DYNAMICS & VIBRATIONS

4730/5730 INTERMEDIATE DYNAMICS

8/27/14

COURSE info on web - ask Q's on Piazza

1) Course intro
2) $\vec{r}, \vec{v}, \vec{a}$ rect & polar

Course intro

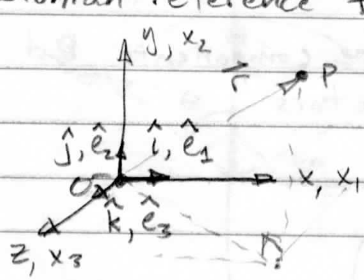
Goal: Given description or picture of system will be able to:
- make description precise
- write governing eq's
- solve equations
  - graph results
  - animate results

  } Matlab
- vibration analysis if appropriate
  - normal modes
  - damping
  - resonance
  - vibration isolation
  - Matrix exponential

$\vec{r}, \vec{v}, \vec{a}$

Assume we are given a good frame = Newtonian = Fixed
- a coordinate system & origin
- in this system Newton's Laws are accurate $(1:10^9)$

Newtonian reference frame



Position vector

$\vec{x} = \vec{r}$ informal

$\vec{r}_{P/O}$ pos. of P relative to O

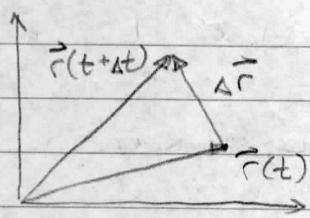
$\vec{r}_{OP}$ vector from O to P

will use all 3

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= r_x\hat{i} + r_y\hat{j} + r_z\hat{k}$$
$$= x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$$
$$= \Sigma x_i \hat{e}_i$$ Einstein summation convention

Velocity (relative to the fixed frame)

given $\vec{r}(t)$



$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

take derivative of components & hold unit vecs. const

Careful notation for $\vec{v}$

$$\vec{v}_{\mathcal{F}} = \frac{d^{\mathcal{F}}\vec{r}_P}{dt}$$ differentiate components but not base vectors

Acceleration

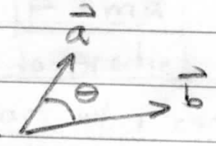
$$\vec{a} = \vec{a}_{P/O} = \frac{d^{\mathcal{F}}}{dt}\left(\frac{d^{\mathcal{F}}\vec{r}_{P/O}}{dt}\right)$$

$$= \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

Need to know

dot product
cross product
mixed triple product

$\Bigg\}$ Ruina & chap 2.



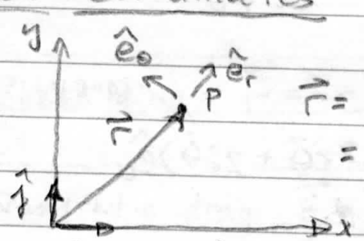
$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z \quad \Big\} \text{ amazing}$$
$$= ab\cos\theta$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\,\hat{\imath} + (a_z b_x - a_x b_z)\,\hat{\jmath} + (a_x b_y - a_y b_x)\,\hat{k}$$
$$= ab\sin\theta\,\hat{n} \qquad \hat{n} \text{ is unit vect } \perp \text{ to } \vec{a} \,\&\, \vec{b} \text{ r.h. rule}$$

Notation

$\vec{v}, \vec{\mathbf{V}}, \underline{V}, \bar{V}, \mathbb{V}$ ← pick one, stick w/ it

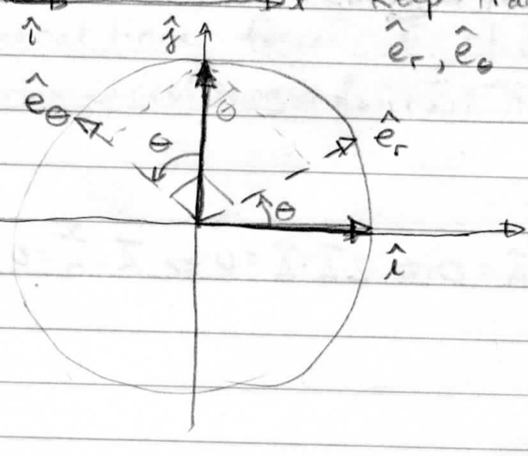
Polar Coordinates



$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$
$$= r\,\hat{e}_r \qquad \hat{e}_r = \frac{\vec{r}}{|\vec{r}|}$$

keep track w/ r,θ instead of x,y

$\hat{e}_r, \hat{e}_\theta$ change



$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$
$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$
$$\hat{\imath} = \cos\theta\,\hat{e}_r + {}^{-}\sin\theta\,\hat{e}_\theta$$
$$\hat{\jmath} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$$

fixed frame

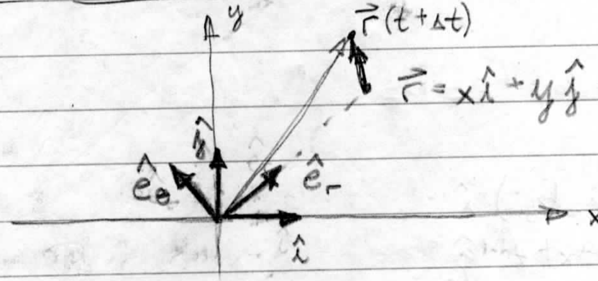
Velocity

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d^{f}}{dt}\left(r\hat{e}_r\right)$$

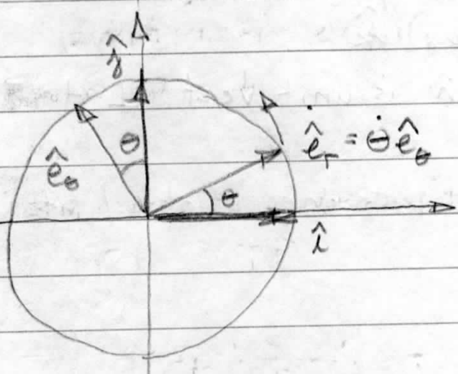
$$= \dot{r}\hat{e}_r + r\dot{\hat{e}}_r \qquad \dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$$

8/29/14 **Polar Coordinates (cont.)**



$\vec{r} = x\hat{i} + y\hat{j} = r\hat{e}_r$



$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$

$|\hat{e}_r| = \text{const}$

$\dot{\hat{e}}_r \neq 0$

CORIOLIS

$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} = \dot{r}\hat{e}_r + \dot{\theta} r\hat{e}_\theta$

$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

$|\vec{A}| = \text{const.} \quad \dot{\vec{A}} \neq 0$

$\vec{A}\cdot\dot{\vec{A}} = ? = \emptyset$

$|\vec{A}| = \text{const}$

$\Rightarrow |\vec{A}|^2 = \text{const}$

$\Rightarrow \{\vec{A}\cdot\vec{A} = \text{const}\}$

$\frac{d}{dt} \Rightarrow \vec{A}\cdot\dot{\vec{A}} + \dot{\vec{A}}\cdot\vec{A} = 0 \Rightarrow 2\vec{A}\cdot\dot{\vec{A}} = 0 \Rightarrow \vec{A}\cdot\dot{\vec{A}} = 0$

Particle Dynamics

Particle - operational def.

neglect (doesn't mean it's not important) rotation + deformation

Only pay attention to position of C.O.M.

$$\vec{F} = m\vec{a} \quad \text{(only)}$$

How to predict the future

Start w/ present state $\vec{r}, \vec{v}$

System description : particle

Rules: (Read Chap. 1)

1. Geometry $\longrightarrow$ $\dot{\vec{r}} = \vec{v}$

3. Newton's Laws $\longrightarrow$ $\vec{F} = m\vec{a}$

2. Material Rules

Constitutive Laws

$$\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$$

Ex. of Constit. Laws for particle Mech.

No Force: $\vec{F} = \vec{0}$

Near earth gravity: $\vec{F} = -mg\hat{j}$

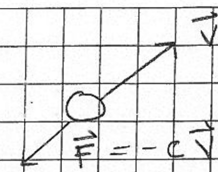
Big G gravity: $\vec{F} = \frac{-mMG}{r^3}\vec{r}$

$$= \frac{-mR_e^2 g}{r^2}\vec{r}$$

Linear Drag: $\vec{F} = -c\vec{v}$

$= -c A \mu \vec{v}$

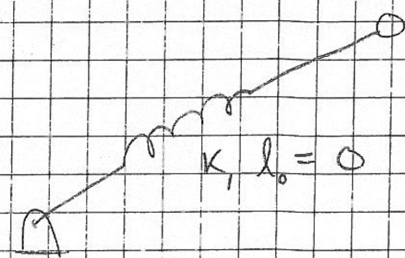
A — representative area $\sim r^2$

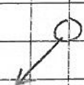


Quadratic Drag: $\vec{F} = -c|\vec{v}|\vec{v}$

Central Linear Force: $\vec{F} = -k\vec{r}$

(zero rest-length spring)





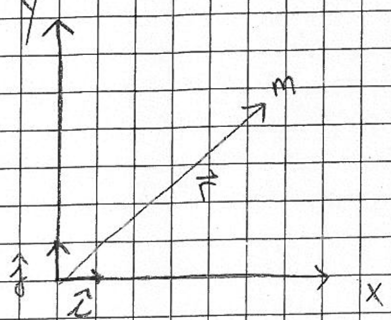
$$\vec{F} = -kr = -k\ell \frac{\vec{r}}{|\vec{r}|} = -k|\vec{r}|\frac{\vec{r}}{|\vec{r}|}$$

Spring w/ rest length l_0:

$$\vec{F} = -k\left(|\vec{r}| - l_0\right)\frac{\vec{r}}{|\vec{r}|}$$

9/3/2014

Particle



$\vec{F} = \sum$ all forces on particle

Three Pillars

I. Constitutive Laws $\qquad \vec{F} = F(\underbrace{\vec{r}, \vec{v}}_{\text{state}}, t)$

II Geometry / Kinematics

$$\vec{v} = \dot{\vec{r}}, \quad \vec{a} = \dot{\vec{v}}$$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$$

III Laws of Mechanics

$$\vec{F} = m\vec{a}$$

How to solve a problem?

(set up ODE)

e.g. R.H.S. file for Matlab

$\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$

$\dot{\vec{r}} = v \qquad \begin{bmatrix} n \\ 2n \end{bmatrix} \begin{bmatrix} 1^{st} \\ 2^{nd} \end{bmatrix}$ order ODEs in n Dimensions

$\dot{\vec{v}} = \vec{F}/m$

Famous Problems

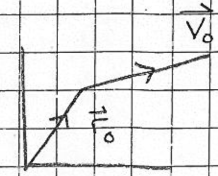
* $\vec{F} = \vec{0}$ "An object tends..."

LMB: $\vec{F} = m\vec{a}$

$$m\dot{\vec{v}} = \vec{0}$$

$$\vec{v} = \vec{v}_0 = \text{const.}$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t$$



* $\vec{F} = \vec{F}_0 = \text{const.}$

LMB: $\vec{F}_0 = m\vec{a}$

$$\vec{F}_0 = m\dot{\vec{v}}$$

$$\vec{v} = \frac{\vec{F}_0}{m} t + \vec{v}_0$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{\vec{F}_0}{m}\frac{t^2}{2}$$

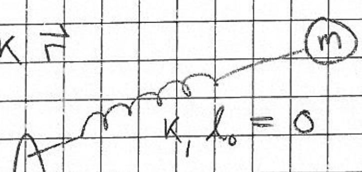
$$\vec{F}_0 = -mg\hat{j}$$

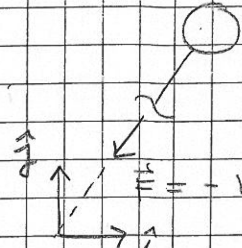
LMB: $-mg\hat{j} = m\vec{a}$

$$x = x_0 + v_{x_0} t$$

$$y = y_0 + v_{y_0} t - gt^2/2$$

* $\vec{F} = -K\vec{r}$





$\vec{F} = -K\vec{r} = -Kr\hat{e}_r$

$$\Sigma \vec{F} = m\vec{a}$$

$$\left\{-K\vec{r} = m\ddot{\vec{r}}\right\}$$

$$\{-k\underbrace{\vec{r}}_{x\hat{\imath}+y\hat{\jmath}} = m\ddot{\vec{r}}\}$$

$$\{\ \}\cdot\hat{\imath} \longrightarrow -kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$x = A\cos(\omega t) + B\sin(\omega t)$$

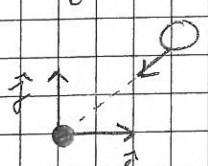
$$\omega = \sqrt{\frac{k}{m}}$$

See appendix ODEs in Ruina / Pratap

$$\left[\begin{array}{l} y = C\cos(\omega t) + D\sin(\omega t) \\ x = A\cos(\omega t) + B\sin(\omega t) \end{array}\right.$$

decoupled equations

* Inverse Square Gravity



$$\vec{F} = -\frac{c}{r^2}\hat{e}_r$$

$$c = \begin{cases} MGm \\ gR_e^2 m \end{cases}$$

LMB: $\vec{F} = m\vec{a}$

$$-\frac{c\hat{e}_r}{r^2} = m\left((\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right)$$

$\{\ \}\cdot\hat{e}_r$

$\{\ \}\cdot\hat{e}_\theta$ $\Longrightarrow$

$$\left.\begin{array}{l} -\dfrac{c}{r^2} = m\ddot{r} - mr\dot{\theta}^2 \\ 0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{array}\right\} *$$

guess circular motion

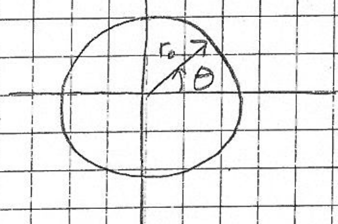
$r = r_0 = \text{const.}$

$$* \longrightarrow \frac{c}{r^2} = m r \dot{\theta}^2 \longrightarrow \dot{\theta}^2 = \frac{c}{m r^3}$$

$$\dot{\theta} = \dot{\theta}_0 = \text{const.}$$

$$\theta = \theta_0 + \dot{\theta}_0 (t)$$

$$\dot{\theta} = \sqrt{\frac{c}{m r^3}}$$



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$$\vec{F} = \frac{-GMm}{r^2}\,\hat{e}_r$$

LMB: $\vec{F} = m\vec{a}$

$$\left\{ \frac{-GMm}{r^2}\,\hat{e}_r = m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta\right] \right\}$$

$$\{\ \}\cdot\hat{e}_r \longrightarrow \frac{-GMm}{r^2} = -m r \dot{\theta}^2$$

$$\frac{GM}{r} = \left(r\dot{\theta}\right)^2 = v^2$$

$$V = \sqrt{\frac{GM}{r}}$$

Bigger r, slower V

Analogy

Rules of arithemetic $\Longrightarrow$ Add two odd numbers, get even #

every # has unique factorization

Soln. of $\vec{F} = m\vec{a}$ $\Longrightarrow$ Conservation of Energy, Mom., Ang. Mom.

Conservation Laws

Linear Momentum: $\vec{L}$

$$\vec{F} = m\vec{a}$$

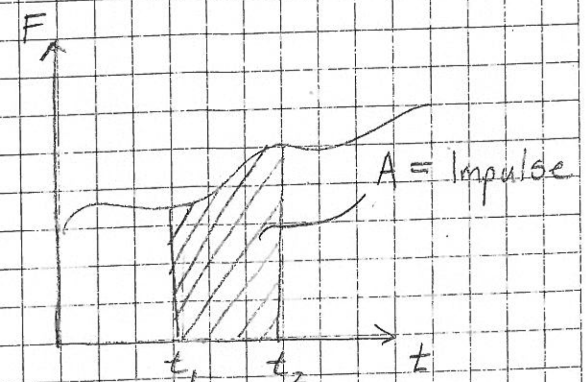
$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{t_1}^{t_2} m\underbrace{\vec{a}}_{\frac{d\vec{v}}{dt}}\,dt$$

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\int d\vec{V}$$

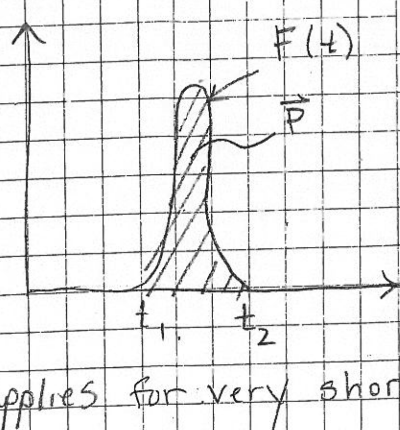
$$= m\,\Delta\vec{V}$$

$$= \Delta\vec{L}$$

$$\vec{L} = m\vec{V}$$



Special:



This applies for very short dur. forces

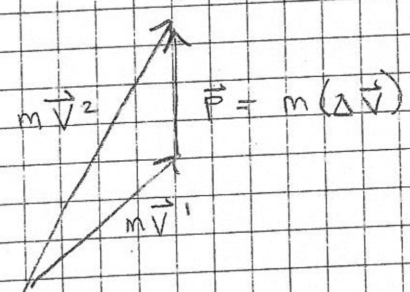
$$F(t) = F\ \delta(t)$$

δ ← delta function (Dirac)

$$\delta(t-t_0) = \begin{cases} 0 & \text{for all } t, \text{ but } t_0 \\ \text{undefined} & \text{for } t = t_0 \end{cases}$$

$$\int_{t_1}^{t_2} \delta(t-t_0) = 1 \qquad \text{if} \quad t_1 < t_0 < t_2$$

Newton love Impulses



Conservation of Momentum

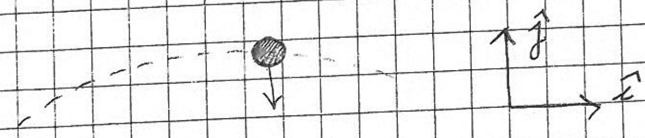
Assume that $\vec{F}\cdot\hat{\lambda} = 0$ for all time t

$$\int F\cdot\hat{\lambda}\,dt = \Delta(m\vec{V})\,\hat{\lambda}$$

$$0 = \Delta\vec{L}\cdot\hat{\lambda}$$

$\vec{L}$ in the $\hat{\lambda}$ is conserved

Ball w/ no air friction



$$\vec{F}\cdot\hat{i} = 0 \quad \text{for all } t$$

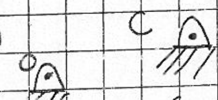
$$L_x = \text{cons}$$

MIQUELRIUS

Angular Momentum

$$\left\{ \boxed{\vec{F} = m\vec{a}} \right\}$$

$* \quad \vec{r}_{/c} \times \{\ \} \Rightarrow \vec{r}_{/c} \times \vec{F} = \vec{r}_{/c} \times (m\vec{a})$



fixed in space

Look at: $\frac{d}{dt}\left(\vec{r}_{/c} \times \vec{v}\right) = \cancelto{0}{\dot{\vec{r}}_{/c} \times \underbrace{\vec{v}}_{\dot{\vec{r}}_{/F}}} + \vec{r}_{/c} \times \underbrace{\dot{\vec{v}}}_{\vec{a}}$

$= \vec{r}_{/c} \times m\vec{a}$

$* \rightarrow \vec{r}_{/c} \times \vec{F} = \frac{d}{dt}\underbrace{\vec{H}_{/c}}$

$\vec{H}_{/c} = \vec{r}_{/c} \times m\vec{v}$

Moment w.r.t. C is $= \frac{d}{dt}$ (Ang. Mom. rel to C)

$\int dt \Rightarrow$ angular impulse $= \Delta\vec{H}_{/c}$

Cons. of Angular Momentum

if $\vec{F} \parallel \vec{r}_{/c}$ for all t

$\rightarrow \vec{r}_{/c} \times \vec{F} = \vec{0}$

$\rightarrow \Delta\vec{H}_{/c} = 0$

$\vec{r}_{/c} \times m\vec{v} = \text{const.}$

Cartesian Coordinates

Cons. of Ang. Mom

$$\left\{ \vec{r} \times \vec{v} = const. \right\}$$

$$\{\ \}\cdot \hat{k} \longrightarrow x\dot{y} + y\dot{x} = const.$$

Polar Coord. Cons. of Ang. Mom.

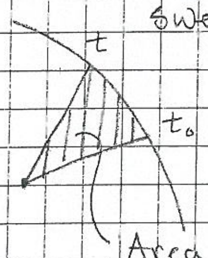
$$f(r, \theta, \dot{r}, \dot{\theta}) = const.$$

$$\dot{r}^2 \dot{\theta} = const.$$

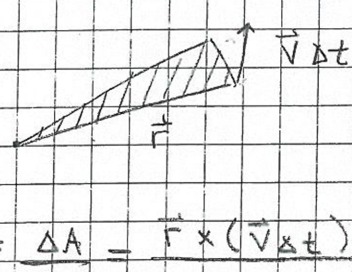
$$\vec{r} = r\hat{e}_r \ , \quad \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{r} \times \vec{v} = r^2 \dot{\theta} \hat{k}$$

Swept Area



$\dot{A}$ - Rate of change of swept area



$$\dot{A} = \frac{\Delta A}{\Delta t} = \frac{\vec{r} \times (\vec{v}\Delta t)\ \tfrac{1}{2}}{\Delta t}$$

$$\vec{H} = const.$$

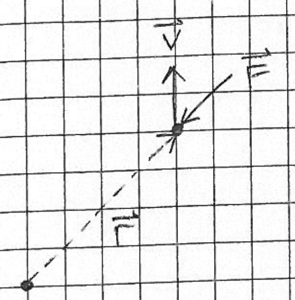
$$\updownarrow$$

$$\dot{A} \underset{\Delta t \to 0}{=} \left| \vec{r} \times \vec{v} \right|$$

Equal Areas in Equal Times

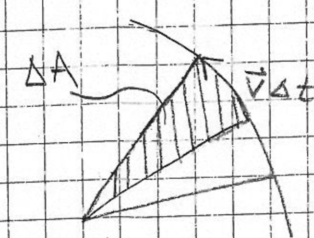
9/8/2014

OH: Mon, 11:15-2



If $\vec{F} = -F(r, \theta, t)\,\hat{e}_r$

$\vec{H}_{/c} = \text{const.} \longrightarrow \vec{r}_{/c} \times \vec{v} = \text{const.}$

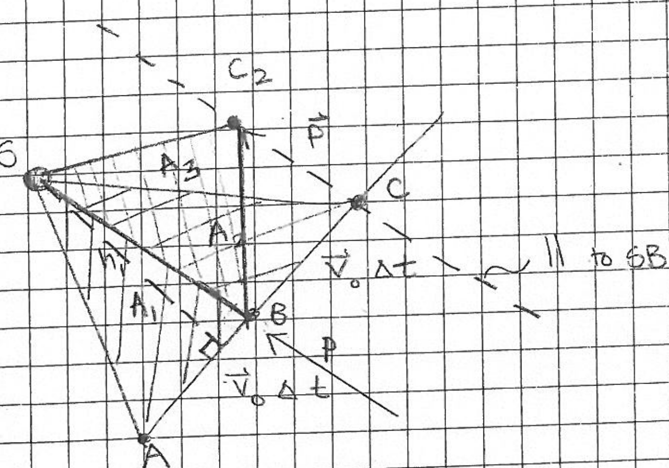


$\dot{A} = \text{const.}$

$\Delta A = \dfrac{|\vec{r} \times \vec{v}\Delta t|}{2}$

$\dot{A} = |\vec{r} \times \vec{v}|/2$

Newton's Argument



$V_o \Delta t = b_1$

min. dist $= h_1$

$A_1 = A_2 = b_1 h_1$

$b_2 = |\vec{r}_2|$

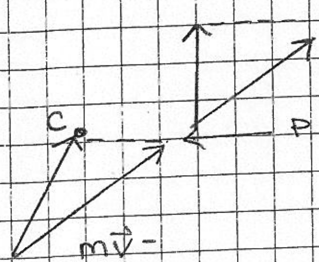
$A_2 = \dfrac{b_1 h_1}{2} = \dfrac{b_2 h_2}{2}$

$A_3 = \dfrac{b_2 h_2}{2}$

$A_1 = A_2 = A_3$

$A_1 = A_3$ Equal Areas in equal times

$m\vec{v} + \vec{p} = m\vec{v}\ +$



↑ Explanation for change in direction

Work, Power, Energy

$$\left\{ \vec{F} = m\vec{a} \right\}$$

$$\{\ \}\cdot\vec{v} \rightarrow \underbrace{\vec{F}\cdot\vec{v}}_{\boxed{P = \dot{E}_K}} = m\,\underbrace{\vec{a}\cdot\vec{V}}$$

$$= \frac{d}{dt}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right)$$

$$= \frac{1}{2}\left(\dot{\vec{v}}\cdot\vec{v} + \vec{v}\,\dot{\vec{v}}\right)$$

$$= \frac{1}{2}\cdot 2\,\vec{v}\cdot\dot{\vec{v}} = \vec{v}\cdot\vec{a}$$

$$\int \{\ \}\, dt$$

$$\int_{t_1}^{t_2} P\, dt = \int_{t_1}^{t_2} \dot{E}_K\, dt$$

$$= \Delta E_K = E_{K2} - E_{K1}$$

$$\int_{t_1}^{t_2} \vec{F}\cdot\underbrace{\vec{v}\, dt}_{d\vec{r}} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \Delta E_K$$

$$\boxed{W = \Delta E_K}$$

$\int P\, dt = W$

$\dot{W} = P$

If $\quad \vec{F} = \vec{F}(\vec{r})$ function of position alone

not $\vec{F}(\vec{r}, \vec{v}, t)$

and if $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(r) \cdot d\vec{r}$ is path independent

then, the force field is "conservative"

Equiv. Statements:

I. $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r}$ is path independent

II. $\oint \vec{F} \cdot d\vec{r} = 0$ for all paths

III. There exists a function $E_p(\vec{r})$ so that

$$\vec{F} = -\vec{\nabla} E_p$$

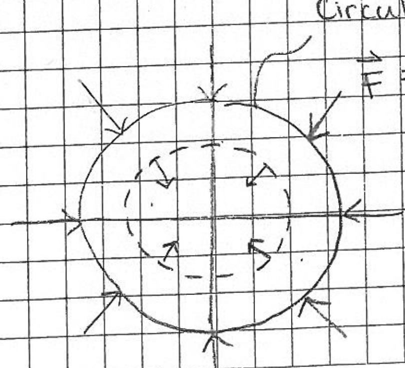
potential energy

$$E_p(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}$$

IV. $\vec{\nabla} \times \vec{F} = \vec{0}$ Everywhere

I – IV are equivalent

Ex. $\vec{F} = -k\vec{r}$



$\vec{F} = -k(x\hat{i} + y\hat{j})$

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} -k\,\vec{r} \cdot d\vec{r}$$

$$= -\int_{\vec{r}_1}^{\vec{r}_2} \frac{1}{2} k\, d(\vec{r} \cdot \vec{r}) = \left. \frac{-Kr^2}{2} \right|_{r_1}^{r_2} = -E_p$$

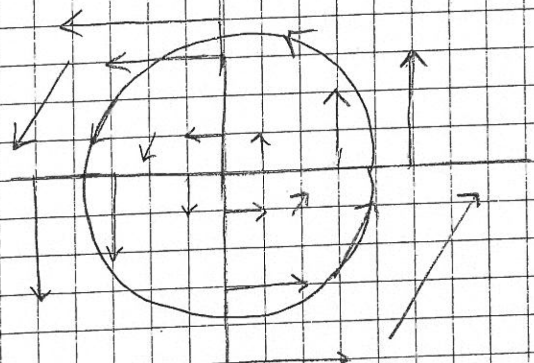
$$\boxed{E_p = \frac{1}{2} K r^2}$$

Ex.'s

$\vec{F} = \sin(\omega t)\,\hat{\imath}$ Not Cons.

$\vec{F} = \vec{B} \times \vec{V}$ Not Cons. force field (even though E is conserved)

$\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$



$\oint dW \neq 0$

$\Rightarrow$ Not Conservative

There is no fn. $E_p(x, y)$

w/ $\vec{\nabla} E_p = \vec{F}$

$\vec{\nabla} \times \vec{F} \neq 0$

MIQUELRIUS

9/10/2014

1) Particle Summary
2) Multiple Particles

Particle Mechanics Summary

I Geometry: $\dot{\vec{r}} = \vec{v}$, $\dot{\vec{v}} = \vec{a}$

II Mechanics: $\vec{F} = m\vec{a}$

III Constitutive: $\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$

I, II, III $\longrightarrow$ ODE $\longrightarrow$ solve

$\vec{r}(t)$, $\vec{F}(t)$

Facts, Properties, Thrms (about solutions of ODE)

Definitions:

- Linear Momentum $\equiv \vec{L} = m\vec{v}$

- Ang. Momentum about C $\equiv \vec{r}_{/c} \times m\vec{v}$

- Kinetic Energy $E_K = \frac{1}{2} m v^2 = \frac{1}{2} m \; \vec{v} \cdot \vec{v}$

$$E_p = - \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}$$

if sensible (conservative field)

Thrms.

- Linear Mom: $\int \{\text{II}\}\, dt$, or $\cdot\, \hat{\lambda}$

$$\int_{t_1}^{t_2} \vec{F}\, dt = \vec{L}_2 - \vec{L}_1$$

if $\vec{F} \cdot \hat{\lambda} = 0 \longrightarrow \left. \begin{array}{l} \vec{L}_2 \cdot \hat{\lambda} = \vec{L}_1 \cdot \hat{\lambda} \\ \vec{L} \cdot \hat{\lambda} = \text{const.} \end{array} \right\}$ Cons. of L.M. in $\hat{\lambda}$ dir

- Ang. Mom.

$$r_{/c} \times \{II\}, \quad \int dt$$

$$\int_{t_1}^{t_2} r_{/c} \times \vec{F} = \vec{H}_{/c}^{2} - \vec{H}_{/c}^{1}$$

if $\vec{r}_{/c} \times \vec{F} = 0$

$\vec{H}_{/c} = \text{const.}$ - Cons. of Angular Mom.

- Energy

$$\{II\} \cdot \vec{v}; \quad \int dt$$

$$P = \dot{E}_K$$

$$W = E_2 - E_1$$

$$W \leftarrow \begin{cases} \int_{t_1}^{t_2} P\,dt \\ \int_{t_1}^{t_2} \vec{F} \cdot d\vec{t} \quad \text{on the actual path} \end{cases}$$

If $\vec{F}$ is conservative

$$\vec{\nabla} \times \vec{F} = \vec{0}, \quad \oint \vec{F} \cdot d\vec{r} = \vec{0}$$

$\int \vec{F} \cdot d\vec{r}$ is path independent

$$E_{tot} = \underset{\text{"T"}}{\underset{\uparrow}{E_K}} + \underset{\text{"V"}}{\underset{\uparrow}{E_p}} = \text{const.}$$

Ex) $\vec{F} = -mg\hat{j} \rightarrow E_p = mgh$

$\vec{F} = -K\vec{r} \longrightarrow E_p = \frac{1}{2} K r^2$

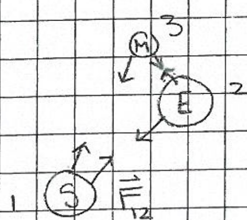
$$\vec{F} = -\frac{c}{r^2}\hat{e}_r \longrightarrow E_p = \frac{c}{r}$$

Think of force as tendency to go down hill on potential

Multiple Particles

use $\vec{F} = m\vec{a}$ for each particle

Ex.



$$\vec{F}_{12} = \frac{-G m_1 m_2}{|r_{1,2}|^3}\vec{r}_{12} = -\vec{F}_{21}$$

$\vec{F}_{12}$: on 1, from 2

$= -\vec{F}_{21}$: princ. of action + reaction

FBD (S)

To solve system, we write ODEs

Ex) w/ particles in 3D $\Longrightarrow$ $\underline{18}$ 1st order scalar ODEs

$$\dot{\vec{r}}_1 = \vec{V}_1, \quad \dot{\vec{r}}_2 = \vec{V}_2, \quad \dot{\vec{r}}_3 = \vec{V}_3$$

$$3 + 3 + 3 = 9$$

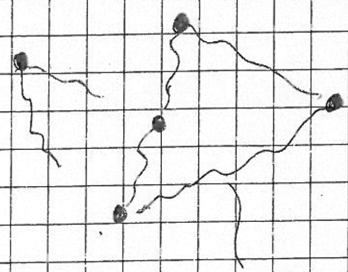
$$\vec{a}_1 = \vec{F}_1/m \quad \ldots \qquad = 9$$

$$+ \overline{\quad 18 \quad}$$

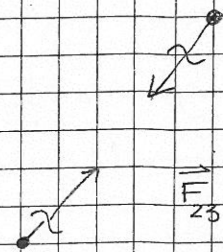
Organize in Matlab:

$$Z = \begin{bmatrix} r \\ V \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \left.\begin{matrix} \left.\begin{matrix} r_{1x} \\ r_{1y} \\ r_{1z} \end{matrix}\right\} r_1 \\ \left.\begin{matrix} r_{2x} \\ r_{2y} \\ r_{2z} \end{matrix}\right\} r_2 \end{matrix}\right\} r \quad \left.\begin{matrix} V_{1x} \\ V_{1y} \end{matrix}\right\} V$$

Ex. Spring Lattice model of solid



$T = K(l - l_0)$



$$\vec{F}_{23} = \frac{K\left(|\vec{F}_{23}\cdot| - l_{23_0}\right)}{|\vec{F}_{23}|}\,\vec{F}_{23}$$

9/12/2014

Axioms of Classical Mechanics

* Space is flat + classical geometry applies $\rightarrow$ x, y, z, $d = \sqrt{x^2+y^2+z^2}$

* Time goes forward (pt. can be localized in space + time)

* Mass is identifiable + non-transient $\approx$ # protons + neutrons
  $\rightarrow$ can identify a (closed) system

* Force + Moment are the measures of interaction

Key Thing is Consistency:

same mass in $F = ma$ + $F = \frac{GMm}{r^2}$

same force in $F = ma$ + $F = \frac{GMm}{r^2}$ + $F = k\Delta l$

0. No FBD $\Longrightarrow$ No Mechanics

Chap. 3.3

I. LMB (linear momentum balance)

$$\sum_{\text{external}} \vec{F} = \sum m_i \vec{a}_i = \dot{\vec{L}}$$

$$\int \vec{a}\, dm$$

$$\vec{L} = \sum m_i \vec{v}_i$$

$v_{I/F}$

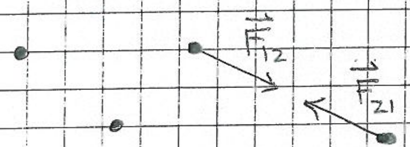
$$\int \vec{v}\, dm$$

III. Internal Forces have no net force + no net moment about any pt.

[For dividing a system in 2 pieces equiv. to princ. of action + reaction]

"Standard" (bad) approach is to assume pairwise equal + opposite forces

Used in (bad) derivations of angular momentum balance.



$\vec{F}_{21} = -\vec{F}_{21}$

What's wrong w/ piecewise equal + opposite assumption?

1) It assumes physics at a scale that we don't know about

2) Bad Physics - Not accurate at microscale (schrödinger Eq. etc.)

3) Makes bad macroscopic predictions

Pairwise theory predicts (incorrectly) that all materials have poisson ratio: $\nu = 1/4$

Various Alternatives to pairwise assumption

1) For any system, $\sum \vec{M}_{/c} = \begin{cases} \sum \vec{r}_{i/c} \times m_i \vec{a}_{i/F} \\ \int \vec{r}_{/c} \times \vec{a}_{/f} \end{cases}$

2) Assume internal forces have no net moment or forces

3) Net internal force + moment do no work in rigid virtual motions

Internal Forces do no work

For every FBD

LMB: $\sum F = \begin{cases} \sum m_i \vec{a}_i = \dot{\vec{L}} \\ \int \vec{a}\, dm \end{cases}$

$\vec{a}_{/F}$

AMB: $\sum \vec{M}_{/C} = \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{a}_i \\ \int \vec{r}_{/C} \times \vec{a}_{/F}\, dm \end{cases}$

C is any pt.

What about $\vec{H}_{/C}$ such that $\frac{d}{dt}\vec{H}_{/C} = \dot{\vec{H}} = \sum \vec{r}_{i/C} \times m_i \vec{a}_{i/F}$

ang. mom.

Define C' to be pt. fixed in F inst. cons. w/ C

Candidate defs of (4 Defs)

$\vec{H}_{/C} = \sum \vec{r}_{i/} \times m_i \vec{v}_{i/}$

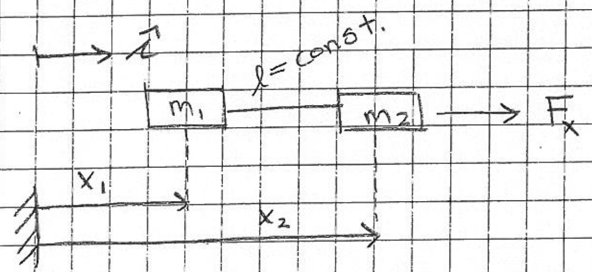
- Falling Cat
- DAEs

The Falling Cat Problem

universal mechanics subconscious vs. above the ears

9/19/2014

Ex. 1D, 2 masses m_1, m_2



"$\ell = $ const." Kinematic / geometric constraint

FBD

LMB$_1$:

$\Sigma F = ma$

$T = m_1 \ddot{x}_1$ ①

LMB$_2$:

$\Sigma F = ma$

$F_A - T = m_2 \ddot{x}_2$ ②

$\ell =$ const. $x_2 - x_1 = \ell_0$ ③

Note: can't solve $\vec{F} = m\vec{a}$ eqs.

①, ②, ③ are a set of

DAE = differential algebraic equations

Why the problem? l = const. is an extreme constitutive law

How to solve?

A simple way: $\frac{d^2}{dt^2}\left\{ x_2 - x_1 = l_0 \right\}$

$$\Rightarrow \ddot{x}_2 - \ddot{x}_1 = 0 \qquad ③'$$

①, ②, ③'

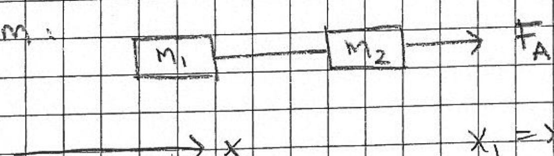
$$\left.\begin{aligned} m_1 \ddot{x}_1 + 0\,\ddot{x}_2 - 1\cdot T &= 0 \\ 0\cdot \ddot{x}_1 + m_2 \ddot{x}_2 + +T &= F_A \\ 1\cdot \ddot{x}_1 - 1\cdot \ddot{x}_2 + 0\cdot T &= 0 \end{aligned}\right\} \text{linear Algebraic Eqs.}$$

$$\underbrace{\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & +1 \\ -1 & +1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} 0 \\ F_A \\ 0 \end{bmatrix}}_{b}$$

$z = A \backslash b \leftarrow$ Matlab

Alternative Approach:

FBD of system:



$x_1 = x$

$x_2 = x + l_0$

LMB: $\Sigma F = m a_G$

$$F_A = (m_1 + m_2)\ddot{x}$$

FBD_1 + LMB_1 $\Longrightarrow$ $T = \dfrac{m_1 F_A}{m_1 + m_2}$

How to deal w/ constrained systems

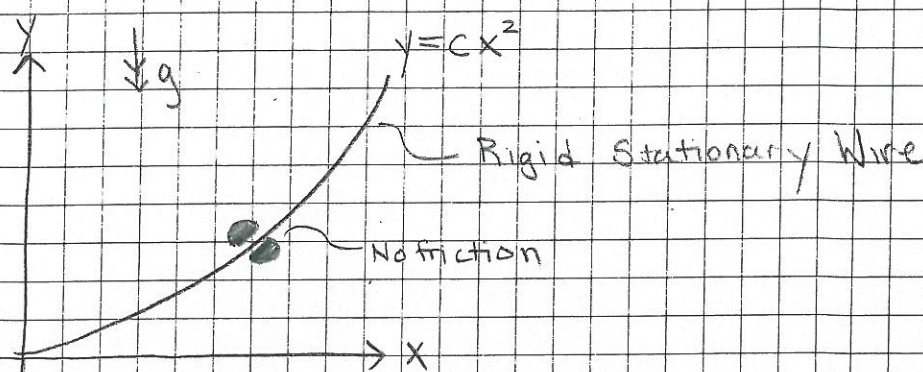
I. $\Sigma \vec{F} = m\vec{a}$ for all parts

$\dfrac{d^2}{dt^2}\{\text{Const. Eqs.}\} \Longrightarrow$ DAEs solve Lin. Eqs. at every time step

II. Pick minimal coordinates. Find Fancy Eqs. of motion. (LMB, AMB, Lag. Eqs, ...)

Solve minimal set of ODEs

Ex)



How does the bead move?

I. Cons. of Energy

minimal coordinate: x

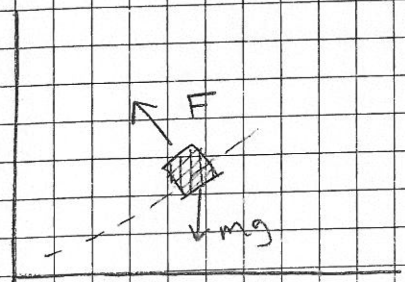
$$E_p = mgh = mgy = mgcx^2$$

$$E_k = \tfrac{1}{2}m\,\dot{x}^2 + \dot{y}^2 = \tfrac{1}{2}m(\dot{x}^2 + (2cx\dot{x})^2)$$

$$\frac{d}{dt}\left\{ E_p + E_k = \text{const.} \right\}$$

$$\ddot{x} = \text{mess}$$

II. FBD



$$\vec{F} = F\hat{n}$$

calc. from geometry + x

LMB: $\Sigma F = ma$

$$F\hat{n} - mg\hat{j} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})$$

Write in terms of $x, \dot{x}, \ddot{x}$

Dot w/ something

$$\Longrightarrow \ddot{x} =$$

$\hat{\lambda} \cdot \{ \quad \}$ kills F!

$$\hat{\lambda} = \frac{\vec{v}}{|\vec{v}|} = \frac{\dot{x}\hat{i} + \dot{y}\hat{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$\dot{y} = 2cx\dot{x}$$

III DAE Approach - due 12 days

9/22/2014

Numerical Optimization or Root Finding

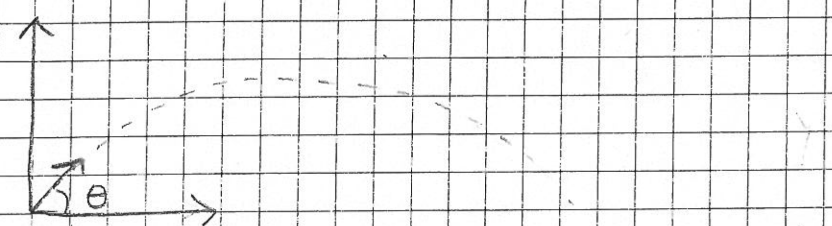
① Why? How? (projectile ex)

② What are the tools?

- FMIN Search
- FMINCON
- CMAES

③ More Complicated

- Periodic Motion
- ways it goes bad



$\vec{V}_o = V_o \cos\theta \, \hat{i} + V_o \sin\theta \, \hat{j}$

FBD

$F = -mg\hat{j}$

LMB: $\ddot{\vec{r}} = -g\hat{j}$

$\dot{\vec{r}} = -gt\hat{j} + \vec{V}_o$

$\vec{r} = -\frac{1}{2}g t^2 \hat{j} + \vec{V}_o t$

$t_{col} = \frac{2V_o^2}{g}\sin\theta$

$x = V_o \cos\theta \, t \longrightarrow x_{col} = \frac{2V_o^2}{g}\sin\theta\cos\theta$

$y = V_o \sin\theta \, t - \frac{1}{2}g t^2$

$$\frac{dX_{col}}{dt} = 0 = \frac{2V_0}{g}\left(\underbrace{-\sin^2\theta + \cos^2\theta}_{0}\right)$$

$$\theta = 45^\circ$$

With Drag

$$F_D = c\,V^2\left(\frac{-\vec{V}}{|\vec{V}|}\right)$$

$$-mg\hat{j}$$

$$\ddot{\vec{r}} = -g\hat{j} + \frac{c}{m}|\vec{V}|^{\cancel{2}}\,\frac{-\vec{V}}{\cancel{|V|}}$$

$$= -g\hat{j} + \frac{-c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,(\dot{x}\hat{i} + \dot{y}\hat{j})$$

Diff Eq. → ODE45 → Solution over time

Initial Cond.

Cost Function → FMIM-search → Minimum of Function

Guess

9/25/2014

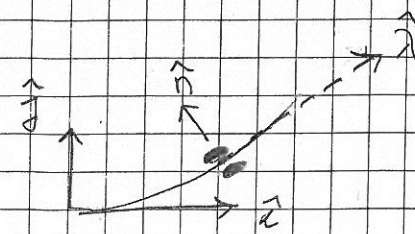
DAEs - differential algebraic Equations

Geometry: $\hat{\lambda} = \dfrac{\vec{v}}{|\vec{v}|}$; $\hat{n} = \hat{k} \times \hat{\lambda}$

Minimal Coord: x

How to find the motion?

EOM?



1) $\dot{E}_{tot} = 0 \Longrightarrow \ddot{x} = \ldots$

2) $\left\{ \vec{F} = m\vec{a} \right\}$

$-mg\hat{j} + N\hat{n} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{j})$

$\hat{n} = \hat{k} \times \hat{\lambda}$, $\quad \ddot{y} = \dfrac{d^2}{dt^2}(cx^2)$

$\hat{\lambda} = \dfrac{\vec{v}}{|\vec{v}|}$, $\quad \vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{j}$

$\dot{y} = \dfrac{d}{dt}(cx^2)$

$\{\ \} \cdot \hat{\lambda} \Longrightarrow \ddot{x} = \ldots$

3) DAE - do not eliminate y using $y = cx^2$

$\vec{F} = m\vec{a}$

$-mg\hat{j} + N\hat{n} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{j})$

$\hat{n} = \dfrac{\vec{\nabla}(y - cx^2)}{|\vec{\nabla}(y - cx^2)|} = \dfrac{\hat{j} - 2cx\hat{\imath}}{\sqrt{1 + 4c^2x^2}}$

$$\frac{d^2}{dt^2}\left(y - cx^2\right) = 0$$

$$\ddot{y} - 2c\dot{x}^2 - 2cx\ddot{x} = 0$$

$\Longrightarrow$ 3 Equations

$\vec{F} = m\vec{a}$ + Constraint Eq.

$$m\ddot{x} + 0\cdot\ddot{y} + -N n_x = 0$$

$$0\cdot\ddot{x} + m\ddot{y} + -n_y N = -mg$$

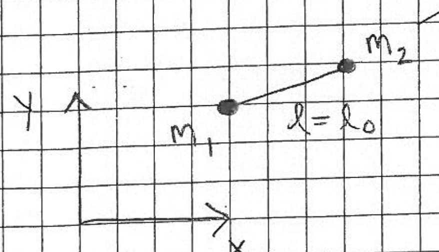
$$-2cx\ddot{x} + 1\cdot\ddot{y} + 0\cdot N = 2c\dot{x}^2$$

$$\underbrace{\begin{bmatrix} m & 0 & -n_x \\ 0 & m & -n_y \\ -2cx & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ N \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} 0 \\ -mg \\ 2c\dot{x}^2 \end{bmatrix}}_{b}$$

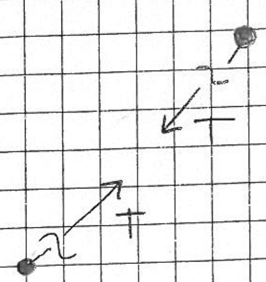
$$\left.\begin{aligned} w &= A \backslash b \\ \ddot{x} &= w(1) \\ \ddot{y} &= w(2) \end{aligned}\right\} \text{MATLAB}$$

Ex) Dumbel

$$\hat{\lambda} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$$



FBD



LMB_1 : $T\hat{\lambda} = m_1 \vec{a}_1$

LMB_2 : $-T\hat{\lambda} = m_2 \vec{a}_2$

$\Big\}$ 4 eqs. for $\ddot{x}_1, \ddot{x}_2, \ddot{y}_1, \ddot{y}_2$

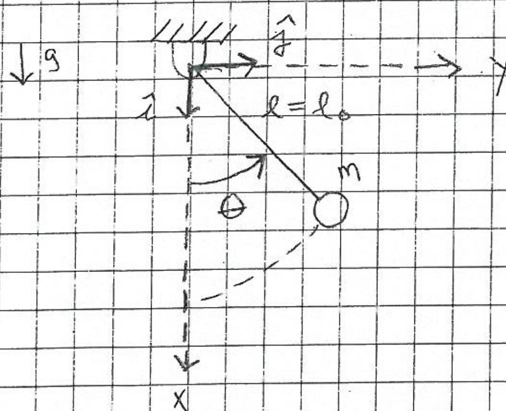
$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = \text{const.}$$

$\frac{d^2}{dt^2}(\ell^2) = 0 \Longrightarrow$ Another eq. w/ $\ddot{x}_1, \ddot{x}_2, \ddot{y}_1, \ddot{y}_2$

5 Algebraic Equations in 5 unknowns.

Can solve for $\ddot{x}_1, \ddot{x}_2, \ddot{y}_1, \ddot{y}_2$

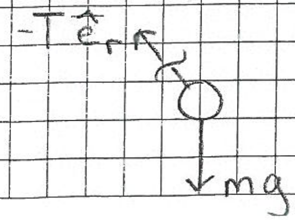
Pendulum:



Minimal Coordinates: θ, y or x

θ: best; y: good for $x > 0$; x: $y > 0$

Equations of Motion



1) $\left\{ \vec{F} = m\vec{a} \right\} \cdot \hat{e}_\theta$

a) using θ

b) using y

A bunch of ways to do this

2) $\frac{d}{dt}\left(E_{tot} \right) = 0 \Longrightarrow \ddot{\theta} = \dots$

OR $\ddot{y} = \dots$

3) $AMB/_0 \Longrightarrow \ddot{\theta} = \dots$

OR $\ddot{y} = \dots$

$$\Sigma \vec{M}/_0 = \dot{\vec{H}}/_c$$

$$\vec{r}/_0 \times \left(-T\hat{e}_r + mg\hat{\imath} \right) = \vec{r}/_0 \times m\vec{a}$$

polar or cartesian

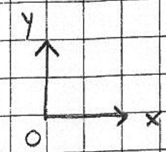
4) DAE

$$\vec{F} = m\vec{a}$$

$$\left.\begin{aligned} -T\hat{e}_r + mg\hat{\imath} &= m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \\ \frac{d^2}{dt^2}\left(x^2 + y^2\right) &= 0 \end{aligned}\right\} \text{3 Eqs. for } \ddot{x}, \ddot{y}, T$$

9/26/2014

Konig (HW)



WRONG

$$KE = \frac{1}{2}\sum m_i V_i^2 = \cancel{\frac{1}{2}\sum m_i (V_{i/G} + V_G)^2}$$

Also Wrong: $(\vec{V}_{i/G} + \vec{V}_G) = \vec{V}_{i/G}^2 + 2\vec{V}_{i/G}\vec{V}_G + \vec{V}_G^2$

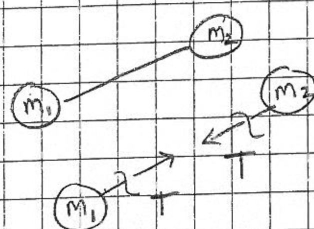
$$|\qquad| = |V_{i/G}|^2 + 2\vec{V}_{i/G}\cdot\vec{V}_G + |V_G|^2 \quad \checkmark$$

$$\vec{H} = \sum \vec{r}_{i/c} \times \vec{V}_{i/c'}\, m$$

c' = stationary but inst. coincides w/ c

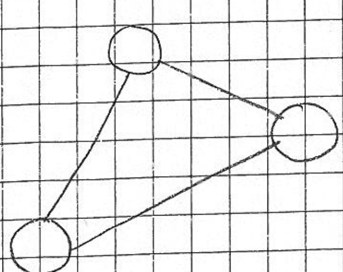
Rigid Objects

A) Rigid object = set of const. particles



$\{ l = \text{const.} \} \leftarrow$ Contrained Eq.

$$\frac{d^2}{dt^2}\{\quad\}$$



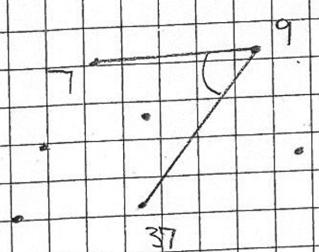
3 times over, 3 constrained Eqs

$\Longrightarrow$ 9 ODE's

$n = 10^{23}$ particles, lots of constrained eqs, at least $2n-3$ const. eq.

B) AMB rigid object approach

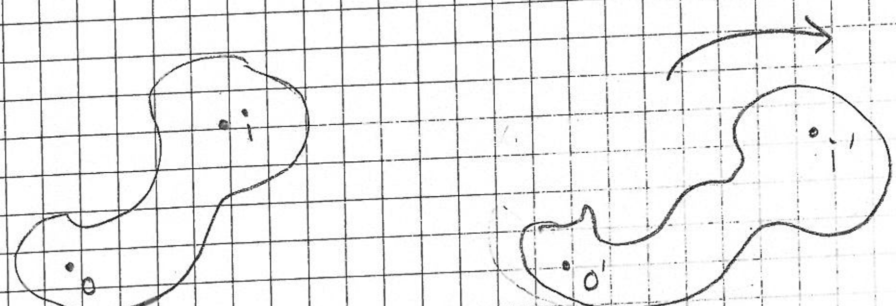
- distances between all pairs of points = const.
- Angles between all triples = const.



$\ell_{79} = \text{const.}$

$\theta_{7,9,37} = \text{const.}$

In motion, all line segments rotate the same amount (RP text)



Ref. Conf. $t = 0$

$$\vec{r}_{i'/o} = r_{o'/o} + (\vec{r}_{i/o} \text{ Rotated})$$

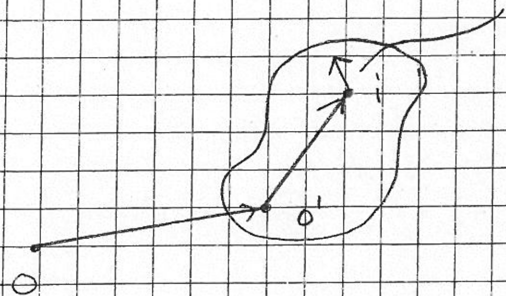
$$\begin{bmatrix} x_{i'/o} \\ y_{i'/o} \end{bmatrix} = \begin{bmatrix} x_{o'/o} \\ y_{o'/o} \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_{i/o} \\ y_{i/o} \end{bmatrix} \quad *$$

Initiated pos. of points $\vec{r}_{i/o}$ + disp. of ref. point: $\vec{r}_{o'/o}$

+ rotation: θ

→ new pos. of pts.: $\vec{r}_{i'/o}$

→ Can calc. $\vec{v}_{i/o}$ and $\vec{a}_{i/o}$



$$V_{i/O'} = \dot{\theta}\hat{k} \times r_{i/O'}\,\hat{e}_r$$
$$= \omega \times r$$
$$\dot{\theta}\hat{k}$$

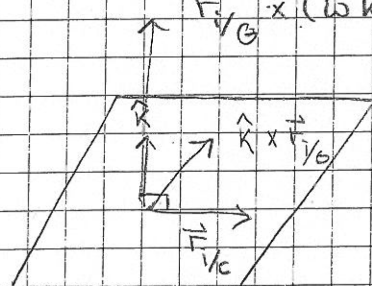
$$\dot{\vec{H}}_{/c} = \frac{d}{dt}\left(\vec{H}_{/c}\right) = \vec{r}_{G/c} \times m_{tot}\,\vec{a}_G + \frac{d}{dt}\Big(\underbrace{\sum \vec{r}_{i/G} \times V_{i/G}\, m}_{\vec{H}_{/G}}\Big)$$

$$\vec{H}_{/G} = \sum \vec{r}_{i/G} \times m_i \vec{v}_{i/G}$$

$$\omega\hat{k}, \quad \omega = \dot{\theta}\hat{k}$$

$$= \sum \vec{r}_{i/G} \times \left(\omega\hat{k} \times \vec{r}_{i/G}\right) m_i$$

$$\vec{r}_{i/G} \times (\omega\hat{k} \times \vec{r}_{i/G})\, m_i = \omega \left|\vec{r}_{i/c}\right|^2 m_i \hat{k}$$



$$\vec{H}_{/G} = \sum m_i \vec{r}_{i/G}^{\,2}\, \omega\hat{k} = \omega I^G \hat{k}$$

$$\left|\vec{r}_{i/G}\right|^2$$

$$I^G = \begin{cases} \sum m_i\, r_{i/G}^2 \\ \int r_{/G}^2\, dm \end{cases}$$

$$\dot{\vec{H}}_{/G} = \frac{d}{dt} \vec{H}_{/G} = I^G \ddot{\theta} \hat{k}$$

$\Longrightarrow$ AMB$_{/G}$

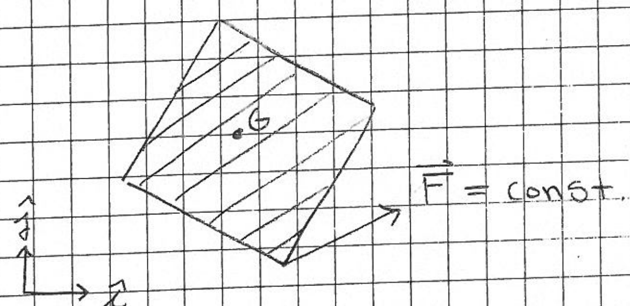
$$\boxed{\sum \vec{M}_G = I^G \ddot{\theta} \hat{k}}$$

Rigid Object Mechanics

$$\sum \vec{F}^{ext} = m_{tot} \vec{a}_G$$

$$\sum \vec{M}_{/G} = I^G \ddot{\theta} \hat{k}$$

also $\sum \vec{M}_{/c} = \vec{r}_{G/c} \times m_{tot} \vec{a}_G + I^G \alpha \hat{k}$



9/29/2014

Review

1) One particle $(n=1)$ $\vec{F} = m\vec{a}$

2) n particles: $\vec{F}_i = m_i \vec{a}_i$ + constit. laws

3) n particles $\vec{F}_i = m_i \vec{a}_i$ + constraint eqs.

4) n (even huge n) General Thrms.

e.g. $\vec{F}_{tot} = m_{tot}\, a_G$

$$E_{tot} = E_{KG} + E_{K/G}$$

$E_{K/G} \leftarrow \frac{1}{2}\sum v_{i/G}^2\, m_i$

$E_{KG} \leftarrow \frac{1}{2} m_{tot} V_G^2$

$$\vec{H}_{/C} = \vec{H}_{G/C} + \vec{H}_{/G}$$

For $\vec{H}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{v}_{i/c}$

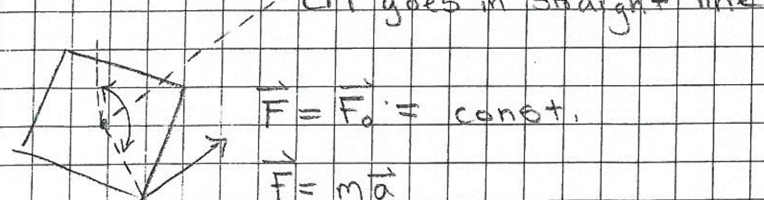
$$\vec{H}_{/c} = \vec{r}_G \times m_{tot} \vec{V}_{G/c} + \vec{H}_{/G}$$

and fixed for C $\dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{a}_{i/F} = \frac{d}{dt}\vec{H}$

For rigid object

$$\vec{H}_{/G} = I^G \omega \hat{k}$$

$$\boxed{\vec{M}_{/c} = \dot{\vec{H}} = \vec{r}_{G/c} \times m_{tot}\, \vec{a}_{G/F} + I^G \dot{\omega} \hat{k}}$$



Oscillations?

$\vec{F} = \vec{F}_0 = $ const.

$\vec{F} = m\vec{a}$

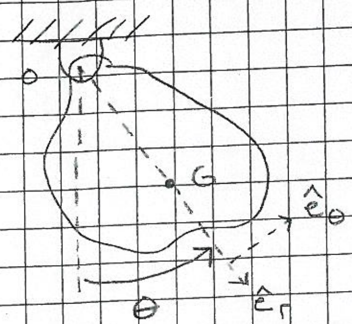
$$\vec{v} = \int \frac{\vec{F}}{m}\, dt = \frac{\vec{F}}{m} t$$

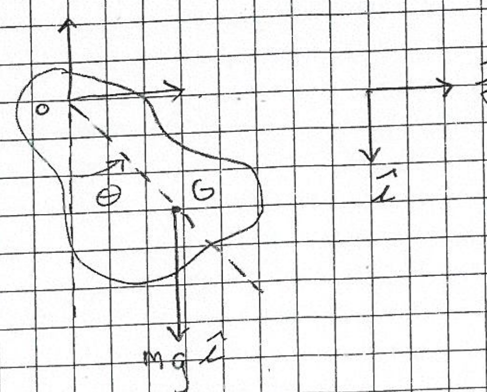
$$\vec{r}_{t/o} = \frac{\vec{F}}{m}\frac{t^2}{2}$$

Ex) Pendulum (Rigid Object)

m, I^G

FBD





AMB/o

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{r}_{G/O} \times (mg\,\hat{\imath}) = \vec{r}_{G/O} \times m_{tot}\,\vec{a}_G + I^G \omega \hat{k}$$

$$-d\sin\theta\, mg\,\hat{k} = d\hat{e}_r \times m_{tot}\left(-d\dot{\theta}^2\,\hat{e}_r + d\ddot{\theta}\,\hat{e}_\theta\right) + I^G \omega \hat{k}$$

$$-d\sin\theta\, mg\,\hat{k} = d\, m_{tot}\, d\ddot{\theta}\,\hat{k} + I^G \omega \hat{k}$$

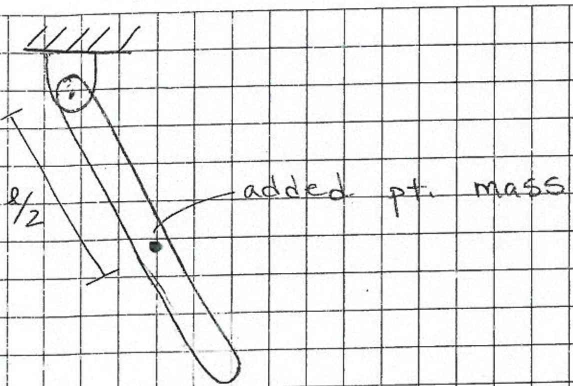
$$\left\{-d\sin\theta\, mg\,\hat{k} = d^2 m_{tot}\,\ddot{\theta}\,\hat{k} + I^G \omega \hat{k}\right\}$$

$$\{\}\cdot\hat{k} \rightarrow -mgd\sin\theta = (I^G + md^2)\,\ddot{\theta}$$

$$\ddot{\theta} = \frac{-mgd\sin\theta}{I^G + md^2}$$

I^O is good short cut for this problem + approximately no others

$$I^O = I^G + md^2$$



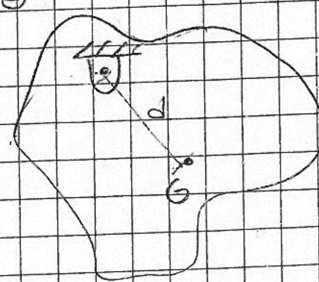
How does the frequency of oscillation change?

Faster as ω goes to 0

10/1/2014

Last Class:

$$\ddot{\theta} = \underbrace{\frac{mgd}{I^G + md^2}}_{-\omega^2} \sin\theta \Rightarrow \ddot{\theta} = -\omega^2 \theta$$



A

$m_A = m_r$

$$I^G = \frac{m(\ell^2 + w^2)}{12}$$

I^G

$$\ddot{\theta} = -\omega^2 \sin\theta$$

$$\ddot{\theta} = -\omega^2 \theta$$

B

$m_B = m_r + m_b$ ← Pt. Mass

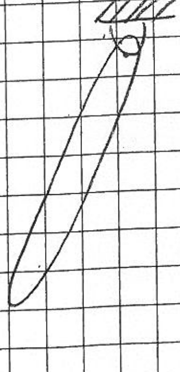
$I^G = I^G$

$$\omega_A^2 = \frac{m_r g (\ell/2)}{I^G + m_r (\ell/2)^2}$$

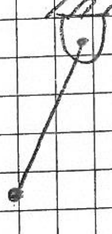
$$\omega_B^2 = \frac{(m_r + m_b)\, g\, (\ell/2)}{I^G + (m_r + m_b)(\ell/2)^2}$$

$$\frac{-mgd}{I^G + md^2} \sin\theta = -\omega^2 \theta$$

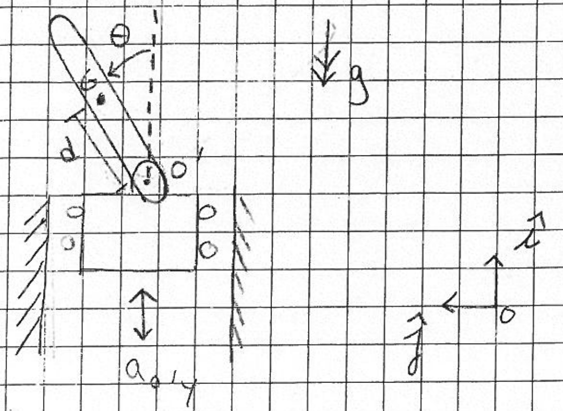
$$\boxed{\omega_A < \omega_B}$$

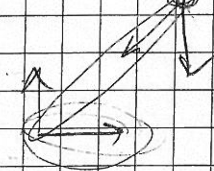


vs.

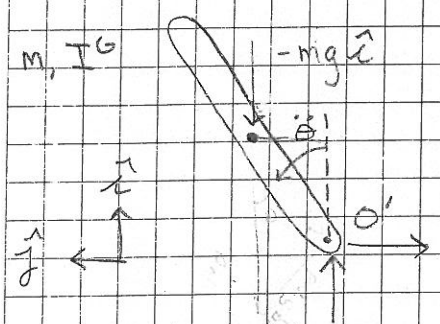


Shaking Inverted Pendulum





FBD



AMB/O'

$$\sum \vec{M}_{/O'} = \dot{\vec{H}}_{/O'}$$

$$mgd\sin\theta\,\hat{k} = \vec{r}_{G/O'} \times m\,\vec{a}_{G/F} + I^G\,\ddot{\theta}\,\hat{k}$$

$$\vec{r}_{G/O'} = d\hat{e}_r$$

Is O same as F ?

$$\vec{a}_{G/F} = \vec{a}_{O'/F} + \vec{a}_{G/O'}$$

$$= \vec{a}_{O'y}\,\hat{\imath} + (\underbrace{\ddot{\theta}\hat{k} \times d\hat{e}_r}_{\ddot{\theta} d\hat{e}_\theta} - \dot{\theta}^2 d\,\hat{e}_r)$$

10/3/2014

Continuation of last class:

θ = the 1 minimal coordinate

$$mgd\sin\theta\,\hat{k} = \vec{r}_{G/O'} \times m_{tot}\,\vec{a}_{G/F} + I^G\ddot{\theta}\,\hat{k} \quad (1)$$

$$a_{G/F} = a\,\hat{\imath} + -d\dot{\theta}^2\hat{e}_r + \ddot{\theta}\,d\,\underset{\uparrow\ \hat{k}\times\hat{e}_r}{\hat{e}_\theta}$$

$$\left\{mgd\sin\theta\,\hat{k} = m\left(-da\sin\theta\,\hat{k} + \vec{0} + \ddot{\theta}d^2\hat{k}\right) + I^G\ddot{\theta}\,\hat{k}\right\}$$

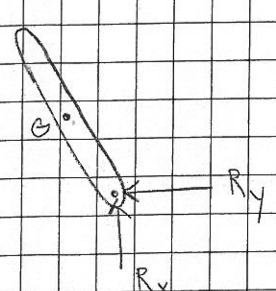
$$\{\ \}\cdot\hat{k} \Rightarrow mgd\sin\theta = -mda\sin\theta + (md^2 + I^G)\ddot{\theta}$$

$$\dot{v}_\theta = \ddot{\theta} = \frac{m(g+a)d}{md^2 + I^G}\sin\theta$$

Can Solve (on computer) $\leftarrow$ $a = \frac{d^2}{dt^2}\left(h\sin(\omega_0 t)\right)$

f right vibes (fast) $\rightarrow$ stable upright

What if we wanted to know R_x, R_y?



Linear Momentum Balance

$$\Sigma F = ma$$

$$R_x\,\hat{\imath} + R_y\,\hat{\jmath} - mg\,\hat{\imath} = m\vec{a}$$

$\uparrow$ found it already

$$\theta, \dot{\theta}, \ddot{\theta} \rightarrow \vec{a}(t)$$

Same Problem using DAE approach

Maximal coord: θ, x_G, y_G

Reaction forces: R_x, R_y } 5 things

Don't know: $\ddot{\theta}, \ddot{x}_G, \ddot{y}_G$

Need $\underbrace{\boxed{\text{5 scalar Equations}}}_{\text{DAE}}$

AMB 1 $\longrightarrow \sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$

LMB 2 $\longrightarrow \sum F = m\vec{a}$

$\vec{a}_{0'} = \vec{a}_{0'}$ 2 $\longrightarrow a(t)\,\hat{\imath} = \vec{a}_G + \vec{a}_{0'/G}$

$+ \frac{}{5}$

$$= \ddot{x}_G\,\hat{\imath} + y_G\,\hat{k} + \ddot{\theta}\,\hat{k} \times \vec{r}_{0'/G} + -\dot{\theta}^2\,\vec{r}_{0'/G}$$

$\downarrow$

2 eqs.

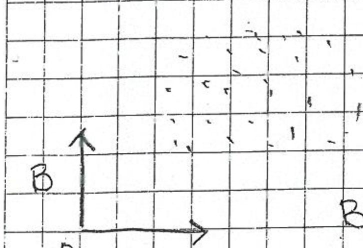
$$\begin{bmatrix} & & \\ & 3\times 3 & \\ & & \end{bmatrix} \begin{bmatrix} \ddot{x}_G \\ \ddot{y}_G \\ \ddot{\theta} \\ R_x \\ R_y \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

forces, v^2 terms

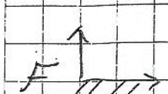
Equivalence of $g + a$

gravity

acceleration of reference frame



B accelerates, but does not rotate



LMB: $\sum \vec{F} = \dot{\vec{L}} = \sum m_i \vec{a}_i$

$$\sum \vec{F}^{appl.} + \sum -m_i g \hat{j} = \sum m_i (\vec{a}_{o'} + \vec{a}_{i/\beta})$$

appl. = all but gravity

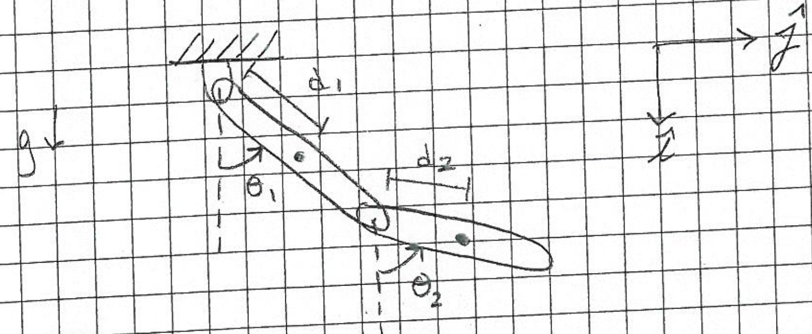
$$\sum \vec{F}^{appl.} = \sum m_i \left[(\vec{a}_o + g\hat{z} + \vec{a}_{i/\beta}) \right]$$

Note: Eq. can't distinguish $\vec{a}_{i/o'}$ from $g\hat{j}$

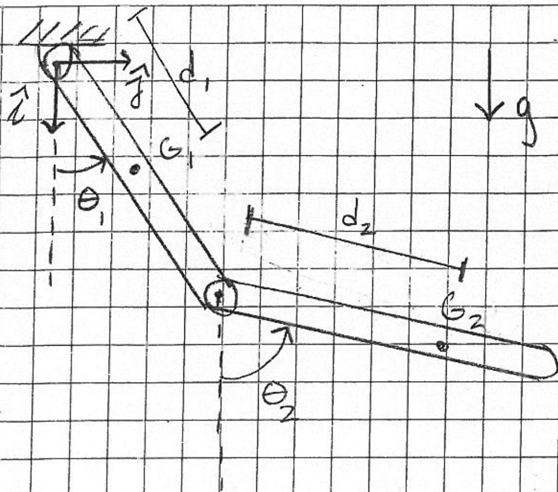
Similar calc. $\rightarrow$ same for AMB$/c$

$\Longrightarrow$ No mechanics eqn. can distinguish g from being in an accel. ref. frame

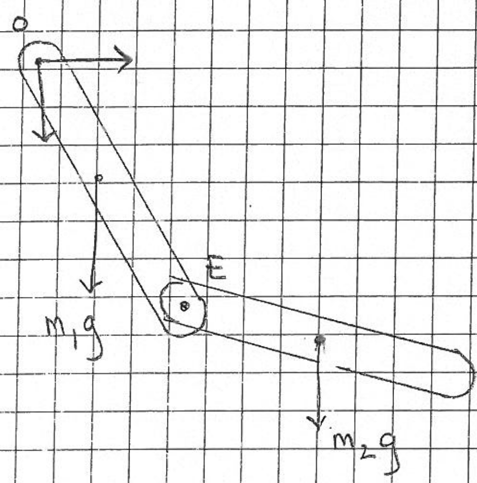
Double Pendulum



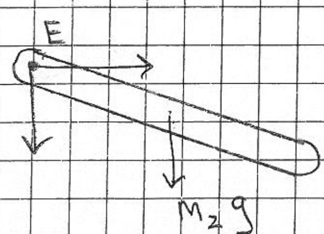
10/6/2014



FBD



Lower Bar: FBD



AMB/E of lower Bar

(1) $\Sigma M_{/E} = \dot{\vec{H}}_{/E}$

$\vec{r}_{G2/E} \times m_2 g \vec{i} = \vec{r}_{G2/E} \times m_2 \vec{a}_{G2/F} + I^{G2} \ddot{\theta}_2 \hat{k}$

$r_{G2/E} = (\cos\theta_2 \vec{i} + \sin\theta_2 \vec{j}) d_2$

$$\vec{a}_{G2/F} = \underbrace{\vec{a}_{E/F}}_{-\dot{\theta}_1^2 \vec{r}_{E/O} + \ddot{\theta}_1 \hat{k} \times \vec{r}_{E/O}} + \underbrace{\vec{a}_{G2/E}}_{-\dot{\theta}_2^2 \vec{r}_{G2/E} + \ddot{\theta}_2 \hat{k} \times \vec{r}_{G2/E}}$$

(2) $\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$

$$\vec{r}_{G1/O} \times m_1 g \hat{i} + \underbrace{\vec{r}_{G2/O}}_{\vec{r}_{E/O} + \vec{r}_{G2/E}} \times m_2 g \hat{i} = \left(\vec{r}_{G1/O} \times m_1 \vec{a}_{G1} + I^{G1} \ddot{\theta}_1 \hat{k}\right) + \left(\vec{r}_{G2/O} \times m_2 \vec{a}_{G2} + I^{G2} \ddot{\theta}_2 \hat{k}\right)$$

$$\left.\begin{array}{l} \{(1)\} \cdot \hat{k} \\ \{(2)\} \cdot \hat{k} \end{array}\right\}$$ 2 Equations in terms of $g, d_1, d_2, l, m_1, m_2, I^{G1}, I^{G2}, \theta_1, \dot{\theta}_1, \ddot{\theta}_1, \theta_2, \dot{\theta}_2, \ddot{\theta}_2$

Solve for $\ddot{\theta}_1$ and $\ddot{\theta}_2$

Note: eqs. are linear in $\ddot{\theta}_1$ and $\ddot{\theta}_2$
(always works out like this)

$$\left.\begin{array}{ll} \frac{d}{dt}\theta_1 = \dot{\theta}_1 & \frac{d}{dt}\dot{\theta}_1 = \ddot{\theta}_1 \\ \frac{d}{dt}\theta_2 = \dot{\theta}_2 & \frac{d}{dt}\dot{\theta}_2 = \ddot{\theta}_2 \end{array}\right\}$$ 4 1st Order ODEs

10/8/2014

Minimal Coords: θ_1, θ_2

$$\left\{ \text{AMB}_{/o} \text{ system} \right\} \cdot \hat{K} \longrightarrow f_1(\theta_1, \theta_2, \underset{\omega_1}{\dot{\theta}_1}, \dot{\theta}_2, \underset{\alpha_1}{\ddot{\theta}_1}, \ddot{\theta}_2, \text{param}) = \tau$$

$$\left\{ \text{AMB}_{/E} \text{ forearm} \right\} \cdot \hat{K} \longrightarrow f_2(\quad " \qquad " \qquad " \quad) = 0$$

Solve$(f_1, f_2, \ddot{\theta}_1, \ddot{\theta}_2)$

$\dot{\omega}_1 = W_1$

$\dot{\omega}_2 = W_2$

$\dot{\theta}_1 = \omega_1$

$\dot{\theta}_2 = \omega_2$

Fact

define $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

f is allways writable in this form

$$\underset{2\times 2}{\begin{bmatrix} & M & \end{bmatrix}} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{b}$$

M has in it expr. w/ θ_1, θ_2, and parameters

How to use:

$$\underset{2\times 2}{M} = \text{Jacobian}(f, \underset{\alpha_1}{\ddot{\theta}_1}, \underset{\alpha_2}{\ddot{\theta}_2})$$

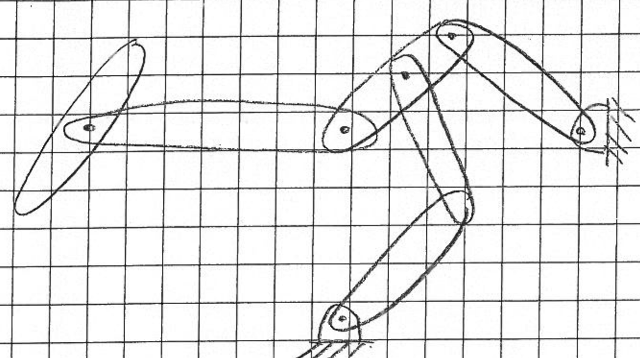
$$M_{ij} = \frac{\partial f}{\partial \ddot{\theta}_1}, \quad M = \begin{bmatrix} \partial f_1/\partial \ddot{\theta}_1 & \partial f_1/\partial \ddot{\theta}_2 \\ \partial f_1/\partial \ddot{\theta}_1 & \partial f_2/\partial \ddot{\theta}_2 \end{bmatrix}$$

$$W = M \backslash \underbrace{\overbrace{\left(-f + M\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}\right)}^{\text{symbolic}}}_{\text{Do numerically}}$$

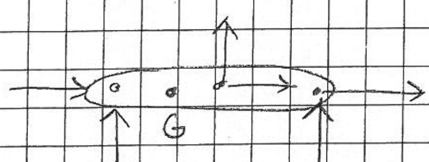
$$W = \begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\end{bmatrix}$$

DAE Approach

Complicated mechanism



FBD of each object



For each object:

$$\left.\begin{aligned} &\text{LMB: } \sum F = m_7 \vec{a}_7 \quad (2 \text{ eqs.}) \\ &\text{AMB}_{/G}\ \sum \vec{M}_{/G7} = \dot{\vec{H}}_{/G7} = I^7 \alpha_7 \hat{k} \end{aligned}\right\} 3 \text{ eqs.}$$

For each pin:

$$\vec{a}_{P19} = \vec{a}_{P19}$$

as figured by other body — as figured by one body

How many equations do we have?

$3N_b + 2N_p$

$\uparrow$ number of bodies

How many unknowns: $\underset{\ddot{x},\ \ddot{y},\ \ddot{\theta}}{3N_b} + \underset{F_x,\ F_y \text{ for each}}{2N_p}$

Collect eqs. always of this form

$$\underbrace{\begin{bmatrix} \begin{matrix} M \\ 3N_b \times 3N_b \end{matrix} & \begin{matrix} 3N_b \times 2N_p \\ \text{Effect of} \\ \text{const. forces} \\ \text{on mom. bal.} \end{matrix} \\ \begin{matrix} \text{const. eqs.} \\ 2N_p \times 3N_b \end{matrix} & 0 \end{bmatrix}}_{N} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\theta}_1 \\ \vdots \\ F_{x_1} \\ F_{y_1} \end{bmatrix} = \begin{bmatrix} \text{Known stuff} \\ \text{given state} \end{bmatrix} = \begin{bmatrix} \text{Applied forces} \\ \text{accel. due to } N^2 \text{ terms} \end{bmatrix}$$

$(3N_b + 2N_b) \times 1$

EoM $\Rightarrow$ $W = N \backslash b;$

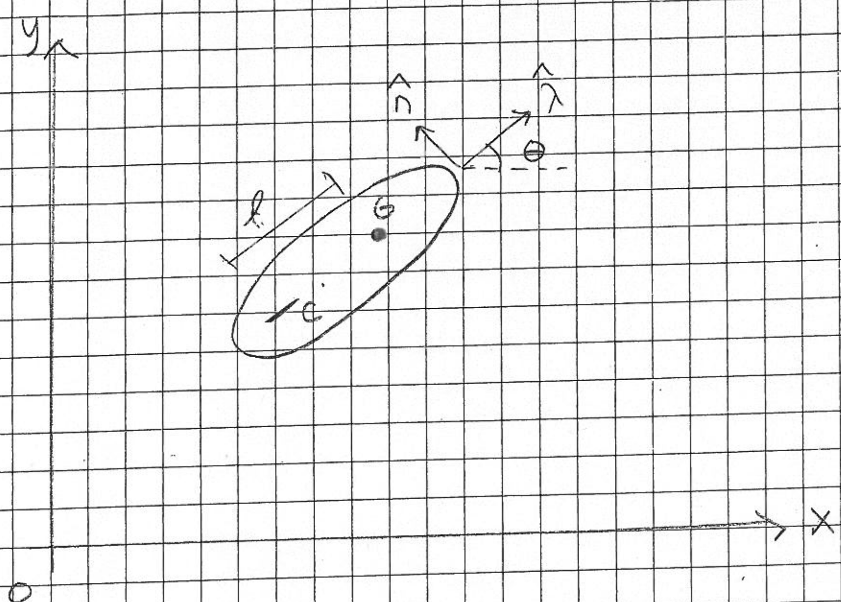
$\uparrow$ first $3N_b$ terms

10/10/2014

Cars, grocery cart, choph Sleigh

The sleigh (2D, looking down) supported, non tipping slippery but for skate constraints

(simplest non-harmonic example)



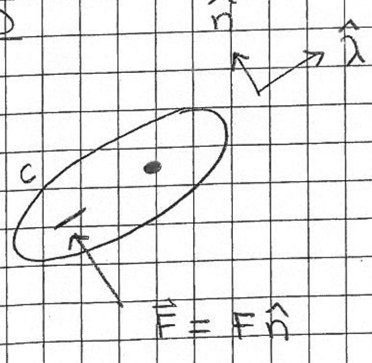
Sleigh Constraint: $\vec{V}_c \cdot \hat{e}_n = 0$

$$-\sin\theta\, \dot{x}_c + \cos\theta\, \dot{y}_c = 0$$

$$f(x, y, \theta, \text{const}) = 0$$

Non-holonomic mechanics

FBD



AMB/c

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$\vec{0} = \vec{r}_{G/c} \times m\vec{a}_G + I\ddot{\theta}\,\hat{k}$$

Coords

$$\vec{V}_c = V_c \hat{\lambda}$$

$$\dot{\theta} = \omega$$

$$\vec{a}_G = \vec{a}_c + \vec{a}_{G/c}$$

$$a_c = \frac{d}{dt}\vec{V}_c = \qquad -d\omega^2\hat{\lambda} + \dot{\omega} d\hat{n}$$

$$= \frac{d}{dt}(v\hat{\lambda}) = \dot{v}\hat{\lambda} + v\dot{\hat{\lambda}}$$

$$= \dot{v}\hat{\lambda} + \dot{\theta} v\hat{n}$$

$$\left\{ \vec{O} = (d\hat{\lambda}) \times \left(\underbrace{\dot{v}\hat{\lambda} + \dot{\theta} v\hat{n}}_{\vec{a}_c} \underbrace{- d\omega^2\hat{\lambda} + \dot{\omega} d\hat{n}}_{\vec{a}_{G/c}} \right) m + I\ddot{\theta}\hat{k} \right\}$$

$$\{\ \} \cdot \hat{k} \rightarrow 0 = m\left(d\dot{\theta} v + \dot{\omega} d^2\right) + I\dot{\omega}$$

$$\dot{\omega} = \frac{d\omega v m}{md^2 + I}$$

$$\{LMB\} \cdot \hat{\lambda} \Rightarrow \dot{v} = \ldots$$

10/15/2014

Lagrange Equations

$F = ma$

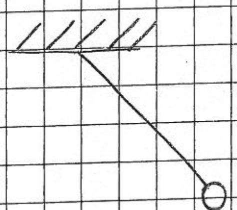
$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\underset{\text{lagrangian}}{\mathcal{L}} = \underset{KE}{T} - \underset{PE}{U}$$

q = generalized coordinate

Pendulum $\longrightarrow \theta$

Cart $\longrightarrow x$



$F_x = m a_x$

$F_y = m a_y$

$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} K x^2$$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 - kx$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} - 0$$

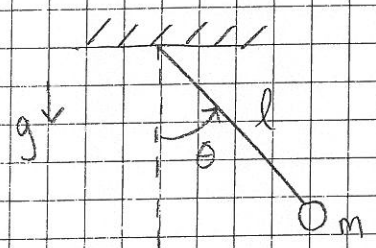
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x}$$

$$-kx = m\ddot{x}$$

$$\ddot{x} = -\frac{k}{m}x$$

$$\dot{x} = v$$

$$\dot{v} = \frac{-k}{m}x$$



$$T = \tfrac{1}{2} m \ell^2 \omega^2$$

$$U = mg\ell(1 - \cos\theta)$$

$$\mathcal{L} = T - U = \frac{1}{2}\ell^2\omega^2 - mg\ell(1 - \cos\theta)$$

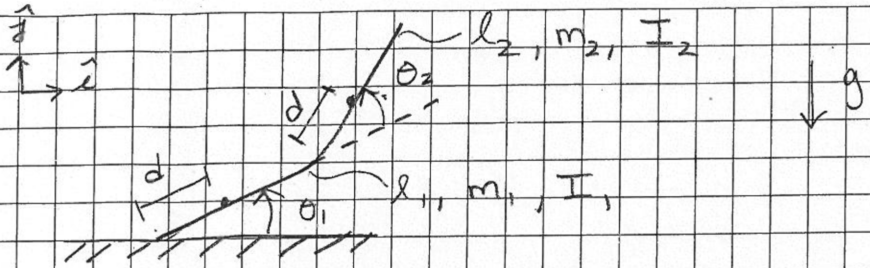
$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \omega}$$

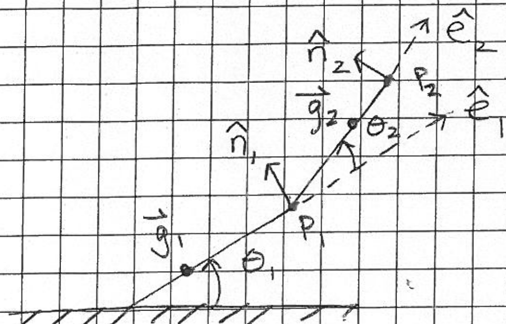
$$\left(0 - mg\ell(0 + \sin\theta)\right) = \frac{d}{dt}\left(m\ell^2\omega - 0\right)$$

$$-mg\ell\sin\theta = m\ell^2\dot{\omega}$$

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -\frac{g}{\ell}\sin\theta$$





$$\hat{e}_1 = \cos\theta_1 + \sin\theta_1 \hat{j}$$

$$\hat{n}_1 = -\sin\theta_1 \hat{i} + \cos\theta_1 \hat{j}$$

$$\hat{e}_2 = \cos\theta_2 \hat{i} + \sin\theta_2 \hat{j}$$

$$\hat{n}_2 = -\sin\theta_2 \hat{i} + \cos\theta_2 \hat{j}$$

$$\dot{\hat{e}}_1 = \dot{\theta}_1 \hat{n}_1, \quad \dot{\hat{n}}_1 = -\dot{\theta}_1 \hat{e}_1$$

$$\dot{\hat{e}}_2 = \dot{\theta}_2 \hat{n}_2, \quad \dot{\hat{n}}_2 = -\dot{\theta}_2 \hat{e}_2$$

$$\vec{g}_1 = d_1 \hat{e}_1, \quad \dot{\vec{g}}_1 = d_1 \dot{\hat{e}}_1$$

$$\vec{g}_2 = l_1 \hat{e}_1 + d_2 \hat{e}_2$$

$$\dot{\vec{g}}_2 = l_1 \dot{\hat{e}}_1 + d_2 \dot{\hat{e}}_2$$

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\varepsilon = \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

$$\varepsilon = M(q, \dot{q})\,\ddot{q} + f(q, \dot{q})$$

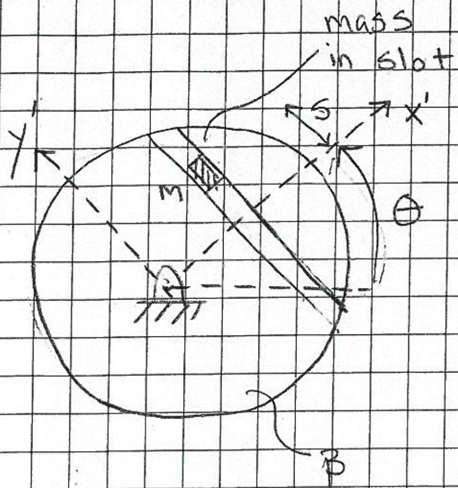
$$M(q, \dot{q}) = J(\varepsilon, \ddot{q})$$

$$f(q, \dot{q}) = \varepsilon(\ddot{q} = 0)$$

10/17/2014

- Rotating Frames
- 3 term $\vec{v}$ formula
- 5 term $\vec{a}$ formula

Ex)



Big Puzzle:

$\vec{a}_m$ in terms of

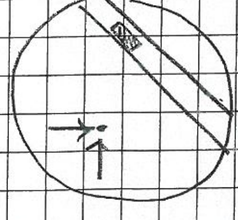
$\theta, \dot{\theta}, \ddot{\theta}, s, \dot{s}, \ddot{s}$

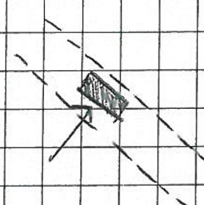
EoM

$\ddot{\theta} = ?$

$\ddot{s} = ?$

FBD





Basic Equations

AMB/o System

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{0} = I^G \ddot{\theta} \hat{k} + \vec{r}_{m/O} \times \vec{a}_m m$$

$\{LMB\} \cdot \hat{j}'$ of mass

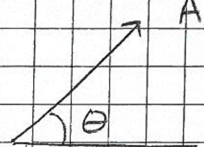
$$\{N \hat{\imath}' = m \vec{a}_m\} \cdot \hat{j}'$$

How to find $\vec{a}_m$?

$$\vec{r}_{m/o} = d\,\hat{i}' + s\,\hat{j}$$

$$\vec{v}_{m/o} = (0\cdot\hat{i}' + d\cdot\dot{\hat{i}}') + (\dot{s}\,\hat{j}' + s\,\dot{\hat{j}}')$$
$$= (0 + d\,\dot{\theta}\,\hat{j}') + (\dot{s}\,\hat{j}' - s\,\dot{\theta}\,\hat{i}')$$

Recall:



$\vec{\omega} = \dot{\theta}\,\hat{k}$

$\dot{\vec{A}} = \vec{\omega}\times\vec{A}$

$$\vec{a}_m = \frac{d^F}{dt}\left(\frac{d^F}{dt}\left(\vec{r}_{m/o}\right)\right) = \frac{d}{dt}(\vec{v}_m) \cancel{= (d\,\ddot{\theta}\,\hat{j}' + d\,\dot{\theta}\,\dot{\hat{j}}') - (\dot{s}\,\dot{\theta}\,\hat{j}' + s\,\ddot{\theta}\,\hat{j}' + s\,\dot{\theta}^2\,\hat{j}') + (\ddot{s}\,\hat{j}' + \dot{s}\,\dot{\hat{j}}')}$$

$-\dot{\theta}\,\hat{i}'$

$-\dot{\theta}\,\hat{i}'$

$$= (d\,\ddot{\theta}\,\hat{j}' + d\,\dot{\theta}\,\dot{\hat{j}}') + \left[(\ddot{s}\,\hat{j}' + \dot{s}\,\dot{\hat{j}}') - (\dot{s}\,\dot{\theta}\,\hat{i}' + s\,\ddot{\theta}\,\hat{i}' + s\,\dot{\theta}\,\dot{\hat{i}}')\right]$$

$$\dot{\hat{i}}' = \dot{\theta}\,\hat{j}', \quad \dot{\hat{j}}' = -\dot{\theta}\,\hat{i}'$$

$\Longrightarrow$ EoM (easy, do on your own)

Other Approach
Quote fancy formulas

3 term $\vec{v}$ formula: $\vec{v}_{/F} = \vec{v}_{o'/F} + \vec{v}_{/B} + \vec{\omega}_{B/F}\times\vec{r}_{/o'}$

5 term $\vec{a}$ formula:

$$\vec{a}_{/F} = \vec{a}_{o'/F} + \vec{a}_{/B} + \underbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r}_{/o'})}_{-\omega^2\,\vec{r}_{/o'}\ \text{in 2D}} + \dot{\vec{\omega}}\times\vec{r}_{/o'} + 2(\vec{\omega}\times\vec{v}_{/B})$$

In the example:

$\vec{V}_{o'} = \vec{0}$ $\quad$ $\vec{\omega}_{B/F} = \dot{\theta}\hat{k}$

$\vec{V}_{/B} = \dot{s}\hat{j}'$ $\quad$ $\vec{r}_{/o'} = s\hat{j}' + d\hat{i}'$

$\vec{a}_{o'/F} = \vec{0}$ $\quad$ $\dot{\vec{\omega}} = \ddot{\theta}\hat{k}$

$\vec{a}_{/B} = \ddot{s}\hat{j}'$

Ex.

$$\vec{a} = \underbrace{\vec{a}_{o'/F}}_{\vec{0}} + \underbrace{\vec{\omega}\times\vec{\omega}_{B/F} \times \vec{r}_{/o'}}_{-\omega^2 \vec{r}_{/o'} \;=\; -\dot{\theta}^2(d\hat{i}+s\hat{j})} + \underbrace{\dot{\vec{\omega}} \times \vec{r}_{/o'}}_{(\ddot{\theta}\hat{k})\times(d\hat{i}'+s\hat{j}')} + \overbrace{\vec{a}_{/B}}^{\ddot{s}\hat{j}'} + \underbrace{2\vec{\omega}\times\vec{V}_{/B}}_{(2\dot{\theta}\hat{k}\times\dot{s}\hat{j}')}$$

Gives same answer as $\ddot{\vec{r}}$ method

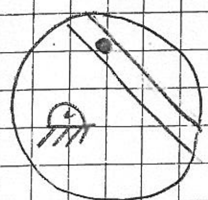
10/20/2014

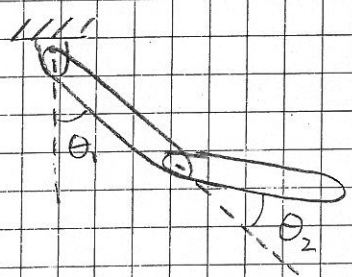
Moving Frames

Big Question: Find $\underbrace{\text{acc. of pts?}}_{\text{in terms of minimal coord. + derivatives}}$

What kinds of problems?

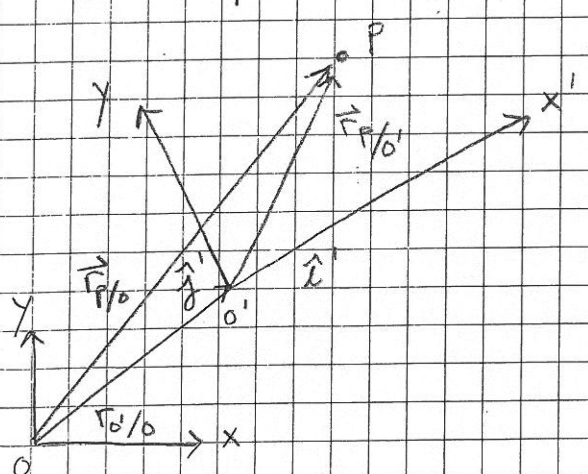
Ex.

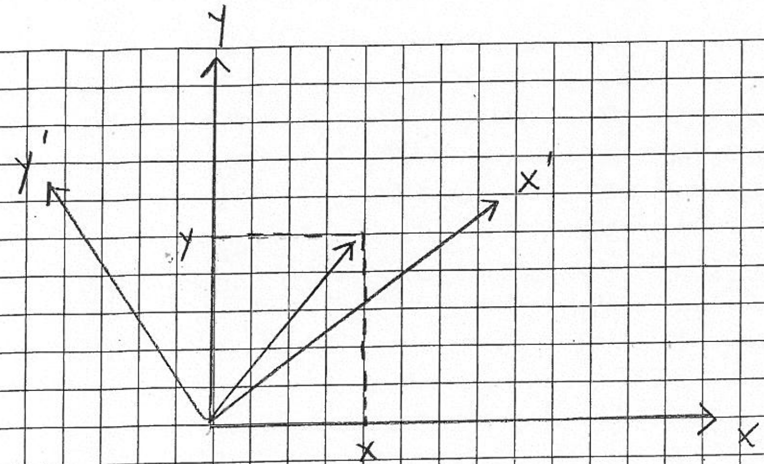




Fixed + Moving Frame

F + B





$$Q_x \hat{i} + Q_y \hat{j} = Q_{x'} \hat{i}' + Q_{y'} \hat{j}'$$

$$\dot{\vec{Q}} = {}^F\dot{\vec{Q}} = \dot{Q}_x \hat{i} + \dot{Q}_y \hat{j}$$

$${}^B\dot{\vec{Q}} = \dot{Q}_{x'} \hat{i}' + \dot{Q}_{y'} \hat{j}'$$

Q-dot Formula/: ${}^F\dot{\vec{Q}} = {}^B\dot{\vec{Q}} + \vec{\omega}_{B/F} \times \vec{Q}$

Transport Thrm.

$\vec{\omega}_{B/F} = \dot{\theta} \hat{k}$

$\vec{\omega}_{F/B} = -\dot{\theta} \hat{k}$

in 2D: $\vec{\omega}_{B/F} = \dot{\theta} \hat{k} = \dot{\theta} \hat{k}'$

ex $\dot{\hat{i}}' = \vec{\omega} \times \hat{i}' = \dot{\theta} \hat{j}'$

$\dot{\hat{j}}' = \vec{\omega} \times \hat{j}' = -\dot{\theta} \hat{i}'$

3 + 5 term Derivation

Method 1:

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

$$= (x_{O'/O} \hat{i} + y_{O'/O} \hat{j}) + (x'_{P/O'} \hat{i}' + y'_{P/O'} \hat{j}')$$

$$\vec{v}_{P/O} = \vec{v}_{P/F} = \frac{d}{dt}\left(\vec{r}_{P/O}\right) = \overbrace{\left(\dot{x}_{O'/O}\,\hat{i} + \dot{y}_{O'/O}\,\hat{j}\right)}^{\vec{v}_{O'/O}}$$

$\omega \times \vec{r}_{P/O'}$

$$+\left[\left(\dot{x}'_{P/O'}\,\hat{i}' + x'_{P/O'}\,\dot{\hat{i}}'\right)\right.$$

$\vec{v}_{P/O}$

$$\left. + \dot{y}'_{P/O'}\,\hat{i}' + y'_{P/O'}\,\dot{\hat{j}}'\right]$$

Recall $\dot{\hat{i}}' = \vec{\omega} \times \hat{i}'$, $\dot{\hat{j}}' = \vec{\omega} \times \hat{j}'$

$$\Longrightarrow \vec{v}_{P/O} = \vec{v}_{O'/O} + \vec{v}_{P/B} + \omega \times \vec{r}_{P/O'}$$

Likewise for accel

$$\vec{a}_{P/O} = \frac{d}{dt}\left(\frac{d}{dt}\left(\vec{r}_{P/O}\right)\right) = 5 \text{ term accel formula}$$

Method 2

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

$$\vec{v}_{P/F} = \frac{{}^F d}{dt}\left(\vec{r}_{P/O}\right) = {}^F\dot{\vec{r}}_{P/O} = {}^F\dot{\vec{r}}_{O'/O} + {}^F\dot{\vec{r}}_{P/O'}$$

$$= \vec{v}_{O'/F} + {}^B\dot{\vec{r}}_{P/O'} + \omega \times \vec{r}_{P/O'}$$

$$\boxed{v_{P/F} = \vec{v}_{O'/O} + \vec{v}_{P/B} + \vec{\omega} \times \vec{r}_{P/O'}}$$

$$\vec{a} = \vec{a}_P = \vec{a}_{P/O} = \vec{a}_{P/F} = \frac{{}^F d}{dt}\left(\frac{{}^F d}{dt}\left(\vec{r}_{P/O}\right)\right)$$

$$\vec{a}_{P/F} = \frac{{}^F d}{dt}\left(\vec{v}_{P/F}\right) = \frac{d}{dt}\left(\vec{v}_{O'/F} + \vec{v}_{O'/B} + \vec{\omega} \times \vec{r}_{P/O'}\right)$$

$$= \frac{{}^F d}{dt}\left(\vec{v}_{P/F}\right) + \frac{{}^F d}{dt}\left(\vec{v}_{P/B}\right) + \frac{{}^F d}{dt}\left(\vec{\omega} \times \vec{r}_{P/O'}\right)$$

$$= \vec{a}_{O'/F} + \frac{d^Q}{dt}\left(\vec{V}_{P/B}\right) + \vec{\omega} \times \vec{V}_{P/B}$$

$$+ \frac{d^Q}{dt}\left(\vec{\omega} \times \vec{r}_{P/O'}\right) + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O'}\right)$$

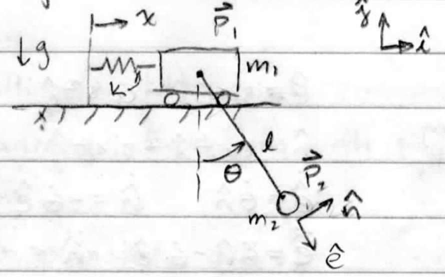
$$= \vec{a}_{O'/F} + \left(\vec{a}_{P/B} + \vec{\omega} \times \vec{V}_{P/B}\right)$$

$$\left[\left(\dot{\vec{\omega}} \times \vec{r}_{P/O'} + \vec{\omega} \times \vec{V}_{P/B}\right) + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O'}\right)\right.$$

$$\vec{a}_{P/F} = \vec{a}_{O'/F} + \vec{a}_{P/B} + \underbrace{\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O'}\right)}_{\text{In 2D } -\omega^2 \vec{r}_{P/O'}}$$

$$+ \dot{\vec{\omega}} \times \vec{r}_{P/O'} + 2\,\vec{\omega} \times \vec{V}_{P/O'}$$

Lagrange 10/22/14



$$\hat{e} = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}$$
$$\hat{n} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\dot{\hat{e}} = \dot{\theta}\hat{n}$$
$$\dot{\hat{n}} = -\dot{\theta}\hat{e}$$

$$\vec{P}_1 = x\hat{i} \qquad \dot{\vec{P}}_1 = \dot{x}\hat{i}$$
$$\vec{P}_2 = \vec{P}_1 + l\hat{e} \qquad \dot{\vec{P}}_2 = \dot{x}\hat{i} + l\dot{\theta}\hat{n}$$

$$\mathcal{L} = T - U \qquad \frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad q_1 = x,\ q_2 = \theta$$

$$T = \frac{1}{2} m_1 \dot{\vec{P}}_1 \cdot \dot{\vec{P}}_1 + \frac{1}{2} m_2 \dot{\vec{P}}_2 \cdot \dot{\vec{P}}_2$$
$$U = \frac{1}{2} k x^2 + m_2 \cdot g \cdot (\vec{P}_2 \cdot \hat{j})$$
$$\dot{\vec{P}}_2 \cdot \dot{\vec{P}}_2 = (\dot{x}\hat{i} + l\dot{\theta}\hat{n}) \cdot (\dot{x}\hat{i} + l\dot{\theta}\hat{n}) = \qquad \hat{i}\cdot\hat{n} = \cos\theta$$
$$= (\dot{x}^2 + 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2)$$

$$\mathcal{L} = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2) - \frac{1}{2} k x^2 + m_2 g (l\cos\theta)$$

$q_1 = x$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m_1\dot{x} + m_2\dot{x} + m_2 l\dot{\theta}\cos\theta$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m_1\ddot{x} + m_2\ddot{x} + m_2 l\ddot{\theta}\cos\theta + -m_2 l\dot{\theta}^2\sin\theta$$

$q_2 = \theta$

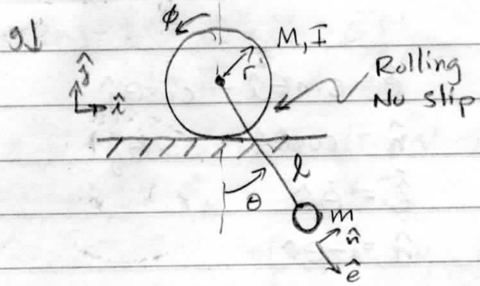
$$\frac{\partial \mathcal{L}}{\partial \theta} = -m_2 l\dot{\theta}\dot{x}\sin\theta - m_2 g l\sin\theta$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_2 l\dot{x}\cos\theta + m_2 l^2\dot{\theta}$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m_2 l\ddot{x}\cos\theta + -m_2 l\dot{x}\dot{\theta}\sin\theta + m_2 l^2\ddot{\theta}$$

① $-kx = m_1\ddot{x} + m_2\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta$

② $-m_2 l\dot{\theta}\dot{x}\sin\theta - m_2 g l\sin\theta = m_2 l\ddot{x}\cos\theta - m_2 l\dot{x}\dot{\theta}\sin\theta + m_2 l^2\ddot{\theta}$

$$\vec{O} = \underline{\underline{M}} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \vec{F}$$

Newton-Euler



$$\hat{e} = \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}$$

$$\hat{n} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

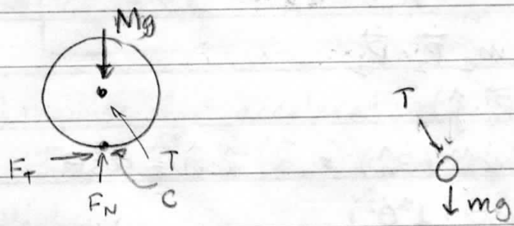
$$\dot{\hat{e}} = \dot{\theta}\hat{n} \qquad \dot{\hat{n}} = -\dot{\theta}\hat{e}$$

$$\ddot{\hat{e}} = \ddot{\theta}\hat{n} - \dot{\theta}^2\hat{e} \qquad \ddot{\hat{n}} = -\ddot{\theta}\hat{e} - \dot{\theta}^2\hat{n}$$

$$\vec{P}_1 = -r\phi\,\hat{\imath} \qquad \dot{\vec{P}}_1 = -r\dot{\phi}\,\hat{\imath} \qquad \ddot{\vec{P}}_1 = -r\ddot{\phi}\,\hat{\imath}$$

$$\vec{P}_2 = \vec{P}_1 + l\hat{e} \qquad \dot{\vec{P}}_2 = -r\dot{\phi}\,\hat{\imath} + l\dot{\theta}\,\hat{n} \qquad \ddot{\vec{P}}_2 = -r\ddot{\phi}\,\hat{\imath} - l(\ddot{\theta}\hat{n} - \dot{\theta}^2\hat{e})$$

FBD



$$\Sigma\vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$(r\hat{\jmath})\times(-T\hat{e}) = I\ddot{\phi}\,\hat{k} + M(r\hat{\jmath})\times(\ddot{\vec{P}}_1) \qquad \{1\}$$

$$\Sigma\vec{F} = \dot{\vec{L}}$$

$$T\hat{e} - mg\hat{\jmath} = m\ddot{\vec{P}}_2 \qquad \{2\}$$

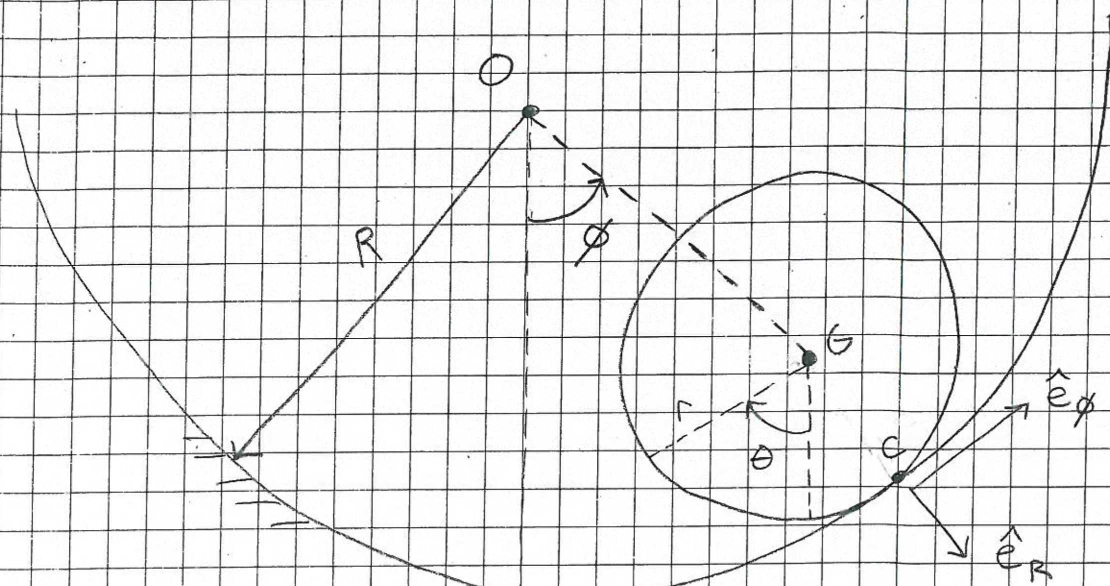
$$\{1\}\cdot\hat{k} = -r\sin\theta = I\ddot{\phi} + Mr(\hat{\jmath}\times\ddot{\vec{P}}_1)\cdot\hat{k}$$

$$\{2\}\cdot\hat{e} = T + mg\cos\theta = m(-r\ddot{\phi}\sin\theta - l\dot{\theta}^2)$$

$$\{2\}\cdot\hat{n} = -mg\sin\theta = m(-r\ddot{\phi}\cos\theta - l\ddot{\theta})$$

10/24/2014

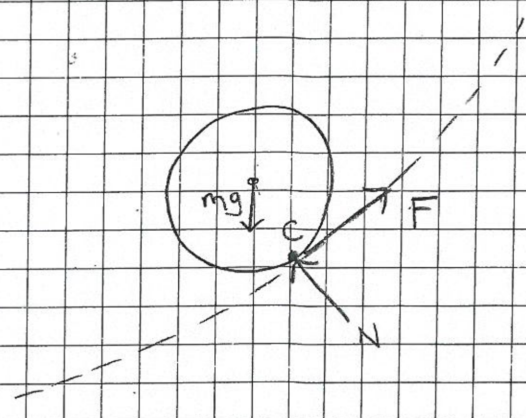
Rolling Example



- Disk rolls inside ring
- 1 DoF

m, I^G

FBD



Kinematics

$$\left\{ \vec{v}_G = \vec{v}_G \right\} \cdot \hat{e}_\phi$$

$$\boxed{(R-r)\,\dot{\phi} = r\,\dot{\theta}}$$

True at all times

$$\Rightarrow \ddot{\theta} = \frac{R-r}{r}\,\ddot{\phi}$$

assume $\theta = \phi = 0$ in Rest Position

AMB/c

$$\left\{ \sum \vec{M}_{/c} = \dot{\vec{H}}_{/c} \right\} \cdot \hat{k} \qquad \ddot{\theta} = \frac{R-r}{r} \ddot{\phi}$$

$$mgr\sin\phi = \left\{ \vec{r}_{G/c} \times m\,\vec{a}_G + I^G \ddot{\theta}\,\hat{k} \right\} \cdot \hat{k}$$

a_G $\qquad -r\hat{e}_R \qquad \vec{a}_G = (R-r)\ddot{\phi}\,\hat{e}_\phi - (R-r)\dot{\phi}^2\,\hat{e}_R$

Lagrange Equations w/ Forcing:

Recall: $\mathcal{L} = E_K - E_P$

took account of conservative forces

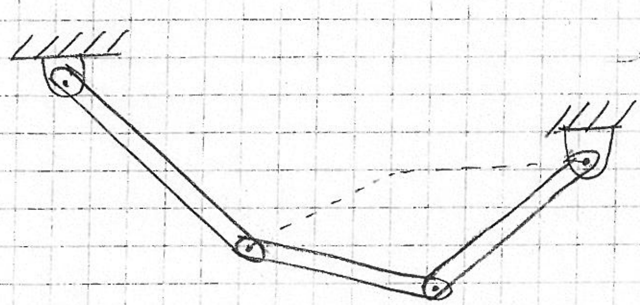
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \cancel{0} \rightarrow Q_i$$

generalized force ass. w/ generalized coord. q_i

What can Q_i be used for?

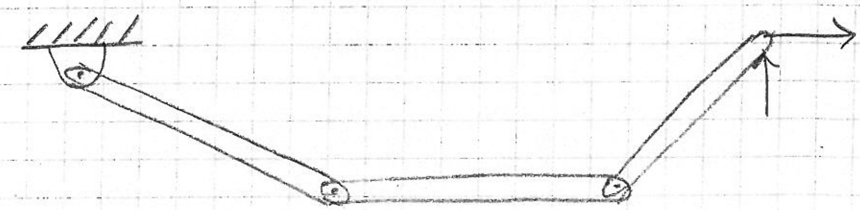
1. Cons. forces
   (then take away from $\mathcal{L}$)
2. Non-cons forces
   eq. $\vec{F}(t)$, or $\vec{F}(\vec{x}) \leftarrow$ w/ $\vec{F} \neq -\vec{\nabla} E_p$
3. Constraint Forces

ex)

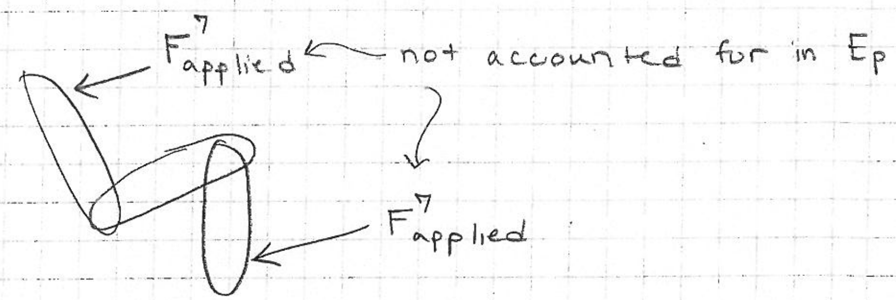


DoF = 3-2 = 1

Avoid Freudensteins Formulas



How to find Q_i



Key Work makes sense

$$dW = dW$$

$$\vec{F}^{\,i}_{appl} \cdot d\vec{r}_i = \sum Q_i \, dq$$

motions of pt. applied of forces

$$Q_i = \sum \vec{F}^{\,j}_{appl.} \cdot \frac{\partial \vec{r}_j}{\partial q_i}$$ "The Jacobian"

all applied forces

2.7315
2.7313

10/27/2014

$q_i = \underbrace{\text{generalized coordinates}}_{\text{minimal coords.}}$

= almost always positions + angles

(exceptions; $q = s$)

n Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$$

generalized forces
(one for each q_i)

$Q_i = 0$ if all forces are conservative ($\vec{F} = \vec{\nabla} E_p$)

What are Q_i?

$dW = dW$

$$\underbrace{\sum \vec{F}_i \cdot d\vec{r}_i}_{\text{all pts. of force application}} = \sum_{\text{all } q\text{'s}} Q_i \, dq_i$$

for any dq_i + corresponding $d\vec{r}_i$

Use: $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots)$

$$Q_7 = \sum_{\substack{\text{all pts.} \\ \text{of force} \\ \text{application}}} \frac{\partial \vec{r}_i}{\partial q_7} \cdot \vec{F}_i$$

The Jacobian

Alt. Equivalent Approach

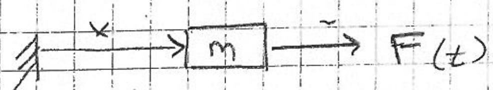
1) Imagine a dq_j

2) Look at $d\vec{r}_i$ for all pts. of force application

3) Calculate $dW = \sum \vec{F}_i \cdot d\vec{r}_i$

4) Set $Q_j = \dfrac{dW}{dq_j}$

Ex) 1D mass



The Simplest Possible Ex:

$E_k = \frac{1}{2} m \dot{x}^2$

$\cancel{E_p = kx}$ ⟵ Not Constant

$\mathcal{L} = E_k - E_p = \frac{1}{2} m \dot{x}^2$

$q = x$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = Q_x$$

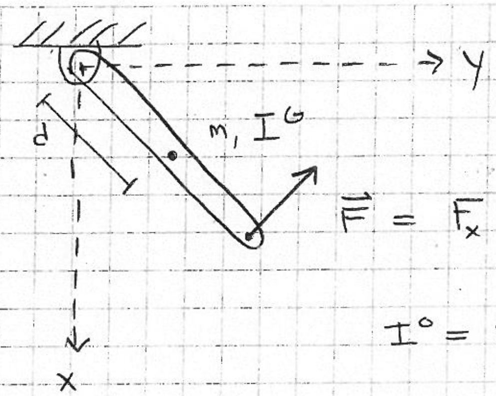
$m\ddot{x} - 0 = Q_x$

$F\,dx = Q_x \, dq$ ⟵ x

$F = Q$

Lagrange ⟶ $F = m\ddot{x}$

Ex)



No gravity

$$\vec{F} = F_x \hat{i} + F_y \hat{j}$$

$$I^o = md^2 + I^G$$

$$\mathcal{L} = E_K - E_P$$

$$= \underbrace{\tfrac{1}{2} I^o \dot{\theta}^2}_{(mv^2 + I^G \dot{\theta}^2)/2} - 0$$

Lag. Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_\theta$$

$$I^o \ddot{\theta} = Q_\theta$$

What is Q_θ?

$$dW = dW$$

$$\vec{F} \cdot \underbrace{d\vec{r}_D}_{\text{arc length}} = Q_\theta \, d\theta$$

$$d\vec{r}_D = (\hat{k}\, d\theta) \times \vec{r}_{D/O}$$

$$\vec{F} \cdot (\hat{k}\, d\theta \times \vec{r}_{r/o}) = Q_\theta \, d\theta$$

(Aside: $\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c}$)

$$\vec{F} \cdot (\hat{k}\, d\theta \times \vec{r}_{r/o}) = Q_\theta\, d\theta$$

$$k\, d\vec{\theta} \cdot \vec{r}_{/o} \times \vec{F} = Q_\theta\, d\theta$$

$$\vec{K} \cdot (\vec{r}_{/o} \times \vec{F})\, d\theta = Q_\theta\, d\theta$$

$$Q_\theta = (\vec{r}_{/o} \times \vec{F}) \cdot \hat{k}$$

(as expected)

Alt. approach

$$Q_\theta = \left(\frac{\partial \vec{r}_o}{\partial \theta}\right) \cdot \vec{F}$$

$$\vec{r}_D = (\cos\theta\, \hat{\imath} + \sin\theta\, \vec{\jmath})\, \ell$$

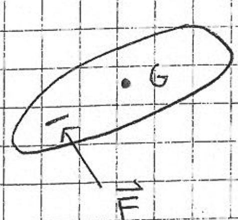
$$\frac{\partial \vec{r}_D}{\partial \theta} = (-\sin\theta\, \hat{\imath} + \cos\theta\, \vec{\jmath})\, \ell$$

$$Q_\theta = \frac{\partial \vec{r}_D}{\partial \theta} \cdot \vec{F} = (-\sin\theta\, F_x + \cos\theta\, F_y)\, \ell$$

$$\{= \vec{r}_{D/o} \times \vec{F}\}$$

An application of gen. forces: Constraint Forces

ex) Chaplygen Sleigh



Write 3 Lagrange Eqs.

θ, x, y

Calc. generalized forces

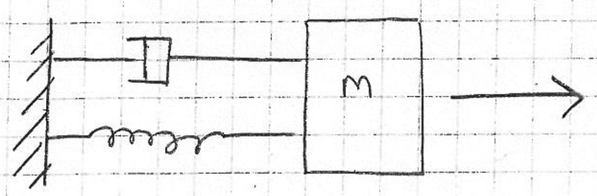
Q_x, Q_y, Q_θ

Now have 4 unknowns

$\ddot{\theta}$, $\ddot{x}$, $\ddot{y}$, F

Add constraint eq: $\frac{d}{dt}$ (velocity constraint)

Vibrations



$F = F_0 \sin(\omega_0 t)$

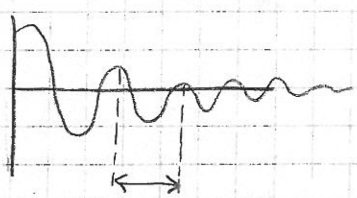
$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega_0 t)$$

4 Frequencies

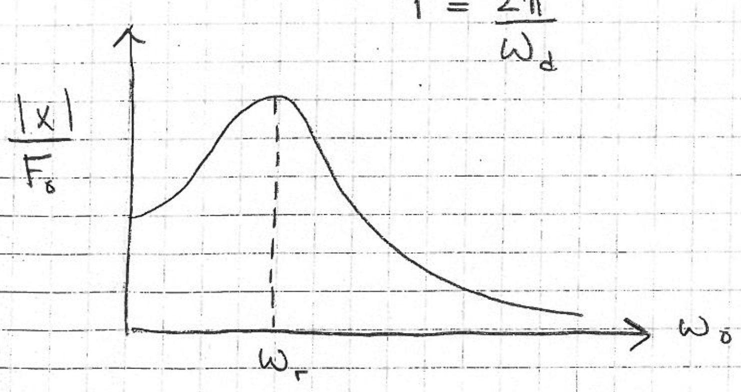
ω_0 = forcing frequency

ω_n = natural frequency = $\sqrt{\frac{k}{m}}$ (eq. $m\ddot{x} + kx = 0$)

ω_d = damped frequency



$T = \frac{2\pi}{\omega_d}$



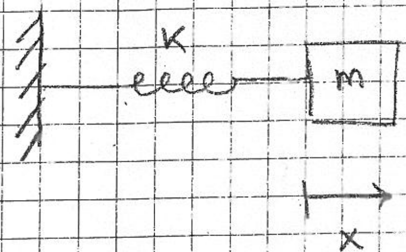
ω_r = peak in appl. response = resonant frequency

10/29/2014

Harmonic Oscillator - Spring Mass
(all review)

Core Problem:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega_f t)$$



$x = 0$, spring relaxed

FBD

$T = kx$

$\Delta(\Delta l)$

$(l - l_0)$

LMB: $F = ma$

$$-kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

Guess Soln:

$$x = x_0 \sin(\omega t)$$

plug in:

$$m x_0 (-\omega^2 \cancel{\sin(\omega t)}) + k x_0 \cancel{\sin(\omega t)} = 0$$

$$-m x_0 \omega^2 + k x_0 = 0$$

Guess was good for all x_0

if $-m\omega^2 + K = 0$

$$\omega^2 = \frac{K}{m}$$

$$\omega = \pm\sqrt{\frac{K}{m}}$$

$$\Longrightarrow \omega = +\sqrt{\frac{K}{m}}$$

$$\boxed{x = x_0 \sin\left(\frac{K}{m} t\right)} \quad ①$$

Another guess:

$$x = x_1 \cos(\omega t) \quad ②$$

is soln. for all x, and $\omega = \sqrt{\frac{K}{m}}$

General Soln: ① + ②

$$x = A \cos\sqrt{\frac{K}{m}}\, t + B \sin\sqrt{\frac{K}{m}}\, t$$

$m\ddot{x} + Kx = 0 \leftarrow$ linear, homogenous $(=0)$

so you can superimpose ①, ②

Are there other solutions?

Rewrite in first order form

define $v = \dot{x}$

$$m\ddot{x} + K\dot{x} + Kx = 0$$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{K}{m} x \end{cases}$$

Parameterize space of all solns. with Initial Conditions

$x(0)$, $v(0)$ and $x(0) = x_0$, $v(0) = v_0$

$A = x_0$

$B = v_0 / \sqrt{\frac{k}{m}}$

Uniqueness of solns. for initial value problems

$\Longrightarrow$ No other solns.

Alternative Approach:

Guess: $x = e^{\lambda t}$

plug in:

$$m\ddot{x} + \cancel{c\dot{x}} + kx = 0$$

$$(m\lambda^2 + k)\cancel{e^{\lambda t}} = 0$$

$$\lambda^2 = k/m$$

$$\lambda = i\sqrt{\frac{k}{m}}$$

$$\Longrightarrow \left\{x = e^{i\sqrt{\frac{k}{m}}t}\right\} \quad \text{or} \quad e^{-i\sqrt{\frac{k}{m}}}$$

2 choices on how to deal w/ complex {} in real world

I. General Complex Soln.

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$

If we give it real initial conditions

$\Longrightarrow$ A and B are such that $x(t)$ is real

II. Thrm.

If real ODE + complex soln.

$\longrightarrow$ Real part alone is a soln.

Thrm. $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$Re\left(e^{i\omega t}\right) = \cos(\omega t)$

$Im\left(e^{i\omega t}\right) = \sin(\omega t)$

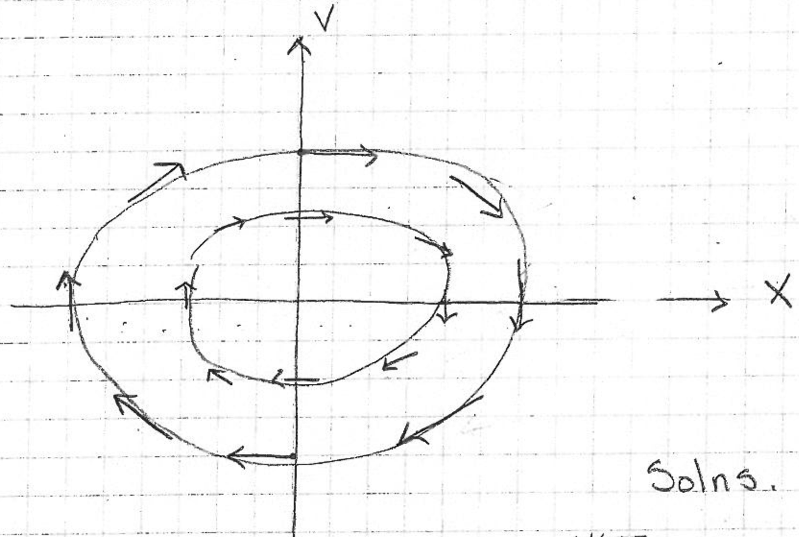
?

$x = A\cos(\omega t) + B\sin(\omega t)$

Phase Plane

$\dot{x} = v$

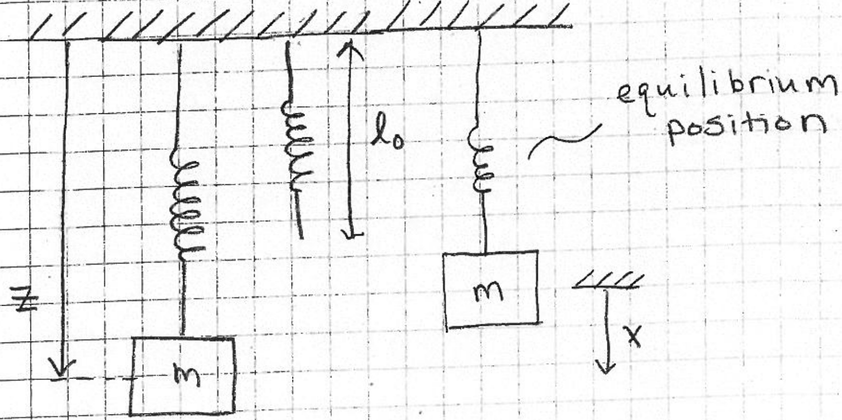
$\dot{v} = -\frac{k}{m}x$



Solns. Elipses

* Frequency is independent of amplitude

ex)



$$T = K(l - l_0) = K(z - l_0)$$

LMB: $F = ma$

$$mg - K(z - l_0) = m\ddot{z}$$

$$m\ddot{z} + Kz = mg + Kl_0$$

$x = Z_{part.}$ = soln. of homogenous eq.: $m\ddot{z} + Kz = 0$

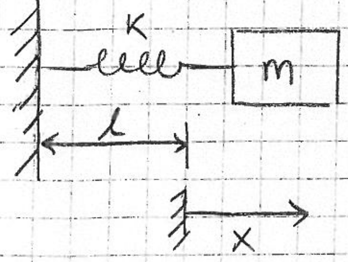
$$Z_{homo.} = \frac{mg}{K} + l_0$$

Another approach: Take ODE + do change change of variables

$$x = z - \frac{mg}{K} - l_0$$

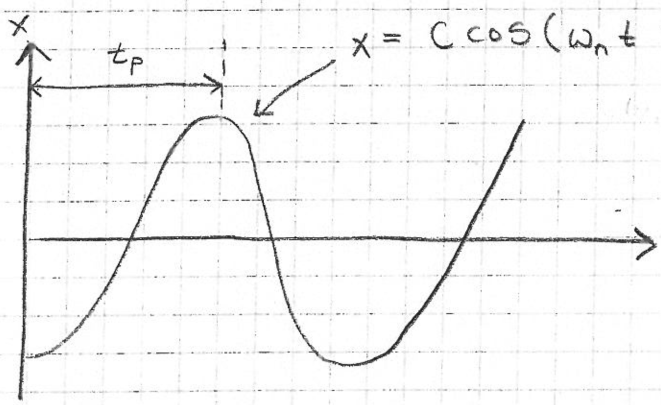
10/31/2014 (Ruina + Pratap Ch. 10-13)

Undamped Motion



$m\ddot{x} + kx = 0$

$x = A\cos(\omega_n t) + B\sin(\omega_n t)$

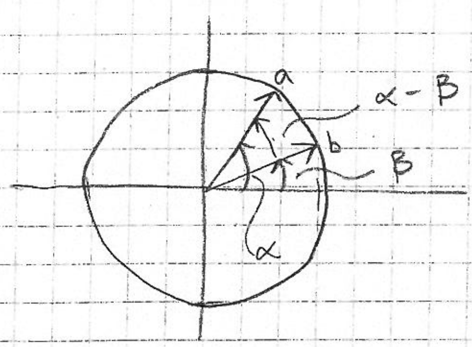


$x = C\cos(\omega_n t - \delta)$, $\omega_n t_p = \delta$

$t_p = \dfrac{\delta}{\omega_n}$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Quick Derivatve:



$\vec{a}\cdot\vec{b} = (\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath})\cdot$
$(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= |\vec{a}|\,|\vec{b}|\cos\theta_{ab}$

$= 1\cdot 1\cdot\cos(\alpha - \beta)$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Back to soln.

$$C\cos(\underbrace{\omega_n t}_{\alpha} - \underbrace{\delta}_{\beta}) = C\cos(\omega_n t - \delta)$$

$$= C\cos\delta\cos(\omega_n t) + C\sin\delta\sin\omega_n t$$

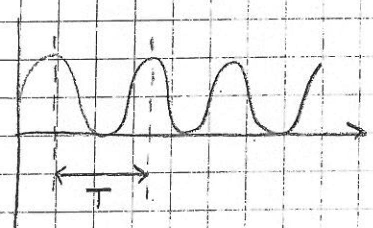
δ = "phase" or "phase lag"

Given $x(t)$, can we find m and k? NO!

$m_2 = 2m$, $k_2 = 2k$

same ODE + same soln.

$$\cancel{2}m\ddot{x} + \cancel{2}kx = 0$$



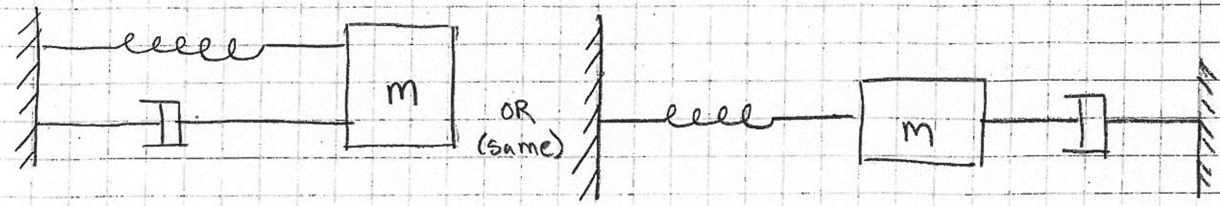
$$x = A\cos(\omega_n t)$$

$$\omega_n = \frac{2\pi}{T}$$

People like to write the governing ODE

$$\ddot{x} + \omega_n^2 x = 0$$

Damped Oscillations



$m\ddot{x} + c\dot{x} + Kx = 0$

$\ddot{x} + \frac{c}{m}\dot{x} + \frac{K}{m}x = 0$

$\ddot{x} + \frac{c}{m}\dot{x} + \omega_n^2 x = 0$

How to solve?

$x(t) = Ce^{\lambda t}$

$\lambda^2 \cancel{C}\cancel{e^{\lambda t}} + \frac{c}{m}\lambda \cancel{C}\cancel{e^{\lambda t}} + \omega_n^2 \cancel{C}\cancel{e^{\lambda t}} = 0$

$\lambda^2 + \frac{c}{m}\lambda + \frac{K}{m} = 0$

$\lambda = \dfrac{-c/m \pm \sqrt{c^2/m^2 - 4K/m}}{2}$

$\lambda_{1,2} = 2$ roots

$x = c_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

Gen., Real Soln.

$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

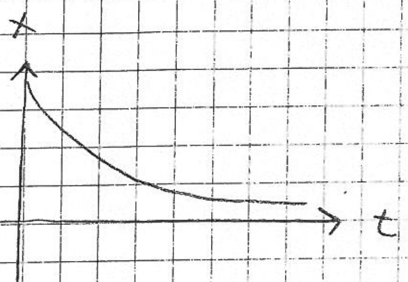
$x = Re\left(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\right)$

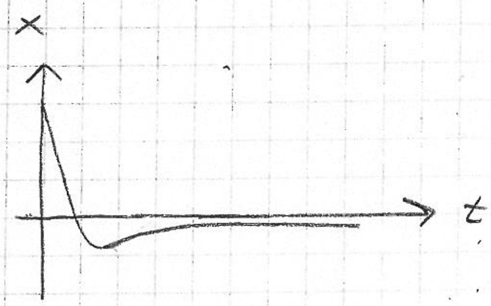
$$x = \mathrm{Re}\left(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \right)$$

<u>Various Cases</u>

1) $\left(\frac{c}{m}\right)^2 > 4 \frac{k}{m} \implies$ 2 real roots

"over damped" $\implies x = C_1 e^{-m t} + C_2 e^{-m t}$





2) $\left(\frac{c}{m}\right)^2 = 4 k m$

Critically Damped

guess is bad because only gives 1 soln.

$\implies x(t) = C_1 e^{-\frac{k}{m} t} + C_2 t e^{-\frac{k}{m} t}$

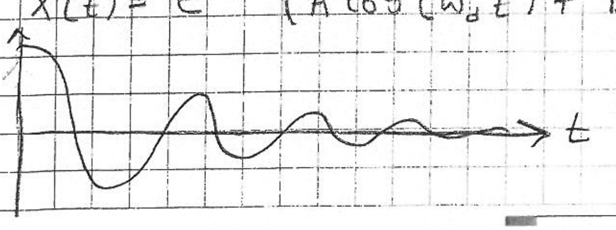
new guess

looks like previous - "fastest possible" decay

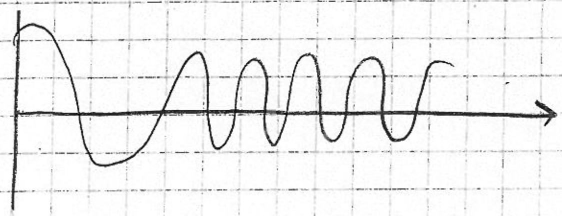
3) $\left(\frac{c}{m}\right)^2 < 4 \frac{k}{m}$

Under Damped

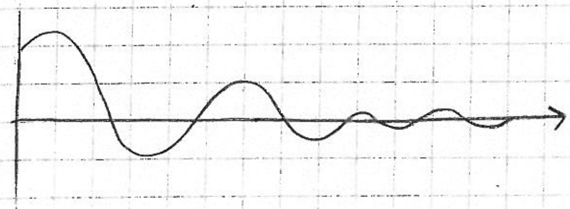
$$x(t) = e^{-\lambda_d t}\left(A \cos(\omega_d t) + B \sin(\omega_d t)\right)$$



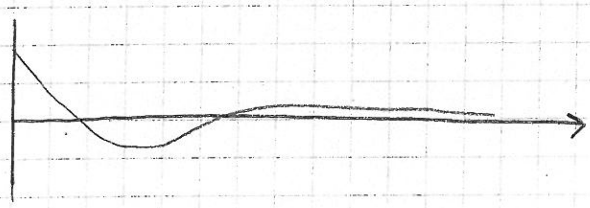
Given k, m, vary damping



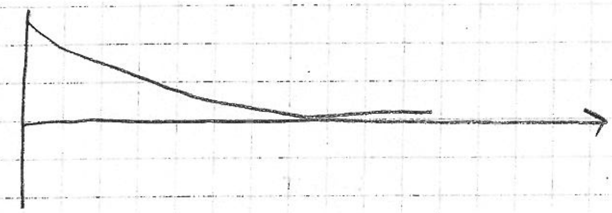
c = 0



c = small



c is bigger

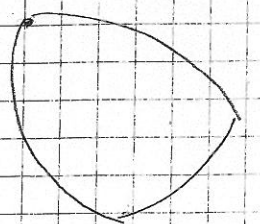


c = big

11/2/2014

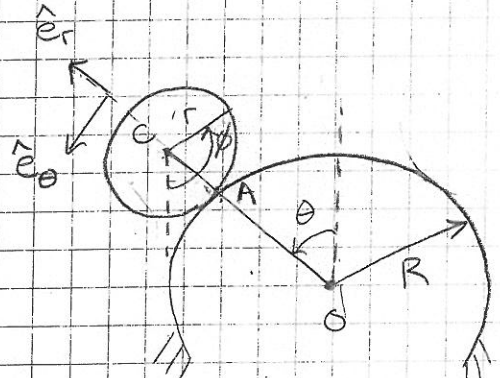
- 1D Vibes
- Damping
- Forcing

Figure w/ Constant Diameter



← 3 circles w/ different centers

Constant Diameter doesn't imply circle / sphere



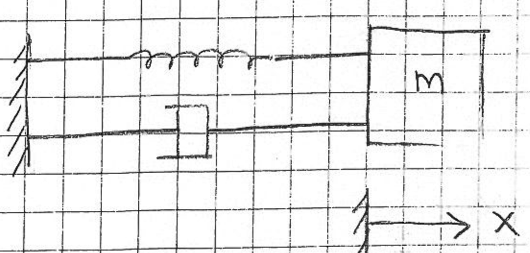
$$\vec{V}_c = (R+r)\,\dot{\theta}\,\hat{e}_\theta$$

$$\vec{V}_c = \vec{V}_A + \vec{V}_{c/A}$$
$$= \vec{0} + r\dot{\phi}\,\hat{e}_\theta$$

$$r\dot{\phi} = (R+r)\,\dot{\theta}$$

$$\dot{\phi} = \frac{R+r}{r}\,\dot{\theta}$$

Oscillations w/ Damping



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$x = \begin{cases} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} \end{cases}$$

λ can be real or complex

$\uparrow$ $c^2 = 4km > 0$ $\qquad$ $\uparrow$ $c^2 = 4km < 0$

$c^2 - 4km = 0$ real, double

Complex Case: the only case of int. in vibes

Real Soln. $\quad x(t) = e^{-\frac{c}{2m}t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right)$

$$\omega_d = \frac{\sqrt{4km - c^2}}{2m} \longleftarrow i \text{ factored out}$$

$$= \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

Crit $\Longrightarrow$ $c_{crit}^2 = 4km$

$$c_{crit} = 2\sqrt{km}$$

$$\omega_d = \sqrt{\underbrace{\left(\frac{k}{m}\right)}_{\omega_n^2} - \left(\frac{c}{2m}\right)^2}$$

$$= \sqrt{\omega_n^2 - \left(\frac{c}{c_{crit}} \cdot \frac{c_{crit}}{2m}\right)^2}$$

$$\omega_d = \sqrt{\omega_n^2 - \left(\frac{c}{c_{crit}}\right)\left(\frac{2\sqrt{Km}}{2m}\right)^2}$$

$$= \sqrt{\omega_n^2 - \underbrace{\left(\frac{c}{c_{crit}}\right)^2}_{\xi = \frac{c}{c_{crit}}} \underbrace{\left(\sqrt{\frac{K}{m}}\right)^2}_{\omega_n}}$$

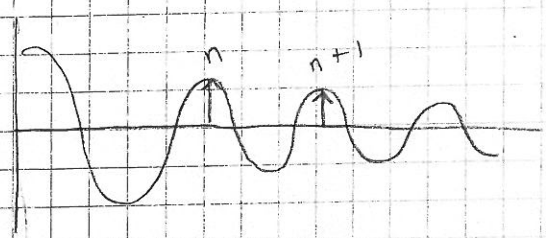
(damping ratio)

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

Equivalent ways to talk about ξ = damping ratio

D = logarithmic decrement

ln((Amplitude at cycle n) / (Amplitude at cycle n+1))



What is relationship between D + ξ ?

$$D = \ln\left(\frac{\cancel{A}\, e^{-\frac{c}{2m} t}}{\cancel{A}\, e^{-\frac{c}{2m} t + T_d}}\right) = \left(\frac{c}{2m} T_d\right) = \frac{2\pi c}{2m\, \omega_n \sqrt{1-\xi^2}}$$

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} \qquad = 2\pi \xi$$

11/5/2014

$* \quad m\ddot{y} + c\dot{x} + Kx = 0$ (unforced)

to keep vibrations going, $c \rightarrow 0$

undamped natural frequency $= \omega_n = \sqrt{\frac{K}{m}}$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{\omega_n}{2\pi}$$

$$T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{K}}$$

Damping

$C_{crit} = 2\sqrt{mK}$

Damping ratio $\xi = \frac{c}{C_{crit}} = \frac{c}{2\sqrt{mK}}$

large $\rightarrow$ very damped

For vibrations, $\xi \ll 1$

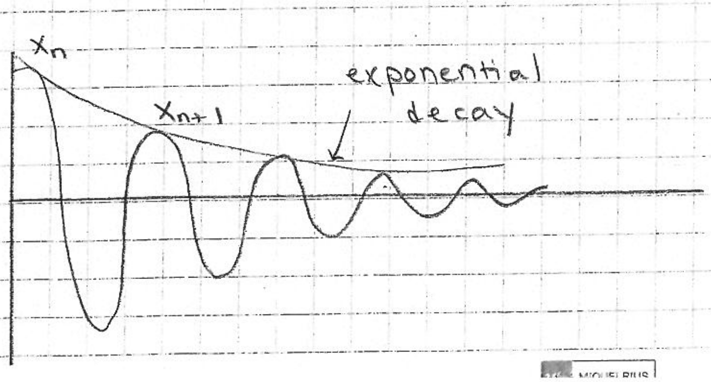
Rewrite $*$ as: $\ddot{x} + 2\omega_n \xi \dot{x} + \omega_n^2 x = 0$

$$2\omega_n \xi = \cancel{2}\sqrt{\frac{\cancel{K}}{m}}\frac{c}{\cancel{2}\sqrt{m\cancel{K}}} = \frac{c}{m}$$

Damped frequency $\omega_d = \frac{\sqrt{4Km - c^2}}{2m} = \omega_n\sqrt{1-\xi^2}$

Note: if c is small ($\xi \ll 1$) then $\omega_d \rightarrow \omega_n -$ (tiny bit)

often neglected



logarithmic Decrement $D = \ln\left(\frac{X_n}{X_{n+1}}\right)$

Can experimentally determine ξ

(can't find c because only rates show up in equation)

$$D = \frac{\xi \cdot 2\pi}{\sqrt{1-\xi^2}} \approx 2\pi\xi \quad \text{for } \xi << 1$$

We usually want to know ξ from D for experiments,

$$\xi = \frac{1}{\sqrt{1+(2\pi/D)^2}} \approx \frac{D}{2\pi}$$

For $\xi << 1$

Can find ω_n, then from measuring $\omega_d = \omega_n\sqrt{1-\xi^2}$

Another measure of damping:

Quality factor $Q = 2\pi\left(\frac{\text{energy at one cycle}}{\text{energy loss at next cycle}}\right)$

$$= 2\pi\left(\frac{X_n}{X_n^2 - X_{n+1}^2}\right)$$

Comes from:

$E_p = \frac{1}{2} k x^2$

$$Q = \frac{1}{1 - e^{-\frac{c}{m}T_d}} \approx \frac{\pi}{D} = \frac{1}{2\xi}$$

ex) If you have a spring mass damper, measure T_d (and x_n, x_{n+1}), find $\frac{K}{m}$ and $\frac{c}{m}$, then if you g on a Known mass, m_{test}, you can measure the new T_d (and x_n, x_{n+1}) and find all the parameters

Forcing

forcing frequency

simple force is $F(t) = F_0 \cos(\omega t)$

$$m\ddot{x} + c\dot{x} + Kx = F_0 \sin(\omega t)$$

soln. Sum of homogenous parts + particular parts

transient — steady state

$$x(t) = X_h + X_p$$

X_h is what we've discussed

for underdamped: $X_h = e^{\frac{-c}{2m}t}\left(A\cos(W_d t) + B\sin(W_d t)\right)$

$$W_d = \omega_n\sqrt{1-\xi^2} = \sqrt{\frac{K}{m}}\sqrt{1-\left(\frac{c}{2\sqrt{mK}}\right)^2}$$

Steady State: the nicest particular soln:

guess that $X_p = C_1 \cos \omega t + C_2 \sin \omega t$

forcing frequency

Rewrite as:

$$X_p = A_0 \sin(\omega t - \delta)$$

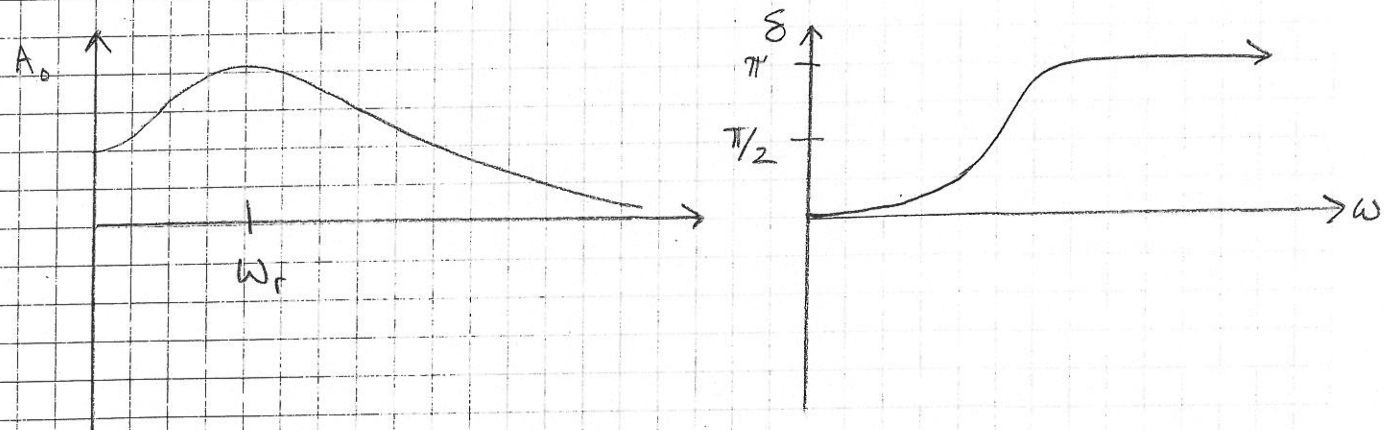
amplitude

phase

$$\delta = \tan^{-1}\left(\frac{C_2}{C_1}\right)$$

$$A_0 = \sqrt{C_1^2 + C_2^2}$$

Can find soln.



Generally, $\omega_r \neq \omega_n \neq \omega_d$

$$\omega_r = \omega_n \sqrt{1 - 2\xi^2}$$

Comes from a long calculation, which you should be able to do

11/7/2014

Misc. Oscillator topics

1) Resonance + near resonance

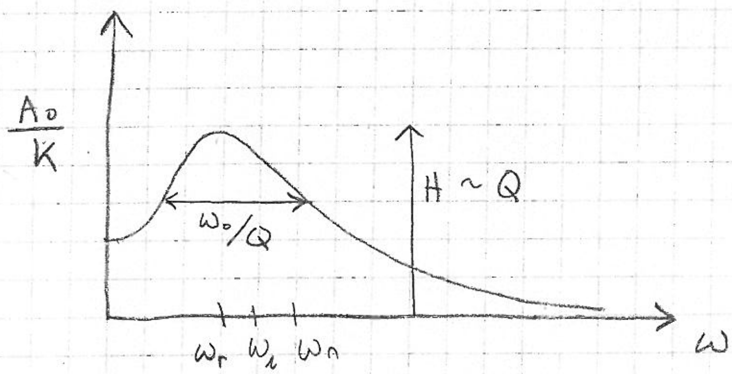
2) Friction

3) Fourier Series

Recall: $m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$

$X_n = X_{ss} = A_0 \sin(\omega t - \delta)$

amplitude of x



Frequency of Forces

Hi Q (Quality Factor)

(low damping)

narrow resonant peak: ω_n/Q

high peak $(A_0/K) \approx Q$

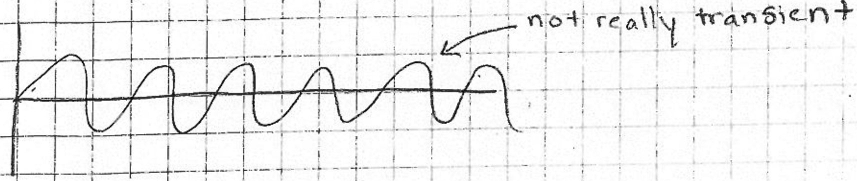
What if $Q = \infty$

$C = 0$

$\xi = 0$

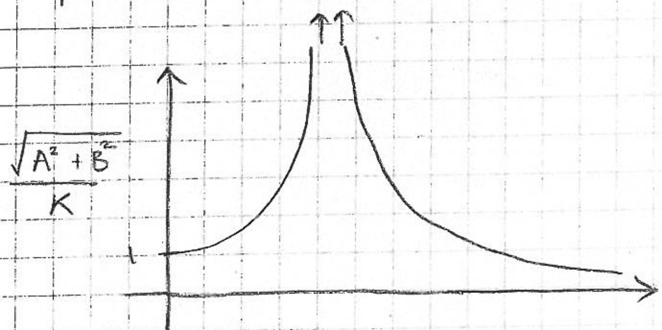
$m\ddot{x} + Kx = F_0 \sin(\omega t)$

$X_n = X_H \quad " = A\cos(\omega_n t) + B\sin(\omega_n t)$



$X_{ss} = ?$ $\quad$ (X_p = particular soln.)

Guess: $X_p = A\cos\omega t + B\sin\omega t$



What happens if $\omega = \omega_n$?

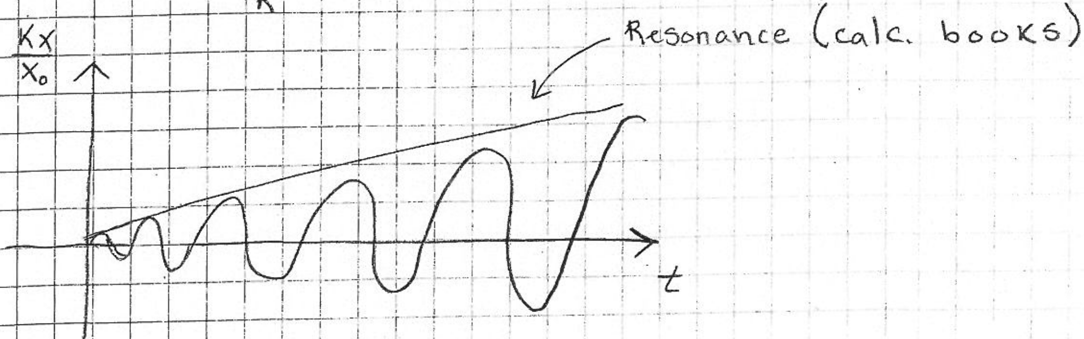
guess $\rightarrow$ $0 = F_o \sin(\omega_n t)$ $\longrightarrow$ Bad Guess

$\Downarrow$

$(X_p = A\cos(\omega_n t) + B\sin(\omega_n t))$

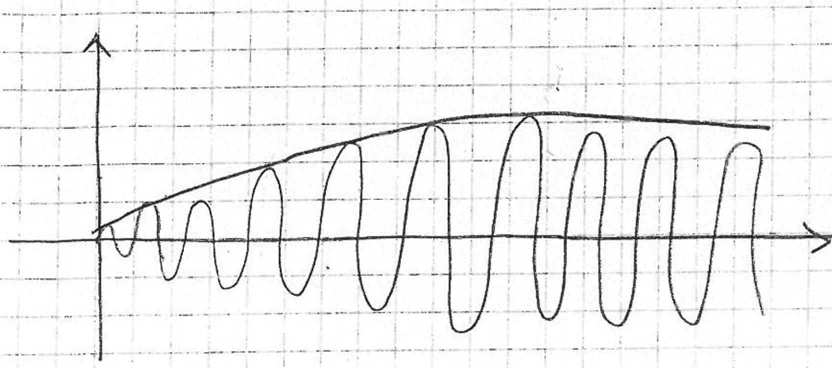
Good Guess:

$$X_p = \frac{-F_o}{K}\cos(\omega_n t)\, t$$



What if $\omega = \omega_r$, but Q = small

I.C. $x(0) = 0$
$\dot{x}(0) = 0$



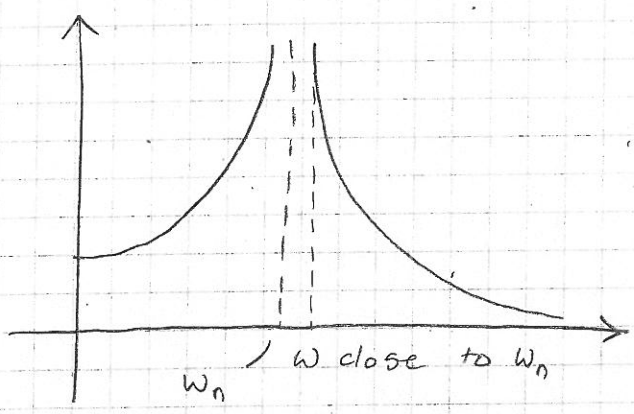
$X = X_n + X_p$

$= (\sin \text{ wave}) + (\text{undamped } \sin \text{ wave})$

Also close to resonance:

$c = 0$

$\omega \approx \omega_n \,, \quad \omega \neq \omega_n$



$X_n = A \cos(\omega_n t) + B \sin(\omega_n t)$

$\omega_n = \sqrt{\frac{K}{m}}$

Can pick A and B to satisfy I.C.'s

$X_p = X_s = A_0 \sin(\omega t)$

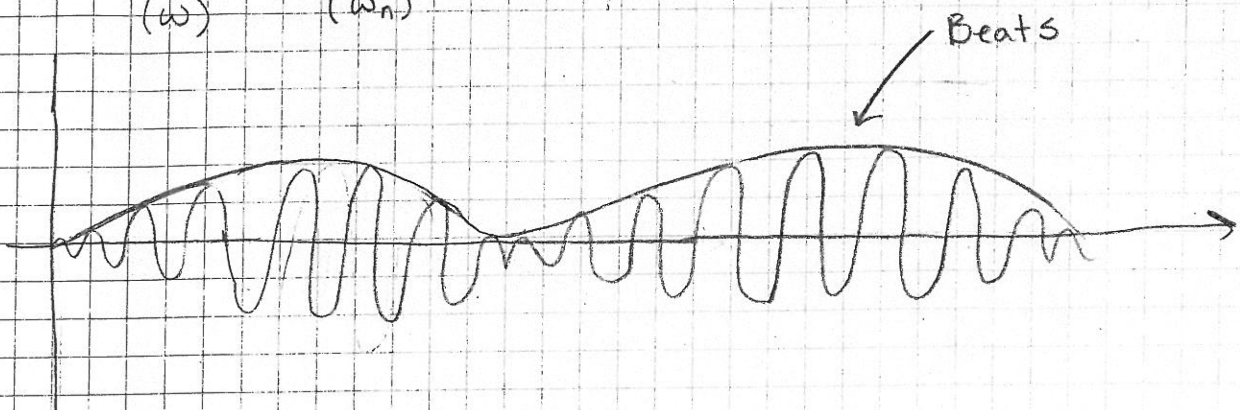
A_0: Big

$\omega \neq \omega_n$

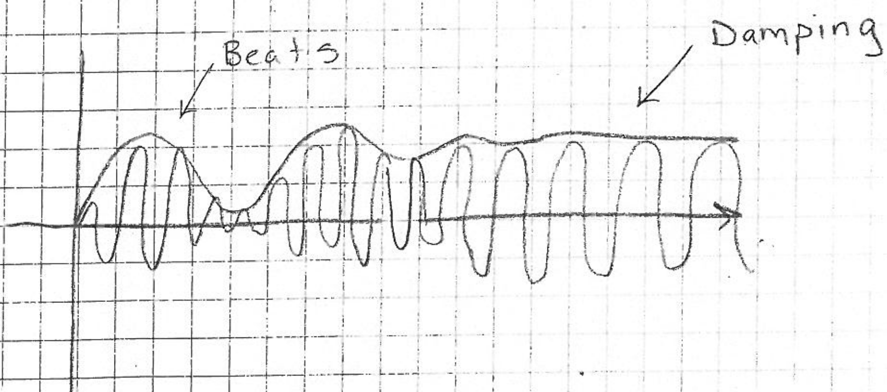
$$X = X_p + X_n$$

X_p: sine wave (ω)

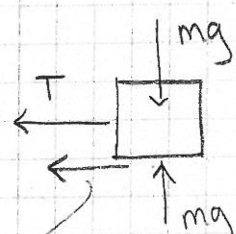
X_n: sine wave (ω_n)



What if $\xi << 1$ and ω is closed to ω_n

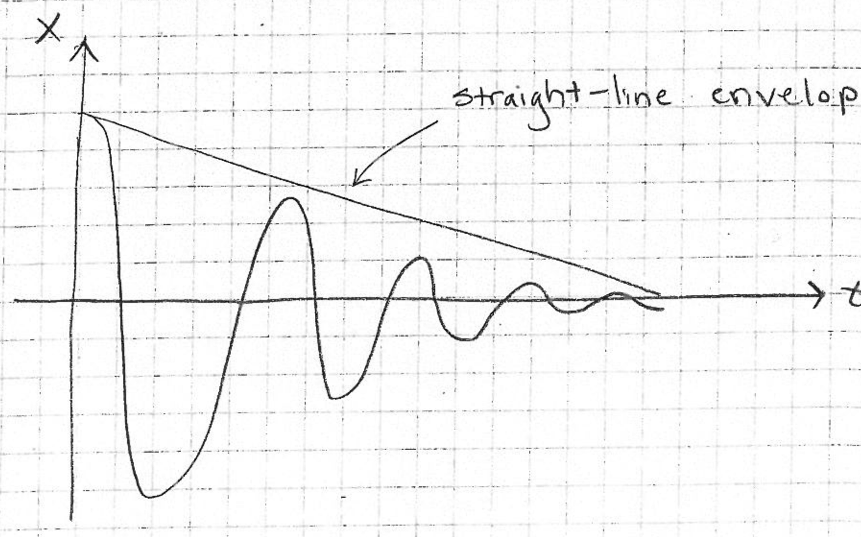


Friction (not dashpot).

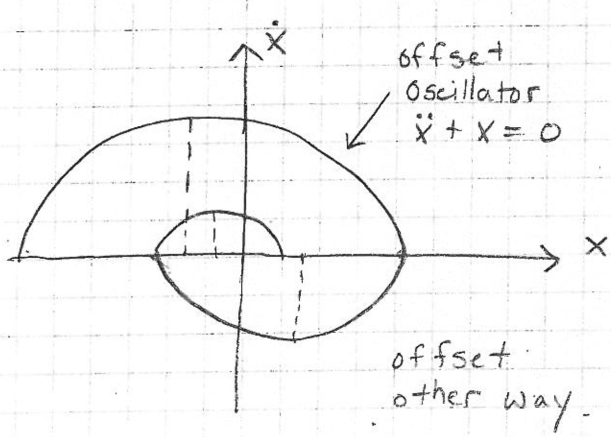


$$F = -\frac{\dot{x}}{|\dot{x}|} \mu m g \qquad \dot{x} \neq 0$$

$$? \qquad x = 0$$

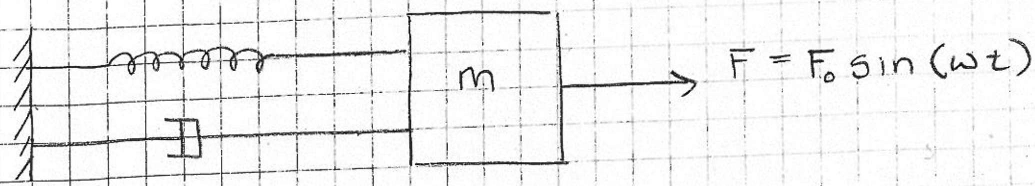


x
straight-line envelop
t



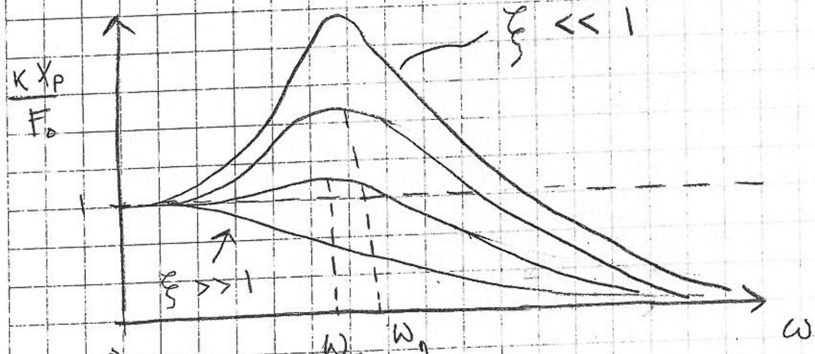
ẋ
offset
Oscillator
ẍ + x = 0
x
offset
other way

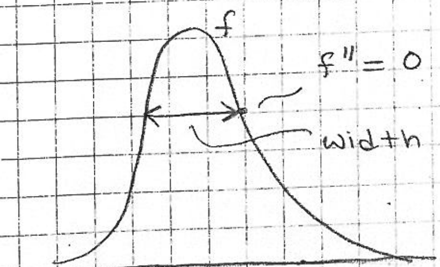
11/ /201

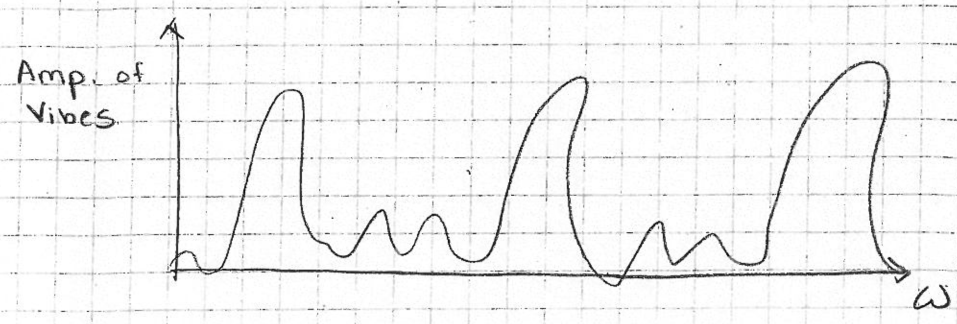


$\omega_n = \sqrt{\frac{k}{m}}$, $\xi = \frac{c}{c_{crit}}$, $c_{rit} = \sqrt{2km}$

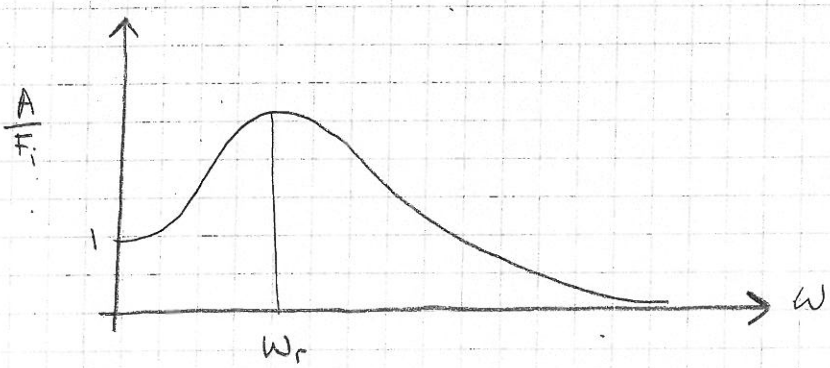
$X = \underbrace{X_n}_{\text{transient}} + \underbrace{X_p}_{\text{steady state}}$







Think of Spectrum as product of 2 functions



$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\uparrow$$

$$\sum F_i \cos(\omega_i t - \delta)$$

Usual problem: F_i are big or have ω_i close to $\omega_n \approx \omega_r$

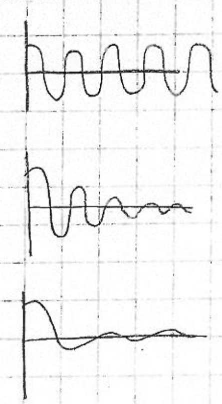
Review of Forcing

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$x = x_h + x_p$$

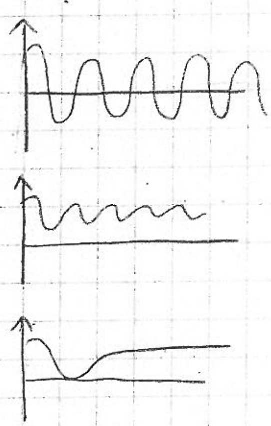
$f(t) =$

- $0 \Longrightarrow x = x_h =$



  $x_p = 0$

- $c \Longrightarrow x_p = \text{const.} = c/k$



$$f(t) = F_0 \sin(\omega t) \Longrightarrow x_p = C \cdot \cos(\omega t - \delta)$$

$$= A \cos(\omega t)$$

$$B \sin(\omega t)$$

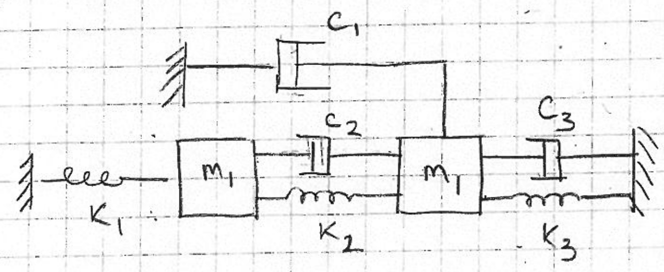
unless: $\omega = \omega_r = \omega_n$

$c = 0$

$f(t) =$ anything $= \sum$ sine waves $\Longrightarrow X_p = \sum$ sine waves

Multi-DoF Vibes

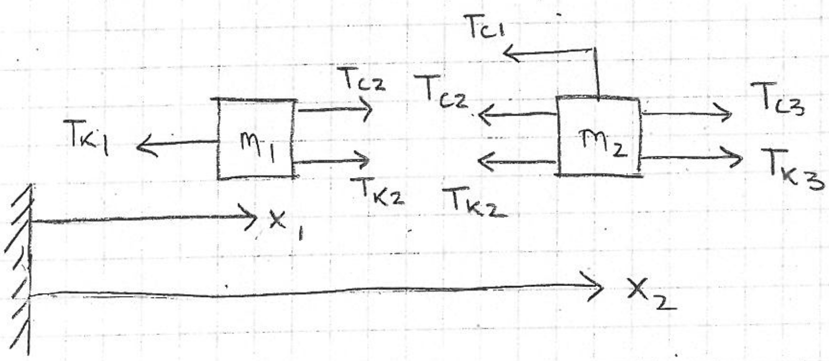
ex)



How to get the right eqs.

1) Put reference point off to right side of pic

2) Tension is always positive



$$T_{c_1} = c_1 \dot{x}_2$$

3) Length changes are always positive

$$T_{c_3} = c_3 \dot{l}_{c3} = c_3(-\dot{x}_2)$$

$$T_{k2} = k_2(l_2 - l_{20}) = k_2((x_2 - x_1) - l_{20})$$

given

$\vdots$

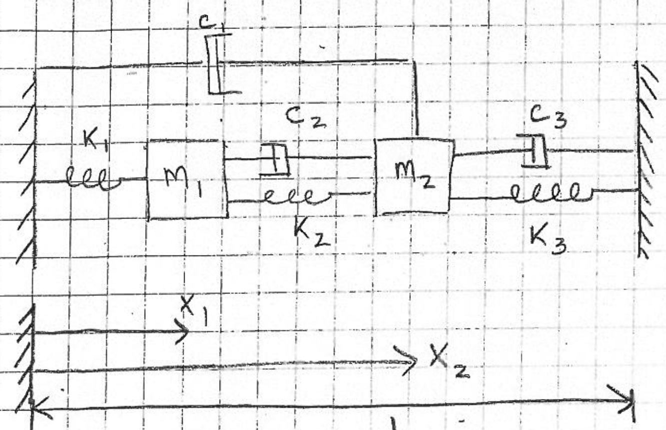
F = ma for each mass

11/12/2014

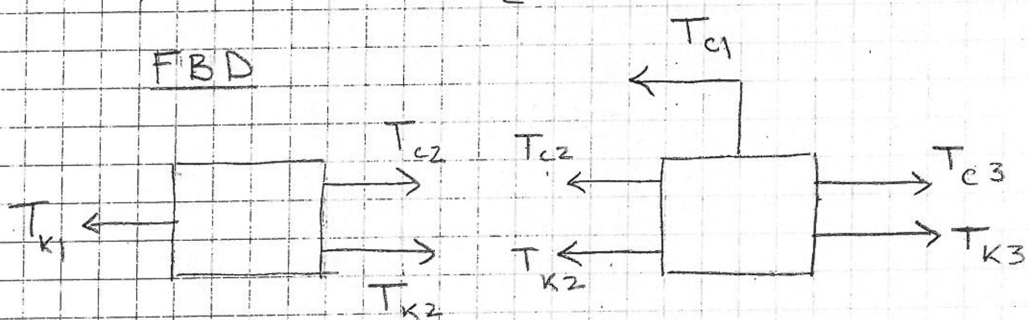
Multi DoF (cont'd)

2^{nd} order matrix form

1^{st} order matrix form



FBD



$$T_{K1} = K_1 \cdot (x_1 - l_{10})$$

$$T_{K2} = K_2((x_2 - x_1) - l_{20})$$

$$T_{c2} = c_2(\dot{x}_2 - \dot{x}_1) \longleftarrow \quad \Delta l = (x_2 - x_1 - l_{20})$$

$$\Delta \dot{l} = (\dot{x}_2 - \dot{x}_1)$$

$$T_{K3} = K_3((L - x_2) - l_{30})$$

$$T_{c3} = c_3(-\dot{x}_2)$$

$$T_{c1} = c_2(\dot{x}_2)$$

LMB

$$m_1 \ddot{x}_1 = T_{c2} + T_{K2} - T_{K1} + F_1$$

$$m_2 \ddot{x}_2 = T_{c3} + T_{K3} - T_{c2} - T_{K2} - T_{c1} + F_2$$

$$m_1 \ddot{x}_1 = -K_1 x_1 - K_2 x_1 + K_2 x_2 - c_2 \dot{x}_1 + c_2 \dot{x}_2$$
$$+ K_1 l_{10} - K_2 l_{20} + F_1$$

$$m_2 \ddot{x}_2 = c_3(-\dot{x}_2) + K_3((L - x_2) - l_{30}) - c_2(\dot{x}_2 - \dot{x}_1)$$
$$- K_2((x_2 - x_1) - l_{20}) - c_2 \dot{x}_2$$
$$= -c_3 \dot{x}_2 + K_3 L - K_3 x_2 - K_3 l_{30} - c_2 \dot{x}_2 + c_2 \dot{x}_1$$
$$- K_2 x_2 + K_2 x_1 + K_2 l_{20} - c_2 \dot{x}_2$$
$$= K_2 x_1 - K_2 x_2 - K_3 x_2 + c_2 \dot{x}_1 - c_3 \dot{x}_2 - c_2 \dot{x}_2$$
$$\vdots$$
$$= K_2 x_1 - K_2 x_2 - K_3 x_2 - c_2 \dot{x}_2 + c_2 \dot{x}_1 - c_1 \dot{x}_2 - c_3 \dot{x}_2$$
$$+ K_2 l_{20} + K_3(L - l_{30}) + F_2$$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\substack{\text{mass} \\ \text{matrix } (M)}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = -\underbrace{\begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_2 + K_3) \end{bmatrix}}_{\text{Stiffness Matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \underbrace{\begin{bmatrix} c_2 & -c_2 \\ -c_2 & (c_1 + c_2 + c_3) \end{bmatrix}}_{\text{Damping Matrix}} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} -K_1 l_{10} - K_2 l_{20} \\ K_2 l_{20} + K_3(L - l_{30}) \end{bmatrix}}_{\text{Forcing}}$$

Notation:

$$\vec{x} = [x], \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

$\vec{r}_i$ for position vectors, if needed

arrow = list of numbers

$$M\ddot{\vec{x}} = -K\vec{x} - C\dot{\vec{x}} + \vec{F}_c + \vec{F}(t)$$

define $\vec{v} = \dot{\vec{x}}$

1st order form:

$$\dot{\vec{v}} = M^{-1}\left(-K\vec{x} - C\dot{\vec{x}} + \vec{F}_c + \vec{F}(t)\right)$$

$$\dot{\vec{x}} = \vec{v} \qquad \text{or} \quad M \backslash [\ldots]$$

Matlab soln.

```
function  zdot = rhs(t, z, p)
    K = p.K;   M = p.M;   C = p.C;   F = p.F;
    n = length(z);
    x = z(1 : n/2);   v = z(n/2+1 : end);

    ẋ = v;
    v̇ = M \ (-K * x - C * v + F);
    ż = [ẋ; v̇];
end
```

1st Order Form

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} 0 & [1] \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right]}_{A} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} [0] \\ [F] \end{bmatrix}$$

11/14/2014

Multi DoF cont'd

$$\dot{X} = Ax + F$$

$$X_p, X_h$$

Damped multi-DoF problem:

$$\underset{n\times n}{M}\underset{n\times 1}{\ddot{x}} + \underset{n\times n}{C}\underset{n\times 1}{\dot{x}} + Kx = \underset{n\times 1}{\tilde{F}(t)}$$

1st order Form

$$\dot{z} = Az + F(t)$$

$$A = \left[\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline M^{-1}K & M^{-1}C \end{array}\right] \qquad z = \left.\begin{bmatrix} x \\ v \end{bmatrix}\begin{matrix} n \\ n \end{matrix}\right\} 2n$$

$$\tilde{F} = \left.\left[\begin{array}{c} 0 \\ \hline M^{-1}F \end{array}\right]\right\} 2n$$

$$X = X_p + X_h$$

no forcing

$$\dot{z} - Az = \tilde{F} = 0$$

$$Z = Z_p + Z_h$$

2nd Order Eqn.

$$F(t) = \underset{\text{const.}}{\underset{\uparrow}{F_c}} + \underset{\text{not const.}}{\underset{\uparrow}{F(t)}}$$

$\vec{F_c} = l_o, L, g$

$\Longrightarrow X_{pc} + X_{P_{interesting}}$

Simple problem: $X_{pc} = ?$

Given M, K, C, F_c, find x_{pc}:

$M\ddot{x} + C\dot{x} + Kx = F_c$

$x = \text{const.}$

$K x_p = F_c$

$x_p = K^{-1} F_c \qquad K \backslash F_c$

From this point on

subtract out x_{pc} and F_c

$x \xleftarrow{\text{replace}} x - x_{pc}$

$X_h = ? \longrightarrow M\ddot{X}_h + C\dot{X}_h + KX_h = 0$

Guess: $\vec{X_h} = \vec{V} e^{\lambda t}$ ← Just 1 homogenous soln.

const. scalar

plug in:

$$\lambda^2 M \vec{V} e^{\lambda t} + \lambda C \vec{V} e^{\lambda t} + K\vec{V} e^{\lambda t} = \vec{O}$$

$$(\lambda^2 M + \lambda C + K)\vec{V} = \vec{O}$$

hard eqn. to solve

$(\lambda^2 M + \lambda C + K)$ has to be singular

$$\det\left[\lambda^2 M + \lambda C + K\right] = 0$$

big polynomial is $\lambda = 0$

Find X_p ?

complex, $\lambda = i\omega$

$$\vec{F}(t) = \vec{F}\, e^{\lambda t}$$

const.

Guess: $X_p = \vec{V}_p\, e^{\lambda t}$

Don't have to find λ (given)

$$\lambda^2 e^{\lambda t} M \vec{V} + \lambda e^{\lambda t} C \vec{V} + e^{\lambda t} K \vec{V} = \vec{F} e^{\lambda t}$$

$$\underbrace{\left(\overset{-\omega^2}{\lambda^2} M + \overset{i\omega}{\lambda} C + K\right)}\vec{V} = \vec{F}$$

$$\vec{V} = \left(\quad \right)^{-1} F$$

complex

Same problem: 1st Order Form

$$\dot{z} - Az = \tilde{F}$$

$\tilde{F} e^{\lambda t}$

const.

$$\dot{z}_p - A z_p = \tilde{F}\, e^{\lambda t}$$

assume: $\vec{z}_p = \vec{V} e^{\lambda t}$

$$e^{\lambda t} \vec{V} - A\vec{V} e^{\lambda t} = \tilde{F} e^{\lambda t}$$

$\uparrow 1\vec{v}$

$$\left[[1] \lambda - A \right] \vec{V} = \tilde{F}$$

$$\vec{V} = \left[[1] \lambda - A \right]^{-1} \tilde{F}$$

$$\vec{z} = \left(\vec{V}_{real} + i \vec{V}_{imaginary} \right) e^{i\omega t}$$

$\uparrow \cos(\omega t) + i \sin(\omega t)$

$z_h = ?$

$\dot{z} = A z$

$\uparrow$ given

Don't Know

guess: $\vec{z} = \vec{v} e^{\lambda t}$

plug in guess:

$$\lambda e^{\lambda t} \vec{V} = e^{\lambda t} A \vec{V}$$

$$\left[A - \lambda I \right] \vec{V} = \vec{0}$$

$\uparrow$ [1]

$\uparrow$ must be singular

if [ ] is singular

$$\Rightarrow \det \left[A - \lambda I \right] = 0$$

char. polynomial $\lambda^{2n} + \underset{\substack{\uparrow \\ \text{trace}(A)}}{} \lambda^{2n-1} + \cdots + \det(A) = 0$

λ is a root $\Rightarrow$ nonzero $\vec{v}$ w/ $[A - \lambda I]\vec{v} = \vec{0}$

General Homogenous Soln.

$$z(t) = \vec{v}_1 e^{\lambda_1 t} + \vec{v}_2 e^{\lambda_2 t} + \cdots C_n \cdots$$

$\vec{v}_i$ and λ_i are eigenvectors

Pick C_i to satisfy I.C.'s $\quad z(0) = C_1 \vec{v}_1 + C_2 \vec{v}_2 + \cdots$

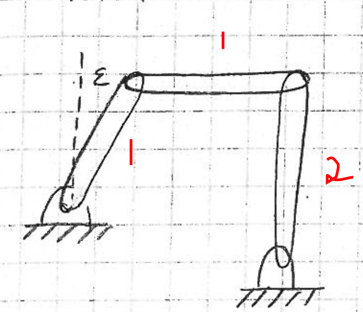
$$\vec{z}_0 = \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \cdots \,\middle|\, \vec{v}_{2n} \right] \begin{bmatrix} C_1 \\ C_2 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \end{bmatrix} = \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \cdots \,\middle|\, \cdots \right]^{-1} \vec{z}_0$$

11/17/2014

Final Project

Animation:



Start from rest

Recall:

$$m\ddot{\vec{x}} + c\dot{\vec{x}} + k\vec{x} = \vec{F}(t)$$

↑ assume

$\vec{F}(t) = \sum$ sine waves

(deal w/ them one at a time)

$\Downarrow$

Sum of solutions, one for each sine wave

$$\tilde{X}_{gen} = \tilde{X}_p + \vec{X}_h$$

Seek "analytic" soln.

one particular sine wave

$$\left\{ m\ddot{\vec{x}}_p + c\dot{\vec{x}}_p + k\vec{x}_p = \vec{V}_0 e^{i\omega t} \right\}$$

Re{ } is still valid soln. $\Rightarrow$ Real + Imag. parts are both Real solns.

Guess: $\vec{x}_p = \vec{V}_p e^{i\omega t}$

↑ const.

$$-\omega^2 \cancel{e^{i\omega t}} \vec{V}_p + i\omega \cancel{e^{i\omega t}} C \vec{V}_p + K \vec{V}_p \cancel{e^{i\omega t}} = \vec{V}_0 \cancel{e^{i\omega t}}$$

$$\boxed{\vec{V}_p = \left(-\omega^2 M + i\omega C + K\right)^{-1} \vec{V}_0}$$

$$\vec{X}_p = \vec{V}_p e^{i\omega t}$$

Homogenous Soln.

Easiest approach $\longrightarrow$ 1st order form

$$\dot{\vec{Z}} = A\vec{Z}$$

$$\vec{Z} = \begin{bmatrix} \vec{X} \\ \vec{V} \end{bmatrix} \qquad A = \left[\begin{array}{c|c} 0 & 1 \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right]$$

Last time:

Guess: $\vec{Z} = \vec{V}_0 e^{\lambda t}$

$\Longrightarrow$ eigen value problem

eigen-values

$$\vec{Z}_h = \sum c_i \vec{V}_i e^{\lambda \cancel{i} t}$$

eigen vectors

(Assume a spanning set of eigen values)

Aside: scalar ODE

$$\dot{X} = aX$$

$$X = Ce^{at}$$

Look for Taylor series Soln.

$$x = x_0\Big|_{t=0} + \dot{x}\Big|_{t=0} t + \ddot{x}\Big|_{t=0} \frac{t^2}{2} + \dots$$

$$x(t) = \underset{\substack{\uparrow \\ x_0}}{x_0} + \underset{\substack{\uparrow \\ \underbrace{\dot{x}_0\big|_{t=0}}_{\text{look at ODE}}}}{\dot{x}_0} t + \ddot{x}_0 \frac{t^2}{2} + \dots$$

$$\dot{x}\Big|_{t=0} = a x\Big|_{t=0}$$

$$\dot{x} = a x$$

$$\ddot{x} = a\dot{x} = a(ax) = a^2 x$$

$$\ddot{x}_0 = a^2 x_0$$

etc.

$$\Longrightarrow x(t) = x_0 + \dot{x}_0 t + \ddot{x}_0 t^2/2 + \dots$$

$$= x_0 + a x_0 t + a^2 x_0 t^2/2 \dots$$

$$= x_0 \left(1 + at + a^2 t^2/2 + a^3 t^3/3! \dots \right)$$

define: $\exp(at) = 1 + at + (at)^2/2 + (at)^3/3! \dots$

Soln. of $\dot{x} = at$ w/ $x(0) = x_0$ is $x(t) = \exp(at) x_0$

Now look at Multi-DoF eqn.

$$\dot{\vec{z}} = A\vec{z}$$

Guess Taylor Series Soln. (term by term in $\vec{z}$)

Same algebra as above

$$\vec{z}(t) = \underbrace{\left(I + At + (At)^2/_2 + \cdots \right)} \vec{z}_0$$

$$\vec{z}(t) = \exp(At)\,\vec{z}_0$$

$\Uparrow$

can use to check HW due Friday

Normal Modes

$$M\ddot{\vec{x}} + \cancel{C\dot{\vec{x}}}^{\,0} + K\vec{x} = \cancel{\vec{F}(t)}^{\,0}$$

$M\ddot{\vec{x}} + K\vec{x} = 0$ ← understand simpler problem first then go back + solve entire one

Approaches:

1) Naive

2) e-vectors w/ $M^{-1}K$

3) e-vectors using $\sqrt{M}$

1) Naive

Governing Equation: $M\ddot{\vec{x}} + K\vec{x} = \vec{0}$

Guess: $\vec{x}(t) = \vec{v}e^{i\omega t}$

$\uparrow$ const.

Plug in guess:

$$-\omega^2 M\vec{v}\,\cancel{e^{i\omega t}} + K\vec{v}\,\cancel{e^{i\omega t}} = 0$$

Good guess only if

$$(-\omega^2 M + K)\vec{v}_0 = \vec{0}$$

Nontrivial solns only if ( ) is singular $\rightarrow$ det( ) = 0

$\Longrightarrow$ polynomial in ω

Roots: $\omega_1, \omega_2, \ldots \omega_n$

Associated vectors: $\vec{V}_1, \vec{V}_2, \ldots \vec{V}_3$

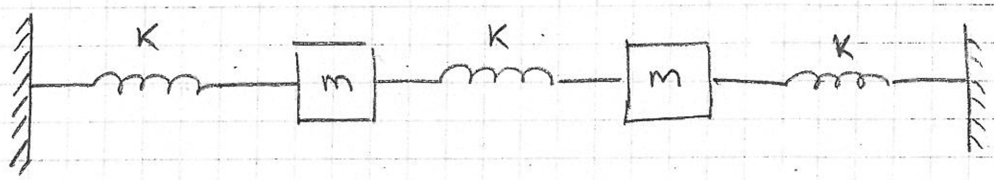
Mode shapes: $\vec{V}_i$

Nat. Frequency: ω_i

gen. soln.: $\vec{X}(t) = \sum C_i \vec{V}_i \, e^{i\omega t}$

const.

ex)



$M\ddot{x} + KX = 0$

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

To find Modes:

Find ω_i

$\det(\omega^2 M + K) = 0$

skip Algebra $\rightarrow$ $\omega_1^2 = 3\ K/m$

$\omega_2^2 = K/m$

From roots, calculate e-vectors

solve lin. eqs $(-\omega_i^2 M + K)\vec{V}_i = \vec{0}$

$\vec{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{X}(t) = \underbrace{C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{\sqrt{\frac{k}{m}}\, t\, i} + C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\sqrt{\frac{3k}{m}}\, t\, i}}_{\text{sum of 2 normal modes}}$$