MAE 5730: Intermediate Dynamics & Vibrations

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Course Summary

I. Dyn. of systems of particles & rigid bodies

1. Uses N-E & maximal coord's (full DOF spec), minimal coord's (minimal DOF to spec. a state), La Grange eqn's
2. Finds governing ODEs for system motion, solve for system motion functions
3. Animate motion using MATLAB

II. Vibrations

1. Use & calculate with concepts such as normal modes, resonance, freq. response, modal damping, vibration isolation

- Will <u>never</u> use Laplace, Fourier, etc., since most ODEs will be nonlinear, solved numerically

Polar Coordinate Kinematics

$\vec{r} = r\hat{e}_r + 0\hat{e}_\theta \qquad \vec{v} = \frac{d}{dt}(r\hat{e}_r) = \frac{dr}{dt}\hat{e}_r + r\dot{\hat{e}}_r = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$

$\vec{a} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$ (with $\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$)

$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

- For numerical solutions, always convert higher order ODEs to 1st order

ex: spring oscillator: $\ddot{x} = \frac{-k}{m}x \Rightarrow \begin{cases} \dot{x} = v_x \\ \dot{v}_x = -\frac{k}{m}x \end{cases}$ $\dot{z} = f(z),\ z(0) = z_0$

$\Delta z = \dot{z}\,\Delta t$

MATLAB

- Pass parameters in single struct ex: `p.m = 5; p.k = 10; res = compute(p)` % passes all struct data members
- Use functions for numerical method, and ODE function

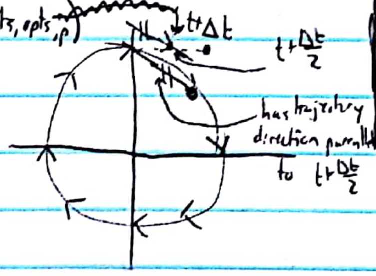
ex: `[t, z] = eulerMethod(@ODEfunc, tspan, initZ, paramStruct)`

% ODEfunc is function containing just computations to find $\dot{z}$ given state vector and parameters

ode45: "opts.RelTol = 1e-6;" ... `ode45(@RHS, tSpan, inits, opts, p)`

Midpoint Method ≡ RK2

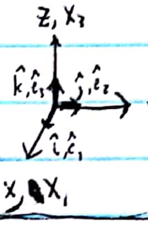
- Measures $\dot{z}(t + \frac{\Delta t}{2})$, uses $z_{n+1} = z_n + \dot{z}(t + \frac{\Delta t}{2})$



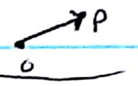
Animation in MATLAB

- Passing a full array of t values to ode45 doesn't affect solution, merely output, which interpolates to give requested values
- MATLAB "tic": starts counting real time; "toc": returns real time since initial "tic"
- Look at examples on website for better methodology

Notation

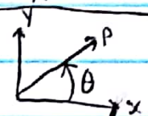


$\vec{r}_{P/O} \equiv$



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \equiv x_i\hat{e}_i$

Components Notation — Index Notation



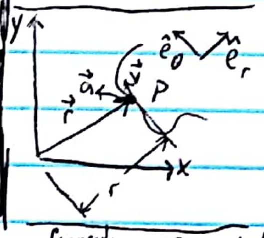
Use arrows on angles to indicate sign convention

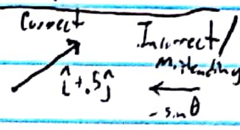
$\vec{v} \equiv \vec{v}_P = \vec{v}_{P/O} \equiv \vec{v}_{P/F}$

$\vec{v}_{P/Q} = \vec{v}_{P/O} - \vec{v}_{Q/O}$

$\vec{v} = \dot{\vec{r}} = v_i e_i$

$\vec{v}_x = \vec{v}\cdot\hat{i} = [\vec{v}]_x$





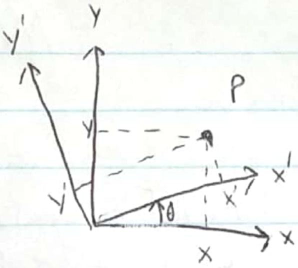
Find $(x,y) = f(x', y')$

$x\hat{i} + y\hat{j} = x'\hat{i}' + y'\hat{j}' \Rightarrow x = x'\hat{i}'\cdot\hat{i} + y'\hat{j}'\cdot\hat{i}$

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix}$$



$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$ Rotation $xy \to x'y'$ by $+\theta$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} = [R] \qquad \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & & y_n \end{bmatrix} = [R] \begin{bmatrix} x_1' & x_2' & \cdots \\ y_1' & y_2' & \end{bmatrix}$$

$\theta \Rightarrow R$

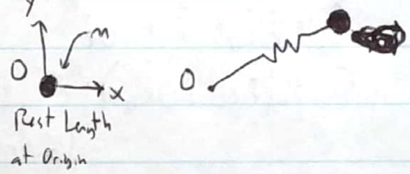
- All rigid motions in 2D described by rotation + displacement $\vec{d}$ $\quad \begin{bmatrix} x_1 & x_2 & \cdots \\ y_1 & y_2 & \cdots \end{bmatrix} = P \quad P' = [R][P] + \vec{d}$
- Break problem into 3 pieces: geometry & kinematics $(m\vec{a})$, force laws & constitutive laws $(\vec{F})$, laws of mechanics $(\vec{F} = m\vec{a})$

I. Geometry: $\vec{v} = \dot{\vec{r}}$, $a = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta \ldots$

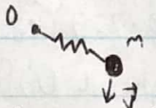
II. Force Laws: $\vec{F} = kx$, $\vec{F} = -bv$, $\vec{F}_{12} = \frac{+Gm_1m_2}{r^2}\hat{e}_{12}$, $m\ddot{\vec{r}} = \frac{-k(r - l_0)\vec{r}}{r}$, dashpot: $\vec{F}_{12} = -c\left(\frac{\dot{\vec{r}}_{12}\cdot\vec{r}_{12}}{|r_{12}|^2}\right)\vec{r}_{12}$

III. Laws of Mechanics: $\vec{F} = m\vec{a}$

- Harmonic oscillators: Only for a zero rest length spring



Fun: 2D spring pivoting about a point



$\dot{\vec{L}} = 0 \Rightarrow \frac{d}{dt}(r\dot{\theta}) = \dot{r}\dot{\theta} + r\ddot{\theta} = 0$

$\vec{F} = -k(r - r_0)\hat{e}_r = m\vec{a} \Rightarrow \frac{-k}{m}(r - r_0) = \ddot{r} - r\dot{\theta}^2 \Rightarrow$

2D Linear Spring Rotating About Point

$\ddot{r} = r\dot{\theta}^2 - \frac{k}{m}(r - r_0)$

$\ddot{\theta} = -\dot{r}\dot{\theta}/r$

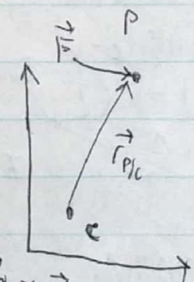
$$x = \begin{Bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{Bmatrix};\ \dot{x} = \begin{Bmatrix} x_3 \\ x_4 \\ x_1x_4^2 - \frac{k}{m}(x_1 - r_0) \\ -x_3x_4/x_1 \end{Bmatrix}$$

Gravitation: $\vec{F}_{12} = Gm_1m_2 \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$

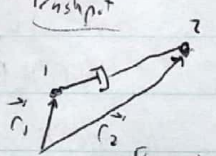
- Take $\vec{F} = m\vec{a}$, $\dot{\vec{r}} = \vec{v}$, $\dot{\vec{v}} = \frac{\vec{F}_{net}}{m}$ as the fundamental mechanical eqns.
- Derive other relationships, facts, etc. from it

Linear Momentum: $\vec{L} \equiv m\vec{v}$, $\boxed{\vec{F} = \dot{\vec{L}}}$ (implies) $\vec{P} = \Delta\vec{L}$

- There is a direction $\hat{\lambda}$ such that $\vec{F}\cdot\hat{\lambda} = 0$, then $\dot{L}_\lambda = 0$



Dashpot



$$\vec{F}_{12} = -c \cdot \frac{\left[(\vec{v}_2 - \vec{v}_1)\cdot(\vec{r}_2 - \vec{r}_1)\right](\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^2}$$

Angular Momentum

$\boxed{\vec{r}_{P/C} \times \vec{F} = \vec{r}_{P/C} \times \vec{a}_{P/F}\, m}$ — Most general form of ang. momentum balance, for any motion of C

Special Case: C is fixed: $\frac{d}{dt}(\vec{r}_{P/C} \times \vec{v}_{P/C}) = \vec{v}_{P/C} \times \vec{v}_{P/C} + \vec{r}_{P/C} \times \vec{a}_{P/C} = \vec{r}_{P/C} \times \vec{a}_{P/F}$

$\sum \vec{M}^{ext}_{/C} = \dot{\vec{H}}_{/C}$ for Newtonian C

For Fixed C: $\underbrace{\vec{r}_{P/C} \times \vec{F}}_{\vec{M}_{/C}} = \frac{d}{dt}\underbrace{(m\,\vec{r}_{P/C} \times \vec{v}_{P/F})}_{\vec{H}_{/C}} \Rightarrow \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

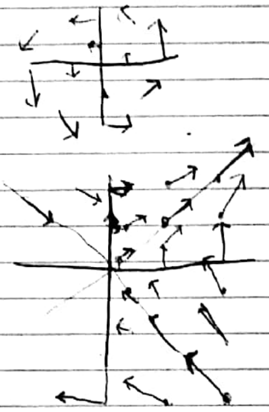
Energy

$\vec{F}\cdot\vec{v} = m\vec{a}\cdot\vec{v} = m\dot{\vec{v}}\cdot\vec{v} = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \vec{F}\cdot\vec{v} = P = \text{power} = \dot{E}_k$

$\int_{t_1}^{t_2} P\,dt = \Delta E_k = \int \vec{F}\cdot\vec{v}\,dt = \int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = E_k$

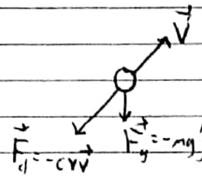
- Not generally solvable at start since $\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$
- If $\vec{F} = \vec{F}(\vec{r})$ then is solvable

- For $\vec{F} = \vec{F}(\vec{r})$, can evaluate $\int_{r_1}^{r_2} \vec{F} d\vec{r}$ without knowing path if $\vec{F}$ field is conservative, and can't if $\vec{F}$ is not conservative
- $\vec{F}$ is conservative if $\nabla \times \vec{F} = \vec{0}$, or in 2D: $\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0 \ \forall \ (x,y) \in$ domain D
- " " $\vec{F} = \nabla f(\vec{r})$, gradient of scalar field, exists a potential energy, equivalent
- Conservative force fields: $\vec{F} = f(r)\hat{e}_r$; $\vec{F} = $ const
- $\vec{F} = \lambda \times \vec{v} \Rightarrow \vec{F} \perp \vec{v} \Rightarrow \dot{E}_k = 0 \Rightarrow E_k =$ const conserves energy, but is not a conservative force
- Non-conservative force fields: $F_x = -ky$, $F_y = kx$



ex. quadratic drag

$$P_{drag} = \frac{d}{dt} E_{tot} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + mgy\right) = \vec{F}_d \cdot \vec{v} = -cv\vec{v}\cdot\vec{v} = -cv^3 = P_{drag, quad}$$



Multi-Particle Systems

define $\vec{L} = \sum \vec{L}_i = \sum m_i \vec{v}_i$ $\quad \vec{H}_{/C} = \sum \vec{H}_{K/C} = \sum \vec{r}_{i/C} \times m_i \vec{v}_i$ (for C fixed in F)

$\dot{\vec{H}}_{/C} = \sum \vec{r}_{i/C} \times m_i \vec{a}_i$ (for C moving arbitrarily)

Assume $\sum \vec{F}_j^{int} = \vec{0}$, e.g., pairwise equal & opposite

$\sum \vec{F}_j^{int} \times \vec{r}_{j/C} = \vec{0}$, i.e., internal moments cancel

- Don't assume all internal forces & moments are pairwise opposite, not strictly true

Linear momentum: $\sum \vec{F}_i = \sum m_i \dot{\vec{v}}_i = \sum_{ij} \vec{F}_{ij}^{int} + \vec{F}_i^{ext} \Rightarrow \sum \vec{F}_i^{ext} = \sum m_i \dot{\vec{v}}_i$ (sums to 0 $\Rightarrow$)

Center of mass: $\sum m_i \vec{r}_i = m_{tot} \vec{r}_G = m_{tot} \vec{r}_{com} \Rightarrow \sum \vec{F}^{ext} = m_{tot} \dot{\vec{v}}_G$

Angular momentum: $\vec{H}_{/C} = \sum \vec{r}_{i/C} \times m_i \vec{v}_i = \sum \vec{r}_{i/C} \times m_i (\vec{v}_{G/C} + \vec{v}_{i/G}) = \sum (\vec{r}_{G/C} + \vec{r}_{i/G}) \times m_i (\vec{v}_{G/C} + \vec{v}_{i/G})$

for C fixed in F

$$\vec{H}_{/C} = m_i \left[\sum \vec{r}_{i/G} \times \vec{v}_{i/G} + \sum \vec{r}_{G/C} \times \vec{v}_{i/G} + \sum \vec{r}_{i/G} \times \vec{v}_{G/C} + \sum \vec{r}_{G/C} \times \vec{v}_{G/C} \right] = \left[\vec{H}_{/G} + \vec{H}_{G/C} + 0 + 0 \right] \Rightarrow \vec{H}_{/C} = \vec{H}_{/G} + \vec{H}_{G/C}$$

Energy

$$\sum P_i = \frac{d}{dt} E_k^{tot} = \sum \tfrac{1}{2} m_i v_i^2 = E_{K,G} + E_{K/G} = \tfrac{1}{2} m_{tot} \vec{v}_G \cdot \vec{v}_G + \tfrac{1}{2} \sum m_i \vec{v}_{i/G} \cdot \vec{v}_{i/G}$$

- Generally, power depends on both external and internal forces, unless no work is done by internal forces

Some Theorems:

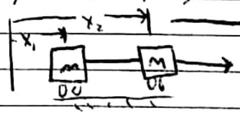
1. For a Newtonian frame F, another frame β is Newtonian if $\vec{a}_{\beta/F} = \vec{0}$ and $\dot{\theta}_{\beta/F} = 0$
2. Can't distinguish between an accelerating reference frame from a fixed frame under a constant gravitational field

For $\vec{F}^{ext}_{tot} = \vec{0}$, $\vec{v}_{G,0} = 0$, then $\vec{x}_G(t) = 0$, but this idea does not apply to rotational motion

For $\vec{T}^{ext}_{tot}(t) = \vec{0}$, and IC $\theta_0 = 0$, $\dot{\theta} = 0$, there can still be changes in θ, i.e., $\theta(t) \neq 0$, in general $\sum \vec{r}_{i/G} \times m_i \dot{\vec{v}}_i = \vec{0}$

non-integrable equation, can't integrate to find that same quantity $= 0$

Constraints



ex. 2 masses tied together and pulled

- Can't solve using $\vec{F}_i = \vec{F}_i(\vec{r}, \vec{v}, t)$ since $T \neq g(\vec{r}, \vec{v}, t)$
- Can replace model with spring-damper, but this sucks
  - Just because spring-damper isn't rigid doesn't mean it's better
  - Adds parameters, complexity, vibration to motion
  - Adds very fast dynamics that we don't care about, solver now must take tiny time steps for two vibration, which sucks.
- STIFF ODEs: ODEs which include unwanted very fast dynamics that make numerical solvers take forever

$\boxed{m} \to T \leftarrow \boxed{m} \to F$

Set $x_2 - x_1 = l_0 \Rightarrow \ddot{x}_2 - \ddot{x}_1 = 0$;

$$\left.\begin{aligned} m\ddot{x}_1 - T &= 0 \\ m\ddot{x}_2 + T &= F \\ \ddot{x}_1 - \ddot{x}_2 &= 0 \end{aligned}\right\} \text{System of DAEs} \quad (**)$$

Algebraic Eqns

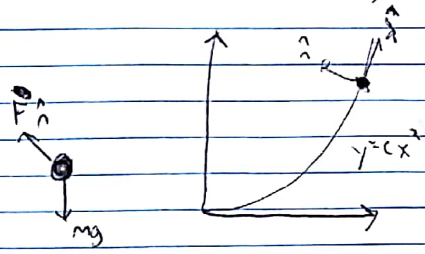
- Put ** in RHS file, solve
- Can always manipulate DAEs that

$$\begin{bmatrix} m & 0 & -1 \\ 0 & m & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix} \Rightarrow (m_1 + m_2)\ddot{x}_1 = F$$

Scanned by CamScanner

- For a 1 DOF system, can always solve for EOM using energy methods
- Constraints are what makes dynamics hard. They ruin the power of $\vec{F}=\vec{F}(\vec{r},\vec{v},t)$, and most of advanced dynamics is coming up with ways to eliminate constraints in [redacted] LAE's

ex: bead on wire



- Can use energy approach to find $|v|=f(y)$, [redacted] solve 1st order ODE for $y(t), x(t)$
- Try using DAE: →

$$\vec{F}=m\vec{a} \Rightarrow F\hat{n}+mg\hat{j}=m\vec{a}=m\ddot{x}\hat{i}+m\ddot{y}\hat{j}$$

$\hat{n}=\hat{k}\times\hat{\lambda}$ $\hat{\lambda}$ in direction of slope, $\hat{\lambda}=\dfrac{\hat{i}+2cx\hat{j}}{\sqrt{1+(2cx)^2}}$

$$\hat{n}=\frac{1}{\sqrt{1+(2cx)^2}}\left[\hat{j}-2cx\hat{i}\right]$$

$$\vec{F}=m\vec{a} \Rightarrow \frac{F(\hat{j}-2cx\hat{i})}{\sqrt{1+(2cx)^2}}-mg\hat{j}=m\ddot{x}\hat{i}+m\ddot{y}\hat{j} \qquad y=cx^2 \Rightarrow \dot{y}=2cx\dot{x} \Rightarrow \ddot{y}=2c\dot{x}^2+2cx\ddot{x}$$

2 DAEs linear in $\ddot{x},\ddot{y},F$

$$\begin{bmatrix} m\hat{i} & m\hat{j} & \frac{2cx\hat{i}-\hat{j}}{\sqrt{1+(2cx)^2}} \\ -2cx & 1 & 0 \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{y}\\ F\end{Bmatrix}=\begin{Bmatrix}-mg\hat{j}\\ 2c\dot{x}^2\end{Bmatrix}$$

Solve [redacted] linear system, plug solution for $\ddot{x},\ddot{y}$ into numerical RHS solution

Alternative Methods for Constraints:
1. Use cons. of energy for 1 DOF systems
2. Manually manipulate LAEs to eliminate [redacted] unknown F term (above) DAE's
3. Use Lagrange eqns
4. Be systematic w/ Newton eqns

[redacted]

Be Systematic with Newton's Equations

ex: same bead on wire

Use minimal coordinates, which in this case is 1 coordinate for 1 DOF

DOF: x $\vec{F}=m\vec{a}=m(\ddot{x}\hat{i}+\ddot{y}\hat{j})$ $y=cx^2 \Rightarrow \ddot{y}=2c\dot{x}^2+2cx\ddot{x}$ $\Rightarrow \{-mg\hat{j}+F\hat{n}=m(\ddot{x}\hat{i}+(2c\dot{x}^2+2cx\ddot{x})\hat{j})\}\cdot\hat{\lambda}$ or $\cdot\vec{v}$ (makes $\hat{n}$ term go to zero)

$\dot{y}=2cx\dot{x}$

$\vec{v}=\dot{x}\hat{i}+2cx\dot{x}\hat{j}$

$$-2mgcx\dot{x}=m\left(\ddot{x}\dot{x}+(2cx\dot{x})(2c\dot{x}^2+2cx\ddot{x})\right)$$

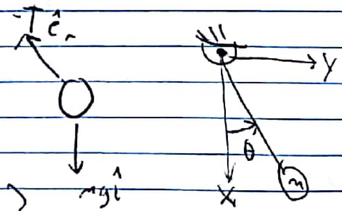
2nd order ODE for x

Limits: An Aside
- When trying to say "when $a\to b$, then $c\to d$", be quantitative in evaluation. Choose a good metric of approaching
  ex: as $a\to b$, $c\to d$, might plot $c-d$, or $c/d-1$, and then can plot on log scale

More Constraints

[redacted] Finding simple single pendulum ODE many ways

Method 1: DAE's $\vec{F}=m\vec{a} = -T\hat{e}_r+mg\hat{i}=m\ddot{x}\hat{i}+m\ddot{y}\hat{j}$; $-T\hat{e}_r=\dfrac{-x\hat{i}-y\hat{j}}{x^2+y^2}T$



Constraint eq: $l=\text{const} \Rightarrow x^2+y^2=\text{const} \Rightarrow \dfrac{d^2}{dt^2}(x^2+y^2)=0 \Rightarrow \dot{x}^2+x\ddot{x}+\dot{y}^2+y\ddot{y}=0$

3 [redacted] LAEs for $\ddot{x},\ddot{y},T$:

$$\begin{bmatrix} x & y & 0 \\ m & 0 & x/\sqrt{x^2+y^2} \\ 0 & m & y/\sqrt{x^2+y^2}\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{y}\\ T\end{Bmatrix}=\begin{Bmatrix}\text{some stuff}\end{Bmatrix}$$

Method 2: [redacted] polar coord, kill $T\hat{e}_r$

$-T\hat{e}_r+mg\hat{i}=\vec{F} \Rightarrow -T\hat{e}_r\cdot\hat{e}_\theta+mg\hat{i}\cdot\hat{e}_\theta=\vec{F}\cdot\hat{e}_\theta$

$mg\hat{i}\cdot\hat{e}_\theta=m\vec{a}\cdot\hat{e}_\theta=m[(\ddot{r}-r\dot{\theta}^2)\hat{e}_r+(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{e}_\theta]\cdot\hat{e}_\theta \Rightarrow mg\hat{i}\cdot\hat{e}_\theta=mr\ddot{\theta}=-mg\sin\theta=mr\ddot{\theta} \Rightarrow \ddot{\theta}=\dfrac{-g\sin\theta}{l}$

Method 3: energy: $\frac{1}{2}m(r\dot{\theta})^2-mgr\cos\theta=E=\text{const}$ 1st order ODE in $\theta,\dot{\theta}$, or take $\frac{d}{dt}$ to find $\ddot{\theta} \Rightarrow r^2\dot{\theta}\ddot{\theta}+gr\sin\theta\dot{\theta}=0$

$r\ddot{\theta}+g\sin\theta=0 \Rightarrow \ddot{\theta}=-\frac{g}{r}\sin\theta$ for $\dot{\theta}\neq 0$

Method 4: Lagrange eqns [redacted] let $\theta=q$, generalized coordinates

Define Lagrangian $\mathcal{L}=KE-PE=\frac{1}{2}m(r\dot{\theta})^2+mgr\cos\theta$

$$\boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)-\frac{\partial\mathcal{L}}{\partial q}=0} \qquad \dot{q}=\dot{\theta} \Rightarrow \frac{\partial\mathcal{L}}{\partial\dot{q}}=mr^2\dot{\theta} \qquad \frac{\partial\mathcal{L}}{\partial q}=-mgr\sin\theta \quad \Rightarrow \frac{d}{dt}(mr^2\dot{\theta})+mgr\sin\theta=0 \Rightarrow \ldots$$

Lagrange Eqn's

Method 5: Angular momentum balance

D'Alembert's Principle

$\vec{F}_i^{tot} = m_i \vec{a}_i$; $\vec{F}_i^{tot} = \vec{F}_i^{applied} + \vec{F}_i^{constraints} \Rightarrow \vec{F}_i^{app} + \vec{F}_i^{constraint} - m\vec{a}_i = \vec{0} \Rightarrow$ ← Any random vector in the world

$\Rightarrow$ Dot product with random velocity vector $\delta \vec{v}^*$ $\Rightarrow (\vec{F}_i^{tot} - m_i\vec{a}_i)\cdot \delta\vec{v}_i^* = 0 = \sum\left[(\vec{F}_i^{constr} + \vec{F}_i^{app}) - m_i\vec{a}_i\right]\cdot\delta\vec{v}_i^* = 0$

- Specify $\delta \vec{v}_i^*$ such that the "velocities" are consistent with the constraints
- Assume that:
  - If restraint forces do no net work in motions that satisfy constraints, then $\sum \vec{F}_i^{constraint}\cdot\delta\vec{v}_i^* = 0$
  - Then: $\boxed{\sum(\vec{F}_i^{applied} - m\vec{a}_i)\cdot\delta\vec{v}_i^* = 0}$ The Fundamental Equation of Analytical Dynamics, only true if ↑ is true

ex: pendulum, minimal parameter θ, so $x = x(\theta) = l\cos\theta$, $y = y(\theta) = l\sin\theta$

Then define $\delta\vec{v}^* = \frac{\partial x}{\partial\theta}\frac{\partial\theta}{\partial t}\hat{i} + \frac{\partial y}{\partial\theta}\frac{\partial\theta}{\partial t}\hat{j} = (-l\sin\theta\hat{i} + l\cos\theta\hat{j})\dot{\theta}$ — Magnitude of $\delta\vec{v}^*$ not important, only direction $\hat{e}_\theta$

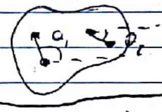
Then $(\vec{F}_i^{applied} - m\vec{a})\cdot\delta\vec{v}^* = 0 \Rightarrow (+mg\hat{j} - m[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta])\cdot\hat{e}_\theta = 0 = (g\hat{j} + \vec{a})\cdot\hat{e}_\theta$

$\Rightarrow g\sin\theta + l\ddot{\theta} = 0$

2D Rigid Objects

– a collection of particles such that d_{ij} = constant ← all distances between all pairs of particles

- Any pair of particles in the object have an angle WRT some reference axis ($\theta_1, \theta_2, \ldots$ potentially infinite)
- After any motion, $\theta_{1,new} - \theta_1$, $\theta_{2,new} - \theta_2 = \ldots$ all are the same $= \theta$ = rigid object rotation



- To find $\vec{v}_{P/O}$ for an arbitrary point on rigid object, similarly, $\vec{\omega} = \dot{\theta}\hat{k}$

$\vec{v}_P = \vec{v}_{O'} + \vec{v}_{P/O'}$ for O' as a reference point also on rigid object, but the $\vec{v}_{P/O'} = \vec{\omega}\times\vec{r}_{P/O'}$ $\Rightarrow$ $\boxed{\vec{v}_P = \vec{v}_{O'} + \vec{\omega}\times\vec{r}_{P/O'}}$ For any 2 points P, O' on rigid object

$\vec{L} = \sum m_i\vec{v}_i = m_{tot}\vec{v}_G$, for a single rigid object, Center of Mass is objective, not dependent on coordinate system

$\vec{H}_{/O} = \vec{H}_{G/O} + \vec{H}_{/G} = \vec{r}_{G/O}\times\vec{v}_{G/O} + \int \vec{r}_{/G}\times\vec{v}_{/G}\,dm = \vec{r}_{G/O}\times\vec{v}_{G/O} + \int\vec{r}_{/G}\times(\vec{\omega}\times\vec{r}_{/G})\,dm = \vec{r}_{G/O}\times\vec{v}_{G/O} + \int|\vec{r}_{/G}|^2\omega\,dm\,\hat{k} = \vec{r}_{G/O}\times\vec{v}_{G/O} + I^G\omega\hat{k}$

$\boxed{\vec{H}_{/G} = I^G\omega\hat{k}}$ $\vec{H}_{/O} = \vec{H}_{/G} + \vec{H}_{G/O}$; $E_k = E_{KG} + E_{K/G}$ $\boxed{\frac{1}{2}m_{tot}v_G^2 + \frac{1}{2}I^G\omega^2 = E_k}$ $I^G \equiv \int r_{/G}^2\,dm$

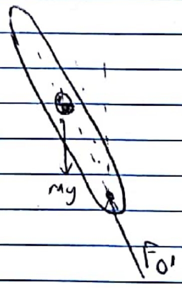
AMB: for a collection of rigid objects, $\boxed{\sum\vec{M}_{/O} = \dot{\vec{H}}_{/O} = \sum\vec{r}_{G_i/O}\times m_i\vec{a}_{G_i} + \sum I^{G_i}\dot{\omega}_i\hat{k}}$

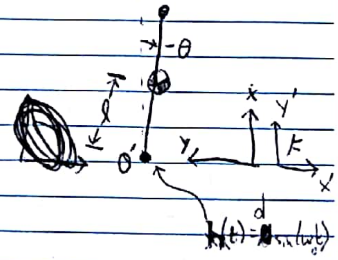
LMB Multiple Rigid Objects: $\sum\vec{F} = \dot{\vec{L}} = \sum m_i\vec{a}_{G_i}$ ← sum over all objects

Power Balance: $P_{total} = \sum\vec{F}_{all}^i\cdot\vec{v}_i = \frac{d}{dt}(E_k)$

- Assume all $\vec{F}_{int}$ do no work, true for forces internal to 1 rigid object

Vertically Actuated Inverted Pendulum





$\vec{F}_{O'}$ direction unknown, try AMB about O'

$\sum\vec{M}_{O'} = \dot{\vec{H}}_{/O'} = \vec{r}_{G/O'}\times(-mg\hat{j}) = (-\sin\theta\hat{i} + \cos\theta\hat{j})\times(-mg\hat{j}) = \ldots$

$\dot{\vec{H}}_{/O'} = \vec{r}_{G/O'}\times m\vec{a}_G + I^G\dot{\omega}\hat{k} = (-\sin\theta\hat{i} + \cos\theta\hat{j})\times m\vec{a}_G + \frac{1}{3}ml^2\ddot{\theta}\hat{k}$

$mgl\sin\theta\hat{k} = \vec{r}_{G/O'}\times m\vec{a}_G + I^G\ddot{\theta}\hat{k}$ $\vec{r}_{G/O'} = l\hat{e}_r$

O' = base of pendulum
O = Fixed at middle of oscillation

$\vec{a}_G = \vec{a}_{G/O'} + \vec{a}_{O'} = l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r - d\omega_0^2\sin(\omega_0 t)\hat{j}$ $\hat{e}_r\times\hat{j} = -\sin\theta\hat{k}$

$\vec{r}_{G/O'}\times\vec{a}_G = (l^2\ddot{\theta} + ld\omega_0^2\sin(\omega_0 t)\sin\theta)\hat{k} \Rightarrow mgl\sin\theta = ml^2\ddot{\theta} + mld\omega_0^2\sin\theta\sin(\omega_0 t) + I^G\ddot{\theta} \Rightarrow ml(g + a_{y_{O'}})\sin\theta = (ml^2 + I^G)\ddot{\theta}$

$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'} \Rightarrow \vec{v}_{P/O} = \vec{v}_{O'/O} + \vec{\omega}_{B/O}\times\vec{r}_{P/O'} = \vec{v}_{O'/O} + |\vec{r}_{P/O'}|\dot{\theta}\hat{e}_\theta \Rightarrow \vec{a}_{P/O} = \vec{a}_{O'/O} + \dot{\vec{\omega}}\times\vec{r}_{P/O'} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/O'})$ where $a_{y_{O'}} = d\omega_0^2\sin(\omega t)$

Can now solve numerically in RHS function

$\vec{a}_{P/O} = \vec{a}_{O'/O} + \ddot{\theta}|\vec{r}_{P/O'}|\hat{e}_\theta - \omega^2|\vec{r}_{P/O'}|$

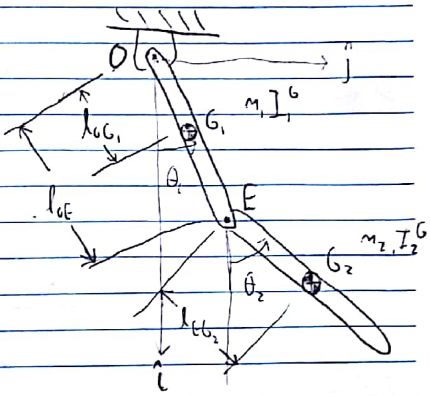
$$\begin{bmatrix}\theta_{11} & \theta_{21}\\ \theta_{12} & \theta_{22}\\ \theta_{13} & \theta_{23}\end{bmatrix} \quad [l_1 \; l_2] \quad \begin{bmatrix} l_1\cos\theta_{11} \\ l_1\cos\theta_{12} \end{bmatrix}$$

L	ω_{NE}	ω_{DAE}
1	4.19	2.089
2	1.04	1.04
3	0.477	0.694
4	~~0.259~~	~~0.521~~

$\omega_{NE} = 1.04$ (

1	L1	2	0	2	1
2	1	1	1	0	0
L1	1/L1	1/2	2	-2	-1

Double Pendulum



$AMB_{system/O} \Rightarrow \sum \vec{M}_{/O} = \dot{\vec{H}}_{/O} = \dot{\vec{H}}_{1/O} + \dot{\vec{H}}_{2/O} = \vec{r}_{G_1/O} \times m_1 g\hat{i} + (\vec{r}_{E/O} + \vec{r}_{G_2/E}) \times m_2 g\hat{i}$

$\vec{r}_{E/O} = l_{OE}\hat{\lambda}_1$; $\vec{r}_{G_2/E} = l_{EG_2}\hat{\lambda}_2$; $\hat{\lambda}_1 = \cos\theta_1\hat{i} + \sin\theta_1\hat{j}$; $\hat{\lambda}_2 = \cos\theta_2\hat{i} + \sin\theta_2\hat{j}$

$\dot{\vec{H}}_{1/O} = \vec{r}_{G_1/O} \times m_1\vec{a}_{G_1} + I_1^G\ddot{\theta}_1\hat{k}$; $\dot{\vec{H}}_{2/O} = \vec{r}_{G_2/O} \times m_2\vec{a}_{G_2} + I_2^G\ddot{\theta}_2\hat{k}$ All known except $\vec{a}_{G_1}, \vec{a}_{G_2}$

$\vec{a}_{G_1} = -\dot{\theta}_1^2\vec{r}_{G_1/O} + \ddot{\theta}_1\hat{k}\times\vec{r}_{G_1/O}$

$\vec{a}_{G_2} = \vec{a}_E + \vec{a}_{G_2/E}$ — $\ddot{\theta}_1 l_{EO}\hat{n}_1$, $\ddot{\theta}_2 l_{G_2E}\hat{n}_2$

$\vec{a}_E = -\dot{\theta}_1^2\vec{r}_{E/O} + \ddot{\theta}_1\hat{k}\times\vec{r}_{E/O}$ $\quad \vec{a}_{G_2/E} = -\dot{\theta}_2^2\vec{r}_{G_2/E} + \ddot{\theta}_2\hat{k}\times\vec{r}_{G_2/E}$

• Combining together, have 2 AMBs for $\ddot{\theta}_1, \ddot{\theta}_2$ as a function of parameters and $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$

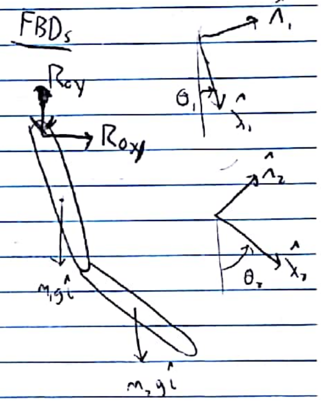
~~$\hat{\lambda}_1 = \cos\theta_1\hat{i} + \sin\theta_1\hat{j}$ $\quad \hat{\lambda}_2 = \cos\theta_2\hat{i} + \sin\theta_2\hat{j}$~~

$\hat{n}_1 = -\sin\theta_1\hat{i} + \cos\theta_1\hat{j}$ $\quad \hat{n}_2 = -\sin\theta_2\hat{i} + \cos\theta_2\hat{j}$

$AMB_{2/E} \Rightarrow \sum \vec{M}_{/E} = \dot{\vec{H}}_{2/E} = \vec{r}_{G_2/E}\times m_2 g\hat{i} = \vec{r}_{G_2/E}\times m\vec{a}_{G_2/O} + I_2^G\ddot{\theta}_2\hat{k}$

FBDs

System:



Use $AMB_{system/O}$

Vertically Actuated Single Pendulum DAE's

$\sum F_x = ma_x = F_{Ox}$

$\sum F_y = ma_y = F_{Oy} - mg$

$\sum M_{cG} = I\ddot{\theta} = F_{Ox}l\cos\theta + F_{Oy}l\sin\theta$

Constraints:

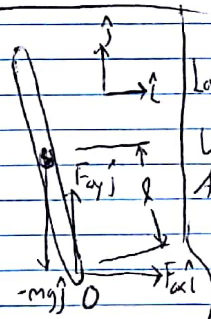
$x = -l\sin\theta \Rightarrow V_x = -l\dot{\theta}\cos\theta$

$y = l\cos\theta + A\sin(\omega_o t) \Rightarrow V_y = -l\dot{\theta}\sin\theta + A\omega_o\cos(\omega_o t)$

$a_x = -l[\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta]$

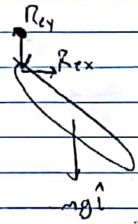
$a_y = -l[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta] + A\omega_o^2\sin(\omega_o t)$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & I & -ml\cos\theta & -ml\sin\theta \\ 1 & 0 & l\cos\theta & 0 & 0 \\ 0 & 1 & l\sin\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \\ \ddot{\theta} \\ F_{Ox}/m \\ F_{Oy}/m \end{Bmatrix} = \begin{Bmatrix} 0 \\ -g \\ 0 \\ l\dot{\theta}^2\sin\theta \\ -l\dot{\theta}^2\cos\theta - A\omega_o^2\sin(\omega_o t) \end{Bmatrix}$$



Lower link:

Use $AMB_{2/E}$



MATLAB Commands/Methods for Making RHS

Symbolic Methods

jacobian

equationsToMatrix - assembles linear eqns into $Ax = B$

matlabFunction - takes symbolic functions to MATLAB functions

• Can also put entire derivation with numbers in RHS file, solve every timestep

• Use print statements or matlabFunction to write RHS file from derivation

• Can spec sym variables as strictly real

"syms x y z real"

Lagrange Equations

Assume: n-DOF system with n minimal coordinates q_i

Holonomic constraints (no rolling or skates)

Method: Conservative forces

1. Find $E_k(q_i, \dot{q}_i)$, $E_p(q_i)$

2. Form the Lagrangian $\mathcal{L} = E_k - E_p$

3. Write Lagrange equations: $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) = 0$ for each q_i and respective $\dot{q}_i$

$$0 = \frac{\partial L}{\partial q_i} - \frac{\partial}{\partial q_i}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i^2}\ddot{q}_i$$

Derivation Flow: Symbolic position vectors

• Unit direction vectors, ~~[illegible]~~

• Simple relative accel vectors

• $\dot{\vec{H}}_{1/O}$, $\dot{\vec{H}}_{2/O}$

$\vec{a}_{P/\mathcal{F}} = \vec{a}_{P/B} + \vec{a}_{o'/o} + \vec{\omega} \times \vec{\omega} \times \vec{r}_{P/o'} + \dot{\vec{\omega}} \times \vec{r}_{P/o'} + 2\vec{\omega} \times \vec{v}_{P/B}$

Jacobian: $J(f(x,y)) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ $\quad J(f(x,y), g(x,y)) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$ $\quad J(\{f(\{x\})\})$, $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \end{Bmatrix}$, $\{f\} = \begin{Bmatrix} f_1(\{x\}) \\ f_2 \\ \vdots \\ f_n \end{Bmatrix}$

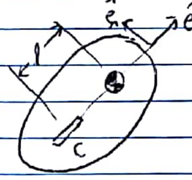
$J(EoM)$ = mass matrix = $[M]$ where $[M]\{\ddot{q}\} = \{b\}$ in a DAE solution

$J(f(x)) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & & \vdots \\ \vdots & & \ddots & \\ \partial f_n/\partial x_1 & \cdots & & \partial f_n/\partial x_n \end{bmatrix}$

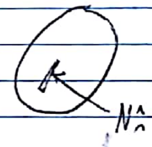
- Jacobian in MATLAB can do derivatives for symbolic vectors the same as scalars, no need to rewrite for expanding vectors, scalable

Chaplygin Sleigh
- skate C can freely rotate, but only translate in the direction of C, $\hat{e}_\lambda$

$\vec{v}_c \cdot \hat{e}_n = 0 \Rightarrow -\sin\theta \dot{x}_c + \cos\theta \dot{y}_c = 0$



- This is non-holonomic because it cannot be integrated to produce $f(x,y,\theta) = \text{const}$
- Compare to $\dot{x} = 0 \Rightarrow x = \text{const}$, which we can't do here and it screws stuff up
- 2 skates/wheels along the same perpendicular axis behave identically kinematically to 1: [ ] - - - [ ] ≡ [ ] - - - o ← frictionless pad
- # of DOF of $\vec{r}_{config} = 3$ } Different # of DOF for $\vec{r}_{config}$ and $\vec{v}$ is why this is weird $\quad \vec{r} = r_x, r_y, \theta$
- # of DOF of $\vec{v} = 2$ $\quad \vec{v} = v_\lambda, \dot{\theta}$



$\vec{v}_c = v_c \hat{\lambda}$, not a direct function of x, y, θ or derivatives

$\dot{\hat{\lambda}} = \dot{\theta}\hat{k} \times \hat{\lambda} = \dot{\theta}\hat{n}$

$\boxed{\vec{a}_c = \dot{v}_c\hat{\lambda} + \dot{\theta} v_c \hat{n}}$ $\quad \vec{a}_G = \vec{a}_{c} + \vec{a}_{G/c} = \vec{a}_c + l\ddot{\theta}\hat{n} - l\dot{\theta}^2\hat{\lambda} = \boxed{(\dot{v}_c - l\dot{\theta}^2)\hat{\lambda} + (\dot{\theta}v_c + l\ddot{\theta})\hat{n} = \vec{a}_G}$

LMB $\cdot \hat{\lambda}$: removes $N\hat{n}$ via "American Way 🇺🇸"

$\sum \vec{F} \cdot \hat{\lambda} = m\vec{a}_G \cdot \hat{\lambda} \Rightarrow 0 = m(\dot{v}_c - l\dot{\theta}^2) \Rightarrow \dot{v}_c = \dot{\theta}^2 l$

AMB$_c$: $\sum M_c = \dot{\vec{H}}_{/c} = \vec{0} = \vec{r}_{G/c} \times m\vec{a}_G + \ddot{\theta} I^G \hat{k} = l\hat{\lambda} \times m((\dot{v}_c - l\dot{\theta}^2)\hat{\lambda} + (\dot{\theta}v_c + l\ddot{\theta})\hat{n}) + \ddot{\theta} I^G \hat{k} = ml\dot{\theta}v_c + ml^2\ddot{\theta} + I^G\ddot{\theta}\ \hat{k}$

$0 = ml\dot{\theta}v_c + (ml^2 + I^G)\ddot{\theta}$

DAE's: LMB: $\sum F = m\vec{a}_G$

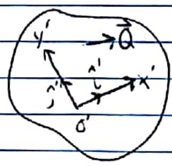
AMB$_{/G}$: $-Nl = I^G\ddot{\theta}$

Constraint: $\frac{d}{dt}[\vec{v}_c \cdot \hat{n} = 0]$

$$\boxed{\ddot{\theta} = \frac{-ml\dot{\theta}v_c}{ml^2 + I^G} \qquad \dot{v}_c = \dot{\theta}^2 l}$$

$\dot{x}_c = v_c\cos\theta$
$\dot{y}_c = v_c\sin\theta$

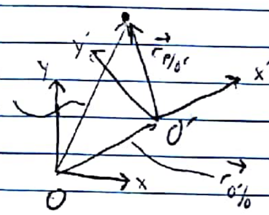
5-Term Acceleration Formula



$\vec{Q} = \vec{Q} = x\hat{i} + y\hat{j} = x'\hat{i}' + y'\hat{j}'$

$\frac{d^F\vec{Q}}{dt} = \overset{F\cdot}{Q} = \dot{x}\hat{i} + \dot{y}\hat{j}$; $\frac{d^B\vec{Q}}{dt} = \overset{B\cdot}{Q} = \dot{x}'\hat{i}' + \dot{y}'\hat{j}' \neq \overset{F\cdot}{Q}$ in general

Apply to $\vec{r}_P$, given $\vec{r}_{P/o'}, \vec{v}_{P/o'}, \vec{a}_{P/o'}, \vec{r}_{o'/o}, \vec{v}_{o'/o}, \vec{a}_{o'/o}, \vec{\omega}_{B/\mathcal{F}}, \dot{\omega}_{B/\mathcal{F}}, \theta_{B/\mathcal{F}}$, find $\vec{r}_{P/o}, \vec{v}_{P/o}, \vec{a}_{P/o}$



$\vec{r}_P = \vec{r}_{P/o'} + \vec{r}_{o'/o} = x\hat{i} + y\hat{j} = x'\hat{i}' + y'\hat{j}'$

$\vec{v}_{P/B} = \frac{d^B}{dt}\vec{r}_{P/o'}$; $\vec{a}_{P/B} = \overset{B\cdot}{\vec{v}}_{P/B}$

- Long way is to calculate with components: $\vec{r}_{P/o} = x_{o'/o}\hat{i} + y_{o'/o}\hat{j} + x'_{P/o'}\hat{i}' + y'_{P/o'}\hat{j}'$ $\Rightarrow \vec{v}_{P/o} = \overset{F\cdot}{\vec{r}}_{P/o} = \vec{v}_{o'/o} + (\dot{x}'\hat{i}' + x'\dot{\hat{i}}') + (\dot{y}'\hat{j}' + y'\dot{\hat{j}}')$, where $\dot{\hat{i}}' = \omega\hat{j}'$, $\dot{\hat{j}}' = -\omega\hat{i}'$

$\vec{a}_{P/\mathcal{F}} = \frac{d^F}{dt}(\vec{v}_{P/\mathcal{F}}) = \frac{d}{dt}(\vec{v}_{o'/o} + \vec{v}_{P/B} + \vec{\omega}_{B/\mathcal{F}} \times \vec{r}_{P/o'})$

$\Rightarrow \vec{v}_{P/\mathcal{F}} = \vec{v}_{o'/o} + \vec{v}_{P/B} + \vec{\omega}_{B/\mathcal{F}} \times \vec{r}_{P/o'}$ — 3-term velocity formula

$= \vec{a}_{o'/o} + \overset{B\cdot}{\vec{v}}_{P/B} + \vec{\omega} \times \vec{v}_{P/B} + (\dot{\vec{\omega}} \times \vec{r}_{P/o'}) + (\vec{\omega} \times (\overset{B\cdot}{\vec{r}}_{P/o'} + \vec{\omega} \times \vec{r}_{P/o'}))$

$$\boxed{\overset{F\cdot}{\vec{Q}} = \overset{B\cdot}{\vec{Q}} + \vec{\omega}_{B/\mathcal{F}} \times \vec{Q}}$$

Q dot Formula
Transport Theorem

- $\overset{B\cdot}{\vec{r}}_{P/o'} = \vec{v}_{P/B}$; $\overset{B\cdot}{\vec{v}}_{P/B} = \vec{a}_{P/B} \Rightarrow$

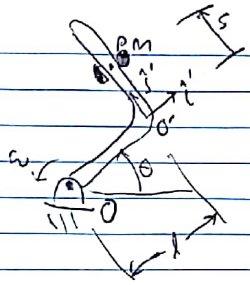
$$\boxed{\vec{a}_{P/\mathcal{F}} = \vec{a}_{o'/o} + \vec{a}_{P/B} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/o'}) + \dot{\vec{\omega}} \times \vec{r}_{P/o'} + 2\vec{\omega} \times \vec{v}_{P/B}}$$

5-term acceleration formula

$a_p = \vec{a}_{p/o} + \vec{a}_{o'/o} \Rightarrow$ ↳ $e^{i\lambda t} = \cos\lambda t + i\sin\lambda t$

ex: Solve $\vec{a}_{p/o}$, bead on rotating rod, use minimal coordinates θ, s

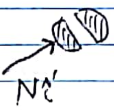
find $\ddot{s}$ given $\theta, \dot\theta, \ddot\theta, s, \dot s$



$\vec{a}_{p/o} = \vec{a}_{o'/o} + \vec{a}_{p/B} - \omega^2 \vec{r}_{p/o'} + \vec{\omega}\times\vec{r}_{p/o'} + 2\vec{\omega}\times\vec{v}_{p/B}$

$\vec{a}_{p/o} =$

FBD



$\vec{a}_{o'/o} = -\omega^2 l \hat{i} + \dot\theta \hat{k}\times l\hat{i}$

$\vec{a}_{p/B} = \ddot{s}\hat{j}'$

$\vec{r}_{p/o'} = s\hat{j}'$

$\vec{\omega} = \dot\theta \hat{k}$

$\vec{v}_{p/B} = \dot{s}\hat{j}'$

American Way

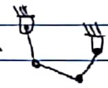
$\{\vec{F} = m\vec{a}_{p/o}\}\cdot\hat{j}' \Rightarrow 0 = m\vec{a}_{p/o}\cdot\hat{j}'$; $m\vec{a}_{p/o} = \ddot\theta l + \ddot{s} - \dot\theta^2 s = 0 \Rightarrow \boxed{\ddot{s} = \dot\theta^2 s - \ddot\theta l}$

Generalized Forces and Lagrange

- It's possible to do Lagrange with non-holonomic and non-conservative forces by using generalized forces and constraint equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q} = \begin{cases} 0 & \text{elementary Lagrange eqs} \\ Q_i & \text{generalized forces} \end{cases}$$

- Q_i is needed if there are non-conservative forces, non-holonomic constraints (can't use minimal coordinates), or if minimal coordinates are too hard



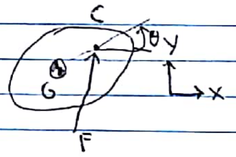
Take $\vec{F}_i$ = forces at $\vec{r}_{i/o}$ not accounted for in E_p and not eliminated by minimal coordinates

- Q_a is the quantity such that for a virtual displacement dq_a, and no other displacements, $dW = Q_a dq_a$

$Q_a dq_a = \sum_i \vec{F}_i\cdot d\vec{r}_i$ (all locations of forces) for $d\vec{r}_i$ when $dq_a \neq 0$ and all other $q_i \neq q_a = 0$ $\Rightarrow d\vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_a} dq_a \Rightarrow \boxed{Q_a = \sum_i \frac{\partial \vec{r}_i}{\partial q_a}\cdot\vec{F}_i}$

Ex: $E_p = 0$; $E_k = \frac{1}{2}(m\dot{x}_G^2 + m\dot{y}_G^2 + I^G\dot\theta^2)$ $\quad q_i = x, y, \theta$

$\mathcal{L} = E_k - E_p = E_k$



$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_i}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q_i$

$m\ddot{x} = Q_x \qquad Q_x = \frac{\partial \vec{r}_i}{\partial x_G}\cdot\vec{F} = F_x$

$m\ddot{y} = Q_y \qquad Q_y = \frac{\partial \vec{r}_i}{\partial y_G} = F_y$

$I^G\ddot\theta = Q_\theta \qquad Q_\theta = \partial\vec{r}_i/\partial\theta\cdot\vec{F}$ $\qquad \partial\vec{r}_i/\partial\theta = \hat{k}\times\vec{r}_{i/G}$

$Q_\theta = (\hat{k}\times\vec{r}_{i/G})\cdot\vec{F} \Rightarrow Q_\theta = \hat{k}\cdot\vec{r}_{i/G}\times\vec{F} = x_{i/G}F_y - y_{i/G}F_x$

$I\ddot\theta = x_{i/G}F_y - y_{i/G}F_x$

ex: $q = \{x_G, y_G, \theta\}$

$Q_x = N\hat{n}\cdot\hat{i}$

$Q_y = N\hat{n}\cdot\hat{j}$ } 3 eqs, 4 unknowns, need to add constraint eqn.

$Q_\theta = -Nd\theta$

$\vec{v}_i\cdot\hat{n} = 0 = (\dot{x}_G\hat{i} + \dot{y}_G\hat{j} + \dot\theta\hat{k}\times\vec{r}_{i/G})\cdot(\cos\theta\hat{j} - \sin\theta\hat{i})$

Now 4 eqs, 4 unknowns, solve DAEs

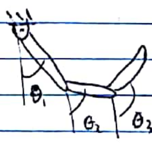
- For >1 non-conservative force or >1 constraint, add forces to RHS of Lagrange eqns, i.e., $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = Q_{i1} + Q_{i2} + \ldots$, and add constraint equations

Further Linkage Approaches:

Maximal Coordinates

- AMB, LMB for each link
- $2\times 4 = 8$ constraint eqns
- 17 ODEs

N-E Minimal coordinates

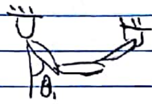


- 3 coordinates in $\ddot\theta$
- 2 constraints on final end position
- 5 ODEs $\ddot\theta_1, \ddot\theta_2, \ddot\theta_3, F_x, F_y$

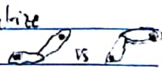
Lagrange + constraint force

- End up with identical EoMs as NE with minimal

Lagrange with 1 DOF



1 Lagrange equation

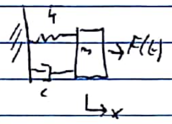
- Really hard to parametrize
- Won't work due to  ambiguity

$V_j = \{$ $\sum A_j v_j \sin \omega t +$ $\{x_1 x_2\} = A_1 \{1\ 1\}\cos(t) + A_2^{\rho} + 0$

$\{x_0\} = \sum B_j v_j = [B_1 B_2 B_3][v_i] = \{x_0\}$ $\{B\} =$

Vibrations

Damped Harmonic Oscillator: $m\ddot{x} + c\dot{x} + kx = F(t)$



Trial solution: $x(t) = Ae^{\alpha t}$ ⇒ $m\alpha^2 + c\alpha + k = 0$ ⇒ $\alpha = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$

Critical damped: $x(t) = Ae^{\beta t} + Ate^{\beta t}$ for $\beta = \frac{-c}{2m}$, $\lambda = \frac{\sqrt{c^2-4mk}}{2m}$

Overdamped: $x(t) = Ae^{\frac{-c+\sqrt{c^2-4mk}}{2m}t} + Be^{\frac{-c-\sqrt{c^2-4mk}}{2m}t} = Ae^{(\beta+\lambda)t} + Be^{(\beta-\lambda)t}$

Underdamped: $x(t) = Ae^{(\beta+\lambda i)t} + Be^{(\beta-\lambda i)t}$ ... $= e^{\beta t}(A\cos\lambda t + B\sin\lambda t) = C_1 e^{\beta t}\sin(\lambda t + \phi)$

$x(t) = Ce^{\frac{-c}{2m}t}\sin\left(\frac{\sqrt{4mk - c^2}}{2m}t + \phi\right)$ $\omega_d = \frac{\sqrt{4mk - c^2}}{2m} = \omega_n\sqrt{1-\zeta^2}$

$x(t) = C_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$ $\zeta = c/\sqrt{4mk}$

- Systems have vibrations near local minima of PE
- Almost always, PE is quadratic with displacement q

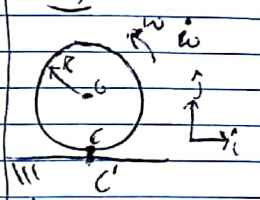
$E_p(q) = E_{p0} + c_1 q + c_2 q^2 + c_3 q^3 \ldots$

arbitrary; $= 0$ at min PE; dominant; small for $q \to 0$

Frequency resp: freq. response peaks, is most resonant

$\omega_n = \sqrt{k/m}$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $\omega_{res} = \omega_n\sqrt{1-2\zeta^2}$

- Quality factor: just a function of ζ, measures height, frequent peak and narrowness



Rolling

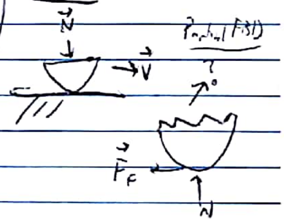
$\vec{V}_G = -\omega R\hat{i}$

$\vec{a}_G = -\dot{\omega}R\hat{i}$

$\vec{a}_{C/G} = \omega^2 R\hat{j} - \dot{\omega}R\hat{i}$

$\vec{a}_C = \omega^2 R\hat{j}$

- Types of fluid-like drag:
  - F_{drag}: cv linear ← Only this works for linear equations, others produce nonlinear equations
  - cv^2 quadratic
  - $c\, v/|v|$ Coulomb (friction)
- Multi-DOF systems of vibrations are just a superposition of normal modes



Sliding

Partial FBD

Matrix Eqns of Motion

$$\begin{bmatrix} m_{11} & m_{12} & \cdots \\ m_{21} & m_{22} & \\ & & m_{nn}\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \vdots \\ \ddot{x}_n\end{Bmatrix} + \begin{bmatrix} c_1 & c_{12} & \cdots \\ & c_{22} & \\ & & c_{nn}\end{bmatrix}\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots \\ \dot{x}_n\end{Bmatrix} + \begin{bmatrix} k_1 & k_2 & \cdots \\ & \ddots & \end{bmatrix}\begin{Bmatrix}x_1\\ \vdots \\ x_n\end{Bmatrix} = \begin{Bmatrix}F_1(t)\\ F_2(t)\\ \vdots \\ F_n(t)\end{Bmatrix} \text{ or } 0$$

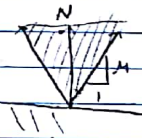
symmetric

$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$

- For simple systems where each DOF represents position of a single mass WRT a reference, $[M]$ is diagonal

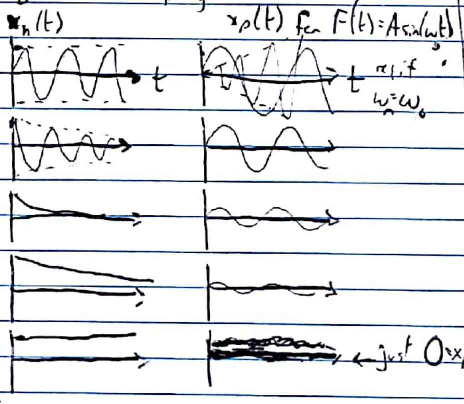
$F_F = \mu N$ $v>0$; $= -\mu N$ $v<0$; $\leq \mu N$ $v=0$

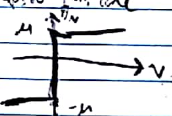
Friction cone



$\vec{F} + \vec{N}$ sweeps out this cone

Effect of Damping





Locus of possible frictional forces

Gets really complicated, friction models at all interfaces

Normal Modes

- Consider undamped system $[A]\{\ddot{x}\} + [k]\{x\} = 0$ (positive definite; positive semidefinite)
- Positive definite: $\{y\}^T[A]\{y\} > 0$ for all $\{y\}$ i.e. $KE > 0$ function
- Semidefinite: $\{y\}^T[A]\{y\} \geq 0$ for all $\{y\}$ i.e. $PE \geq 0$ for any displacement from the equilibrium
- All motions of a system are a superposition of normal modes
- For a normal mode i, $x(t) = \begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix} = [v^i]\sin(\omega_i t)$ or $[v^i]\cos(\omega_i t)$ where $[v_i]$ is a mode shape, $n \times 1$, real numbers, constant in time; ω_i is mode frequency

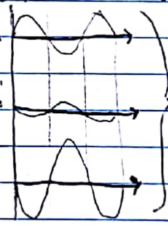
Collisions

- Short, non-negative forces
- Collisional FBD
  - Ignore non-collision forces, small in comparison

$\vec{L}^+ - \vec{L}^- = \vec{F}\Delta t$

$\vec{H}_c^+ - \vec{H}_c^- = \vec{r}_{i/c} \times \vec{F}\Delta t$

- In one mode, all x_i move exactly in/out of phase
- For a n-DOF systems, 2n linearly independent solutions exist to satisfy any initial conditions, n mode shapes, each with a $\sin(\omega_i t)$ and $\cos(\omega_i t)$ solution



Behavior of each DOF for a particular mode, all in phase

- In one normal mode, each mass has the same $\omega_i = \sqrt{k_{eff}/m_j}$, where $k_{eff} = F_{eff}/x_j$

General solution of undamped free oscillator: $\{x(t)\} = \sum_i A_i V_i \sin(\omega_i t) + B_i V_i \cos(\omega_i t)$ (superpose over i for n modes; e'vectors)

Sort of like $[A]\{x\} = \lambda\{x\}$ Solving multiple e'value problems in parallel

$[M]\{\ddot{x}\} + [K]\{x\} = 0$; assume $\vec{x} = \vec{V}\cos(\omega t)$ ⇒ $(-\omega^2[M] + [K])\vec{V} = 0$ ⇒ $([K] - \omega^2[M])\vec{V} = 0$; multiply by $[M]^{-1}$ ⇒ $[M]^{-1}[K]\{v\} = \omega^2[I]\{v\}$

- Taking $eig([M]^{-1}[K])$ not mathematically guaranteed to result in sufficient modes, so modify using change of variable

$eig([M]^{-1}[K])$ $eig([K],[M])$

$\{y\} = [M]^{1/2}\{y\}$ ⇒ $[M][M]^{-1/2}\{\ddot{y}\} + [K][M]^{-1/2}\{y\} = 0$ ⇒ $[M]^{-1/2}[M][M]^{-1/2}\{\ddot{y}\} + [M]^{-1/2}[K][M]^{-1/2}\{y\} = 0$ ⇒ $\{\ddot{y}\} + [\tilde{K}]\{y\} = 0$

Now, in new basis, guess $\{y\} = \{w\}\cos(\omega t)$, $-\omega^2\{w\} + [\tilde{K}]\{w\} = 0$ (classic symmetric e'value problem)   where $[\tilde{K}] = [M]^{-1/2}[K][M]^{-1/2}$ is always symmetric

$eig(1) = [x_{10}; \dot{x}_{20}] = \ldots$

- Since symmetric e'value problems always have n orthogonal e'vectors, $\{y(t)\} = \sum_i \left(A_i\cos(\omega_i t) + B_i\sin(\omega_i t)\right)\{w\}$, $\{x(t)\} = [M]^{-1/2}\{y(t)\}$

$eig(2) = [\dot{x}_{1o}; \dot{x}_{2o}] \ldots$

$I_{z,2D}$ [m⁴] $= \int_A y^2 dA$

$I_{zz,3D}$ [kg m²] $= \int_V r^2 dm = \rho \int_V r^2 dV = \rho \int_0^{2\pi} \int_A y^2 dA \, d\theta = 2\pi\rho \int_A r^3 dA$: 3rd moment of area about z

∴ Cannot determine $I_{zz,3D}$ from $I_{z,2D}$

$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} x_0 - Im(x_{p0}) \\ v_0 - Im(i\omega x_{p0}) \end{Bmatrix}$

Matrix Exponential

$e^{At} = \sum_{n=1}^{\infty} \frac{(At)^n}{n!} = [I] + \frac{([A]t)^1}{1} + \frac{([A]t)^2}{2!} + \ldots$

- For the IVP $\dot{\vec{x}}(t) = [A]\vec{x}$; $\vec{x}(0) = \{x_0\}$, solution is $\vec{x}(t) = e^{[A]t}\{x_0\}$
- For a diagonal matrix [D], $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \\ 0 & e^{\lambda_2 t} & \\ & & \ddots \end{bmatrix}$ where $[D] = \begin{bmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \\ & & \ddots \end{bmatrix}$
- ~~If A is diagonalizable~~, can write $[A] = [P][D][P]^{-1}$ and $e^{At} = [P]e^{Dt}[P]^{-1}$
- [A] need not be diagonalizable, but is much simpler if it is, don't have to expand e^{Dt} into stupid polynomials
- For $[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\}$, can reduce to 1st order system

$\{y\} = \begin{Bmatrix} \{x\} \\ \{\dot{x}\} \end{Bmatrix} \Rightarrow \{\dot{y}\} = \begin{bmatrix} 0 & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \{y\}$

$M\ddot{x} + C\dot{x} + Kx = fe^{\lambda t}$

$[P] = M\lambda^2 + C\lambda + K \Rightarrow x = fe^{\lambda t}[P]^{-1}$

Lagrange Eqns. & Vibrations

- Consider a system near equilibrium, $q_i \approx 0$

$E_p = E_{p0} + C_i q_i + \sum_{i}\sum_{j} \frac{C_{ij} q_i q_j}{2!} + \sum_{ijk} \frac{C_{ijk} q_i q_j q_k}{3!}$

datum, set to 0 @ equilibrium — General Taylor series for a potential energy — higher order

$\Rightarrow E_p \approx \frac{1}{2} \sum_i \sum_j \frac{\partial^2 E_p}{\partial q_i \partial q_j} q_i q_j$

$E_p = \frac{1}{2} \vec{q}^T [K] \vec{q}$

$E_k \approx \frac{1}{2} \frac{\partial^2 E_k}{\partial \dot{q}_i \partial \dot{q}_j} \dot{q}_i \dot{q}_j$ — K_{ij} = entry in stiffness matrix

$[M] \Rightarrow E_k = \frac{1}{2} \dot{\vec{q}}^T [M] \dot{\vec{q}}$

$\mathcal{L} = E_k - E_p = \frac{1}{2} \dot{\vec{q}}^T [M] \dot{\vec{q}} - \frac{1}{2} \vec{q}^T [K] \vec{q}$

$\Rightarrow$ ... math ... $[M]\ddot{\vec{x}} + [K]\vec{x} = 0$

do this, it will be on final exam

- For solving $[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = F_0 \vec{V} \sin(\omega_0 t)$
  - Use $[M]^{-1/2}$ change of basis
  - If $[M]^{-1/2}[C][M]^{-1/2}$ is not diagonal, then can't use normal modes to find exact solution
  - Can neglect off-diagonal terms, now you have n uncoupled 1-DOF equations

1 DOF ~~nonhomogeneous~~ general solution

Find $x(t)$ for given x_0, v_0, F_0, ω

$x = x_h + x_p$ — $= F_0 \sin(\omega t)$

$x_h = ae^{\lambda_1 t} + be^{\lambda_2 t}$, $\lambda = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$

$x_p = x_{p0} e^{i\omega t} \Rightarrow m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} = x_{p0}(-m\omega^2 + i c\omega + k) = F_0$

$x_{p0} = F_0 / (-m\omega^2 + ic\omega + k)$, and $x_p = imag(x_{p0} e^{i\omega t})$ ← because F(t) is sin, would be real if cos

$x(0) = x_0 = x_p(0) + x_h(0) = Im(x_{p0}) + ae^{\lambda_1 t} + be^{\lambda_2 t} = Im(x_{p0}) + a + b$

$v(0) = v_0 = \dot{x}_p(0) + \dot{x}_h(0) = Im(i\omega x_{p0}) + a\lambda_1 + b\lambda_2$

Solve 2 eqns for a, b, get $x_h(t)$, set $x(t) = x_p + x_h$

Multi-DOF general solution

$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{F}e^{i\omega t}$; $\vec{x}(0) = \vec{x}_0$; $\dot{\vec{x}}(0) = \vec{v}_0$

Guess $\vec{x}_p = \vec{x}_{p0} e^{i\omega t}$, sub into ODE:

$-\omega^2 M \vec{x}_{p0} e^{i\omega t} + i\omega C \vec{x}_{p0} e^{i\omega t} + K \vec{x}_{p0} e^{i\omega t} = \vec{F} e^{i\omega t}$

$(-\omega^2 M + i\omega C + K)\vec{x}_{p0} = \vec{F} \Rightarrow \vec{x}_{p0} = \vec{F} / (K - \omega^2 M + i\omega C)$

- Take $\vec{x}_{p0} = Re(\vec{x}_{p0})$ for $Fe^{i\omega t}$ = cos

  $\vec{x}_{p0} = Im(\vec{x}_{p0})$ for $Fe^{i\omega t}$ = sin

Solving for homogeneous $\vec{x}_h(t)$

- Change of variable $\vec{x} = M^{-1/2}\vec{y}$ — normalize mass
- Change of variable $\vec{y} = [V]\vec{r}$ — e'vector transform
-

$MM^{-1/2}\ddot{\vec{y}} + CM^{-1/2}\dot{\vec{y}} + KM^{-1/2}\vec{y} = F_0 e^{i\omega t}$; premultiply by $M^{-1/2}$

$[I]\ddot{\vec{y}} + M^{-1/2}CM^{-1/2}\dot{\vec{y}} + M^{-1/2}KM^{-1/2}\vec{y} = M^{-1/2}Fe^{i\omega t}$ — $\vec{y}(0) = M^{-1/2}\vec{x}_0$, $\dot{\vec{y}}(0) = M^{-1/2}\vec{v}_0$

$\tilde{K}$ is a pos. def. symmetric matrix

Find e'solution of $\tilde{K}$, $[V, D] = eig(\tilde{K})$

Change of variable $\vec{y} = [V]\vec{r}$ — $\vec{r}(0) = V^{-1}M^{-1/2}\vec{x}_0$

$[I][V]\ddot{\vec{r}} + M^{-1/2}CM^{-1/2}V\dot{\vec{r}} + \tilde{K}V\vec{r} = M^{-1/2}Fe^{i\omega t}$ — $\dot{\vec{r}}(0) = V^{-1}M^{-1/2}\vec{v}_0$

Premultiply by $V^{-1} = V^T$

$[I]\ddot{\vec{r}} + V^{-1}M^{-1/2}CM^{-1/2}V\dot{\vec{r}} + V^{-1}\tilde{K}V\vec{r} = V^{-1}M^{-1/2}Fe^{i\omega t}$

$\tilde{\tilde{C}}$ = last damping, non-diagonal term

[D], matrix of e'values $\begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \; \omega_n^2 \end{bmatrix}$

- Can deal with $\tilde{\tilde{C}}$ in 2 ways:
  - $\tilde{\tilde{C}} \rightarrow \tilde{\tilde{C}}^* = \tilde{\tilde{C}}$ with all off-diagonal terms set to 0 ← modal damping
  - Rayleigh damping: $C = \alpha M + \beta K \Rightarrow \tilde{\tilde{C}} = \alpha[I] + \beta[D]$ — is diagonal
- Now the n-DOF is an approximated series of uncoupled 1-DOF equations

Vibration Isolation

- Goal is to reduce vibration amplitude of a particular mass in a system, m_7
- Attach spring-mass k, m to m_7 such that $k/m = \omega_f^2$
- This results in a particular solution with zero amplitude for x_7

Lagrange & Vibrations

ex: simple pendulum $\ddot{\theta} = \frac{-g\sin\theta}{L} \Rightarrow \sin\theta \approx \theta$ — $\ddot{\theta} = -\frac{g}{L}\theta$

or

$E_k = \frac{1}{2}Lm\dot{\theta}^2$, $E_p = -mgL\cos\theta \approx -mgL\left(1 - \frac{\theta^2}{2} + \ldots\right) \approx mgL\theta^2/2$ — constant

$\mathcal{L} = \frac{1}{2}mL\dot{\theta}^2 - mgL\theta^2/2 \Rightarrow$ Lag. eqn. $\Rightarrow \ddot{\theta} = -g\theta/L$

Matrix Exponential

$\dot{\vec{z}} = [A]\vec{z}$; $A = \begin{bmatrix} 0 & 1 \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \Rightarrow \vec{z} = e^{[A]t} \cdot \vec{z}_0$ is the solution

$x_0 = x_{p0} + a + b$ — $x(t) = x_p(t) + x_h(t)$ — $x(0) = x_0$, $v_0 = v(0)$

$v_0 = i\omega x_{p0} + a\lambda_1 + b\lambda_2$ — $x_p(t) = Im(x_{p0} e^{i\omega t})$ — $x_p(0) = Im(x_{p0})$

$x_{p0} = F/(-m\omega^2 + k + ic\omega)$ — $x_h(t) = ae^{\lambda_1 t} + be^{\lambda_2 t}$ — $\dot{x}_p(0) = i\omega \, Im(x_{p0})$

$a + b = x_0 - Im(x_{p0})$ — $x_h(0) = a + b$

$a\lambda_1 + b\lambda_2 = v_0 - i\omega \, Im(x_{p0})$ — $\dot{x}_h(0) = a\lambda_1 + b\lambda_2$