MAE 5710: Applied Dynamics

Prof Andy Ruina

Change room in calendar

- Einstein notation convention: $x\hat{i}+y\hat{j}+z\hat{k}=\vec{r}=\sum_{i=1}^{3} r_i\hat{e}_i = r_i\hat{e}_i$ ← implied summation over i when 2 instances of i are present on one side of an equation
- $\underline{a}\cdot\underline{b} = a_i b_i = a_i b_j \hat{e}_i \cdot \hat{e}_j = a_i b_j \delta_{ij}$; $\delta_{ij} \equiv \hat{e}_i \cdot \hat{e}_j$
- $A = \begin{bmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots \end{bmatrix}$; $A_{ii} = tr(A)$
- ϵ_{ijk} = alternating epsilon $= \begin{cases} 1 & ijk \text{ encode RH CSYS e.g. } (1,2,3), (3,1,2) \text{ even} \\ -1 & ijk \text{ encode LH CSYS e.g. } (3,2,1) \\ 0 & \text{otherwise} \end{cases}$
- $\underline{a}\times\underline{b} = \underline{c} = c_i\hat{e}_i = (a_j\hat{e}_j)\times(b_k\hat{e}_k) = a_j b_k(\hat{e}_j\times\hat{e}_k)$
  $= a_j b_k \hat{e}_i \epsilon_{ijk}$

Dyads

- $\underline{a}\,\underline{b} = \underline{a}\otimes\underline{b}$
- Obey associative, distributive properties
- $(\underline{a}\,\underline{b})\cdot\underline{c} = \underline{a}(\underline{b}\cdot\underline{c})$
- $\underbrace{\hat{e}_i\hat{e}_i}_{\underline{\underline{I}}}\cdot\underline{a} = \hat{e}_i\hat{e}_i\cdot(a_j\hat{e}_j)$
  $= a_j\hat{e}_i\delta_{ij} = a_i\hat{e}_i = \underline{a}$
- $\underline{d} = \underline{a}\,\underline{b}\cdot\underline{c} = \hat{e}_i d_i = a_i\hat{e}_i b_j\hat{e}_j \cdot c_k\hat{e}_k$
  $= a_i b_j c_k \hat{e}_i \hat{e}_j \cdot \hat{e}_k$
  $= a_i b_j c_k \hat{e}_i \delta_{jk}$
  $= a_i b_j c_j \hat{e}_i = \underline{a}(\underline{b}\cdot\underline{c})$
  $\therefore \underline{a}\,\underline{b}\cdot\underline{c} = \underline{a}(\underline{b}\cdot\underline{c})$
- $\underline{a}^{\times}\underline{b} = (\epsilon_{ijk} a_i \hat{e}_j \hat{e}_k)\cdot\underline{b}$
  $\underline{a}^{\times} = (-\epsilon_{ijk} a_i \hat{e}_j \hat{e}_k) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

$-\epsilon_{ijk} a_i \hat{e}_j \hat{e}_k \cdot b_l \hat{e}_l = -a_i b_l \epsilon_{ijk} \delta_{kl} \hat{e}_j = -a_i b_k \hat{e}_j \epsilon_{ijk}$
$= +a_j b_k \hat{e}_i \epsilon_{ijk} = \underline{a}\times\underline{b}$

Index Notation Cheat Sheet

$\underline{a}\cdot\underline{b} = a_i b_i$

for $[A]$, $A_{ii} = tr(A)$

$\underline{a}\times\underline{b} = a_j b_k \hat{e}_i \epsilon_{ijk}$

$c = (\underline{a}\times\underline{b})\cdot\hat{e}_l \Rightarrow c_l = a_j b_k \epsilon_{jkl}$

$\underline{\underline{1}} = \hat{e}_i\hat{e}_i = \hat{e}_i \otimes \hat{e}_i$

${}^{N}\underline{\underline{Q}}{}^{B} = \hat{b}_i \hat{n}_i$ $\quad$ Ruina: $R_{ij} = \hat{e}_i \cdot \underline{\underline{R}} \cdot \hat{e}_j$

Ruina: $\underline{\underline{R}} = \hat{a}_i \hat{b}_i$

$${}^{N}Q^{B} = \begin{bmatrix} \underline{a}_1\cdot\underline{b}_1 & \underline{a}_1\cdot\underline{b}_2 & \underline{a}_1\cdot\underline{b}_3 \\ \underline{a}_2\cdot\underline{b}_1 & \underline{a}_2\cdot\underline{b}_2 & \underline{a}_2\cdot\underline{b}_3 \\ \underline{a}_3\cdot\underline{b}_1 & \underline{a}_3\cdot\underline{b}_2 & \underline{a}_3\cdot\underline{b}_3 \end{bmatrix}$$

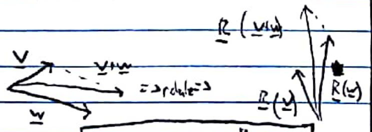
Linear Functions

- Linear functions obey superposition: $f(ax+by) = af(x)+bf(y)$
  - Also applies to vector functions, vector-valued functions, etc.
- Most general linear function $f(\underline{v}) = \underline{v}\cdot\underline{c}$, for some constant $\underline{c}$ ← scalar function of vector
- Most general linear function $\underline{f}(\underline{v})$: $\underline{\underline{F}}\cdot\underline{v}$
  - $\underline{f}_i = \underline{f}(\hat{e}_i) \Rightarrow$ ~~$\underline{f}(\underline{v}) = v_i \underline{f}_i$~~
  - $\underline{v} = v_i\hat{e}_i$; $\underline{f}(\underline{v}) = v_i \underline{f}_i = \underline{\underline{F}}\cdot\underline{v} = \underline{f}_i\hat{e}_i\cdot v_k\hat{e}_k = \underline{f}_i v_k \delta_{ik}$
    $= \underline{f}_i v_i$
- Dyad vs. dyadic: dyadic is sum of dyads
  - Dyad: $\underline{a}\,\underline{b}$ can be represented by a 3×3 matrix, but only has 6 DOF
  - Not all dyadics can be represented by a single dyad since 6 DOF < 9 DOF
  - $T_{ij}\hat{e}_i\hat{e}_j$ is most general dyadic, can always be written as sum of 3 dyads $\underline{a}\,\underline{b}$

Rotation

$\underline{\underline{R}}(\underline{v}) + \underline{\underline{R}}(\underline{w}) = \underline{\underline{R}}(\underline{v}+\underline{w})$

- Treat rotation as a vector-valued function

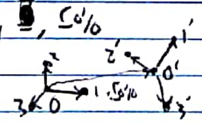


$\underline{\underline{R}} = \hat{b}_i\hat{e}_i$, where $\hat{b}_i = \underline{\underline{R}}(\hat{e}_i) \Rightarrow \boxed{\underline{\underline{R}} = \hat{b}_i\hat{e}_i \;;\; \hat{b}_i = \underline{\underline{R}}\cdot\hat{e}_i}$

$R_{ij} = \hat{b}_i\cdot\hat{e}_j = \text{DCM } {}^{B}Q^{E}$

Animation

- Define rigid object by set of points
- Use ref pt. O as center of rotations, reference for displacements
- Define reference rotation $\underline{r}_{1/0}, \underline{r}_{2/0}, \ldots$ $\quad {}^{B}Q^{ref}$
- An arbitrary transformation defined by ${}^{B}Q^{ref}$, $\underline{r}_{0'/0}$



$${}^{ref}R_0 = \begin{bmatrix} x_{10} & x_{20} \\ y_{10} & y_{20} & \cdots \\ z_{10} & z_{20} \end{bmatrix} = \begin{bmatrix} {}^{ref}X_1 & {}^{ref}X_2 & \ldots & {}^{ref}X_n \end{bmatrix}$$

MATLAB

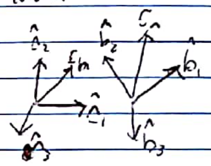
$3\times n$

$R = \underset{3\times1}{{}^{ref}r_{0'/0}} + \underset{3\times3}{{}^{B}Q^{ref}} * \underset{3\times n}{R_0}$ where $\underset{3\times1}{{}^{ref}r_{0'/0}}$ is added to each column

$\underline{r}_b = {}^{B}\underline{\underline{Q}}{}^{N}\cdot\underline{r}_n$; $\underline{b}_1 = {}^{B}\underline{\underline{Q}}{}^{N}\,\underline{a}_1$; ...

${}^{B}\underline{\underline{Q}}{}^{N} = \hat{b}_1\hat{n}_1 + \hat{b}_2\hat{n}_2 + \hat{b}_3\hat{n}_3 = \hat{n}_i\hat{b}_i$

$Q_{ij} = \hat{b}_j\cdot\hat{n}_i$



Axis-Angle Formula

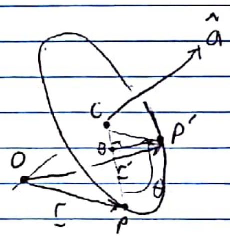
Given $\hat{a}, \theta, \underline{r}$, find $\underline{r}'$

$\underline{r}' = \underline{r}_{OC} + \underline{r}_{CB} + \underline{r}_{BP'}$

$\underline{r}_{OC} = \underline{r}\cdot\hat{a}\hat{a}$

$\underline{r}_{CB} = (\underline{r} - (\underline{r}\cdot\hat{a}\hat{a}))\cos\theta$

$\underline{r}_{BP'} = \hat{a}\times(\underline{r} - (\underline{r}\cdot\hat{a}\hat{a}))\sin\theta = (\hat{a}\times\underline{r})\sin\theta$



$$\boxed{\underline{r}' = \underline{r}\cdot\hat{a}\hat{a} + (\underline{r} - (\underline{r}\cdot\hat{a}\hat{a}))\cos\theta + \hat{a}\times\underline{r}\sin\theta}$$

Axis-Angle Vector Form

$$\underline{r}' = \underbrace{\left[\hat{a}\hat{a} + \cos\theta(\underline{\underline{1}} - \hat{a}\hat{a}) + \sin\theta\,\underline{\underline{a}}^{\times}\right]}_{Q}\cdot\underline{r}$$

Index Notation Form

$$r'_i = [a_i a_j + \cos\theta(\delta_{ij} - a_i a_j) + \sin\theta\,\epsilon_{kij} a_k]\cdot r_j$$

Matrix Form

$$\underset{3\times1}{{}^{N}r'} = \left[{}^{N}a\,{}^{N}a^T + \cos\theta\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - {}^{N}a\,{}^{N}a^T\right) + \sin\theta\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\right]\cdot{}^{N}r$$

Small angles: $\underline{r}' \approx \underline{\underline{1}} + \theta\hat{a}\times\underline{r} \Rightarrow Q \approx \underline{\underline{1}} - \theta\,\underline{\underline{a}}^{\times}$

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DCM Kinematics

- For moving basis $\hat{b}_i$ and a fixed basis $\hat{a}_i$,

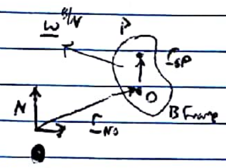
${}^{N}\dot{\hat{b}}_i = ({}^{N}\underline{\omega}^{B/N}\times\hat{b}_i) = {}^{N}(\underline{\omega}^{\times}\cdot\underline{b}_i)$

${}^{B}\dot{\underline{\underline{Q}}}{}^{N} = \underline{\omega}^{B/N\times}\cdot{}^{B}\underline{\underline{Q}}{}^{N} = \underline{\omega}^{B/N}\times{}^{B}\underline{\underline{Q}}{}^{N} \Rightarrow$ $\boxed{\underset{3\times3}{{}^{B}\dot{Q}^{N}} = \underset{3\times1}{{}^{N}\underline{\omega}}\times\underset{3\times3}{{}^{B}Q^{N}} = \left[\underset{3\times1}{\underline{\omega}\times\underline{b}_1}\;\underset{3\times1}{\underline{\omega}\times\underline{b}_2}\;\underset{3\times1}{{}^{N}\underline{\omega}\times\underline{b}_3}\right]}$

Transport Theorem: ${}^{N}\dot{\underline{v}} = {}^{B}\dot{\underline{v}} + \underline{\omega}^{B/N}\times\underline{v}$

$\underline{r}_P = \underline{r}_{NP} = \underline{r}_{NO} + \underline{r}_{OP}$

$\underline{v}_P = {}^{N}\dot{\underline{r}}_P = \underline{v}_{NO} + \underline{v}_{OP} = \underline{v}_{NO} + \underline{\omega}\times\underline{r}_{OP} + \cancel{{}^{B}\dot{\underline{r}}_{OP}}$



$\underline{a}_P = \underline{a}_{O/N} + \frac{{}^{N}d}{dt}(\underline{\omega}\times\underline{r}_{P/O})$

$$\boxed{\underline{a}_P = \underline{a}_{O/N} + {}^{N}\dot{\underline{\omega}}^{B/N}\times\underline{r}_{P/O} + \underline{\omega}\times(\underline{\omega}\times\underline{r}_{P/O})}$$

${}^{N}\dot{\underline{\omega}} = {}^{B}\dot{\underline{\omega}} \Rightarrow {}^{N}\dot{\omega}_i\hat{n}_i = {}^{B}\dot{\omega}_i\hat{b}_i$

Dynamics

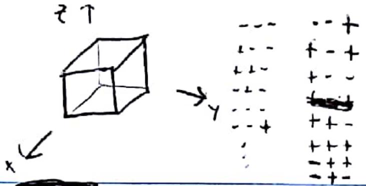
$\sum \underline{F}_i^{ext} = \sum m_i\underline{a}_i = m_{tot}\underline{a}_G$ ← acceleration of COM

$\sum \underline{M}_{/C}^{ext} = \dot{\underline{H}}_{/C}$ for C = COM of system, or fixed in N frame, or accelerating towards G

$\underline{H}_{/C} = \sum \underline{r}_{i/C}\times m\underline{v}_{i/C} = \underline{r}_{G/C}\times m_{tot}\underline{a}_{0/C} + \sum \underline{r}_{i/G}\times m_i\underline{\dot{v}}_{i/G}$

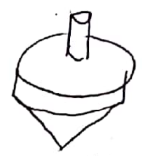
$$\boxed{{}^{N}\dot{\underline{H}}_{/C} = {}^{N}\dot{\underline{H}}_{/G} + \underline{r}_{G/C}\times\underline{a}_{G/N}\,m_{tot}}$$

← inertial frame



$$^N Q^{B} \, ^B b_i = {}^N b_i \qquad {}^N b_1 = \begin{bmatrix} b_1 \cdot a_1 \\ b_1 \cdot a_2 \\ b_1 \cdot a_3 \end{bmatrix} \qquad {}^N Q^B = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & a_1 \cdot b_3 \\ a_2 \cdot b_1 & & \end{bmatrix}$$

$$^N Q^B = [\,^N b_1 \; ^N b_2 \; ^N b_3\,]$$



3D Moment of Inertia

$\underline{H}_{/G} = \sum \underline{r}_i \times m_i \underline{v}_{i/G}$

$d\underline{H} = \underline{r}_{/G} \times \underline{v}_{/G}\, dm$

- For rigid objects, $\underline{v}_{/G} = \underline{\omega} \times \underline{r}_{/G}$

$\Rightarrow d\underline{H} = \underline{r}_{/G} \times (\underline{\omega} \times \underline{r}_{/G})$

- Use vector identity $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$

$\Rightarrow d\underline{H} = \underline{r} \times (\underline{\omega} \times \underline{r}) = \underline{\omega}(\underline{r} \cdot \underline{r}) - \underline{r}(\underline{r} \cdot \underline{\omega}) = \underline{\omega}(\underline{\underline{1}}(\underline{r} \cdot \underline{r}) - \underline{r}\underline{r}) = \underline{\underline{I}}\,\underline{\omega}$

$\underline{\underline{I}} = \int \underline{r}_{/G} \cdot \underline{r}_{/G}\, dm\, \underline{\underline{1}} - \int \underline{r}\underline{r}\, dm = \int (\delta_{ij} r_k r_k - r_i r_j)\, dm$

$$\boxed{\underline{H}_{/G} = \sum \underline{r}_{i/G} \times m_i \underline{v}_{i/G} = \underline{\underline{I}} \cdot \underline{\omega}}$$

$$\underline{\underline{I}} = \underline{\underline{1}} \int x^2 + y^2 + z^2\, dm - \int \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} dm = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm$$

where $x = \underline{r} \cdot \hat{a}_1$, $y = \underline{r} \cdot \hat{a}_2$, $z = \underline{r} \cdot \hat{a}_3$, where $\hat{a}_i$ are initially fixed, and $\underline{r} = \underline{r}_{/G}$

This is problematic since $\underline{\underline{I}}$ is constantly changing for rotating objects

$\underline{\underline{I}} = I_{ij}\, \hat{a}_i \hat{a}_j = {}^B I_{ij}\, \hat{b}_i \hat{b}_j$

constant for rigid objects

${}^N\underline{\underline{I}} = {}^N Q^B \, {}^B I \, {}^B Q^N$ where ${}^B I$ is constant in time for rigid objects

AMB

$\sum \underline{M}_{/G} = {}^N\dot{\underline{H}}_{/G} = (\underline{\underline{I}} \cdot \underline{\omega})^{\cdot} = {}^B(\underline{\underline{I}} \cdot \underline{\omega})^{\cdot} + \underline{\omega} \times (\underline{\underline{I}} \cdot \underline{\omega})$

$= \underline{\underline{I}} \cdot \dot{\underline{\omega}} + \underline{\omega} \times (\underline{\underline{I}} \cdot \underline{\omega}) = \underline{\underline{I}} \cdot \underline{\alpha} + \underline{\omega} \times (\underline{\underline{I}} \cdot \underline{\omega})$

${}^B\dot{\underline{\omega}} = {}^N\dot{\underline{\omega}} = \underline{\alpha}$ = angular accel

$$\boxed{\underline{\alpha} = \underline{\underline{I}}^{-1} \cdot \left[\underline{M}_{/G} - \underline{\omega} \times (\underline{\underline{I}} \cdot \underline{\omega})\right]}$$

but for rotating objects, $\frac{{}^N d}{dt}\underline{\underline{I}} \neq \underline{\underline{0}}$, so this eqn is problematic

~~Form~~ EOMs:

Given ${}^N Q^B$, $\underline{\omega}$, ${}^B I_{3\times3}$, ${}^N M_{/G\;3\times1}$

$\rightarrow$ ${}^N I = {}^N Q^B \, {}^B I \, {}^B Q^N$

${}^N\alpha_{3\times1} = {}^N I^{-1}_{3\times3} \left[{}^N M_{/G\,3\times1} - {}^N\omega_{3\times1} \times ({}^N I_{3\times3} \cdot {}^N\omega)\right]$ — N, B for I? basis

${}^N\dot{\omega}_{3\times1} = {}^N\alpha_{3\times1}$

${}^N\dot{Q}^B_{3\times3} = {}^N\omega^{\times}_{3\times3} \cdot {}^N Q^B_{3\times3}$

RHS files: Rotation | Translation

${}^N I = {}^N Q^B \, {}^B I \, {}^B Q^N$ $\qquad \dot{\underline{u}} = \underline{F}/m$

${}^N\alpha = {}^N I^{-1} \left({}^N M_{/G} - {}^N\omega \times ({}^N I \cdot {}^N\omega)\right)$ $\qquad \dot{\underline{v}} = \underline{a}$

${}^N\dot{\omega} = {}^N\alpha$ $\qquad \dot{\underline{r}} = \underline{v}$

${}^N\dot{Q}^B = {}^N\omega^{\times} \, {}^N Q^B$

Parallel axis theorem: $\boxed{\underline{\underline{I}}_{tot} = \sum_{i=1}^{n} \underline{\underline{I}}_i - (\underline{r}_{i/G})^{\times}(\underline{r}_{i/G})^{\times} m_i}$

$I^{tot}_{ij} = \sum_{p=1} (r_k r_k \delta_{ij} - r_i r_j) m_p + I^p_{ij}$ — $\underline{\underline{I}}$ of each part about its own COM

Perpendicular axis theorem for planar objects in xy plane: $I_{xx} + I_{yy} = I_{zz}$

Euler Equations

${}^B I = \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{bmatrix} \Rightarrow \underline{\underline{I}} = I_1 \hat{b}_1 \hat{b}_1 + I_2 \hat{b}_2 \hat{b}_2 + I_3 \hat{b}_3 \hat{b}_3$

$\underline{\omega} = {}^B\omega_i \hat{b}_i$; $\underline{\alpha} = {}^B\alpha_i \hat{b}_i$

For $\underline{M}_{/G} = \underline{0}$: $\underline{\alpha} = \underline{\underline{I}}^{-1} \cdot (-\underline{\omega} \times \underline{\underline{I}} \cdot \underline{\omega}) = {}^N\dot{\underline{\omega}}$ ← no ${}^B Q^N$ dependency

${}^B\dot{\omega}_i \hat{b}_i = \left(\frac{1}{I_j} \hat{b}_j \hat{b}_j\right) \cdot \left(-{}^B\omega_i \hat{b}_i \times (I_k {}^B\omega_k \hat{b}_k)\right)$ — strictly, can't do Einstein summation over 3 terms, but it seems OK here

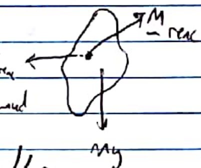
${}^B\dot{\omega}_1 = {}^B\omega_2 {}^B\omega_3 \left(\frac{I_2 - I_3}{I_1}\right) + {}^B\tau_1 / I_1$

${}^B\dot{\omega}_2 = {}^B\omega_1 {}^B\omega_3 \left(\frac{I_3 - I_1}{I_2}\right) + {}^B\tau_2 / I_2$

${}^B\dot{\omega}_3 = {}^B\omega_1 {}^B\omega_2 \left(\frac{I_1 - I_2}{I_3}\right) + {}^B\tau_3 / I_3$

Dynamic and Static Balance –

- In this context, balance = doesn't cause vibrations when rotating
- Consider fixed-axis rotation at constant $\underline{\omega}$
- A rotating object is balanced if F_{reac} and M_{reac} are constant while an object rotates



$\underline{F}_{net} = m\underline{a}_G = -mg\hat{k} + \underline{F}_{reac} = \underline{\omega} \times (\underline{\omega} \times \underline{r}_{G/O})$

- If $\underline{r}_{G/O} \neq \underline{0}$, then F_{reac} is periodic and is statically imbalanced

Static balance – G is on the axis of rotation

Dynamic balance – spin axis aligns with a principal axis, $\underline{\omega} \parallel \hat{p}_i$

- The equations describing free rotation about G for a rigid object also apply for that object with an arbitrary point fixed to the object also fixed in an inertial frame

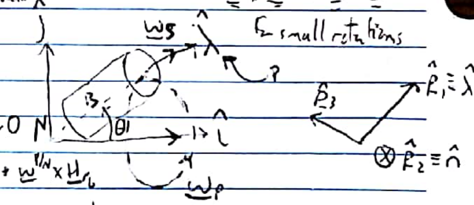
Wobbling Plate

- Small rotations add linearly: ${}^N Q^B = \underline{\underline{1}} + ({}^N Q^B - \underline{\underline{1}})$ ⇒ ${}^N Q^B \, {}^B Q^C = (\underline{\underline{1}} + ({}^N Q^B - \underline{\underline{1}}))(\underline{\underline{1}} + ({}^B Q^C - \underline{\underline{1}}))$

$= \underline{\underline{1}} + ({}^N Q^B - \underline{\underline{1}}) + ({}^B Q^C - \underline{\underline{1}}) + \underbrace{\cancel{({}^N Q^B - \underline{\underline{1}})({}^B Q^C - \underline{\underline{1}})}}_{\text{higher order}} \approx {}^N Q^B + {}^B Q^C - \underline{\underline{1}}$, so order of ${}^N Q^B$ and ${}^B Q^C$ doesn't matter

${}^N Q^B \, {}^B Q^C \approx {}^B Q^C \, {}^N Q^B$ for small rotations

Symmetric Object Small Steady Precession



${}^{B/N}\underline{\omega} = \omega_s \hat{\lambda} + \omega_p \hat{k}$

- Assume motion and find necessary torques to produce this

$\underline{\tau}_{/G} = \frac{{}^N d}{dt}(\underline{\underline{I}}_G \underline{\omega}^{B/N}) = \frac{{}^P d}{dt}(\underline{\underline{I}}_G \underline{\omega}^{B/N}) + \underline{\omega}^{P/N} \times \underline{H}_{/G}$

precessing frame derivative, $\underline{H}$ is constant in that frame since no difference in that frame as object precesses

$\underline{\tau}_{/G} = \underline{\omega}^P \times (\underline{\underline{I}} \cdot \underline{\omega}^B)$

${}^B I^C = \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_2 \end{bmatrix}$

$\underline{\underline{I}}^C = I_1 \hat{\lambda}\hat{\lambda} + I_2 \hat{m}\hat{m} + I_2 \hat{n}\hat{n}$ for $\hat{m}, \hat{n} \perp \hat{\lambda}$

$= I_1 \hat{\lambda}\hat{\lambda} + I_2(\underline{\underline{1}} - \hat{\lambda}\hat{\lambda})$

${}^B\omega = {}^B Q^N \, {}^N\omega \qquad {}^B Q^N = \hat{a} \cdot \hat{b}$

$\underline{\omega} = \underline{\omega}_s^{S/N} + \underline{\omega}_p^{P/N}$, $\quad \underline{\omega} \cdot \hat{\lambda} = \underline{\omega}_s \cdot \hat{\lambda} + \underline{\omega}_p \cdot \hat{\lambda} = {}^B\omega_1 = \omega_s + \omega_p \cos\theta$

$\underline{\tau}_{/G} = \underline{\omega}^P \times \underline{H}_{/G} = \dot{\underline{H}}_{/G} = \dot{\underline{H}}_{/G} + \underline{\omega}^{P/N} \times \underline{H}_{/G}$

$= 0$ since $\underline{H}$ is constant in precessing frame

$\underline{H}_{/G} = \underline{\underline{I}}_{/G} \cdot \underline{\omega} = [I_1 \hat{\lambda}\hat{\lambda} + I_2(\underline{\underline{1}} - \hat{\lambda}\hat{\lambda})] \cdot (\omega_p \hat{k} + \omega_s \hat{\lambda})$; $\hat{\lambda} \cdot \hat{k} = \cos\theta$

$= I_1 \omega_p \cos\theta\, \hat{\lambda} + I_2 \omega_p \hat{k} - I_2 \omega_p \cos\theta\, \hat{\lambda} + (I_1 \omega_s \hat{\lambda} + \cancel{I_2 \omega_s \hat{\lambda}} - \cancel{I_2 \omega_s \hat{\lambda}})$

$= \omega_p (I_1 - I_2)\cos\theta + I_1 \omega_s \hat{\lambda} + I_2 \omega_p \hat{k}$

$\underline{\omega}^{P/N} \times \underline{H}_{/G} = \omega_p \hat{k} \times [\omega_p (I_1 - I_2)\cos\theta + I_1 \omega_s \hat{\lambda} + I_2 \omega_p \hat{k}] = (\omega_p^2 (I_1 - I_2)\sin\theta\cos\theta + I_1 \omega_s \omega_p \sin\theta)\, \hat{p}_2$

$$\boxed{\underline{\tau}_{/G} = \sin\theta\, \omega_p (\omega_p (I_1 - I_2)\cos\theta + I_1 \omega_s)\, \hat{p}_2 = \text{required torque to cause steady precession}}$$

For Wobbling Feynman plate, $I_1 = 2I_2$, $\theta \ll 1$, $\underline{\tau}_{/G} = \underline{0}$ $\quad \{\,\} \cdot \hat{p}_2$

$\Rightarrow 0 = \theta\omega_p(\omega_p I_2 + I_1 \omega_s) = \omega_p \theta I_2 + 2 I_2 \omega_s \Rightarrow \omega_p = -2\omega_s$ — result from homework

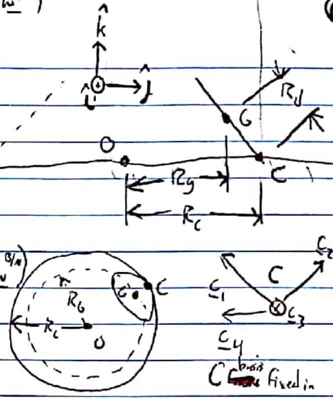
- For $\hat{k} \approx \hat{\lambda}$, $\omega = \omega \cdot \hat{k} \approx \omega_s + \omega_p = \frac{-1}{2}\omega_p + \omega_p = {}^1\!/_2\, \omega_p \approx \omega$
- For axisymmetric motions, $\underline{H}$, $\underline{\omega}$, and symmetry axis are coplanar

$$\boxed{\sum \underline{\tau}_{/C} = \underline{r}_{G/C} \times m\underline{a}_{G/N} + \underline{\underline{I}} \cdot {}^B\dot{\underline{\omega}}^{B/N} + \underline{\omega}^{B/N} \times (\underline{\underline{I}} \cdot \underline{\omega}^{B/N}) = \underline{r}_{G/C} \times m\underline{a}_{G/N} + (\underline{\underline{I}} \cdot \underline{\omega}^{B/N})^{\cdot}}$$

$\sum \underline{M}_{/C} = \underline{r}_{G/C} \times m \underline{a}_{G/N} + (\underline{\underline{I}}^G \cdot {}^N\underline{\dot{\omega}}^{B/N})$

$= '' \quad '' + \underline{\underline{I}}^G \cdot \underline{\dot{\omega}}^{CB/N} + \underline{\omega}^{C/N} \times (\underline{\underline{I}}^G \cdot \underline{\omega}^{B/N})$

Rolling Disk



B frame fixed to the disk

C basis fixed in precessing frame

$\underline{\omega}^{B/N} = \underline{\omega}^{B/C} + \underline{\omega}^{C/N} = -\omega^{B/C} \underline{c}_2 + \omega^C \hat{k}$

$\sum \underline{M}_{/C} = \underline{r}_{G/C} \times m \underline{a}_{G/N} + \underline{\underline{I}}^G \cdot {}^N\underline{\dot{\omega}}^{C/N} + \underline{\omega}^{C/N} \times (\underline{\underline{I}}^G \cdot \underline{\omega}^{B/N})$

C basis fixed in precessing frame

Aerodynamic Applications: Airplanes, Rockets, Drones

- Exact same problem of rigid object in 3D, just with new forces/moments

$\sum \underline{\tau}_{/G} = {}^N\underline{\dot{H}}_{/G} = \underline{\underline{I}} \cdot \underline{\dot{\omega}}^{NB/N} + \underline{\omega}^{B/N} \times (\underline{\underline{I}} \cdot \underline{\omega}^{B/N})$

${}^B\underline{\dot{Q}}^N = \underline{\omega}^{\times} \cdot {}^B\underline{Q}^N$; $\sum \underline{F} = \sum \underline{F}(\underline{\omega}, {}^N\underline{Q}^B, \underline{v}, \underline{r}, \rho)$

$\sum \underline{\tau}_{/G} = \underline{\tau}(\underline{\omega}, {}^N\underline{Q}^B, \underline{v}, \underline{r})$

RHS file: function $\dot{\underline{z}} = RHS(t, \underline{r}, \underline{v}, {}^N\underline{Q}^B, \underline{\omega}^{B/N})$

general case — $\underline{F}_{net} = \underline{F}_{net}(\underline{r}, \underline{v}, {}^N\underline{Q}^B, \underline{\omega})$

$\underline{\tau}_{net/G} = \underline{\tau}_{net}('' \quad '')$

$\dot{\underline{r}} = \underline{v}$; $\dot{\underline{v}} = \frac{1}{m} \cdot \underline{F}_{net}$;

${}^B\underline{\dot{Q}}^N = \underline{\omega}^{\times} \cdot {}^B\underline{Q}^N$; $\dot{\underline{\omega}} = \underline{\underline{I}}^{-1}(\underline{\tau}_{net/G} - \underline{\omega} \times (\underline{\underline{I}} \cdot \underline{\omega}))$

Thrust: rocket, propeller, engine

$\underline{F}_T = F_T \hat{\lambda}$ for $\hat{\lambda}$ fixed in B

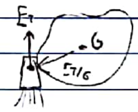
$\underline{r}_{T/G}$ = position of $\underline{F}_T$ wrt G

$\underline{\tau}_{T/G} = \underline{r}_{T/G} \times \underline{F}_T$

Propellers:

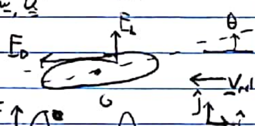
$\underline{F}_T = T\hat{\lambda} = c_T \omega^2 \hat{\lambda}$

$\underline{\tau}_T = \tau \hat{\lambda} = c_\tau \omega^2 \hat{\lambda}$



Aerodynamics: Lift and Drag

- Assume $\underline{F}_{lift}$, $\underline{F}_{drag}$ depend only on instantaneous $\underline{r}, \underline{v}, \underline{\omega}, {}^N\underline{Q}^B$ and not on time history
- Assume $\underline{F}_L, \underline{F}_D \propto |\underline{v}|^2$, $\propto A$ = X-section area $\perp \underline{v}_{rel}$

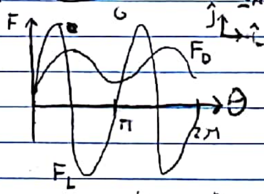


$\sum \underline{F}$ = gravity + lift + drag + ... + $\sum_i \underline{F}_{T_i}$

$\sum \underline{\tau}_{/G} = \underline{r}_{L/G} \times \underline{F}_L + ... + \sum_i \underline{r}_i \times \underline{F}_T + \sum_i \tau_i \lambda_i$

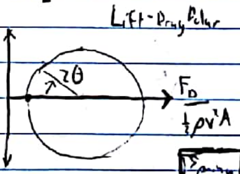
$\underline{F}_L = f(\theta) \cdot \frac{1}{2}\rho A |\underline{v}_{rel}|^2 \hat{j}$

$\underline{F}_D = g(\theta) \cdot \frac{1}{2}\rho A |\underline{v}_{rel}|^2 \cdot (-\hat{i})$



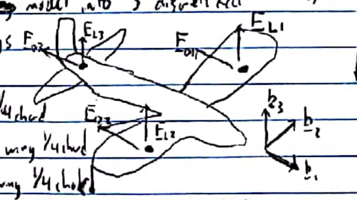
- Lift drag polar model is bad at modelling θ: behavior outside of typical flight regime $(-15°, 20°)$

Lift-Drag Polar



$\boxed{\text{Spring Break}}$

- Lump plane model into 3 discretized "point wings"



Center of tail, 1/4 chord

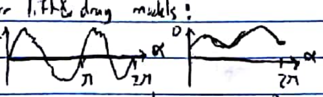
Center of right wing 1/4 chord

Center of left wing 1/4 chord

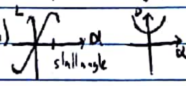
- To add control to plane, adjust the AoA according to some control law

4 approaches for lift & drag models:

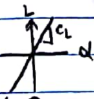
- 1. full polar



- 2. Partial polar (NACA empirical data)

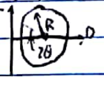


- 3. Linear model

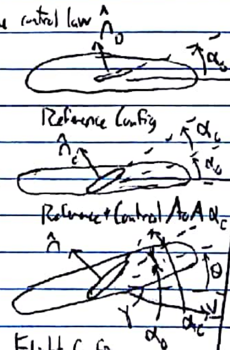


Drag = const

- 4. Circular polar



see above



Reference Config

Reference + Control AoA α_c

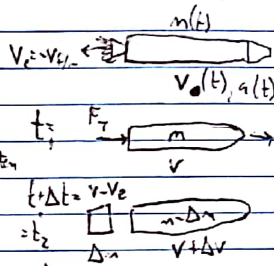
Flight Config

α_0 = wing AoA in level flight, no control

α_c = added AoA from control

θ = plane body AoA relative to horizontal

Rockets (1-D)



- ~~Neglect external forces~~
- Use sloppy pseudo-calculus derivation

$L_1 = mv$; $L_2 = L_1 + F_{ext}\Delta t$ } whole system

$F_{ext}\Delta t = L_2 - L_1 = (m - \Delta m)(v + \Delta v) + (\Delta m)(v - v_e) - mv$

Neglect higher order: $F_{ext}\Delta t = -v\Delta m + m\Delta v - v_e \Delta m + \Delta m v$

$F_{ext}\Delta t = m\Delta v - v_e \Delta m + O(\Delta^2)$

$F_{ext} = m\frac{\Delta v}{\Delta t} - v_e \frac{\Delta m}{\Delta t}$; $\dot{m} \triangleq \frac{dm}{dt} = -\frac{\Delta m}{\Delta t}$; $\dot{v} \triangleq \frac{dv}{dt} = \frac{\Delta v}{\Delta t}$ } sloppy

$F_{ext} = m\dot{v} + v_e \dot{m} \implies \dot{v} = (F_{ext} - v_e \dot{m})/m$

Neglecting F_{ext}: $m\dot{v} = -v_e \dot{m} \implies \frac{\dot{m}}{\dot{v}} = -\frac{m}{v_e} = \frac{dm}{dv} \implies -\frac{1}{v_e}dv = \frac{1}{m}dm$

$-v/v_e = \ln(m) + c \implies -v/v_e + c = \ln(m) \implies m = c_1 e^{-v/v_e}$

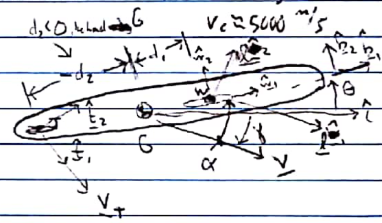
IC: $m = m_0$ @ $v = 0 \implies c_1 = m_0 \implies m/m_0 = e^{-v/v_e} \implies v = v_e \ln\left(\frac{m_0}{m}\right)$

- Energy of fuel $= E_f = (m_0 - m)\left(\frac{1}{2}v_e^2\right)$

specific KE of fuel ≈ 3 kcal/gm $= 12\times10^6$ J/kg $= \frac{1}{2}v_e^2$

$v_e \approx 5000$ m/s

Airplane Geometry



$\underline{L} = L \hat{l}_2$ $\quad \hat{l}_1 \parallel \underline{v}_{w/N}$

$\underline{v} = v\hat{\underline{w}}_1$

$\underline{v}_{w/N} = \underline{v}_G + \omega\hat{k} \times d_1 \hat{b}_1$

$\underline{v}_{T/N} = \underline{v}_G + \omega\hat{k} \times d_2 \hat{b}_1$

$\hat{l}_2 = \hat{\underline{\omega}} \times \hat{\underline{v}}_{w/N}$ $\quad \hat{l}_1 = \hat{l}_2 \times \hat{k}$

$\hat{t}_2 = \hat{k} \times \hat{\underline{v}}_{T/N}$ $\quad \hat{t}_1 = \hat{t}_2 \times \hat{k}$

$\underline{F}_w = L_w \hat{l}_2 - D_w \hat{l}_1$

$\underline{F}_T = L_T \hat{t}_2 - D_T \hat{l}_2$ (or more generally, $\hat{w}_1$)

α = AoA = angle from $\hat{v}_w$ to $\hat{b}_1$

$\alpha = \text{atan2}(\hat{w}_1 \cdot \hat{l}_2, \hat{\underline{w}}_1 \cdot \hat{\underline{l}}_1)$

- Ruina also talked of modeling plane as a Chaplygin sleigh where the tail is the skate

$\underline{v}_c = \underline{v}_T = (\omega\hat{k} \times d_2\hat{b}_1) + \underline{v}_G = \underline{v}_G + \omega d_2 \hat{\underline{b}}_2$

$\underline{v}_c \cdot \underline{b}_2 \{0 = (\underline{v}_G \cdot \hat{b}_2) + (\omega\hat{k} \times d_2\hat{b}_1) \cdot \hat{b}_2\}$

$\frac{d}{dt}\{\ \} \implies 0 = \dot{\underline{v}}_G \cdot \hat{b}_2 + \underline{v}_G \cdot \dot{\hat{b}}_2 + \dot{\omega} d_2$

Chaplygin Sleigh

$\ddot{\theta} = \frac{-lm\dot{\theta}v_c}{ml^2 + I^G}$

$\dot{v}_c = \dot{\theta}^2 l$

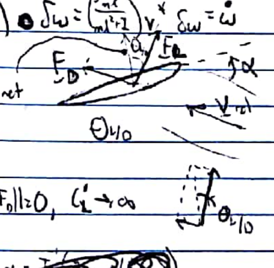
Chaplygin Sleigh Linearization

- Linearize Ch. Sl. about $\dot{\theta} = 0$, $v_c = \text{const}$

$v_c = v_c^* + \delta v_c \implies \dot{v}_c = d\,\delta\omega^2 \quad O(n^2) \quad O(n^2) = 0 = \dot{v}_c$

$\omega = \cancel{\omega^*} + \delta\omega \implies \dot{\omega} = \left(\frac{-ml}{ml^2 + I}\right)(v_c^* + \cancel{\delta v_c})\delta\omega = \left(\frac{-ml}{ml^2+I}\right)v_c^* \delta\omega = \dot{\omega}$

Sailboats



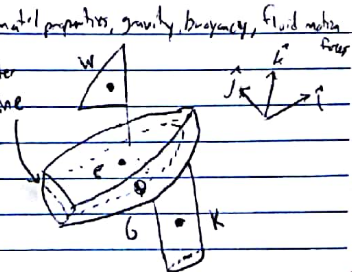
- In sailboats, generally want L/D to be large, so $\theta_{L/D} = \text{atan2}(F_D, F_L) \ll 1$
- Ruina ideal wing: $\|\underline{C}_d\| = 0$, $\|\underline{C}_L\| = C_L' \alpha$
- Ruina super-ideal wing: non-holonomic constraint, $\|F_D\| = 0$, $C_L' \to \infty$
- Treat sailboat as a single 3D rigid body
- Kinematics: $\dot{\underline{r}} = \underline{v}$; $\dot{\underline{v}} = \underline{a}$; ${}^N\underline{\dot{Q}}^B = \underline{\omega}^{\times N} \underline{Q}^B$; $\dot{\underline{\omega}} = \underline{\alpha}$
- Geometry: submerged volume found using ${}^N\underline{Q}^B$, $\underline{r}_G \cdot \hat{k} = z_G \implies$ centroid of submerged volume $\forall_{sub}$
- Laws of Motion: $\underline{a}_G = \underline{F}_{net}/m$; $\underline{\alpha} = \underline{\underline{I}}^{-1}(\underline{\tau} - \underline{\omega} \times (\underline{\underline{I}} \cdot \underline{\omega}))$
- Force Laws / Constitutive Laws: material properties, gravity, buoyancy, fluid motion forces

W: center of sail

C: centroid of submerged volume

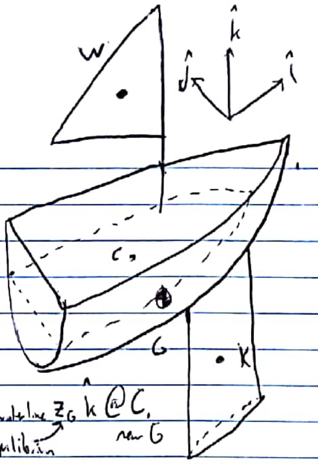
G: CoM

K: center of keel



$\underline{F}_{buoyancy} = \rho g \forall_{sub} \hat{k}$ @ C, $\underline{F}_g = -mg\hat{k}$ @ G

Sailboats CONT'D



Buoyancy:

- Consider finding the submerged volume $\forall_{sub}$ linearizing about equilibrium:

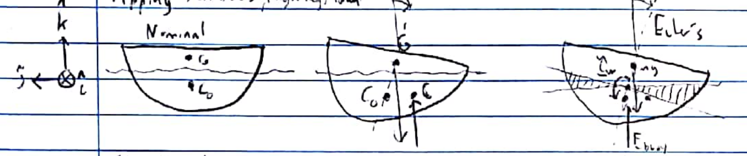
$\underline{F}_{buoy} = \rho g \forall_{sub} \hat{k}$

- In equilibrium, $\underline{F}_{buoy} = mg\hat{k}$

Small perturbation: $\underline{F}_{buoy} = mg\hat{k} - \rho g A_{waterline} z_G \hat{k}$ @ C, near G

measured from equilibrium

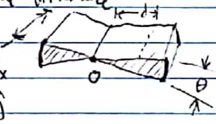
Tipping Sailboats, Righting Moment



- Instead of calculating $\underline{r}_{C/G}$ for every θ, apply $\underline{F}_{buoy}$ at C_0 and correct with a torque $\underline{\tau}_w$ to account for the difference

$\underline{\tau}_w = \int \rho g \, d\forall = \rho g \, l_x \int d \, d\theta dy$

$\underline{\tau}_w = -\rho g \theta \int d^2 \, d\theta dy \, \hat{i} = -\rho g \theta I_{w\hat{i}}$ ← moment of area of waterline wedges



- Same process used to calculate moment in $\hat{j}$

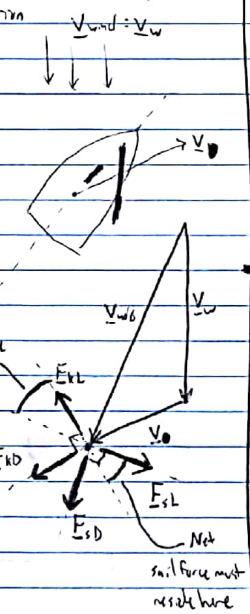
$\underline{\tau}_w = -\rho g \theta_x I_{w,xx} \hat{i} - \rho g \theta_y I_{w,yy} \hat{j}$

- Treat keel and sail just like airplane wings, similar aero model and allow for control of sail angle and position

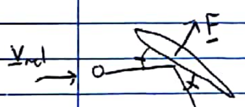
2D Sailboat: top-down

- Assume still water
- Demonstrate a static equilibrium where $(\underline{v}_w \cdot \underline{v}) < 0$, ignore full dynamic model

$\underline{v}_{w/G} = \underline{v}_w - \underline{v}$



Elastic Lift-Drag Model



- Model as particles bouncing off of wing elastically

$C_L = 4 \cos\theta \sin^2\theta$

$C_D = 4 \sin^3\theta$

Hurt right wrist
Sorry future reader.

Robot Arms

- Model as a tree structure of rigid objects connected by kinematic constraints (e.g. hinges), actuators, and fixed
- In series, parallel tree structure, closed loop, etc.

- 6 DOF of "hand" → usually 6 or 7 1DOF joints in arm

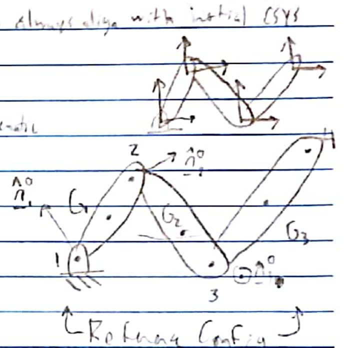
- Types of problems: given geometry, inertia props, robot dims.
  - Given all θ_i of each joint, find "pose" of hand & all parts
  - Given $\theta_i(t)$, find trajectories of parts
  - Given $\theta_i, \dot{\theta}_i, \ddot{\theta}_i$, find $\underline{v}_{Gi}, \underline{a}_{Gi}, \underline{\omega}_i, \underline{\alpha}_i, {}^N\underline{\underline{Q}}^{B_i}$
  - Given $\theta_i, \dot{\theta}_i, \ddot{\theta}_i$, find τ_i = torques of joint actuators
  - Given $\tau_i(t)$, ICs, find $\underline{v}_{Gi}, \underline{a}_{Gi}$...
  - Given ${}^N\underline{\underline{Q}}^H(t)$ = hand DCM & $\underline{r}_H(t)$ = hand pos'n, find $\theta_i(t)$, $\tau_i(t)$.

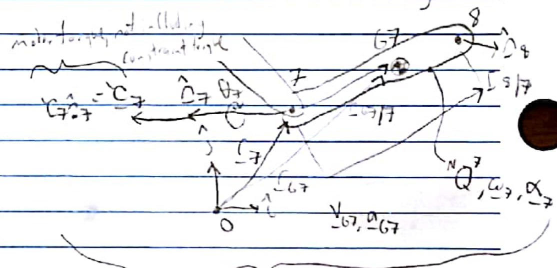
- Servos follow kinematic control more reliably than torque control
- Now, better motors leading to better torque control
- Model multi-DOF joints as a series of 1DOF joints
  - Ball & socket ≡ 3 hinge joints with infinitesimal links between

Variables: $\theta_i(t)$, $\dot{\theta}_i$, $\ddot{\theta}_i$, ${}^N\underline{\underline{Q}}^{B_i}$, $\underline{r}_{i/i-1}$, $\underline{r}_{G_i/i}$, coordinates of $\underline{r}_{G_i/O}$, $\underline{v}_{G_i/O}$, $\underline{a}_{G_i/O}$

Parameters: all position vectors at $\theta_i = 0 \; \forall i$, ${}^{B_i}\underline{\underline{I}}_i$ = constant inertias in each body frame, $\underline{n}_i$ = hinge axes in reference config, where $\theta_i = 0 \; \forall i$

- In reference config (arbitrarily chosen), $\theta_i = 0$, $\underline{n}_i = \underline{n}_i^0$
- Kane's method: use CSYS's which always align with inertial CSYS
- τ_7 is scalar component of total joint torque in direction of $\hat{n}_7$
  - Also torques to maintain kinematic constraints, not keeping track in this formulation

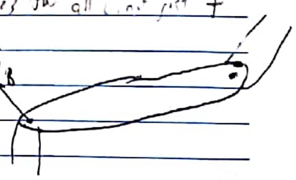




Applies for all i, not just 7

$\underline{r} = {}^B\underline{\underline{Q}}^N \underline{r}_0$; ${}^B\underline{\underline{Q}}^N = \hat{n}\hat{n} + (\underline{\underline{1}} - \hat{n}\hat{n})\cos\theta + \sin\theta \, \hat{n}^\times$

where $\hat{n}, \theta$ are the Euler parameters (Axis-Angle)



- For small rotations by $\delta\theta$, ${}^B\underline{\underline{Q}}^N \approx \underline{\underline{1}} + \delta\theta \, \hat{n}^\times$

$\delta\theta = \omega \delta t \Rightarrow {}^B\underline{\underline{Q}}^N \approx \underline{\underline{1}} + \omega \delta t \, \hat{n}^\times$

$\underline{r} = {}^B\underline{\underline{Q}}^N \cdot \underline{r}_0 = \underline{r}_0 + \omega \delta t \, \hat{n}^\times \cdot \underline{r}_0 = \underline{r}_0 + \omega \hat{n} \times \underline{r}_0 \, \delta t = \underline{r}_0 + \underline{\omega} \times \underline{r}_0 \, \delta t \Rightarrow \dot{\underline{r}} = \underline{\omega} \times \underline{r}_0$

- For $\underline{r} = {}^B\underline{\underline{Q}}^A \cdot {}^A\underline{\underline{Q}}^N \cdot \underline{r}_0$, for small rotations, $\dot{\underline{r}} = (\underline{\omega}^{B/A} + \underline{\omega}^{A/N}) \times \underline{r}$

$\underline{\omega}^{B/N} = \underline{\omega}^{B/A} + \underline{\omega}^{A/N}$

${}^N\dot{\underline{\omega}}^{B/N} = {}^N\dot{\underline{\omega}}^{B/A} + {}^N\dot{\underline{\omega}}^{A/N} = {}^A\dot{\underline{\omega}}^{B/A} + \underline{\omega}^{A/N} \times \underline{\omega}^{B/A} + {}^N\dot{\underline{\omega}}^{A/N}$

- For ${}^N\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\underline{f} = [u \; v \; w]$, $\underline{\underline{J}} = \partial \underline{f} / \partial {}^N\underline{r} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \ddots & \\ \partial w/\partial x & & \end{bmatrix}$

$\dot{\underline{f}} = \underline{\underline{J}} \, {}^N\dot{\underline{r}}$ ← column vector

- In robotics, the Jacobian is relating end effector pose rate to joint rotation rates

$\underline{\chi} = [\dot{x}_f \; \dot{y}_f \; \dot{z}_f \; {}^N\omega_{1f} \; {}^N\omega_{2f} \; {}^N\omega_{3f}]^T$; $\underline{\Omega} = [\dot{\theta}_1 \; \dot{\theta}_2 \cdots \dot{\theta}_f]^T$

$\underline{\underline{J}} = \dfrac{\partial \underline{\chi}}{\partial \underline{\Omega}} = \begin{bmatrix} \dfrac{\partial \dot{x}_f}{\partial \dot{\theta}_1} & \dfrac{\partial \dot{x}_f}{\partial \dot{\theta}_2} & \cdots \\ \dfrac{\partial \dot{y}_f}{\partial \dot{\theta}_1} & \ddots & \end{bmatrix}$

- For some configurations, the Jacobian is singular, which means some DOF directions can't be instantaneously achieved

$\underline{\chi} = \underline{\underline{J}} \, \underline{\Omega}$

Calculating $\underline{\underline{J}}$: after solving the forward kinematics, apply $\underline{\Omega}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

- Get $\underline{\chi}_1$ from $\underline{\Omega}_1$, then $\underline{\chi}_1$ = 1st column of $\underline{\underline{J}}$
- Define $FK(\underline{\Omega}, \underline{\theta}) = \underline{\chi}$ = forward kinematics

$\underline{\underline{J}} = FK(\underline{\underline{1}}_{6\times6}, \underline{\theta})$, evaluating for each column of $\underline{\underline{1}}$

"bias term, mostly $\underline{\omega} \times \underline{\omega} \times$" terms

- Also, $\underline{\underline{J}} \, \dot{\underline{\Omega}} = \dot{\underline{\chi}} + \underline{B}$

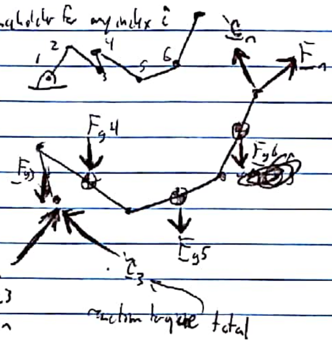
Inverse Dynamics of Robot Arm: given motion, applied loads, and $\underline{\tau}_n, \underline{F}_n$ hand force/torque, find joint torques

- Forward kinematics to find all $\underline{v}_i, \underline{a}_i, \underline{\alpha}_i$, etc.
- Inverse dynamics to find $\underline{\tau}_i(t)$

- Forward dynamics: given joint torques τ_i, find motion (harder)
- Typical problem: given motion of hand, find joint torques τ_i
  - Do inverse kinematics to go from $\underline{r}_f(t), {}^{N}Q^{H}(t) \rightarrow \theta_i(t)\ \forall i$
  - Do inverse dynamics to go from $\theta_i(t) \rightarrow \tau_i(t)$

Inverse dynamics: in this section, "3" is a placeholder for any index i

- Draw FBD from joint 3 and up
- $AMB_{/3} \Rightarrow \underline{\tau}_3$
- $\tau_3 = \underline{\tau}_3 \cdot \hat{\underline{n}}_3$
- Start either from link 1 or link n, doesn't matter
- Go down/up the joints, draw FBD for each, cutting at all joints, find τ_i



$$\sum_{i}^{3} \underline{\tau}_{/3} = \underbrace{\underline{\tau}_n}_{\text{known}} + \underline{r}_n \times \underline{F}_n + \underbrace{\langle \text{gravity terms} \rangle}_{\text{Known: } \underline{r}_{Gi/3} \times \underline{F}_{gi}} + \underline{\tau}_3 \} \text{unknown} = \underbrace{\sum \underline{r}_{Gi} \times m_i \underline{a}_{Gi} + \underline{\underline{I}}_i^{G} \cdot \underline{\alpha}_i + \underline{\omega}_i \times (\underline{\underline{I}}_i^{G} \cdot \underline{\omega}_i)}_{\text{All these kinematics solved earlier}}$$

all alone: $\{\hat{\underline{n}}_3\}$

Forward Dynamics of Robot Arm

- Given joint torques $\tau_i, \underline{F}_n, \underline{\tau}_n$, find $\theta_i(t)$
- The n equations from $AMB_{/i}$ are linear in $\ddot{\theta}_i$
- Define $\{E\}$ as col. vec. of AMB equations
- Define $\{F\}$ as col. vec. of terms not multiplying any $\ddot{\theta}_i$
- Define $\{\ddot{\theta}\}$ as col. vec. of $\ddot{\theta}_i$ terms

$\{F(\theta, \dot{\theta}, \underline{F}_n, \underline{\tau}_n)\} = [A]\{\ddot{\theta}\}$ where $[A]$ = a jacobian of $\dfrac{\partial \{E\}}{\partial \{\ddot{\theta}\}}$

- Evaluating jacobian is computationally expensive, will use other method
- Inverse kinematics: given ${}^{N}Q^{f}(t), \underline{r}_f(t)$, find $\theta_i(t)$
  - Evaluate $J = \partial \underline{\chi} / \partial \underline{\Omega}$ at ICs, invert, solve $\underline{\Omega} = J^{-1} \underline{\chi}$
  - Propagate state with numerical solver, iterate
- Find the jacobian J s.t. $\underline{\chi} = J \underline{\Omega}$
  - $\underline{\chi} = FK(\underline{\Omega}, \underline{\theta}) \Rightarrow J = FK(\underline{1}_{6\times6}, \underline{\theta})$, use probing method to extract J by solving forward kinematics 6 times
  - $J \dot{\underline{\Omega}} = \dot{\underline{\chi}} + \underline{B}$ ← bias = pure acceleration of hand if $\ddot{\theta}_i = 0$
- Find $\underline{B}$ the kinematic bias term given $\theta_i, \dot{\theta}_i$
  - Use $J(\underline{\Omega}, \underline{\theta})$, set $\dot{\underline{\Omega}} = 0 \Rightarrow 0 = \dot{\underline{\chi}} + \underline{B} \Rightarrow \underline{B} = -\dot{\underline{\chi}}$
- Find mass matrix M
  - $M \ddot{\underline{\theta}} = \hat{\underline{Q}}_{6\times1} + \underline{\tau}_{6\times1}$ ← joint torques; $\hat{Q}$ = dynamic bias $Q(\theta_i, \dot{\theta}_i, g, \underline{F}_n, \underline{\tau}_n)$
  - $\underline{\tau} = ID(\theta_i, \dot{\theta}_i, \ddot{\theta}_i)$ = inverse dynamics for given joint angles and derivatives
  - Evaluate $\underline{\tau}_0 = ID(\theta_i, \dot{\theta}_i, 0)$ with $\ddot{\theta}_i = 0$, then $M \cdot \underline{0} = \hat{\underline{Q}} + \underline{\tau}_0$; $\hat{\underline{Q}} = -\underline{\tau}_0$
  - Use probe method: $\underline{\tau} = ID(\theta_i, \dot{\theta}_i, \underline{1}_{6\times6})$
  - For each column j in $\underline{1}_{6\times6}$, $M_j = \hat{\underline{Q}} + \underline{\tau}_j$; $\underline{\tau}_j = ID(\theta_i, \dot{\theta}_i, j^{\text{th}}$ column of $\underline{1}_{6\times6})$

$M = [\hat{Q}\ \hat{Q} \ldots \hat{Q}]_{6\times6} + ID(\theta_i, \dot{\theta}_i, \underline{1}_{6\times6})$

- Inverse kinematics: given $\underline{r}_f(t), {}^{N}Q^{B}(t)$, find $\theta_i(t), \dot{\theta}_i(t), \ddot{\theta}_i(t)$
  - Differentiate $\underline{\chi}(t)$, find $\underline{v}_f, \underline{a}_f, \underline{\omega}_f$, etc. at each time step
  - Calculate $\dot{\underline{\theta}}_i = J(\theta_i)^{-1} \underline{\chi}$
  - Since θ_i unknown for given $\underline{r}_f, {}^{N}Q^{B}_f$, start with a guess of θ_i (reference config works), integrate kinematics to go from guess to given $\underline{r}_f, {}^{N}Q^{B}_f$
  - Now have θ_i at $\underline{r}_f, {}^{N}Q^{B}$, evaluate $J(\theta_i)$, $\dot{\theta}_i = J(\theta_i)^{-1} \underline{\chi}$
- Compute torque at time t given $\underline{r}_f(t), {}^{N}Q^{B}(t)$, find $\underline{\tau}_i$
  - Inverse kinematics to find $\theta_i, \dot{\theta}_i, \ddot{\theta}_i$