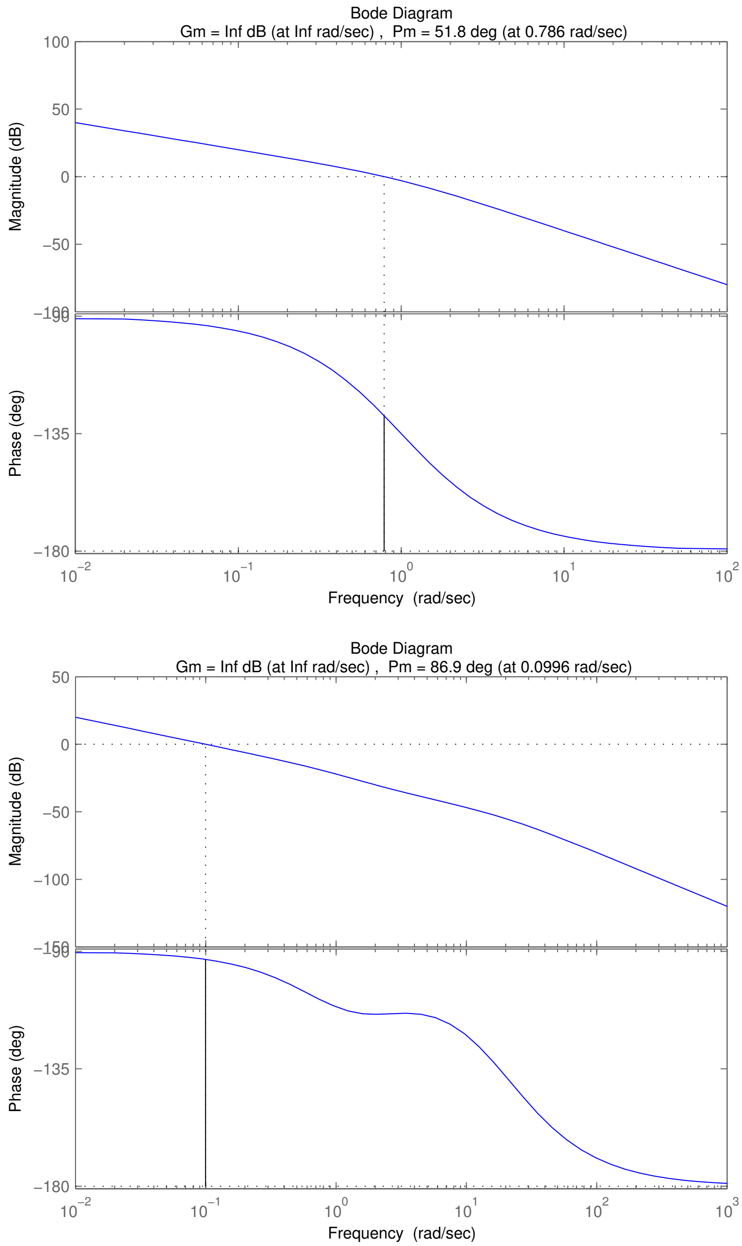

Figure 18.0.12: Problem 16.8.11(ii). Phase margin for $L(s) = \frac{1}{s(s+1)}$ and $L(s) = \frac{s+2}{s(s+1)(s+20)}$

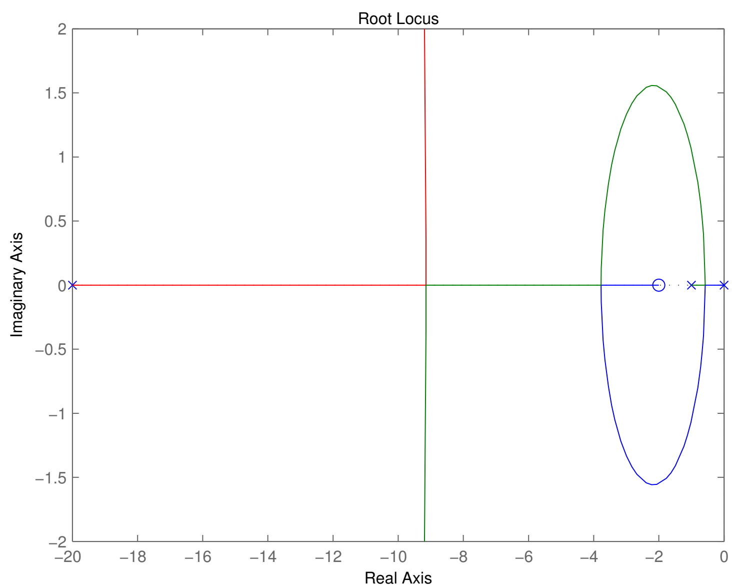

Figure 18.0.13: Problem 16.8.11(v). Root Locus for $L(s) = \frac{k(s+2)}{s(s+1)(s+20)}$

Problem 16.8.12. The lateral dynamics of an experimental aircraft are modeled by the transfer function $1/(\tau s + 1)$, where $\tau > 0$ is a time constant. For this transfer function a basic servo loop is closed with the integral controller K_I/s.

i) Determine the values of K_I for which the closed-loop system is asymptotically stable.

ii) For which values of K_I is the steady-state error to a unit-slope ramp command less than 0.025?

iii) For a value of K_I such that the closed-loop system is asymptotically stable, determine the amplitude of the harmonic steady-state response to the command $r(t) = r_0 \cos(\omega t)$.

Solution 16.8.12.
The open loop transfer function is given by

$$L = \frac{1}{\tau s + 1} \frac{K_I}{s} = \frac{K_I}{s(\tau s + 1)}. \tag{18.0.1}$$

The sensitivity transfer function is given by

$$S = \frac{1}{1 + L} = \frac{s^2 + \frac{1}{\tau} s}{s^2 + \frac{1}{\tau} s + \frac{K_I}{\tau}}, \tag{18.0.2}$$

and the closed loop transfer function is given by

$$G_c = \frac{L}{1 + L} = \frac{\frac{K_I}{\tau}}{s^2 + \frac{1}{\tau} s + \frac{K_I}{\tau}}, \tag{18.0.3}$$

i) Applying the Routh criterion, the closed-loop system is asymptotically stable if and only if $\frac{1}{\tau} > 0$ and $\frac{K_I}{\tau} > 0$, that is, $K_I > 0$ and $\tau > 0$.

ii) Now, use the FVT to evaluate the steady-state error. For a unit ramp input we have,

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\, \hat{e}(s) = \lim_{s\to 0} s\, S\, \frac{1}{s^2} = \lim_{s\to 0} s\, \frac{s^2 + \frac{1}{\tau} s}{s^2 + \frac{1}{\tau} s + \frac{K_I}{\tau}} \frac{1}{s^2} = \frac{1}{K_I}. \tag{18.0.4}$$

The steady state error $\lim_{t\to\infty} e(t) < 0.025$ if and only if $K_I > 40$

iii) Applying the fundamental theorem of linear systems, the amplitude of the harmonic steady-state response to the command $r(t) = r_0 \cos(\omega t)$ is given by

$$y_0 = |G_c(\jmath\omega)| r_0 = \left| \frac{\frac{K_I}{\tau}}{-\omega^2 + \jmath\frac{\omega}{\tau} + \frac{K_I}{\tau}} \right| r_0 = \frac{K_I r_0}{\sqrt{\omega^2 + (K_I - \tau\omega^2)^2}}. \tag{18.0.5}$$

Problem 16.8.13. At a given Mach number, the open-loop longitudinal dynamics of an experimental aircraft are given by

$$L(s) = \frac{1}{s(s+1)^2}.$$

i) Sketch the Bode plot of L (magnitude and phase plots).

ii) Determine the phase crossover frequency ω_{pco} and the gain margin in dB of the closed-loop system and illustrate them on the Bode plot. (Hint: $\log_{10} 2 = 0.3$)

iii) Sketch the Nyquist plot of L.

iv) Indicate the phase crossover frequency ω_{pco} and the gain margin on the Nyquist plot. Be sure that what you show on the Nyquist plot is consistent with the Bode plot.

Solution 16.8.13.
i) Due to the pole at zero, the Bode magnitude plot has a slope of -20 dB/dec and the phase plot has $-90°$ phase for low values of ω. Due to the poles at 1, the Bode magnitude plot has a slope of -60 dB/dec and the phase plot has $-270°$ phase for $\omega > 1$. Figure 18.0.14 shows the Bode magnitude and phase plots for $L(s) = \frac{1}{s(s+1)^2}$.

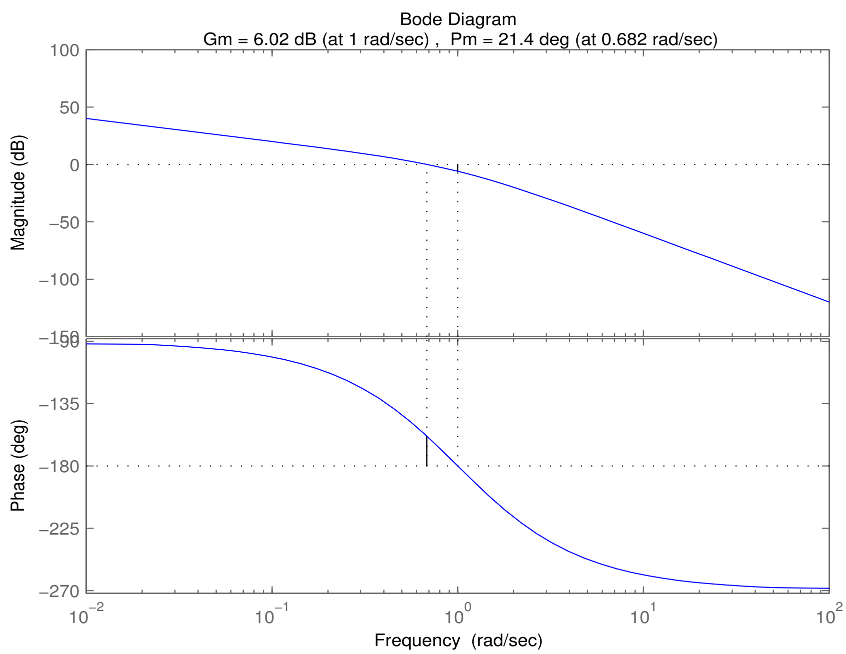

Figure 18.0.14: Problem 16.8.13: Bode magnitude and phase plots for $L(s) = \frac{1}{s(s+1)^2}$. The phase and gain margins are also shown.

ii) The phase crossover frequency $\omega_{\text{pco}} = 1$ rad/s and the gain margin GM = 6.02 dB. The gain crossover frequency $\omega_{\text{gco}} = 0.682$ rad/s and the phase margin PM = 21.5 deg.

iii) Consider the open loop transfer function

$$L(\jmath\omega) = \frac{1}{\jmath\omega(\jmath\omega + 1)^2} = -\frac{2}{(1+\omega^2)^2} - \jmath\frac{1-\omega^2}{\omega(1+\omega^2)^2}. \tag{18.0.6}$$

As $\omega \to 0$, $L(\jmath\omega) \to -2 + \jmath\infty$ and as $\omega \to \infty$, $L(\jmath\omega) \to 0$. Also note that $Im(L(\jmath\omega)) = 0$ for $\omega = 1$, that is, the Nyquist plot crosses the negative real axis at -0.5 which corresponds to $\omega = 1$. The Nyquist plot is shown in Figure 18.0.15.

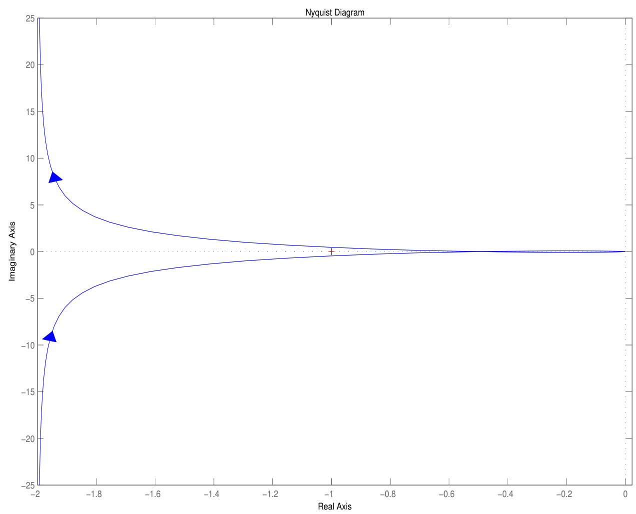

Figure 18.0.15: Problem 16.8.13: Nyquist plot for $L(s) = \frac{1}{s(s+1)^2}$.

iv) The phase crossover frequency $\omega_{\text{pco}} = 1$ rad/s and $L(\jmath) = -0.5$. Hence, the gain margin GM $= 20\log_{10}(1 - |L(\jmath)|) = 6.02$ dB. The gain crossover frequency ωgco $= 0.683$ rad/s and the phase margin PM = arctan $\frac{0.377}{0.936} = 21.9°$. Figure 18.0.16 shows ω_{pco} and ω_{gco} on the Nyquist plot.

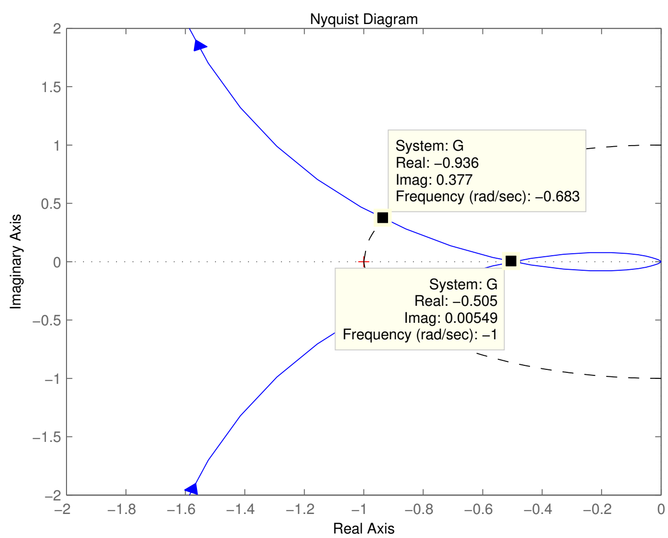

Figure 18.0.16: Problem 16.8.13: The ω_{pco} and ω_{gco} are shown on the Nyquist plot for $L(s) = \frac{1}{s(s+1)^2}$.

Problem 16.8.14. At a given Mach number, the open-loop longitudinal dynamics of an unstable experimental aircraft are given by

$$L(s) = \frac{4(s+10)}{(s-1)(s-2)}.$$

i) Sketch the Bode plot of L (magnitude and phase plots).

ii) Sketch the Nyquist plot of L.

iii) Apply the Nyquist test to this system and use it to assess closed-loop stability.

Solution 16.8.14.
i) The bode plot of L is shown on Fig. 0.0.9.

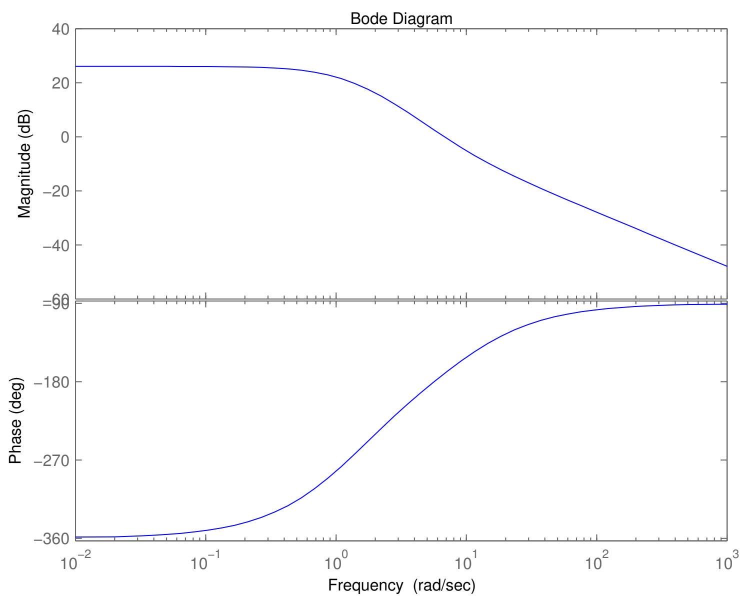

Figure 18.0.17: Problem 16.8.14. Bode Plot for $L(s) = \frac{4(s+10)}{(s-1)(s-2)}$

ii) The Nyquist plot of L is given by Fig. 0.0.10.

iii) The curve meets the real axis at 0 and -1.33. There is 2 counterclockwise encirclements of -1 so $N = -2$. Since $L(s)$ has 2 open loop poles in the right half plane, $P = 2$. Then, $Z = N + P = -2 + 2 = 0$ which indicates that the system is stable.

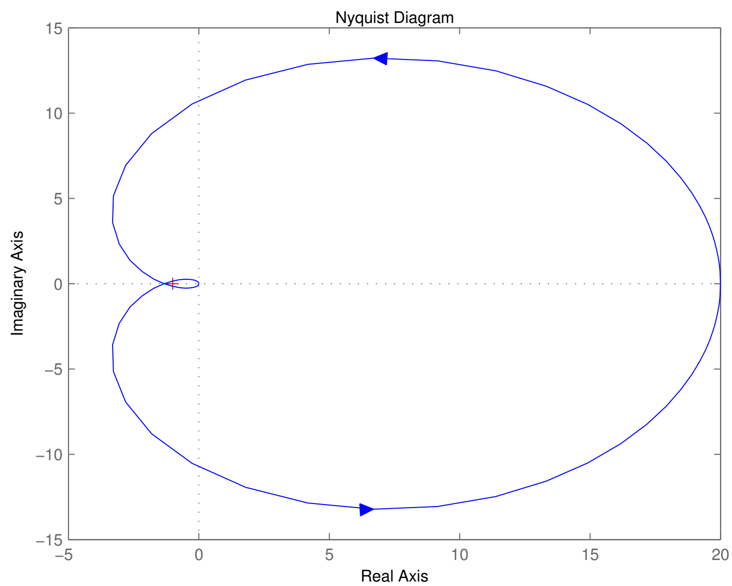

Figure 18.0.18: Problem 16.8.14. Bode Plot for $L(s) = \frac{4(s+10)}{(s-1)(s-2)}$

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