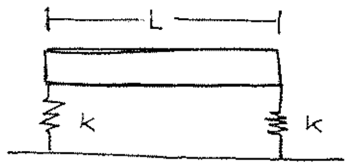
DYNAMICS Q - EXAM

AUG 25, 1995

JOHN WEISENFELD

1.)

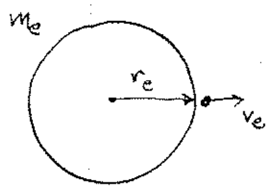


M, L

(strogatz)

discuss dynamics of small amplitude motion,

2.) Find escape velocity from the earth's surface (Burns)



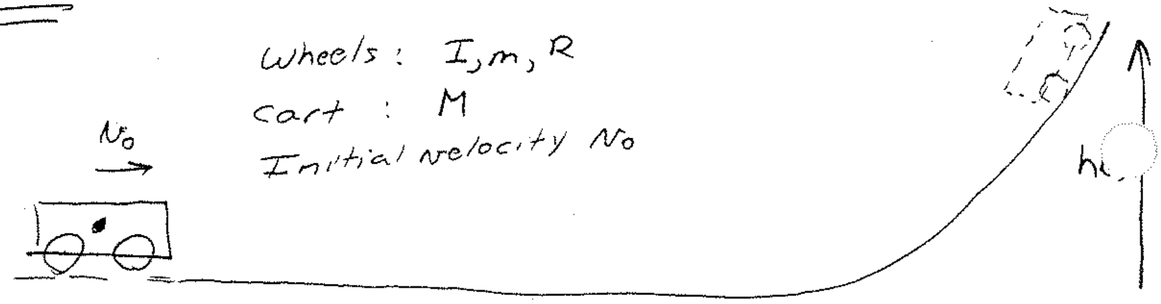
3.) A ball rolls on a plane in <u>3-D</u>, discuss dynamics, (Ruina)
make any assumptions that you need

Dawson, MacDonald

'92
Summer

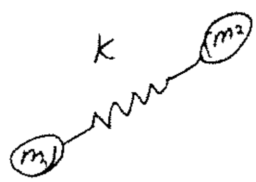
Dynamics

Wheels : I, m, R
cart : M
Initial velocity V_0



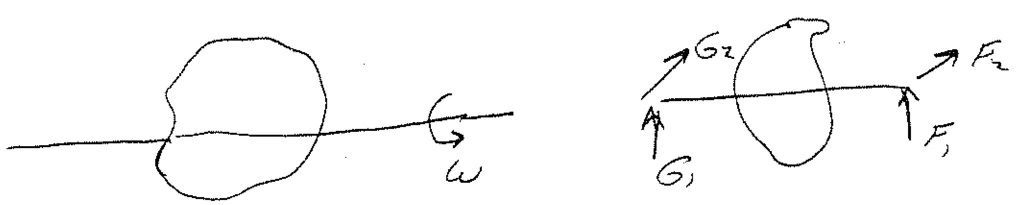
How far up does it go?

What is a conservative force?
Which of the forces are conservative?
General free body diagram.
Why do use energy?



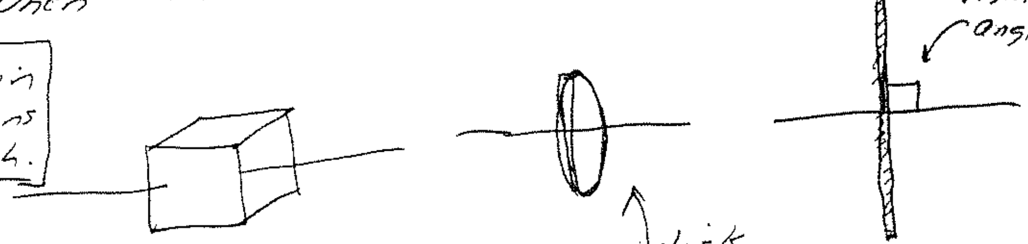
How many ~~[illegible]~~ degrees of freedom?
If $k \to \infty$, again.
Equations of motion.
Any other ways?

Sipinning Objects



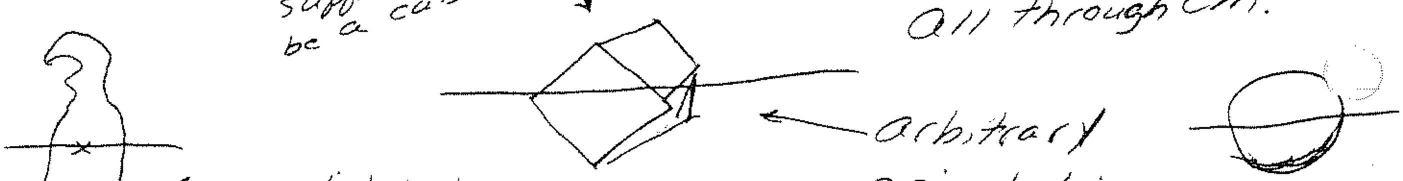
When are reactions zero?

Explain rotations of each.



That's supposed to be a cube

All through cm.



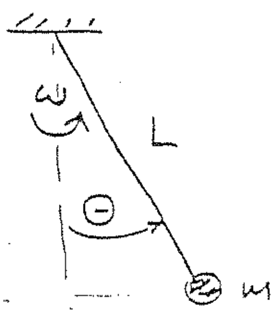
arbitrary planar object

arbitrary orientation

Sphere

(over)

(50) Spherical pendulum ($\dot{\theta} = 0$)
Given an I.C. on v, find the period.
If the mass is given a slight disturbance
($\dot{\theta}$ no longer $= 0$), find the period!



$$\left[\begin{array}{l} \omega^2 = \frac{g}{L\cos\theta} \\ v = \omega(L\sin\theta). \end{array}\right.$$

1

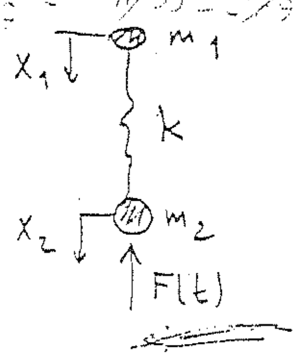
Period $T = T(\theta)$.

Determine new steady state.

θ', v'

$\left[\begin{array}{l} \text{Energy Conservation.} \\ v = v(\theta). \end{array}\right.$

(51) Set up eq.'s of motion.
Given $(\ddot{x}_1 - \ddot{x}_2)$ how would
you find $F(t)$?



1

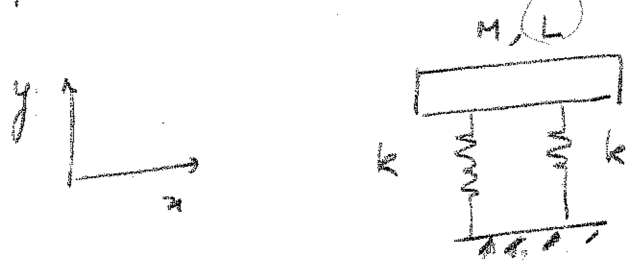
... not well-posed.

(52) What is the difference between LE' and
NE's? 1) No superposition
2) Solutions depend on i.c.

1

(B) DYNAMICS (Aug 95)

1. Find the natural frequencies of the following system.



(a) Assume no motion in x-direction.

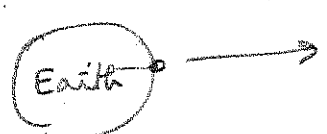
$\frac{2k}{m}$ (translational), $\frac{kL^2}{2I}$ (rotational).

(b) (Lucia) What is the frequency in the x-direction? Is this ~~not~~ a valid question?

If the motion is large. For small vibration I don't think there is any motion in the x-direction (1st order)

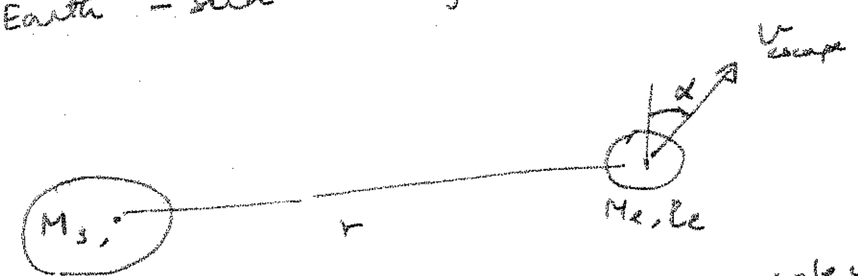
2. (J. Burns). (a) Find the escape velocity of a particle from the Earth. (Mass M_e, radius R_e) Note that $\frac{M_e G}{R_e^2} = g$.

$$-\frac{M_e m G}{R_e} + \frac{1}{2} m V_{esc}^2 = 0.$$



$$\Rightarrow \frac{1}{2} m V_{esc}^2 = \frac{M_e m G}{R_e} \Rightarrow V_{esc} = \sqrt{\frac{2 M_e G}{R_e}} = \sqrt{2gR_e}.$$

(b) Find the escape velocity of a particle from the Earth - Sun system?



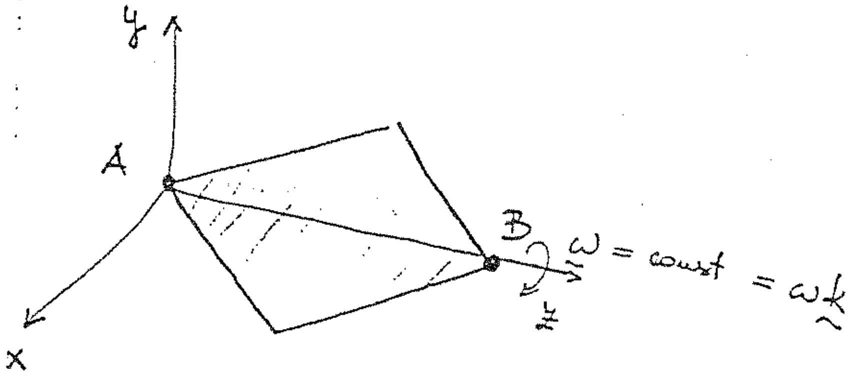
(c) In the first case one neglected the spin of the Earth. Why? Is this justified? What about the second case?

3. (Alusia) A ball is set rolling on the floor with general initial conditions. (3 component of $\underline{\omega}$, and velocity in plane parallel to the ground).

(a) What is the general motion (qualitative).

(b) Once the transient slip dies out what is the final pure rolling state of motion (qualitative)

(63) Thin plate in y-z plane!



How do you find reactions at A and B?
(dynamic reactions) reactions

1

(1) Define $\underset{\sim}{F} = m\underset{\sim}{a}$

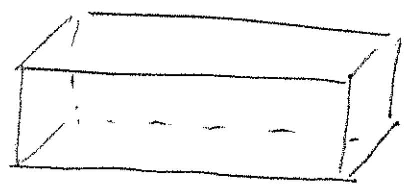
(2) Given a tube of paper which is then subjected to torsion. At what angle will the paper buckle and why?

(3) Given Fig 1 below with no friction in the walls. Find the stresses

(4) Given Fig 1 again, but now the walls have friction. State the shearing stress distribution on the walls

Solids:

(1)

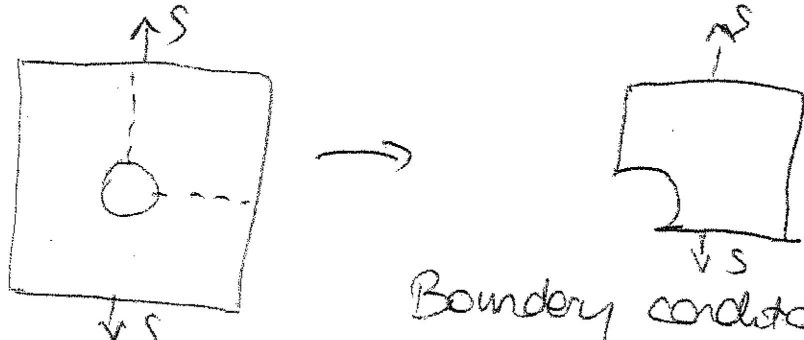


Steel contanter with elastic material inside.
Suppose no friction

What is stress field?

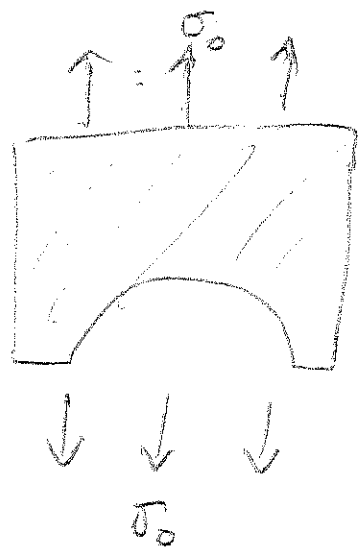
What if the steel and elastic material are glued together?

(2)



Boundery conditions on traction?
Stress concentration?

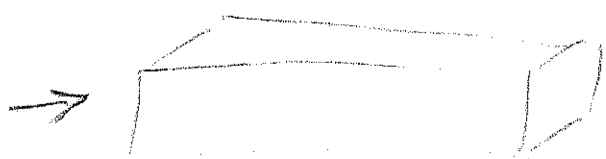
Solid Mechanics Q-exam · January 1998



Boundary conditions on plate ?

(Mukherjee)

2.



(Hui)

elastic solid ~~is~~ is forced into metal box.

discuss BC.

What if elastic solid is glued to the plates that make up the box - what BC will be different.

3. Back to question 1 ! What is the maximum load you could apply before the plate would break, given that an intact plate (without hole) can withstand a load S_{max}. (They want you to talk about stress concentrations at the hole).

(Moon)

3. Solid Mechanics.

Examiners - Conway, Jenkins, Ruina, Healey, Mcm...
& a visiting Prof.

Questions - i) What are the steps in solving a 3-D probl... in linear, isotropic, small strain elasticity?. What are the required equations, explain.

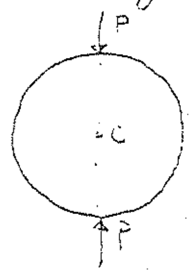
a) Is there any other kind of boundary condition? (i.e other than $\sigma_{ij} n_j = T_i$. Ans - Displ. b.c.s)

b) How do you take care of such boundary condition

c) Why compatibility conditions are not required in displacement formulation? What does that tell you about compatibility conditions.

d) give an example of a problem with displ. b.c.s

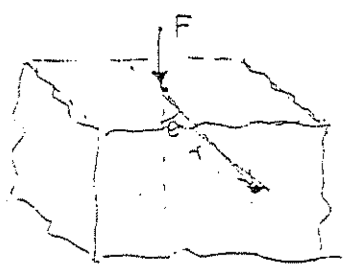
ii)



A ring is compressed as shown by two forces. Draw free body diagram about a vertical section and a horizontal section. Show all the forces and solve for them.

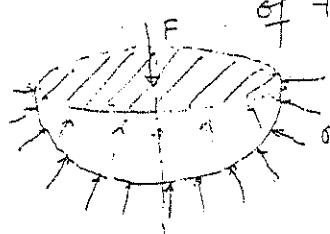
What are the required boundary conditi...

iii)



A point load is applied on a half-space. Derive an expression for stresses at a point (r, θ). How do the stress vary? Get an estimate of the stress variations (σ_{rr}).

(Hint :



integrate σ_{rr} over area and check equ...

σ_{rr}. note: σ_{rr} is θ independent.

$\therefore \; 2\pi r^2 . \sigma_{rr} = F . \Rightarrow \sigma_{rr} = \frac{K}{r^2}$. $\leftarrow$ con...

Thus, $\sigma_{rr} \propto \frac{1}{r^2}$.

General Comments :— Rule - 1 → Don't panic. Try to smile throughout even if the examiners look serious.

2. → Look at the guy who nods and smile a lot. It helps. Try to read their respons... from the face.

3 → As a general rule - they are all ver... helpful. They try to get you to the righ... answer through hints.

GOOD LUCK

Lance, Drew, Conway, Rosakis
(Visiting Prof).

Continuum

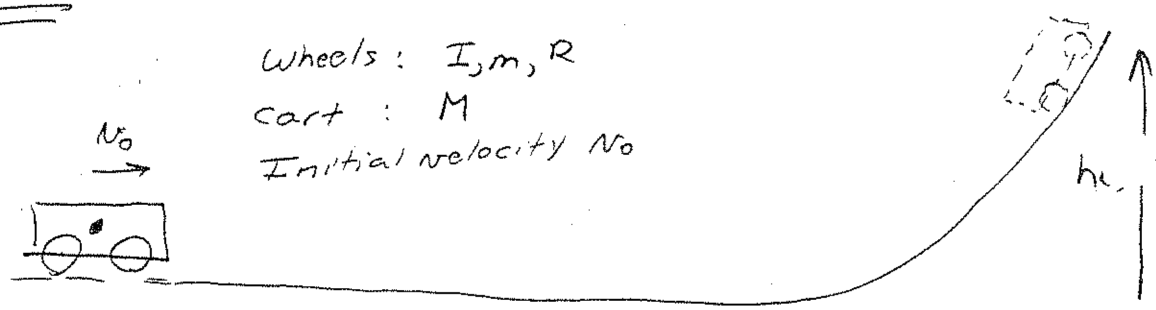
1. What is stress
~~[illegible]~~ State the force balance equation.
Derive the equation for force balance in a continuum in 2-D in cartesian coordinates

2. What is circulation.
Describe lift (how you calculate it)
What is lift on aerofoil
Is there lift on a circular body
Why or why not?

3. What is viscosity?
Describe an experiment to measure viscosity.
What does viscosity do in dashpot (like shock absorber on car)

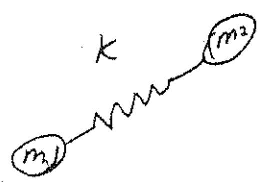
Dynamics

Wheels : I, m, R
cart : M
Initial velocity v_0



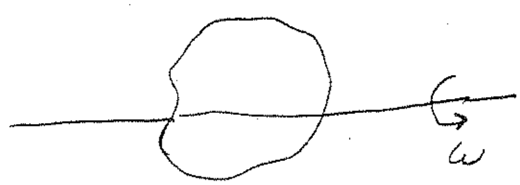
How far up does it go?

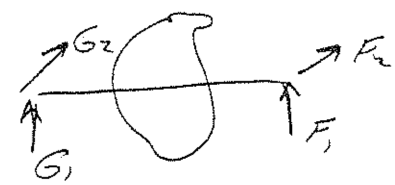
What is a conservative force?
Which of the forces are conservative?
General free body diagram.
Why do use energy?



How many ~~[illegible]~~ degrees of freedom?
If $k \to \infty$, again.
Equations of motion.
Any other ways?

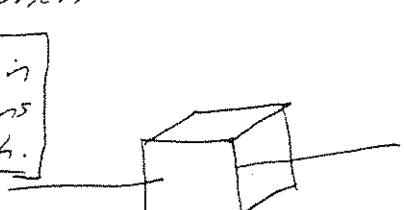
Spinning Objects

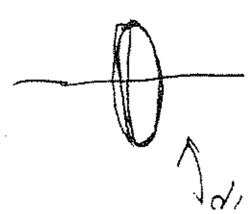


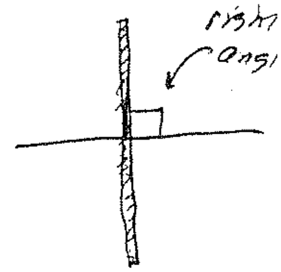


When are reactions zero?

Explain rotations of each.

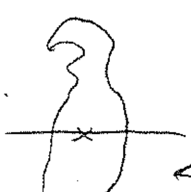


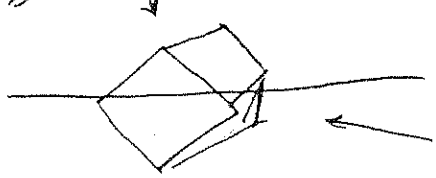




That's supposed to be a cube

All through cm.





← arbitrary planar object

← arbitrary orientation

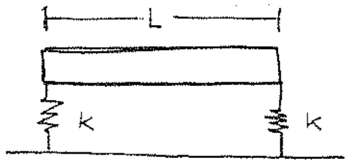
Sphere

DYNAMICS Q - EXAM

AUG 25, 1995

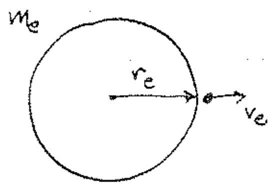
JOHN WEISE[illegible]

1.

1.)  M, L (strogatz)

discuss dynamics of small amplitude motion,

2.) Find escape velocity from the earth's surface (Burns)



3.) A ball rolls on a plane in <u>3-D</u>, discuss dynamics, (Ruina)
make any assumptions that you need

Dawson, MacDonald

8.

- equilibrium eqs + kinematic ~~constitutive~~ equations in polar coords

- Difference between plate theory & plane stress.

- Work $\equiv^m$ eqns, constitutive eqns, kinematics, compatibility.

- If we use a stress-function formulation, we have to enforce compatibility. (ie if we specify 6 stresses then we get 6 strains which must reduce to 3 displacements)

- Strain Energy Density $= \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$

- Simply-connected domain ⇒ stresses & displacements are single-valued.
- Multiply-connected domain ⇒ stresses single valued but either
  - displacement single valued & force enclosed (curl-free vector field)
  - or - NO enclosed force (ie a dislocation) this is divergence-free field.

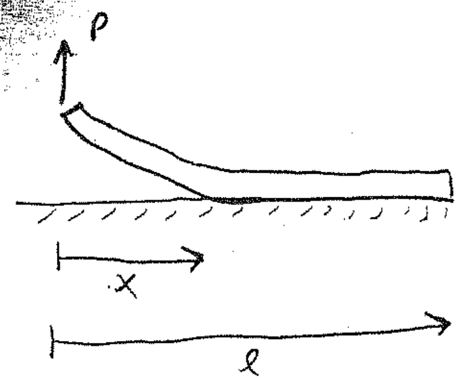
'92
Summer

Scott Adams
Kevin Duprey
Rich Bake
~~[illegible]~~
Harry Dankou
Brianno Coll
Dane Quinn

Continuum (Solids)

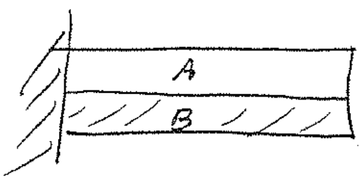
... per unit length

Find X.



What is stress?

Why is it not $\underset{\sim}{t} = \underline{\underline{\sigma}}\, \underset{\sim}{n}^2$?



2 material beam.
Describe $\underset{\sim}{u}$, $\underset{\sim}{\gamma}$, and $\underline{\underline{\sigma}}$ near and at the interface?
Continuous / Discontinuous?

yes no yes

$t(n) = \underline{\underline{\sigma}} \cdot \underset{\sim}{n}$ cts ✓

- equilibrium eqs + ~~constitutive~~ kinematic equations in polar coords

- Difference between plate theory & plane stress.

- Work Ξ^m eqns, constitutive eqns, kinematics, compatibility.

- If we use a stress-function formulation, we have to enforce compatibility. (ie if we specify 6 stresses then we get 6 strains which must reduce to 3 displacements)

- Strain Energy Density $= \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$

- Simply-connected domain $\Rightarrow$ stresses & displacements are single-valued.
- Multiply-connected domain $\Rightarrow$ stresses single valued but either
  - displacement single valued & force enclosed (curl-free vector field)

  or
  - NO enclosed force (ie a dislocation) thus is divergence-free field.

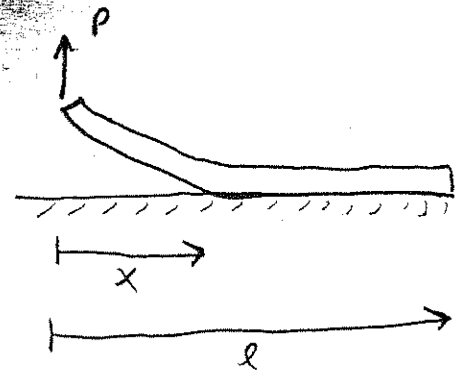
'92
Summer

Scott Adams
Kevin Duprey
Rich Bake
~~[illegible]~~
Harry Dankou
Brianno Coll
Dane Quinn

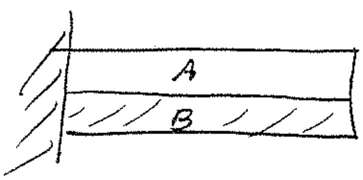
Continuum (Solids)

[illegible] per unit length

Find X.



What is stress?
Why is it not $\underset{\sim}{t} = \underset{=}{\sigma}\, \underset{\sim}{n}^2$?

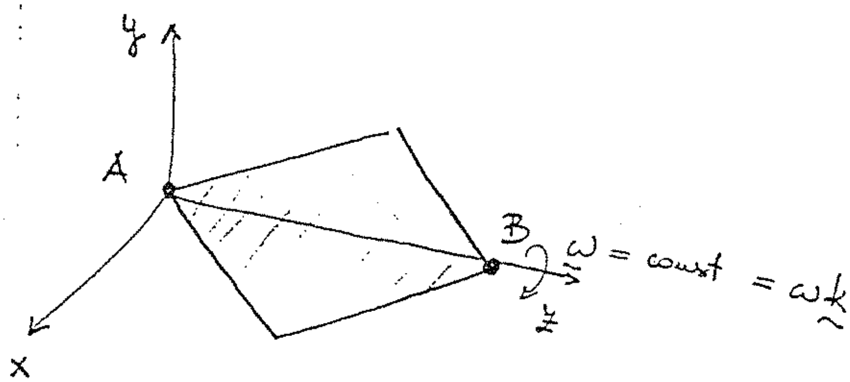


2 Material beam.
Describe $\underset{\sim}{u}$, $\underset{=}{\gamma}$, and $\underset{=}{\sigma}$ near and at the interface?
Continuous / Discontinuous?

yes no yes

$\underset{\sim}{t}(\underset{\sim}{n}) = \underset{=}{\sigma} \cdot \underset{\sim}{n}$ cts ✓

(63) Thin plate in y-z plane!



How do you find reactions at A and B?
(dynamic reactions) reactions

1

(1) Define $\underset{\sim}{F} = m\underset{\sim}{a}$

(2) Given a tube of paper which is then subjected to torsion. At what angle will the paper buckle and why?

(3) Given Fig 1 below with no friction in the walls. Find the stresses

(4) Given Fig 1 again, but now the walls have friction. State the shearing stress distribution on the walls

CORNELL UNIVERSITY

SCHOOL OF CIVIL ENGINEERING

PROBLEM

NAME

DATE

INSTRUCTOR

COURSE

SHEET NO. OF

Applied Math

i) Extremize $F(x,y,z) = xy + 2z^3$ the function F on the unit sphere.

ii) Given the harmonic function $u(x,y)$ on some simply connected region Ω, find its conjugate harmonic function.

iii) Phase plane stuff

a) What is the phase plane

b) Write down a general 2^{nd} order O.D.E. What can you say about it.

c) What are some general features of the phase plane

d) How are critical points classified?

e) Given a linear system of two first order O.D.E's, what is the condition on the coefficients for the origin to be a saddle.

'92
Summer

Math

i^i (again)

$\int_C \frac{1}{z}\, dz$

compare results

$\int_C \bar{z}\, dz = ?$

$|z| = 1 \leftarrow$ contour

what if you choose $|z| = R$?

$\ddot{x} + x = 1$

$x(0) = 0$
$x(1) = 0$

Find solution.
why didn't we get a non-trivial solution?
Existence / Uniqueness

$(1+x)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ What is this? (physically)

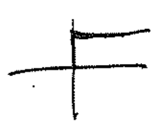
$u(\cancel{x}=0, t) = u(x=1, t) = 0$

$u(x, 0) = f(x)$

Input

step function in

OUTPUT



$\boxed{\text{Black Box}} \rightarrow h(t)$ out

How do you solve for the response to a general input?

3. Appl. math :

- Define $\boxed{z^{\alpha}}$ where z & α complex. (many other complex questions.
- Solve $y''' = y$
- When does sol to $Ax = B$ exist ?
- Convergence of inf. series. (power series)

(5) Calculate various contour integrals (principal value, etc.)

(6) Eigenvalue problems

(7) Determine the existence and uniqueness of the equation $\underset{\approx}{A}\underset{\sim}{x} = \underset{\sim}{b}$ when $\underset{\approx}{A}$ is a matrix, $\underset{\sim}{b}$ is known, and one is solving for $\underset{\sim}{x}$.

Moon :

(1) What is an ODE, PDE, IVP, BVP ?

(2) Fredholm Alternative

(3) Defn of analytic function - ~~state~~ state some ~~many~~ theorems from $\mathbb{C}A$

Q Math

1 (1) What is the definition of an analytic function
What is the condition of differentiable of complex function ?

1 (2) Write the equation of any kind of surface ? $f(x,y,z) = 0$
What is the physical meaning of the direction of derivative ?

1 (3) What is the definition of ∇ operator ?

1 (4) What is the definition of curl, divergence of Cartesian coordinates ?

1 (5) What is the physical meaning of gradient, divergence, curl ?

1 (6) What does it mean for vectors to be linearly independent.

2 (7) If x, y and z are linearly independent, are $x+y$, $y+z$, $x+z$ also linearly independent ?

1 (8) Cauchy integral formula and theorem !

(9) Consider $\ddot{x} + \omega^2 x = 0$, what is the solution ?
... $Ax + B$)

the relation between the two answers?

1

(10) Write down the wave equation! $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

What type of equation is it? hyperbolic

How do you solve it? (explain all 3 methods)?

1) sep. var
2) D'Alembert sol.
3) Sol. of characteristics

(11) Define z where z & α are complex ...

$$z^\alpha = z^{x+iy} = z^x(\cos y + i\sin y)$$

(12) Solve $\boxed{y''' = y}$ $\quad y''' - y = 0$

$\cancel{y(t) = Ay^3 + By^2 + Cy + D}$

$\cancel{y' = 3Ay^2 + 2By + C} \quad \cancel{y'' = 6Ay + 2B} \quad y''' = 6A$

(13) When does solution to $\boxed{Ax = b}$ exists? Uniqueness?

$\boxed{Ax = b}$ $\quad A^{-1}$ = not singular

(14) Convergence of inf. power series?

1

(15) Calculate various contour integrals!

2

(16) Principal value, eigenvalue problem!

1

(17) State the Residue theorem!

1

(18) Define the derivative of a complex function!

$\cancel{y = Ay^3 + By^2 + Cy + D}$

(19) Give a brief lecture on Fourier series!

2

(20) What is Euler's constant γ?

3 Show $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n} - \log(n)\right)$

converges, give a bound on this limit!

(21) How do you solve the Cauchy-Eulers eq.

1 $x^2y'' + xy' - y = 0$?

How do you solve the inhomogeneous case?

1 $x^2y'' + xy' - y = f(x)$?

(22) What is curl $(f(\underline{u}))$? $f: \mathbb{R}^3 \to \mathbb{R}$

2

~~(23) Poisson's Integral Representation:~~

1 → Consider the Dirichlet problem on a unit disc.
Derive Poisson's Integral representation and discuss applications.

3

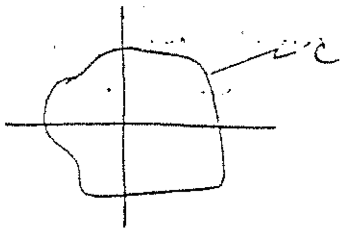
(Ref. Zachmanoglou - Thoe; - Intro to PDE)

a) Identify the denominator in the integral representation

b) What happens when $r = 1$? $[\delta f^n]$

c) Why did you neglect the eigenvalues?

d) Solve $\nabla^2\phi = 0$ in the ...

(24)  $f(z)$ analytic outside C

z_0 a point outside C

$f(\infty) = a$ What is $\oint_C \frac{f(z)}{z - z_0}\, dz$ (1)

(25) What can you say about the following:

$$L(u) + \alpha u = 0$$

$$u(0) = u_0$$
$$u(a) = u_1$$

L is a linear operator ! (1)

~~(26) Steepest descents~~ ! (1)

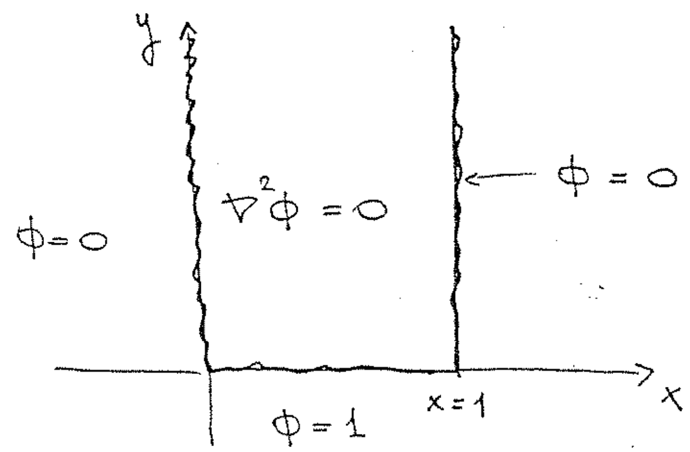
(27) Evaluate $\int_C \frac{dz}{z^{2n} + 1}$, where C is : $|z| = 3$ (2)

(There is a ~~trick~~ for evaluating the residues of a function that is a quotient of two polynomials – such as that above,

[illegible]

(28) Solve the following BVP by elementary methods (no complex variables).

2



$|\phi|$ is bounded at $y \to \infty$

(29) A is a real $n \times n$ matrix.
What are the least restrictive constraints on A that will allow you to diagonalize it?

1

Hint: The diagonal matrix: rank

$$\boxed{D = C^{-1} A C}$$ where C is a special matrix.

(30) Solve $\displaystyle\int_0^\infty \frac{\sin \lambda x}{x}\, dx$

1

$\cancel{\underset{\sim}{v} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3}$

(31) Are the square matrix and the 2nd order tensors the same thing?

1

(32) Describe the solvability of

$$y'' + \lambda y = g(x)$$

1

a) $\lambda > 0$ (h.o.)
b) $\lambda < 0$ (exp.)
c) $\lambda = 0$ (linear)

$y(0) = y(1) = 0$

$0 \le x \le 1$

$y'' = 0 \rightarrow \underline{y = Ax + B}$

for various values of $\underline{\underline{\lambda}}$ ($\lambda \in \mathbb{R}$)

(33) In some space exists 3 vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ each can be expressed as a linear combination of unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$!
What conditions must be satisfied that $\underline{\underline{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3}}$ are linearly independent.

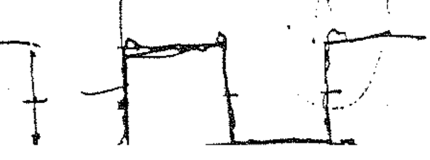
1

(34) $$y(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 < x < 1 \end{cases}$$

What is the value of the Fourier Series of $y(x)$ at the points:

a) $x = -1 \rightarrow \cancel{0} \tfrac{1}{2}$ d) $x = 96.256$

b) $x = +1 \rightarrow \cancel{1} \tfrac{1}{2}$



(35) $f(x) \geq 0$ Col. var.

$0 < x < 1$ $\frac{\partial}{\partial \alpha} \int f(x)\,dx = 0$

1 $f(0) = f(1) = 0$

Maximize area under the curve $f(x)$ subject to the curve f and constraint length

$$L = \frac{\pi}{2}$$

1 (36) What is the second order tensor?

How is it represented?

Given an 2^{nd} order tensor $\underset{\sim}{T}$ and a set of 3 vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ on $\mathbb{R}^n$ with the $\underset{\sim}{v}_i$ LI and the following:

$$\underset{\sim}{T} \cdot \underset{\sim}{v}_1 = -\underset{\sim}{v}_1 + \underset{\sim}{v}_3$$

$$\underset{\sim}{T} \cdot \underset{\sim}{v}_2 = 2\underset{\sim}{v}_2$$

$$\underset{\sim}{T} \cdot \underset{\sim}{v}_3 = \underset{\sim}{v}_2 + \underset{\sim}{v}_3$$

What are the elements of $\underset{\sim}{T}$?

1 (37) $$\int_0^\infty \frac{\cos x}{x^2+1}\,dx$$

(38) What is a differential equation?

Discussion about solutions, existence, uniqueness, difference between PDE & ODE, difference between I.V. problems and B.V. problems, constants on λ in $y'' + \lambda y = 0$ with different I.C. and diff. B.C., methods of solutions, numerical methods.

1

(39) i^i Simplify! $\left(e^{i\frac{\pi}{2}}\right)^i = \underline{\underline{e^{-\frac{\pi}{2}}}}$

Solved

1

(40) $$\int_{|z|=1} \frac{dz}{z^n - a}$$ n is pos. int > 1, $a \in \mathbb{R}$, $0 < a < 1$

(Ans. 0)

(41) Solve: $y''' + y = 0$; $y(0) = y(\pi) = y'(\pi) = 0$

(Ans $y(x) = 0$)

How does this differ (only zero solution) from most math problems?

(want something about e. value problems)

1

(42) $$\int_{-\pi}^{\pi} f(x)\,dx = 0$$ Let $f(x) = ax$ $$\int_{-\pi}^{\pi} f^2\,dx = \frac{a^2}{3} 2\pi^3$$

$$\int_{-\pi}^{\pi} [f(x)]^2\,dx = \int_{-\pi}^{\pi} [f'(x)]^2\,dx$$

$f(x)$ is not the zero function!

What are the restrictions on f?

Use fourier Series!

(Ans. $f(x) = A\cos x + B\sin x$ is only possibility

NO

(43) Given: $\lim_{z \to 0} f(z) = 0$

and $g(z)$ is bounded at neighborhood of z

find : $\lim_{z \to 0} f(z)\,g(z)$ and explain!

(44) A, B, C are LI variables!

How about $A+B$, $B+C$, $A+C$

Prove!

(45) Solve: $y'' + y' = a$ $y' = \cancel{z}$ $y' = a$ $y = ax$

(45) $A_{n\times n}$ has eigenvalues: $\lambda_1, \dots, \lambda_n$

a) What are the e. values of A^{-1}?
Prove!

(46) a) Wave equation (write down, 1-D)

b) Given initial conditions, If the system is a circular string, what are the B.C.?

c) Solve it!

d) Any other way to solve the equation?

e) How can a traveling wave represent a standing wave?

(47) Why does: $\dot{x} = \sqrt{x}$ have two solutions?

(48) $\displaystyle\int_0^\infty \frac{1}{x^3+1}\,dx$ ✓

(49) Tell about Contour ~~Integrals~~, Residue Theorem, Indented contours!

(50) How do Runge-Kutta, Euler methods work? Error estimate?

(51)

How do you get from:

$$I = \int_{t_1}^{t_2} f(x, \dot{x}, t)\, dt \quad \text{to Euler equation}$$

(Calculus of Variations)

let $X = x + \epsilon\eta$
$X' = x' + \epsilon\eta'$

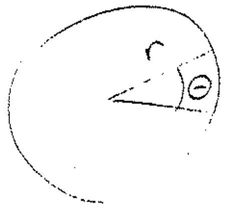
$\frac{\partial f}{\partial X}\frac{\partial X}{\partial \epsilon} + \frac{\partial f}{\partial X'}\frac{\partial X'}{\partial \epsilon}$

(52) $\int_C \frac{dz}{z^m}$ over $|z| = 1$

($z = e^{i\Theta}$, $\therefore$ $I = 0$ for $m \neq 1$
& $I = 2\pi i$ for $m = 1$)

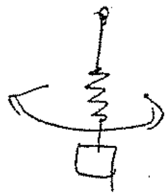
(53) Write a differential equation for heat conduction in a circular ring. Solve the equation given:

$$\Theta(t=0) = g(\Theta)$$



$u_{\Theta\Theta} = \frac{1}{\alpha^2} u_t$ take $\alpha^2 = 1$

Only initial cond. is given. Find out B.C.
What are e. values, e. functions?
What will be the temp. distrib. when $t \to \infty$
How does your solution take care of that?



THEORETICAL AND APPLIED MECHANCIS

QUALIFYING EXAMINATION

January 22, & 23 1987

Dynamics

D.1 Dynamics of a Swing

Discuss the dynamics of pumping a swing:

Taking the rider to be a point mass, derive the equation of motion for a variable length pendulum. Find an approximate equation of motion by considering the length variations to be small and sinusoidal. Discuss the stability of this equation, perhaps using a perturbation analysis to find an approximate solution to it.

Give some discussion of how conservation principles may be used to understand the increasing amplitude of the vibration.

References: T.E. Stern, Theory of Non-Linear Networks and Systems-Introduction;
P. Tea and H. Falk, Am. Jnl. Physics, Dec. 1968.

D.2 Yo-Yo

Describe, using mechanics, the operation of a yo-yo.

D.3 Rotating Celestial Bodies

a) A spacecraft, composed of an inelastic material, is spinning and tumbling freely in space with an initially randomly chosen orientation of the angular momentum to the body's principal axes. Describe the final mode of spin that the body will adopt and explain why.

b) Most natural satellites rotate so as to present approximately the same face to their planets all around their orbits. Discuss what orientation the principal axes of the satellite will have in relation to its orbit and its planet.

D.4 Spinning Egg

Discuss the dynamics of a spinning egg. Consider four cases: raw egg, soft-boiled, hard-boiled, empty shell (but intact).

D.5 Collisions of Two Rigid Bodies

Two rigid bodies of mass m_1 amd m_2 collide in space. At the moment of impact, they contact along a common tangent plane having a normal, $\underset{\sim}{n}$, to the plane. Body #1 moves with angular velocity $\underset{\sim}{\omega}_1$ and linear velocity $\underset{\sim}{v}_1$ (of its center of mass), and Body #2 moves with $\underset{\sim}{\omega}_2$ and $\underset{\sim}{v}_2$.

a) Based on the dynamical principles involving linear momentum and angular momentum and an empirical law for collisions, derive the equations of motion for the unkown velocities $\underset{\sim}{v}_1'$, $\underset{\sim}{v}_2'$ and angular velocities $\underset{\sim}{\omega}_1'$, $\underset{\sim}{\omega}_2'$ of the two bodies after impact. Assume the surfaces of contact are so rough that the bodies collide without slipping at the contact point.

b) Illustrate the solution of the equations of motion for the simplified case when one body is a sphere of radius r_o and the second body is a stationary half space. The sphere hits the rough surface of the half space with linear velocity $\underset{\sim}{v}_1 = v\cos\theta\underset{\sim}{i} + v\sin\theta\underset{\sim}{j}$ and zero angular velocity ($\underset{\sim}{\omega}_1 = 0$).

Reference: Greenwood, Principles of Dynamics, Chs. 4 and 8.
E.T. Whittaker; Analytical Dyanmics, Sections 95-97.

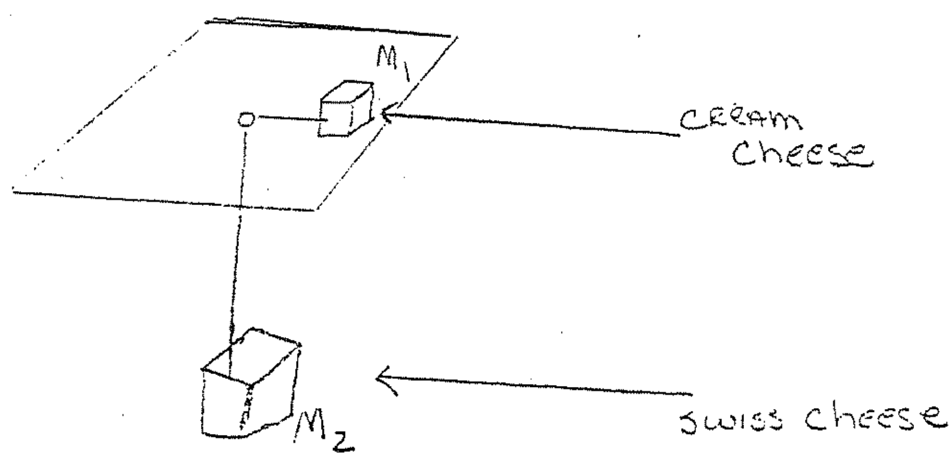
D.6 Billiards

Discuss the mechanics of billiards. Consider the effects of impacts, friction, impure roll, and wall interactions.

References: Greenwood; Principles of Dynamics
Summerfield, Mechanics

D.7 Bodies Connected by a String

Consider the physical problem to two moving masses connected by a massless inextensible string as shown in figure:



Derive the equations of motion of the system, stating clearly all your assumptions. Can you solve the equations?

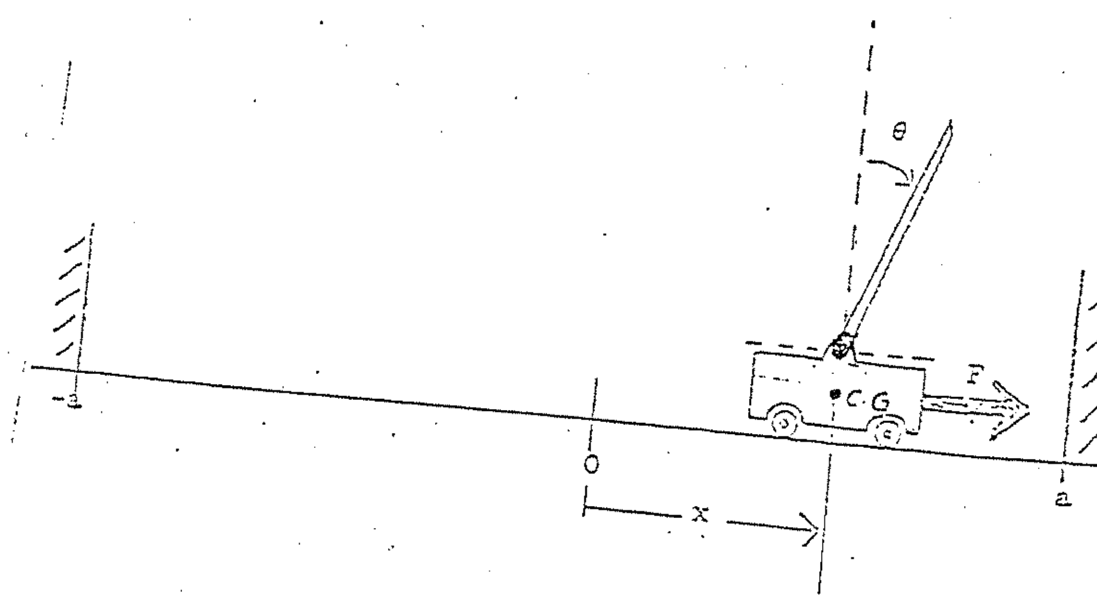
D.8 Control of Dynamical System

The figure shows a cart on which is mounted a rod, hinged as shown. T designer has at his disposal the forcing function $F = F(x,\dot{x},\theta,\dot{\theta})$. He would like choose this function (design the system) so that the motion of the cart will ke the rod in balance. The track extends only from $x = -a$ to $x = +a$. Thus the id is to choose $F(x,\dot{x},\theta,\dot{\theta})$ so that

$$-a < x < a$$
$$-\pi/2 < \theta < \pi/2$$

hold for all t.

1. Set up the differential equations of motion.
2. Even if you can't come up with a design, discuss how one might go about it, the difficulties involved (both practical and theoretical), what effect it would have if F were restricted to be "bang-bang" i.e. F can only be $+F_o$ or $-F_o$, range of initial conditions, etc.



I. COLLISIONS OF RIGID BODIES.

Two bodies of mass m_1 and m_2 collide as shown in the figure. At the moment of collision, they contact at point P with a common tangent plane and a normal, $\mathbf{n}$, to that plane. We are in general given:

$$\mathbf{v}_1\ ,\ \mathbf{v}_2\ ,\ \boldsymbol{\omega}_1\ ,\ \boldsymbol{\omega}_2 \text{ before impact,}$$

and wish to determine:

$$\mathbf{v}_1'\ ,\ \mathbf{v}_2'\ ,\ \boldsymbol{\omega}_1'\ ,\ \boldsymbol{\omega}_2' \text{ after impact.}$$

The twelve unknown quantities (4 vectors) are related to the given quantities by the empirical law of collision and equations for impulsive motion:

$$\int_0^{\Delta t} \mathbf{F}\, dt = \left[m\, \mathbf{v}_c\right]_0^{\Delta t} \qquad \text{Linear momentum}$$

$$\int_0^{\Delta t} \mathbf{N}\, dt = \left[\mathbf{H}\right]_0^{\Delta t} \qquad \text{Angular momentum}$$

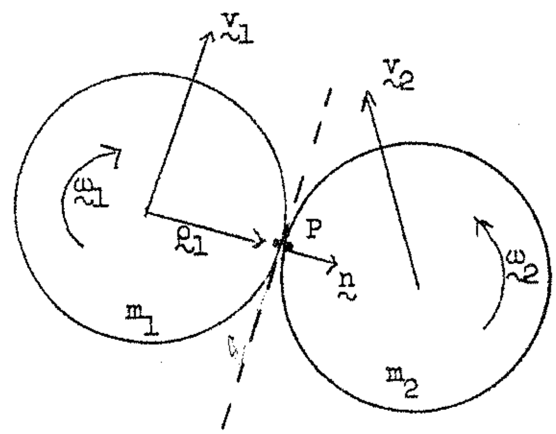
The twelve equations are:

(1) Law of Collisions - When two bodies collide, the values of the normal component of the relative velocity of the surfaces in contact at instants immediately after and immediately before the impact bear a definite ratio to each other; this ratio, denoted by $-e$, depends only on the material of which the bodies are composed (one eq.).

$$-e = \frac{\mathbf{v}_{1p}'\cdot\mathbf{n} - \mathbf{v}_{2p}'\cdot\mathbf{n}}{\mathbf{v}_{1p}\cdot\mathbf{n} - \mathbf{v}_{2p}\cdot\mathbf{n}}$$

where

$$\mathbf{v}_{1p} = \mathbf{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{\rho}_1 \quad \text{etc.}$$



common tangent plane

(2) Constancy of angular momentum of each body about the point of contact, p, because of zero moment about p (six eqs.).

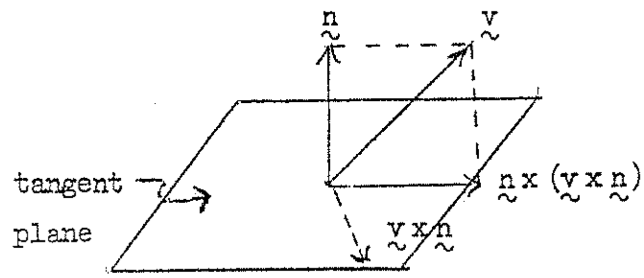
$$\mathbf{L}_p = 0\ .$$

(3) Constancy of linear momentum of the system (two bodies) normal to the surface of contact because of equal and opposite normal impulses (one eq.)

$$(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)\cdot\mathbf{n} = (m_1\mathbf{v}_1' + m_2\mathbf{v}_2')\cdot\mathbf{n}\ .$$

(4A) For smooth surfaces - No change in tangential components of the linear momentum for each body because of zero tangential forces (four eqs.)

$$\underset{\sim}{n} \times (m\underset{\sim}{v}' \times \underset{\sim}{n}) - \underset{\sim}{n} \times (m\underset{\sim}{v} \times \underset{\sim}{n}) = 0$$



(4B) For Rough Surfaces without Slipping at the Contact Point

(i) Constancy of tangential components of the linear momentum of the system because of equal and opposite tangential impulses (two eqs.)

$$\underset{\sim}{n} \times [(m_1\underset{\sim}{v}_1 + m_2\underset{\sim}{v}_2) \times \underset{\sim}{n}] = \underset{\sim}{n} \times [(m_1\underset{\sim}{v}_1' + m_2\underset{\sim}{v}_2') \times \underset{\sim}{n}]$$

(ii) Vanishing of the tangential components of the relative velocity of the two bodies after impact because of no slipping constraint (two eqs.)

$$\underset{\sim}{n} \times [(\underset{\sim}{v}_{2p}' - \underset{\sim}{v}_{1p}') \times \underset{\sim}{n}] = 0 .$$

(4C) For Rough Surfaces with Slipping at the Contact Point

(i) Same as (4Bi) (two eqs.)

(ii) Change of the components of the linear momentum by the tangential impulse which equals, in magnitude to μ (coefficient of friction) times the normal impulse for each body (two eqs.)

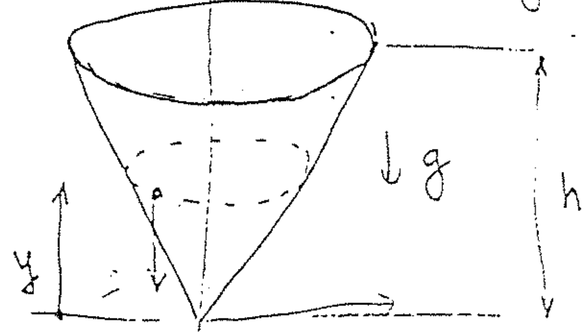
$$|\underset{\sim}{n} \times [m(\underset{\sim}{v}' - \underset{\sim}{v}) \times \underset{\sim}{n}]| = \mu m(\underset{\sim}{v}' - \underset{\sim}{v}) \cdot \underset{\sim}{n}$$

Note that there are two equations of the above form for each body; a total of 4 equations is obtainable. However, only two (for either body) are independent equations.

DYNAMICS QUESTIONS

(9) A particle travels in a circular path on the inside of a cone. Find its speed v as a function of y.

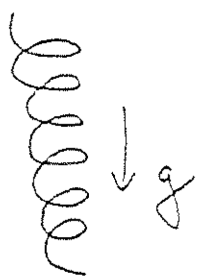
1



(Newton)

(10) A particle travels along a helical spring, until it reaches the end of the spring and falls off. What is its trajectory?

1



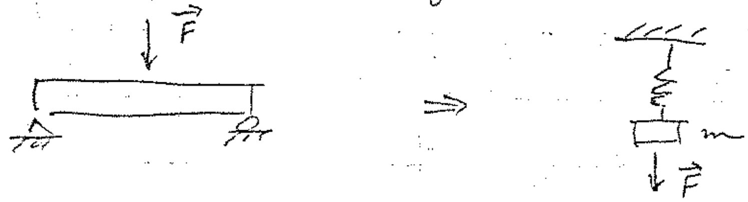
1

(11) a) State Newtons 3 laws
b) D'Alembert's Principle (Notes)
extension of P.V.W. from Statics to Dynamics

(12) Find the altitude of a geosynchronous orbit.

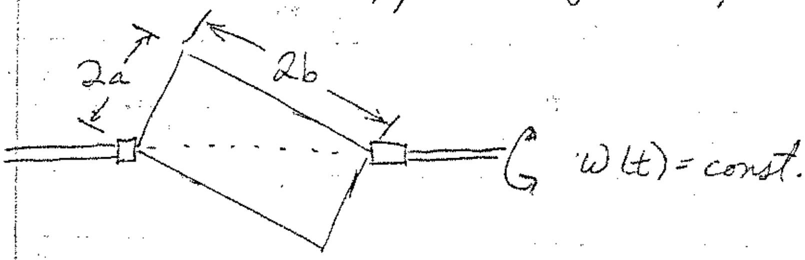
DYNAMICS

1. An elastic beam with simple support can be modeled as a harmonic oscillator (mass supported by a spring)
The beam is subjected to the time varying force $\vec{F} = \mathcal{F}\,\delta(t)\,\hat{j}$

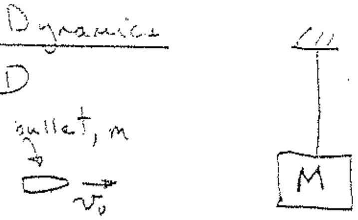


What is the equation of motion?
Solve it for $y(t)$ when $\vec{F}(t) = \mathcal{F}\,\delta(t-\tau)\,\hat{j}$

2. A rectangular plate of constant thickness spins about an axis. What moment is exerted on the support by the plate.



Dynamics

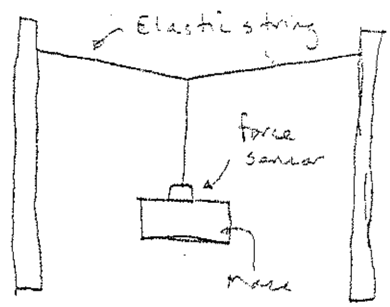
(1) 

Bullet strikes pendulum and is embedded. What is maximum angle θ of subsequent motion of the pendulum? (Karel)

(2) Discuss nature of angular momentum and relevant equations for a spacecraft. What are the equations? What about other formulations? (Moon)

(3) What constraints are there on the choice of reference frames for using the equations $\Sigma \underline{M} = \dot{\underline{H}}$, $\Sigma \underline{F} = m\underline{a}$? What is an inertial reference frame? (Burns)

(4) There was an apparatus: (SACHSE)



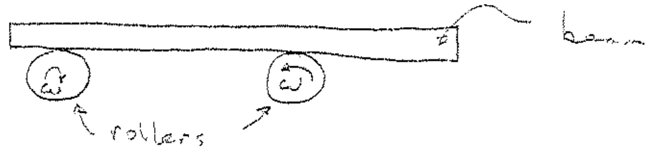
Prof Sachse pulled the mass down and released it.

What is the nature of the motion?

What is the equation of motion?

When is force in the sensor of maximum magnitude?

How would you verify that the motion obeys the solution to the equation of motion?

(5) 

(Zehnder)

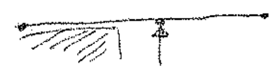
There is friction between rollers and beam. What is the nature of the motion and what is the equation of motion?

Jan 21.

1. Dynamics.

Examinee C. Zhang

Questions: 1) A rule initially lies on a table supported by hand, then withdraw the hand. ~~Descript~~ Derive equations of motion.



What is the relationship between the angular velocity $\vec{\omega}$ and the velocity of mass center?

2)

Derive the velocity and acceleration.

3)

given constant $\vec{\omega}$, find θ.

How many method can you use?

4). There ~~was~~ was an experimental set-up. An oscilloscope and a hammer attached to it. Prof. Sachse gives a small blow to the cable with hammer and asks to explain what the pulse on the screen indicates?

What is inside the hammer?

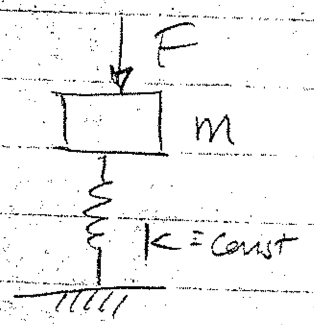
Write down the equation for the mechanism inside the hammer. Explain how to measure the force?

5). What is the difference between Newton's method ~~equation~~ and Lagrange ~~equation~~ method.

What is constrain?

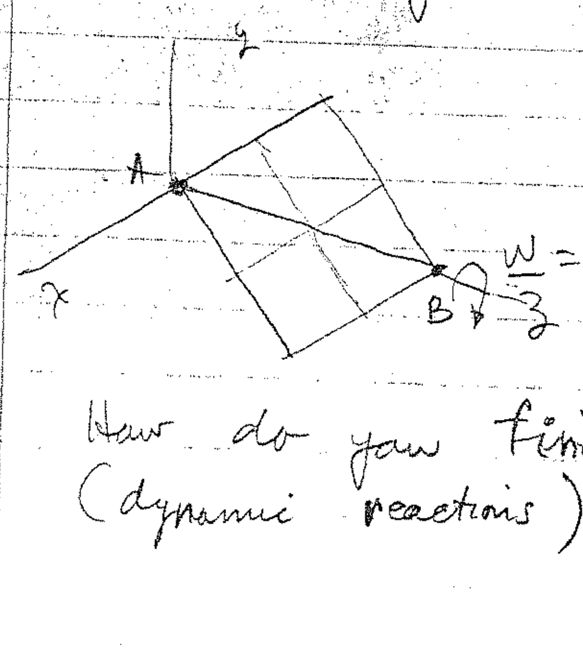
Dynamics

(1)



- Impact load is applied
- How do you solve for displacement of mass?
  ($m\ddot{x} + kx = \underbrace{F}_{\text{const.}}\delta(t-t_0)$ ← supplied by student)
- what are initial conditions?
- what is the solution?
- How do you apply boundary conditions?
  How do you solve for arbitrary excitation

(2)



Thin plate in y-z plane

$\underline{w} = \text{const} = \omega\,\underline{k}$

How do you find reactions at A and B?
(dynamic reactions)

Dynamics

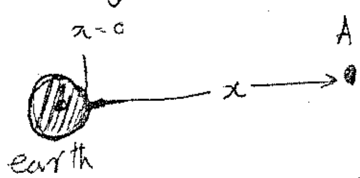
1. Derive a formula for escape velocity in terms of "known" quantities for
   1. earth - 'satellite'
   2. Sun - Earth - satellite systems.

(known $\rightarrow g$, year, 24 hrs, distances : no masses, G)

Kepler's law.

2. Express angular velocity as a matrix (not vector)

3. Why does potential engy have a $-ve$ sign?



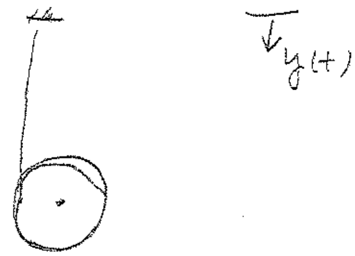
What is Pot. En. at A for $x \gtreqless 0$ & $x = \infty$?

---

general $\rightarrow$ • You should be able to ~~[illegible]~~ fully justify every statement you make.
• Better to write your answers rather than just say it.

Dynamics

1)

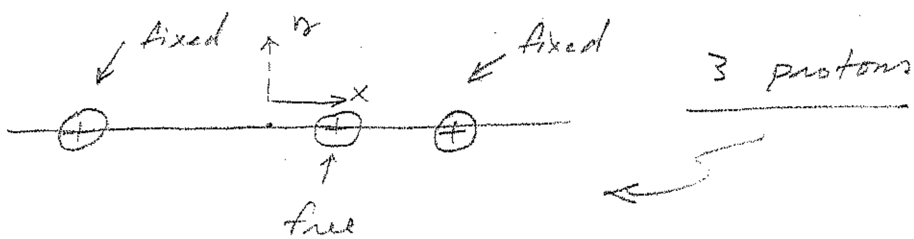


Spool with string falling down

a) find $y(t)$

b) will it swing?

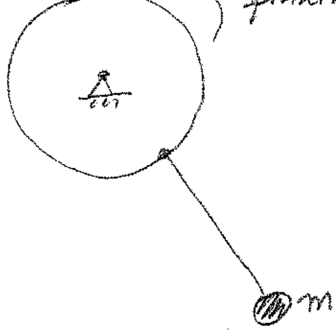
2)



Discuss the stability of the free proton?

3)

Spinning w/ a constant ω

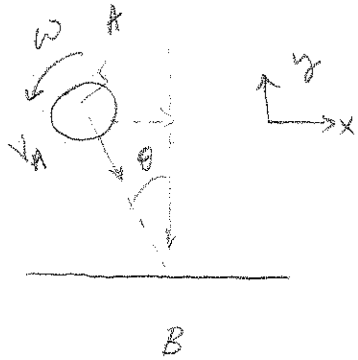


Find eqn. of motion of the mass m.

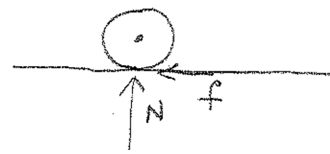
Dynamics

4) Suppose there's a chair How can one find the principal moment of inertia of the chair?

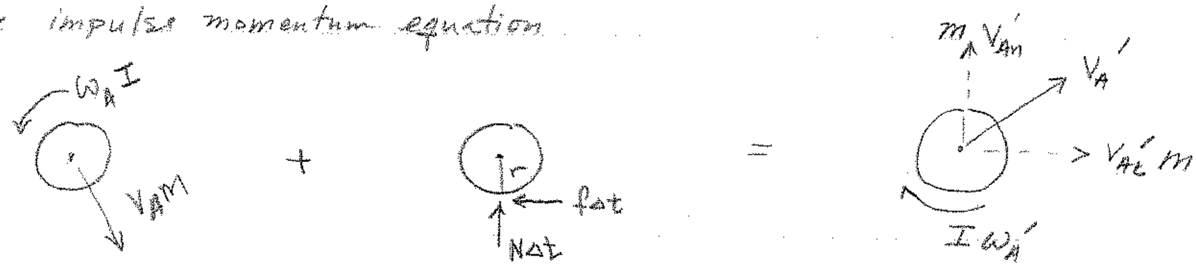
Rocket Puck



at impact



write impulse momentum equation



given normal component is elastic

$$(V_{Bn}' - V_{An}') = e (V_{An} - V_{Bn})$$ memorize this thing

elastic $\Rightarrow$ $e = 1$ $\qquad$ $V_{Bn} = V_{Bn}' = 0, \rightarrow$ wall.

$$\Rightarrow \quad -V_{An}' = V_{An} \quad \text{or} \quad V_{An}' = -V_{An} = + V_A \cos\theta$$

$$x: \quad mV_A \sin\theta - f\Delta t = mV_{At}'$$
$$y: \quad -mV_A \cos\theta + N\Delta t = mV_{An}' = mV_A \cos\theta$$
$$+\circlearrowleft: \quad I\omega_A - f\Delta t\, r = -I\omega_A'$$

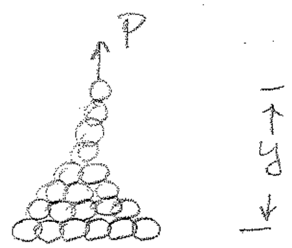
V_{At}', ω_A', $f\Delta t$, $N\Delta t$ $\qquad$ 4 unknowns $\qquad$ 3 equations.

need one more equation

Assume friction force related to N, i.e, $\quad f = \mu N$

$\Rightarrow \quad f\Delta t = \mu N\Delta t$ $\quad$ This gives the 4th equation needed to solve for everything.

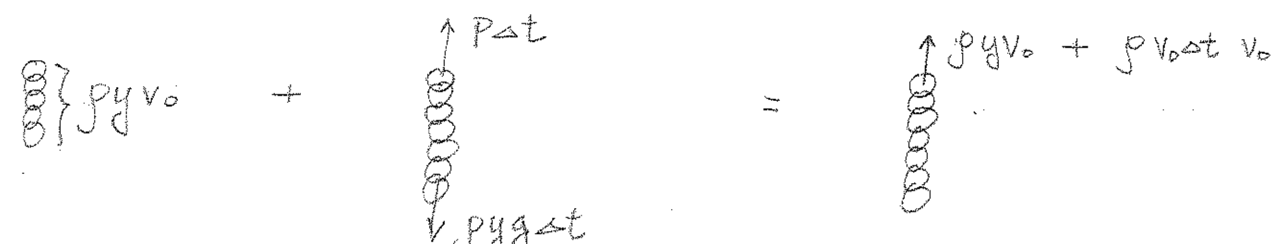
Chain-pulling problem



constant v_o upward, determine P as a function of time to maintain constant v_o.

impulse momentum technique. ↗ use impulse momentum.

each link on the floor acquires its velocity ~~abruptly~~ through the impact w/ its upper link which lifts it off the floor. Thus, the lifting of such a chain involves an energy loss.

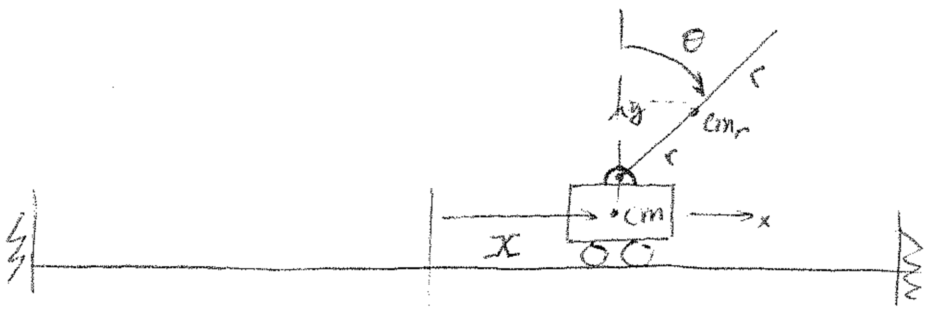


$$\cancel{\rho y v_o} + P\Delta t - \rho y g \Delta t = \cancel{\rho y v_o} + \rho v_o^2 \Delta t$$

$$\Rightarrow P \cancel{\Delta t} = \rho (yg + v_o^2) \cancel{\Delta t}$$

$$\Rightarrow \boxed{P = \rho (yg + v_o^2)}$$

ok.



Length of rod is $2r$. Attach a coordinate system to the cart.

$$\underset{\sim}{r}_{cm_r} = X\underset{\sim}{E}_1 + r\sin\theta\,\underset{\sim}{e}_1 + r\cos\theta\,\underset{\sim}{e}_2$$

note that $\underset{\sim}{E}_1 = \underset{\sim}{e}_1$

$$\Rightarrow \underset{\sim}{r}_{cm_r} = (X + r\sin\theta)\underset{\sim}{e}_1 + r\cos\theta\,\underset{\sim}{e}_2$$

$$\dot{\underset{\sim}{r}}_{cm_r} = (\dot{X} + r\dot{\theta}\cos\theta)\underset{\sim}{e}_1 - (r\dot{\theta}\sin\theta)\underset{\sim}{e}_2$$

$$\dot{\underset{\sim}{r}}_{cm_r}\cdot\dot{\underset{\sim}{r}}_{cm_r} = (\dot{X} + r\dot{\theta}\cos\theta)\cdot(\dot{X} + r\dot{\theta}\cos\theta) + (r\dot{\theta}\sin\theta)(r\dot{\theta}\sin\theta)$$
$$= \dot{X}^2 + 2\dot{X}r\dot{\theta}\cos\theta + r^2\dot{\theta}^2$$

$$T = \frac{1}{2}m_{rod}(\dot{X}^2 + 2\dot{X}r\dot{\theta}\cos\theta + r^2\dot{\theta}^2) + \frac{1}{2}m_{cart}\dot{X}^2$$
$$+ \frac{1}{2}I_{rod}\dot{\theta}^2 \qquad \text{(Kinetic energy of the system)}$$

$$V = \cancel{mgr\cos\theta(1-\sin\theta)}\quad mgr\,\cancel{(1-\cos)}\,(\cos\theta - 1)$$

$$\mathcal{L} = T - V = \frac{1}{2}m_r(\dot{X}^2 + 2\dot{X}r\dot{\theta}\cos\theta + r^2\dot{\theta}^2) + \frac{1}{2}m_c\dot{X}^2 + \frac{1}{2}I_r\dot{\theta}^2$$
$$- mgr\,\cancel{(1-\cos\theta)}\,(\cos\theta - 1)$$

$\dot{x}$

$$\frac{\partial L}{\partial \dot{x}} = m_r \dot{x} + m_r r \dot{\theta} \cos\theta + m_c \dot{x}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

$$\Rightarrow \quad m_r \ddot{x} + m_r r \ddot{\theta} \cos\theta - m_r r \dot{\theta}^2 \sin\theta + m_c \ddot{x} = F$$

or

$$\boxed{(m_r + m_c)\ddot{x} + m_r r(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) = F} \quad (1)$$

$\dot{\theta}$

$$\frac{\partial L}{\partial \dot{\theta}} = m_r \dot{x} r \cos\theta + m r^2 \dot{\theta} + I_r \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = \cancel{mgx\cos\theta}\ mgr\sin\theta$$

$$\Rightarrow \quad m_r \ddot{x} r \cos\theta - m_r \dot{x} r \dot{\theta} \sin\theta + m r^2 \ddot{\theta} + I_r \ddot{\theta} - mgr\sin\theta = 0$$

$$\boxed{m_r r(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta) + (I_r + m_r r^2)\ddot{\theta} - mgr\sin\theta = 0} \quad (2)$$

note x doesn't appear explicitly in the equations ∴ x is an ignorable coordinate.

Look for steady motion.

(i) all nonignorable coordinates are constant

(ii) all ignorable coordinates' velocity are constant.

~~$I_a^2 \Omega^2 \sin\theta_0 \cos\theta_0 + mga\sin\theta = 0$~~

Steady motion

$\Rightarrow F = 0$ from eqn. (1)

$\Rightarrow mgr\sin\theta = 0 \quad \Rightarrow \quad \theta = 0, \pi, .. p\pi,$

$\theta = 0$ is the only allowable steady-motion in this system due to design requirement.

Linearize equation of motion about $\theta = 0$

$\theta = \varepsilon(t) \qquad \varepsilon << 1$

$$(m_r + m_c)\ddot{x} + m_r r \ddot{\varepsilon} = F \qquad (1)$$

$$m_r r \ddot{x} + (I_r + m_r r^2)\ddot{\varepsilon} - mgr\varepsilon = 0 \qquad (2)$$

Solve equation (1) for $\ddot{x}$

$\Rightarrow \qquad (m_r + m_c)\ddot{x} = F - m_r r \ddot{\varepsilon}$

$\Rightarrow \qquad \ddot{x} = \dfrac{F - m_r r \ddot{\varepsilon}}{(m_r + m_c)} \qquad (3)$

Substitute (3) into (2)

$$\Rightarrow m_r r \left(\frac{F - m_r r \ddot{\varepsilon}}{(m_r + m_c)} \right) + (I_r + m_r r^2)\ddot{\varepsilon} - mgr\varepsilon = 0$$

$$\Rightarrow \frac{m_r r}{(m_r + m_c)} F - \frac{m_r^2 r^2}{(m_r + m_c)} \ddot{\varepsilon} + (I_r + m_r r^2)\ddot{\varepsilon} - mgr\varepsilon = 0$$

$$\Rightarrow \frac{m_r r}{(m_r + m_c)} F + \left(\frac{m_r r^2 (m_r + m_c) + I_r (m_r + m_c) - m_r^2 r^2}{(m_r + m_c)} \right) \ddot{\varepsilon} - mgr\varepsilon = 0$$

$$\Rightarrow \frac{m_r r}{(m_r+m_c)} F + \left(\frac{\cancel{m_r^2 r^2} + m_r m_c r^2 + I_r(m_r+m_c) \cancel{- m_r^2 r^2}}{(m_r+m_c)} \right) \ddot{\varepsilon}$$

$$- mgr\varepsilon = 0$$

$$\Rightarrow \boxed{ \underbrace{\left(\frac{m_r m_c r^2 + I_r (m_r+m_c)}{(m_r+m_c)} \right)}_{>0} \ddot{\varepsilon} - \underbrace{mgr}_{>0} \varepsilon + \frac{m_r r F}{(m_r+m_c)} = 0 }$$

$F = F_0 \delta(\varepsilon - \varepsilon_0)$

if $F = 0$

stability equation.

Bang-Bang type.

$$\Rightarrow \quad c_1 \ddot{\varepsilon} - c_2 \varepsilon = 0$$

$$c_1 > 0 \quad c_2 > 0$$

$$\varepsilon = e^{rt} \qquad r^2 e^{rt} \quad - e^{rt}$$

$$c_1 r^2 - c_2 = 0$$

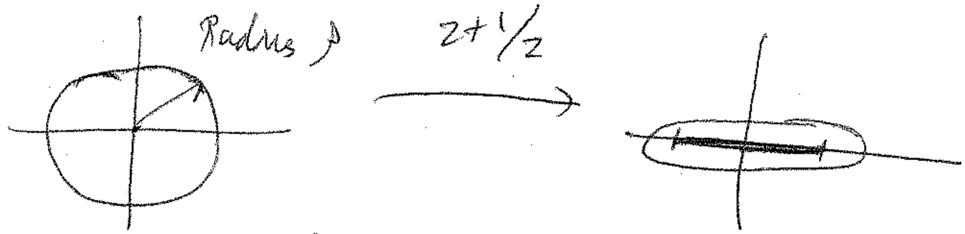
$$r^2 - \frac{c_2}{c_1} = 0 \quad \Rightarrow \quad r = \pm \frac{c_2}{c_1}$$

We have 1 positive root $\left(r = \frac{c_2}{c_1} \right)$ so the system will become unstable as $t \to \infty$ if we don't use an appropriate forcing function.

Qualifying Exam - Aug 23, 2002 - MATHEMATICS

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

(or)



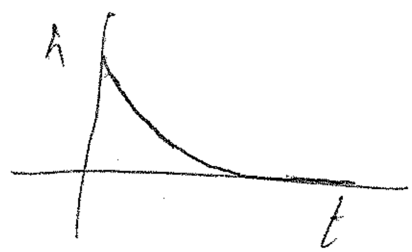
2. $h(t) - h_0 = -a \int_0^t d\tau \sqrt{h(\tau)}$

a) Is the eqn. Linear or Non-Linear?

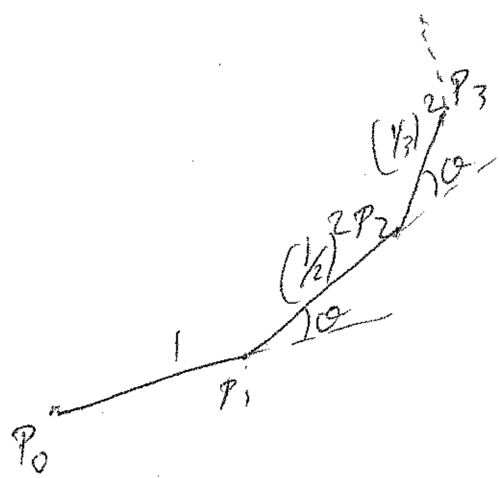
b) Equivalent Differential eqn.?

c) Sketch $h(t)$ without solving

d) Numerical Scheme



3.



$P_n \to ?$ as $n \to \infty$

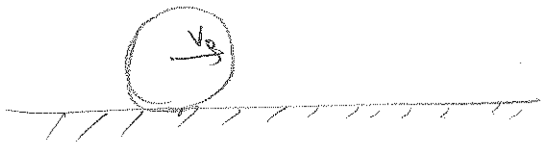
$$\sum \frac{e^{in\theta}}{n^2}$$

Dynamics

Jan '06

Carlos
Erik
Sanjay

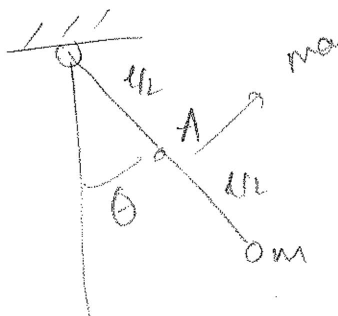
1)



a) Find time at which rolling (w/o slipping) begins

2)

massless rod

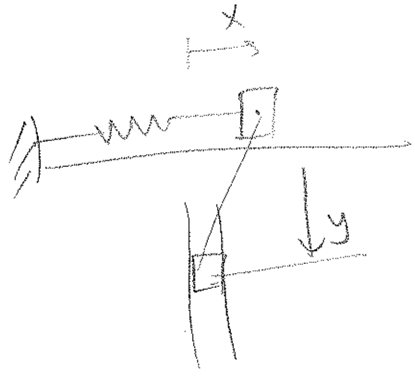


a) @ $\theta_0 = \frac{-\pi}{2}$ find acceleration pt. A

b) what is the tension in the rod at $\theta = 0$

(c) find the period of oscillation

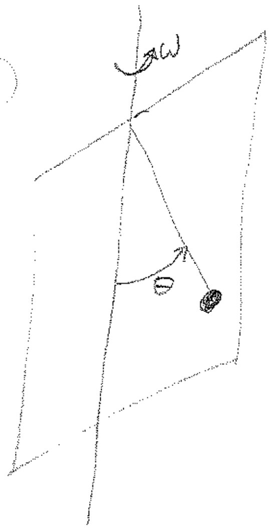
3)



Find the equations of motion.

2000 Q-EXAM
08/18/00
DYNAMICS - MORNING

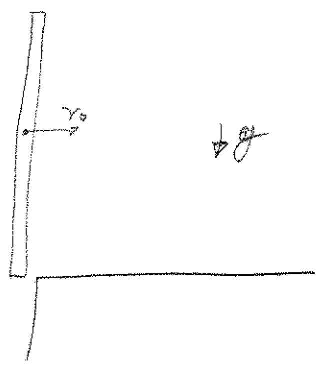
(1)



PENDULUM CONFINED TO A PLANE ROTATING AT ANGULAR VELOCITY ω.

FIND EQUATIONS OF MOTION USING LAGRANGE.

(2)



A THIN ROD MOVING HORIZONTALLY WITH VELOCITY v_0 BARELY STRIKES A TABLE IN AN INELASTIC COLLISION - THEN ROTATES ABOUT THE IMPACT POINT UNDER INFLUENCE OF GRAVITY.

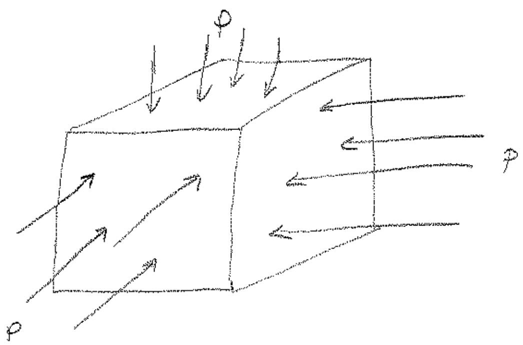
WHAT IS THE FINAL MOTION OF THE ROD?

(3) WRITE OUT EQUATIONS OF MOTION FOR A RIGID BODY ROTATING ABOUT ITS CENTER OF MASS.

2000 - QEXAM
08/18/00
CONTINUUM MECHANICS

1. a) WHAT IS THE DIFFERENCE BETWEEN A SOLID AND A LIQUID?
b) WRITE DOWN A CONSTITUTIVE MODEL FOR A SOLID.
c) WRITE DOWN A CONSTITUTIVE MODEL FOR A LIQUID (STATIC).
d) WRITE DOWN A CONSTITUTIVE MODEL FOR A LIQUID (DYNAMIC).
e) WRITE DOWN THE MOST GENERAL CONSTITUTIVE MODEL YOU CAN THINK OF.

---

2. 

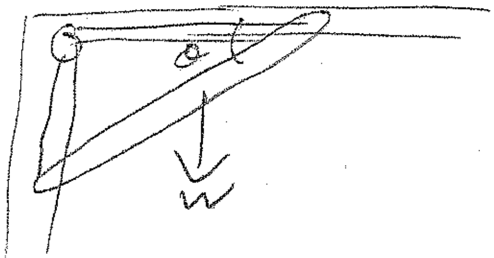
SAME PRESSURE ON ALL SIX SIDES OF CUBE

3D ELASTICITY

a) FIND THE DISPLACEMENT FIELD
b) IS THE ANSWER UNIQUE?
c) WHAT ASSUMPTIONS DID YOU MAKE IN ADDITION TO 3D ELASTICITY?
d) IS THE ANSWER THE SAME IF THE BODY HAD A DIFFERENT SHAPE?
e) WHAT IF THERE WAS A HOLE IN IT?
f) SUPPOSE THERE ARE TWO MATERIALS - ONE INSIDE THE OTHER - CAN YOU COME UP WITH A RESTRICTION ON E, ν SUCH THAT THE DISPLACEMENT FIELD REMAINS THE SAME?
g) HOW WOULD YOU GET RID OF THE NON-UNIQUE TERMS IN THE DISPLACEMENT FIELD TO SOLVE IT ON THE COMPUTER?

SOLIDS

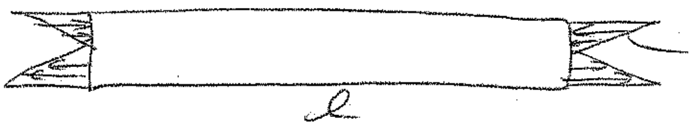
①



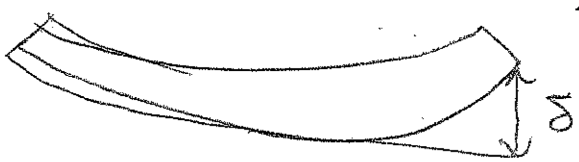
FIND ⇒ TENSION IN ROPE
INTE

FIND θ WHEN T = 3W

②



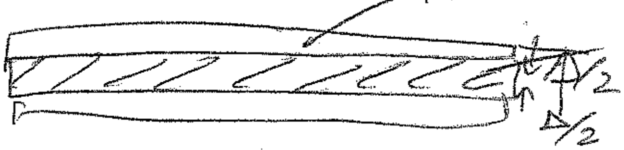
$\sigma_0 = \frac{\sigma_y}{w}$



FIND δ

IS THIS SOLN THE 3D EXACT SOLN

②



L, H, E, I

WHAT IS THE STATE OF STRESS

w/ ADHESION/FRICT

elastic moduli of a microcracked solid by a different approach. They do not start with eqn (4), but utilize the potential energy balance and the relationship between potential energy change and the crack energy release rate. Their approach is particularly useful for solids with microcracks embedded since they avoid the difficulty of evaluating inclusion strain $\bar{\varepsilon}_{ij}$ for microcracks in eqn (4). Penny-shaped cracks were assumed to be randomly distributed in the matrix material such that the crack material behaves like an isotropic solid. The moduli of the cracked solid were found, depending not on the porosity volume concentration, but on a newly introduced parameter, the crack density,

$$\varepsilon = N\langle a^3 \rangle, \tag{5}$$

where N is the number of cracks per unit volume, a is the radius of the penny-shaped crack, and $\langle \cdot \rangle$ is the average of the argument. A critical value of crack density, $\varepsilon = 9/16$, was established at which the effective elastic moduli of the cracked solid vanish.

Although Budiansky and O'Connell's (1976) analysis of a cracked solid was based on the potential energy balance, it is shown in the following that their results still fall into the general framework of self-consistent mechanics of composite materials (Budiansky, 1965; Hill, 1965). A penny-shaped crack can be considered as the limit of an oblate spheroidal cavity with $a_1 = a_2 = a$ and $a_3 \to 0$, where a_1, a_2 and a_3 are the half-axes of the spheroid. The basic equation [eqn (4)], for a solid with spheroidal cavities embedded, becomes

$$\frac{1+\bar{\nu}}{\bar{E}}\sigma^0_{ij}\sigma^0_{ij} - \frac{\bar{\nu}}{\bar{E}}\sigma^0_{kk}\sigma^0_{ll} = \frac{1+\nu_N}{E_N}\sigma^0_{ij}\sigma^0_{ij} - \frac{\nu_N}{E_N}\sigma^0_{kk}\sigma^0_{ll} + c\sigma^0_{ij}\bar{\varepsilon}_{ij}, \tag{6}$$

where $\bar{E}$ and E_N are the Young's modulus and $\bar{\nu}$ and ν_N are Poisson's ratio of the cracked solid and matrix material, respectively; c is the volume concentration of the cavity. Note that

$$c\bar{\varepsilon}_{ij} = \frac{1}{V}\int_{V_{\text{cavity}}} \varepsilon_{ij}\,\mathrm{d}V = \frac{1}{V}\sum_{\text{all cavities}} \varepsilon_{ij}\cdot\tfrac{4}{3}\pi a^2 a_3, \tag{7}$$

where V_{cavity} is the total volume of the cavities, V is the total volume of the solid, $4/3(\pi a^2 a_3)$ is the volume of each cavity, and ε_{ij}, which is uniform within each cavity, was given by Eshelby (1957). In the case of remote hydrostatic tension, $\sigma^0_{ij} = \sigma^0\delta_{ij}$, the strain, ε_{ij}, within the cavity has the asymptotic form (Eshelby, 1957; Mura, 1982)

$$\varepsilon_{33} = \frac{4}{\pi}\frac{1-\bar{\nu}^2}{\bar{E}}\frac{a}{a_3}\sigma_0 + O(1), \quad \text{others} = O(1). \tag{8}$$

Thus, eqn (7), in the limit of cracks ($a_3/a \to 0$), gives

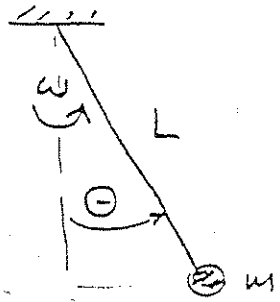
$$c\bar{\varepsilon}_{kk} = \frac{1}{V}\Sigma a^3 \cdot \frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\sigma_0 = N\langle a^3\rangle \frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\sigma_0. \tag{9}$$

Substituting into the basic equation [eqn (4)], one finds

$$\frac{3(1-2\bar{\nu})}{\bar{E}} = \frac{3(1-2\nu_N)}{E_N} + \frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\varepsilon, \tag{10}$$

which is exactly one of the governing equations for determination of effective moduli given by Budiansky and O'Connell (1976). If other kinds of the remote loading σ^0_{ij} are applied, one can similarly derive other governing equations of effective moduli identical to Budiansky and O'Connell's. This shows that their analysis of a cracked solid is consistent with the self-consistent mechanics of composite materials (Budiansky, 1965; Hill, 1965). One can

(50) Spherical pendulum ($\dot{\theta} = 0$)
Given an I.C. on v, find the period.
If the mass is given a slight disturbance
($\dot{\theta}$ no longer $= 0$), find the period!



$$\left[\begin{array}{l} \omega^2 = \dfrac{g}{L\cos\theta} \\ v = \omega (L\sin\theta). \end{array}\right.$$

1

Period $T = T(\theta)$. $\dot{\theta} \neq 0$

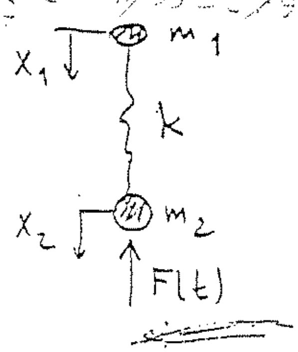
Determine new steady state.

θ', v'

$$\left[\begin{array}{l} \text{Energy Conservation.} \\ v = v(\theta). \end{array}\right.$$

---

(51)



Set up eq.'s of motion.
Given $(\ddot{x}_1 - \ddot{x}_2)$ how would you find $\underline{F(t)}$?

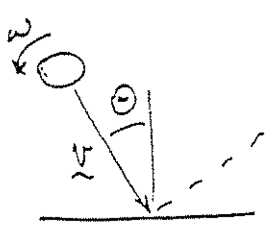
1

... not well-posed.

(52) What is the difference between LE' and NE's? 1) No superposition
2) Solutions depend on i.c.

1

(39) Rocket puck impacts a wall.
What is its direction, velocity and spin when it leaves?
Normal collision is elastic.



(2)

$v_n' = -v_n$

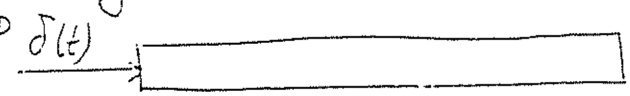
$v_t' = v_t$

$I\omega' = I\omega$

(40) Bang on the end of a rod with a hammer.
Listen to the tone. What is the bar mode of? How can you tell?

$\sqrt{\frac{E}{\rho}}$

(1)



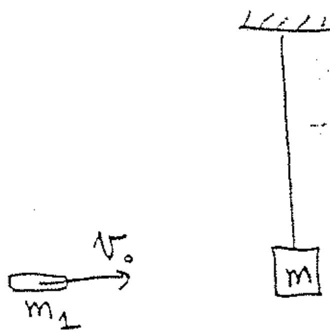
@ $x=0$: $u(X)=0$

@ $x=c$: $u'(x=L)=0$

$u(x,t) = f(L, c, \omega$

(41) Bullet strikes pendulum and is embedded.
What is maximum angle θ of subsequent motion of the pendulum?

(1)



$m_1 v_0 = (m + m_1) v$

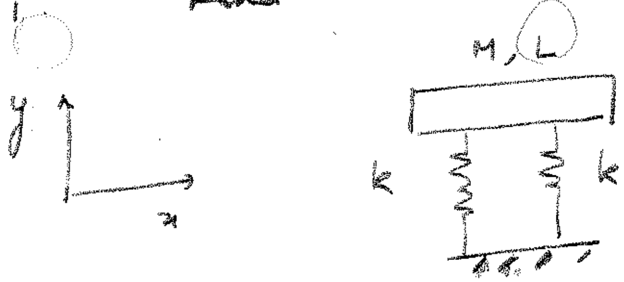
$v = \frac{m_1}{m + m_1} v_0$

$\frac{m v^2}{2} = m g R (1 - \cos\theta)$

$\cos\theta = 1 - \frac{v^2}{2gR} = 1 - \left(\frac{m_1}{m + m_1}\right)^2 \frac{v_0^2}{2g}$

(B) DYNAMICS (Aug 95)

1. Find the natural frequencies of the following system.



(a) Assume no motion in x-direction.

$\frac{2k}{m}$ (translational), $\frac{kL^2}{2I}$ (rotational).

(b) (Russia) What is the frequency in the x-direction? Is this a valid question?

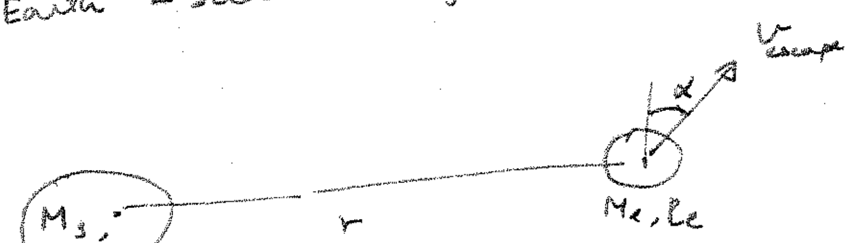
If the motion is large. For small vibration I don't think there is any motion in the x-direction (1st order).

2. (J. Burns). (a) Find the escape velocity of a particle from the Earth. (Mass M_e, radius R_e) Note that $\frac{M_e G}{R_e^2} = g$.

$$-\frac{M_e m G}{R_e} + \frac{1}{2} m v_{esc}^2 = 0.$$

$$\Rightarrow \frac{1}{2} m v_{esc}^2 = \frac{M_e m G}{R_e} \Rightarrow v_{esc} = \sqrt{\frac{2 M_e G}{R_e}} = \sqrt{2 g R_e}.$$

(b) Find the escape velocity of a particle from the Earth - Sun system?



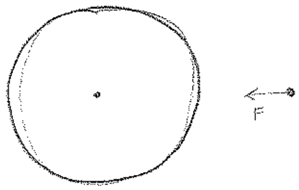
(c) In the first case one neglected the spin of the Earth. Why? Is this justified? What about the second case?

3. (Russia) A ball is set rolling on the floor with general initial conditions. (3 component of $\underline{\omega}$, and velocity in plane parallel to the ground).

(a) What is the general motion (qualitative).

(b) Once the transient slip dies out what is the final ... state of motion (qualitative).

Find escape velocity for object on earth.



$$F = -\frac{G m M_e}{r^2}\,\hat{e}_r$$

$$W = \int_{R_E}^{\infty} \left(-\frac{G m M_e}{r^2}\,\hat{e}_r\right)\cdot\left(\hat{e}_r\right) dr$$

$$W = \left.\frac{G m M_e}{r}\right|_{R_E}^{\infty}$$

$$W = \frac{G m M_e}{R_e}$$

$$\frac{1}{2} m v_e^2 - W = 0$$ no velocity, no potential energy

$$v_e = \sqrt{\frac{2 G M_e}{R_e}}$$

Note contribution due to rotation of the earth, actual v_e is less if rocket is launched at equator, for instance.

MATH

(1)

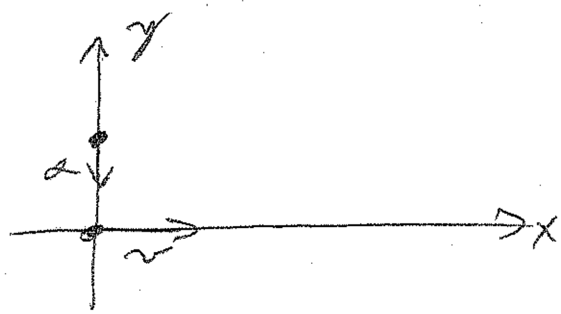
$$E(x) = \int_0^x dt\, e^{-t^2} \longrightarrow$$

FIND $\dfrac{dE(x)}{dx}$

FIND $\displaystyle\int_0^x E(t)\,dt \longrightarrow \int_0^x \left(\int_0^t e^{-z^2}\,dz\right) dt$

---

(2)



FIND THE DIFF. EQN.

FIND $y(x)$

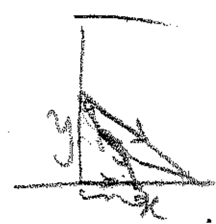
PURSUIT EQN.

---

(3)

$$\int_0^\infty \frac{\cos x}{(x^2-1)}\,dx$$

FIND CPV.



$y/x = \dfrac{dy}{dx}$

$y = \alpha t \qquad x$

$x = vt$

$x = vt$

$\dfrac{dy}{dt} =$

$\dot{x} = v$

$\dot{y} = \alpha$

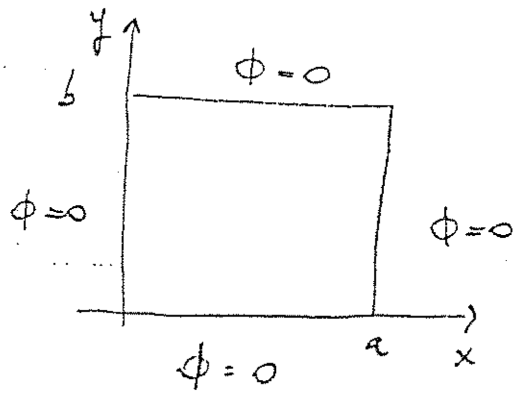
$\dfrac{dy}{dx} =$

(54) What is the solution of:

$$\boxed{y^{IV} + y'' = 0}$$

(55) $$\int_0^\infty \frac{1}{1+x^4}\,dx$$

(56) Solve $\nabla^2 \phi = 0$ on



Show technique of choice!

($\phi(x,y) = 0$ is the only solution)

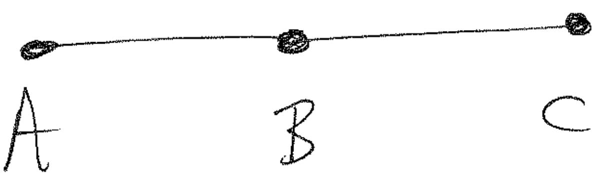
(1)

$$\dot{x} = 8x - 4y$$

$$\dot{y} = x + 4y$$

(not used)

People migrate every day between 3 cities A, B, C as follows:



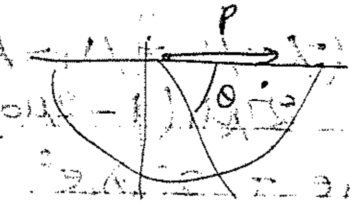
All people at A at time t, move to B, at time t+1.

Same is true for people starting at C. Furthermore,

Half the population at B moves to A, half move to C.

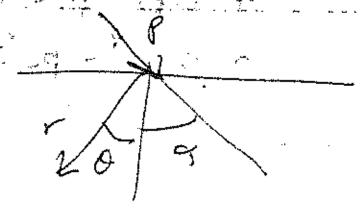
Q: What happens in the long run?

SOLID MECHANICS STUDY AIDS



$$\sigma_r = \frac{-2P}{\pi r} \cos\theta$$

By superposition we can get any angled p



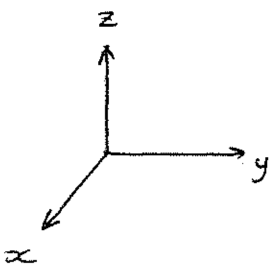
$$\sigma_r = \frac{-2P}{\pi r} \cos(\varphi + \theta)$$

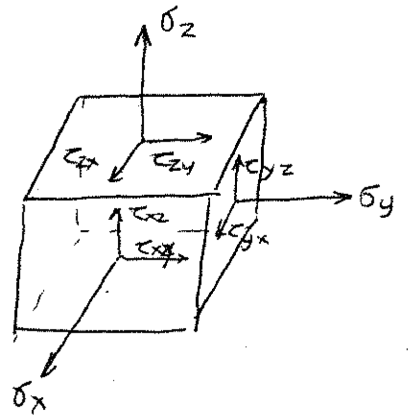
$$\sigma = \frac{My}{I}$$

$$\tau = \frac{Tr}{J}$$

Elastic – material comes back to original shape after removal of ext. force.

Isotropic – (elastic) properties same in all dir.



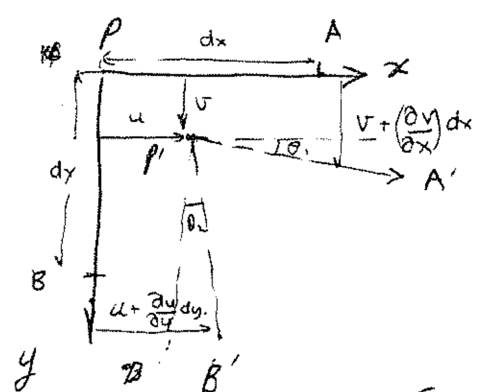


$\sigma_x, \ldots$ etc. are components of stress.

Taking moments about CM, we find

$$\tau_{ij} = \tau_{ji}$$

(For small elements, body forces don't come into the picture, cause body forces $\propto r^3$ where as surface forces $\propto r^2$)



Unit elongation along x-dir is $\frac{\partial u}{\partial x}$,

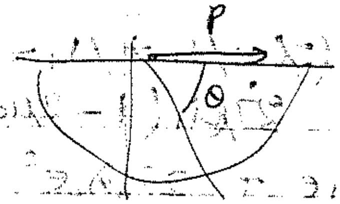
$$\theta_1 \approx \frac{\partial v}{\partial x}$$

$$\theta_2 \approx \frac{\partial u}{\partial y}.$$

$$\therefore \; \epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z}$$

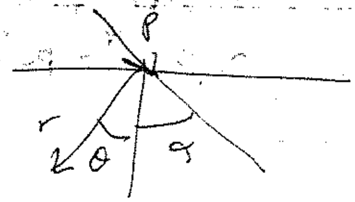
Shearing strain $\quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

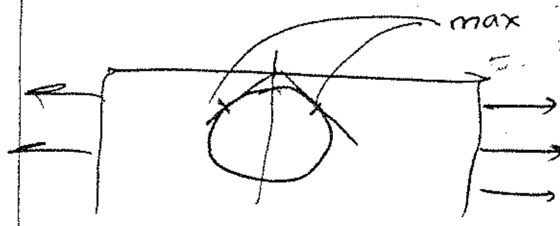


$$\sigma_r = \frac{-2P}{\pi r} \cos\theta$$

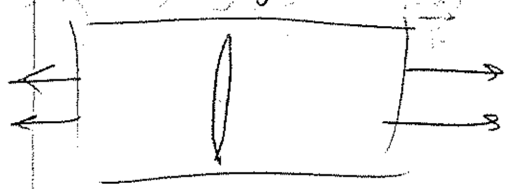
By superposition we can get any angled P



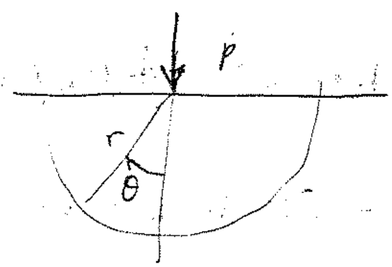
$$\sigma_r = \frac{-2P}{\pi r} \cos(\alpha+\theta)$$



Slender hole $\perp$ direction of tension causes high stress conc.



Conc. force at a pt. of straight bdry:

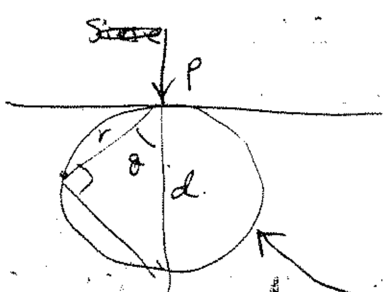


Simple radial distribution.

$$\sigma_r = -\frac{2P}{\pi}\frac{\cos\theta}{r}$$

$$\sigma_\theta = \tau_{r\theta} = 0$$

$$\phi = -\frac{P}{\pi} r\,\theta \sin\theta .$$



$\because d\cos\theta = r$

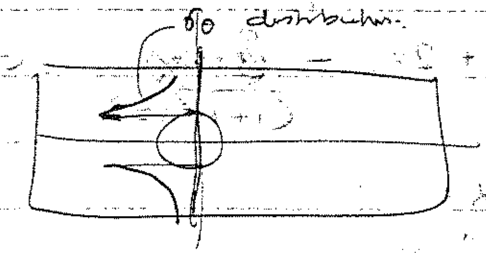
$$\sigma_r = -\frac{2P}{\pi}\left(\frac{r}{d}\right)\frac{1}{r} = -\frac{2P}{\pi d}.$$

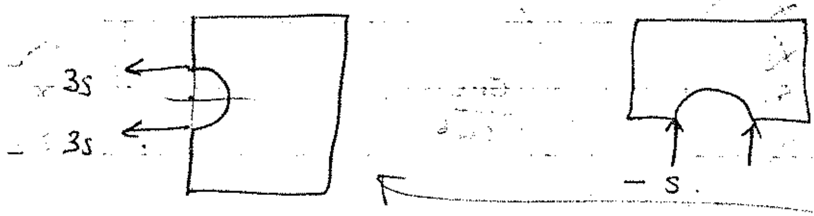
$\therefore$ on this circle of dia d,

$\sigma_r = \frac{-2P}{\pi d}$ except at pt. of applica at hole

σ_θ / greatest when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$(\sigma_\theta)_{max} = 3S$



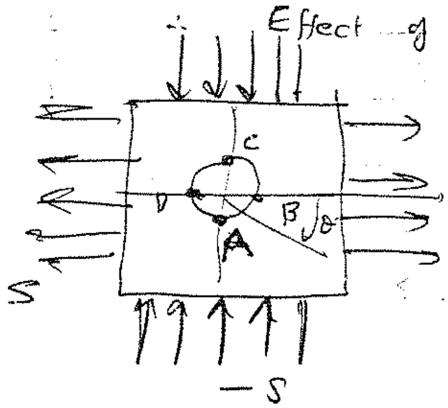


for $\theta = \pi$ or 0, $\sigma_\theta = -S$ ⇒ compression.

Also $\tau_{r\theta} = 0$ throughout the plane $\theta = \pi/2$.

and $\sigma_\theta = \frac{S}{2}\left(2 + \frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)$ (for $r = a$, $\sigma_\theta = 3S$)

Effect of hole is local.



Pure shear mode

Using superposition.

$\sigma_\theta = S - 2S\cos 2\theta - [S - 2S\cos(2\theta - \pi)]$

at A & C, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$,

$\sigma_\theta = +4S$

and at B & D, $\theta = 0, \pi$,

$\sigma_\theta = -4S$

SCHAUM'S OUTLINE
CHAPTER 2 ; p. 1

DENSITY (SCALAR PROPERTY):

$$\rho_{(\Delta V)} = \frac{\Delta M}{\Delta V} \qquad \rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV}$$

STRESS :

TAKING n_i AS THE OUTWARD UNIT NORMAL AT POINT P OF A SMALL ELEMENT OF SURFACE ΔS OF S, LET Δf_i BE THE RESULTANT FORCE EXERTED ACROSS ΔS UPON THE MATERIAL WITHIN V BY THE MATERIAL OUTSIDE OF V. Δf_i WILL DEPEND ON ΔS & n_i

THE AVERAGE FORCE PER UNIT AREA ON ΔS IS GIVEN BY $\Delta f_i / \Delta S$

CAUCHY STRESS PRINCIPLE : $\displaystyle \lim_{\Delta S \to 0} \frac{\Delta f_i}{\Delta S^{(n)}} = \frac{df_i}{dS} = t_i^{(n)}$

STRESS VECTOR $\underline{t}^{(n)}$ AT A GIVEN POINT P DEPENDS EXPLICTLY ON THE PARTICULAR SURFACE ELEMENT CHOOSEN, AS REPRESENTED BY $\underline{n}$

ALL POSSIBLE PAIRS OF $\{\underline{t}^{(n)}, \underline{n}\}$ REPRESENT THE STATE OF STRESS AT A GIVEN POINT. THIS CAN BE REPRESENTED BY THE STRESS VECTOR $\underline{t}^{(n)}$ ON THREE MUTUALLY ORTHOGONAL PLANES AT A POINT.

$$\underline{t}^{(e_1)} = t_1^{(e_1)}\underline{e}_1 + t_2^{(e_1)}\underline{e}_2 + t_3^{(e_1)}\underline{e}_3 = t_j^{(e_1)}\underline{e}_j$$
$$\underline{t}^{(e_2)} = t_1^{(e_2)}\underline{e}_1 + t_2^{(e_2)}\underline{e}_2 + t_3^{(e_2)}\underline{e}_3 = t_j^{(e_2)}\underline{e}_j$$
$$\underline{t}^{(e_3)} = t_1^{(e_3)}\underline{e}_1 + t_2^{(e_3)}\underline{e}_2 + t_3^{(e_3)}\underline{e}_3 = t_j^{(e_3)}\underline{e}_j$$

$$\underline{\text{OR}} \qquad t_j^{(e_i)} = \sigma_{ij}$$

$[\sigma_{ij}]$: STRESS TENSOR

THROUGH FORCE EQUILIBRIUM ON A PLANE OF ARBITRARY ORIENTATION

$$t_i^{(n)} = \sigma_{ji} n_j \qquad \underline{\text{OR}} \qquad \underline{t}^{(n)} = \underline{n} \cdot \underline{\sigma}$$

FORCE & MOMENT EQUILIBRIUM:

FOR STATIC SYSTEMS, FORCE EQUILIBRIUM REQUIRES

$$\int_S \underline{t}^{(n)} dS + \int_V \rho \underline{b}\, dV = 0$$

REPLACING $\underline{t}^{(n)} = \underline{n} \cdot \underline{\sigma}$
& USING THE DIVERGENCE THEOREM:

PROBLEMS

12 (CONT.)

b. ASSUME FRICTION IN THE WALLS: $\hat{t} = \mu \bar{t}$

$u_r = u_\theta = 0$ $\qquad$ $\sigma_{rz} = \mu \sigma_{rr}$

$u_z = u_z(r, z)$

$\underline{u} = 0 \quad (\underline{e}_r \cdot \underline{u} = 0)$

$\leftarrow \hat{t} = \mu \bar{t}$

$(t \cdot e_z = \mu t \cdot e_r)$

STRAIN - DISPLACEMENT

$\gamma_{rr} = 0 \qquad \gamma_{r\theta} = 0$

$\gamma_{\theta\theta} = 0 \qquad \gamma_{rz} = 1/2\, u_{z,r}$

$\gamma_{zz} = u_{z,z} \qquad \gamma_{\theta z} = 0$

CONSTITUTIVE LAW

$\sigma_{rr} = \lambda u_{z,z} \qquad \sigma_{r\theta} = 0$

$\sigma_{\theta\theta} = \lambda u_{z,z} \qquad \sigma_{rz} = \lambda/2\; u_{z,r}$

$\sigma_{zz} = (2\mu + \lambda) u_{z,z} \qquad \sigma_{\theta z} = 0$

EQUILIBRIUM EQUATIONS

$$\sigma_{rr,r} + \sigma_{rz,z} = 0$$

$$\sigma_{rz,r} + \sigma_{zz,z} + r^{-1} \sigma_{rz} = 0$$

$$\lambda u_{z,zr} + \lambda/2\, u_{z,rz} = 0 \qquad \lambda/2\, u_{z,rr} + (2\mu + \lambda) u_{z,zz} + \frac{\lambda}{2r} u_{z,r} = 0$$

$$u_{z,zr} = 0$$

$$u_z(r,z) = \varphi(r) + \psi(z)$$

$$\lambda/2\left(\varphi_{,rr} + \frac{\varphi_{,r}}{r}\right) + (2\mu + \lambda)\psi_{,zz} = 0$$

PROBLEMS

12 (CONT.)

b. ASSUME FRICTION IN THE WALLS $\underline{f} = \mu \underline{N}$

$\sigma_{r\theta} = \sigma_{z\theta} = 0 \qquad \underline{\sigma} = \underline{\sigma}(r,z) \qquad \underline{u} = u_z(r,z)\, \underline{e}_z$

$\gamma_{rr} = \gamma_{\theta\theta} = 0$

$$\sigma_{zz} = (\lambda + 2\mu)\gamma_{zz}$$
$$\sigma_{rr} = \lambda \gamma_{zz}$$
$$\sigma_{\theta\theta} = \lambda \gamma_{zz}$$
$$\sigma_{rz} = \lambda \gamma_{rz}$$

EQUILIBRIUM EQUATIONS $\quad \underline{\nabla} \cdot \underline{\sigma} = 0$

$$\sigma_{rr,r} + \sigma_{rz,z} - r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$
$$\sigma_{rz,r} + \sigma_{zz,z} + r^{-1}\sigma_{rz} = 0$$

$$\sigma_{rr,r} + \sigma_{rz,z} = 0 \qquad \lambda\gamma_{zz,r} + \lambda\gamma_{rz,z} = 0$$
$$\sigma_{rz,r} + \sigma_{zz,z} + r^{-1}\sigma_{rz} = 0 \qquad \lambda\gamma_{rz,r} + (\lambda + 2\mu)\gamma_{zz,z} + r^{-1}\lambda\gamma_{rz} = 0$$

$\gamma_{zz} = u_{z,z} \qquad \gamma_{rz} = 1/2(u_{z,r})$

$$\lambda u_{z,zr} + \lambda/2\, u_{z,rz} = 0$$
$$u_{z,rz} = 0$$

$\underline{t} \cdot \underline{e}_z = \mu\, \underline{t} \cdot \underline{e}_r$
$\sigma_{rz} = \mu \sigma_{rr}$

$$\sigma_{rr,r} + \mu\sigma_{rr,z} = 0 \qquad \lambda\gamma_{zz,r} + \mu\lambda\gamma_{zz,z} = 0$$
$$\mu\sigma_{rr,r} + \sigma_{zz,z} + r^{-1}\mu\sigma_{rr} = 0 \qquad \mu\lambda\gamma_{zz,r} + (\lambda + 2\mu)\gamma_{zz,z} + r^{-1}\mu\lambda\gamma_{zz} = 0$$

$$\mu\lambda\gamma_{zz,r} - \frac{\lambda + 2\mu}{\mu}\gamma_{zz,r} + \mu\lambda\frac{\gamma_{zz}}{r} = 0$$
$$\gamma_{zz,r}\left[\frac{\lambda(\mu^2 - 1) - 2\mu}{\mu}\right] + \mu\lambda\frac{\gamma_{zz}}{r} = 0$$

LET $\gamma_{zz} = C(z) r^{\alpha}$

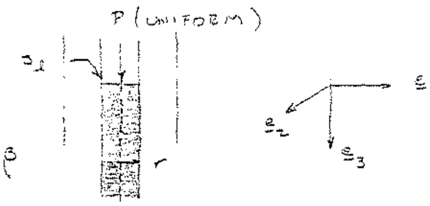
$$C\alpha\left[\lambda(\mu^2 - 1) - 2\mu\right] r^{\alpha} + C\mu^2\lambda r^{\alpha} = 0$$

$$\alpha = \frac{\mu^2\lambda}{\lambda(\mu^2 - 1) - 2\mu}$$

PROBLEMS

12. FIND THE STRESS :

a. NO FRICTION ON THE WALLS



$\sigma_{ij,j} = 0$

$$\int_{s_\ell} \sigma_{33}\, dA = -P \qquad \int_{s_\ell} x_2 \sigma_{33}\, dA = \int_{s_\ell} x_1 \sigma_{33}\, dA = 0$$

$$\int_{s_\ell} (x_1 \sigma_{23} - x_2 \sigma_{13})\, dA = 0$$

LET $\sigma_{33} = k \qquad \sigma_{ij} = 0 \quad (i \neq j)$

$$\sigma_{33} = k = \frac{-P}{A}$$

$$\gamma_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right] \qquad \sigma_{ij} = 2\mu\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij}$$

$\gamma_{11} = \gamma_{22} = 0$

$\sigma_{11} = \lambda\gamma_{33} = \sigma_{22} \qquad \sigma_{11,1} = 0$
$\qquad\qquad\qquad\qquad\sigma_{22,2} = 0$

$$\gamma_{33} = \frac{1}{2\mu}\left[\sigma_{33} - \frac{\lambda}{3\lambda - 2\mu}\left(\sigma_{33} + 2\lambda\gamma_{33}\right)\right]$$

$$2\mu\gamma_{33} = \frac{2\lambda + 2\mu}{3\lambda + 2\mu}\sigma_{33} - \frac{2\lambda^2}{3\lambda + 2\mu}\gamma_{33}$$

$$\gamma_{33}\left[\frac{2\lambda^2 + 6\lambda\mu + 4\mu^2}{3\lambda + 2\mu}\right] = \frac{2\lambda + 2\mu}{3\lambda + 2\mu}\sigma_{33}$$

$$(\lambda^2 + 3\lambda\mu + 2\mu^2)\gamma_{33} = (\lambda + \mu)\sigma_{33}$$
$$(\lambda + 2\mu)(\lambda + \mu)\gamma_{33} = (\lambda + \mu)\sigma_{33}$$

$$\gamma_{33} = \frac{-P}{A(\lambda + 2\mu)}$$

OR

$$\sigma_{33} = 2\mu\gamma_{33} + \lambda\gamma_{33}$$

$$\gamma_{33} = \frac{\sigma_{33}}{2\mu + \lambda} = \frac{-P}{A(\lambda + 2\mu)}$$

PROBLEMS

1. WHAT IS WRONG HERE? WHY?

A MOMENT MUST BE PLACED @ $x = l$
TO OBTAIN THIS CONFIGURATION

663 NOTES

PURE SHEAR

$$\gamma_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ij}\right].$$

$$\gamma_{33} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_{33}$$

$$\gamma_{22} = \frac{-\lambda}{3\lambda+2\mu}\sigma_{33} = \gamma_{11}$$

$$\gamma_{13} = \frac{\sigma_{13}}{2\mu}$$

$$\sigma_{31,13} + \sigma_{33,33} = 0$$
$$= (\sigma_{13,3})_{,1} + \sigma_{33,33} = 0$$
$$\sigma_{33,33} = 0$$

σ_{33} IS LINEAR IN x_3

$$\sigma_{33} = x_3\hat{\sigma}_{33}(x_1, x_2) + c(x_1, x_2)$$

$$\sigma_{31,1} + \sigma_{33,3} = 0 \qquad \text{LET} \quad \sigma_{31} = \psi(x_1, x_2)$$
$$\psi_{,1} + \hat{\sigma}_{33} = 0$$

$$\hat{\sigma}_{33} = -\psi_{,1}$$

$$\sigma_{13} = \psi(x_1, x_2)$$
$$\sigma_{33} = -x_3\psi_{,1} + c(x_1, x_2)$$

ASSUME:

$$\sigma_{33} = -(x_3 - l)\psi_{,1}$$

$$\int_{S_l} \psi \, dA = V \qquad \int_{S_l} x_2 \psi \, dA = 0 \qquad n_\alpha \psi = 0 \text{ ON } \mathcal{C}$$

Plane Stress

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$$
$$\varepsilon_{xz} = \varepsilon_{yz} = 0$$

Constitutive

$$\sigma_{xx} = \frac{E}{(1-\nu^2)} \left[\varepsilon_{xx} + \nu \varepsilon_{yy} \right]$$

$$\sigma_{\alpha\beta} = \frac{2G\lambda}{\lambda+2G} \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2G \varepsilon_{\alpha\beta}$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)} \left[\varepsilon_{yy} + \nu \varepsilon_{xx} \right]$$
$$\sigma_{xy} = 2G\, \varepsilon_{xy}$$

Equilibrium

$$\sigma_{xx,x} + \sigma_{xy,y} = -F_x$$
$$\sigma_{xy,x} + \sigma_{yy,y} = -F_y$$

$$\boxed{\sigma_{\alpha\beta,\beta} = -F_\alpha}$$

Kinematics

$$\varepsilon_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right)$$

Stress Function

$$\phi'(z) \qquad \psi'(z)$$

ϕ' ψ' analytic

$$\sigma_{11} + \sigma_{22} = 4 \operatorname{Re} \left[\phi'(z) \right]$$
$$(\sigma_{22} - \sigma_{11}) + 2i\sigma_{12} = 2 \left[\bar{z} \phi''(z) + \psi'(z) \right]$$
$$2G (u_1 + iu_2) - \left(K\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \right]$$

$$K = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

Displacement Formulation (16 Feb)

$$\nabla^2 u_\alpha + \left(\frac{\lambda+G}{G} \right) u_{\beta,\beta\alpha} = -F_\alpha$$

Plane Strain

$$u_x(x,y) \qquad u_y(x,y) \qquad u_z = 0$$
$$\Rightarrow \quad \varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$$
$$\sigma_{xz} = \sigma_{yz} = 0$$

Displacement Formulation

$$\nabla^2 u_\alpha + \left(\frac{\lambda+G}{G} \right) u_{\beta,\beta\alpha} = -F_\alpha.$$

Compatibility
(exact for plane strain & approximate " stress)

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - \frac{2\, \partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0$$

16 Feb.

$$\epsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\gamma_{r\theta} = \frac{\partial u}{r\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{1}{G}\tau_{r\theta}$$

plane stress

For symmetric stress distributions, we get

$$\frac{\partial v}{\partial \theta} = \frac{4Br}{E} - f(\theta)$$

$$\Rightarrow v = \frac{4Br\theta}{E} - \int f(\theta)\,d\theta + f_1(r)$$

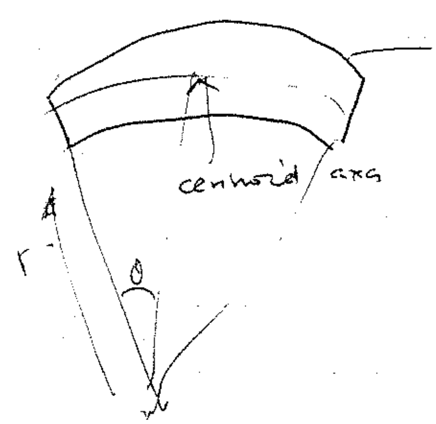
Since $\tau_{r\theta} = 0$ for symm. stress distr.

$\Rightarrow \gamma_{r\theta} = 0$.

$\Rightarrow$ one can solve for $f(\theta)$ and $f_1(r)$,

we get

$$v = \frac{4Br\theta}{E} + Fr + H\cos\theta - K\sin\theta$$



conditions of constraint

$u = 0$, $v = 0$, $\frac{\partial v}{\partial r} = 0$ at centroid axis.

663 NOTES

30 SEPT 91 (CONT.)

LOCALIZATION OF TRANSPORT THEOREM

$$\frac{d}{dt} g(\underline{x},t) = \frac{\partial}{\partial t} g(\underline{y},t) + \nabla_y g(\underline{y},t)\, \underline{v}(\underline{y},t)$$

9 OCT 91

TRANSFORMATION OF STRESS COORDINATES

PRINCIPLE NORMAL STRESSES ARE EIGENVALUES OF $\underline{\sigma}$

PRINCIPLE DIRECTIONS ARE THE CORRESPONDING EIGENVECTORS

21 OCT 91

ELASTIC MATERIAL

- $\exists$ AN UNSTRESSED REFERENCE CONFIGURATION
- $\exists$ A SCALAR FUNCTION W SUCH THAT

$$\sigma_{ij} = \frac{\partial W}{\partial F_{ij}}$$

23 OCT 91

LINEAR THEORY OF ELASTICITY

- $\exists$ A REFERENCE CONFIGURATION SUCH THAT $\underline{\sigma} = \underline{0}$
- DISPLACEMENTS ARE SMALL $\|\nabla_x \underline{u}\| \ll 1$

$$\sigma_{ij}(x,t) = c_{ijk\ell}(x)\, u_{k,\ell}(x,t)$$

$$u_{k,\ell} = \Omega_{k\ell} + \gamma_{k\ell}$$

$$\Omega_{k\ell} = \tfrac{1}{2}(u_{k,\ell} - u_{\ell,k}) \qquad \text{ANTI-SYMMETRIC}$$

$$\gamma_{k\ell} = \tfrac{1}{2}(u_{k,\ell} + u_{\ell,k}) \qquad \text{SYMMETRIC}$$

$$c_{ijk\ell} = c_{k\ell ij} \qquad \text{MAJOR SYMMETRY}$$

$$c_{ijk\ell} = c_{ij\ell k} = c_{jik\ell} \qquad \text{MINOR SYMMETRY}$$

$$c_{ijk\ell}\Omega_{k\ell} = 0 \quad \rightarrow \quad \sigma_{ij} = c_{ijk\ell}\gamma_{k,\ell}$$

LINEAR STRESS-STRAIN RELATIONSHIP

LAGRANGIAN STRAIN TENSOR:

$$\underline{E} = \tfrac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T + \nabla\underline{u}^T\nabla\underline{u}) \; ; \quad \nabla\underline{u}^T\nabla\underline{u} \rightarrow 0$$

$$= \tfrac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T) = \underline{\gamma}$$

LINEARIZATION OF MASS BALANCE

$$\rho(\underline{y}(x,t),x) = J(x,t)\,\rho_0(x)$$

$$= (1 + u_{k,k})\,\rho_0(x)$$

663 NOTES

23 SEPT 91

KINEMATICS

POSITION $\underline{y}$ OF PARTICLE $\underline{x}$ AT TIME t

$$\underline{y} = \hat{\underline{y}}(\underline{x}, t) = \underline{x} + \underline{u}(\underline{x}, t)$$

A PERMISSABLE MOTION MEETS THE FOLLOWING REQUIREMENTS

- ONE-TO-ONE MAPPING
- JACOBIAN > 0
- MAPPING MUST BE TWICE CONTINUOUSLY DIFFERENTIABLE

DEFORMATION GRADIANT TENSOR

$$\underline{F}(\underline{x}, t) = \nabla \underline{y}(\underline{x}, t)$$

$$= \nabla \underline{u}(\underline{x}, t) + \underline{1}$$

JACOBIAN : $J(\underline{x}, t) = \det(\underline{F}(\underline{x}, t))$

LOCALLY, J MEASURES THE VOLUME CHANGE OF A DEFORMATION

$J(\underline{x}, t) = 1$ CORRESPONDS TO A (LOCALLY) VOLUME PRESERVING DEFORMATION

25 SEPT 91

RIGID MOTION IS ONE IN WHICH $|\hat{y}(x) - \hat{y}(z)| = |\underline{x} - \underline{z}|$

$\underline{F}$: CONSTANT, PROPER ORTHOGONAL TENSOR

$\underline{c}$: CONSTANT VECTOR

HOMOGENEOUS (UNIFORM) DEFORMATION TAKES THE FORM $\hat{y}(x) = \underline{F}\,\underline{x} + \underline{c}$

$\underline{F}$: CONSTANT TENSOR ; $\det(\underline{F}) > 0$

$\underline{c}$: CONSTANT VECTOR

LAGRANGIAN STRAIN TENSOR

$$\underline{E} = \frac{1}{2}(\underline{F}^T\underline{F} - \underline{1}) = \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T - \nabla\underline{u}^T\nabla\underline{u})$$

30 SEPT 91

EVERY DEFORMATION IS LOCALLY HOMOGENEOUS

KINETICS OF A DEFORMING CONTINUUM

MASS BALANCE

$$\int_A \left(J(\underline{x}, t)\,\rho(\hat{\underline{y}}(\underline{x}, t), t) - \rho_0(\underline{x}) \right) dV_x = 0$$

TRANSPORT THEOREM :

$$\frac{d}{dt}\int_A \rho(y,t)\,g(y,t)\,dt = \int_A \rho(y,t)\,\frac{\partial}{\partial\tau} g(\hat{x}(y,t),\tau)\Big|_{\tau=t} dV$$

$$= \int_A \rho(y,t)\left\{ \frac{\partial}{\partial t} g(y,t) + \nabla_y g(y,t)\,\underline{q}(y,t) \right\} dV$$

663 NOTES

WAVE SPEEDS

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

$$\sigma_{ij} = 2\mu \gamma_{ij} + \lambda \gamma_{kk} \delta_{ij}$$

$$= \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}$$

$$\sigma_{ij,j} = \mu(u_{i,jj} + u_{j,ij}) + \lambda u_{k,ki}$$

$$\mu u_{i,jj} + (\mu + \lambda) u_{j,ij} = \rho \ddot{u}_i$$

$$\operatorname{div}\left(\mu u_{i,jj} + (\mu + \lambda) u_{j,ij} = \rho \ddot{u}_i\right)$$

$$\mu u_{i,ijj} + (\mu + \lambda) u_{j,jii} = \rho \ddot{u}_{i,i}$$

$$(2\mu + \lambda) \eta_{,jj} = \rho \ddot{\eta} \qquad \eta = u_{i,i}$$

NOW LET $\eta = f((x_1 - vt), (x_2 - vt), (x_3 - vt))$

$$\ddot{\eta} = v^2 \eta_{,jj}$$

$$(2\mu + \lambda) \eta_{,jj} = \rho v^2 \eta_{,jj}$$

$$v^2 = \frac{2\mu + \lambda}{\rho}$$

LONGITUDINAL WAVE SPEED
SO $2\mu + \lambda > 0$

$$\operatorname{curl}\left(\mu u_{i,jj} + (\mu + \lambda) u_{j,ij} = \rho \ddot{u}_i\right)$$

$$\varepsilon_{ijk}\left(\mu u_{k,jrr} + (\mu + \lambda) u_{r,kjr}\right) = \varepsilon_{ijk} \rho \ddot{u}_{k,j}$$

$$\varepsilon_{ijk} \mu u_{k,jrr} = \varepsilon_{ijk} \rho \ddot{u}_{k,j}$$

$$\mu \eta_{rr} = \rho \ddot{\eta} \qquad \eta = \varepsilon_{ijk} u_{k,j}$$

AGAIN LET $\eta = f((x_1 - vt), (x_2 - vt), (x_3 - vt))$

$$\ddot{\eta} = v^2 \eta_{rr}$$

$$\mu \eta_{rr} = \rho v^2 \eta_{rr}$$

$$v^2 = \frac{\mu}{\rho}$$

SHEAR WAVE SPEED
SO $\mu > 0$

663 NOTES

23 OCT 91 (CONT.)

COMPATIBILITY RELATIONS FOR SMALL STRAINS INSURE THAT ANY STRAIN FIELD THAT SATISFIES THESE RELATIONS CORRESPONDS TO A SINGLE-VALUED DISPLACEMENT FIELD

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

$$\gamma_{ij,kl} = \tfrac{1}{2}(u_{i,jkl} + u_{j,ikl}) \qquad \gamma_{kl,ij} = \tfrac{1}{2}(u_{k,lij} + u_{l,kij})$$

$$\gamma_{ik,jl} = \tfrac{1}{2}(u_{i,kjl} + u_{k,ijl}) \qquad \gamma_{jl,ik} = \tfrac{1}{2}(u_{j,lik} + u_{l,jik})$$

$$\gamma_{ij,kl} - \gamma_{ik,jl} - \gamma_{jl,ik} + \gamma_{kl,ij} = 0$$

81 EQUATIONS REDUCE TO 6:

$$\gamma_{11,23} = \gamma_{12,13} + \gamma_{13,12} - \gamma_{23,11}$$

$$\gamma_{22,31} = \gamma_{23,21} + \gamma_{21,23} - \gamma_{31,22}$$

$$\gamma_{33,12} = \gamma_{31,32} + \gamma_{32,31} - \gamma_{12,33}$$

$$2\gamma_{12,12} = \gamma_{11,22} + \gamma_{22,11}$$

$$2\gamma_{23,23} = \gamma_{22,33} + \gamma_{33,22}$$

$$2\gamma_{31,31} = \gamma_{33,11} + \gamma_{11,33}$$

FOR AN ISOTROPIC MATERIAL, HOOKE'S LAW REDUCES TO

$$\sigma_{ij} = 2\mu\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij}$$ LINEAR, ISOTROPIC STRESS-STRAIN LAW

$$\gamma_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right]$$ INVERSE STRESS-STRAIN LAW

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

DISPLACEMENT EQUATION FOR STATIC EQUILIBRIUM

$$c_{ijks}u_{k,sj} = 0$$

663 NOTES

$$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\, u_{1,11} = \rho \ddot{u}_1$$

$$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\, \varphi_{,11} = \rho\ddot{\varphi} \qquad \varphi = u_1$$

LET $\varphi = f(x_1 + vt) + g(x_1 - vt)$

$$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\,(f'' + g'') = \rho v^2 (f'' + g'')$$

$$v^2 = \frac{\mu(3\lambda+2\mu)}{\rho(\lambda+\mu)}$$

$$\dot{u}_1 = \sqrt{v}\, f'$$
$$u_{1,1} = f' \qquad f' = \frac{v_*}{\sqrt{v}}$$

$$\sigma_{11} = \left[\frac{\rho\mu(3\lambda+2\mu)}{(\lambda+\mu)}\right]^{1/2} v_*$$

TO MAINTAIN $\underline{u} = 0$ ON IMPACT FACE

$$v_0 \underline{e}_1 + v_* \underline{e}_1 = \underline{0}$$

$$v_* = -v_0$$

$$\sigma_{11} = -\left[\frac{\rho\mu(3\lambda+2\mu)}{(\lambda+\mu)}\right]^{1/2} v_0$$ STRESS IN THE RESULTING WAVE

BECAUSE THE EQUATIONS OF LINEAR ELASTICITY ARE LINEAR, THE PRINCIPLE OF SUPERPOSITION HOLDS

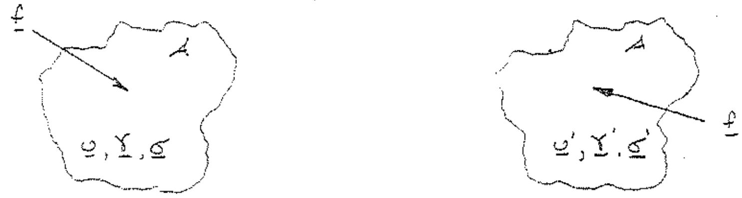
SOLUTIONS TO THE GENERAL ELASTOSTATIC PROBLEM ARE UNIQUE

ST. VENANT'S PRINCIPLE

FOR LOCATIONS SUFFICIENTLY REMOTE FROM THE AREA OF APPLICATION OF THE LOADINGS, THE DIFFERENCE BETWEEN TWO SEPARATE BUT STATICALLY EQUIVALENT SYSTEMS OF SURFACE TRACTIONS BEING APPLIED TO SOME PORTION OF THE BOUNDARY ARE NEGLIGIBLE.

663 NOTES

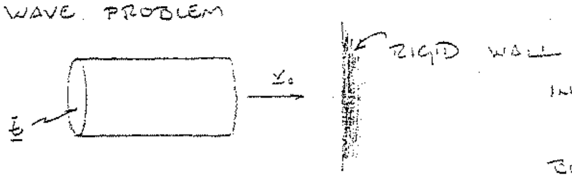
RECIPROCAL THEOREM

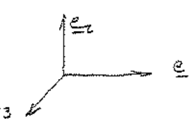


$$\int_A (\underline{f}\cdot\underline{u}')\,dV + \int_{\partial A}(\underline{t}\cdot\underline{u}')\,dA = \int_A(\underline{f}'\cdot\underline{u})\,dV + \int_{\partial A}(\underline{t}\cdot\underline{u}')\,dA =$$

$$= \int_A \sigma_{ij}\gamma'_{ij}\,dV = \int_A \sigma'_{ij}\underline{\gamma}_{ij}\,dV$$

WAVE PROBLEM





INITIAL CONDITIONS

$\underline{u} = \underline{0}\ ,\ \dot{\underline{u}} = v_0\underline{e}_1$

BOUNDARY CONDITIONS

$\hat{\underline{t}} = \underline{0} \quad \hat{\underline{u}} = 0$

ASSUME $\sigma_{ij} = 0 \quad (i \neq j\ ,\ i = j \neq 1)$

EQUILIBRIUM EQUATIONS:

$\sigma_{ij,j} = \rho\ddot{u}_i$

STRESS - STRAIN LAW:

$$2\mu\gamma_{ij} = \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}$$

$$\sigma_{11,1} = \rho\ddot{u}_1 \qquad \ddot{u}_2 = \ddot{u}_3 = 0$$

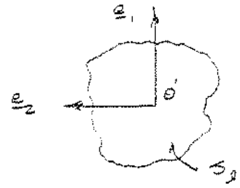
$$2\mu\gamma_{11} = \sigma_{11}\left[\frac{2\lambda+2\mu}{3\lambda+2\mu}\right] \qquad \sigma_{11} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}u_{1,1}$$

$$2\mu\gamma_{22} = -\sigma_{11}\left[\frac{\lambda}{3\lambda+2\mu}\right]$$

$$2\mu\gamma_{33} = -\sigma_{11}\left[\frac{\lambda}{3\lambda+2\mu}\right]$$

663 NOTES

PURE EXTENSION



AREA $\int_{S_l} dA = A$

ASSUME UNIFORM TRACTION

$$\int_{S_l} \sigma_{i3}\, dA = L_i \qquad \sigma_{i3} A = L_i$$

$$\sigma_{33} = \frac{F}{A} \qquad \sigma_{23} = \sigma_{13} = 0$$

$\underline{\sigma} \cdot \underline{n} = 0$ ON $\mathcal{L}$

$$\sigma_{\alpha\beta} = 0 \qquad (\alpha, \beta) \cdot \{1, 2\}$$

$\underline{M}_{O'} = 0$ ASSUME O' IS AT THE CENTROID

$$\int_{S_l} x_2 \sigma_{33}\, dA = \sigma_{33} \int x_2\, dA = 0 \qquad \text{DEFINITION OF } O'$$

$$\int_{S_l} -x_1 \sigma_{33}\, dA = -\sigma_{33} \int x_1\, dA = 0 \qquad \text{DEFINITION OF } O'$$

$$\int_{S_l} (x_1 \sigma_{23} - x_2 \sigma_{13})\, dA = 0 \qquad \sigma_{13} = \sigma_{23} = 0$$

EQUILIBRIUM EQUATIONS

$\sigma_{ij,j} = 0$ SATISFIED

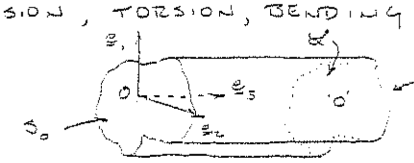
$$\gamma_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} \sigma_{kk} \delta_{ij} \right]$$

$$\gamma_{11} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_{33} = \gamma_{22}$$

$$\gamma_{33} = \frac{(\lambda + \mu)}{\mu(3\lambda + 2\mu)} \sigma_{33} \qquad \gamma_{ij} = 0 \quad (i \neq j)$$

663 NOTES

EXTENSION, TORSION, BENDING, + SHEARING OF CYLINDERS



LATERAL SURFACE α

BODY FORCES $\underline{f} = \underline{0}$

UNRELAXED PROBLEM STATEMENT

FIND $\{ \underline{u}, \underline{\gamma}, \underline{\sigma} \}$ SUCH THAT

$$\underline{t} = \underline{\sigma} \cdot \underline{n} = \underline{0} \quad \text{ON } \alpha \qquad \int_{S_o} \underline{t}^{*(0)} dA + \int_{S_\ell} \underline{t}^{*(\ell)} dA = \underline{0}$$

$$\underline{t} = \underline{t}^{*(0)} \quad \text{ON } S_o$$

$$\underline{t} = \underline{t}^{*(\ell)} \quad \text{ON } S_\ell \qquad \int_{S_o} (\underline{x} \times \underline{t}^{*(0)}) dA + \int_{S_\ell} (\underline{x} \times \underline{t}^{*(\ell)}) dA = \underline{0}$$

RELAXED PROBLEM STATEMENT (APPLY ST. VENANT'S PRINCIPLE)

FIND $\{ \underline{u}, \underline{\gamma}, \underline{\sigma} \}$ SUCH THAT

$$\underline{t} = \underline{\sigma} \cdot \underline{n} = \underline{0} \quad \text{ON } \alpha$$

$$\int_{S_\ell} \underline{t}\, dA = \underline{L} \qquad \int_{S_\ell} ((\underline{x} - \ell \underline{e}_3) \times \underline{t})\, dA = \underline{M}_{o'}$$

+ OVERALL EQUILIBRIUM EQUATIONS

$$\sigma_{ij} = 2\mu \gamma_{ij} + \lambda \gamma_{kk} \delta_{ij} \qquad \gamma_{ij} = \frac{1}{2\mu} \left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} \sigma_{kk} \delta_{ij} \right]$$

$$\sigma_{ij,j} = 0$$

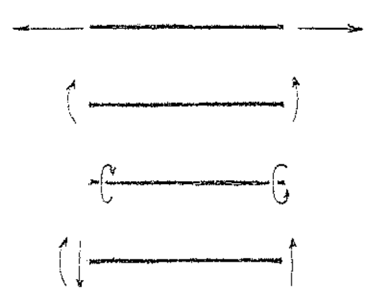
$$\int_{S_\ell} \underline{\sigma} \cdot \underline{e}_3\, dA = \int_{S_\ell} \sigma_{i3}\, dA = L_i$$

$$\int_{S_\ell} ((\underline{x} - \ell \underline{e}_3) \times \underline{\sigma} \cdot \underline{e}_3)\, dA = \int_{S_\ell} \varepsilon_{ijk} \xi_j \sigma_{k3}\, dA \qquad \underline{\xi} = \underline{x} - \ell \underline{e}_3 = x_1 \underline{e}_1 + x_2 \underline{e}_2$$

$$M_1 = \int_{S_\ell} x_2 \sigma_{33}\, dA$$

$$M_2 = \int_{S_\ell} - x_1 \sigma_{33}\, dA$$

$$M_3 = \int_{S_\ell} (x_1 \sigma_{23} - x_2 \sigma_{13})\, dA$$



PURE EXTENSION

$\underline{L} = F \underline{e}_3 \qquad \underline{M} = \underline{0}$

PURE BENDING

$\underline{L} = \underline{0} \qquad \underline{M} = B \underline{e}_2$

PURE TORSION

$\underline{L} = \underline{0} \qquad \underline{M} = T \underline{e}_3$

PURE SHEAR

$\underline{L} = V \underline{e}_1 \qquad \underline{M} = \underline{0}$

443 NOTES

PURE EXTENSION (CONT.)

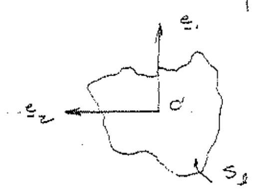
$$C_1 = C_2 = C_3 = 0$$

$$a_1 = -a_2 = -\omega_3$$
$$b_2 = -b_3 = -\omega_1$$
$$b_1 = -a_3 = \omega_2$$

$$\underline{u} = \left[-C_1 x_1 - \omega_3 x_2 + \omega_2 x_3 + d_1\right] \underline{e}_1 + \left[-C_1 x_2 + \omega_3 x_1 - \omega_1 x_3 + d_2\right] \underline{e}_2 + \left[C_2 x_3 - \omega_2 x_1 + \omega_1 x_2 + d_3\right] \underline{e}_3$$

$$C_1 = \frac{F\lambda}{2A\mu(3\lambda + 2\mu)} \qquad C_2 = \frac{F(\lambda + \mu)}{A\mu(3\lambda + 2\mu)}$$

PURE BENDING



$(\underline{e}_1, \underline{e}_2)$ ARE PRINCIPLE AXIS

O' IS LOCATED @ CENTROID

$$\int_{S_l} x_1 x_2 \, dA = 0$$

$$\int_{S_l} x_1 \, dA = \int_{S_l} x_2 \, dA = 0$$

$\underline{L} = \underline{0}$ $\qquad \underline{t} = \underline{0}$ ON ∀

$\underline{M} = B \underline{e}_2$

$$\int_{S_l} \sigma_{i3} \, dA = 0 \qquad \int_{S_l} x_2 \sigma_{33} \, dA = 0 \qquad \int_{S_l} -x_1 \sigma_{33} \, dA = B \qquad \int_{S_l} (x_1 \sigma_{23} - x_2 \sigma_{13}) \, dA = 0$$

LET $\sigma_{33} = k x_1 \qquad \sigma_{ij} = 0 \qquad ij \neq 33$

$$\int_{S_l} \sigma_{33} \, dA = \int_{S_l} k x_1 \, dA = 0$$ DEFINITION OF CENTROID

$$\int_{S_l} x_2 \sigma_{33} \, dA = \int_{S_l} k x_1 x_2 \, dA = 0$$ DEFINITION OF PRINCIPLE AXIS

$$\int_{S_l} -x_1 \sigma_{33} \, dA = \int_{S_l} -k x_1^2 \, dA = -k I_{22} \qquad I = \int_{S_l} x_1^2 \, dA$$ MOMENT OF INERTIA

$$B = -k I_{22} \qquad \sigma_{33} = \frac{-B x_1}{I}$$

663 NOTES

PURE EXTENSION (CONT.)

$$2\gamma_{ij} = u_{i,j} + u_{j,i}$$

$$\gamma_{11} = u_{1,1}$$

$$= \frac{-F\lambda}{2A\mu(3\lambda+2\mu)} \qquad u_1 = \frac{-F\lambda x_1}{2A\mu(3\lambda+2\mu)} + v_1(x_2,x_3)$$
$$= -C_1 x_1 + v_1(x_2,x_3)$$

$$\gamma_{22} = u_{2,2}$$

$$= \frac{-F\lambda}{2A\mu(3\lambda+2\mu)} \qquad u_2 = \frac{-F\lambda x_2}{2A\mu(3\lambda+2\mu)} + v_2(x_1,x_2)$$
$$= -C_1 x_2 + v_1(x_2,x_3)$$

$$\gamma_{33} = u_{3,3}$$

$$= \frac{(\lambda+\mu)F}{A\mu(3\lambda+2\mu)} \qquad u_3 = \frac{(\lambda+\mu)F x_3}{A\mu(3\lambda+2\mu)} + v_3(x_1,x_2)$$
$$= C_2 x_3 + v_3(x_1,x_2)$$

$$\gamma_{12} = \tfrac{1}{2}(u_{1,2} + u_{2,1}) \qquad \gamma_{23} = \tfrac{1}{2}(u_{2,3} + u_{3,2}) \qquad \gamma_{13} = \tfrac{1}{2}(u_{1,3} + u_{3,1})$$
$$= \tfrac{1}{2}(v_{1,2} + v_{2,1}) \qquad = \tfrac{1}{2}(v_{2,3} + v_{3,2}) \qquad = \tfrac{1}{2}(v_{1,3} + v_{3,1})$$

$$v_{1,2} = -v_{2,1} \qquad v_{2,3} = -v_{3,2} \qquad v_{1,3} = -v_{3,1}$$

$$v_{1,23} = -v_{2,13} \qquad v_{1,32} = -v_{3,12}$$

$$v_{3,12} - v_{2,13} = 0$$

$$(v_{3,2} - v_{2,3})_{,1} = 0$$

$$2v_{3,21} = 0$$ $v_3(x_1,x_2)$ IS LINEAR IN x_1 & x_2

SIMILAR CALCULATIONS SHOW
$v_2(x_1,x_3)$ IS LINEAR IN x_1 & x_3
$v_1(x_2,x_3)$ IS LINEAR IN x_2 & x_3

$$v_1 = a_1x_2 + b_1x_3 + c_1x_2x_3 + d_1$$
$$v_2 = a_2x_1 + b_2x_3 + c_2x_1x_3 + d_2$$
$$v_3 = a_3x_1 + b_3x_2 + c_3x_1x_2 + d_3$$

$$v_{1,2} = -v_{2,1} \qquad a_1 + c_1x_3 = -a_2 - c_2x_3 \qquad a_1 = -a_2 \qquad c_1 = -c_2$$
$$v_{2,3} = -v_{3,2} \qquad b_2 + c_2x_1 = -b_3 - c_3x_1 \qquad b_2 = -b_3 \qquad c_2 = -c_3$$
$$v_{1,3} = -v_{3,1} \qquad b_1 + c_1x_2 = -a_3 - c_3x_2 \qquad b_1 = -a_3 \qquad c_1 = -c_3$$

663 NOTES

PURE BENDING (CONT.)

$$\gamma_{13} = \tfrac{1}{2}(u_{1,3} + u_{3,1}) = 0$$

$$= \tfrac{1}{2}\left[v_{1,3} + \frac{B(\lambda+\mu)x_3}{I(3\lambda+2\mu)} + v_{3,1}\right]$$

$\gamma_{12,1} = 0$

$= \tfrac{1}{2}(v_{2,11})$

$\gamma_{12,2} = 0$

$= \tfrac{1}{2}\left[v_{1,22} - \frac{B\lambda}{I(3\lambda+2\mu)}\right]$

$\gamma_{23,2} = 0$

$= \tfrac{1}{2}(v_{3,22})$

$\gamma_{23,3} = 0$

$= \tfrac{1}{2}(v_{2,33})$

$\gamma_{13,1} = 0$

$= \tfrac{1}{2}(v_{3,11})$

$\gamma_{13,3} = 0$

$= \tfrac{1}{2}\left[v_{1,33} + \frac{B(\lambda+\mu)}{I(3\lambda+2\mu)}\right]$

v_2 IS LINEAR IN x_1 & x_3

v_3 IS LINEAR IN x_1 & x_2

$$v_1 = \frac{B\lambda x_2^2}{2I(3\lambda+2\mu)} - \frac{B(\lambda+\mu)x_3^2}{2I(3\lambda+2\mu)} + a_1x_2 + b_1x_3 + c_1x_2x_3 + d_1$$

$$v_2 = a_2x_1 + b_2x_3 + c_2x_1x_3 + d_2$$
$$v_3 = a_3x_1 + b_3x_2 + c_3x_1x_2 + d_3$$

$$v_{1,2} - \frac{B\lambda x_2}{I(3\lambda+2\mu)} + v_{2,1} = 0$$

$$\frac{B\lambda x_2}{I(3\lambda+2\mu)} + a_1 + c_1x_3 - \frac{B\lambda x_2}{I(3\lambda+2\mu)} + a_2 + c_2x_3 = 0$$

$$v_{2,3} + v_{3,2} = 0$$

$$b_2 + c_2x_1 + b_3 + c_3x_1 = 0$$

$$v_{1,3} + \frac{B(\lambda+\mu)x_3}{I(3\lambda+2\mu)} + v_{3,1} = 0$$

$$\frac{-B(\lambda+\mu)x_3}{I(3\lambda+2\mu)} + b_1 + c_1x_2 + \frac{B(\lambda+\mu)x_3}{I(3\lambda+2\mu)} + a_3 + c_3x_2 = 0$$

663 NOTES

PURE BENDING (CONT.)

EQUILIBRIUM EQUATIONS

$$\sigma_{ij,j} = 0 \qquad \sigma_{33,3} = 0 \qquad \text{SATISFIED}$$

$$\gamma_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{(3\lambda+2\mu)}\sigma_{kk}\delta_{ij}\right]$$

$$\gamma_{11} = -\frac{\lambda}{3\lambda+2\mu}\sigma_{33} = \gamma_{22}$$

$$\gamma_{33} = \frac{(\lambda+\mu)}{\mu(3\lambda+2\mu)}\sigma_{33} \qquad \gamma_{ij} = 0 \quad (i \neq j)$$

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

$$\gamma_{11} = u_{1,1} = \frac{-B\lambda x_1}{I(3\lambda+2\mu)}$$

$$u_1 = \frac{-B\lambda x_1^2}{2I(3\lambda+2\mu)} + v_1(x_2, x_3)$$

$$\gamma_{22} = u_{2,2} = \frac{-B\lambda x_1}{I(3\lambda+2\mu)}$$

$$u_2 = \frac{-B\lambda x_1 x_2}{I(3\lambda+2\mu)} + v_2(x_1, x_3)$$

$$\gamma_{33} = u_{3,3} = \frac{B(\lambda+\mu)x_1}{I\mu(3\lambda+2\mu)}$$

$$u_3 = \frac{B(\lambda+\mu)x_1 x_3}{I\mu(3\lambda+2\mu)} + v_3(x_1, x_2)$$

$$\gamma_{12} = \tfrac{1}{2}(u_{1,2} + u_{2,1}) = 0$$

$$= \tfrac{1}{2}\left[v_{1,2} - \frac{B\lambda x_2}{I(3\lambda+2\mu)} + v_{2,1}\right]$$

$$\gamma_{23} = \tfrac{1}{2}(u_{2,3} + u_{3,2}) = 0$$

$$= \tfrac{1}{2}(v_{2,3} + v_{3,2})$$

643 NOTES

PURE TORSION (CONT.)

$$\int_{S_\ell} (x_1\sigma_{23} - x_2\sigma_{13})\,dA = T \quad : \quad \begin{array}{ll} \sigma_{13} = \sigma_{13}(x_1, x_2) & \sigma_{23} = \sigma_{23}(x_1, x_2) \\ \sigma_{13}n_1 + \sigma_{23}n_2 = 0 & \text{ON } \alpha \\ \sigma_{31,1} + \sigma_{32,2} = 0 & \end{array}$$

$$\gamma_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right]$$

$$\gamma_{13} = \frac{\sigma_{13}}{2\mu} \qquad \gamma_{23} = \frac{\sigma_{23}}{2\mu} \qquad \gamma_{ij} = 0 \quad (ij \neq \{13, 23\})$$

$$\gamma_{ij} = 1/2\,(u_{i,j} + u_{j,i})$$

$$\begin{array}{ll} u_{1,1} = 0 & u_1 = u_1(x_2, x_3) \\ u_{2,2} = 0 & u_2 = u_2(x_1, x_3) \\ u_{3,3} = 0 & u_3 = u_3(x_1, x_2) \end{array}$$

$$\gamma_{13} = 1/2\,(u_{1,3} + u_{3,1})$$

$$\gamma_{13,3} = 0 = 1/2\,(u_{1,33} + u_{3,13}) = u_{1,33} \qquad \gamma_{12,2} = 0 = 1/2\,(u_{1,22} + u_{2,12}) = u_{1,22}$$

$$\gamma_{23,3} = 0 = 1/2\,(u_{2,33} + u_{3,23}) = u_{2,33} \qquad \gamma_{12,1} = 0 = 1/2\,(u_{1,21} + u_{2,11}) = u_{2,11}$$

$$\begin{array}{l} u_1 = a_1x_2 + b_1x_3 + c_1x_2x_3 + d_1 \\ u_2 = a_2x_1 + b_2x_3 + c_2x_1x_3 + d_2 \\ u_3 = a_3x_1 + b_3x_2 + d_3 + c_3\varphi(x_1, x_2) \end{array} \qquad \varphi(x^2)$$

$$\gamma_{12} = 1/2\,(u_{1,2} + u_{2,1}) = 0$$
$$= a_1 + c_1x_3 + a_2 + c_2x_3 \qquad a_1 + a_2 = 0, \quad c_1 + c_2 = 0$$

$$\gamma_{13} = 1/2\,(u_{1,3} + u_{3,1})$$
$$= b_1 + c_1x_2 + a_3 + c_2\varphi_{,x_1} \qquad b_1 + a_3 = 0$$

$$\gamma_{23} = 1/2\,(u_{2,3} + u_{3,2})$$
$$= b_2 + c_2x_1 + b_3 + c_2\varphi_{,x_2} \qquad b_2 + b_3 = 0$$

$$\underline{u} = \left[-\omega_3x_2 + \omega_2x_3 - \Theta x_2x_3 + d_1\right]\underline{e}_1 + \left[\omega_3x_1 - \omega_1x_3 + \Theta x_1x_3 + d_2\right]\underline{e}_2 +$$
$$+ \left[\omega_1x_2 - \omega_2x_1 + \Theta\varphi(x_1, x_2) + d_3\right]\underline{e}_3$$

663 NOTE

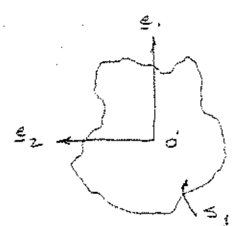
PURE BENDING (CONT.)

$$a_1 + a_2 = 0 \qquad b_2 + b_3 = 0 \qquad b_1 + a_3 = 0$$
$$c_1 + c_2 = 0 \qquad c_2 + c_3 = 0 \qquad c_1 - c_3 = 0$$

$$a_1 = -a_2 = -\omega_3 \qquad\qquad c_1 = c_2 = c_3 = 0$$
$$b_2 = -b_3 = -\omega_1$$
$$b_1 = -a_3 = \omega_2$$

$$\underline{u} = \left[\frac{-B\lambda x_1^2}{2I(3\lambda+2\mu)} + \frac{B\lambda x_2^2}{2I(3\lambda+2\mu)} - \frac{B(\lambda+\mu)x_3^2}{2I(3\lambda+2\mu)} - \omega_3 x_2 + \omega_2 x_3 + d_1 \right] \underline{e}_1 +$$
$$+ \left[\frac{-B\lambda x_1 x_2}{I(3\lambda+2\mu)} + \omega_3 x_1 - \omega_1 x_3 + d_2 \right] \underline{e}_2 + \left[\frac{B(\lambda+\mu) x_1 x_3}{I(3\lambda+2\mu)} - \omega_2 x_1 + \omega_1 x_2 + d_3 \right] \underline{e}_3$$

PURE TORSION



$(\underline{e}_1, \underline{e}_2)$ ARE PRINCIPLE AXIS

O' IS LOCATED @ CENTROID

$$\int_{S_\ell} x_1 x_2 \, dA = 0$$
$$\int_{S_\ell} x_1 \, dA = \int_{S_\ell} x_2 \, dA = 0$$

$\underline{L} = \underline{0}$ $\qquad \underline{t} = \underline{0}$ ON Σ

$\underline{M} = T \underline{e}_3$

$$\int_{S_\ell} \sigma_{i3} \, dA = 0 \qquad \int_{S_\ell} -x_1 \sigma_{33} \, dA = 0 \qquad \int_{S_\ell} x_2 \sigma_{33} \, dA = 0 \qquad \int_{S_\ell} (x_1 \sigma_{23} - x_2 \sigma_{13}) \, dA = T$$

ASSUME $\sigma_{ij} = 0$ $\quad (ij \neq \{13, 23\})$

$$\int_{S_\ell} (x_1 \sigma_{23} - x_2 \sigma_{13}) \, dA = T$$

$$\int_{S_\ell} \sigma_{i3} \, dA = 0 \qquad \sigma_{33} \int_{S_\ell} -x_1 \, dA = \sigma_{33} \int x_2 \, dA = 0$$

$$n_1 \sigma_{13} + n_2 \sigma_{23} = 0$$

EQUILIBRIUM EQUATIONS

$$\sigma_{ij,j} = 0$$
$$\sigma_{31,1} + \sigma_{32,2} = 0 \qquad \sigma_{13,3} = 0 \qquad \sigma_{23,3} = 0$$

$$\sigma_{13} = \hat{\sigma}_{13}(x_1, x_2)$$
$$\sigma_{23} = \hat{\sigma}_{23}(x_1, x_2)$$

663 NOTES

PURE TORSION (CONT.)

$$\underline{u} = \left[-\omega_3 x_2 - \omega_2 x_3 - \Theta x_2 x_3 - d_1 \right] \underline{e}_1 + \left[\omega_3 x_1 - \omega_1 x_3 + \Theta x_1 x_3 + d_2 \right] \underline{e}_2 +$$

$$+ \left[\omega_1 x_2 - \omega_2 x_1 + \Theta \varphi(x_1, x_2) + d_3 \right] \underline{e}_3$$

SUBJECT TO:

$$\varphi_{,1} n_1 + \varphi_{,2} n_2 = x_2 n_1 - x_1 n_2 \qquad \text{ON } \alpha$$

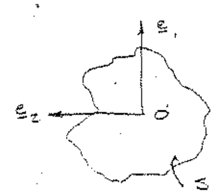
$$T = \mu \Theta \int_{S_\ell} (x_1^2 + x_2^2) \, dA + \mu \Theta \int (\varphi_{,2} x_1 - \varphi_{,1} x_2) \, dA$$

$$\varphi_{,\alpha\alpha} = 0$$

+ HENCE:

$$\begin{aligned} \sigma_{13} &= 2\mu \gamma_{13} \\ &= \mu (u_{1,3} + u_{3,1}) \\ &= \mu (\omega_2 - \Theta x_2 - \omega_2 + \Theta \varphi_{,1}) \\ &= \mu \Theta (\varphi_{,1} - x_2) \end{aligned} \qquad \begin{aligned} \sigma_{23} &= 2\mu \gamma_{23} \\ &= \mu (u_{2,3} + u_{3,2}) \\ &= \mu (-\omega_1 + \Theta x_1 + \omega_1 + \Theta \varphi_{,2}) \\ &= \mu \Theta (\varphi_{,2} + x_1) \end{aligned}$$

PURE SHEAR



$(\underline{e}_1, \underline{e}_2)$ ARE PRINCIPLE AXIS
O' IS LOCATED @ CENTROID

$$\int_{S_\ell} x_1 x_2 \, dA = 0$$

$$\int_{S_\ell} x_1 \, dA = \int_{S_\ell} x_2 \, dA = 0$$

$\underline{L} = V \underline{e}_1 \qquad \underline{t} = \underline{0}$ ON α

$\underline{M} = \underline{0}$

$$\int_{S_\ell} \sigma_{i3} \, dA = \begin{cases} 0 & i \neq 1 \\ V & i = 1 \end{cases} \qquad \int_{S_\ell} x_2 \sigma_{33} \, dA = 0 \qquad \int_{S_\ell} -x_1 \sigma_{33} \, dA = 0 \qquad \int_{S_\ell} (x_1 \sigma_{23} - x_2 \sigma_{13}) \, dA = 0$$

ASSUME $\sigma_{ij} = 0 \qquad (ij \neq \{13, 33\})$

EQUILIBRIUM EQUATIONS $\qquad \sigma_{ij,j} = 0$

$$\sigma_{31,1} + \sigma_{33,3} = 0$$

$$\sigma_{13,3} = 0 \qquad\qquad \sigma_{13} = \hat{\sigma}_{13}(x_1, x_2)$$

$$\sigma_{33} = \hat{\sigma}_{33}(x_1, x_2, x_3)$$

BOUNDARY CONDITIONS

$$n_1 \sigma_{13} = 0 \qquad \text{ON } \alpha$$

663 NOTES

PURE TORSION (CONT.)

EQUILIBRIUM EQUATION : $\sigma_{13,1} + \sigma_{23,2} = 0$

$$\begin{aligned} 0 &= 2\mu(\gamma_{13,1} + \gamma_{23,2}) \\ &= \mu(u_{1,31} + u_{3,11} + u_{2,32} + u_{3,22}) \\ &= \mu(\Theta\varphi_{,11} + \Theta\varphi_{,22}) \end{aligned}$$

$$\varphi_{,11} + \varphi_{,22} = 0$$

BOUNDARY CONDITION ON $\mathcal{L}$ $\quad \sigma_{13}n_1 + \sigma_{23}n_2 = 0$

$$\begin{aligned} 0 &= 2\mu(\gamma_{13}n_1 + \gamma_{23}n_2) \\ &= \mu(u_{1,3}n_1 + u_{3,1}n_1 + u_{2,3}n_2 + u_{3,2}n_2) \\ &= \mu\left[(\omega_2 - \Theta x_2 - \omega_2 + \Theta\varphi_{,1})n_1 + (-\omega_1 + \Theta x_1 + \omega_1 + \Theta\varphi_{,2})n_2\right] \end{aligned}$$

$$\begin{aligned} 0 &= (\varphi_{,1} - x_2)n_1 + (\varphi_{,2} + x_1)n_2 \\ \varphi_{,1}n_1 + \varphi_{,2}n_2 &= x_2 n_1 - x_1 n_2 = \frac{\partial\varphi}{\partial n} \end{aligned}$$

FORCE / MOMENT EQUATIONS ON S_ℓ $\quad \int_{S_\ell}(x_1\sigma_{23} - x_2\sigma_{13})\,dA = T$

$$T = \int_{S_\ell}\Theta\mu\left[(\varphi_{,2} + x_1)x_1 - (\varphi_{,1} - x_2)x_2\right]dA$$

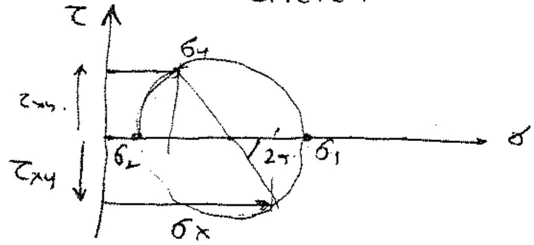
$$= \mu\Theta\int_{S_\ell}(x_1^2 + x_2^2)\,dA + \mu\Theta\int_{S_\ell}(\varphi_{,2}x_1 - \varphi_{,1}x_2)\,dA$$

$$0 = \int_{S_\ell}\sigma_{i3}\,dA$$

$$0 = \int_{S_\ell}\sigma_{13}\,dA = \int_{S_\ell}\mu\Theta(\varphi_{,1} - x_2)\,dA \qquad\qquad 0 = \int_{S_\ell}\sigma_{23}\,dA = \int_{S_\ell}\mu\Theta(\varphi_{,2} + x_1)\,dA$$

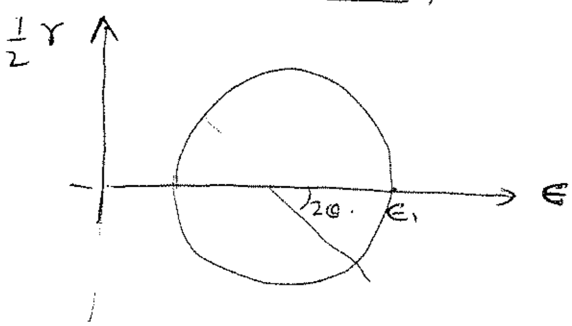
$$\begin{aligned} \int_{S_\ell}\varphi_{,\alpha}\,dA &= \int_{S_\ell}(x_\alpha\varphi_{,\beta})_{,\beta}\,dA \\ &= \oint_C(x_\alpha\varphi_{,\beta}n_\beta)\,ds \\ &= \oint_C(x_\alpha(x_2n_1 - x_1n_2))\,ds \\ &= \int_{S_\ell}((x_\alpha x_2)_{,1} - (x_\alpha x_1)_{,2})\,dA \\ &= \int_{S_\ell}x_2\,dA - \int_{S_\ell}x_1\,dA = 0 \end{aligned}$$

Mohr Circle.

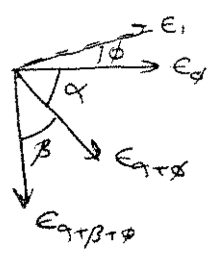


If τ_{xy} is +ve then σ_x is measured below and σ_y above.

Strain at a point.

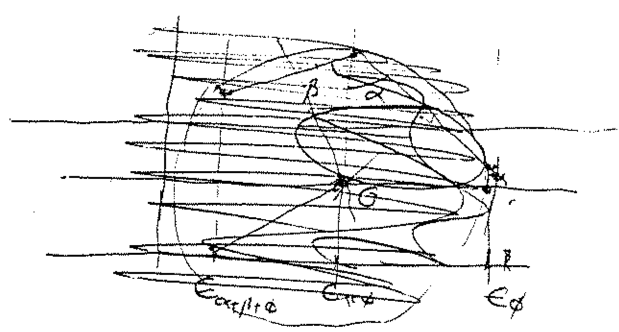


Strain rosette.

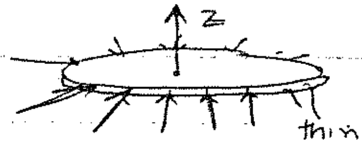


~~[illegible]~~ Unit elongations measured along ϵ_ϕ, $\epsilon_{\alpha+\phi}$, $\epsilon_{\alpha+\beta}$. by strain gauges.
α, β known. ϕ = angle ϵ_1 makes w/ ϵ_ϕ unknown. To get circle do:

(pg 25)



Plane stress –

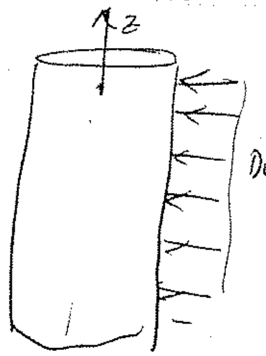


$\sigma_z, \tau_{xz}, \tau_{yz}$ are zero on both sides of the plate & tentatively throughout

$\therefore \quad \sigma_z = \tau_{xz} = \tau_y = 0.$

Also assume $\sigma_x, \sigma_y, \tau_{xy} = f(x,y)$ only.

Plane-strain:



$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0, \quad \omega = 0$ also.

$\Rightarrow \tau_{xz} = \tau_{xy} = 0.$

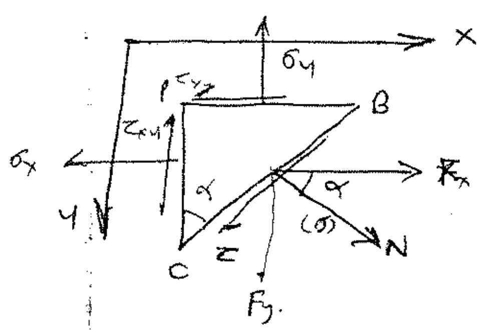
Assume u, v are $f(x,y)$ only.

$\therefore \epsilon_z = 0 = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right)$

$\Rightarrow \sigma_z = \nu(\sigma_x + \sigma_y).$

$\sigma_x, \sigma_y, \tau_{xy}$ are $f(x,y)$ only.

Stress at a point:



Let Area BC dA.

$\therefore$ Area PB $= dA \sin\alpha$

" PC $= dA \cos\alpha$

$\therefore F_x dA = \sigma_x dA \cos\alpha + \tau_{xy} dA \sin\alpha$

$\Rightarrow F_x = \sigma_x \cos\alpha + \tau_{xy} \sin\alpha$

$F_y = \sigma_y \sin\alpha + \tau_{xy} \cos\alpha.$

$\therefore \sigma = F_x \cos\alpha + F_y \sin\alpha = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy} \sin\alpha \cos\alpha$

cont.

Plane strain. $\sigma_z = \nu(\sigma_x+\sigma_y)$ or

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$

ϕ'

If body forces are absent are are constant, both come down to $\nabla^2(\sigma_x+\sigma_y) = 0$.

If so letting
$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} - f(y).$$
← ~~take into account~~ take into account const body forc in y-dir.

$$\sigma_y = \frac{\partial^2\phi}{\partial y^2} - f(y)$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}$$

gives
$$\frac{\partial^4\phi}{\partial x^4} + \frac{2\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \quad \text{as COMP. COND.}$$

Saint-Venant's Principle: ??

If the distribution of forces at the ends have zero resultant force and couple it's ok for not have to this distribution.

Polar coordinates:

Letting
$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

remember:

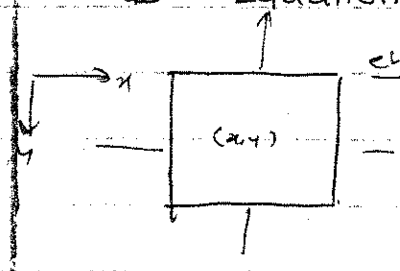
$\sigma_r = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$

$\sigma_\theta = \sigma_y\cos^2\theta + \sigma_x\sin^2\theta - 2\tau_{xy}\sin\theta\cos\theta$

$\tau_{r\theta} = (\sigma_y-\sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$

$\therefore \sigma_x = \sigma_r\cos^2\theta + \sigma_\theta\sin^2\theta - 2\tau_{r\theta}\sin\theta\cos\theta$ etc.

Equations of equilibrium:



eh.

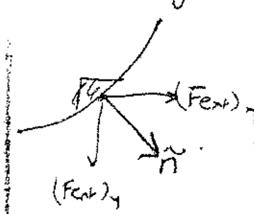
Balancing forces

body force

equilibrium eq^{ns}.

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + F_x = 0 \\[2ex] \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + F_y = 0. \end{cases}$$

Bdry conditions:



$$\boxed{\begin{aligned} (F_{ext})_x &= \sigma_x n_x + \tau_{xy} n_y \\ (F_{ext})_y &= \sigma_y n_y + \tau_{xy} n_x \end{aligned}}$$

Compatability Eq^{n}.

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

u, v are 2 f^{ns}. but have 3 strain comp i.e. $\epsilon_x, \epsilon_y, \gamma_{xy}$.

$\therefore$ need cond. of compatability :

$$\boxed{\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}}$$

Plane stress $\Rightarrow$ $\epsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y)$ etc.

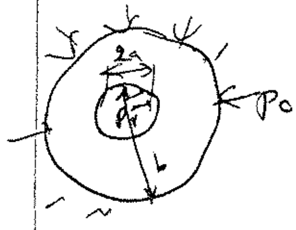
substituting in compat. cond. & using equilibrium

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$

~~radial displacement~~

$$E\epsilon_\theta = \sigma_\theta - \nu\sigma_r$$ for plane stress

Now for cylinder with p_o = outside $p_r = 0$.



$$\Rightarrow \sigma_r = \frac{a^2 p_i}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{a^2 p_i}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right)$$

$\therefore$ σ_r = compressive

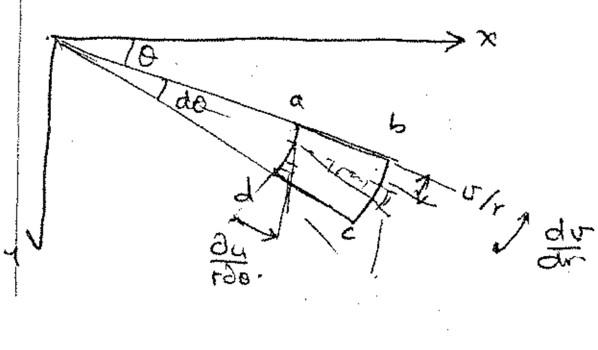
σ_θ = tensile.

$$(\sigma_\theta)_{max} = \sigma_\theta\big|_{r=a} = \frac{p_i(a^2+b^2)}{b^2-a^2}$$

$\therefore (\sigma_\theta)_{max} > p_i$ and $(\sigma_\theta)_{max} \to p_i$ as $b \to \infty$.

In pure bending, cross-sections DO REMAIN PLANE. Discrepancies between actual and elementary sol^n is in neglecting σ_r in the elementary and assuming longitudinal fibres of bent bar are in simple tension or compression.

Strain.



u = radial disp

v = tangential

we get

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0$$

for const body forces.

Stress Distribution symmetric about an Axis.

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) = 0.$$

$$\therefore \quad \phi = A\ln r + Br^2 \ln r + Cr^2 + D.$$

$$\therefore \sigma_r = \frac{1}{r}\frac{d\phi}{dr} = \frac{A}{r^2} + B(1 + 2\ln r) + 2C.$$

$$\sigma_\theta = \frac{d^2\phi}{dr^2} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C.$$

$$\tau_{r\theta} = 0.$$

No hole at origin $\Rightarrow$ A and B vanish

$\therefore \sigma_r = \sigma_\theta = \text{const.}$

If hole at origin then -

(i) $B = 0$ anyway * since as'll be shown on pg 9, ~~...~~
if not, then the tangential displacement'll be ~~...~~ multi-valued. which cannot be allowed.

$$\text{(ii)} \quad \left.\begin{array}{l} \sigma_r = \dfrac{A}{r^2} + 2C \\[2ex] \sigma_\theta = -\dfrac{A}{r^2} + 2C \end{array}\right\} \Rightarrow \sigma_r + \sigma_\theta = \text{constant.}$$

$$\Rightarrow \epsilon_z = \frac{1}{E}\left[\overset{=0}{\sigma_z} - \nu(\sigma_r + \sigma_\theta)\right] = \text{const}$$

$\Rightarrow$ plane stress assumption valid

MCGILL + KING - KINETICS

NEWTONIAN MECHANICS

LINEAR MOMENTUM $\underline{L} = m\underline{v}_c$

$$\Sigma \underline{F}_{cm} = {}^{\mathcal{F}}\dot{\underline{L}}$$

ANGULAR MOMENTUM $\underline{H}_p = \int \underline{r} \times \underline{v}\, dm$

$$\underline{v} = \underline{v}_p + \underline{\omega} \times \underline{r}$$

$$= \int \underline{r} \times (\underline{v}_p + \underline{\omega} \times \underline{r})\, dm$$

$$= \int \underline{r}\, dm \times \underline{v}_p + \int \underline{r} \times (\underline{\omega} \times \underline{r})\, dm$$

$$= m\underline{r}_{pc} \times \underline{v}_p + \int \underline{r} \times (\underline{\omega} \times \underline{r})\, dm$$

IF 1. $\underline{r}_{pc} = \underline{0}$ SUCH THAT P IS THE MASS CENTER

2. $\underline{v}_p = \underline{0}$ SUCH THAT P IS STATIONARY IN A INERTIAL FRAME

3. $\underline{r}_{pc} \parallel \underline{v}_p$ SUCH THAT P MOVES IN THE DIRECTION OF C

THEN

$$\underline{H}_p = \int \underline{r} \times (\underline{\omega} \times \underline{r})\, dm$$

$$= \int \left((\underline{r} \cdot \underline{r})\underline{\omega} - (\underline{r} \cdot \underline{\omega})\underline{r} \right) dm$$

$$= \begin{bmatrix} \int (y^2 + z^2)\, dm\, \omega_x + \int -xy\, dm\, \omega_y + \int -xz\, dm\, \omega_z \\ \int -xy\, dm\, \omega_x + \int (x^2 + z^2)\, dm\, \omega_y + \int -yz\, dm\, \omega_z \\ \int -xz\, dm\, \omega_x + \int yz\, dm\, \omega_y + \int (x^2 + y^2)\, dm\, \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I^p_{xx} & I^p_{xy} & I^p_{xz} \\ I^p_{xy} & I^p_{yy} & I^p_{yz} \\ I^p_{xz} & I^p_{yz} & I^p_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underline{\underline{I}}^p \underline{\omega}$$

IF NOT THEN AN ADDITIONAL $(m\underline{r}_{pc} \times \underline{v}_p)$ TERM EXISTS

$\Sigma \underline{M}_p = {}^{\mathcal{F}}\dot{\underline{H}}_p$ ABOUT ANY POINT P

ASSUME P IS ONE OF THE THREE ABOVE POINTS

$$= {}^{\mathcal{F}}\dot{(\underline{\underline{I}}^p \underline{\omega}_{B/\mathcal{F}})}$$

$$= {}^{B}\dot{\underline{H}}_p + \omega_{B/\mathcal{F}} \times \underline{H}_p$$

MCGILL & KING - KINETICS

TRANSFORMATION OF INERTIAL PROPERTIES

$$I_{zz}^{P} = \int (x^2 + y^2)\,dm \qquad \text{LET} \quad x = x_1 - \bar{x} \quad \bar{x} = x_P - x_{cm}$$
$$\qquad\qquad\qquad\qquad\qquad y = y_1 - \bar{y} \quad \bar{y} = y_P - y_{cm}$$

$$= \int \left((x_1 - \bar{x})^2 + (y_1 - \bar{y})^2\right) dm$$

$$= \int (x_1^2 + y_1^2)\,dm - 2\bar{x}\int x_1\,dm - 2\bar{y}\int y_1\,dm + (\bar{x}^2 + \bar{y}^2)\int dm$$

$$= I_{zz}^{C} + (\bar{x}^2 + \bar{y}^2)\,m$$

& SIMILARLY...

$$I_{yy}^{P} = I_{yy}^{C} + (\bar{x}^2 + \bar{z}^2)\,m$$

$$I_{xx}^{P} = I_{xx}^{C} + (\bar{y}^2 + \bar{z}^2)\,m$$

$$I_{xy}^{P} = -\int xy\,dm$$

$$= -\int (x_1 - \bar{x})(y_1 - \bar{y})\,dm$$

$$= -\int x_1 y_1\,dm + \bar{x}\int y_1\,dm + \bar{y}\int x_1\,dm - \bar{x}\bar{y}\int dm$$

$$= I_{xy}^{C} - \bar{x}\bar{y}\,m$$

& SIMILARLY...

$$I_{xz}^{P} = I_{xz}^{C} - \bar{x}\bar{z}\,m$$

$$I_{yz}^{P} = I_{yz}^{C} - \bar{y}\bar{z}\,m$$

ANALYTICAL MECHANICS

CONSERVATIVE SYSTEMS

$\exists$ V SUCH THAT $\dfrac{\partial V}{\partial q_i} = -Q_i$

$$\delta W = \sum_{i=1}^{n} -\frac{\partial V}{\partial q_i}\,\delta q_i = 0 \qquad \text{SO} \qquad \frac{\partial V}{\partial q_i} = 0$$

A CONSERVATIVE SYSTEM IN EQUILIBRIUM HAS A CONSTANT POTENTIAL V $\forall$ δq

NON-CONSERVATIVE SYSTEMS

$Q_i = -\dfrac{\partial V}{\partial q_i} + R_i$ $\qquad$ R_i IS THE NON-CONSERVATIVE COMPONENT OF THE GENERALIZED FORCE

EQUILIBRIUM CONDITION $Q_i = 0$

$$R_i = \frac{\partial V}{\partial q_i}$$

CONSTRAINTS

CONSTRAINT FORCES DO NO WORK. HENCE THEY DO NOT HAVE TO BE TAKEN IN TO ACCOUNT WHEN CALCULATING THE GENERALIZED FORCE

$$Q_i^T = \sum_{j=1}^{3N} F_j^T \frac{\partial x_j}{\partial q_i}$$
$$= \sum_{j=1}^{3N} \left(F_j^{(a)} + f_j\right)\frac{\partial x_j}{\partial q_i}$$
$$= Q_i + Q_i^{(c)}$$
$$= Q_i \qquad Q_i^{(c)} = 0 \text{ UNDER THE ASSUMPTION ABOVE}$$

HOLONOMIC CONSTRAINTS

CONSERVATIVE SYSTEM

$V(q_1, q_2, \dots, q_n)$ SUBJECT TO $f(q_1, q_2, \dots, q_n) = 0$

a. SOLVE $f(q) = 0$ FOR q_n

$$V(q) = V(q_1, q_2, \dots \bar{q}_n(q_1, q_2, \dots, q_{n-1}))$$
$$= V(q_1, q_2, \dots, q_{n-1})$$

b. LAGRANGE MULTIPLIER -

CONSTRUCT $V^*(q, \lambda) = V(q) - \lambda f(q)$ $\qquad$ λ: LAGRANGE MULTIPLIER

$\nmid$ PROCEED AS BEFORE W/ $\lambda = q_{n+1}$

$$\delta W = \sum_{i=1}^{n+1} -\frac{\partial V^*}{\partial q_i}\,\delta q_i$$
$$= \sum_{i=1}^{n} -\frac{\partial V^*}{\partial q_i}\,\delta q_i - \frac{\partial V^*}{\partial q_{n+1}}\,\delta q_{n+1} \qquad -\frac{\partial V^*}{\partial q_{n+1}} = -\frac{\partial V^*}{\partial \lambda} = f(q) = 0$$
$$= \sum_{i=1}^{n} -\left[\frac{\partial V}{\partial q_i} - \lambda \frac{\partial f}{\partial q_i}\right]\delta q_i$$

ANALYTICAL MECHANICS

TRANSFORMATION FROM D'ALEMBART'S PRINCIPLE TO LAGRANGE'S EQUATIONS

D'ALEMBART'S PRINCIPLE

$$\sum_i \left(\underline{F}_i^{(a)} - m\underline{a}_i\right) \cdot \delta \underline{r}_i = 0$$

$$\sum_i \underline{F}_i^{(a)} \cdot \delta \underline{r}_i = \sum_j Q_j \delta q_j$$

$$\sum_i m\underline{a}_i \cdot \delta \underline{r}_i = \sum_i m\ddot{\underline{r}}_i \cdot \delta \underline{r}_i$$

$$= \sum_j m\ddot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(m\dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) - m\dot{\underline{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j} \right) \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(m\underline{v}_i \cdot \frac{\partial \underline{v}_i}{\partial \dot{q}_j} \right) - m\underline{v}_i \cdot \frac{\partial \underline{v}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left[m \frac{\partial}{\partial \dot{q}_j}\left(\frac{\underline{v}_i \cdot \underline{v}_i}{2} \right)\right] - m\frac{\partial}{\partial q_j}\left(\frac{\underline{v}_i \cdot \underline{v}_i}{2} \right) \right] \delta q_j$$

$$\frac{m}{2} \underline{v}_i \cdot \underline{v}_i = T$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

$$0 = \sum_j \left\{ \frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right\} \delta q_j$$

LAGRANGE'S EQUATIONS

$$Q_j = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j} \right] - \frac{\partial T}{\partial q_j}$$

HOLONOMIC CONSTRAINTS

$$Q_j = -\frac{\partial V}{\partial q_j} + \lambda \frac{\partial f}{\partial q_j} + R_j$$

NON HOLONOMIC CONSTRAINTS

$$Q_j = -\frac{\partial V}{\partial q_j} + \lambda g_j + R_j$$

ANALYTICAL MECHANICS

GENERALIZED COORDINATES - ANY SET OF COORDINATES SUFFICIENT TO DESCRIBE ALL CONFIGURATIONS IN A SIMPLY-CONNECTED OPEN SUBSET OF A CONFIGURATION SPACE

GENERALIZED FORCE - VIRTUAL WORK PERFORMED PER UNIT VIRTUAL DISPLACEMENT OF THE ASSOCIATED GENERALIZED COORDINATE WHEN ALL REMAINING GENERALIZED COORDINATES REMAIN FIXED (INCLUDES WORK DONE BY CONSTRAINT FORCES)

$$x_1 = f_1(q_1, q_2, \ldots, q_n, t)$$
$$x_2 = f_2(q_1, q_2, \ldots, q_n, t)$$
$$x_{3N} = f_{3N}(q_1, q_2, \ldots, q_n, t)$$

$$\delta W = \sum_{j=1}^{3N} F_j \delta x_j \qquad \delta x_j = \sum_{i=1}^{n} \frac{\partial x_j}{\partial q_i} \delta q_i$$
$$= \sum_{j=1}^{3N} \sum_{i=1}^{n} F_j \frac{\partial x_j}{\partial q_i} \delta q_i$$
$$= \sum_{i=1}^{n} Q_i \delta q_i \qquad Q_i = \sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i}$$

$\underline{Q}_i$: GENERALIZED FORCE ASSOCIATED W/ δq_i

HOLONOMIC CONSTRAINTS : REAL-VALUED FUNCTION $f(q_1, q_2, \ldots, q_n) = 0$ RELATING <u>ONLY</u> THE GENERALIZED COORDINATES

NONHOLONOMIC CONSTRAINTS : NON-INTEGRABLE CONSTRAINT; NO COORDINATES CAN BE ELEMINATED THROUGH THIS CONDITION

PRINCIPLE OF VIRTUAL WORK

A MECHANICAL SYSTEM WITHOUT INEQUALITY CONSTRAINTS IS IN EQUILIBRIUM IF & ONLY IF THE VIRTUAL WORK IS ZERO FOR ALL POSSIBLE VIRTUAL DISPLACEMENTS.

VIRTUAL DISPLACEMENT - A KINEMATICALLY ADMISSABLE, INFINITESIMAL DISPLACEMENT FIELD

VIRTUAL WORK - WORK PERFORMED BY THE FORCES ACTING THROUGH THE VIRTUAL DISPLACEMENT FIELD

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i$$

NOTE: Q_i NEED NOT BE AN ACTUAL FORCE, & δq_i NEED NOT BE AN ACTUAL DISPLACEMENT, RATHER $Q_i \delta q_i$ MUST HAVE UNITS OF WORK

HOLONOMIC CONSTRAINTS IMPLY THAT A SET OF INDEPENDENT GENERALIZED COORDINATES CAN BE FOUND

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i = 0 \quad \forall \delta q_i$$
$$= (\underline{Q}, \delta \underline{q})$$

BECAUSE THIS MUST HOLD $\forall \delta \underline{q}$ THEN $\underline{Q} = \underline{0}$, $Q_i = 0$

IN EQUILIBRIUM, $\underline{F} = 0$ $\quad Q_i = \sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i} = 0$

ANALYTICAL MECHANICS

λ IS THE GENERALIZED FORCE THE ENFORCES THE CONSTRAINT

NON-CONSERVATIVE SYSTEMS

$$Q_i = -\frac{\partial V^*}{\partial q_i} + R_i$$

$$\delta W = \sum_{i=1}^{n+1} Q_i \delta q_i$$

$$= \sum_{i=1}^{n}\left[-\frac{\partial V^*}{\partial q_i} + R_i\right]\delta q_i + \left[-\frac{\partial V^*}{\partial q_{n+1}} + R_{n+1}\right]\delta q_{n+1}$$

$$-\frac{\partial V^*}{\partial q_{n+1}} = f(q) = 0 \qquad R_{n+1} = 0$$

$$= \sum_{i=1}^{n}\left[-\frac{\partial V}{\partial q_i} + \lambda\frac{\partial f}{\partial q_i} + R_i\right]\delta q_i$$

NON-HOLONOMIC CONSTRAINTS

q SUBJECT TO $g_1(q)\dot{q}_1 + g_2(q)\dot{q}_2 + \dots + g_n(q)\dot{q}_n = 0$

REPLACE $\frac{\partial f}{\partial q_i}$ BY $g_i(q)$ $\qquad \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\dot{q}_i = 0 = \sum_{i=1}^{n} g_i \dot{q}_i$

$$\delta W = \sum_{i=1}^{n}\left[-\frac{\partial V}{\partial q_i} + \lambda g_i + R_i\right]\delta q_i$$

D'ALEMBERT'S PRINCIPLE

$$\underline{F}^T = \underline{F}^{(a)} + \underline{f}^{(c)} - m\underline{a} = 0$$

$$\sum_{j=1}^{3N}\left(F_j^{(a)} + f_j^{(c)} - ma_j\right)\delta x_j = 0$$

BUT WE ASSUME CONSTRAINT FORCES DO NO WORK

$$\delta W = \sum_{j=1}^{3N}\left(F_j^{(a)} - ma_j\right)\delta x_j = 0 \qquad \text{FOR A PARTICLE}$$

$$= \sum_i \left(\underline{F}_i^{(a)} - m\underline{a}_i\right)\cdot\delta\underline{r}_i = 0 \qquad \text{FOR A SYSTEM}$$

MCGILL & KING - KINEMATICS

ANGULAR VELOCITY - RELATION BETWEEN DERIVATES

$${}^{B}\underline{Q} = q_x e_1 + q_y e_2 + q_z e_3$$

$${}^{A}\dot{\underline{Q}} = \dot{q}_x e_1 + \dot{q}_y e_2 + \dot{q}_z e_3 + q_x \dot{e}_1 + q_y \dot{e}_2 + q_z \dot{e}_3$$

$$e_1 \cdot e_1 = e_2 \cdot e_2 = e_3 \cdot e_3 = 1$$

$$\dot{e}_1 \cdot e_1 = \dot{e}_2 \cdot e_2 = \dot{e}_3 \cdot e_3 = 0$$

$$\dot{e}_1 = \underline{\alpha} \times e_1 = \alpha_z e_2 - \alpha_y e_3$$
$$\dot{e}_2 = \underline{\beta} \times e_2 = -\beta_z e_1 + \beta_x e_3$$
$$\dot{e}_3 = \underline{\gamma} \times e_3 \quad \gamma_y e_1 - \gamma_x e_2$$

$\{\alpha_x, \beta_y, \gamma_z\}$ ARBITRARY

$$e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0$$

$$\dot{e}_1 \cdot e_2 + e_1 \cdot \dot{e}_2 = (\underline{\alpha} \times e_1) \cdot e_2 + e_1 \cdot (\underline{\beta} \times e_2) = 0$$

$$= \alpha_z - \beta_z \longrightarrow \begin{array}{l} \alpha_z = \beta_z \\ \alpha_y = \gamma_y \\ \beta_x = \gamma_x \end{array}$$

SO LET
$$\alpha_x = \beta_x = \gamma_x$$
$$\alpha_y = \beta_y = \gamma_y$$
$$\alpha_z = \beta_z = \gamma_z \qquad \text{OR} \qquad \underline{\alpha} = \underline{\beta} = \underline{\gamma}$$

& SO WE CALL $\underline{\alpha} = \underline{\beta} = \underline{\gamma} = \underline{\omega}_{B/A}$ ANGULAR VELOCITY OF FRAME B WITH RESPECT TO FRAME A

$${}^{A}\dot{\underline{Q}} = {}^{B}\dot{\underline{Q}} + q_x(\underline{\omega}_{B/A} \times e_1) + q_y(\underline{\omega}_{B/A} \times e_2) + q_z(\underline{\omega}_{B/A} \times e_3)$$
$$= {}^{B}\dot{\underline{Q}} + \underline{\omega}_{B/A} \times \underline{Q}$$

ANGULAR VELOCITY RELATES THE MOTION OF TWO FRAMES ... MEANINGLESS TO REFER TO ANGULAR VELOCITY OF A POINT

$$\underline{\omega}_{B/A} = -\underline{\omega}_{A/B}$$

$$\underline{\omega}_{C/A} = \underline{\omega}_{C/B} + \underline{\omega}_{B/A}$$

$${}^{A}\dot{\underline{\omega}}_{B/A} = {}^{B}\dot{\underline{\omega}}_{B/A}$$

MCGILL + KING - KINEMATICS

ANGULAR ACCELERATION

$$\underline{\alpha}_{\beta/A} = {}^{A}\dot{\underline{\omega}}_{\beta/A}$$

$$\underline{\alpha}_{c/A} = {}^{A}\dot{\underline{\omega}}_{c/A} = {}^{A}\left(\dot{\underline{\omega}_{c/\beta} + \underline{\omega}_{\beta/A}}\right)$$

$$= {}^{\beta}\dot{\underline{\omega}}_{c/\beta} + \underline{\omega}_{\beta/A} \times \underline{\omega}_{c/\beta} + {}^{A}\dot{\underline{\omega}}_{\beta/A}$$

$$= \underline{\alpha}_{c/\beta} + \underline{\alpha}_{\beta/A} + \underline{\omega}_{\beta/A} \times \underline{\omega}_{c/\beta}$$

SIMPLE ADDITION THEOREM DOES NOT HOLD

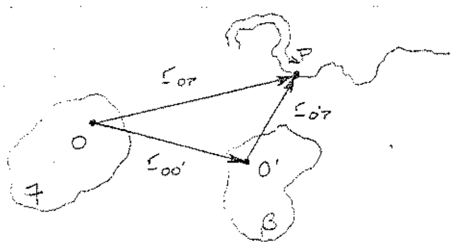
NOTE THAT ${}^{A}\dot{\underline{\omega}}_{\beta/A}$ MUST BE TAKEN IN THE BASE FRAME A, BUT THE CALCULATIONS CAN BE PERFORMED IN ANY FRAME USING

$$^{A}\dot{\underline{Q}} = {}^{\beta}\dot{\underline{Q}} + \underline{\omega}_{\beta/A} \times \underline{Q}$$

$$\underline{\alpha}_{c/A} = {}^{A}\dot{\underline{\omega}}_{c/A}$$

$$= {}^{\beta}\dot{\underline{\omega}}_{c/A} + \underline{\omega}_{\beta/A} \times \underline{\omega}_{c/A} = \underline{\alpha}_{c/\beta} + \underline{\alpha}_{\beta/A} + \underline{\omega}_{\beta/A} \times \underline{\omega}_{c/\beta}$$ FROM ABOVE

VELOCITY IN A MOVING FRAME OF REFERENCE



$$\underline{r}_{OP} = \underline{r}_{OO'} + \underline{r}_{O'P}$$

$$^{\mathcal{F}}\dot{\underline{r}}_{OP} = {}^{\mathcal{F}}\dot{\underline{r}}_{OO'} + {}^{\mathcal{F}}\dot{\underline{r}}_{O'P}$$

REFERENCE FRAME : $\mathcal{F}$

MOVING BODY : β

$$\underline{v}_{P/\mathcal{F}} = \underline{v}_{P/\beta} + \underline{v}_{O'/\mathcal{F}} + \underline{\omega}_{\beta/\mathcal{F}} \times \underline{r}_{O'P}$$

IF $\underline{v}_{P/\beta} = 0$ (P FIXED IN β)

$$\underline{v}_{P/\mathcal{F}} = \underline{v}_{O'/\mathcal{F}} + \underline{\omega}_{\beta/\mathcal{F}} \times \underline{r}_{O'P}$$

ACCELERATION IN A MOVING FRAME OF REFERENCE

$$\underline{a}_{P/\mathcal{F}} = {}^{\mathcal{F}}\dot{\underline{v}}_{P/\mathcal{F}}$$

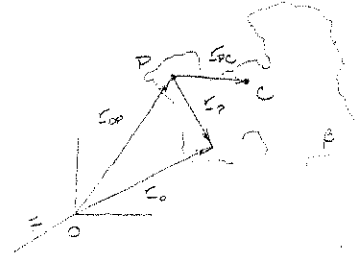
$$= {}^{\mathcal{F}}\left(\dot{\underline{v}_{O'/\mathcal{F}} + \underline{v}_{P/\beta} + \underline{\omega}_{\beta/\mathcal{F}} \times \underline{r}_{O'P}}\right)$$

$$= {}^{\mathcal{F}}\dot{\underline{v}}_{O'/\mathcal{F}} + {}^{\mathcal{F}}\dot{\underline{v}}_{P/\beta} + {}^{\mathcal{F}}\dot{\underline{\omega}}_{\beta/\mathcal{F}} \times \underline{r}_{O'P} + \underline{\omega}_{\beta/\mathcal{F}} \times {}^{\mathcal{F}}\dot{\underline{r}}_{O'P}$$

$$= \underline{a}_{O'/\mathcal{F}} + \underline{a}_{P/\beta} + \underline{\omega}_{\beta/\mathcal{F}} \times \underline{v}_{P/\beta} + \underline{\alpha}_{\beta/\mathcal{F}} \times \underline{r}_{O'P} + \underline{\omega}_{\beta/\mathcal{F}} \times \left({}^{\beta}\dot{\underline{r}}_{O'P} + \underline{\omega}_{\beta/\mathcal{F}} \times \underline{r}_{O'P}\right)$$

$$= \underline{a}_{P/\beta} + \underline{a}_{O'/\mathcal{F}} + \underline{\alpha}_{\beta/\mathcal{F}} \times \underline{r}_{O'P} + \underline{\omega}_{\beta/\mathcal{F}} \times \left(\underline{\omega}_{\beta/\mathcal{F}} \times \underline{r}_{O'P}\right) + 2\,\underline{\omega}_{\beta/\mathcal{F}} \times \underline{v}_{P/\beta}$$

ASIDE: REDERIVATION OF ANGULAR MOMENTUM



O: FIXED IN INERTIAL SPACE
P: FIXED IN BODY β
β: MOVING IN INERTIAL SPACE
$\mathcal{I}$: INERTIAL SPACE

$$\underline{H}_O = \int \underline{r}_O \times \underline{v}\, dm$$

$$\underline{r}_O = \underline{r}_{OP} + \underline{r}_P$$

$$\underline{v} = \underline{v}_P + \underline{\omega}_{\beta/\mathcal{I}} \times \underline{r}_P$$

$$= \int (\underline{r}_{OP} + \underline{r}_P) \times (\underline{v}_P + \underline{\omega}_{\beta/\mathcal{I}} \times \underline{r}_P)\, dm$$

$$= (\underline{r}_{OP} \times \underline{v}_P)\int dm + (\underline{r}_{OP} \times (\underline{\omega}_{\beta/\mathcal{I}} \times \int \underline{r}_P\, dm)) + \int \underline{r}_P\, dm \times \underline{v}_P + \int (\underline{r}_P \times (\underline{\omega}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$= \underline{r}_{OP} \times m\underline{v}_P + \underline{r}_{OP} \times (\underline{\omega}_{\beta/\mathcal{I}} \times m\underline{r}_{PC}) + m\underline{r}_{PC} \times \underline{v}_P + \int (\underline{r}_P \times (\underline{\omega}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$= \underline{r}_{OP} \times m\underline{v}_C + m\underline{r}_{PC} \times \underline{v}_P + \int (\underline{r}_P \times (\underline{\omega}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$= \underline{r}_{OC} \times m\underline{v}_C - \underline{r}_{PC} \times m\dot{\underline{r}}_{PC} + \int (\underline{r}_P \times (\underline{\omega}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$\dot{\underline{H}}_O = \dot{\underline{r}}_{OC} \times m\underline{v}_C + \underline{r}_{OC} \times m\dot{\underline{v}}_C - \dot{\underline{r}}_{PC} \times m\dot{\underline{r}}_{PC} - \underline{r}_{PC} \times m\ddot{\underline{r}}_{PC} + \int (\underline{r}_P \times (\dot{\underline{\omega}}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$= \underline{r}_{OC} \times m\dot{\underline{v}}_C - \underline{r}_{PC} \times m\ddot{\underline{r}}_{PC} + \int (\underline{r}_P \times (\dot{\underline{\omega}}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$= \underline{r}_{OC} \times m\underline{a}_C - \underline{r}_{PC} \times m\ddot{\underline{r}}_{PC} + \int (\underline{r}_P \times (\dot{\underline{\omega}}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$\underline{M}_O = \int (\underline{r}_O \times \underline{a})\, dm$$

$$= \int ((\underline{r}_{OP} + \underline{r}_P) \times \underline{a})\, dm$$

$$= \underline{r}_{OP} \times \int \underline{a}\, dm + \int (\underline{r}_P \times \underline{a})\, dm$$

$$= \underline{r}_{OP} \times \underline{F} + \underline{M}_P$$

$$\underline{M}_O = \dot{\underline{H}}_O$$

$$(\underline{r}_{OP} \times \underline{F}) + \underline{M}_P = (\underline{r}_{OP} \times m\underline{a}_C) + (\underline{r}_{PC} \times m\underline{a}_C) - (\underline{r}_{PC} \times m\ddot{\underline{r}}_{PC}) + \int (\underline{r}_P \times (\dot{\underline{\omega}}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$\underline{M}_P = \underline{r}_{PC} \times m\ddot{\underline{r}}_{OP} + \int (\underline{r}_P \times (\dot{\underline{\omega}}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

$$\underline{M}_P = (\underline{r}_{PC} \times m\underline{a}_P) + \int (\underline{r}_P \times (\dot{\underline{\omega}}_{\beta/\mathcal{I}} \times \underline{r}_P))\, dm$$

ASIDE: REDERIVATION OF ANGULAR MOMENTUM (CONT.)

IN TERMS OF $\underline{H}_P$

$$\underline{H}_O = \underline{r}_{OP} \times m\underline{v}_c + \underline{H}_P$$

$$\dot{\underline{H}}_O = \dot{\underline{r}}_{OP} \times m\underline{v}_c + \underline{r}_{OP} \times m\dot{\underline{v}}_c + \dot{\underline{H}}_P$$

$$= \underline{v}_P \times m\underline{v}_c + \underline{r}_{OP} \times m\underline{a}_c + \dot{\underline{H}}_P$$

$$\underline{M}_O = \underline{r}_{OP} \times \underline{F} + \underline{M}_P$$

$$(\underline{r}_{OP} \times \underline{F}) + \underline{M}_P = (\underline{r}_{OP} \times m\underline{a}_c) + (\underline{v}_P \times m\underline{v}_c) + \dot{\underline{H}}_P$$

$$\underline{M}_P = (\underline{v}_P \times m\underline{v}_c) + \dot{\underline{H}}_P$$

$$= \underline{v}_P \times m(\underline{v}_P + \underline{\omega}_{B/I} \times \underline{r}_{PC}) + \dot{\underline{H}}_P$$

$$= \underline{v}_P \times m(\underline{\omega}_{B/I} \times \underline{r}_{PC}) + \dot{\underline{H}}_P$$

$\underline{M}_P = \dot{\underline{H}}_P$ IF:

1. $\underline{v}_P = \underline{0}$ — P IS STATIONARY IN AN INERTIAL FRAME
2. $\underline{v}_c = \underline{0}$ — MASS CENTER IS STATIONARY IN AN INERTIAL FRAME
3. $\underline{v}_P \parallel \underline{v}_c$ — VELOCITY OF P IS PARALLEL TO VELOCITY OF C (INCLUDES CASE P = C)

$$\underline{H}_P = (m\underline{r}_{PC} \times \underline{v}_P) + \int (\underline{r}_P \times (\underline{\omega}_{B/I} \times \underline{r}_P))\, dm$$

$\underline{M}_P = \int (\underline{r}_P \times (\dot{\underline{\omega}}_{B/I} \times \underline{r}_P))\, dm$ IF:

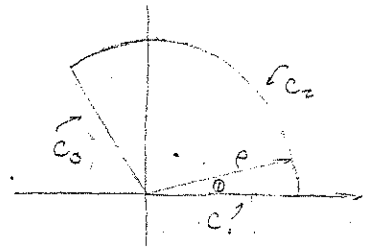
1. $\underline{a}_P = \underline{0}$ — P IS FIXED IN AN INERTIAL SPACE
2. $\underline{r}_{PC} = \underline{0}$ — P IS LOCATED AT THE MASS CENTER
3. $\underline{r}_{PC} \parallel \underline{a}_P$ — P ACCELERATES TOWARD THE MASS CENTER

APPLIED MATHEMATICS QUESTIONS

APPLIED
MATH
STUDY
AIDS

PROBLEMS

48. $\displaystyle\int_0^\infty \frac{dx}{x^3-1}$



LET $\displaystyle J = \int_C \frac{dz}{z^3+1}$

ON C_2 $\displaystyle\int_{C_2} \frac{i\rho\, d\theta}{\rho^3 e^{i3\theta}+1} = i\int_{C_2} \frac{d\theta}{\rho^2 e^{i3\theta}+1/\rho} \cdot O(1/\rho) \to 0$ AS $\rho\to\infty$

ON C_3 $\displaystyle\int_\infty^0 \frac{e^{i2\pi/3}\, d\rho}{\rho^3+1} = -e^{i2\pi/3}\int_0^\infty \frac{d\rho}{\rho^3+1}$

$$\frac{J}{2\pi i} = \lim_{z\to e^{i\pi/3}} \left[\frac{z-e^{i\pi/3}}{z^3+1}\right] = \frac{1}{(e^{i\pi/3}-e^{i\pi})(e^{i\pi/3}-e^{-i\pi/3})}$$

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi i}{(e^0-e^{i2\pi/3})(e^{i\pi/3}-e^{i\pi})(e^{i\pi/3}-e^{-i\pi/3})}$$

$$= \left[(e^{i\pi/3}-e^{i\pi}-e^{i\pi}+e^{i5\pi/3})(e^{i\pi/3}-e^{-i\pi/3})\right]^{-1} 2\pi i$$

$$= \left[e^{i2\pi/3}-2e^{i4\pi/3}+e^{i2\pi}-e^0+2e^{i2\pi/3}-e^{i4\pi/3}\right]^{-1} 2\pi i$$

$$= \frac{2\pi}{3\sqrt{3}}$$

49. TELL ABOUT CONTOUR INTEGRALS, RESIDUE THEOREM, + INDENTED CONTOURS

50. HOW DO RUNGE-KUTTA, EULER METHODS WORK? ERROR ESTIMATE?

51. HOW DO YOU GET FROM $\displaystyle I = \int_{t_1}^{t_2} f(x,\dot{x},t)\,dt$ TO EULER EQUATIONS?

$$I[x] = \int_{t_1}^{t_2} f(x,\dot{x},t)\,dt$$

$$\frac{d}{d\alpha} I[x+\alpha\eta]\Big|_{\alpha=0} = \frac{d}{d\alpha}\left[\int_{t_1}^{t_2} f(x+\alpha\eta,\ \dot{x}+\alpha\dot{\eta},\ t)\,dt\right]_{\alpha=0} \qquad \eta(t_1)=\eta(t_2)=0$$

PROBLEMS

51. (CONT.)

$$\frac{d}{d\alpha} I\left[x+\alpha\eta\right]\bigg|_{\alpha=0} = \int_{t_1}^{t_2}\left[\frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial \dot{x}}\dot{\eta}\right]dt = 0$$

$$= \frac{\partial f}{\partial x'}\eta\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left[\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\eta\, dt = 0$$

$$= \int_{t_1}^{t_2}\left[\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\eta\, dt$$

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

52. SOLVE $J = \int_c \frac{dz}{z^m}$ OVER $|z| = 1$

FOR $m \leq 0$, ARGUEMENT IS ANALYTIC INSIDE c, $J = 0$

FOR $m > 1$, RESIDUE @ $z=0$ IS ZERO, $J = 0$

FOR $m = 1$, RESIDUE @ $z=0$ IS ONE, $J = 2\pi i$

NOTES

ORDINARY DIFFERENTIAL EQUATIONS

$$g(x, y, y', y'', \dots, y^{(n)}) : \text{ ODE OF ORDER } (n)$$

METHODS OF SOLUTION - HOMOGENEOUS EQUATIONS

SEPARATION OF VARIABLES

$$y'(x) = g(x) h(y)$$

$$\int \frac{dy}{h(y)} = \int g(x)\, dx$$

HOMOGENEOUS OF DEGREE ZERO

$F(x_1, x_2, \dots)$ IS HOMOGENEOUS OF ORDER k IF
$$F(\lambda x_1, \lambda x_2, \dots) = \lambda^k F(x_1, x_2, \dots)$$

OF ORDER ZERO:
$$y' = f(x,y) = f(\lambda x, \lambda y) = f(1, y/x) = F(y/x)$$
$$= F(v)$$

$$v + xv' = F(v)$$

$$v' = \frac{F(v) - v}{x} \quad : \text{ VARIABLE SEPERABLE}$$

EXACT DIFFERENTIALS + INTEGRATING FACTORS

$P(x,y)\,dx + Q(x,y)\,dy = 0$ THIS EQUATION IS EXACT IF

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad + \text{ SO } \quad \frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q$$

IF THIS EQUATION IS NOT EXACT, WE SEEK TO MAKE IT EXACT BY

$$\mu P(x,y)\,dx + \mu Q(x,y)\,dy$$

SUCH THAT $(\mu P)_y = (\mu Q)_x$

$$P\mu_y - Q\mu_x = (Q_x - P_y)\mu$$

LET $\mu = \mu(x)$
$$\mu(x) = \exp\left\{ \int \frac{P_y - Q_x}{Q}\, dx \right\}$$

$\mu = \mu(y)$
$$\mu(y) = \exp\left\{ \int \frac{Q_x - P_y}{P}\, dy \right\}$$

NOTES

VARIATION OF PARAMETERS

IF ONLY ONE LI SOLUTION CAN BE FOUND BY STANDARD METHODS WE CAN INTRODUCE THE SECOND SOLUTION

$$y(x) = A(x)\, e^{\lambda x}$$

CAUCHY-EULER EQUATION

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-2} x^2 y'' + a_{n-1} x y' + a_n = 0$$

SEEK A SOLUTION $y(x) = Cx^{\alpha}$

METHODS OF SOLUTION - INHOMOGENEOUS CASE

METHOD OF UNDETERMINED COEFFICIENTS

GUESS

VARIATION OF PARAMETERS

TAKE $y_P(x)$ TO BE OF THE SAME FORM AS $y_H(x)$ BUT VARIABLE COEFFICIENTS

$$y' + M(x)y = N(x)$$

$$y_H(x) = C_1 \exp\left\{-\int M(x)\,dx\right\}$$

$$y_P(x) = A(x) \exp\left\{-\int M(x)\,dx\right\}$$

$$A'(x)\exp\left\{-\int M(x)\,dx\right\} - M(x)A(x)\exp\left\{-\int M(x)\,dx\right\} + M(x)A(x)\exp\left\{-\int M(x)\,dx\right\} = N(x)$$

$$A'(x) = N(x)\exp\left\{\int M(x)\,dx\right\}$$

$$A(x) = \int N(x)\exp\left\{\int M(\eta)\,d\eta\right\}dx$$

$$y(x) = \exp\left\{-\int M(x)\,dx\right\}\left[\int N(x)\exp\left\{\int M(\eta)\,d\eta\right\}dx + C_1\right]$$

NOTES

PARTIAL DIFFERENTIAL EQUATIONS

CLASSIFICATIONS OF $Au_{xx} + 2Bu_{xy} + Cu_{yy} = F$

$(B^2 - AC) = 0$ PARABOLIC
$(B^2 - AC) > 0$ HYPERBOLIC
$(B^2 - AC) < 0$ ELLIPTIC

HEAT EQUATION $\alpha^2 u_{xx} = u_t$ PARABOLIC $\longrightarrow$ $v_{\eta\eta} = 0$

WAVE EQUATION $a^2 u_{xx} = u_{tt}$ HYPERBOLIC $\longrightarrow$ $v_{\eta\xi} = 0$

LAPLACE EQUATION $u_{xx} + u_{yy} = 0$ ELLIPTIC $\longrightarrow$ $v_{\eta\eta} + v_{\xi\xi} = 0$

METHODS OF SOLUTION

SEPARATION OF VARIABLES
$$u(x,t) = X(x)\,T(t)$$

TRANSFORM METHODS - HEAT EQUATION, LAPLACE EQUATION

FOURIER TRANSFORM $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$

LAPLACE TRANSFORM $\bar{f}(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt \qquad f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s)\, e^{st}\, ds$

TRAVELLING WAVES - WAVE EQUATION

ANY FUNCTION $u(x,t) = f(x - \alpha t)$ SOLVES THE WAVE EQUATION GIVEN ABOVE

$$y(x,t) = \frac{f(x+\alpha t) + f(x-\alpha t)}{2} + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(\xi)\, d\xi\,; \qquad \begin{array}{l} y(x,0) = f(x) \\ y_t(x,0) = g(x) \end{array}$$

IS THE GENERAL SOLUTION TO THE WAVE EQUATION

CHARACTERISTICS - WAVE EQUATION, HEAT EQUATION
CORRESPOND TO TRAVELLING WAVE SOLUTIONS OF THE WAVE EQUATION
SOLUTIONS REMAIN CONSTANT ALONG THE CHARACTERISTICS

HYPERBOLIC EQUATIONS HAVE TWO REAL CHARACTERISTICS
PARABOLIC EQUATIONS HAVE ONE REAL CHARACTERISTICS
ELLIPTIC EQUATIONS HAVE NO REAL CHARACTERISTICS

NOTES

CHARACTERISTICS (CONT)

CONSIDER

$$A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} = F$$

LET $u = \varphi_x$

$v = \varphi_y$

$$Au_x + 2Bu_y + Cv_y = F \qquad\qquad u_s = u_x x_s + u_y y_s$$

$$u_y - v_x = 0 \qquad\qquad v_s = v_x x_s + v_y y_s$$

$$\begin{bmatrix} A & 2B & 0 & C \\ x_s & y_s & 0 & 0 \\ 0 & 0 & x_s & y_s \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} F \\ u_s \\ v_s \\ 0 \end{bmatrix} \qquad \underline{A}\,\underline{x} = \underline{f}$$

ALONG CHARACTERISTICS SOLUTIONS ARE NOT ANALYTIC ∴ $\{u_x, u_y, v_x, v_y\}$ IS NOT UNIQUE & HENCE THE MATRIX MUST BE SINGULAR

$\det(\underline{A}) = 0$ PROVIDES THE CHARACTERISTIC EQUATION

BEHAVIOR OF $\{u, v\}$ ALONG CHARACTERISTICS IS FOUND FROM COMPATABILITY RELATIONSHIP: $u_s = u_x x_s + u_y y_s$ OR

$v_s = v_x x_s + v_y y_s$

$$\frac{du}{dx} = u_x + u_y\frac{dy}{dx} \qquad\qquad \frac{dv}{dx} = v_x + v_y\frac{dy}{dx}$$

GREEN'S FUNCTION - LAPLACE EQUATION

USING VARIATION OF PARAMETERS, SOLVE THE EQUATION TO OBTAIN

$$y_p(x) = \int_{\xi} g(x,\xi)\, f(\xi)\, d\xi \qquad\qquad g(x,\xi) : \text{GREEN'S FUNCTION}$$

$g(x,\xi)$ IS THE RESPONSE TO A UNIT FORCE @ $x = \xi$

FINDING GREENS FUNCTIONS (LAPLACIAN)

SEEK $g = g_1 + g_2$ AS A SUM, WHERE $\Delta g_1 = \delta(\underline{x} - \underline{\xi})$ BUT WITH NO PARTICULAR BOUNDARY CONDITIONS & $\Delta g_2 = 0$ WITH $(g_1 + g_2)_{\partial\Omega} = \underline{u}|_{\partial\Omega}$ SO g SATISFIES THE DE & BC

METHOD APPLIES TO LINEAR OPERATORS AS WELL

PROBLEMS

1. WHAT IS THE DEFINITION OF AN ANALYTIC FUNCTION?

IF $f(z)$ IS DIFFERENTIABLE AT z_0 + IN ADDITION, DIFFERENTIABLE THROUGHOUT SOME NEIGHBORHOOD OF z_0 ($|z - z_0| < \varepsilon$, $\varepsilon > 0$) THEN $f(z)$ IS ANALYTIC AT z_0

WHAT IS THE CONDITION OF DIFFERENTIABILITY OF A COMPLEX FUNCTION?

CAUCHY-RIEMANN CONDITIONS:

$$u_x = v_y$$
$$u_y = -v_x$$

2. WRITE DOWN THE EQUATION OF ANY KIND OF SURFACE.

$f(x, y, z) = 0$

WHAT IS THE PHYSICAL REASONING OF THE DIRECTIONAL DERIVATIVE OF A FUNCTION?

RATE OF CHANGE OF THE GIVEN FUNCTION AS ONE MOVES IN THE DESIRED DIRECTION

3. WHAT IS THE DEFINITION OF $\underline{\nabla}$ OPERATOR?

$L(g(\underline{v})) = \underline{\nabla}_{\underline{v}}\, g$ SUCH THAT $L(g(\underline{v})) \cdot \underline{n} = \frac{d}{d\alpha}\left[g(\underline{v} + \alpha \underline{n}) \right]\Big|_{\alpha = 0}$

4. WHAT IS THE DEFINITION OF CURL + DIVERGENCE?

DIVERGENCE: CONTRACTION OF $\underline{\nabla}_{\underline{v}}\, g$

$$\underline{\nabla}_{\underline{v}} \cdot g$$

CURL: $\underline{\nabla}_{\underline{v}} \times g$

5 WHAT IS THE PHYSICAL MEANING OF GRADIANT, DIVERGENCE, + CURL

GRADIANT: LINEAR OPERATOR CONTAINING ALL INFORMATION ABOUT THE CHANGE OF ITS ARGUMENT

DIVERGENCE: MEASURE OF VOLUME EXPANSION + CONTRACTION

CURL: MEASURE OF ANGULAR VELOCITY OF A FLOW

PROBLEMS

6. WHAT DOES IT MEAN FOR VECTORS TO BE LINEARLY INDEPENDENT?

NO VECTOR CAN BE WRITTEN IN TERMS OF THE OTHERS

7. IF $\{x, y, z\}$ ARE LINEARLY INDEPENDENT, ARE $\{x+y, y+z, x+z\}$ ALSO LINEARLY INDEPENDENT?

$$(c_1+c_3)x + (c_1+c_2)y + (c_2+c_3)z = 0$$

$$\left.\begin{array}{l} c_1+c_3=0 \\ c_1+c_2=0 \\ c_2+c_3=0 \end{array}\right\} \quad \left.\begin{array}{l} c_1-c_2=0 \\ c_2-c_1=0 \end{array}\right\} \quad 2c_1=0 \Big\} \quad \begin{array}{l} c_1=0 \\ c_2=0 \\ c_3=0 \end{array}$$

$\{x+y, y+z, x+z\}$ ARE LINEARLY INDEPENDENT

8. WHAT ARE THE CAUCHY INTEGRAL FORMULA + THEOREM

CAUCHY'S THEOREM:
IF $f(z)$ IS ANALYTIC INSIDE + ON A CLOSED CURVE C, THEN:

$$\int_C f(z)\,dz = 0$$

CAUCHY INTEGRAL FORMULA:
IF $f(z)$ IS ANALYTIC INSIDE + ON A SIMPLE, CLOSED CCW CURVE C + IF THE POINT a IS INSIDE (BUT NOT ON) C, THEN:

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz = f(a)$$

9. CONSIDER $\ddot{x} + \omega^2 x = 0$, WHAT IS THE SOLUTION?

$\omega^2 > 0$ $\quad x_1(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$

CONSIDER $\omega = 0$, WHAT IS THE RELATIONSHIP BETWEEN THE TWO SOLUTIONS?

$\omega = 0$: $\quad x_2(t) = c_3 t + c_4$

TAYLOR EXPANDING $x_1(t)$:

$$x_1(t) = c_1\left(\omega t - \frac{(\omega t)^3}{6} + \dots\right) + c_2\left(1 - \frac{(\omega t)^2}{4} + \dots\right)$$

+ SO WE SEE AS $\omega \to 0^+$, $x_1(t) \to (c_1\omega)t + c_2$
$\to c_3 t + c_4$
$\to x_2(t)$

PROBLEMS

10. WRITE DOWN THE WAVE EQUATION.

$$u_{xx} = \frac{u_{tt}}{c^2}$$

WHAT TYPE OF EQUATION IS IT?

HYPERBOLIC

HOW DO YOU SOLVE IT?

SEPARATION OF VARIABLES
D'ALEMBART'S SOLUTION
METHOD OF CHARACTERISTICS

11. DEFINE $\underline{z}^{\underline{\alpha}}$, WHERE $\underline{z}$ & $\underline{\alpha}$ ARE COMPLEX

$$\underline{z} = \rho \exp\{i\theta\}$$
$$\underline{\alpha} = u + iv$$

$$\begin{aligned}
\underline{z}^{\underline{\alpha}} &= \left(\rho e^{i\theta}\right)^{(u+iv)} \\
&= \rho^{(u+iv)} \cdot e^{(-\theta v + i\theta u)} \\
&= \rho^{u} \left(\exp\{ i v \ln(\rho)\} \right) \exp\{-\theta v + i\theta u\} \\
&= \rho^{u} \exp\{-\theta v\} \left[\cos(v \ln(\rho) + u\theta) + i \sin(v \ln(\rho) + u\theta) \right]
\end{aligned}$$

12. SOLVE $y''' = y$

$$y''' - y = 0 \qquad \text{CHARACTERISTIC EQN.:} \quad r^3 - 1 = 0$$
$$r^3 = 1$$

$$\begin{aligned}
y(x) &= c_1 e^{x} \\
&+ C_2 \exp\{ e^{2\pi/3\, i}\, x\} \\
&+ C_3 \exp\{ e^{-2\pi/3\, i}\, x\}
\end{aligned}$$

13. WHEN DOES THE SOLUTION TO $\underline{A}\,\underline{x} = \underline{b}$ EXIST?

WHEN $\underline{b} \in R(\underline{A})$

UNIQUENESS?

$\underline{b} \in R(\underline{A})$ & $\underline{A}$ IS NON-SINGULAR

PROBLEMS

14. CONVERGENCE OF AN INFINITE POWER SERIES?

IFF IT IS A CAUCHY SEQUENCE, THAT IS, IF FOR EACH $\varepsilon > 0$, THERE CORRESPONDS AN $N(\varepsilon)$ SUCH THAT $|s_m - s_n| < \varepsilon$ + $m + n > N$

TESTS OF CONVERGENCE

15. CALCULATE VARIOUS CONTOUR INTEGRALS

16. PRINCIPLE VALUE, EIGENVALUE PROBLEMS

17. STATE THE RESIDUE THEOREM.

$$I = \int_C f(z)\,dz = 2\pi i \sum_j c_{-1}^{(j)}$$

CONTOUR INTEGRAL I IS EQUAL TO $(2\pi i)$ TIMES THE SUM OF THE RESIDUES OF $f(z)$ CONTAINED WITHIN C

18. DEFINE THE DERIVATIVE OF A COMPLEX FUNCTION

$$\frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

19. GIVE A BRIEF LECTURE ON FOURIER SERIES.

EXPANSION OF A FUNCTION IN A GIVEN SET OF BASIS (FUNCTIONS)

20. WHAT IS EULER'S CONSTANT γ?

$$\gamma = \lim_{n\to\infty}\left[\sum_{x=1}^{n} x^{-1} - \ln(n)\right] \cong .577$$

SHOW $\lim_{n\to\infty}\left[1 + 1/2 + 1/3 + \dots + 1/n - \ln(n)\right]$ CONVERGES + GIVE A BOUND ON THIS LIMIT

$$\lim_{n\to\infty}\left[\sum_{x=1}^{n}\frac{1}{x}\right] \le \lim_{n\to\infty}\left[\int_1^n \frac{1}{x}\,dx + 1\right] = \lim_{n\to\infty}\left[\ln(x)\Big|_1^n + 1\right]$$

$$= \lim_{n\to\infty}\left(\ln(n) + 1\right)$$

$$\lim_{n\to\infty}\left[\sum_{x=1}^{n}\frac{1}{x} - \ln(n)\right] \le \lim_{n\to\infty}\left(\ln(n) + 1 - \ln(n)\right)$$

$$\le \lim_{n\to\infty}(1)$$

$$\le 1$$

PROBLEMS

21. HOW DO YOU SOLVE THE CAUCHY-EULER EQUATION $x^2y'' + xy' - y = 0$

ASSUME
$$y(x) = x^r$$
$$y'(x) = rx^{r-1}$$
$$y''(x) = r(r-1)x^{r-2}$$

$$r(r-1)x^r + rx^r - x^r = 0$$
$$r^2 - r + r - 1 = 0$$
$$r^2 - 1 = 0$$
$$r = \pm 1 \longrightarrow \quad y(x) = \{ c_1 x^{(1)} + c_2 x^{(-1)} \}$$

22. WHAT IS CURL $(f(\underline{u}))$? $\quad f: \mathbb{R}^3 \to \mathbb{R}$

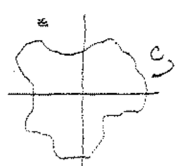
BECAUSE $f: \mathbb{R}^3 \to \mathbb{R}$ INSTEAD OF $f: \mathbb{R}^3 \to \mathbb{R}^3$, DOES THE CURL EXIST?

23. CONSIDER THE DIRICHLET PROBLEM ON A UNIT DISK. DERIVE POISSON'S INTEGRAL REPRESENTATION & DISCUSS APPLICATIONS

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad c: \text{ CCW CIRCLE } |\zeta| = R, \quad \zeta = Re^{i\varphi}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\zeta}{\zeta - z} f(\zeta)\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\zeta}{\zeta - z} + \frac{\bar z}{\bar\zeta - \bar z}\right] f(\zeta)\,d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{|\zeta - z|^2} f(\zeta)\,d\varphi$$

24. 

$f(z)$ IS ANALYTIC OUTSIDE C

z_0 IS A POINT OUTSIDE C $\qquad f(\infty) = a$

$$\oint_c \frac{f(z)}{z - z_0}\,dz$$

$$\frac{1}{2\pi i}\int_{c_T} \frac{f(z)}{z - z_0}\,dz = f(z_0); \quad \lim_{z\to\infty}\int_{c_\infty} \frac{f(z)\,dz}{z - z_0} = \lim_{\rho\to\infty}\int_0^{2\pi} \frac{i\rho e^{i\theta} f(z)\,d\theta}{\rho e^{i\theta}}$$
$$= (2\pi i)\,a$$

$$\oint_c \frac{f(z)}{z - z_0}\,dz = 2\pi i\,(a - f(z_0))$$

PROBLEMS

25. WHAT CAN YOU SAY ABOUT THE FOLLOWING:

$$L(u) - \alpha u = 0 \quad ; \quad u(0) = u_0$$
$$u(1) = u_1$$

L IS A LINEAR OPERATOR

α_n IS AN EIGENVALUE WITH CORRESPONDING EIGENVECTOR u, SUBJECT TO BOUNDARY CONDITIONS

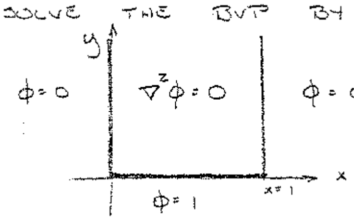
SOLUTION EXISTS IF $(u, z) = 0$ FOR ALL z SUCH THAT $L^*(z) = 0$

27. EVALUATE $\displaystyle\int_c \frac{dz}{z^{2n}+1}$, WHERE $c: |z| = 3$

POLES @ $z_m = e^{i\pi(1+2m)/2n}$

$$\text{RES} = \frac{(z_0 - z_1) - (z_0 - z_2) + (z_1 - z_2)}{(z_0 - z_1)(z_0 - z_2)(z_1 - z_2)} = 0$$

28. SOLVE THE BVP BY ELEMENTARY METHODS



$\phi_{xx} + \phi_{yy} = 0$ SUBJECT TO $\phi(x=0) = 0$

$\phi(y=0) = 1$

$\phi(x,y) = X(x)Y(y)$ $\phi(x=1) = 0$

$|\phi(y \to \infty)| < \infty$

$$\nabla^2\phi = X_{xx}Y + XY_{yy} = 0$$

$X(0) = 0$

$X(1) = 0$

$$-\frac{X_{xx}}{X} = \frac{Y_{yy}}{Y} = \lambda$$

$$X_{xx} + \lambda X = 0$$

$\lambda < 0 \quad \lambda = -\mu^2$

$$X_{xx} - \mu^2 X = 0$$

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$c_1 + c_2 = 0$$
$$c_1 e^{\mu} + c_2 e^{-\mu} = 0$$
$$c_1(e^{\mu} - e^{-\mu}) = 0$$

DOES NOT EXIST FOR $\mu \neq 0$

$\lambda = 0$

$$X_{xx} = 0$$

$$X(x) = c_1 x + c_2$$

$c_2 = 0$

$c_1 = 0$ TRIVIAL SOLUTION

PROBLEMS

28. (CONT)

$\lambda > 0$

$$X_{xx} + \lambda X = 0$$

$$X(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

$$C_1 = 0$$
$$C_2 \sin(\sqrt{\lambda}) = 0$$

$$\sqrt{\lambda} = n\pi$$
$$\lambda = n^2\pi^2 \qquad n = 1, 2, 3, \ldots$$

$$Y_{yy} - (n\pi)^2 Y = 0 \qquad |Y(y \to \infty)| < \infty$$

$$Y(y) = C_3 e^{n\pi y} + C_4 e^{-n\pi y}$$

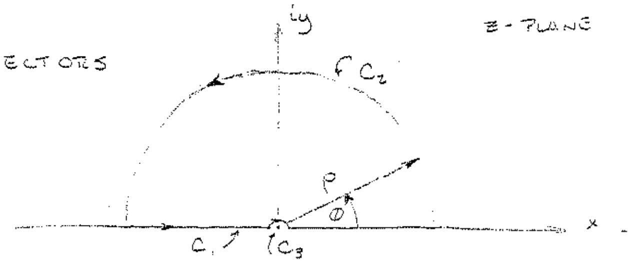
$$C_3 + C_4 = 1$$
BUT FOR $|Y(y \to \infty)| < \infty$ $\quad C_3 = 0$

$$\phi(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi y} (\sin(n\pi x))$$

C_n IS DETERMINED BY A FOURIER SERIES TO FIT BC.

29. WHEN A IS A REAL $(n \times n)$ MATRIX, WHAT ARE THE LEAST RESTRICTIVE CONSTRAINTS ON A SUCH THAT IT WILL BE DIAGONALIZABLE?

A HAS n UNIQUE EIGENVECTORS



30. SOLVE
$$\int_0^{\infty} \frac{\sin(\lambda x)}{x}\, dx$$

LET $J = \oint_C \dfrac{e^{i\lambda z}}{z}\, dz$

$$\left| \int_{C_2} \frac{e^{i\lambda z}}{z}\, dz \right| = \left| \int_0^{\pi} \frac{\exp\{i\lambda(x+iy)\}}{\rho e^{i\Phi}}\, i\rho e^{i\Phi}\, d\Phi \right|$$

$$\le \int_0^{\pi} \left|\exp\{i\lambda x\}\right| \left|\exp\{-\lambda y\}\right| d\Phi$$

$$\le \int_0^{\pi} \left|\exp\{-\lambda y\}\right| d\Phi \qquad y = \rho\sin\Phi \;\le \frac{2\rho\Phi}{\pi} \quad \text{ON } \Phi \in [0, \pi/2]$$

$$\le 2\int_0^{\pi/2} \exp\left\{-\lambda \frac{2\rho}{\pi}\Phi\right\} d\Phi$$

2) $x^2y'' + xy' - y = 0$ Let $y = x^m$

$m(m-1) + m - 1 = 0 \Rightarrow m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y(x) = c_1 x + \frac{c_2}{x}$

Consider, $x^2y'' + xy' - y = g(x)$ then $y'' + \frac{y'}{x} - \frac{y}{x^2} = \frac{g(x)}{x^2}$

$y(x) = y_h(x) + y_p(x)$

$= c_1 x + \frac{c_2}{x} + u(x)\cdot x + \frac{v(x)}{x}$

where $u(x) = -\int \frac{g(x)}{x^2}\frac{1}{x}\frac{dx}{W}$

$v(x) = \int \frac{g(x)}{x^2}\cdot\frac{x}{W}\,dx$

$W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix}$

$W = -\frac{2}{x}$

$\Rightarrow u(x) = \int \frac{g(x)}{x^2}\frac{dx}{2}$

$v(x) = -\int \frac{g(x)}{2}\,dx$

3) Curl $(f\underline{r}) = ?$ $f(r)$: continuous function of r.

$\text{curl}(f\underline{r})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(f x_k) = \varepsilon_{ijk}\frac{\partial f}{\partial x_j}x_k + \varepsilon_{ijk} f\frac{\partial x_k}{\partial x_j}$

$= \varepsilon_{ijk}\frac{\partial f}{\partial r}\frac{\partial r}{\partial x_j}x_k + \varepsilon_{ijk} f(r)\delta_{kj}$

$= \varepsilon_{ijk}\frac{\partial f}{\partial r}\frac{x_j}{r}x_k + \varepsilon_{ijj} f(r) = 0 + 0 = 0$

4) <u>Residue Theorem</u>: If $f(z)$ is an analytic function on a simple closed curve C & also inside C, except at finitely many points then

$$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \text{Res}\, f(z_i)$$

C: Integration taken in the counterclockwise sense.

$\underline{\underline{A}}_{n\times n}$: real matrix.

$\Rightarrow$ Least restrictions so as to diagonalize $\underline{\underline{A}}$:

If $\underline{\underline{A}}$ has 'n' independent eigenvectors then possible to diagonalize $\underline{\underline{A}}$.

$$Q^{-1}\underline{\underline{A}}\,Q = \underline{\underline{D}}$$

Diagonal matrix consisting of diagonal elements as eigenvalues.

also, $\underline{\underline{A}}\,Q = QD$

$$\underline{\underline{Q}} = \begin{bmatrix} \vdots & \vdots & \vdots & \\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

eigenvectors

12> Square matrix & 2nd Rank Tensors are <u>not</u> the same thing.

A square matrix obeys linear transformation properties like a Tensor i.e.

$\underline{\underline{A}}$: $n\times n$ matrix then

$$\underline{\underline{A}}\,(\alpha\underline{u} + \beta\underline{v}) = \alpha\underline{\underline{A}}\,\underline{u} + \beta\underline{\underline{A}}\,\underline{v}$$

However, ~~Tensors are related under various~~ the components of ~~a~~ a tensor ~~which~~ are related under various orthogonal transformation i.e.

$$T'_{ijkl\ldots} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}\cdots T_{pqrs\ldots}$$

17) Second-order Tensor: A 3x3 matrix which obeys certain transformation laws under orthogonal transformations is a second-order Tensor i.e.

$$T'_{ij} = Q_{ip} Q_{jq} T_{pq}$$

where T'_{ij} & T_{pq} are elements the the second-order Tensor in two set of axes.

$$\underset{\sim}{T}\, \hat{e}_1 = -\hat{e}_1 + \hat{e}_3$$

$$\underset{\sim}{T}\, \hat{e}_2 = 2\hat{e}_2$$

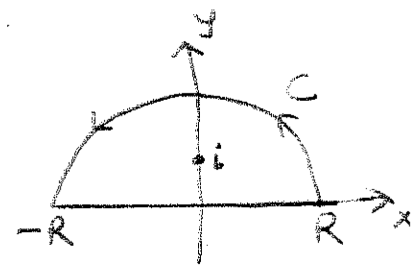
$$\underset{\sim}{T}\, \hat{e}_3 = \hat{e}_2 + \hat{e}_3$$

Now,

$$\underset{\sim}{T}\, \underset{\sim}{e}_i = T_{ji}\, \underset{\sim}{e}_j$$

$$\Rightarrow \quad \underset{\sim}{T} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

18) $$\int_0^\infty \frac{\cos x}{x^2+1}\, dx = ?$$



Consider: $$\oint_C \frac{e^{iz}}{z^2+1}\, dz$$

$z^2+1=0 \Rightarrow z = \pm i$

$$= 2\pi i \,\text{Res}\, f(z=i) = \frac{2\pi i\, e^{-1}}{2i} = \pi e^{-1}$$

$$f(z) = \frac{e^{z}}{z^2+1}$$

$$\oint \frac{e^{z}}{z^2+1}\, dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+1}\, dx + \int_{C_R} \frac{e^{iz}}{z^2+1}\, dz$$

$|e^{iz}| = |e^{ix}|e^{-y} = e^{-y} \underset{y\to\infty}{\to} 0$

$$= \int_{-R}^{R} \frac{\cos x}{x^2+1}\, dx + i \underbrace{\int_{-R}^{R} \frac{\sin x}{x^2+1}\, dx}_{=0} + \underbrace{\int_{C_R} \frac{e^{iz}}{z^2+1}\, dz}_{=0}$$

even function   odd function $R\to\infty$ using ML estimate.

$$= 2\int_0^R \frac{\cos x}{x^2+1} \Rightarrow \boxed{\int_0^\infty \frac{\cos x}{x^2+1} = \frac{\pi}{2e}}$$

Now, $\lim_{z\to 0} f(z) = 0 \rightarrow |f(z)| < \varepsilon$ for $|z| < \delta$

$\Rightarrow M|f(z)| < M\varepsilon$ for $|z| < \delta$

Now, $|f(z)||g(z)| \leq M|f(z)| < M\varepsilon$

$\Rightarrow |f(z)||g(z)| < M\varepsilon \Rightarrow |f(z)g(z)| < M\varepsilon$

Let $M\varepsilon = \varepsilon_1 \Rightarrow |f(z)g(z)| < \varepsilon_1$ for $|z| < \delta$

$\Rightarrow \lim_{z\to 0} f(z)g(z) = 0.$

21) A, B, C are LI: $\Rightarrow c_1 A + c_2 B + c_3 C = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0$ only solution.

Now, $d_1(A+B) + d_2(B+C) + d_3(A+C) = 0$

$\Rightarrow (d_1+d_3)A + (d_1+d_2)B + (d_2+d_3)C = 0$

But, since, A, B, C are LI:

$$\left.\begin{aligned} d_1 + d_3 &= 0 \\ d_2 + d_3 &= 0 \\ d_1 + d_2 &= 0 \end{aligned}\right\} d_1 = d_2 = d_3 = 0$$

$\Rightarrow A+B;\ B+C;\ A+C$ are LI.

22) $y''' + y' = a$ consider: $y''' + y' = 0 \Rightarrow \lambda^3 + \lambda = 0$

$\Rightarrow y(x) = e^{\lambda x}$ $\quad \lambda_1 = 0;\ \lambda_{2,3} = \pm i$

$y_h(x) = c_1 + c_2\cos x + c_3 \sin x$

$y_p(x) = ax \Rightarrow \boxed{y(x) = c_1 + c_2 \cos x + c_3 \sin x + ax}$

27.) $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \Rightarrow \underline{\underline{A}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^{-1}(\lambda \underline{x})$

$\Rightarrow \underset{\sim}{I}\,\underline{x} = \lambda \underline{\underline{A}}^{-1}\underline{x} \Rightarrow \underline{\underline{A}}^{-1}\underline{x} = \frac{1}{\lambda}\underline{x}.$

28.) One-Dimensional Wave Equation: $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

(Hyperbolic in nature).

~~29.) $\dot{x} = \sqrt{x} \Rightarrow \frac{dx}{\sqrt{x}} = dt$~~

31.) Contour Integrals: Useful in evaluating improper real integrals.

PROBLEMS

33. IN SOME SPACE EXISTS 3 VECTORS $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$, EACH CAN BE EXPRESSED AS A LINEAR COMBINATION OF UNIT VECTORS $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$. WHAT CONDITION MUST BE SATISFIED THAT $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ ARE LINEARLY INDEPENDENT?

WRONSKIAN $\quad w(\underline{v}_i) = \det\left((\underline{v}_i, \underline{e}_j)\right) = 0$

34. $y(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 < x < 1 \end{cases}$

WHAT IS THE VALUE OF THE FOURIER SERIES OF $y(x)$ @:

a) $x = 1 \quad y(x)_F = 1/2$

b) $x = 0 \quad y(x)_F = 1/2$

c) $x = -1 \quad y(x)_F = 1/2$

d) $x = 96.328 \quad y(x)_F = 0$

35. $f(x) \ge 0 \quad 0 < x < 1$

$f(0) = f(1) = 0$

MAXIMIZE AREA UNDER THE CURVE $f(x)$ SUBJECT TO THE CURVE $f(x)$ & CONSTANT LENGTH

$y = f(x) \qquad I = \int y\,dx \quad$ SUBJECT TO $\quad \int \sqrt{1 + y'^2}\,dx = \ell$

$$I = \int \left(y - \lambda\sqrt{1+y'^2}\right)dx$$

$$\frac{dI}{d\alpha}(y + \alpha h) = 0$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad f(x) = y - \lambda\sqrt{1+y'^2}$$

$$y'\frac{\partial f}{\partial y'} - f = c_1$$

$$y'\left[\frac{-\lambda y'}{\sqrt{1+y'^2}}\right] - y + \lambda\sqrt{1+y'^2} = c_1$$

$$-\lambda y'^2 + \lambda(1+y'^2) - y\sqrt{1+y'^2} = c_1\sqrt{1+y'^2}$$

$$\lambda = (c_1 + y)(1+y'^2)^{1/2}$$

$$y' = \left[\left[\frac{\lambda}{c_1+y}\right]^2 - 1\right]^{1/2}$$

$$\int \frac{dy}{\left[\left[\frac{\lambda}{c_1+y}\right]^2 - 1\right]^{1/2}} = x + c_2 \qquad \text{SUBJECT TO}$$

$$\int \frac{\lambda}{c_1+y}\,dx = \ell$$

PROBLEMS

35. (CONT.)

$$\int \frac{dy}{\left[\left[\frac{\lambda}{c_1+y}\right]^2 - 1\right]^{1/2}} = x + C_2 \qquad \text{LET} \quad \xi = \lambda / c_1+y$$

$$d\xi = \frac{-\lambda}{(c_1+y)^2}\, dy$$

$$-\lambda \int \frac{d\xi}{\xi^2\,(\xi^2-1)^{1/2}} = x + C_2$$

$$-\lambda \left[\frac{(\xi^2-1)^{1/2}}{\xi}\right] = x + C_2$$

$$(\xi^2-1)^{1/2} = \frac{-\xi\,(x+C_2)}{\lambda}$$

$$\xi^2 - 1 = \frac{\xi^2}{\lambda^2}\,(x+C_2)^2$$

$$\xi = \left(1 - \frac{(x+C_2)^2}{\lambda^2}\right)^{-1/2}$$

$$y = \lambda\left(1 - \frac{(x+C_2)^2}{\lambda^2}\right)^{1/2} - C_1$$

$$(y+C_1)^2 = \lambda^2 - (x+C_2)^2 \qquad \text{SUBJECT TO}$$

$$\int_0^1 \frac{\lambda}{c_1+y}\,dx = l$$

$$= \int_0^1 \frac{\lambda\,dx}{(\lambda^2 - (x+C_2)^2)^{1/2}} \qquad \text{LET} \quad \eta = x + C_2,\quad d\eta = dx$$

$$= \int_{C_2}^{1+C_2} \frac{\lambda\,dx}{(\lambda^2 - \eta^2)^{1/2}} \qquad \text{LET} \quad \eta = \lambda \sin(\varphi),\quad d\eta = \lambda\cos(\varphi)\,d\varphi$$

$$= \int \lambda\,d\varphi = \lambda\varphi = \lambda\,\mathrm{ASIN}\left(\frac{\eta}{\lambda}\right)$$

$$= \lambda\,\mathrm{ASIN}\left(\frac{1+C_2}{\lambda}\right) - \lambda\,\mathrm{ASIN}\left(\frac{C_2}{\lambda}\right)$$

$$y(0) = y(1) = 0$$

$$= \lambda^2 - (c_1^2 + c_2^2)$$

$$= \lambda^2 - (c_1^2 + (1-c_2)^2) \qquad c_2 = -1/2$$

$$\sin\left[\frac{l}{2\lambda}\right] = \frac{1}{2\lambda} \quad \rightarrow \quad \sin(\mu) = \frac{\mu}{l}$$

$$\lambda = \frac{l}{2\mu}$$

$$c_1 = (\lambda^2 - 1/4)^{1/2}$$

$$\left(y + (\lambda^2 - 1/4)^{1/2}\right)^2 + (x - 1/2)^2 = \lambda^2$$

PROBLEMS

36. WHAT IS A SECOND ORDER TENSOR?

LINEAR FUNCTION $\underline{T} : V \to V$

HOW IS IT REPRESENTED?

$$T = T_{ij}\,\underline{e}^i \underline{e}^j$$

GIVEN A SECOND ORDER TENSOR $\underline{T}$ & A SET OF 3 VECTORS $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ ON $\mathbb{R}^n$ WITH THE $\underline{v}_i$ LINEARLY INDEPENDENT & THE FOLLOWING:

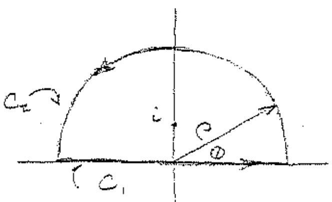
$\underline{T} \cdot \underline{v}_1 = -\underline{v}_1 + \underline{v}_3$

$\underline{T} \cdot \underline{v}_2 = 2\underline{v}_2$

$\underline{T} \cdot \underline{v}_3 = \underline{v}_2 + \underline{v}_3$ WHAT ARE THE ELEMENTS OF T?

$$\underline{T} = \begin{bmatrix} -1 & 0 & 0 & \vdots \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & \vdots \\ \cdots & 0 & \cdots & \ddots \end{bmatrix}$$

37. SOLVE $\displaystyle\int_0^\infty \frac{\cos(x)}{x^2+1}\,dx = \frac{1}{2}\,\mathrm{Re}\left\{ \int_{C_1} \frac{e^{iz}\,dz}{z^2+1} \right\}$ $C = C_1 + C_2$



$$J = \int_C \frac{e^{iz}}{z^2+1}\,dz$$

$$\int_{C_2} \frac{e^{iz}\,dz}{z^2+1} = \int_{C_2} \frac{i\exp\{ix-y\}\,\rho\,d\theta}{\rho^2 e^{i2\theta}+1} \to 0 \quad \text{AS } \rho \to \infty$$

$$J = 2\pi i \lim_{z \to i} \left[(z-i) \frac{e^{iz}}{(z-i)(z+i)} \right] = \pi e^{-1}$$

$$\int_{C_1} \frac{e^{iz}\,dz}{z^2+1} = \pi e^{-1} \qquad \int_0^\infty \frac{\cos(x)}{x^2+1}\,dx = \frac{\pi}{2e}$$

38. WHAT IS A DIFFERENTIAL EQUATION?
DISCUSS SOLUTIONS, EXISTENCE, UNIQUENESS, DIFFERENCE BETWEEN PDE & ODE, DIFFERENCE BETWEEN INITIAL VALUE PROBLEMS & BOUNDARY VALUE PROBLEMS, CONSTRAINTS ON λ IN $y'' + \lambda y = 0$ WITH DIFFERENT IC & BC, METHODS OF SOLUTIONS, NUMERICAL METHODS

$$f(\underline{x};\, u, u_{\underline{x}}, u_{\underline{xx}}, \ldots, u_{n\underline{x}}) = 0$$

PROBLEMS

39. SIMPLIFY $i^i = \exp\{i\pi/2\}^i$

$$= \exp\{-\pi/2\}$$

40. $\displaystyle\int_{|z|=1} \frac{dz}{z^n - a}$ n IS POS. INTEGER > 1, $a \in \mathbb{R}$, $0 < a < 1$

LET $|z| \to \infty$ $z = \rho e^{i\theta}$, $\rho \to \infty$

$$\int_0^{2\pi} \frac{i\rho\, d\theta}{\rho^n e^{in\theta} - a} \to 0 \quad \text{AS } \rho \to \infty$$

41. SOLVE $y''' + y = 0$

$$y(0) = y(\pi) = y'(\pi) = 0$$

$y(x) = C e^{rx}$

NOTE: $L[y] = \left(\frac{d^3}{dx^3} + 1\right)$ IS NOT SELF - ADJOINT

$y'(x) = C r e^{rx}$
$y''(x) = C r^2 e^{rx}$
$y'''(x) = C r^3 e^{rx}$

$$C r^3 e^{rx} + C e^{rx} = 0 \longrightarrow r^3 + 1 = 0$$
$$r = \{e^{i\pi/3}, e^{-i\pi/3}, e^{i\pi}\}$$

$$y(x) = C_1 \exp\{-x\} + C_2 \exp\{e^{i\pi/3} x\} + C_3 \exp\{e^{-i\pi/3} x\}$$

$y(0) = C_1 + C_2 = 0$
$y(\pi) = C_1 \exp\{-\pi\} + C_2 \exp\{e^{i\pi/3}\pi\} + C_3 \exp\{e^{-i\pi/3}\pi\}$
$y'(\pi) = -C_1 \exp\{-\pi\} + e^{i\pi/3} C_2 \exp\{e^{i\pi/3}\pi\} + e^{-i\pi/3} C_3 \exp\{e^{-i\pi/3}\pi\}$

$$\underline{A} = \begin{bmatrix} 1 & 1 & 1 \\ \exp\{-\pi\} & \exp\{e^{i\pi/3}\pi\} & \exp\{e^{-i\pi/3}\pi\} \\ -\exp\{-\pi\} & e^{i\pi/3}\exp\{e^{i\pi/3}\pi\} & e^{-i\pi/3}\exp\{e^{-i\pi/3}\pi\} \end{bmatrix}$$

$$\det(\underline{A}) = e^{-i\pi/3} \exp\{\pi(e^{i\pi/3} + e^{-i\pi/3})\} - \exp\{\pi(e^{-i\pi/3} + e^{i\pi})\} +$$
$$+ e^{-i\pi/3} \exp\{\pi(e^{i\pi} + e^{-i\pi/3})\} + \exp\{\pi(e^{i\pi} + e^{i\pi/3})\} -$$
$$- e^{i\pi/3} \exp\{\pi(e^{i\pi/3} + e^{-i\pi/3})\} - e^{i\pi/3} \exp\{\pi(e^{i\pi} + e^{-i\pi/3})\}$$

PROBLEMS

42. $\int_{-\pi}^{\pi} f(x)\,dx = 0$; $\int_{-\pi}^{\pi} (f(x))^2 dx = \int_{-\pi}^{\pi} (f'(x))^2 dx$ $\qquad f(x) \not\equiv 0$

WHAT ARE THE RESTRICTIONS ON $f(x)$?

EXPAND $f(x)$ IN A FOURIER SERIES

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$f'(x) = \sum_{n=1}^{\infty} n\left(-A_n \sin(nx) + B_n \cos(nx)\right)$$

$A_0 = 0$ BY DEFN.

IN $\int_{-\pi}^{\pi} [f(x)]^2 dx + \int_{-\pi}^{\pi} [f'(x)]^2 dx$, THE ONLY NON-ZERO TERMS WILL BE $\int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} A_n^2 \cos^2(nx) + B_n^2 \sin^2(nx)\right) dx$ +

$$\int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} n^2 A_n^2 \sin^2(nx) + n^2 B_n^2 \cos(nx)\right) dx$$

+ SO

$$\sum_{n=1}^{\infty} \pi (A_n^2 + B_n^2) = \sum_{n=1}^{\infty} n^2 \pi (A_n^2 + B_n^2)$$

$$\sum_{n=1}^{\infty} \pi (A_n^2 + B_n^2)(n^2 - 1) = 0$$

$n = 1$ FOR THIS CONDITION TO HOLD

$$f(x) = A\cos(x) + B\sin(x)$$

PROBLEMS

43. GIVEN: $\lim_{z\to 0} f(z) = 0$ + $g(z)$ IS BOUNDED IN A NEIGHBORHOOD OF $z=0$, FIND $\lim_{z\to 0} f(z)g(z)$

FOR ANY $\varepsilon > 0$, $|z| < \delta_1$ SUCH THAT $|f(z)| < \varepsilon$

$|z| < \delta_2$ SUCH THAT $|g(z)| < \infty$

BECAUSE $g(z)$ IS BOUNDED NEAR $z=0$, $\exists$ SOME NUMBER N, SUCH THAT $\left|\lim_{z\to 0} g(z)\right| < N$

$$|f(z)g(z)| = |f(z)g(z) - Nf(z) + Nf(z)|$$

$$\leq |f(\ )||g(z) - N| + N|f(z)|$$

$$\leq \varepsilon |N - g(z)| + \varepsilon N \qquad \text{FOR ANY } \varepsilon > 0,\ 0 < |z| < \min(\delta_1, \delta_2)$$

THUS, AS $\varepsilon \to 0$ $|f(z)g(z)| \to 0$ FOR $|z| < \min(\delta_1, \delta_2)$

44. $\{A, B, C\}$ ARE LINEARLY INDEPENDENT VARIABLES, SHOW LINEAR (IN)DEPENDENCE FOR $\{A+B, B+C, A+C\}$

$$c_1(A+B) + c_2(B+C) + c_3(A+C) = 0$$

$$\begin{aligned} A(c_1 + c_3) &= 0 \\ B(c_1 + c_2) &= 0 \\ C(c_2 + c_3) &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underline{0} \ ; \quad \underline{\underline{A}}\,\underline{c} = \underline{0}$$

$$\det(\underline{A}) = (1) + (1) = 2 \neq 0$$

$\{A+B, B+C, A+C\}$ IS LINEARLY INDEPENDENT

45. SOLVE $y''' + y' = 2$

y_H: $(y'' + y)' = 0$

$$y_H = C_1 + C_2 \sin(x) + C_3 \cos(x)$$

y_P: $y''' + y' = 2$

$$y_P = 2x$$

$$\begin{aligned} y(x) &= y_P + y_H \\ &= 2x + C_1 + C_2 \sin(x) + C_3 \cos(x) \end{aligned}$$

III FLUID MECHANICS:

1. EQUATIONS OF A POTENTIAL FLOW
2. EQUATIONS OF VORTICES

$\boxed{\text{: Fluid}}$: <u>Boundary layer</u>:

- Separation
- some questions on <u>order of mag</u>.

Focus shifted from <u>viscosity</u> → potential flow

2-D Potential flow in details.

eg. flow past around circular cylinder

etc. so many. <u>basic</u> questio

(some of them are def. type).

eg. What's the <u>ideal</u> fluid?

Qual. questions 1986 Emily Stone

Fluids

talk - Deriving N-S for Incompressible Constant Property Newtonian Fluid

- Some questions on steps in derivation
- how does viscosity depend on temp.?
- in what cases would derivation be different? (stratified flows, non-isotropic fluids)
- what's kinematic viscosity?
- define circulation.

Topic - Student - Abe

Fluids: (talk given in fluids)

Be careful of general questions on talk topic.

e.g. Balance of: Energy, Momentum, Angular Momentum; constitutative relations.

PROBLEMS

45. $\underline{\underline{A}}_{m\times m}$ HAS EIGENVALUES $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. WHAT ARE THE EIGENVALUES OF $\underline{\underline{A}}^{-1}$?

$$1 = \det(\underline{\underline{I}}) = \det(\underline{\underline{A}}\,\underline{\underline{A}}^{-1})$$
$$= \det(\underline{\underline{A}})\det(\underline{\underline{A}}^{-1})$$

$$= (\lambda_1 \cdot \lambda_2 \cdots \lambda_m)(\lambda_1^{(-1)} \cdot \lambda_2^{(-1)} \cdots \lambda_m^{(-1)})$$

$$\lambda_1^{(-1)} = 1/\lambda_1 \,,\quad \lambda_2^{(-1)} = 1/\lambda_2 \,, \ldots, \lambda_m^{(-1)} = 1/\lambda_m .$$

$\underline{\underline{A}}^{-1}_{m\times m}$ HAS EIGENVALUES $= \{1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_m\}$

46. a. WHAT IS THE (1-D) WAVE EQUATION?

$$a^2 u_{\phi\phi} = u_{tt}$$

b. GIVEN INITIAL CONDITIONS, IF THE SYSTEM IS A CIRCULAR STRING, WHAT ARE THE BOUNDARY CONDITIONS?

$$u(\phi, t) = u(\phi + 2\pi, t)$$
$$u_\phi(\phi, t) = u_\phi(\phi + 2\pi, t)$$

c. SOLVE THIS SYSTEM

$$u(\phi, t) = \Theta(\phi)T(t)$$

$$a^2\Theta_{\phi\phi}T = \Theta T_{tt} \rightarrow \frac{\Theta_{\phi\phi}}{\Theta} = \frac{T_{tt}}{a^2T} = \lambda$$

$$\Theta_{\phi\phi} - \lambda\Theta = 0 \qquad\qquad \Theta(\phi) = \Theta(\phi + 2\pi)$$
$$\Theta_\phi(\phi) = \Theta_\phi(\phi + 2\pi)$$

$\lambda < 0 \quad \lambda = -\mu^2$

$$\Theta_{\phi\phi} + \mu^2\Theta = 0$$

$$\Theta(\phi) = C_1 \sin(\mu\phi) + C_2\cos(\mu\phi)$$

$$C_1 \sin(\mu\phi) + C_2\cos(\mu\phi) = C_1\sin(\mu(\phi + 2\pi)) + C_2\cos(\mu(\phi + 2\pi))$$

$$\mu = n$$
$$\lambda = -n^2 \,, \quad \Theta(\phi) = C_1 \sin(n\phi) + C_2\cos(n\phi)$$

$\lambda = 0$

$$\Theta_{\phi\phi} = 0$$

$$\Theta(\phi) = C_3\phi + C_4$$

$$C_3\phi + C_4 = C_3(\phi + 2\pi) + C_4 \qquad C_3 = 0$$

$$\lambda = 0, \quad \Theta(\phi) = C_3$$

PROBLEMS

46 (CONT.)

$\lambda > 0$

$$\Theta_{\phi\phi} - \lambda\Theta = 0$$

$$\Theta(\phi) = C_5 \exp\{\sqrt{\lambda}\phi\} + C_6 \exp\{-\sqrt{\lambda}\phi\}$$

CAN NOT FIT BC

$$T_{tt} + (n\alpha)^2 T = 0$$

$n = 0$

$$T_{tt} = 0$$

$$T(t) = C_7 t + C_8 \qquad \text{ASSUME BOUNDED BEHAVIOR} \rightarrow C_7 = 0$$

$n \neq 0$

$$T_{tt} + (n\alpha)^2 T = 0$$

$$T(t) = C_9 \sin(n\alpha t) + C_{10} \cos(n\alpha t)$$

$$u(\phi, t) = C_1 + \sum_{n=1}^{\infty} (C_2 \sin(n\phi) + C_3 \cos(n\phi))(C_4 \sin(n\alpha t) + C_5 \cos(n\alpha t))$$

$$= C_1 + \sum_{n=1}^{\infty} \sin(n\alpha t)\left[C_2 \sin(n\phi) + C_3 \cos(n\phi)\right] + \cos(n\alpha t) \cdot$$

$$\cdot \left[C_4 \sin(n\phi) + C_5 \cos(n\phi)\right]$$

d. COULD IT BE SOLVED BY OTHER METHODS?

- D'ALEMBERTS SOLUTION
- METHOD OF CHARACTERISTICS

e. HOW CAN A TRAVELLING WAVE REPRESENT A STANDING WAVE?

$u(\phi, t)$ IS EXPRESSED IN TERMS OF $\{\sin(x \pm \alpha t)\}, \{\cos(x \pm \alpha t)\}$

47. WHY DOES $\dot{x} = \sqrt{x}$ HAVE TWO SOLUTIONS?

THE D.E. IS NOT LIPSCHITZ @ $x = 0$

FLUID DYNAMICS QUESTIONS

Topic –

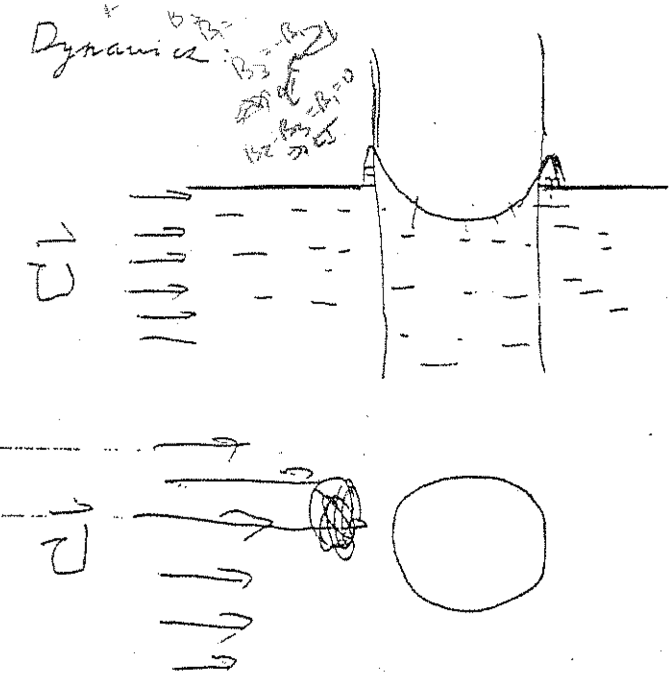
Student – Abe

Fluids: (talk given in Fluids)

Be careful of general questions on talk topic.

e.g. Balance of: Energy, Momentum, Angular Momentum; constitutative relations.

3. Fluid Dynamics



FLUID MECHANICS:

1. EQUATIONS OF A POTENTIAL FLOW
2. EQUATIONS OF VORTICES

1- Presentation in Fluid Mechanics:

Derivation of Navier Stoke's equation

Questions:

a- How is the derivation different for stratified flows
b- Does density depend on P, T, ... (Thermodynamics)
c- Write down Bernoulli's equation, (explain the law).
d- Relate Navier when airplane flies.
e- How is the lift on the wings
f- Can you have Vorticity without Viscosity?
g- Can you draw the velocity field of a common household fan

Continuum mechanics

1) What is stress?

2) What is circulation?

3) Does a wing have circulation? Does a cylinder?
Does each have lift?

III FLUID MECHANICS:

1. EQUATIONS OF A POTENTIAL FLOW

2. EQUATIONS OF VORTICES

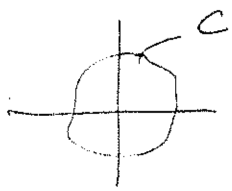
(1) What would $v_{ij} \cdot v_{ij}$ mean?

$$v_{1j} \cdot v_{1j} + v_{2j} \cdot v_{2j} + v_{3j} \cdot v_{3j}$$

$$= v_{11} \cdot v_{11} + v_{12}\, v_{12} + v_{13} \cdot v_{13} + v_{21} \cdot v_{21} + v_{22} \cdot v_{22} + \dots$$

$$v_{ii} \cdot v_{jj} = (v_{11} + v_{22} + v_{33}) \cdot (v_{11} + v_{22} + v_{33})$$

Math.

1. 

f(z) analytic outside C

z_0 a point outside C

$$f(\infty) = a$$

What is $$\oint_C \frac{f(z)}{z - z_0}\, dz$$

[~~I think~~ something is wrong

For ~~consider~~ $f(z) = \frac{z}{z - \alpha}$ $\quad |\alpha| < 1$

~~C ≡ unit~~ circle

the integral is $= \frac{2\pi i\, \alpha}{z_0 - \alpha}$, $\quad f(z_0) = \frac{z_0}{z_0 - \alpha}$

Take mapping $\zeta = \frac{1}{z}$ $\quad f(\infty) =$]

2. What can you say about the following

$$L(u) + \alpha u = 0$$

$u(0) = u_0$

$u(a) = u_1$

L is a linear operator.

1) a) Identify the denominator in the integral representation.

b) What happens when $r = 1$? [δ f^n]

2) Why did you neglect the -ve eigenvalues?

3) Solve $\nabla^2 \phi = 0$ in the annulus given ϕ on the boundaries, by separation of variables.

4) What are the necessary conditions for the Neuman problem to be well posed?

MATH (STROGATZ - MODERATOR, RAND, GUCKENHEIMER, MUKHERJEE, HEALEY, CHAVEZ)

1) $\dot{x} = Ax$ WHERE A IS 3×3 MATRIX

DISCUSS THE BOUNDEDNESS OF THE SOLUTIONS FOR $t \to \pm\infty$

2) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a - \sin\theta}$ SOLVE IT (SEE GREENBERG P. 1251)

3) $\ddot{y} + \alpha^2 y = \sin 2x$, $y = y(x)$

DISCUSS EXISTENCE OF SOLUTIONS

DISCUSS UNIQUENESS OF SOLUTIONS

Bimal Poddar:

Topic: Poission's Integral Representation.

Consider the Dirichlet problem on a unit disc. ~~Derive~~ Poission's integral representation and discuss applications.

Ref: Zachmanoglon-thoe Intro to PDE.

1. (a) Identify the denomenator in the integral representation.

(b) What happens when $r = 1$? [δ fn.]

2. Why did you neglet the -ve eigenvalues?

3. ~~Arsstas~~. Solve $\nabla^2 \varphi = 0$ in the analus given ϕ on the boundries. by separation of variables.

4 What are the necessary conditions for the Neuman problem to be well pased?

b) Can the Neuman problem be solved if $\frac{\partial \phi}{\partial \theta}$ is specified on the boundry?

Ans $\int_0^\theta \frac{\partial \phi}{\partial \theta}\Big|_{r=r_0} d\theta = \phi(\theta)\Big|_{r=r_0} - \phi(0)\Big|_{r=r_0}$ — a Dirchlet prob

Hence know solⁿ to within a constant.

However if the region is an analus, the constant solution is $\phi(r_0,\theta) \ln r$ ....

~~Hence~~

5. What is the Green's function for this problem?

Green's function is the solution to the axisymmetric problem.

Ans 2-D case $\ln r$.

(b) what is it for the 3-D case?

6 Can you give a physical problem with Neuman bd. cond.

39) What is the definition of an analytic function?
What is the condition of differentiability of a complex function?

40) Write the equation of any kind of surface?
What is the physical meaning of directional derivative?

41) What is the definition of $\underset{\sim}{\nabla}$ operator?

42) What is the definition of curl, divergence of cartesian co-ordinates?

43) What is the physical meaning of gradient, divergence curl?

44) State Cauchy's Integral formula & Integral Theorem.

45) Consider $\ddot{x} + \omega^2 x = 0$. What is the solution?

46) Write down the wave equation?
What type of equation is it? How do you solve it?
(Explain all 3 methods).

47) When does the solution to $Ax = b$ exist? Uniqueness?

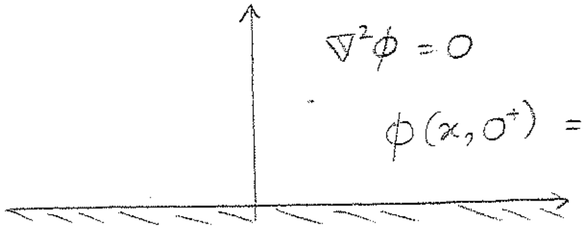
48) Convergence of infinite Power series?

49) Solve the following ODE:

$$x^2 y'' + 2x y' + 4y = \sin x \quad , x > 0$$

Is there more than one way to solve this equation?

50> Solve the following BVP:



$$\nabla^2 \phi = 0 \qquad y > 0$$

$$\phi(x, 0^+) = f(x) \qquad \phi, \phi_{,x}, \phi_{,y} \to 0 \text{ as } (x^2 + y^2) \to \infty.$$

51> Solve the following BVP:

$$\ddot{x} + x = \cos 17t$$

B.C.'s: $x(0) = x(2\pi) = 0$

(Periodic BC's) $x'(0) = x'(2\pi) = 0$ $\Big\} \Rightarrow x(t + 2\pi) = x(t)$

a> Does a solution exist?

b> Is this solution unique?

c> For the general problem: $\ddot{x} + x = g(t)$.
What conditions must $g(t)$ satisfy to ensure existence of a solution.

52> Talk was on steepest descents, & all questions were on the talk.

53> Define: z^α where z & α are complex.

54> Poisson's Integral Representation:

Consider the Dirichlet problem on a unit disc. Derive Poisson's integral representation & discuss applications.

<u>Ref.:</u> Zachmanoglou - thoe. Intro to PDE.

23> What is a differential equation?

Discuss about solution, existence, uniqueness, difference between PDE & ODE, difference between IV problems & B.V. problems. Constraints on λ in $y'' + \lambda y = 0$ with different I.C & different B.C., methods of solutions. Numerical methods.

24> Solve : i^i

25> $\displaystyle\int_{|z|=1} \frac{dz}{z^n - a}$ n is positive integer >1, $a \in \mathbb{R}$, $0 < a < 1$.

26> Solve : $y''' + y = 0$; $y(0) = y(\pi) = y'(\pi) = 0$

How does this differ (only zero solution) from most math problems?

(Want something about eigenvalue problems).

27> $\underline{\underline{A}}_{n\times n}$ has eigenvalues $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$. What are the eigen-values of A^{-1}?

28> Write down 1-D Wave Equation.

29> Why does $\dot{x} = \sqrt{x}$ have two solutions?

30> $\displaystyle\int_0^{\infty} \frac{1}{x^3+1}\, dx = ?$ (Hint: $\displaystyle\oint_C \frac{\ln z}{z^3+1}\, dz$).

31> Tell about Contour Integrals, Residue Theorem, Indented Contours.

32> How do Runge-Kutta, Euler methods work?
Error estimate?

33> How do you get from:

$$I = \int_{t_1}^{t_2} f(x; \dot{x}; t)\, dt$$ to Euler Equations.

(Calculus of variations).

34> $\int_C \frac{dz}{z^m}$ over $|z| = 1$

35> Write a differential equation for heat conduction in a circular ring. Solve the equation given:

$\theta(t=0) = g(\theta)$ $\qquad u_{\theta\theta} = \frac{1}{\alpha^2} u_t$ take $\alpha^2 = 1$.

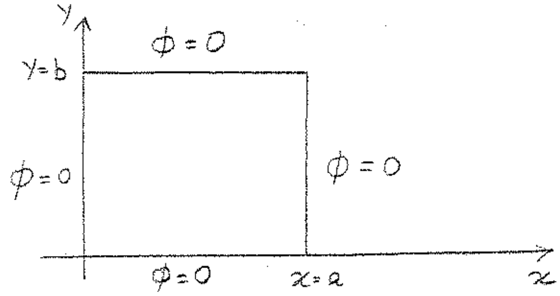
Only Initial conditions given. Find the B.C.?
What are the eigen values? eigen functions?
What will the temp. distribution be at $t \to \infty$

36> What is the solution of:

$$y^{iv} + y'' = 0$$

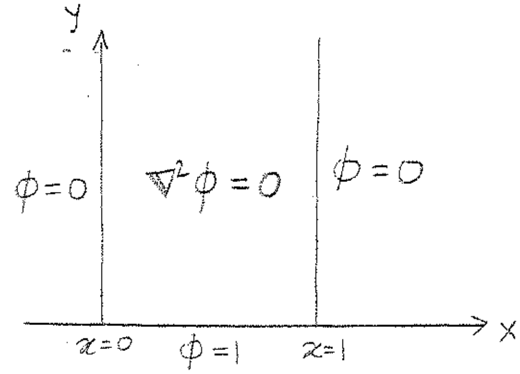
37> $\int_0^\infty \frac{dx}{1+x^4} = ?$

38> Solve $\nabla^2 \phi = 0$ on:



Show technique of choice!

9) Solve the following BVP by elementary variables (as complex variables).



$|\phi|$ is bounded at $y\to\infty$.

10) $\underline{\underline{A}}$ is a real $n\times n$ matrix. What are the least restrictive constraints on $\underline{\underline{A}}$ that will allow you to diagonalize it?

11) Solve: $\displaystyle\int_0^{\infty} \frac{\sin \lambda x}{x}\, dx$

12) Are the square matrix and the 2nd order tensors the same thing?

13) Describe the solvability of:

$$y'' + \lambda y = g(x) \qquad y(0) = y(1) = 0 \qquad 0 \le x \le 1$$

for various values of λ ($\lambda \in \mathbb{R}$)

14) In some there exists 3 vectors $\underset{\sim}{v}_1; \underset{\sim}{v}_2; \underset{\sim}{v}_3$ each can be expressed as a linear combination of unit vectors $\underset{\sim}{e}_1; \underset{\sim}{e}_2; \underset{\sim}{e}_3$. What conditions must be satisfied that $\underset{\sim}{v}_1; \underset{\sim}{v}_2; \underset{\sim}{v}_3$ are linearly independent.

15) $y(x) = \begin{cases} 0 \ldots\ldots & -1 \le x \le 0 \\ 1 \ldots\ldots & 0 < x < 1 \end{cases}$ What is the value of the Fourier series of $y(x)$ at the points: a) $x = -1$ b) $x = +1$ c) $x = 70.256$

16) $f(x) \geq 0 \qquad 0 < x < 1 \qquad f(0) = f(1) = 0$

Maximize area under the curve $f(x)$ subject to the constraint that length of the curve is $L = \pi/2$.

17) What is a second order tensor? How is it represented? Given an 2nd order tensor $\underset{\sim}{T}$ and a set of 3 vectors $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ & $\underset{\sim}{v}_3$ on $\mathbb{R}^n$ with $\underset{\sim}{v}_i$ LI and the following

$$\underset{\sim}{T}\,\underset{\sim}{v}_1 = -\underline{v}_1 + \underline{v}_3$$

$$\underset{\sim}{T}\,\underset{\sim}{v}_2 = 2v_2$$

$$\underset{\sim}{T}\,\underline{v}_3 = \underline{v}_2 + \underline{v}_3$$

What are the elements of $\underset{\sim}{T}$?

18) $$\int_0^{\infty} \frac{\cos x}{x^2+1}\, dx = ?$$

19) $$\int_{-\pi}^{\pi} f(x)\, dx = 0 \qquad \int_{-\pi}^{\pi} [f(x)]^2\, dx = \int_{-\pi}^{\pi} [f'(x)]^2\, dx$$

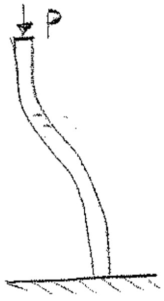
$f(x)$ is not the zero function. What are the restrictions on f? (Hint: use fourier series).

20) Given $\lim_{z \to 0} f(z) = 0$ and $g(z)$ is bounded at neighbourhood of $z=0$. Find $\lim_{z \to 0} f(z)\,g(z) = 0$ and explain!

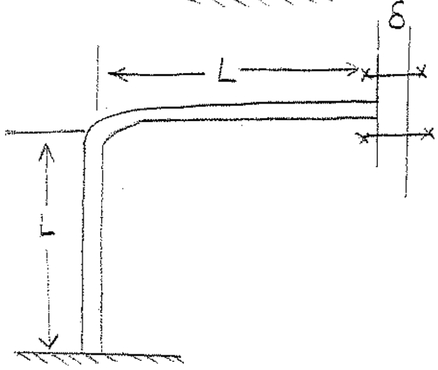
21) A, B, C are LI variables. Are A+B, B+C, A+C are also LI?

22) Solve: $y''' + y' = a$

4> What is wrong here? Why?



5>



Bolts are tightened to close the gap.

In the configuration shown, stresses in the pipe are zero. If the two bolts are tightened.

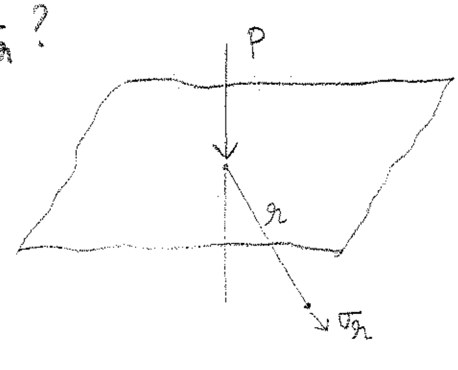
a) Draw the free body diagram.

b) How would you go about solving for the unknown reactions?

6> What is needed to solve an elasticity problem?
(equil., eqn's ; compat. eqn's, constitutive reln's ; BC's ....).

7> What is the meaning of constitutive relations?
Are they always satisfied?
Are the compatibility equations always needed?
Why does different problem have different solutions?

8> Point force 'P' on a half space. What is the singularity of σ_r?



Hint: Look at half sphere of radius "r".

$P = \sigma(r) A$

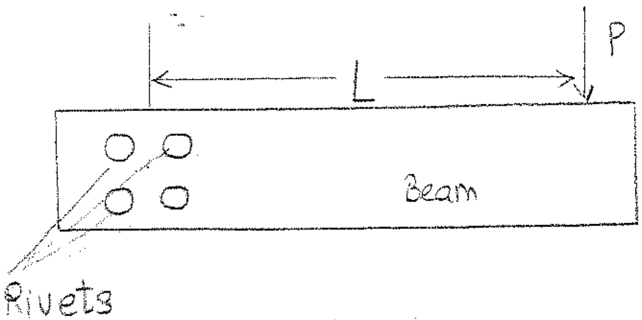
$\sigma_r \propto \frac{1}{r^2}$.

How is σ_r varying with r?

9.> What is a gage factor? $c =$ [illegible]

Can a strain gage be used in dynamic problems? ~~No.~~ Yes

10.>



a.) Find Rivets load

b.) What if Rivets are at different sites?

c.> What about Friction?

11.> What are the steps in solving a 3-D problem in Linear, isotropic, small strain elasticity? What are the required equations, explain?

Also,

a.) Is there different possibilities for B.C.? (other than: $\sigma_{ij} n_j = S_i$)

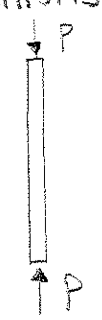
b.> How do you take care of such B.C.?

c.> Why are compatibility conditions not required in displacement formulation?

d.> What does this tell us about compatibility conditions?

12.> Buckling of a slender elastic bar:

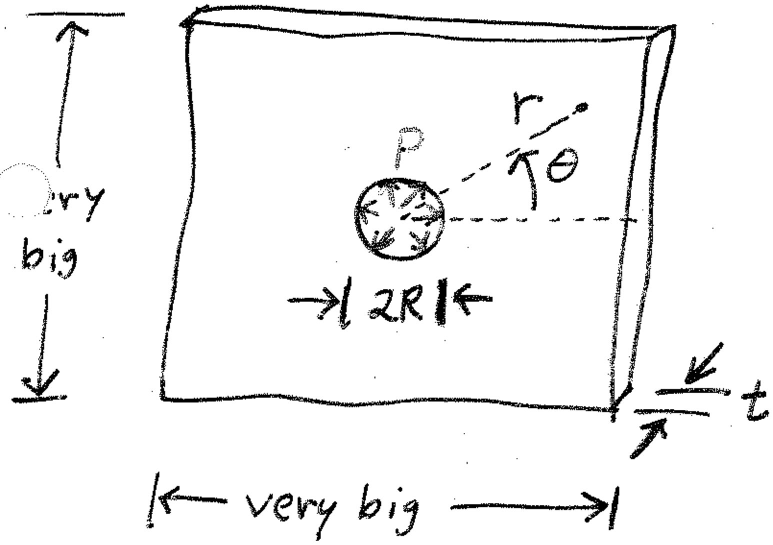
a.) Different modes of buckling w.r.t. different boundary conditions. Describe them!



b.) What is the governing differential equation for this phenomena?

c.> What is buckling?

d.> What are the other effects which affect this phenomena?



Pressure P in hole with radius R in a thin linear-isotropic-elastic plate with thickness t.

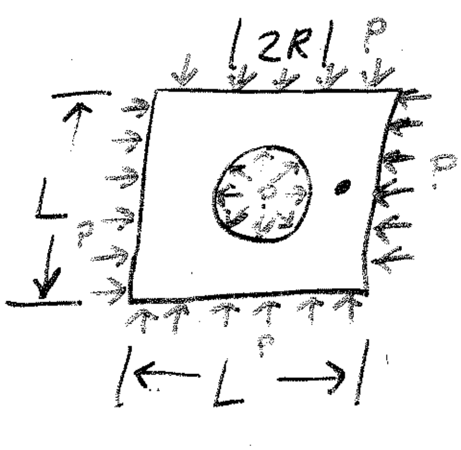
a) Draw a FREE-BODY-DIAGRAM of a differential element showing σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$. (No equations, no calculations).

b) What is $\sigma_{r\theta}$ for the problem above? Why?

c) Roughly (no detailed calculation) how do σ_{rr} and $\sigma_{\theta\theta}$ depend on r and θ?

d) How might you find an exact solution for σ_{rr} and $\sigma_{\theta\theta}$ (assuming small strain, isotropy, etc.)?

e)



1) What is (approximately) $\sigma_{\theta\theta}$ at the point shown?

2) How accurate is your guess?

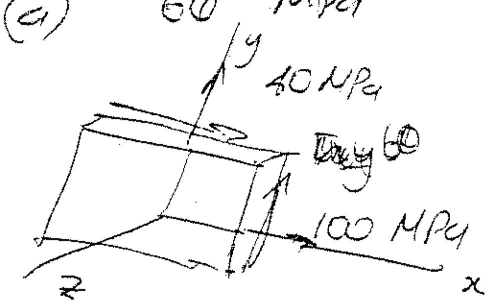
3) What if the material was non-linear, anisotropic or inhomo- genius?

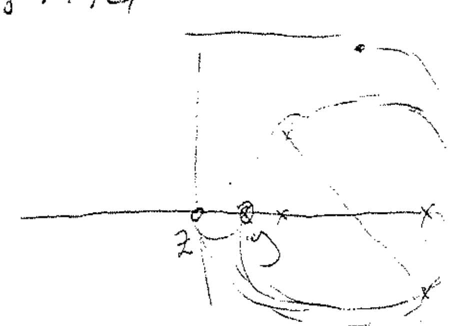
Questions : Solid Mechanics Duncan O.

(1) A point load is applied on a half space. Derive an expression for stresses at a point (r, θ). What happens to stresses at $r = 0$, $r = \infty$?

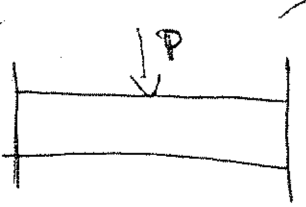
(2) What is needed to solve ~~a~~ a 3-D elasticity problem?
Are compatability equations always needed? If not under what conditions are / aren't they required?

(3) For state of plane stress shown, determine value of τ_{xy} for which maximum shearing stress is (a) 60 MPa (b) 78 MPa





(4)



(i) What degree of indeterminacy?
(ii) Can we solve for equation of deflection
(iii) Where do remaining equations needed to determine unknowns come from?

II. Solid Mech!

1) Interpret cpts of $\underline{E} = \frac{1}{2}(\underline{C} - \underline{I})$

talk about chgs in length volume etc.

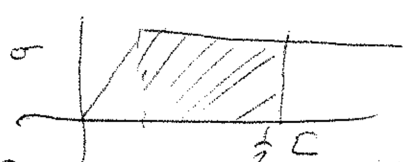
2) Elastodynamic state – no body forces

$$t_{ij,j} = \rho \ddot{u}_i$$

Suppose $\underline{u} = u\underline{e}_2$ where $u = u(x_1)$. Interpret?

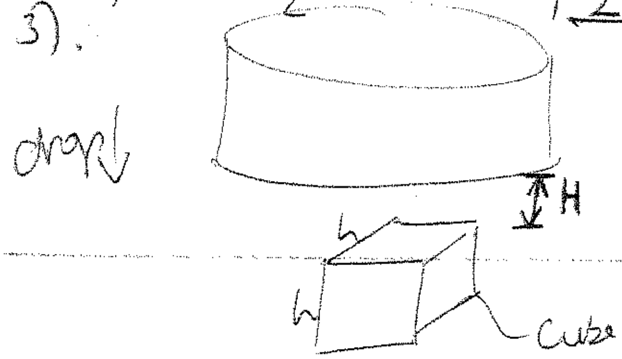
(ans: equil eqns + constitutive eqn give $u_{xx} = \frac{\rho}{\mu} u_{tt}$

– wave eqn).

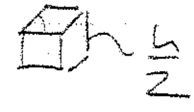


$\boxed{\frac{1}{2}\sigma_{11}\varepsilon_{11} = mgH}$ $\varepsilon_{11} = \frac{1}{2} = \Delta$

3).



⇒ squashes cube!



Calculate initial height H required to squash cube into one with exactly half the dimensions.

(15 min) <u>SIMPLE</u> <u>COMPOSITE</u>

What, approximately, is the increase in length of the composite rod shown?

F = applied load

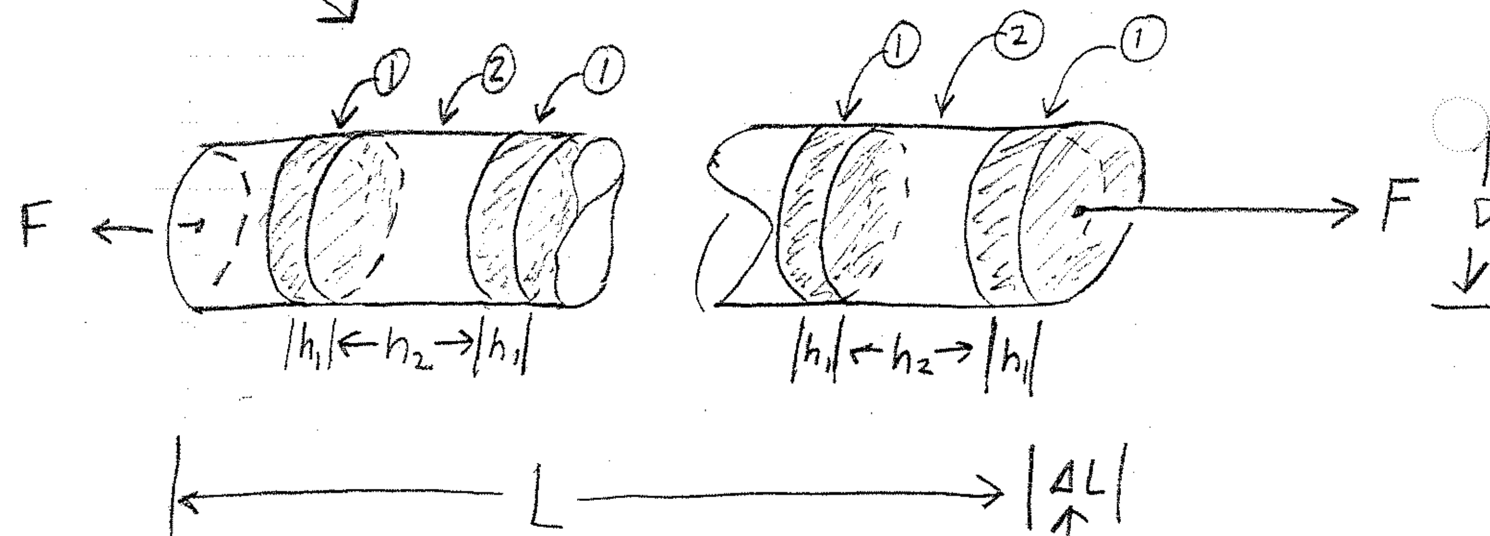
E_1, ν_1, G_1 = moduli of material ①

E_2, ν_2, G_2 = moduli of material ②

D = diameter of disks (both)

L = length of total rod $\gg h_1$, $\gg h_2$

Stack of Disks

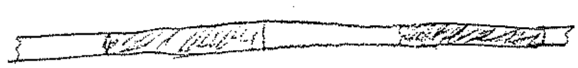


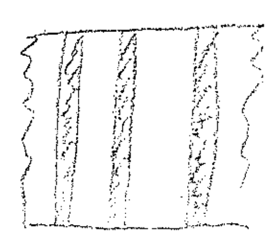
?

Consider two special cases:

a) $D \ll h_1$
$D \ll h_2$

a) $D \gg h_1$
$D \gg h_2$





1

0

(16) Given the displacement field $w = f(x,y)$ $\boxed{u = v = 0}$

What are the governing equations? (Ans. $\nabla^2 w = 0$

Which σ, γ are zero?

What is the physical meaning of the existence condition for the Neumann problem?

(Ans. equilibrium

How do you measure E and G experimentally

Elastic Waves

(17) Two equal rods of length L are moving longitudinally with velocities $\pm U$ along the x axis. At $t = 0$ they collide at the origin.

a) Determine the subsequent motion (assuming it is longitudinal) when the rods are elastic and do not adhere?

b) What happens when the deformation has a lateral component or there is adhesion?

c) What are the different approaches to derive the equations of motion and to solve them?

d) What if the rods are not one dimensional?

e) What if the rods are not of equal length?

f) What can you say about the problem of a rod colliding against a wall?

(20) What is the momentum Balance Law?
What do the terms mean?
From what principle is it derived?
What is balance of angular momentum
What does it tell us about stress tensor?

1

663

(21) State the force balance equation.

Derive the equat. for force balance in continuum in 2-D in cartesian coord.

(22) What is continuum?

What parameters are involved in continuum mechanics?

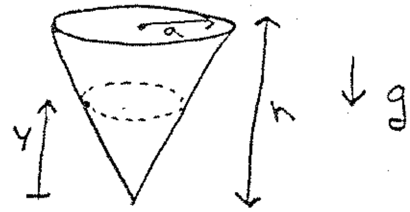
Give an example of a boundary value problem and the equations necessary to solve it.

1

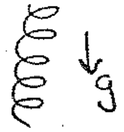
(23) In a 2-D body with a point load at the origin, how do stress, displacements vary with r?

In a 3-D body?

Sample Dynamics Questions (Not from any exam)

1. 

A particle travels in a circular path on the inside of a cone. Find its speed v as a function of y.

2. 

A particle travels along a helical spring, until it reaches the end of the spring and falls off. What is its trajectory?

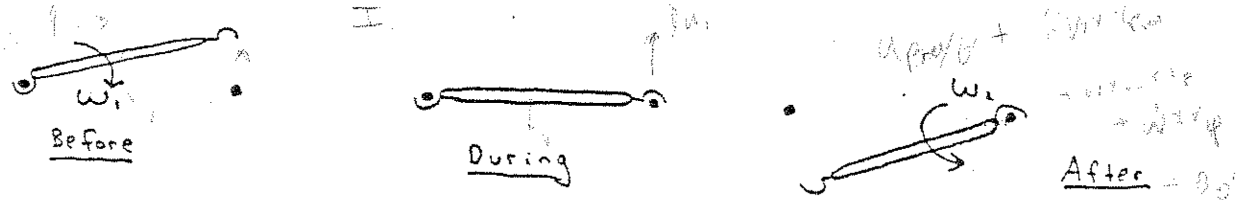
3. State Newton's 3 laws; D'Lambert's Principle.

4. Find the altitude of a geosynchronous orbit.

5. A single propeller airplane is making a right turn. The propeller is spinning counter-clockwise as viewed from behind. What is the gyroscopic effect? How could you compensate on a multi-engine craft.
Answers: Nose rises. Have the propellers spin in opposite directions.

6. Explain the 5 term acceleration formula. What is the Coriolis acceleration?

7. A motor is sitting on a turntable. It spins at ω_1, and its shaft is horizontal. The turntable spins at ω_2. Find $\underline{\omega}$, $\underline{\alpha}$ of the shaft.

8. 

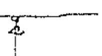
Show that $\omega_2 = \frac{1}{2}\omega_1$. Rigid rod, $\bar{I} = \frac{1}{3} m L^2$, $I_{end} = \frac{1}{12} m L^2$

Jan 21.

1. Dynamics.

Examinee G Zhang

Questions: 1) A rule initially lies on a table supported by hand, then withdraw the hand. ~~Descript~~ Derive equations of motion.



What is the relationship between the angular velocity $\vec{\omega}$ and the velocity of mass center?

2)  Derive the velocity and acceleration!

3)  given constant $\vec{\omega}$, find θ.

How many method can you use

4). There was an experimental set-up. an oscilloscope and a hammer attached to it. Prof Sachse gives a small blow to the table with hammer and asks to explain what the pulse on the screen indicates?

What is inside the hammer?

Write down the equation for the mechanism inside the hammer. Explain how to measure the force?

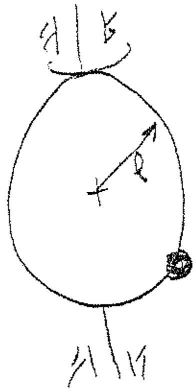
5). What is the difference between Newton's method and Lagrange equation method.

What is constraint?

Dynamics

i) Rand had a baseball book that says ~~for~~ maximum distance, one must hit the ball at an angle ___ .
What is the angle.
How would drag affect this angle.
(The book says 35°, why?,

ii)



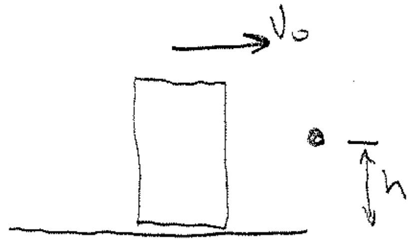
Bead on a hoop which is free to spin about axis shown

How many D.O.F.?

Find Equations of Motion

Is anything Conserved?

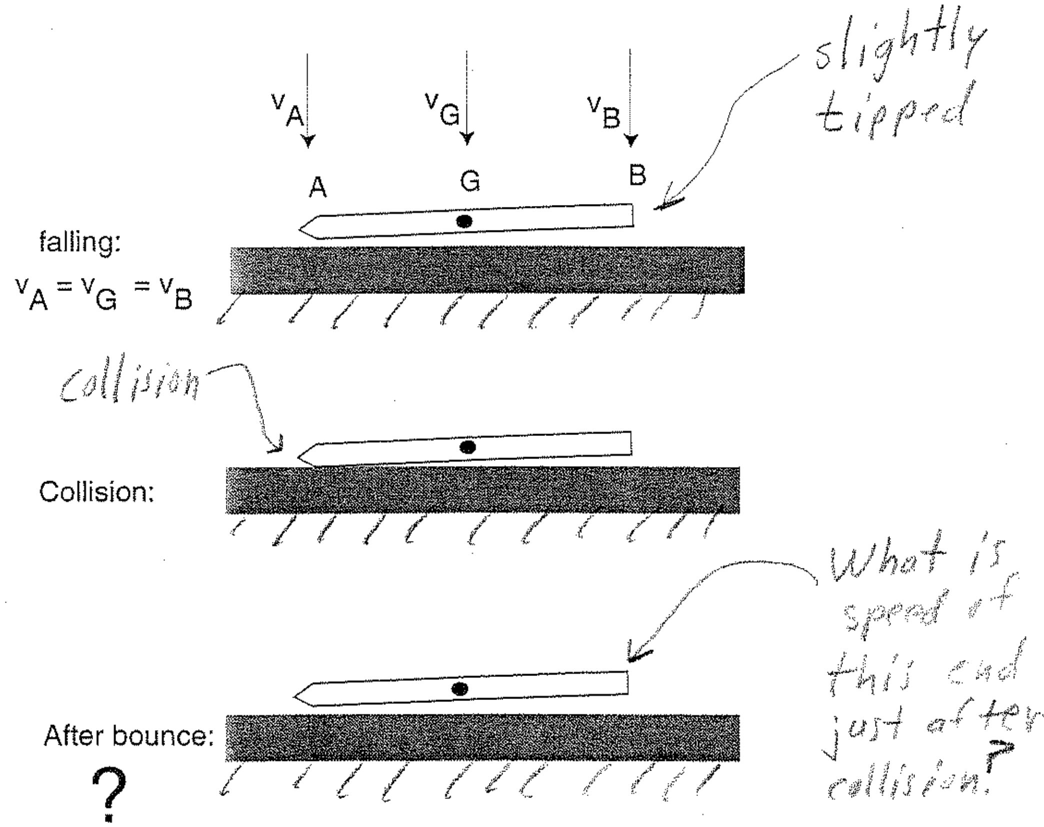
iii)



A cylinder slides into a horizontal rod at height h with velocity V_0.
What should h be so the can does not tip over.

...Dr. Goyal runs an animated picture of a pencil dropping: The entire pencil falls at the same velocity but when one end hits first, the follow-up blow on the other end can occur at up to twice the speed. That's because the end that hits first bounces back at the same velocity, *thereby doubling the speed of the opposite end.* — Wall Street Journal, December 9, 1993

True or False? Why?

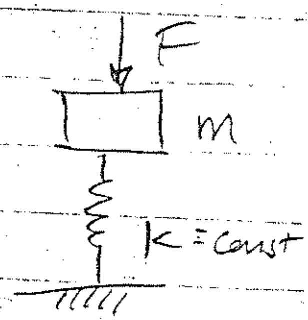


"follow up blow can occur at up to twice the speed" means

speed of V_B can increase by up to a factor of 2.

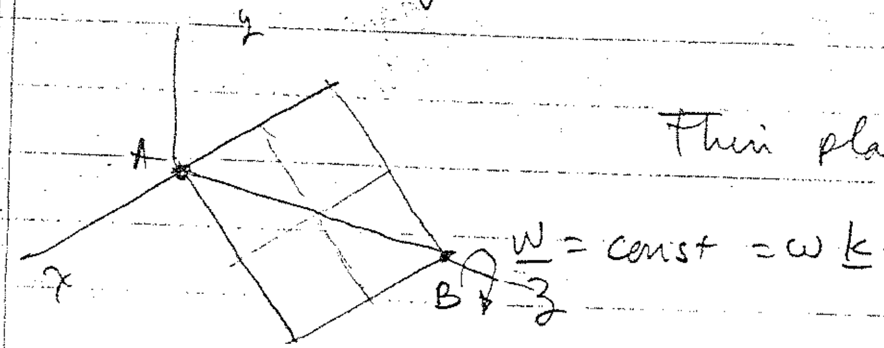
Dynamics

(1)



- Impact load is applied
- How do you solve for displacement of mass?
  ($m\ddot{x} + kx = F\delta(t - t_0)$ ← supplied by student) (F → const.)
- what are initial conditions?
- what is the solution?
- How do you apply boundary conditions?
  How do you solve for arbitrary excitation

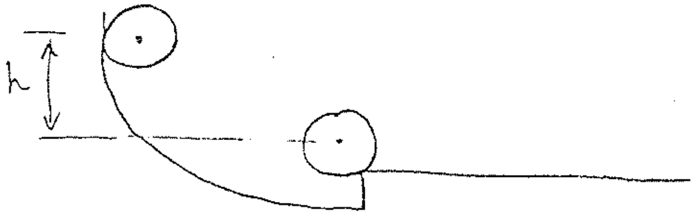
(2)



Thin plate in y-z plane

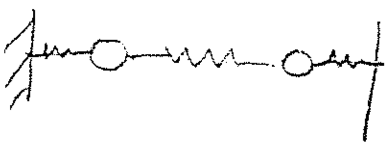
How do you find reactions at A and B?
(dynamic reactions)

Dynamic



talk about how
you'de solve this
problem

- Describe what happens when you try to turn a spinning wheel - (gyroscopes)

- 

describe motion
how to solve?

Define $\quad G = \dfrac{E}{2(1+\nu)}$

G - modulus of elasticity in shear
- " " rigidity

$\therefore \quad \gamma = \dfrac{\tau}{G}$

$\gamma_{xy} = \dfrac{1}{G}\tau_{xy}$ etc.

Now volume expansion $\dfrac{\Delta V}{V} = e = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z) -$

$\approx \epsilon_x + \epsilon_y + \epsilon_z + H.O.T.$

$\therefore \quad e = \epsilon_x + \epsilon_y + \epsilon_z = \dfrac{1}{E}[\sigma_x - \nu(--)] + \dfrac{1}{E}[\sigma_y - \nu(--)] + \dfrac{1}{E}[\sigma_z - \nu(--)]$

$\Rightarrow \quad e = \left(\dfrac{1-2\nu}{E}\right)\Theta \qquad [\Theta = \sigma_x + \sigma_y + \sigma_z]$.

Uniform hydrostatic pressure $\Rightarrow \sigma_x = \sigma_y = \sigma_z = -p$,

$\Rightarrow \quad e = -3\dfrac{(1-2\nu)}{E}p$.

$K = \dfrac{E}{3(1-2\nu)}$ = modulus of vol. expansion.

From $\epsilon_x = \dfrac{1}{E}[\sigma_x - \nu(--)]$ etc. and $e = \epsilon_x + \epsilon_y + \epsilon_z$

we get

$\sigma_x = \left(\dfrac{\nu E}{(1+\nu)(1-2\nu)}\right)e + \left(\dfrac{E}{1+\nu}\right)\epsilon_x$ etc.

Let $\quad \lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$, also $\dfrac{E}{1+\nu} = 2G$.

$\Rightarrow \quad \sigma_x = \lambda e + 2G\epsilon_x$ etc.

$\therefore \quad \Theta = (\sigma_x + \sigma_y + \sigma_z) = \underbrace{(3\lambda + 2G)}_{K}e$

Hooke's Law: Linear relations between comp. of stress & strain.

(E - modulus of elasticity in tension.

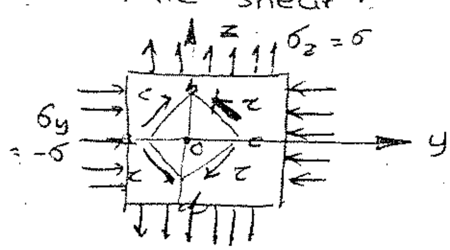
$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$, ν - Poisson's ratio.

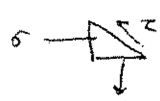
etc.

Method of Superposition - Valid as long as deformations are small & do NOT affect forces causing them. Then calc. are based on initial dimensions & shape of body. Then resultant displacements g... by superposing lin. f^{ns} of ext. forces.

(eg where superpos can't be used: Simultaneous axial & late... forces on thin bar. Effect of deformation on moment of external forces must be considered though def. is small. Then def $\neq$ lin. f^{n} of fo... $\Rightarrow$ cannot get by simple sup.

Pure shear: no normal stress on face.





$\leftarrow$ balancing, $\tau = \sigma$.

Angle bet. bc & ab changes After de...

$$\frac{Oc}{Ob} = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{1+\epsilon_y}{1+\epsilon_z}$$

But
$$\epsilon_y = \frac{1}{E}\left[-\sigma - \nu\sigma\right] = -\frac{(1+\nu)\sigma}{E}$$
$$\epsilon_z = \frac{1}{E}\left[\sigma - \nu(-\sigma)\right] = \frac{(1+\nu)}{E}\sigma$$

$$\therefore \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\gamma/2}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma}{2}} \approx \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}$$

Comparing $\frac{\gamma}{2} = \frac{(1+\nu)}{E}\sigma^{=\tau} \Rightarrow \gamma = \frac{2(1+\nu)}{E}\tau$

II DYNAMICS

✓1. DERIVE THE EQUATIONS OF A PENDULUM

✓2. DERIVE THE EQUATIONS OF A DAMPED PENDULUM

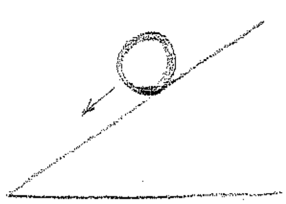
✓3. DERIVE THE EQUATION OF A FALLING CHAIN

✓4. STABILITY OF AXES OF ROTATION OF A BOOK

Generally level of T&AM 203

B. Dynamics

1.



1.1 Solve the problem. assume

(i) with slippage (ii) without slippage

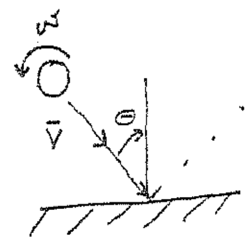
1.2 What are the restrictions on

$$\underset{\sim}{T} = \frac{dL}{dt}?$$

1.3 Where is the above equation derived from?

Dynamics

1) Hockey puck impacts a wall
what is its direction, velocity
and spin when it leaves?
normal collision is elastic.



2) Bang on the end of a rod with a hammer. Listen to
the tone. What is the bar made of? How can
you tell.

2 - Dynamics

a - What is the equation for simple pendulum
b - write down the equation motion for a pendulum
whose length varies with time.
c - suppose you have a Rope of constant mass, you lift
it from the table with Constant F, neglect g, calculate
V in time.

Topic -

Student - Tapesh

Dynamics:

③ Explain Foucault's Gyroscope

Q-Exam Questions — Jeffrey Nussbaum

Dynamics

Presentation – Dynamics of the spinning top

Questions essentially on talk:

Euler's equations – what are they, where do they come from, why use Euler angles instead of Euler equations?
What if top is not axi-symmetric? What if there is friction
Is conservation assumed or derived? Show how to derive.
Why doesn't a stabilizing gyro slow down?

Tunnel drilled through the earth. Describe motion
(Simple harmonic, period = 84 minutes.)

Math

Why does $\dot{x} = \sqrt{x}$ have 2 solutions
$\int_0^\infty \frac{1}{x^3+1}\,dx$

Tell about contour integrals, Residue Theorem, Indented contours.

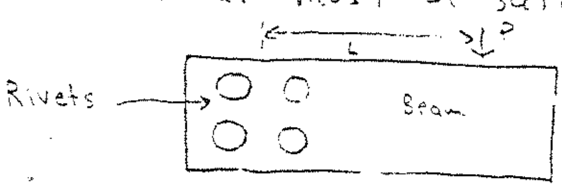
How do Runge-Kutta, Euler methods work? Error estimate?

How do you get from $I = \int_{t_1}^{t_2} f(x, \dot{x}, t)\,dt$ to Euler equation
(Calculus of Variations)

Elasticity

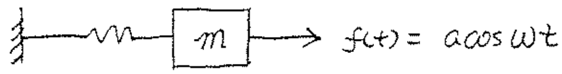
What is a gage factor? Can a strain gage be used in dynamic prob

Suppose $\underline{\underline{\sigma}} = \overset{\text{given}}{\text{constant}}$ is the stress tensor. Is this OK?
What must be satisfied? Why? (Sym., Equil., $T_i = \sigma_{ij} n_j$)



Find rivet loads, what if rivets are different sizes, what about friction?

2.



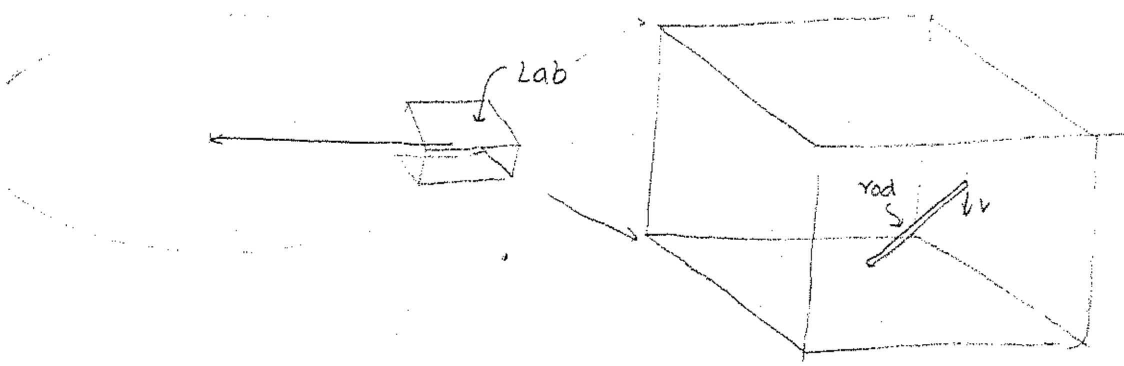
2.1. What does "resonance" means?

2.2. If resonance occures, what does the response look like?

2.3. If w is a little bit different from w_c (natural frequency), what would the response look like?

2.4. If add a damper to the system, what is the relationship between the phase lag and the driving frequency ω?

3.



3.1. Does the rod have angular accerlation?

3.2 How to measure angular accerlation?

3.3 How to measure linear accerlation by using a force sensor?

4.) $\int_{-\pi}^{\pi} f(x)\,dx = 0$

$$\int_{-\pi}^{\pi} [f(x)]^2\,dx = \int_{-\pi}^{\pi} [f'(x)]^2\,dx$$

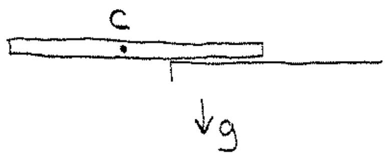
f is not the zero function

What are the restrictions on f ?
Use fourier series

(Answer is $f(x) = A\cos x + B\sin x$ is only possibility)

Dynamics

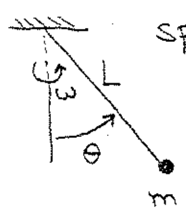
1.)



Ruler is released from position above.
Initially, the point of the ruler touching the corner of the table sticks, but at some time it starts to slide.

Describe how you would set up this problem.

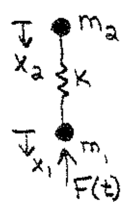
2.)



spherical pendulum ($\dot{\theta} = 0$)
m is given an initial velocity v. find the period.

If the mass is given a slight disturbance ($\dot{\theta}$ no longer $= 0$) find the period.

3.)



set up equations of motion.

Given $(\ddot{x}_1 - \ddot{x}_2)$ how would you find $F(t)$?

4.) What is the difference between Lagrange's Eqns. and Newton's Eqns. ?

TAM 1370

December 1973

I. COLLISIONS OF RIGID BODIES.

Two bodies of mass m_1 and m_2 collide as shown in the figure. At the moment of collision, they contact at point P with a common tangent plane and a normal, $\underset{\sim}{n}$, to that plane. We are in general given:

$$\underset{\sim}{v}_1 , \underset{\sim}{v}_2 , \underset{\sim}{\omega}_1 , \underset{\sim}{\omega}_2 \text{ before impact,}$$

and wish to determine:

$$\underset{\sim}{v}_1' , \underset{\sim}{v}_2' , \underset{\sim}{\omega}_1' , \underset{\sim}{\omega}_2' \text{ after impact.}$$

The twelve unknown quantities (4 vectors) are related to the given quantities by the empirical law of collision and equations for impulsive motion:

$$\int_0^{\Delta t} \underset{\sim}{F}\, dt = [m \underset{\sim}{v}_c]_0^{\Delta t} \qquad \text{Linear momentum}$$

$$\int_0^{\Delta t} \underset{\sim}{M}\, dt = [\underset{\sim}{H}]_0^{\Delta t} \qquad \text{Angular momentum}$$

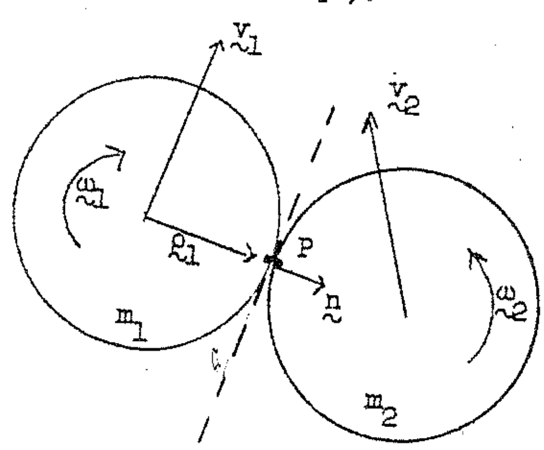
The twelve equations are:

(1) Law of Collisions - When two bodies collide, the values of the normal component of the relative velocity of the surfaces in contact at instants immediately after and immediately before the impact bear a definite ratio to each other; this ratio, denoted by -e, depends only on the material of which the bodies are composed (one eq.).

$$-e = \frac{\underset{\sim}{v}_{1p}' \cdot \underset{\sim}{n} - \underset{\sim}{v}_{2p}' \cdot \underset{\sim}{n}}{\underset{\sim}{v}_{1p} \cdot \underset{\sim}{n} - \underset{\sim}{v}_{2p} \cdot \underset{\sim}{n}}$$

where

$$\underset{\sim}{v}_{1p} = \underset{\sim}{v}_1 + \underset{\sim}{\omega}_1 \times \underset{\sim}{\rho}_1 \quad \text{etc.}$$



common tangent plane

(2) Constancy of angular momentum of each body about the point of contact, P, because of zero moment about p (six eqs.).

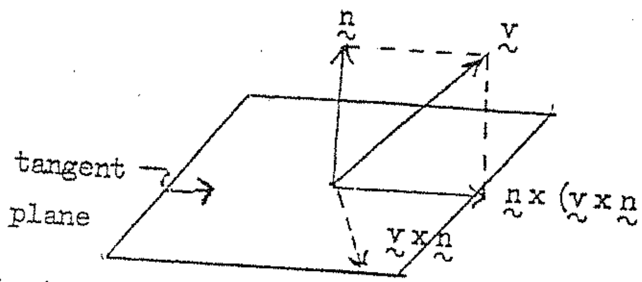
$$\underset{\sim}{L}_p = 0 .$$

(3) Constancy of linear momentum of the system (two bodies) normal to the surface of contact because of equal and opposite normal impulses (one eq.)

$$(m_1 \underset{\sim}{v}_1 + m_2 v_2) \cdot \underset{\sim}{n} = (m_1 \underset{\sim}{v}_1' + m_2 \underset{\sim}{v}_2') \cdot \underset{\sim}{n} \quad .$$

(4A) For smooth surfaces - No change in tangential components of the linear momentum for each body because of zero tangential forces (four eqs.)

$$\underset{\sim}{n} \times (m\underset{\sim}{v}' \times \underset{\sim}{n}) - \underset{\sim}{n} \times (m\underset{\sim}{v} \times \underset{\sim}{n}) = 0$$



(4B) For Rough Surfaces without Slipping at the Contact Point

(i) Constancy of tangential components of the linear momentum of the system because of equal and opposite tangential impulses (two eqs.)

$$\underset{\sim}{n} \times [(m_1\underset{\sim}{v}_1 + m_2\underset{\sim}{v}_2) \times \underset{\sim}{n}] = \underset{\sim}{n} \times [(m_1\underset{\sim}{v}_1' + m_2\underset{\sim}{v}_2') \times \underset{\sim}{n}]$$

(ii) Vanishing of the tangential components of the relative velocity of the two bodies after impact because of no slipping constraint (two eqs.)

$$\underset{\sim}{n} \times [(\underset{\sim}{v}_{2p}' - \underset{\sim}{v}_{1p}') \times \underset{\sim}{n}] = 0 .$$

(4C) For Rough Surfaces with Slipping at the Contact Point

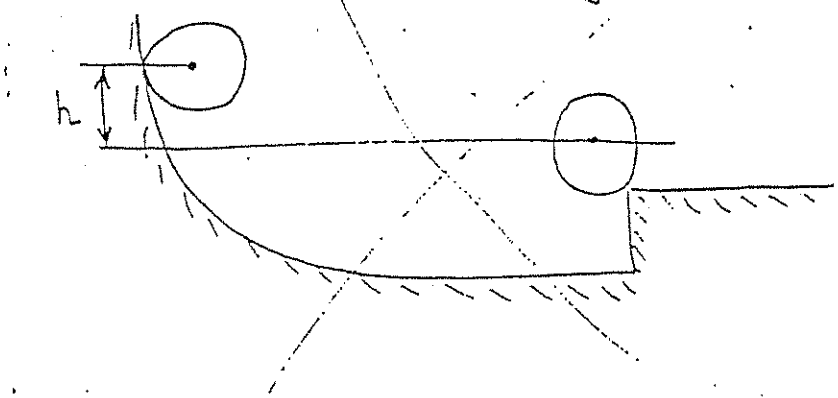
(i) Same as (4Bi) (two eqs.)

(ii) Change of the components of the linear momentum by the tangential impulse which equals, in magnitude to μ (coefficient of friction) times the normal impulse for each body (two eqs.)

$$|\underset{\sim}{n} \times [m(\underset{\sim}{v}' - \underset{\sim}{v}) \times \underset{\sim}{n}]| = \mu m(\underset{\sim}{v}' - \underset{\sim}{v}) \cdot \underset{\sim}{n}$$

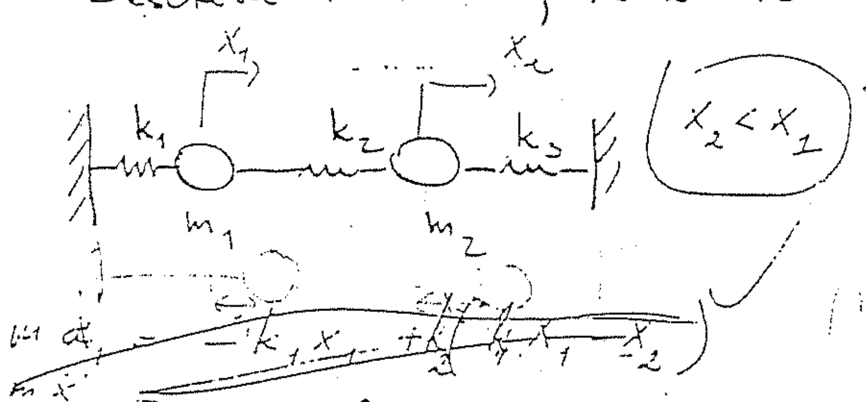
Note that there are two equations of the above form for each body; a total of 4 equations is obtainable. However, only two (for either body) are independent equations.

(46) How would you solve this problem?



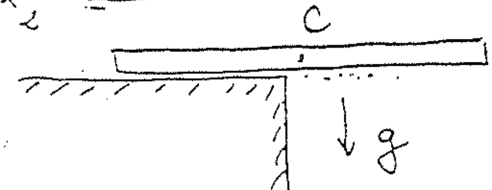
(47) Describe what happens when you try to turn a spinning wheel (gyroscopes)

(48) Describe motion, how to solve?



$m_1\ddot{x}_1 = -k_1 x_1 - k_2(x_2 - x_1)$

(49)



Ans: P.T.O.

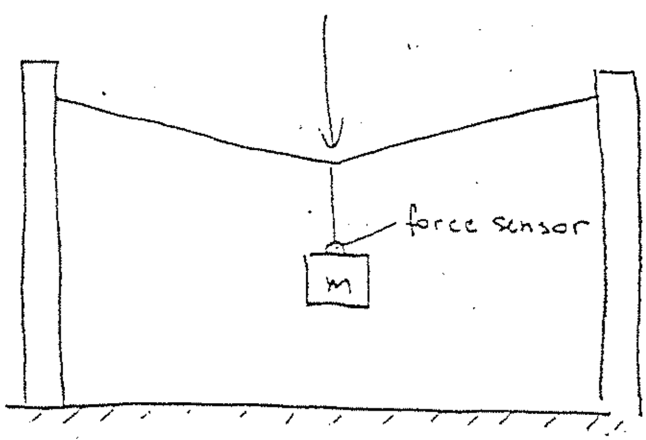
What is the relationship between the ang. vel. $\underset{\sim}{\omega}$ and $\underset{\sim}{v}$ of C
Ruler is released from position above.
Initially, the point of the ruler touching the corner of the table sticks, but at some time it starts to slide.
?

Discuss nature of angular momentum and relevant equations for a spacecraft.
What are the equations? What about other formulations?

2

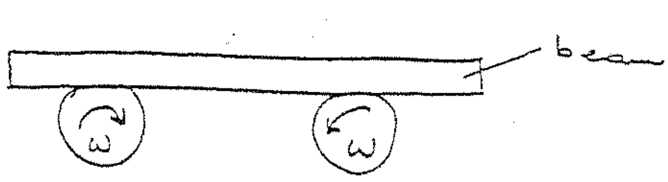
43 What constraints are there on the choice of reference frames for using the equations
$\Sigma H = \dot{H}$, $\Sigma F = ma$, ?
What is inertial reference frame? 2

44



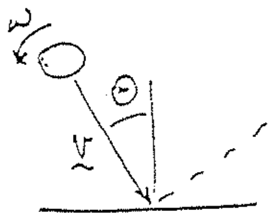
One pulls mass and released it,
What is the nature of the motion?
What is the equation of motion?
When force has max. amplitude?
How would you verify that the motion obeys the solution to the equation of motion?

45



There is a friction between rollers and beam.
What is the nature of the motion and eq. of

(39) Rocket puck impacts a wall.
What is its direction, velocity and spin when it leaves?
Normal collision is elastic.



2

$v_n' = -v_n$

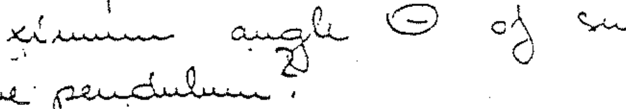
$v_t' = v_t$

$\bar{I}\omega' = \omega$

(40) Bang on the end of a rod with a hammer.
Listen to the tone, What is the tone made of? How can you tell?

$\sqrt{\frac{E}{\rho}}$

1



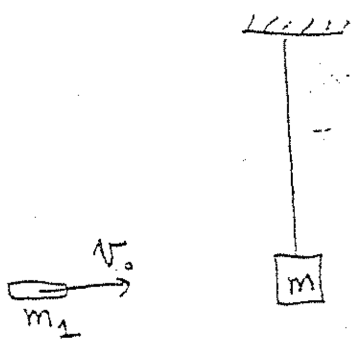
@ $x = 0$: $u(X) = 0$

@ $x = c$: $u'(x = L) = 0$

$u(x, t) = f(L, c, \omega$

(41) Bullet strikes pendulum and is embedded.
What is maximum angle Θ of subsequent motion of the pendulum?

1



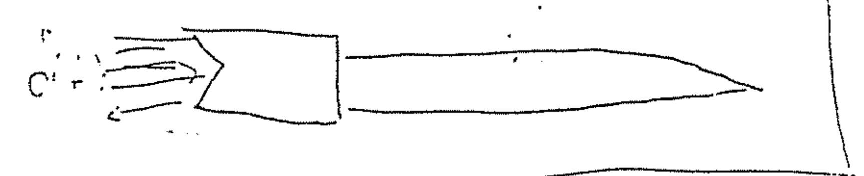
$m_1 v_0 = (m + m_1) v$

$v = \frac{m_1}{m + m_1} v_0$

$\frac{m v^2}{2} = mgR(1 - \cos\theta)$

$\cos\theta = 1 - \frac{v^2}{2gR} = 1 - \left(\frac{m_1}{m+m_1}\right)^2 \frac{v_0^2}{2g}$

$= \sqrt{\frac{E}{\rho}}$



(34) In order to study the vibrations of a beam, assume we model the beam as a linear - mass spring system subject to a time varying force $F(t)$,

$F(t)$ is taken to be large, so that the motion is essentially 1-D.
What is motion of the system for the cases:

(a) $F(t) = F\delta(t)$ (an impulse force)

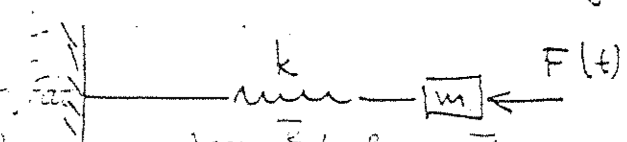
$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t + \sqrt{\frac{m}{k}}\int F(t)\sin\ldots$

(b) $F(t) =$ an arbitrary function in time;

(c) How do you apply B.C. ?

Hints:
In (a) you need to consider balance of some quantity.

a) $mv(0^+) - mv(0^-) = \hat{F}$



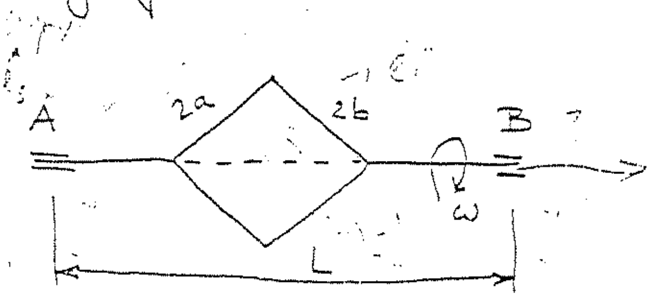
$m\ddot{x} + kx = F\delta(t)$

$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$

$x(0) = 0 \Rightarrow A = 0$

(35) A rectangular ~~square~~ plate rotates at a constant angular velocity about one of its diagonals.
What are reaction forces at supports A + B ?
What is the pertinent formula, how would you go about deriving it if you didn't remember it ?

(2)



$\omega_1 = -\omega\cos\alpha$

$\omega_2 = -\omega\sin\alpha$

$\omega_3 = 0$

$M_C = \ldots$

$= \hat{e}_1 \frac{L}{2}\cos\alpha(B_3 - A_3) - \hat{e}_2\frac{L}{2}\cos\alpha(B_3 - \ldots) + \hat{e}_3\frac{L}{2}\sin\alpha(-B_1 + A_1) \ldots$

$\omega \times H = \ldots = \hat{e}_3\,\omega^2\sin\alpha\cos\alpha\ldots$

$F = ma \Rightarrow B_1 + A_1 = 0$

(7)

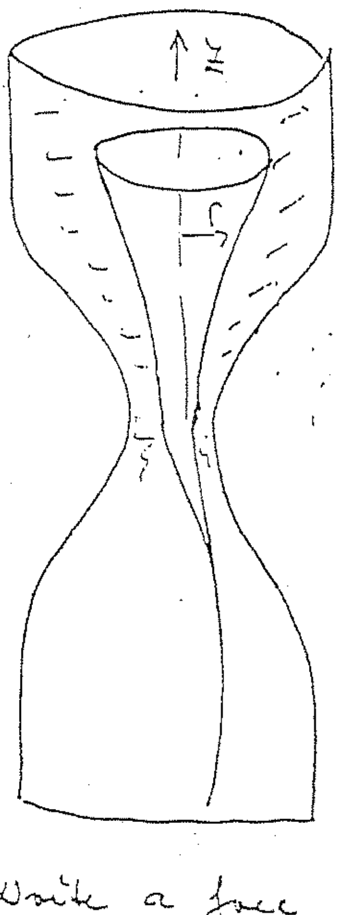
6) What is generalized force ?
if we have non-cons forces
$\delta W = \sum_i F_i^{nc} \cdot \delta q_i$

1

7) Lagrangian equations !

1

8) Water inside the hour glass. If we spin it, it forms a curve as it goes down. Find this curve as : $z(r)$.



3

a) Write a force body diagram for an surface element
b) Force balances

skew matrix

6

$$\underline{\underline{W}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

(29) Express angular velocity as a matrix (not vector)

$\underset{\sim}{W}\,\underset{\sim}{v} = \underset{\sim}{\omega} \times \underset{\sim}{v}$

(30) What are restrictions on Euler Equations?

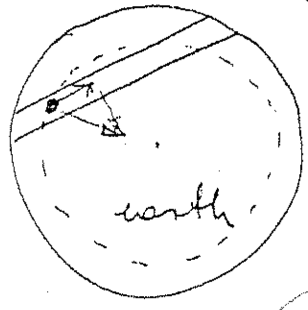
time dependent body-fixed coordinates

difficult for inertial fixed observer.

2

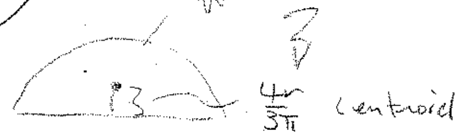
(31) Drill a tunnell through the earth. How long does it take for a body to fall from one end to the other?

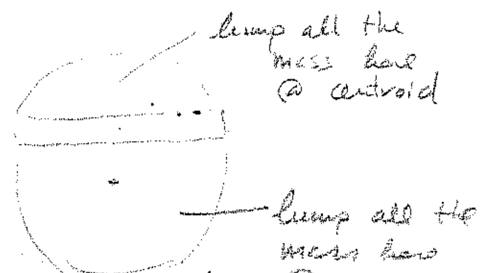
Give the number for the time if the hole passes through the earth's center.



$T = ?$

half a circle





then the force on the particle can be derived!

(32) Derive the equation of a falling chain.

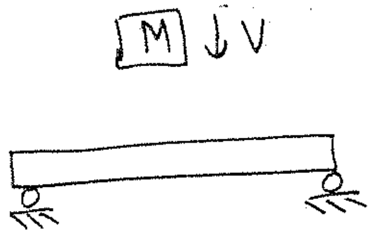
2

(33) Stability of axes of rotation of a book.

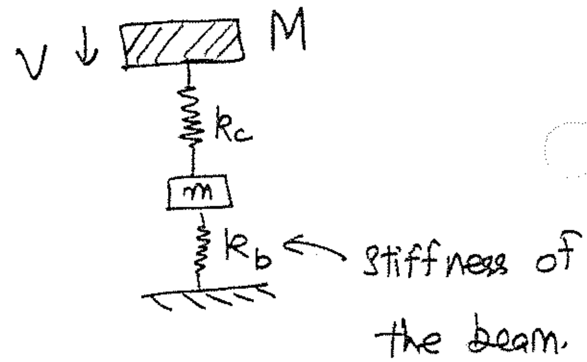
1

Dynamics

1) (Zehnder) A mass (M) is dropped on a beam.



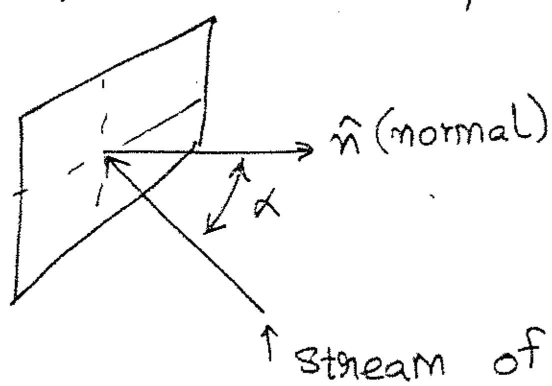
Modelled as:



Obtain the equations of motion.

How would you get (m) & k_b.

2) (Lance) Jet on a plate.



Formulate the problem.

What forces would be required? to hold the plate in place?

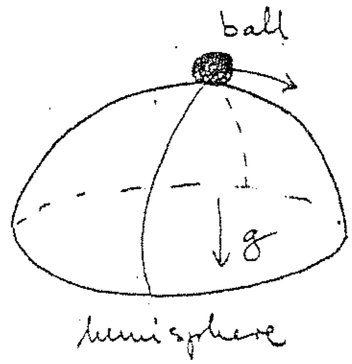
3) How do musical instruments work?

Discussed a lot on that. Pick an instrument of your choice and explain it.

(25) Describe the motion of a simple pendulum. Change the rigid swing-arm to a spring, how does this affect the system? What if the pendulum is moving in a fluid?

1

(26)



Describe the motion of the ball.

1

(27) Explain <u>Foucault's</u> Gyroscope

(Healy)

(28) Derive a formula for escape velocity in terms of "known" quantities for

1 a) earth - satellite

2 b) Sun - earth - satellite system

$$\mathcal{E} = \frac{m_s V^2}{2} - \frac{m_s M_{earth}}{r_{se}} - \frac{m_s M_{sun}}{r_{ss}} = 0 \quad \to \text{solve for } (V).$$

known: g, year, 24 hours, distances, no masses, G.

Kepler's law 2

THEORETICAL AND APPLIED MECHANCIS

QUALIFYING EXAMINATION

January 22, & 23 1987

Dynamics

D.1 Dynamics of a Swing

Discuss the dynamics of pumping a swing:

Taking the rider to be a point mass, derive the equation of motion for a variable length pendulum. Find an approximate equation of motion by considering the length variations to be small and sinusoidal. Discuss the stability of this equation, perhaps using a perturbation analysis to find an approximate solution to it.

Give some discussion of how conservation principles may be used to understand the increasing amplitude of the vibration.

References: T.E. Stern, Theory of Non-Linear Networks and Systems-Introduction;
P. Tea and H. Falk, Am. Jnl. Physics, Dec. 1968.

D.2 Yo-Yo

Describe, using mechanics, the operation of a yo-yo.

Dynamics

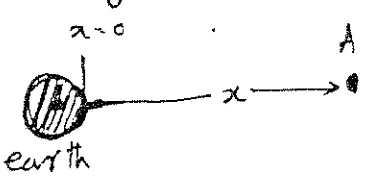
1. Derive a formula for escape velocity in terms of "known" quantities for
   1. earth - satellite
   2. Sun - Earth - satellite systems.

(Known → g, year, 24 hrs, distances ; no masses, G)

Kepler's law.

2. Express angular velocity as a matrix (not vector)

3. Why does potential energy have a -ve sign?



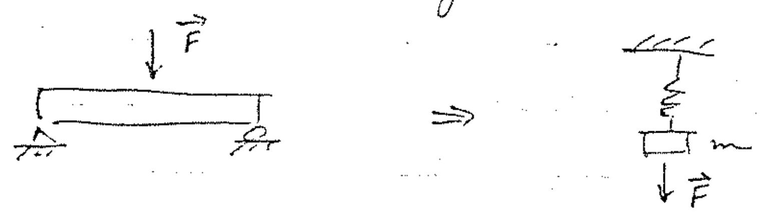
What is Pot. En. at A for $x \gtrless 0$ & $x = \infty$?

---

general → • You should be able to ~~satisfy~~ fully justify every statement you make.
• Better to write your answers rather than just say it.

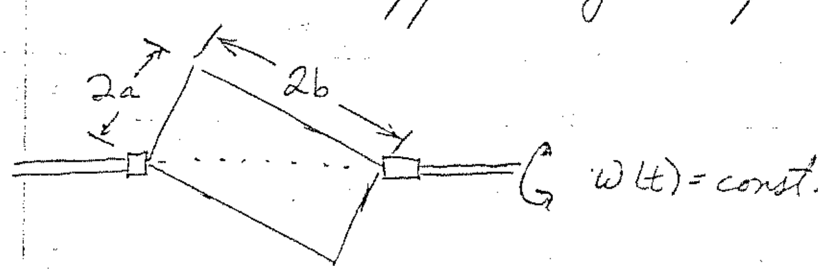
DYNAMICS

1. An elastic beam with simple support can be modeled as a harmonic oscillator (mass supported by a spring.) The beam is subjected to the time varying force $\vec{F} = \mathcal{F}\,\delta(t)\,\hat{j}$



What is the equation of motion? Solve it for $y(t)$ when $\vec{F}(t) = \mathcal{F}\,\delta(t-\tau)\,\hat{j}$

2. A rectangular plate of constant thickness spins about an axis. What moment is exerted on the supports by the plate.



(58) Euler's equations. Dynamics of the spinning top.

a) What are they?

b) Where do they come from?

c) Why use Euler angles, instead of Euler equations? — because ω's ... do not correspond to the ...

d) What if top is not axi-symmetric?

(1) e) What if there is a friction?

f) Is conservation assumed or derived?

g) Show how to derive!

h) Why doesn't a stabilizing gyro slow down?

(59) In the game of bowling a player throws the ball on the surface which slides upto certain distance and then starts to roll.

(2) Derive an expression for the time taken before rolling starts, given initial velocity v_0.

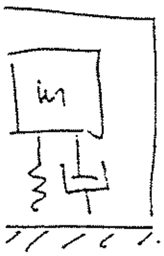
Does the result depends on the geometry or mass of the object if v_0 is the same?

$t = \frac{v_0}{\ldots}$ $\qquad -\mu g t = -\mu \frac{m}{I} g r^2 t - v_0$

(53) Puls hammer ? Discuss !

How to measure the force ?

Equation of the mechanism inside the hammer ?



$m\ddot{x} + kx + c\dot{x} = F(t)$

2

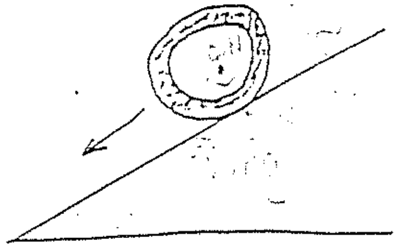
(54) What is constraint ?

holonomic

nonholonomic

1

(55)



a) Solve the problem :

(i) with slippage

(ii) without slippage

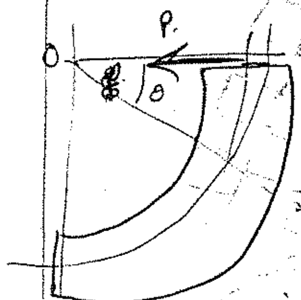
b) What are restrictions on $\underset{\sim}{T} = \dfrac{d\underset{\sim}{L}}{dt}$

c) Where is the above equation derived from?

$N = mg\cos\alpha$

$F = ma$

$I\ddot{\theta} + m\ddot{x} \ldots$



Bending of a curved bar by a force at end

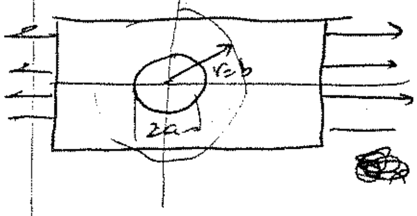
∴ we must have ~~beada~~

bending mom ∝ $\sin\theta$ at any cross-section.

∴ in $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$

take $\phi = f(r)\sin\theta$.

<u>Effects of circular holes on stress distributions in plates:</u>



$(\sigma_r)_{r=b} = \sigma_x \cos^2\theta = S\cos^2\theta = \frac{1}{2}S(1+\cos 2\theta)$

$(\tau_{r\theta})_{r=b} = -\frac{1}{2}S\sin 2\theta$.

$\tau_{r\theta} = \sigma_r = 0$ at the edge of hole.

stress f^n $\phi = f(r)\cos 2\theta$ put in

gen sol^n

$f(r) = Ar^2 + Br^4 + \frac{C}{r^2} + D$.

$\sigma_r = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$

$\sigma_\theta = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$

$\tau_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$

∴ at edge of hole, $r = a$.

$\sigma_r = \tau_{r\theta} = 0 \quad ; \quad \sigma_\theta = S(1 - 2\cos 2\theta)$

SCHAUM'S OUTLINE
CHAPTER 2 ; p. 2

$$\int_V (\nabla \cdot \underline{\sigma} + \rho \underline{b})\, dV = 0$$

SINCE V IS ARBITRARY THE INTEGRAND MUST VANISH

$$\nabla \cdot \underline{\sigma} + \rho \underline{b} = 0$$

EQUILIBRIUM EQUATIONS

EQUILIBRIUM OF MOMENTS REQUIRES THAT

$$\int_S (\underline{x} \times \underline{t}^{(n)})\, dS + \int_V (\underline{x} \times \rho \underline{b})\, dV = 0$$

$$\int_S (\underline{x} \times \underline{t}^{(n)})\, dS = \int_S (\underline{x} \times (\underline{n} \cdot \underline{\sigma}))\, dS$$

$$= \int_V (\varepsilon_{ijk}\, x_j\, \sigma_{rk})_{,r}\, dV$$

$$= \int_V (\varepsilon_{ijk}\, x_{j,r}\, \sigma_{rk} + \varepsilon_{ijk}\, x_j\, \sigma_{rk,r})\, dV$$

$$= \int_V (\varepsilon_{ijk}\, \sigma_{jk} + \underline{x} \times (\nabla \cdot \underline{\sigma}))\, dV$$

$$\int_V (\varepsilon_{ijk}\, \sigma_{jk} + \underline{x} \times (\nabla \cdot \underline{\sigma} + \rho \underline{b}))\, dV = 0$$

$$\int_V \varepsilon_{ijk}\, \sigma_{jk}\, dV = 0$$

$\sigma_{jk} = \sigma_{kj}$ OR $\underline{\sigma} = \underline{\sigma}^T$

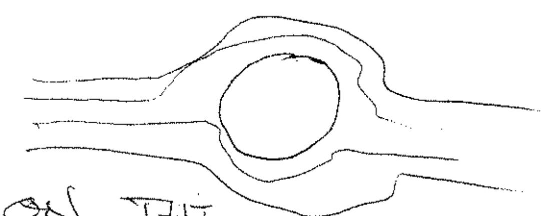
STRESS TENSOR IS SYMMETRIC

CONTINUUM MECHANICS (FLUID)

26 AUGUST 92, BRIANNO COLLER

1.) WHAT IS THE PRESSURE DISTRIBUTION IN A HURRICANE. (A TOPIC OF DISCUSSION LATELY SINCE HURRICANE ANDREW JUST HIT)

2.) INVISCID, INCOMPRESSIBLE FLOW OVER CYLINDER



ANY FORCE ON THE BODY? DRAG? LIFT?

3.) WHAT IS THE RELATION BETWEEN STREAM FUNCTION & POTENTIAL FUNCTION.

4.) WHY NO SHEAR STRESSES?

5.) ANY GENERATION OF CIRCULATION WHY?

6.) WHAT IS THE RELATION BETWEEN CIRCULATION & LIFT.

7.) HOW CAN YOU CREATE CIRCULATION & HENCE LIFT ON THE CYLINDER

PROBLEM [illegible]

FLUID CONT.

8.) THE KUTTA CONDITION.

9) CAUCHEY STRESS PRINCIPLE

10.) WHY ARE YOU TAKING FLUIDS INSTEAD OF SOLIDS.

11.) WHAT IS SEPARATION OF BOUNDARY LAYER?

12.) WHAT RELATION BETWEEN LIFT & ANGLE OF ATTACK OF AIRFOIL? QUALITATIVELY?

13.) WHY DOES AN AIRPLANE FLY.

Dynamics

Continuum

Mechanics

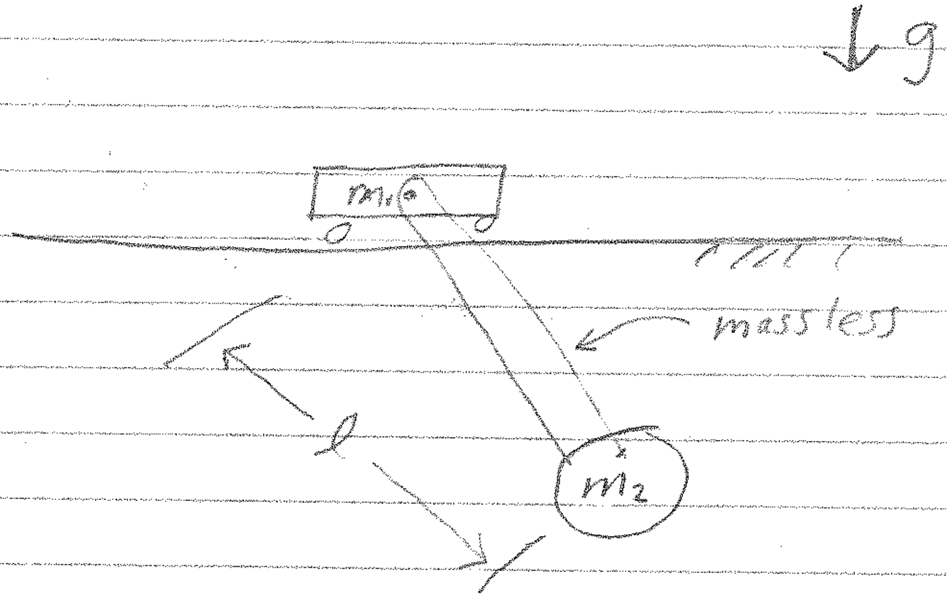
Includes ts taught by all personnel, not just faculty.

DYNAMICS

STUDY

AIDS

Find equations of motion for this system. Use any coordinates or method you like.



2) $x^2y'' + xy' - y = 0$ Let $y = x^m$

$m(m-1) + m - 1 = 0 \Rightarrow m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$y(x) = c_1 x + \frac{c_2}{x}$$

Consider, $x^2y'' + xy' - y = g(x)$ then $y'' + \frac{y'}{x} - \frac{y}{x^2} = \frac{g(x)}{x^2}$

$$y(x) = y_h(x) + y_p(x)$$

$$= c_1 x + \frac{c_2}{x} + u(x)\cdot x + \frac{v(x)}{x}$$

where $u(x) = -\int \frac{g(x)}{x^2}\frac{1}{x}\frac{dx}{W}$

$v(x) = \int \frac{g(x)}{x^2}\cdot\frac{x\,dx}{W}$

$W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix}$

$W = -\frac{2}{x}$

$$\Rightarrow u(x) = \int \frac{g(x)}{x^2}\frac{dx}{2}$$

$$v(x) = -\int \frac{g(x)}{2}\,dx$$

3) Curl $(f\underline{r}) = ?$ $f(r)$: continuous function of r.

$$\text{curl}\,(f\underline{r})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(f x_k) = \varepsilon_{ijk}\frac{\partial f}{\partial x_j}x_k + \varepsilon_{ijk} f \frac{\partial x_k}{\partial x_j}$$

$$= \varepsilon_{ijk}\frac{\partial f}{\partial r}\frac{\partial r}{\partial x_j}x_k + \varepsilon_{ijk} f(r)\,\delta_{kj}$$

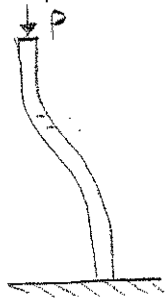
$$= \varepsilon_{ijk}\frac{\partial f}{\partial r}\frac{x_j}{r}x_k + \varepsilon_{ijj} f(r) = 0 + 0 = 0$$

4) <u>Residue Theorem</u>: If $f(z)$ is an analytic function on a simple closed curve C & also inside C, except at finitely many points then

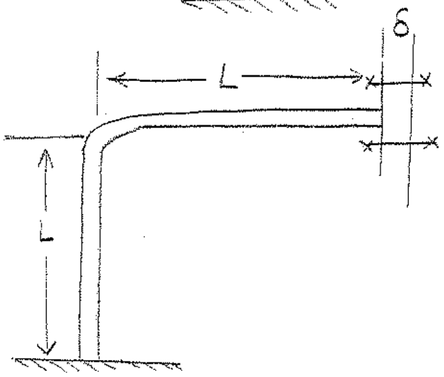
$$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \text{Res}\, f(z_i)$$

C: Integration taken in the counterclockwise sense.

4> What is wrong here? Why?



5>



Bolts are tightened to close the gap.

In the configuration shown, stresses in the pipe are zero. If the two bolts are tightened.

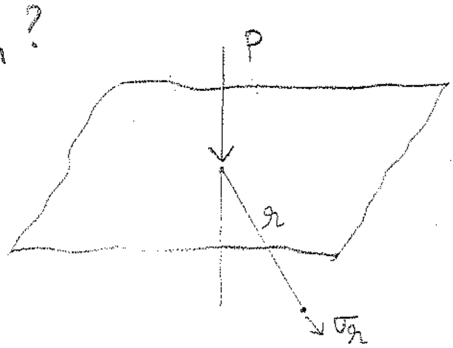
a> Draw the Free body diagram.

b) How would you go about solving for the unknown reactions

6> What is needed to solve an elasticity problem?
(equil., eqn's ; compat. eqn's, constitutive reln's ; BC's ....).

7> What is the meaning of constitutive relations?
Are they always satisfied?
Are the compatibility equations always needed?
Why does different problem have different solutions?

8> Point force 'P' on a half space. What is the singularity $\cancel{\sigma}$ σ_r ?



Hint: Look at half sphere of radius 'r'.

$P = \sigma(r) A$

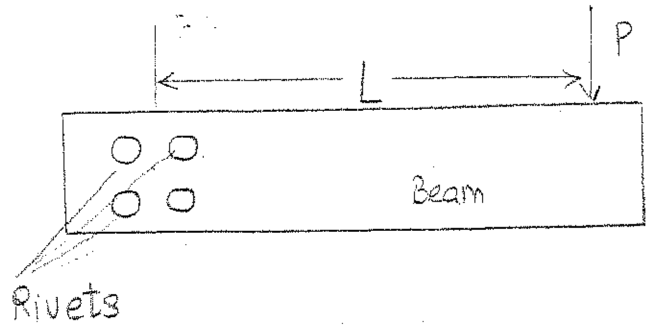
$\sigma_r \propto \frac{1}{r^2}$.

How is σ_r varying with r?

9.) What is a gage factor:

Can a strain gage be used in dynamic problems? ~~No.~~ Yes

10.)



a.) Find Rivets load

b.) What if Rivets are at different sites?

c.) What about friction?

11.) What are the steps in solving a 3-D problem in linear, isotropic, small strain elasticity? What are the required equations, explain?

Also,

a.) Is there different possibilities for B.C.?
(other than: $\sigma_{ij} n_j = S_i$)

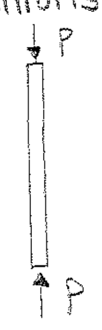
b.) How do you take care of such B.C.?

c.) Why are compatibility conditions not required in displacement formulation?

d.) What does this tell us about compatibility conditions?

12.) Buckling of a slender elastic bar:

a.) Different modes of buckling w.r.t. different boundary conditions. Describe them!



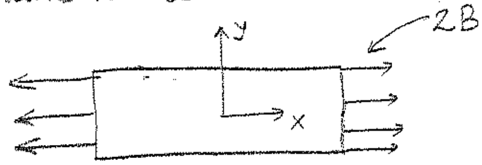
b.) What is the governing differential equation for this phenomena?

c.) What is buckling?

d.) What are the other effects which affect this phenomen

13) Derivation of the governing equations:

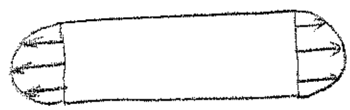
a) solutions to:



$$\phi = By^2$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2B$$

b) Solution to:



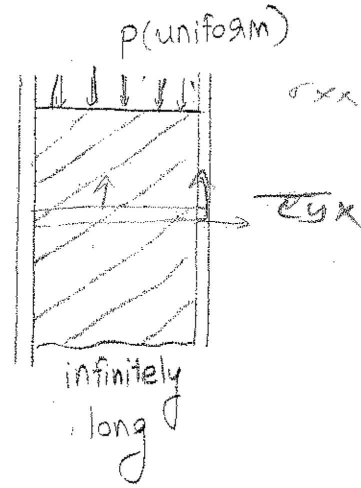
$$\phi = S\left(\frac{y^2}{2} - \frac{y^4}{12b^2}\right) + \cdots\cdots$$

$$\sigma_x = S\left(1 - \frac{y^2}{b^2}\right)$$

need addition terms. So, that $\nabla^2 \nabla^2 \phi = 0$ is satisfied.

14) Given a tube of paper which is then subjected to <u>torsion</u>. At what angle will the paper buckle & why?

15) Find the stress?



a) No friction on the walls?

b) Now with friction. State the Shearing stress distribution on the walls.

16) Why does buckling not come under the classical theory of Elasticity? — (Ans.: by the assumption of smallness).

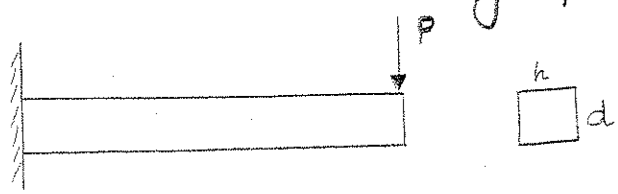
17) What are compatibility equations? What do they mean?

18) Given the displacement field $w = f(x,y)$? $u = v = 0$.

What are the governing equations.
Which of $\underset{\sim}{\sigma}$; $\underset{\sim}{\varepsilon}$ are zero!

✓ How do you measure E & G experimentally?

* 19> Consider the following problem :

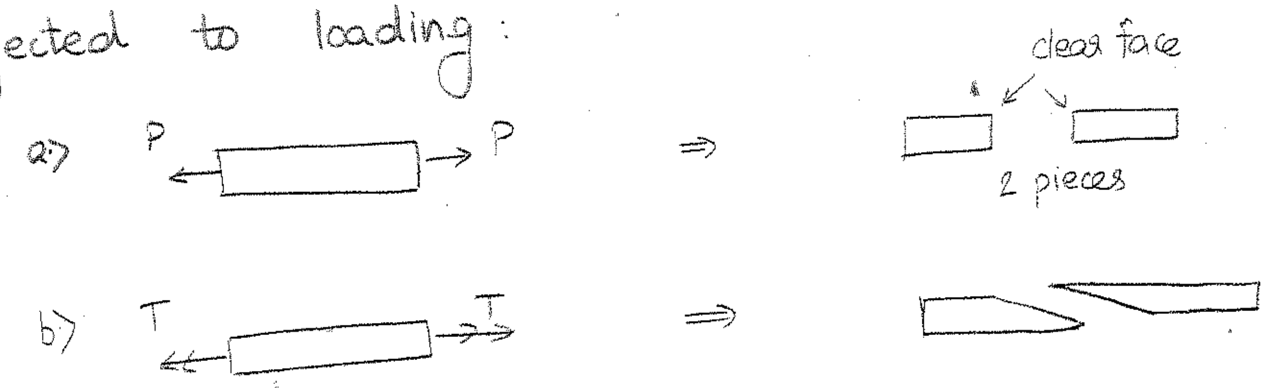


a> Draw the shape of the transverse cross-section of the beam after it has been loaded.

b> What assumptions are being made? In which region of the beam is the kinematic ~~constraint~~ approximation of beam theory good. Are there regions where the cross-section warps?

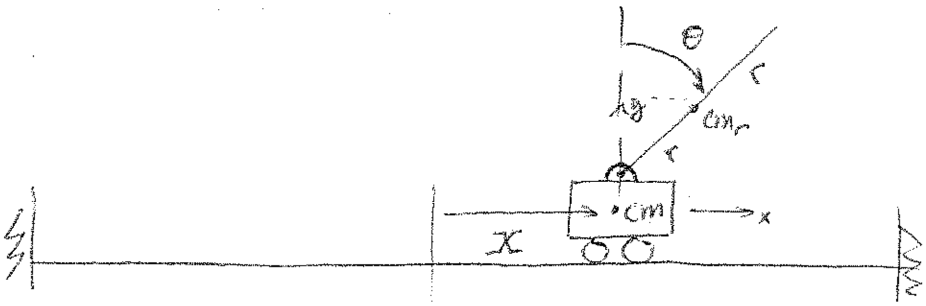
c> What is the relation between the anti-elastic radius of curvature & the bending radius of curvature? Is there a value of ν (Poisson's ratio that immediately tells you what the relation should be?)

✓ 20> A piece of chalk (a brittle material) fails when subjected to loading:



Failure criterion : Max. normal stress.

Explain different failure modes.



Length of rod is $2r$. Attach a coordinate system to the cart.

$$\underset{\sim}{r}_{cm_r} = X \underset{\sim}{E}_1 + r\sin\theta\, \underset{\sim}{e}_1 + r\cos\theta\, \underset{\sim}{e}_2$$

note that $\underset{\sim}{E}_1 = \underset{\sim}{e}_1$

$$\Rightarrow \underset{\sim}{r}_{cm_r} = (X + r\sin\theta)\underset{\sim}{e}_1 + r\cos\theta\, \underset{\sim}{e}_2$$

$$\dot{\underset{\sim}{r}}_{mr} = (\dot{X} + r\dot{\theta}\cos\theta)\underset{\sim}{e}_1 - (r\dot{\theta}\sin\theta)\underset{\sim}{e}_2$$

$$\dot{\underset{\sim}{r}}_{cm_r} \cdot \dot{\underset{\sim}{r}}_{cm_r} = (\dot{X} + r\dot{\theta}\cos\theta)\cdot(\dot{X} + r\dot{\theta}\cos\theta) + (r\dot{\theta}\sin\theta)(r\dot{\theta}\sin\theta)$$
$$= \dot{X}^2 + 2\dot{X}r\dot{\theta}\cos\theta + r^2\dot{\theta}^2$$

$$T = \frac{1}{2}m_{rod}(\dot{X}^2 + 2\dot{X}r\dot{\theta}\cos\theta + r^2\dot{\theta}^2) + \frac{1}{2}m_{cart}\dot{X}^2$$
$$+ \frac{1}{2}I_{rod}\dot{\theta}^2 \qquad \text{(Kinetic energy of the system)}$$

$$V = \cancel{mgr\cos\theta\,(1-\sin\theta)}\quad mgr\,\cancel{(1-\cos)}\,(\cos\theta - 1)$$

$$\mathcal{L} = T - V = \frac{1}{2}m_r(\dot{X}^2 + 2\dot{X}r\dot{\theta}\cos\theta + r^2\dot{\theta}^2) + \frac{1}{2}m_c\dot{X}^2 + \frac{1}{2}I_r\dot{\theta}^2$$
$$- mgr\,\cancel{(1-\cos\theta)}\,(\cos\theta - 1)$$

$\dot{x}$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m_r \dot{x} + m_r r \dot{\theta} \cos\theta + m_c \dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F$$

$$\Rightarrow \quad m_r \ddot{x} + m_r r \ddot{\theta} \cos\theta - m_r r \dot{\theta}^2 \sin\theta + m_c \ddot{x} = F$$

or

$$\boxed{(m_r + m_c)\ddot{x} + m_r r(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) = F} \qquad (1)$$

$\dot{\theta}$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_r \dot{x} r \cos\theta + m_r r^2 \dot{\theta} + I_r \dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \cancel{mgx\cos\theta}\ mgr\sin\theta$$

$$\Rightarrow \quad m_r \ddot{x} r \cos\theta - m_r \dot{x} r \dot{\theta} \sin\theta + m_r r^2 \ddot{\theta} + I_r \ddot{\theta} - mgr\sin\theta = 0$$

$$\boxed{m_r r(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta) + (I_r + m_r r^2)\ddot{\theta} - mgr\sin\theta = 0} \qquad (2)$$

note x doesn't appear explicitly in the equations.
$\therefore$ x is an ignorable coordinate.

Look for steady motion.

(i) all nonignorable coordinates are constant

(ii) all ignorable coordinates' velocity are constant.

$\cancel{r^2\Omega^2 \sin\theta_0 \cos\theta_0 + mga\sin\theta = 0}$

<u>Steady motion</u>

$\Rightarrow F = 0$ from eqn. (1)

$\Rightarrow mgr\sin\theta = 0 \quad \Rightarrow \quad \theta = 0, \pi, .. p\pi,$

$\theta = 0$ is the only allowable steady-motion in this system due to design requirement.

Linearize equation of motion about $\theta = 0$

$\theta = \varepsilon(t) \qquad \varepsilon \ll 1$

$$(m_r + m_c)\ddot{x} + m_r r \ddot{\varepsilon} = F \qquad (1)$$

$$m_r r \ddot{x} + (I_r + m_r r^2)\ddot{\varepsilon} - mgr\varepsilon = 0 \qquad (2)$$

Solve equation (1) for $\ddot{x}$

$$\Rightarrow \quad (m_r + m_c)\ddot{x} = F - m_r r \ddot{\varepsilon}$$

$$\Rightarrow \quad \ddot{x} = \frac{F - m_r r \ddot{\varepsilon}}{(m_r + m_c)} \qquad (3)$$

Substitute (3) into (2)

$$\Rightarrow m_r r \left(\frac{F - m_r r \ddot{\varepsilon}}{(m_r + m_c)} \right) + (I_r + m_r r^2)\ddot{\varepsilon} - mgr\varepsilon = 0$$

$$\Rightarrow \frac{m_r r}{(m_r + m_c)} F - \frac{m_r^2 r^2}{(m_r + m_c)} \ddot{\varepsilon} + (I_r + m_r r^2)\ddot{\varepsilon} - mgr\varepsilon = 0$$

$$\Rightarrow \frac{m_r r}{(m_r + m_c)} F + \left(\frac{m_r r^2 (m_r + m_c) + I_r (m_r + m_c) - m_r^2 r^2}{(m_r + m_c)} \right) \ddot{\varepsilon} - mgr\varepsilon = 0$$

$$\Rightarrow \frac{m_r r}{(m_r+m_c)} F + \left(\frac{\cancel{m_r^2 r^2} + m_r m_c r^2 + I_r (m_r+m_c) \cancel{- m_r^2 r^2}}{(m_r+m_c)} \right) \ddot{\varepsilon}$$

$$- mgr\varepsilon = 0$$

$$\Rightarrow \boxed{\underbrace{\left(\frac{m_r m_c r^2 + I_r (m_r+m_c)}{(m_r+m_c)} \right)}_{>0} \ddot{\varepsilon} - \underbrace{mgr}_{>0} \varepsilon + \frac{m_r r F}{(m_r+m_c)} = 0}$$

$F = F_0 \delta(\varepsilon - \varepsilon_c)$

if $F = 0$

stability equation.

Bang-Bang type.

$$\Rightarrow c_1 \ddot{\varepsilon} - c_2 \varepsilon = 0$$

$$c_1 > 0 \quad c_2 > 0$$

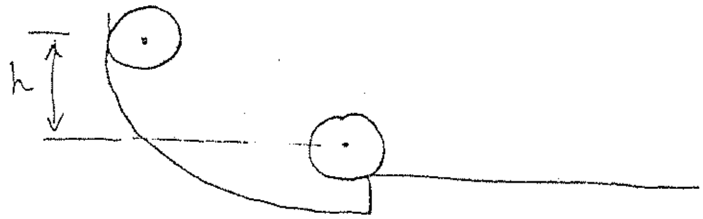
$$\varepsilon \sim e^{rt} \qquad r^2 e^{rt} - e^{rt}$$

$$c_1 r^2 - c_2 = 0$$

$$r^2 - \frac{c_2}{c_1} = 0 \quad \Rightarrow \quad r^* = \pm \frac{c_2}{c_1}$$

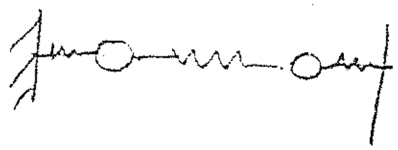
We have 1 positive root $\left(r = \frac{c_2}{c_1}\right)$ so the system will become unstable as $t \to \infty$ if we don't use an appropriate forcing function.

Dynamic



talk about how
you'd solve this
problem

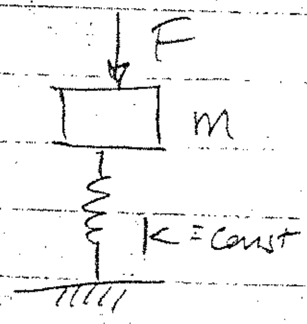
? - Describe what happens when you try to turn a spinning wheel - (gyroscopes)

- 

describe motion
how to solve?

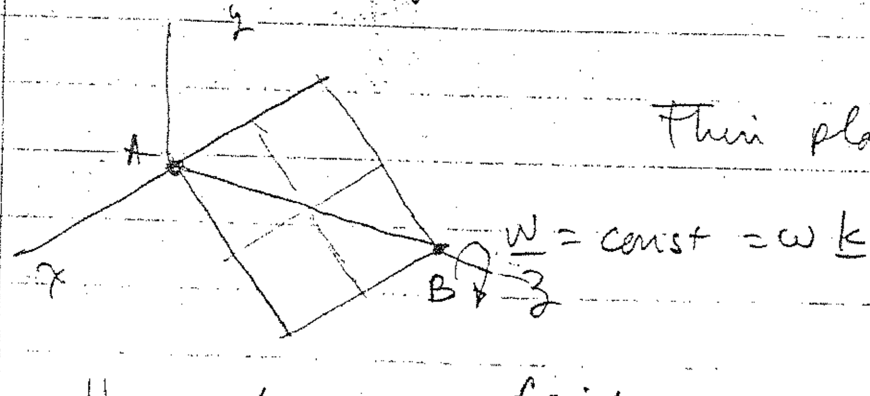
Dynamics

(1)



- Impact load is applied
- How do you solve for displacement of mass?
  ($m\ddot{x} + kx = F\delta(t - t_0)$ ← supplied by student)
  (F ↳ const.)
- What are initial conditions?
- What is the solution?
- How do you apply boundary conditions?
  How do you solve for arbitrary excitation

(2)



Thin plate in y-z plane

How do you find reactions at A and B?
(dynamic reactions)

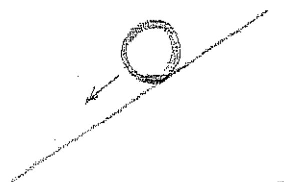
II DYNAMICS

1. DERIVE THE EQUATIONS OF A PENDULUM
2. DERIVE THE EQUATIONS OF A DAMPED PENDULUM
3. DERIVE THE EQUATION OF A FALLING CHAIN
4. STABILITY OF AXES OF ROTATION OF A BOOK

Generally level of T&AM 203

B. Dynamics

1.



1.1. Solve the problem. assume

(i) with slippage (ii) without slippage

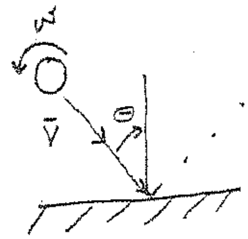
1.2 What are the restrictions on

$\underset{\sim}{T} = \frac{dL}{dt}$?

1.3 Where is the above equation derived from?

Dynamics

1) Hockey puck impacts a wall



what is its direction, velocity and spin when it leaves? normal collision is elastic.

2) Bang on the end of a rod with a hammer. Listen to the tone. What is the bar made of? How can you tell.

2 - Dynamics

a - What is the equation for simple pendulum

b - write down the equation motion for a pendulum whose length varies with time.

c - Suppose you have a Rope of constant mass, you lift it from the table with Constant F, neglect g, calculate V in time.

Topic -

Student - Tapesh

Dynamics:

3 Explain Foucault's Gyroscope

4.) $\int_{-\pi}^{\pi} f(x)\,dx = 0$

$$\int_{-\pi}^{\pi} [f(x)]^2\,dx = \int_{-\pi}^{\pi} [f'(x)]^2\,dx$$

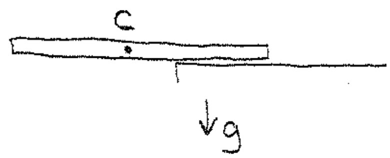
f is not the zero function

What are the restrictions on f ?
Use fourier series

(Answer is $f(x) = A\cos x + B\sin x$ is only possibility)

Dynamics

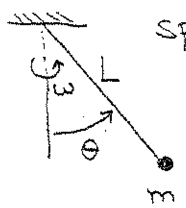
1.)



Ruler is released from position above.
Initially, the point of the ruler touching the corner of the table sticks, but at some time it starts to slide.

Describe how you would set up this problem.

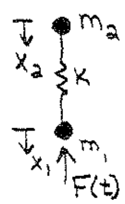
2.)



spherical pendulum ($\dot{\theta} = 0$)
m is given an initial velocity v. find the period.

If the mass is given a <u>slight</u> disturbance ($\dot{\theta}$ no longer $= 0$) find the period.

3.)



set up equations of motion.

Given $(\ddot{x}_1 - \ddot{x}_2)$ how would you find $F(t)$?

4.) What is the difference between Lagrange's Eqns. and Newton's Eqns. ?

TAM 1370

December 1973

I. COLLISIONS OF RIGID BODIES.

Two bodies of mass m_1 and m_2 collide as shown in the figure. At the moment of collision, they contact at point P with a common tangent plane and a normal, $\underset{\sim}{n}$, to that plane. We are in general given:

$$\underset{\sim}{v}_1 , \underset{\sim}{v}_2 , \underset{\sim}{\omega}_1 , \underset{\sim}{\omega}_2 \text{ before impact,}$$

and wish to determine:

$$\underset{\sim}{v}_1' , \underset{\sim}{v}_2' , \underset{\sim}{\omega}_1' , \underset{\sim}{\omega}_2' \text{ after impact.}$$

The twelve unknown quantities (4 vectors) are related to the given quantities by the empirical law of collision and equations for impulsive motion:

$$\int_0^{\Delta t} \underset{\sim}{F}\, dt = [m \underset{\sim}{v}_c]_0^{\Delta t} \qquad \text{Linear momentum}$$

$$\int_0^{\Delta t} \underset{\sim}{M}\, dt = [\underset{\sim}{H}]_0^{\Delta t} \qquad \text{Angular momentum}$$

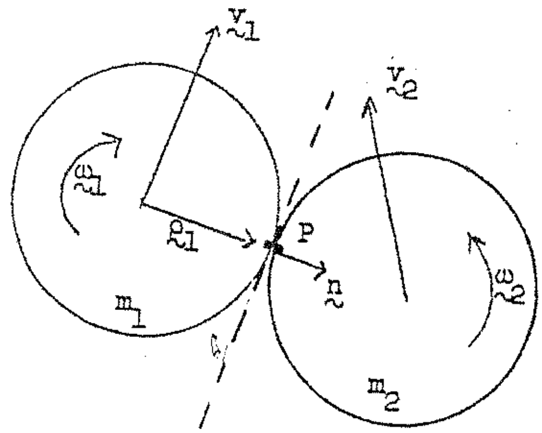
The twelve equations are:

(1) Law of Collisions - When two bodies collide, the values of the normal component of the relative velocity of the surfaces in contact at instants immediately after and immediately before the impact bear a definite ratio to each other; this ratio, denoted by -e, depends only on the material of which the bodies are composed (one eq.).

$$-e = \frac{\underset{\sim}{v}_{1p}' \cdot \underset{\sim}{n} - \underset{\sim}{v}_{2p}' \cdot \underset{\sim}{n}}{\underset{\sim}{v}_{1p} \cdot \underset{\sim}{n} - \underset{\sim}{v}_{2p} \cdot \underset{\sim}{n}}$$

where

$$\underset{\sim}{v}_{1p} = \underset{\sim}{v}_1 + \underset{\sim}{\omega}_1 \times \underset{\sim}{\rho}_1 \quad \text{etc.}$$



(2) Constancy of angular momentum of each body about the point of contact, P, because of zero moment about p (six eqs.).

$$\underset{\sim}{L}_p = 0 .$$

(3) Constancy of linear momentum of the system (two bodies) normal to the surface of contact because of equal and opposite normal impulses (one eq.)

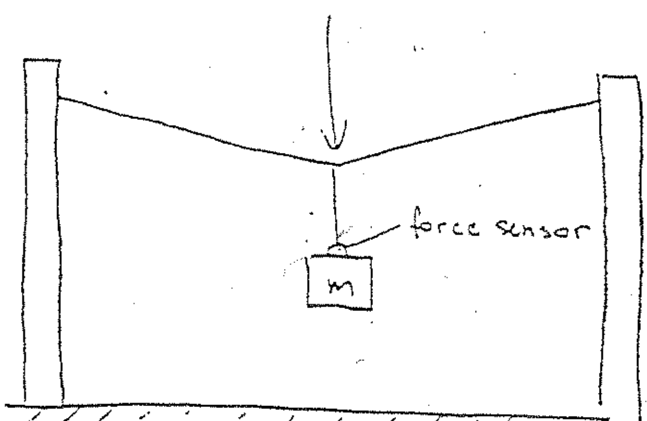
$$(m_1\underset{\sim}{v}_1 + m_2 v_2)\cdot\underset{\sim}{n} = (m_1\underset{\sim}{v}_1' + m_2\underset{\sim}{v}_2')\cdot\underset{\sim}{n} .$$

Discuss nature of angular momentum and relevant equations for a spacecraft. What are the equations? What about other formulations?

2

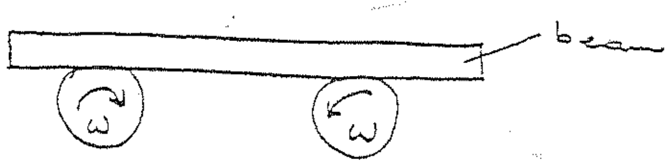
(43) What constraints are there on the choice of reference frames for using the equations
$\sum M = \dot{H}$, $\sum F = ma$, ?
What is inertial reference frame? 2

(44)



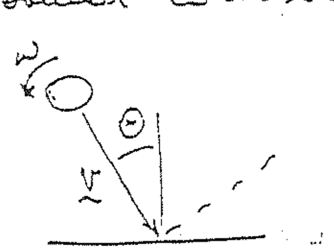
One pulls mass and released it.
What is the nature of the motion?
What is the equation of motion?
When force has max. amplitude?
How would you verify that the motion obeys the solution to the equation of motion?

(5)



There is a friction between rollers and beam.
What is the nature of the motion ...

(39) Rocket puck impacts a wall.
What is its direction, velocity and spin when it leaves?
Normal collision is elastic.



(2)

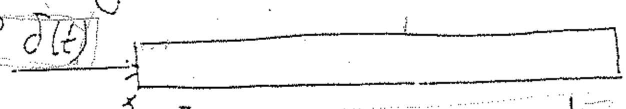
$v_n' = -v_n$

$v_t' = v_t$

$I\omega' = I\omega$

(40) Bang on the end of a rod with a hammer.
Listen to the tone. What is the low mode of? How can you tell?

$\sqrt{\frac{E}{\rho}}$ (1)

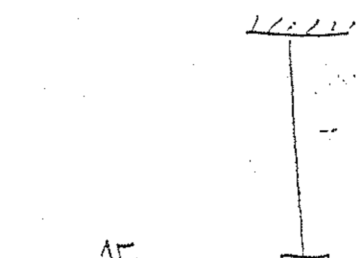


@ $x=0$: $u(x)=0$ $\quad u(x,t) = f(L, c, \omega,$

@ $x=c$: $u'(x{=}L)=0$

(41) Bullet strikes pendulum and is embedded.
What is maximum angle Θ of subsequent motion of the pendulum?

(1)



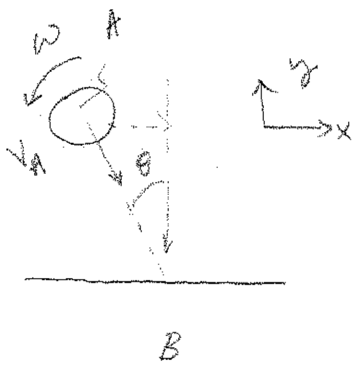
$$m_1 v_0 = (m+m_1)v$$

$$v = \frac{m_1}{m+m_1} v_0$$

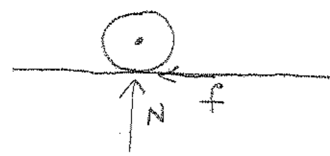
$$\frac{m v^2}{2} = m g R(1-\cos\theta)$$

$$\cos\theta = 1 - \frac{v^2}{2gR} = 1 - \left(\frac{m_1}{m+m_1}\right)^2 \frac{v_0^2}{2g}$$

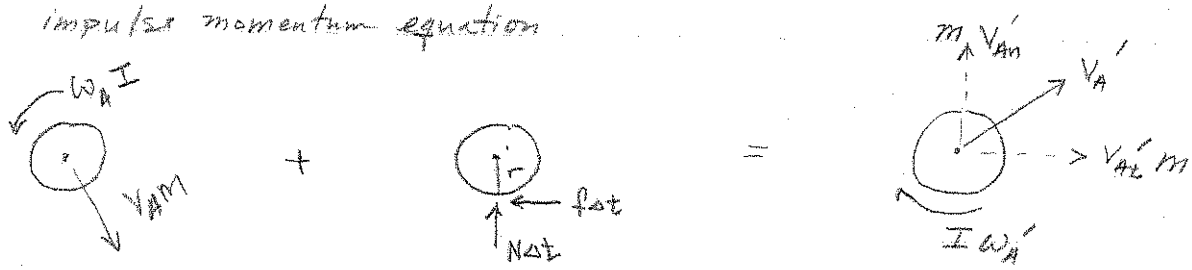
Rocket Pock



at impact



write impulse momentum equation



given normal component is elastic

$$(V_{Bn}' - V_{An}') = e\,(V_{An} - V_{Bn}) \quad \text{memorize this thing}$$

elastic $\Rightarrow$ $e = 1$ $\qquad V_{Bn} = V_{Bn}' = 0 \rightarrow$ wall.

$$\Rightarrow \; -V_{An}' = V_{An} \quad \text{or} \quad V_{An}' = -V_{An} = +V_A \cos\theta$$

$$x: \; mV_A \sin\theta - f\Delta t = mV_{At}'$$

$$y: \; -mV_A \cos\theta + N\Delta t = mV_{An}' = mV_A \cos\theta$$

$$+\circlearrowleft: \; I\omega_A - f\Delta t\, r = -I\omega_A'$$

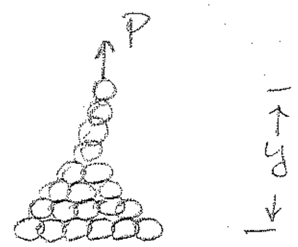
V_{At}', ω_A', $f\Delta t$, $N\Delta t$ $\quad$ 4 unknowns $\quad$ 3 equations.

need one more equation

Assume friction force related to N, i.e, $f = \mu N$

$\Rightarrow$ $f\Delta t = \mu N\Delta t$ This gives the 4th equation needed to solve for everything.

Chain-Pulling Problem

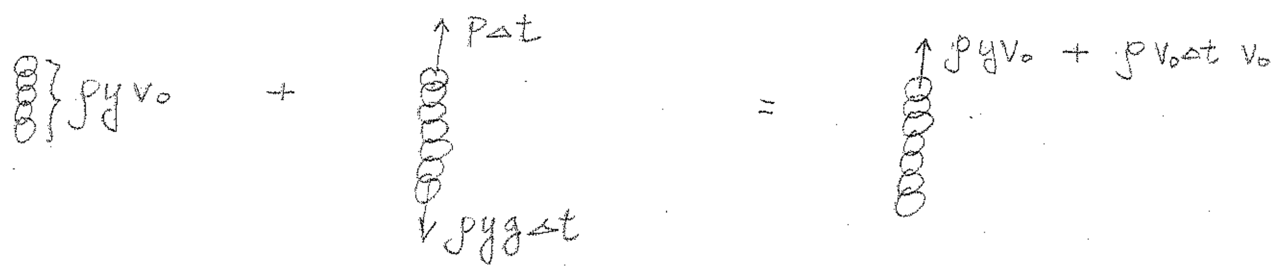


constant v_0 upward, determine P as a function of time to maintain constant v_0.

impulse momentum technique.

each link on the floor acquires its velocity <u>abruptly</u> through the impact w/ its upper link which lifts it off the floor. Thus, the lifting of such a chain involves an energy loss.

↗ use impulse momentum.



$$\cancel{\rho y v_0} + P\Delta t - \rho y g \Delta t = \cancel{\rho y v_0} + \rho v_0^2 \Delta t$$

$$\Rightarrow P \cancel{\Delta t} = \rho (yg + v_0^2) \cancel{\Delta t}$$

$$\Rightarrow \boxed{P = \rho(yg + v_0^2)}$$

ok.

the relation between the two answers?

1 (10) Write down the wave equation! $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

What type of equation is it? hyperbolic

How do you solve it? (explain all 3 methods)?

1) sep. var
2) D'Alembert sol.
3) Sol. of characteristics

(11) Define z^α where z & α are complex ...

$$z^\alpha = z^{x+iy} = z^x(\cos y + i\sin y)$$

(12) Solve $\boxed{y''' = y}$ $y''' - y = 0$

$\cancel{y(x) = Ay^3 + By^2 + Cy + D}$

$\cancel{y' = 3Ay^2 + 2By + C}$ $\cancel{y'' = 6Ay + 2B}$ $y''' = 6A$

$\cancel{A + Ay^3 + By^2 + Cy + D}$

(13) When does solution to $\boxed{Ax = B}$ exists?

Uniqueness?

$\boxed{Ax = b}$ A^{-1} = not singular

(14) Convergence of inf. power series?

1

(15) Calculate various contour integrals!

2

(16) Principal value, eigenvalue problem!

1

(17) State the Residue theorem!

1

(18) Define the derivative of a complex function!

(19) Give a brief lecture on Fourier series!

2

(20) What is Euler's constant γ?

3

Show $\lim_{n \to \infty} \left(1 + 1/2 + 1/3 + \dots 1/n - \log(n) \right)$

converges, give a bound on this limit!

(21) How do you solve the Cauchy-Eulers eq.

1

$$x^2 y'' + x y' - y = 0 \;?$$

How do you solve the inhomogeneous case?

1

$$x^2 y'' + x y' - y = f(x) \;?$$

(22) What is curl$(f(\underset{\sim}{u}))$? $f: \mathbb{R}^3 \to \mathbb{R}$

2

~~(23) Poisson's Integral Representation:~~

1 → Consider the Dirichlet problem on a unit disc.
Derive Poisson's Integral representation and discuss applications.

3

(Ref. Zachmanoglou - Thoe; - Intro to PDE)

a) Identify the denominator in the integral representation

b) What happens when $r = 1$? $[\delta f^{u}]$

c) Why did you neglect the eigenvalues?

d) S[illegible]

(not used)

Consider the equation

$$2f(x) + \frac{d}{dx} f(x) + \int_0^x f(x')\,dx' = 0 .$$

a) Find a solution.

b) Find the most general solution.

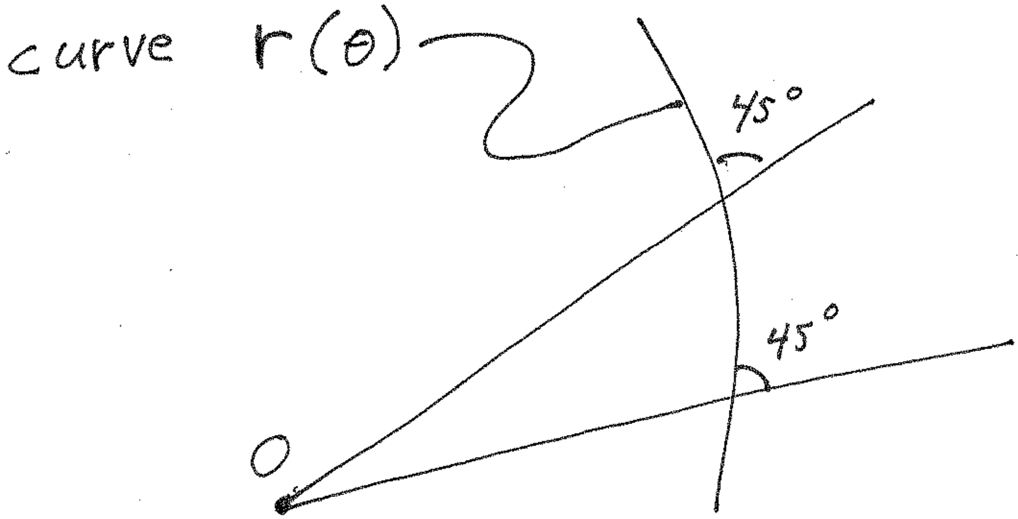
(4)

Show

$$\int_0^{\infty} \frac{\cos x}{\sqrt{x}}\,dx = \frac{1}{\sqrt{2}} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\,dt$$

(2)

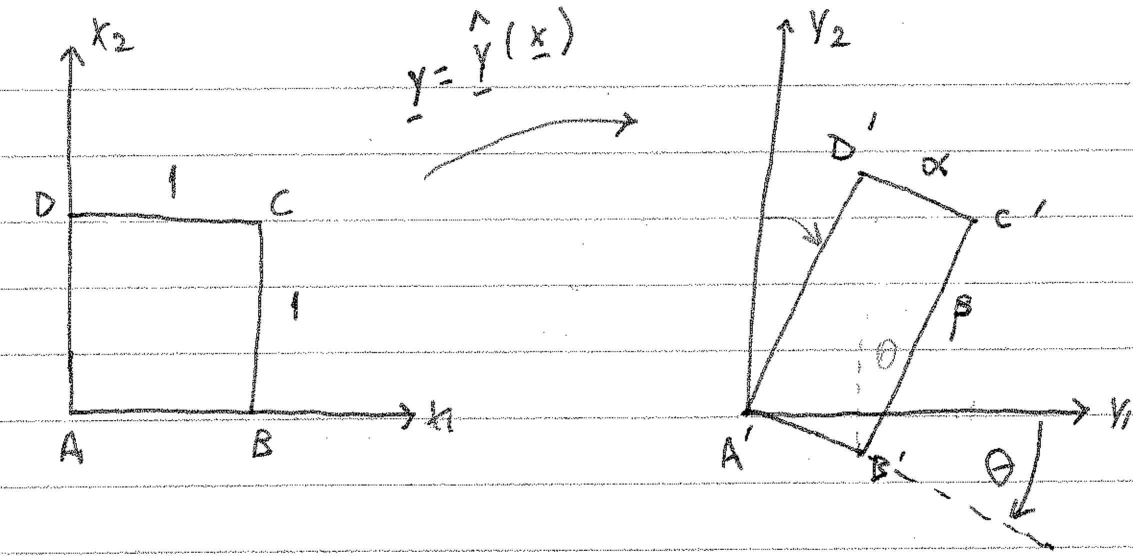
Find the curve $r(\theta)$ that makes a $45°$ angle with every radial line from the origin.



a) Any such $r(\theta)$.

b) Most general $r(\theta)$.

(1)



Find F, Q, U

What are principal stretches ?

principal directions ?

2. $$x^2 u_{xx} - y^2 u_{yy} = 0$$

Canonical form for this hyperbolic PDE is $A = C = 0$

$$x^2 \phi_x^2 - y^2 \phi_y^2 = 0 \quad \text{where } \phi = \xi, \eta$$

$$\phi = \text{const} \Rightarrow \phi_x dx + \phi_y dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\phi_x}{\phi_y}$$

But $$\frac{\phi_x^2}{\phi_y^2} = \frac{y^2}{x^2} \Rightarrow \frac{\phi_x}{\phi_y} = \pm \frac{y}{x}$$

$$\frac{dy}{dx} = \pm \frac{y}{x}$$

or

$$\ln y = \pm \ln x + \text{const}$$

$$\Rightarrow y = C_1 x, \quad y = \frac{C_2}{x}$$

$$\therefore C_1 = \frac{y}{x} = \xi, \quad C_2 = xy = \eta$$

$$\Rightarrow B = c\,\eta_y \xi_y + a\,\eta_x \xi_x = -y^2 x \frac{1}{x} + x^2 y \left(\frac{-y}{x^2}\right)$$

$$B = -2y^2$$

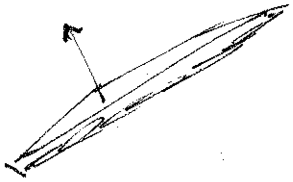
$$D = c\,\xi_{yy} + a\,\xi_{xx} = 0 + x^2\left(\frac{2y}{x^3}\right) = 2\frac{y}{x}$$

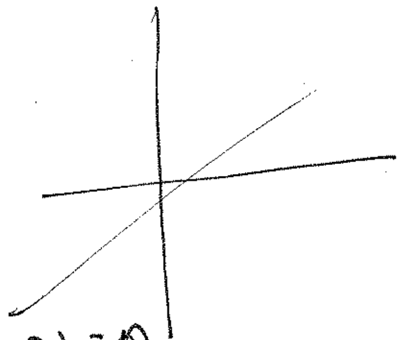
$$E = 0$$

PDE becomes $2Bu_{\xi\eta} + Du_\xi = 0$

$$-4y^2 u_{\xi\eta} + 2\frac{y}{x} u_\xi = 0$$

$$y^2 = \xi\eta \qquad \frac{y}{x} = \xi$$





$A(x-x_0) + B(\cancel{x}y-y_0) + C(z-z_0) = \cancel{D}$

$ax + by + cz = \cancel{d}\ 0$ $\quad \bar{v} \quad (4, -3, 7)$

(through origin)

$\cancel{X_1} = -3 + 4t \quad \cancel{Y_2} = 2 - 3t \quad z \quad -3, 7t$

$A(0,0,1) \quad B(2,0,0) \quad C(0,3,0)$

$AB\ (2, 0, -1) \quad AC\ (0, 3, -1)$

$$\begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} \qquad i(3) - j(-2) + k(6)$$

$3\hat{i} + 2\hat{j} + 6\hat{k}$

$9 + 4 + 36 = 49 \quad \sqrt{49} = 7$

$\hat{n} = \frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$

$\frac{3}{7}(x) + \frac{2}{7}\cancel{x}y + \frac{6}{7}(y-1) = 0$

$3x - 6y - 2z = 15 \qquad 2x + y - 2z = 15$

$\hat{n}_1 = \langle 3, -6, -2 \rangle \qquad \hat{n}_2 = \langle 2, 1, -2 \rangle$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix}$$

$i(-2 - 12) - j(-4 + 6) + k(-12 - 3)$

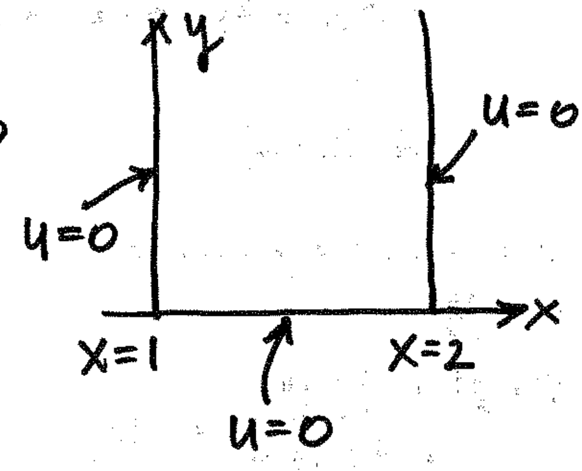
$-14i - 2j - 15k$

Solution to last year's Final Exam question:

$$x^2 u_{xx} + x u_x + u_{yy} = 0$$

$u = 0$ for $x=1$, $x=2$, $y=0$

Assume soln $u = X(x)\,Y(y)$



$$x^2 X'' Y + x X' Y + X Y'' = 0$$

$$\frac{x^2 X'' + x X'}{X} = \frac{-Y''}{Y} = -\lambda^2$$

$$x^2 X'' + x X' + \lambda^2 X = 0 \,, \quad Y'' - \lambda^2 Y = 0$$

Assume $X = x^k$

$$k(k-1) + k + \lambda^2 = 0$$

$$k^2 + \lambda^2 = 0, \quad k = \pm i\lambda$$

$$X(x) = c_1 \sin(\lambda \ln x) + c_2 \cos(\lambda \ln x)$$

B.C. $X(1) = 0 \Rightarrow c_2 = 0$

$X(2) = 0 = c_1 \sin(\lambda \ln 2)$

$\Rightarrow \lambda \ln 2 = n\pi, \quad n = 1,2,3,\ldots$

$$X_n(x) = \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$$

$$Y = c_3 \sinh \lambda y + c_4 \cosh \lambda y$$

$Y(0) = 0 \Rightarrow c_4 = 0$

$$Y_n(y) = \sinh \lambda_n y = \sinh\left(\frac{n\pi y}{\ln 2}\right)$$

General solution $u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi \ln x}{\ln 2} \sinh\frac{n\pi y}{\ln 2}$

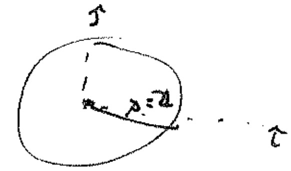
$x = \frac{8}{3} + 2t \qquad y = -2t \qquad z = 1 + t$

$3x + 2y + 6z = 6$

$A\ (x - x_0)$

circle

$r\theta = s$



$\bar{r} : r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$

$\bar{v} = \frac{d\bar{r}}{ds}$

$\bar{T} = \frac{d\bar{r}}{ds} = \frac{dr}{dt}\frac{dt}{ds}$

$= \frac{dr}{d\theta}\frac{d\theta}{ds} \quad = \frac{1}{r}$

$\bar{T} = \frac{1}{\cancel{r}}\left[-\cancel{r}\sin\theta\,\hat{\imath} + \cancel{r}\cos\theta\,\hat{\jmath}\right]$

$|\bar{T}| = \sqrt{2}$

$K = \left|\frac{d\bar{T}}{ds}\right| = \left|\frac{d\bar{T}}{d\theta}\frac{d\theta}{ds}\right|$

$= \left|(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})\frac{1}{r}\right|$

$= \frac{1}{r}$

$x^2 + y^2 = 4$

$r\cos\theta = x$

$R = 2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath}$

$\bar{V} = \cancel{-2\sin\theta}$

$= -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}$

$|\bar{V}| = \sqrt{r^2\sin^2 + r^2\cos^2\theta} = \sqrt{r^2}$

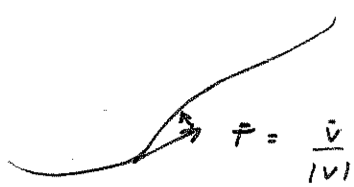
$\bar{T} = \cancel{\sin\theta}$

$\bar{T} = \frac{-\sin\theta}{r}\hat{\imath} + \frac{\cos\theta}{r}\hat{\jmath}$

$K = \left|\frac{d\bar{T}}{ds}\right| = \left|\frac{dT}{d\theta}\frac{d\theta}{ds}\right| \qquad \frac{d\theta}{ds} = \frac{1}{r}$

$-\frac{\cos\theta}{r^2}\hat{\imath} - \frac{\sin\theta}{r^2}\hat{\jmath}$

$\sqrt{\frac{1}{r^4}} = \frac{1}{32} = \frac{1}{r} = \frac{1}{2} \quad \rho = 2$

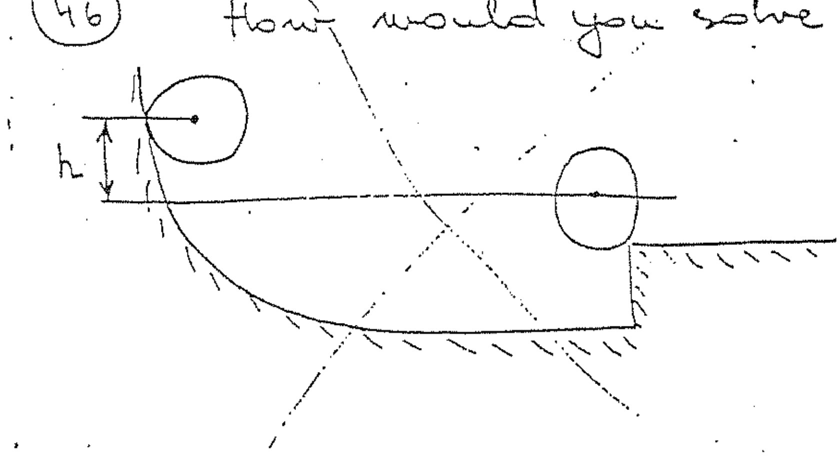


$\bar{T} = \frac{\bar{v}}{|v|}$

$\bar{N} = \left|\frac{d\bar{T}}{ds}\right|$

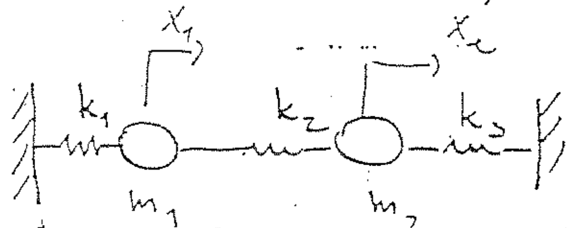
$K = \left|\frac{d\bar{T}}{ds}\right|$

(46) How would you solve this problem?



(47) Describe what happens when you try to turn a spinning wheel (gyroscopes)

(48) Describe motion, how to solve?



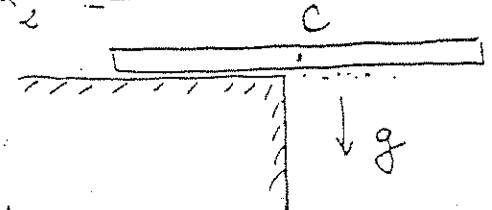
$x_2 < x_1$

$m_1\ddot{x}_1 = -k_1 x_1 - k_2(x_2 - x_1)$

1

Ans: P.T.O.

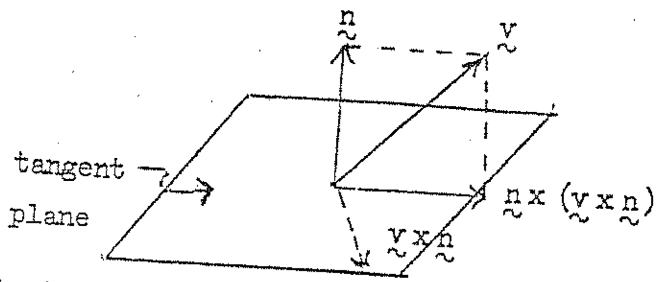
(49)



What is the relationship between the ang. vel. ω and v of C.
Ruler is released from position above,
Initially, the point of the ruler touching the corner of the table sticks, but at some time it starts to slide,

(4A) For smooth surfaces - No change in tangential components of the linear momentum for each body because of zero tangential forces (four eqs.)

$$\underset{\sim}{n} \times (m\underset{\sim}{v}' \times \underset{\sim}{n}) - \underset{\sim}{n} \times (m\underset{\sim}{v} \times \underset{\sim}{n}) = 0$$



(4B) For Rough Surfaces without Slipping at the Contact Point

(i) Constancy of tangential components of the linear momentum of the system because of equal and opposite tangential impulses (two eqs.)

$$\underset{\sim}{n} \times [(m_1\underset{\sim}{v}_1 + m_2\underset{\sim}{v}_2) \times \underset{\sim}{n}] = \underset{\sim}{n} \times [(m_1\underset{\sim}{v}_1' + m_2\underset{\sim}{v}_2') \times \underset{\sim}{n}]$$

(ii) Vanishing of the tangential components of the relative velocity of the two bodies after impact because of no slipping constraint (two eqs.)

$$\underset{\sim}{n} \times [(\underset{\sim}{v}_{2p}' - \underset{\sim}{v}_{1p}') \times \underset{\sim}{n}] = 0 .$$

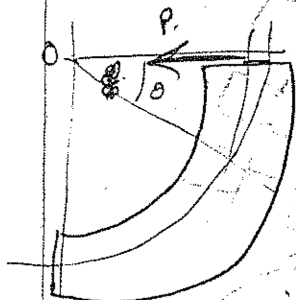
(4C) For Rough Surfaces with Slipping at the Contact Point

(i) Same as (4Bi) (two eqs.)

(ii) Change of the components of the linear momentum by the tangential impulse which equals, in magnitude to μ (coefficient of friction) times the normal impulse for each body (two eqs.)

$$|\underset{\sim}{n} \times [m(\underset{\sim}{v}' - \underset{\sim}{v}) \times \underset{\sim}{n}]| = \mu m(\underset{\sim}{v}' - \underset{\sim}{v}) \cdot \underset{\sim}{n}$$

Note that there are two equations of the above form for each body; a total of 4 equations is obtainable. However, only two (for either body) are independent equations.



Bending of a curved bar by a force at end

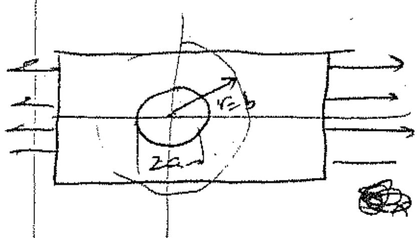
$\therefore$ we must have ~~be~~

bending mom $\propto \sin\theta$ - at any cross-section.

$\therefore$ in $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$

take $\phi = f(r)\sin\theta$.

Effects of circular holes on stress distributions in plates:



$(\sigma_r)_{r=b} = \sigma_x \cos^2\theta = S\cos^2\theta = \frac{1}{2}S(1+\cos 2\theta)$

$(\tau_{r\theta})_{r=b} = -\frac{1}{2}S\sin 2\theta.$

$\tau_{r\theta} = 0$ $\sigma_r = 0$ at the edge of hole.

Stress f^n $\phi = f(r)\cos 2\theta$ put in

gen sol^n:

$f(r) = Ar^2 + Br^4 + \frac{C}{r^2} + D.$

$\therefore \sigma_r = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$

$\sigma_\theta = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$

$\tau_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$

$\therefore$ at edge of hole, $r = a$.

$\sigma_r = \tau_{r\theta} = 0$; $\sigma_\theta = S(1 - 2\cos 2\theta)$

SCHAUM'S OUTLINE
CHAPTER 2 ; p. 2

$$\int_V (\nabla \cdot \underline{\sigma} + \rho \underline{b})\, dV = 0$$ SINCE V IS ARBITRARY THE INTEGRAND MUST VANISH

$$\nabla \cdot \underline{\sigma} + \rho \underline{b} = 0$$ EQUILIBRIUM EQUATIONS

EQUILIBRIUM OF MOMENTS REQUIRES THAT

$$\int_S (\underline{x} \times \underline{t}^{(n)})\, dS + \int_V (\underline{x} \times \rho \underline{b})\, dV = 0$$

$$\int_S (\underline{x} \times \underline{t}^{(n)})\, dS = \int_S (\underline{x} \times (\underline{n} \cdot \underline{\sigma}))\, dS$$

$$= \int_V (\varepsilon_{ijk} x_j \sigma_{rk})_{,r}\, dV$$

$$= \int_V (\varepsilon_{ijk} x_{j,r} \sigma_{rk} + \varepsilon_{ijk} x_j \sigma_{rk,r})\, dV$$

$$= \int_V (\varepsilon_{ijk} \sigma_{jk} + \underline{x} \times (\nabla \cdot \underline{\sigma}))\, dV$$

$$\int_V (\varepsilon_{ijk} \sigma_{jk} + \underline{x} \times (\nabla \cdot \underline{\sigma} + \rho \underline{b}))\, dV = 0$$

$$\int_V \varepsilon_{ijk} \sigma_{jk}\, dV = 0$$

$$\sigma_{jk} = \sigma_{kj} \quad \text{OR} \quad \underline{\sigma} = \underline{\sigma}^T$$

STRESS TENSOR IS SYMMETRIC

Hooke's Law: Linear relations between comp. of stress & strain.

(E - modulus of elasticity in tension.

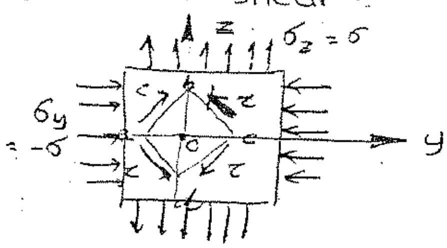
$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right],$$ ν - Poisson's ratio.

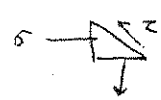
etc.

Method of superposition - Valid as long as deformations are small & do NOT affect forces causing them. Then calc. are based on initial dimensions & shape of body. Then resultant displacements given by superposing lin. f^{n}s of ext. forces.

(eg where superpos can't be used: Simultaneous axial & lateral forces on thin bar. Effect of deformation on moment of external forces must be considered though def. is small. Then def $\neq$ lin. f^n of fo... $\Rightarrow$ cannot get by simple sup.

Pure shear: no normal stress on face.





$\leftarrow$ balancing, $\tau = \sigma$.

Angle bet. bc & ab changes. After def,

$$\frac{Oc}{Ob} = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{1+\epsilon_y}{1+\epsilon_z}$$

But $$\epsilon_y = \frac{1}{E}\left[-\sigma - \nu\sigma\right] = -\frac{(1+\nu)}{E}\sigma$$

$$\epsilon_z = \frac{1}{E}\left[\sigma - \nu(-\sigma)\right] = \frac{(1+\nu)}{E}\sigma$$

$$\therefore \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\gamma/2}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma}{2}} \approx \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}$$

Comparing $$\frac{\gamma}{2} = \frac{(1+\nu)}{E}\sigma^{=\tau} \Rightarrow \gamma = \frac{2(1+\nu)}{E}\tau.$$

Define $G = \dfrac{E}{2(1+\nu)}$ G - modulus of elasticity in shear

$\therefore \gamma = \dfrac{\tau}{G}$ - " " rigidity

$\gamma_{xy} = \dfrac{1}{G}\tau_{xy}$ etc.

Now volume expansion $\dfrac{\Delta V}{V} = e = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z) -$

$\approx \epsilon_x + \epsilon_y + \epsilon_z + H.O.T.$

$\therefore e = \epsilon_x + \epsilon_y + \epsilon_z = \dfrac{1}{E}[\sigma_x - \nu(\cdot -)] + \dfrac{1}{E}[\sigma_y - \nu(- -)] + \dfrac{1}{E}[\sigma_z - \nu(- -)]$

$\Rightarrow e = \left(\dfrac{1-2\nu}{E}\right)\Theta \qquad [\Theta = \sigma_x + \sigma_y + \sigma_z]$

Uniform hydrostatic pressure $\Rightarrow \sigma_x = \sigma_y = \sigma_z = -p$,

$\Rightarrow e = -3\dfrac{(1-2\nu)}{E}p$.

$K = \dfrac{E}{3(1-2\nu)}$ = modulus of vol. expansion.

From $\epsilon_x = \dfrac{1}{E}[\sigma_x - \nu(-)]$ etc. and $e = \epsilon_x + \epsilon_y + \epsilon_z$

we get

$\sigma_x = \left(\dfrac{\nu E}{(1+\nu)(1-2\nu)}\right)e + \left(\dfrac{E}{1+\nu}\right)\epsilon_x$ etc.

Let $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$, also $\dfrac{E}{1+\nu} = 2G$.

$\Rightarrow \sigma_x = \lambda e + 2G\epsilon_x$ etc.

$\therefore \Theta = (\sigma_x + \sigma_y + \sigma_z) = \underbrace{(3\lambda + 2G)}_{K} e$

Q-Exam Questions Jeffrey Nussbaum

Dynamics

Presentation- Dynamics of the spinning top

Questions essentially on talk:

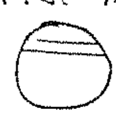
Euler's equations - what are they, where do they come from, why use Euler angles instead of Euler equations?

What if top is not axi-symmetric? What if there is friction

* Is conservation assumed or derived? Show how to derive.

Why doesn't a stabilizing gyro slow down?

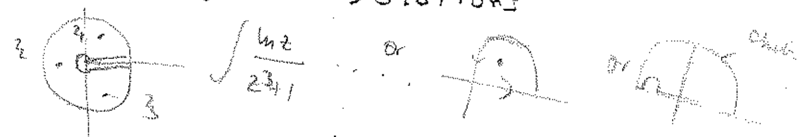
✓ Tunnel drilled through the earth. Describe motion (Simple harmonic, period = 84 minutes,)



Math

✓* Why does $\dot{x} = \sqrt{x}$ have 2 solutions

$\int_0^\infty \frac{1}{x^3+1}\, dx$



*✓ Tell about contour integrals, Residue Theorem, Indented contours.

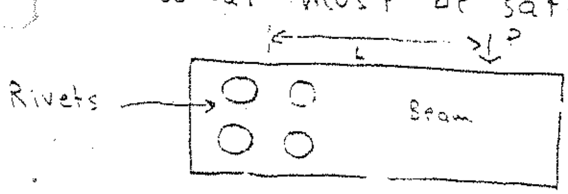
✓ How do Runge-Kutta, Euler methods work? Error estimate?

✓ How do you get from $I = \int_{t_1}^{t_2} f(x, \dot{x}, t)\, dt$ to Euler equation (Calculus of Variations)

Elasticity

What is a gauge factor? Can a strain gauge be used in dynamic prob

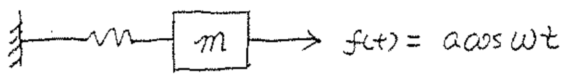
Suppose $\underline{\underline{\sigma}}$ = constant (given) is the stress tensor. Is this ok? What must be satisfied? Why? (Sym, Equil, $T_i = \sigma_{ij} n_j$)



Find rivet loads, what if rivets are different sizes, what about friction?

P2) 3Ps

2.



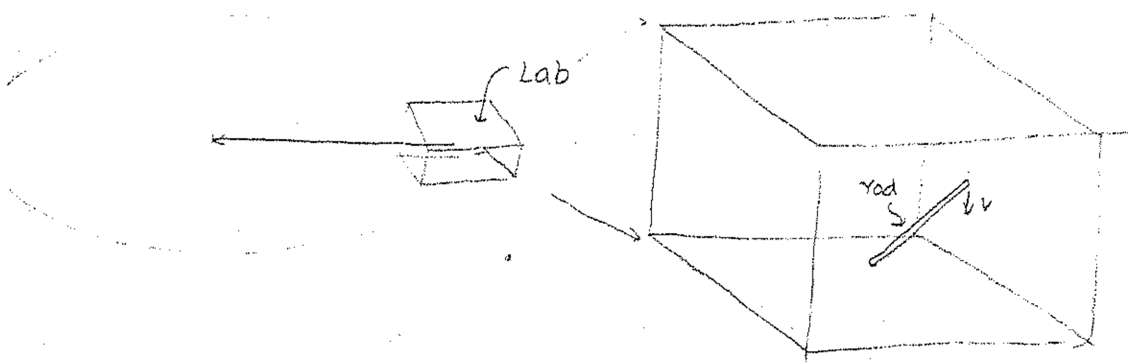
2.1. What does "resonance" means?

2.2. If resonance occures, what does the response look like?

2.3. If ω is a little bit different from ω_c (natural frequency), what would the response look like?

2.4. If add a damper to the system, what is the relationship between the phase lag and the driving frequency ω?

3.



3.1. Does the rod have angular accerlation?

3.2 How to measure angular accerlation?

3.3 How to measure linear accerlation by using a force sensor?

~~$\underset{\sim}{v} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3$~~

(31) Are the square matrix and the 2^{nd} order tensors the same thing ?

(1)

(32) Describe the solvability of

$$y'' + \lambda y = g(x)$$

(1)

a) $\lambda > 0$ (h.o.)
b) $\lambda < 0$ (exp.)
c) $\lambda = 0$ (linear)

$y(0) = y(1) = 0$

$0 \le x \le 1$ $\qquad y'' = 0 \rightarrow \underline{y = Ax + B}$

for various values of λ ($\lambda \in \mathbb{R}$)

(33) In some space exists 3 vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ each can be expressed as a linear combination of unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$! What conditions must be satisfied that $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ are linearly independent.

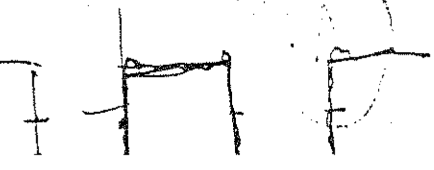
(1)

(34) $$y(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 < x < 1 \end{cases}$$

What is the value of the fourier series of $y(x)$ at the points :

a) $x = -1 \rightarrow \cancel{0}\,\cancel{2}\ 1/2$ $\qquad$ d) $x = 96.256 \rightarrow 1$

b) $x = +1 \rightarrow \cancel{1}\ 1/2$



(35) $f(x) \geq 0$

Col. var.

$0 < x < 1$

$\frac{\partial}{\partial x} \int f(x)\,dx = 0$

$f(0) = f(1) = 0$

Maximize area under the curve $f(x)$ subject to the curve f and constraint length

$$L = \frac{\pi}{2}$$

1

(36) What is the second order tensor?

How is it represented?

Given an 2^{nd} order tensor $\underset{\sim}{T}$ and a set of 3 vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ on $\mathbb{R}^n$ with the $\underset{\sim}{v}_i$ L I and the following:

$$\underset{\sim}{T} \cdot \underset{\sim}{v}_1 = -\underset{\sim}{v}_1 + \underset{\sim}{v}_3$$

$$\underset{\sim}{T} \cdot \underset{\sim}{v}_2 = 2\underset{\sim}{v}_2$$

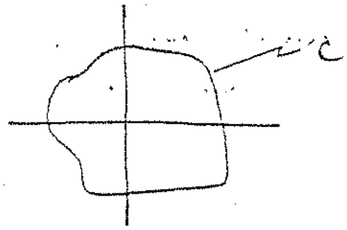
$$\underset{\sim}{T} \cdot \underset{\sim}{v}_3 = \underset{\sim}{v}_2 + \underset{\sim}{v}_3$$

What are the elements of $\underset{\sim}{T}$?

(37) $$\int_0^\infty \frac{\cos x}{x^2+1}\,dx$$

1

(24)



$f(z)$ analytic outside C

z_0 a point outside C

$f(\infty) = a$ What is $\oint_C \frac{f(z)}{z - z_0}\, dz$

1

(25) What can you say about the following:

$$L(u) + \alpha u = 0$$

$$u(0) = u_0$$
$$u(a) = u_1$$

L is a linear operator !

1

~~(26) Steepest descents~~ !

1

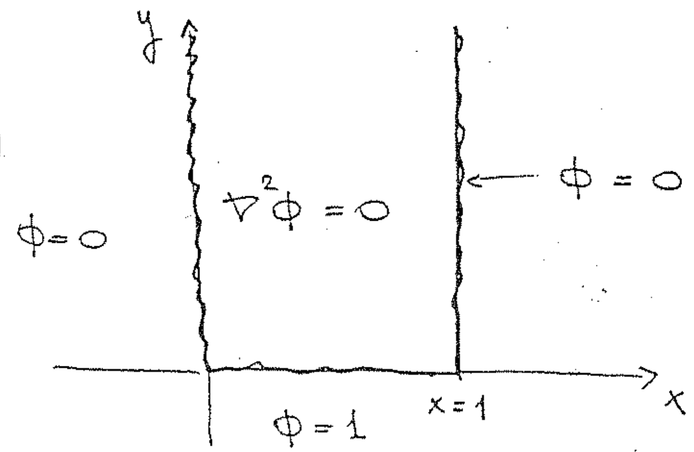
(27) Evaluate $\int_C \frac{dz}{z^{2n} + 1}$, where C is : $|z| = 3$

2

(There is a ~~trick~~ for evaluating the residues of a function that is a quotient of two polynomials - such as that above,

(28) Solve the following BVP by elementary methods (no complex variables).

(2)



$|\phi|$ is bounded at $y \to \infty$

(29) A is a real $n \times n$ matrix. What are the least restrictive constraints on A that will allow you to diagonalize it?

(1)

Hint: The diagonal matrix: rank

$\boxed{D = C^{-1} A C}$ where C is a special matrix.

(30) Solve $\displaystyle\int_0^\infty \frac{\sin \lambda x}{x}\,dx$

(1)

(16) Given the displacement field $w = f(x, y)$. $\boxed{u = v = 0}$
What are the governing equations? (Ans. $\nabla^2 w = 0$
Which τ, γ are zero?

What is the physical meaning of the existence condition for the Neumann problem?

(Ans. equilibrium

How do you measure E and G experimentally

Elastic Waves

(17) Two equal rods of length L are moving longitudinally with velocities $\pm u$ along the x axis. At $t = 0$ they collide at the origin.

a) Determine the subsequent motion (assuming it is longitudinal) when the rods are elastic and do not adhere?

b) What happens when the deformation has a lateral component or there is adhesion?

c) What are the different approaches to derive the equations of motion and to solve them?

d) What if the rods are not one dimensional!

e) What if the rods are not of equal length!

f) What can you say about the problem of a rod colliding against a wall?

(20) What is the momentum Balance Law?
What do the terms mean?
From what principle is it derived?
What is balance of angular momentum
What does it tell us about stress tensor?

1

663

(21) State the force balance equation.
Derive the equat. for force balance in continuum in 2-D in cartesian coord.

(22) What is continuum?
What parameters are involved in continuum mechanics?
Give an example of a boundary value problem and the equations necessary to solve it.

1

(23) In a 2-D body with a point load at the origin, how do stress, displacements vary with r?
In a 3-D body?

II. Solid Mech:

1) Interpret cpts of $\underline{E} = \frac{1}{2}(\underline{C} - \underline{I})$?

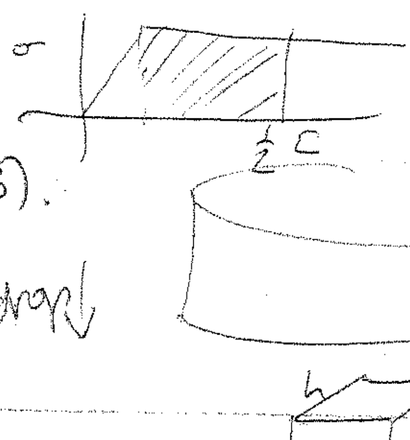
talk about chgs in length volume etc.

2) Elastodynamic state – no body forces

$$t_{ij,j} = \rho \ddot{u}_i$$

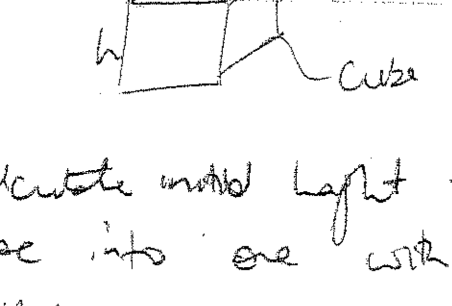
Suppose $\underline{u} = u\underline{e}_2$ where $u = u(x_1)$. Interpret?

(ans: equil eqns + constitutive eqn give $u_{xx} = \frac{\rho}{\mu} u_{tt}$ – wave eqn).



$\boxed{\frac{1}{2}\sigma_{11}\varepsilon_{11} = mgH}$ $\quad \varepsilon_{11} = \frac{1}{2} = \Delta$

3). drop ↓



$\Rightarrow$ squashes cube!

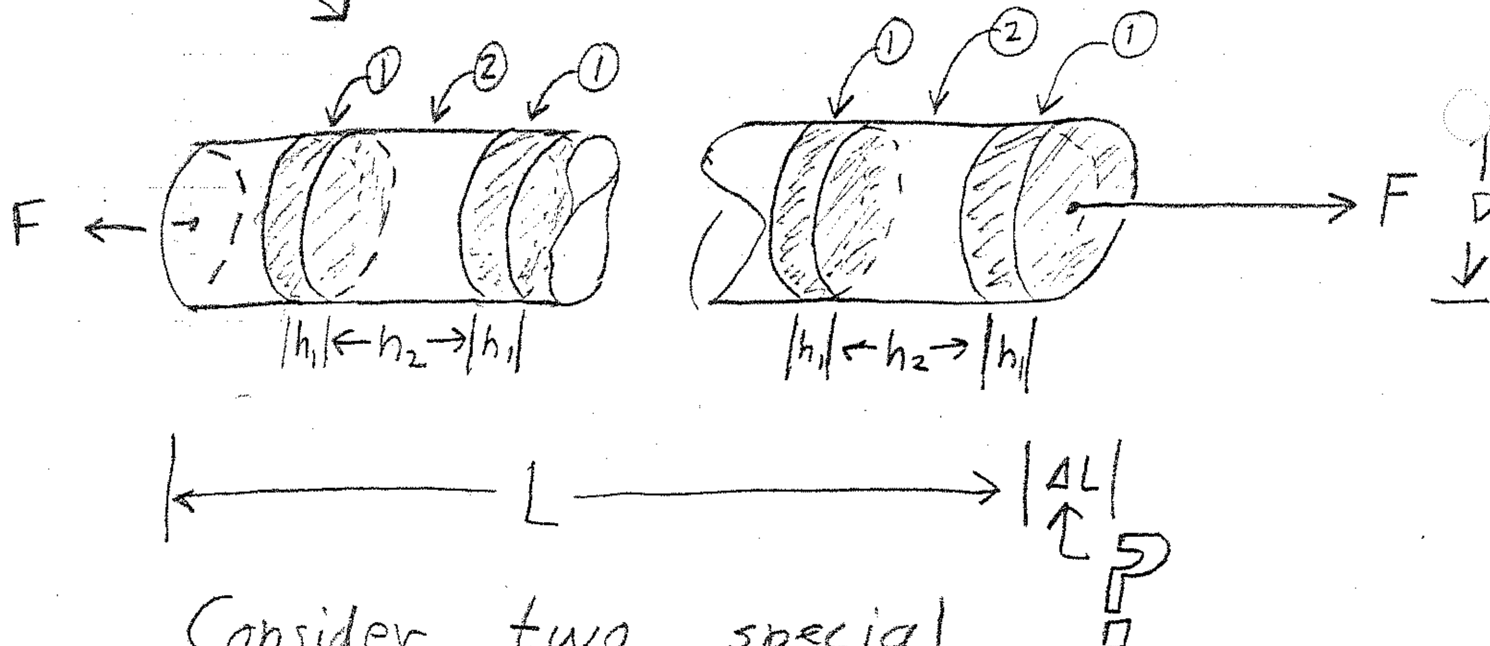
Calculate initial height H required to squash cube into one with exactly half the dimensions.

(15 min) <u>SIMPLE COMPOSITE</u>

What, approximately, is the increase in length of the composite rod shown?

F = applied load
E_1, ν_1, G_1 = moduli of materiel 1
E_2, ν_2, G_2 = moduli of material 2
D = diameter of disks (both)
L = length of total rod
$\gg h_1$
$\gg h_2$

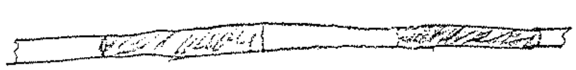
Stack of Disks

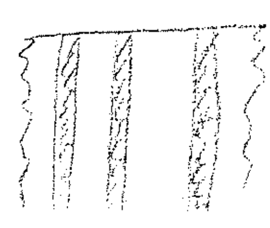


Consider two special cases:

a) $D \ll h_1$
$D \ll h_2$

a) $D \gg h_1$
$D \gg h_2$





elastic moduli of a microcracked solid by a different approach. They do not start with eqn (4), but utilize the potential energy balance and the relationship between potential energy change and the crack energy release rate. Their approach is particularly useful for solids with microcracks embedded since they avoid the difficulty of evaluating inclusion strain $\bar{\varepsilon}_{ij}$ for microcracks in eqn (4). Penny-shaped cracks were assumed to be randomly distributed in the matrix material such that the crack material behaves like an isotropic solid. The moduli of the cracked solid were found, depending not on the porosity volume concentration, but on a newly introduced parameter, the crack density,

$$\varepsilon = N\langle a^3 \rangle, \tag{5}$$

where N is the number of cracks per unit volume, a is the radius of the penny-shaped crack, and $\langle \cdot \rangle$ is the average of the argument. A critical value of crack density, $\varepsilon = 9/16$, was established at which the effective elastic moduli of the cracked solid vanish.

Although Budiansky and O'Connell's (1976) analysis of a cracked solid was based on the potential energy balance, it is shown in the following that their results still fall into the general framework of self-consistent mechanics of composite materials (Budiansky, 1965; Hill, 1965). A penny-shaped crack can be considered as the limit of an oblate spheroidal cavity with $a_1 = a_2 = a$ and $a_3 \to 0$, where a_1, a_2 and a_3 are the half-axes of the spheroid. The basic equation [eqn (4)], for a solid with spheroidal cavities embedded, becomes

$$\frac{1+\bar{\nu}}{\bar{E}}\sigma^0_{ij}\sigma^0_{ij} - \frac{\bar{\nu}}{\bar{E}}\sigma^0_{kk}\sigma^0_{ll} = \frac{1+\nu_N}{E_N}\sigma^0_{ij}\sigma^0_{ij} - \frac{\nu_N}{E_N}\sigma^0_{kk}\sigma^0_{ll} + c\sigma^0_{ij}\bar{\varepsilon}_{ij}, \tag{6}$$

where $\bar{E}$ and E_N are the Young's modulus and $\bar{\nu}$ and ν_N are Poisson's ratio of the cracked solid and matrix material, respectively; c is the volume concentration of the cavity. Note that

$$c\bar{\varepsilon}_{ij} = \frac{1}{V}\int_{V_{\text{cavity}}} \varepsilon_{ij}\,\mathrm{d}V = \frac{1}{V}\sum_{\text{all cavities}} \varepsilon_{ij} \cdot \tfrac{4}{3}\pi a^2 a_3, \tag{7}$$

where V_{cavity} is the total volume of the cavities, V is the total volume of the solid, $4/3(\pi a^2 a_3)$ is the volume of each cavity, and ε_{ij}, which is uniform within each cavity, was given by Eshelby (1957). In the case of remote hydrostatic tension, $\sigma^0_{ij} = \sigma^0\delta_{ij}$, the strain, ε_{ij}, within the cavity has the asymptotic form (Eshelby, 1957; Mura, 1982)

$$\varepsilon_{33} = \frac{4}{\pi}\frac{1-\bar{\nu}^2}{\bar{E}}\frac{a}{a_3}\sigma_0 + O(1), \quad \text{others} = O(1). \tag{8}$$

Thus, eqn (7), in the limit of cracks ($a_3/a \to 0$), gives

$$c\bar{\varepsilon}_{kk} = \frac{1}{V}\Sigma a^3 \cdot \frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\sigma_0 = N\langle a^3 \rangle \frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\sigma_0. \tag{9}$$

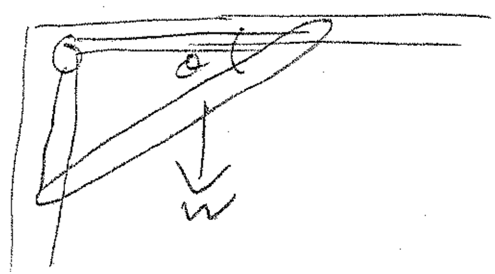
Substituting into the basic equation [eqn (4)], one finds

$$\frac{3(1-2\bar{\nu})}{\bar{E}} = \frac{3(1-2\nu_N)}{E_N} + \frac{16}{3}\frac{1-\bar{\nu}^2}{\bar{E}}\varepsilon, \tag{10}$$

which is exactly one of the governing equations for determination of effective moduli given by Budiansky and O'Connell (1976). If other kinds of the remote loading σ^0_{ij} are applied, one can similarly derive other governing equations of effective moduli identical to Budiansky and O'Connell's. This shows that their analysis of a cracked solid is consistent with the self-consistent mechanics of composite materials (Budiansky, 1965; Hill, 1965). One can

SOLIDS

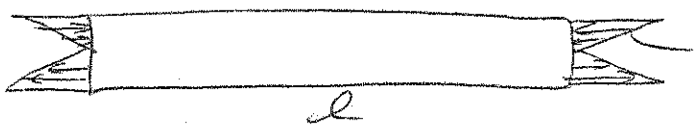
(1)



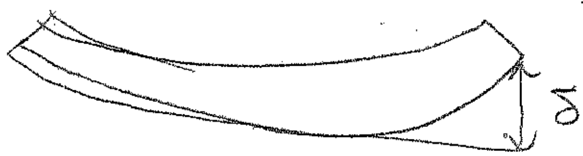
FIND ⇒ TENSION IN ROPE
INTL

FIND θ WHEN T = 3W

(2)



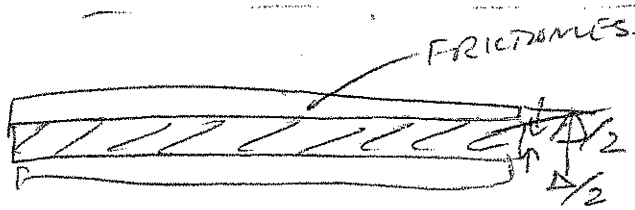
$\sigma_0 = \frac{\sigma_y}{w}$



FIND δ

IS THIS SOLN THE 3D EXACT SOLN

(2)



L, H, E, I

WHAT IS THE STATE OF STRESS

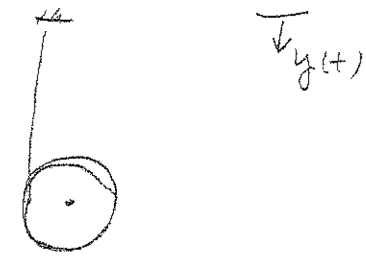
w/ ADHESION/FRICT

Dynamics

4) Suppose there's a chair how can one find the principal moment of inertia of the chair?

Dynamics

1)

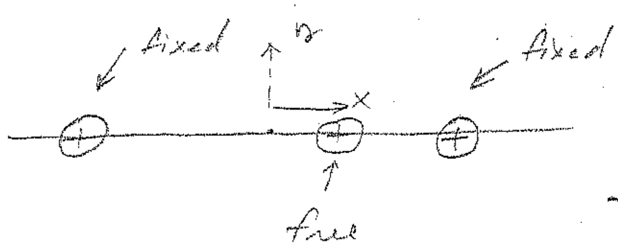


Spool with string falling down

a) find $y(t)$

b) will it swing?

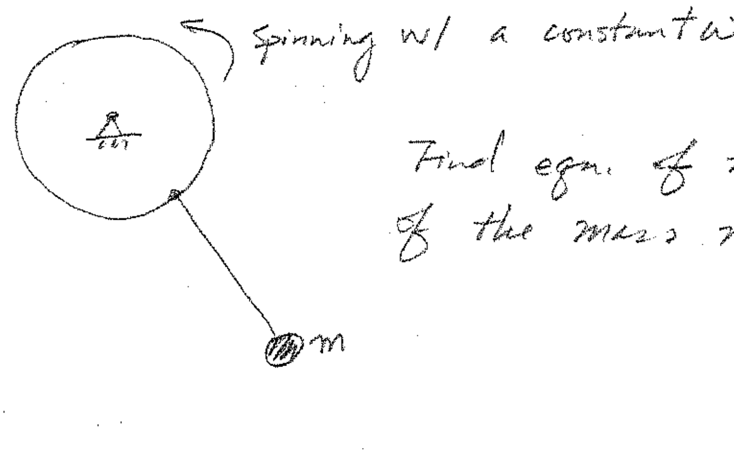
2)



3 protons

Discuss the stability of the free proton?

3)



Find eqn. of motion of the mass m.

Alternate form

Using balance of linear momentum, derive the equations of motion for a solid body B with surface S.

~~Describe quantities~~

What, exactly, are traction, stress, and how are they related?

~~Write down the~~

(2)

~~Starting from N~~

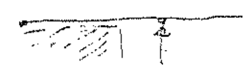
Consider a body B of volume V and surface S. Based on Newton second law, Write down the equations of motion in integral form over the whole body. Now find a local form of this equation valid at any point in the body.

Do you know how to write the virtual work principle?

1. Dynamics. Jan 21.

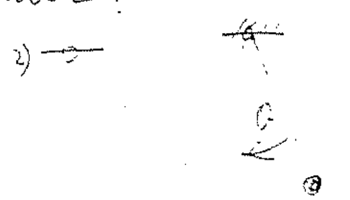
Examinee: G. Zhang

Questions: 1) A rule initially lies on a table supported by hand. Then withdraw the hand. ~~Describe~~ Derive equations of motion.



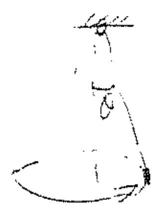
What is the relationship between the angular velocity ω and the velocity of mass center?

2)



Derive the velocity and acceleration.

3)



given constant ω, find θ.

How many method can you use

4). There was an experimental set-up, an oscilloscope and a hammer attached to it. Prof Sachse gives a small blow to the table with hammer and asks to explain what the pulse on the screen indicates?

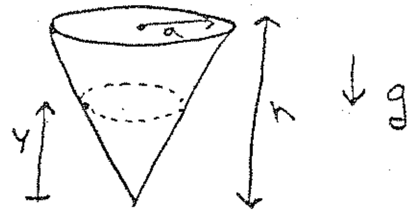
What is inside the hammer?

Write down the equation for the mechanism inside the hammer. Explain how to measure the force?

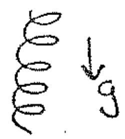
5). What is the difference between Newton's method ~~[illegible]~~ and Lagrange equation method.

What is constraint?

Sample Dynamics Questions (Not from any exam)

1. 

A particle travels in a circular path on the inside of a cone. Find its speed v as a function of y.

2. 

A particle travels along a helical spring, until it reaches the end of the spring and falls off. What is its trajectory?

3. State Newton's 3 laws; D'Lambert's Principle.

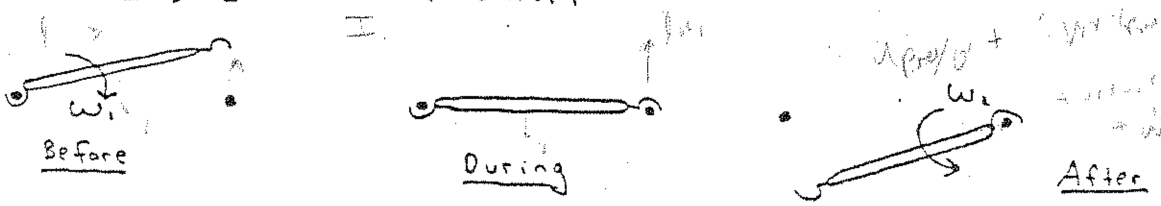
4. Find the altitude of a geosynchronous orbit.

5. A single propeller airplane is making a right turn. The propeller is spinning counter-clockwise as viewed from behind. What is the gyroscopic effect? How could you compensate on a multi-engine craft.

Answers: Nose rises. Have the propellers spin in opposite directions

6. Explain the 5 term acceleration formula. What is the Coriolis acceleration?

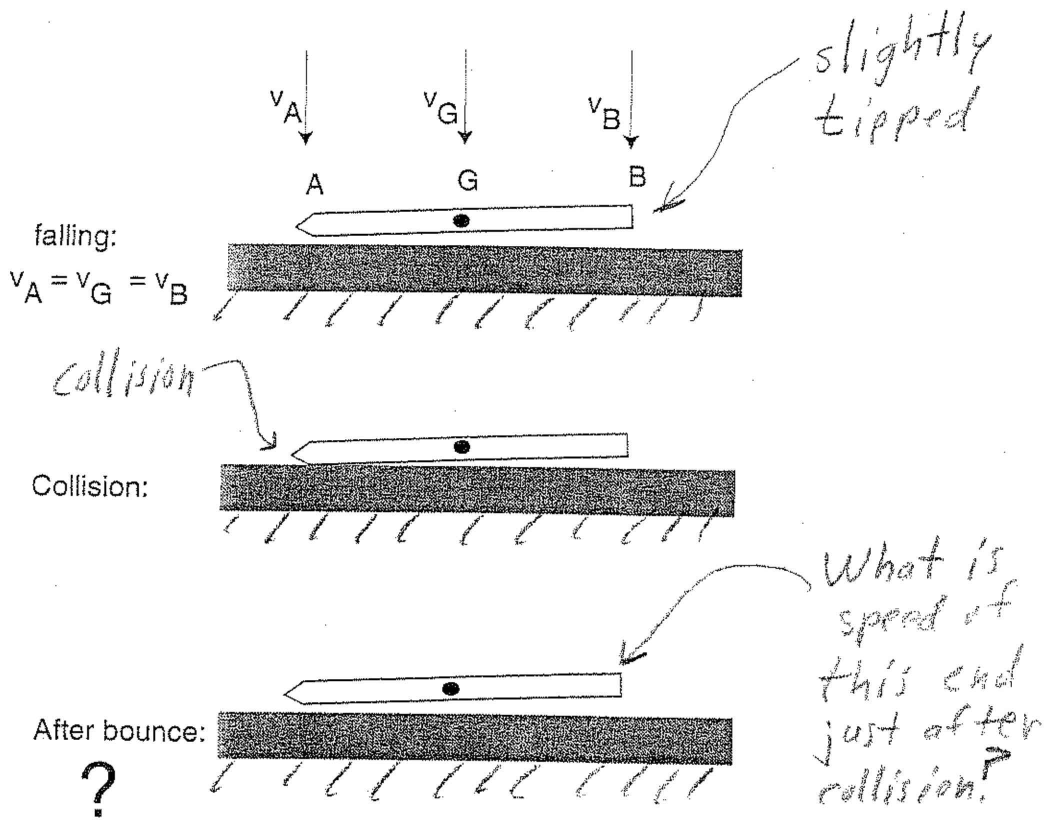
7. A motor is sitting on a turntable. It spins at ω_1, and its shaft is horizontal. The turntable spins at ω_2. Find $\underline{\omega}$, $\underline{\alpha}$ of the shaft.

8. 

Show that $\omega_2 = \frac{1}{2}\omega_1$. Rigid rod, $\bar{I} = \frac{1}{3} m L^2$, $I_{end} = \frac{1}{12} m L^2$

...Dr. Goyal runs an animated picture of a pencil dropping: The entire pencil falls at the same velocity but when one end hits first, the follow-up blow on the other end can occur at up to twice the speed. That's because the end that hits first bounces back at the same velocity, *thereby doubling the speed of the opposite end.* — Wall Street Journal, December 9, 1993

True or False? Why?



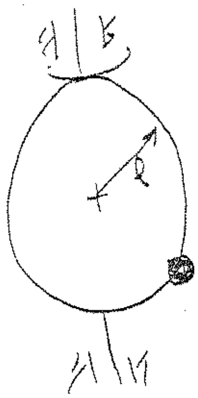
"follow up blow can occur at up to twice the speed" means

speed of V_B can increase by up to a factor of 2.

Dynamics

i) Rand had a baseball book that says ~~for~~ maximum distance, one must hit the ball at an angle ___.
What is the angle.
How would drag affect this angle.
(The book says 35°, why?

ii)



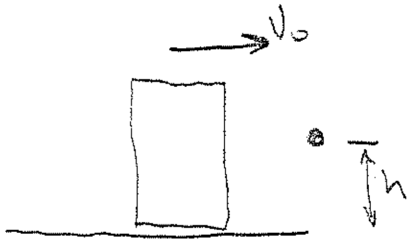
Bead on a hoop which is free to spin about axis shown

How many D.O.F.?

Find Equations of Motion

Is anything Conserved?

iii)

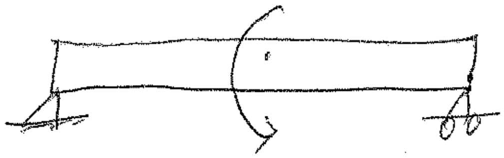


A cylinder slides into a horizontal rod at height h with velocity Vo.
What should h be so the can does not tip over.

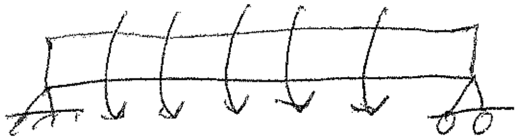
SOLID MECHANICS

1 a)

Shear & Moment Diagram



b)

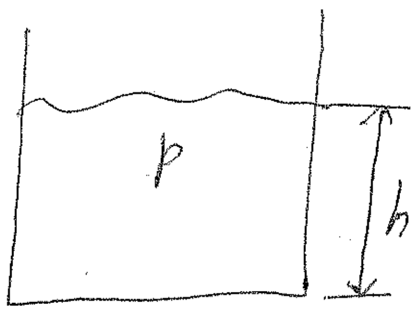


Distributed Moment

2. a) What is Stress (Traction etc.)

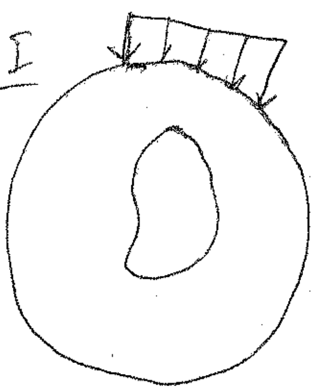
b) Compare stress state in a fluid vs. Solid

c)



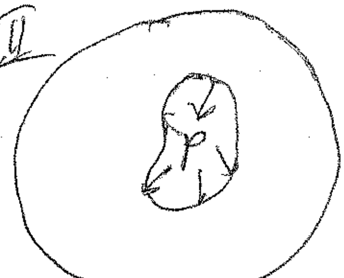
Find stress tensor ~~of~~ at every point.

3) Prob I



p constant

Prob. II

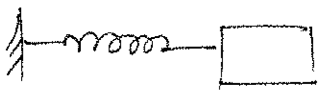


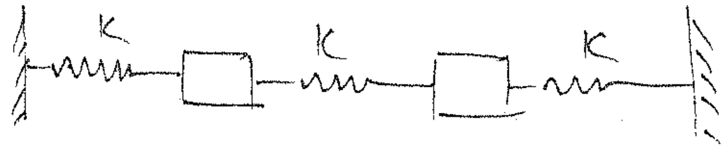
1. Number of solutions for Problem I
2. I Known, find soln. to II
3. What happens if matl. is inhomogeneous?

Dynamics

08/23/2002

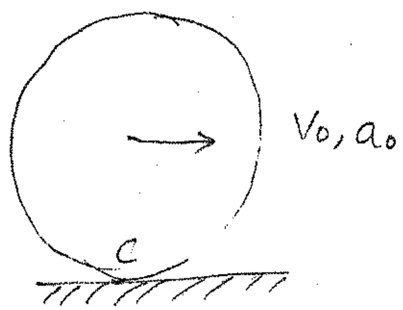
1)





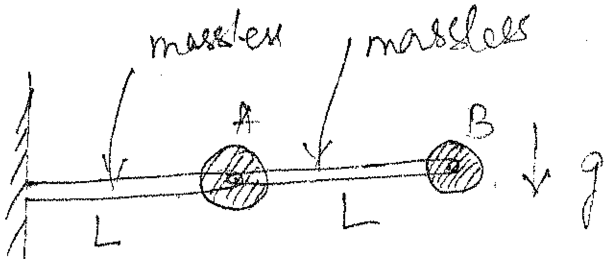
what is resonance, natural frequency, normalmodes...

✓2)



find ω, a_c, V_c etc

✓3)



$a_A(t=0^+) = ?$