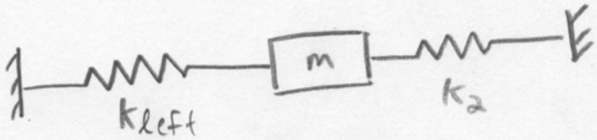
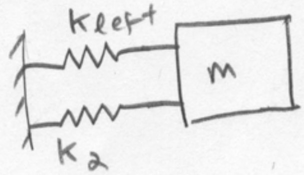
HW1 SOLUTIONS

(1) $k_{BEAM} = k_B = \dfrac{AE}{L}$

k_{left} for k_B & k_1

$$k_{left} = \frac{1}{\left(\frac{1}{k_B} + \frac{1}{k_1}\right)} = \frac{k_1 k_B}{k_1 + k_B}$$

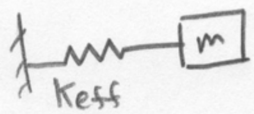




k_{left} & k_2 act in parallel

$$k_{eff} = k_{left} + k_2 = k_2 + \frac{k_1 AE}{L\left(k_1 + \frac{AE}{L}\right)}$$

$$k_{eff} = k_2 + \frac{k_1 AE}{Lk_1 + AE}$$



Sys Eqn

$$m\ddot{x} + k_{eff} x = 0$$

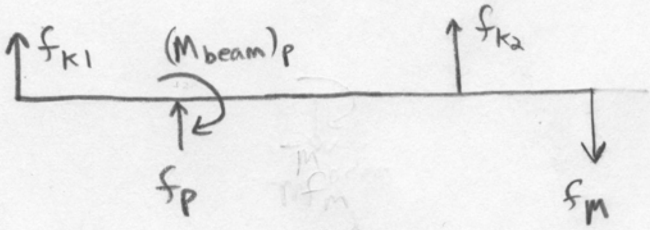
$$\ddot{x} + \frac{k_{eff}}{m} x = 0$$

$$\omega_n = \sqrt{\frac{k_{eff}}{m}}$$

$$\boxed{\omega_n = \sqrt{\frac{\left(k_2 + \frac{k_1 AE}{Lk_1 + AE}\right)}{m}}}$$

small ∡ approx

② ⓐ Draw system as



$$f_{k_1} = k a \theta$$

$$(M_{beam})_p = (I_{beam})_p \ddot{\theta}$$

$$f_{k2} = -k \cdot 2a \cdot \theta$$

$$f_M = M\, 3a\, \ddot{\theta}$$

Positive θ CCW ↺

$$\sum M_p = 0 = -f_{k_1} \cdot a - (M_{beam})_p + f_{k2} \cdot 2a - f_m \cdot 3a$$

$$0 = -k\cancel{a}^2\theta - \frac{7}{3} m \cancel{a}^2 \ddot{\theta} - k \cdot 4\cancel{a}^2 \cdot \theta - 9\cancel{a}^2 M \ddot{\theta}$$

$$\left(\frac{7}{3} m + 9M\right)\ddot{\theta} + (5k)\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{5k}{\frac{7}{3}m + 9M}\theta = 0}$$

about CM of beam

$$(I_{beam})_p = \underbrace{(I_{beam})_{cm} + m a^2}_{\text{parallel axis theorem}}$$

$$(I_{beam})_{cm} = \frac{1}{12} m \cdot (4a)^2$$

$$(I_b)_p = (I_{beam})_p = \frac{7}{3} m a^2$$

ⓑ

	T	V
beam	$\frac{1}{2}(I_b)_p \dot{\theta}^2$	0
left k	0	$\frac{1}{2}k(a\theta)^2$
right k	0	$\frac{1}{2}k(2a\theta)^2$
mass M	$\frac{1}{2}M(3a\dot{\theta})^2$	0

$$L = T - V \qquad L = \frac{1}{2}(I_b)_p \dot{\theta}^2 + \frac{9}{2}a^2 M \dot{\theta}^2 - \frac{1}{2}k a^2 \theta^2 - \frac{1}{2}k\, 4a^2\theta^2$$

$$L = \left(\frac{1}{2}(I_b)_p + \frac{9}{2}a^2 M\right)\dot{\theta}^2 - \left(\frac{5}{2}k a^2\right)\theta^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left[2\left(\tfrac{1}{2}(I_b)_p + \tfrac{9}{2}a^2 M\right)\dot\theta\right] + 5ka^2\theta = 0$$

$$\left((I_b)_p + 9a^2 M\right)\ddot\theta + 5ka^2\theta = 0$$

$$\left(\tfrac{7}{3}m\cancel{a^2} + 9M\cancel{a^2}\right)\ddot\theta + 5k\cancel{a^2}\theta = 0$$

$$\boxed{\ddot\theta + \frac{5k}{\left(\frac{7}{3}m + 9M\right)}\theta = 0}$$

Note! For KE of beam, can alternatively use

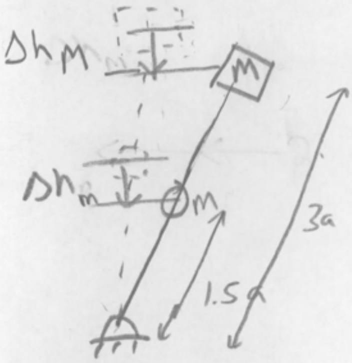
$$T_{beam} = \frac{1}{2}(I_{beam})_{cm}\dot\theta^2 + \frac{1}{2}m(a\dot\theta)^2$$

as K.E. rigid body $= \frac{1}{2}I_{cm}\dot\theta^2 + \frac{1}{2}mV_{cm}^2$

Keep in mind both rotational & linear motion

③ Use small ∡ approx. Do not neglect gravity.

$I_{beam} = \frac{1}{12} m (3a)^2$



	T	V
M	$\frac{1}{2} M (3a\dot{\theta})^2$	$-Mg3a\frac{\theta^2}{2}$
beam $V_{cm} = \frac{3}{2}a\dot{\theta}$	$\frac{1}{2} I_{beam}\dot{\theta}^2 + \frac{1}{2} m V_{cm}^2$	$-mg\frac{3}{2}a\frac{\theta^2}{2}$
Springs	0	$2 \cdot \frac{1}{2} k (2a\theta)^2$

PE loss by rotation of beam by ∡ θ

Mass M: $M \cdot g \cdot \Delta h_M$

beam: $m \cdot g \cdot \Delta h_m$

$\Delta h_M = (3 \cdot a) \cdot (1 - \cos\theta)$

$\Delta h_m = (\frac{3}{2} a) \cdot (1 - \cos\theta)$

small ∡ → $\cos\theta \simeq 1 - \frac{\theta^2}{2}$

$\Delta h_M = 3a\frac{\theta^2}{2}$ $\quad \Delta h_m = \frac{3}{2} a \frac{\theta^2}{2}$

$L = T - V$

$$L = \frac{9}{2} M a^2 \dot{\theta}^2 + \frac{9}{24} m a^2 \dot{\theta}^2 + \frac{1}{2} m \left(\frac{9}{4}\right) a^2 \dot{\theta}^2 + Mg3a\frac{\theta^2}{2} + mg\frac{3}{2}a\frac{\theta^2}{2} - 4ka^2\theta^2$$

Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$0 = \frac{d}{dt}\left[\left(9Ma^{\cancel{2}} + \frac{3}{4} ma^{\cancel{2}} + \frac{9}{4} ma^{\cancel{2}}\right)\dot{\theta}\right] - \left(Mg3\cancel{a} + mg\frac{3}{2}\cancel{a} - 8ka^{\cancel{2}}\right)\theta$$

$$0 = (9Ma + 3ma)\ddot{\theta} + \left(-Mg3 - mg\frac{3}{2} + 8ka\right)\theta$$

Effective Stiffness is coefficient in front of θ

(as in $m\ddot{x} + kx = 0$)

System is unstable (by divergence) if effective stiffness < 0

Stability:

$$8ka - \left(3Mg + \frac{3}{2}mg\right) > 0$$

$$8ka > 3Mg + \frac{3}{2}mg$$

$$\boxed{k > \frac{\left(3Mg + \frac{3}{2}mg\right)}{8a}}$$

In order to see how the $x = c_1 \cos \omega t + c_2 \sin \omega t$ solution agrees with the $x = 1 + t$ solution in the limit as ω approaches zero, we observe that the initial conditions associated with $x = 1 + t$ are:

$$\begin{cases} x(0) = 1 \\ x'(0) = 1 \end{cases}$$

applying these initial conditions to $x = c_1 \cos \omega t + c_2 \sin \omega t$, we get:

$$\begin{cases} x(0) = 1 \\ x'(0) = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ \omega c_2 = 1 \Rightarrow c_2 = \frac{1}{\omega} \end{cases}$$

so the solution becomes:

$$x = \cos \omega t + \frac{1}{\omega} \sin \omega t$$

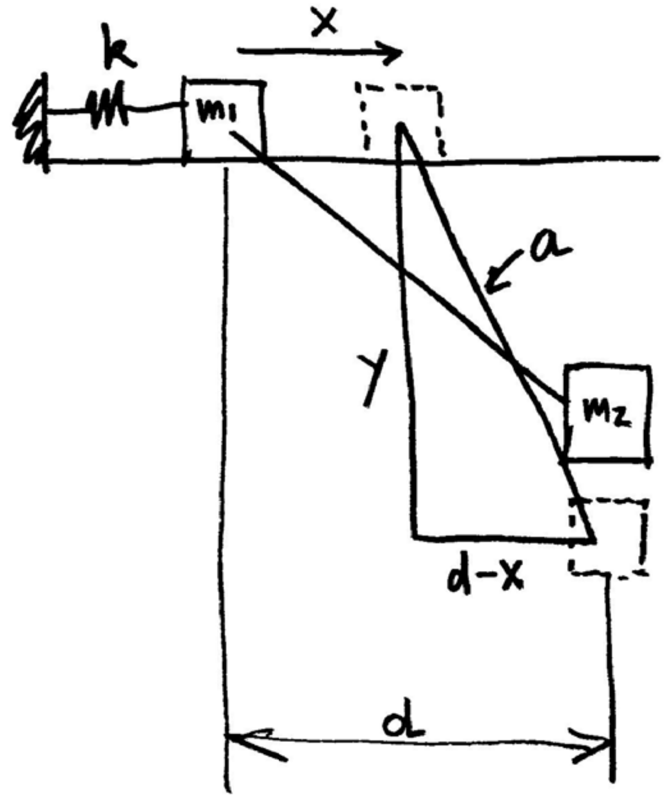
now, we take the limit of this solution as $\omega \to 0$:

$$\lim_{\omega \to 0} x = \lim_{\omega \to 0} \cos \omega t + \lim_{\omega \to 0} \frac{1}{\omega} \sin \omega t = 1 + \lim_{\omega \to 0} \frac{t \cos \omega t}{1} = 1 + t$$

where we use l'hopital's rule to evaluate the second limit.

In summary, since one of the arbitrary constants actually depends on ω, it is essential to account for it when taking the limit $\omega \to 0$.

#1.



$$y^2 + (d-x)^2 = a^2$$

$$y = \sqrt{a^2 - (d-x)^2}$$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 , \quad V = \frac{1}{2} k x^2$$

$$2y\dot{y} - 2(d-x)\dot{x} = 0 \Rightarrow \dot{y} = \frac{d-x}{\sqrt{a^2-(d-x)^2}} \dot{x}$$

$$\therefore T = \frac{1}{2} f(x) \dot{x}^2, \quad f(x) = m_1 + \frac{(d-x)^2 m_2}{a^2-(d-x)^2}$$

$$\mathcal{L} = T - V = \frac{1}{2} f(x) \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2} f' \dot{x}^2 - kx$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = f\dot{x} \quad , \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = f\ddot{x} + f'\dot{x}^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow f\ddot{x} + f'\dot{x}^2 - \frac{f'\dot{x}^2}{2} + kx$$

#1 continued

So Lagrange's equation is

$$\boxed{f\ddot{x} + f'\frac{\dot{x}^2}{2} + kx = 0}$$ Ans. to part a

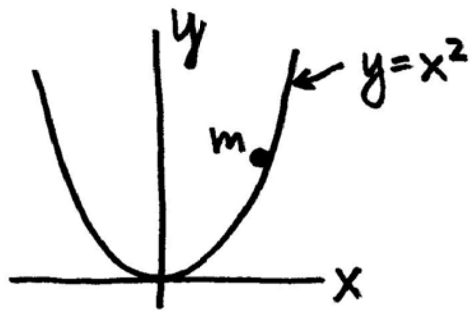
where $f(x) = m_1 + \dfrac{(d-x)^2 m_2}{a^2 - (d-x)^2}$

b) To find natural frequency, keep only linear terms.

This gives $f(0)\ddot{x} + kx = 0$

$$\boxed{\omega^2 = \frac{k}{f(0)} = \frac{k}{\dfrac{d^2 m_2}{a^2 - d^2} + m_1}}$$

#2



$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, $V = mgy$

$\dot{y} = 2x\dot{x}$

$T = \frac{1}{2}m(1+4x^2)\dot{x}^2$, $V = mgx^2$

$\mathcal{L} = T - V = \frac{1}{2}m(1+4x^2)\dot{x}^2 - mgx^2$

$\mathcal{L} = \frac{1}{2}f(x)\dot{x}^2 - mgx^2$, $f(x) = m(1+4x^2)$

Similar to problem #1,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow f\ddot{x} + f'\frac{\dot{x}^2}{2} + 2mgx = 0$$

where $f(x) = m(1+4x^2)$

linearize around $x=0$:

$$f(0)\ddot{x} + 2mgx = 0, \quad f(0) = m$$

$$\ddot{x} + 2gx = 0$$

$$\omega^2 = 2g, \quad \boxed{\omega = \sqrt{2g}}$$

#3. $x_p = R\cos(\Omega t - \phi)$

$\dot{x}_p = -R\Omega \sin(\Omega t - \phi)$

Amplitude of velocity $= R\Omega$

from class notes, $\dfrac{R}{\delta_{st}} = \dfrac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2 + 4\left(\frac{n}{\omega}\right)^2\left(\frac{\Omega}{\omega}\right)^2}}$

$$\therefore \quad \frac{R\Omega}{\delta_{st}} = \frac{\Omega}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2 + 4\left(\frac{n}{\omega}\right)^2\left(\frac{\Omega}{\omega}\right)^2}}$$

dimensionless amplitude of velocity $= \dfrac{R}{\delta_{st}}\dfrac{\Omega}{\omega} = \dfrac{\Omega/\omega}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2 + 4\left(\frac{n}{\omega}\right)^2\left(\frac{\Omega}{\omega}\right)^2}}$

$$= \frac{q}{\sqrt{(1-q^2)^2 + 4p^2q^2}}$$

Set $\frac{d}{dq}\left(\frac{q}{\sqrt{(1-q^2)^2+4\beta^2q^2}}\right) = 0$ for max

Obtain

$$\frac{q^4-1}{\left((1-q^2)^2+4\beta^2q^2\right)^{3/2}} = 0 \Rightarrow q=1$$

$\therefore$ Max velocity amplitude occurs at

$$\boxed{\Omega=\omega}$$

#4.

$$\dot\theta = \sqrt{\frac{2mgR}{mR^2+I_o}}\,\sqrt{\theta}\,, \quad \theta(0)=0$$

has two exact solutions:

One of them is $\theta \equiv 0$ as stated.
The other is obtained by separation of variables

$$\frac{d\theta}{\sqrt{\theta}} = \sqrt{\cdots}\; dt$$

$$2\theta^{1/2} = \sqrt{\cdots}\; t + \cancel{C} \quad (C=0 \text{ for } \theta(0)=0)$$

$$\therefore \theta(t) = \frac{1}{2}\sqrt{\cdots}\; t^2$$

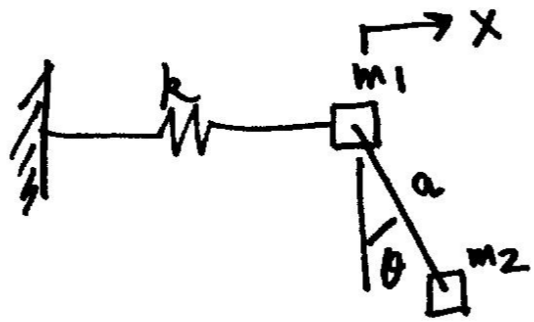
#4 continued

Comment: An equation with two (or more) Solutions is <u>bad news</u>: how do you know if the solution you have found is the right one for your application?

That is why the mathematicians have given us a <u>uniqueness proof</u> which gives conditions on the right hand side of the differential equation ($f(\theta)$)

$$\frac{d\theta}{dt} = f(\theta)$$

which guarantee that there is only one solution.



$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 V_2^2$$

$$\bar{r}_2 = x\,\hat{e}_x + a\sin\theta\,\hat{e}_x + a\cos\theta\,\hat{e}_y$$

$$\bar{V}_2 = (\dot{x} + a\dot{\theta}\cos\theta, -a\dot{\theta}\sin\theta)$$

$$\bar{V}_2^2 = (\dot{x} + a\dot{\theta}\cos\theta)^2 + a^2\dot{\theta}^2\sin^2\theta$$

$$= \dot{x}^2 + 2a\dot{x}\dot{\theta}\cos\theta + a^2\dot{\theta}^2$$

Let $m_1 = M$

$m_2 = m$

$$V = \frac{1}{2}kx^2 - amg\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m_1\dot{x} + m_2(\dot{x} + a\dot{\theta}\cos\theta)$$

$$\frac{d}{dt}\,'' \quad \boxed{(m_1+m_2)\ddot{x} + m_2(a)(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + kx = 0}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -amg\sin\theta - a\dot{x}\dot{\theta}\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (a\dot{x}\cos\theta + a^2\dot{\theta})m_2$$

$$\frac{d}{dt}\,'' = (a\ddot{x}\cos\theta - a\dot{x}\dot{\theta}\sin\theta + a^2\ddot{\theta})m_2$$

$$\boxed{m_2(a\ddot{x}\cos\theta + a^2\ddot{\theta}) + amg\sin\theta = 0}$$

$$(m+M)\ddot{x} + ma\ddot{\theta} + kx = 0$$

$$\ddot{x} + a\ddot{\theta} + g\theta = 0$$

(b) Linearize about $x = \theta = 0$ $\cos\theta \to 1$ $\sin\theta \to \theta$ $\dot{\theta}^2 \to 0$ $\dot{x}\dot{\theta} \to 0$

1) $\boxed{(M+m)\ddot{x} + ma\ddot{\theta} + kx = 0}$

2) $\boxed{ma\ddot{x} + ma^2\ddot{\theta} + mga\theta = 0}$

(c) $M\ddot{z} + Kz = 0$ $z = \begin{bmatrix} x \\ \theta \end{bmatrix}$ $\ddot{z} = \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}$

$$M = \begin{bmatrix} (M+m) & ma \\ ma & ma^2 \end{bmatrix} \qquad K = \begin{bmatrix} k & 0 \\ 0 & mga \end{bmatrix}$$

(d) $$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

frequencies ω_1, ω_2

$$\det[K - \omega^2 M] = 0$$

$$\det\begin{bmatrix} 1-2\omega^2 & -\omega^2 \\ -\omega^2 & 1-\omega^2 \end{bmatrix} = 0$$

$$(1-2\omega^2)(1-\omega^2) - \omega^4 = 0$$

$$1 - 3\omega^2 + 2\omega^4 - \omega^4 = 0$$

$$(\omega^2)^2 - 3(\omega^2) + 1 = 0$$

$$\omega^2 = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\omega_1^2 = \frac{3-\sqrt{5}}{2} \qquad \boxed{\omega_1 = 0.6180}$$

$$\omega_2^2 = \frac{3+\sqrt{5}}{2} \qquad \boxed{\omega_2 = 1.6180}$$

Modal vectors ${}_1z$ ${}_2z$

$$M^{-1}Kz = \omega^2 z$$

Solve by hand or in MATLAB

$$\boxed{{}_1z = \begin{bmatrix} -0.5257 \\ -0.3249 \end{bmatrix} \quad {}_2z = \begin{bmatrix} -0.8507 \\ 1.3764 \end{bmatrix}}$$

MATLAB: `[v,d]=eig(K,M)`

$$v = \begin{bmatrix} {}_1z & {}_2z \end{bmatrix}$$

$$d = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

1 ⓔ Orthogonality condition

$X_j^T M X_i = 0$ $\quad X_1 = [{}_1z \;\; {}_1z]$ $\quad X_2 = [{}_2z \;\; {}_2z]$

$X_j^T K X_i = 0$

Solving in MATLAB, all combinations of above equations yield zero.

ⓕ $R = [{}_1z \;\; {}_2z]$

$$R^T M R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

diagonal

ⓖ x, θ as linear combinations of principal coordinates p_1, p_2

principal coordinates $p = P^{-1} z$ (pg 181)

Modal matrix $P = [{}_1z \;\; {}_2z]$ $\quad z = Pp$ $\quad z = \begin{bmatrix} x \\ \theta \end{bmatrix}$

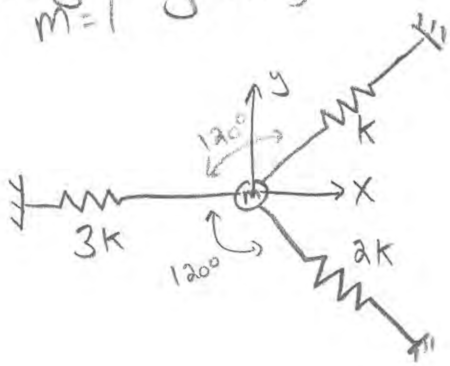
$x = -0.5257\, p_1 - 0.8507\, p_2$

$\theta = -0.3249\, p_1 + 1.3764\, p_2$

ⓗ $R^T M R\, \ddot{p} + R^T K R\, p = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{p} + \begin{bmatrix} 0.3820 & 0 \\ 0 & 2.6180 \end{bmatrix} p = 0$$

(2) Neglect gravity
$m = 1$

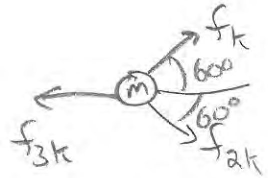


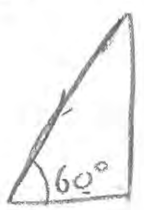
2 approaches

1) Model changing angles in creating EOM

2) Assume small vibrations in generation of EOM because M, K matrices will be linearized about small vibrations in calculation of ω_1, ω_2.

Approach 2





Component	T KE	V PE
m	$\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$	—
3k	—	$\frac{1}{2}3kx^2$
2k	—	$\frac{1}{2}2k(\Delta l_{2k})^2$
k	—	$\frac{1}{2}k(\Delta l_k)^2$

Compression of k

$$\Delta l = [\cos 60, \sin 60]\cdot[x, y]$$

$$\Delta l_k = x\cos\frac{\pi}{3} + y\sin\frac{\pi}{3}$$

Compression of 2k

$$\Delta l_{2k} = x\underbrace{\cos\frac{\pi}{3}}_{0.5} - y\underbrace{\sin\frac{\pi}{3}}_{\sqrt{3}/2}$$

$$L = T - V$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{3}{2}kx^2 - k\Delta l_{2k}^2 - \frac{1}{2}k\Delta l_k^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad (2)$$

$$\Delta l_k^2 = \frac{1}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2$$

$$\Delta l_{2k}^2 = \frac{1}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2$$

(1) $m\ddot{x} + 3kx + k\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right) + \frac{1}{2}k\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right) = 0$

$m\ddot{x} + x\left(3k + \frac{1}{2}k + \frac{1}{4}k\right) + y\left(-\frac{\sqrt{3}}{2}k + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}k\right)$

$\rightarrow\ m\ddot{x} + \frac{15}{4}kx - \frac{\sqrt{3}}{4}ky = 0$

(2) $m\ddot{y} + k\left(-\frac{\sqrt{3}}{2}x + \frac{3}{2}y\right) + \frac{1}{2}k\left(\frac{\sqrt{3}}{2}x + \frac{3}{2}y\right) = 0$

$\rightarrow\ m\ddot{y} - \frac{\sqrt{3}}{4}kx + \frac{9}{4}ky = 0$

$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad K = \begin{bmatrix} 15/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 9/4 \end{bmatrix}$ (setting $k=1$)

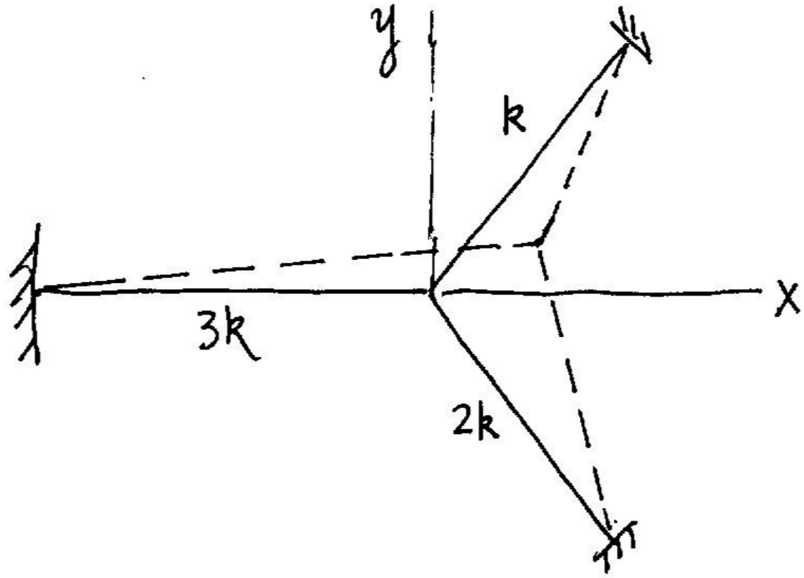
(setting $m=1$)

MATLAB $[v,d] = \text{eig}(K,M)$

$d = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$

$$\boxed{\begin{aligned} \omega_1 &= 1.4608 \\ \omega_2 &= 1.9662 \end{aligned}}$$

2.



$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

Lagrange's Eqs give

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

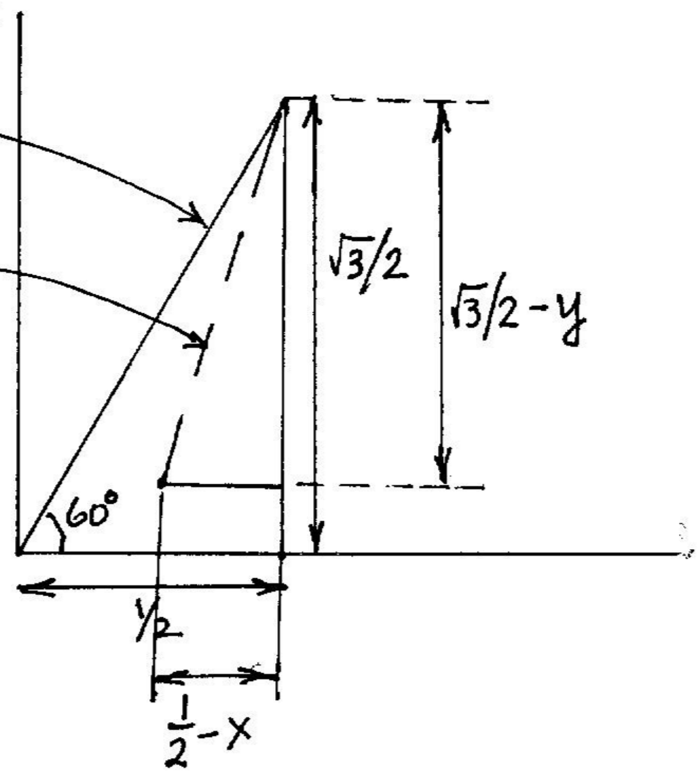
$$m\ddot{y} = -\frac{\partial V}{\partial y}$$

Blow up view:

original length = 1

deformed length =

$\sqrt{\left(\frac{1}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}-y\right)^2}$



$$V = \frac{1}{2} k \left(\sqrt{\left(\frac{1}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}-y\right)^2} - 1 \right)^2$$

$$+ \frac{1}{2}(2k) \left(\sqrt{\left(\frac{1}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}+y\right)^2} - 1 \right)^2$$

$$+ \frac{1}{2}(3k) \left(\sqrt{(1+x)^2 + y^2} - 1 \right)^2$$

Expand V using the Taylor series for $\sqrt{1+z}$

$$\sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \cdots$$

$$\sqrt{\left(\tfrac{1}{2}-x\right)^2 + \left(\tfrac{\sqrt{3}}{2}-y\right)^2} = \sqrt{\tfrac{1}{4} - x + x^2 + \tfrac{3}{4} - \sqrt{3}y + y^2}$$

$$= \sqrt{1 + \left(-x + x^2 - \sqrt{3}y + y^2\right)}$$

$$= 1 + \frac{\left(-x + x^2 - \sqrt{3}y + y^2\right)}{2}$$

$$- \frac{\left(-x + \cancel{x^2} - \sqrt{3}\,y + \cancel{y^2}\right)^2}{8} + \cdots$$

$$= 1 - \frac{x}{2} - \sqrt{3}\frac{y}{2} + \frac{x^2}{2} + \frac{y^2}{2}$$

$$- \frac{1}{8}\left(x^2 + 3y^2 + 2\sqrt{3}\,xy\right) \cancel{+\cdots}$$

$$= \underbrace{1 - \frac{x}{2} - \frac{\sqrt{3}}{2}y + \frac{3}{8}x^2 + \frac{1}{8}y^2 - \frac{\sqrt{3}}{4}xy}$$

$$\left(\sqrt{\left(\tfrac{1}{2}-x\right)^2 + \left(\tfrac{\sqrt{3}}{2}-y\right)^2} - 1\right)^2 = \left(\tfrac{1}{2}-x\right)^2 + \left(\tfrac{\sqrt{3}}{2}-y\right)^2 + 1 - 2\left(\quad\right)$$

$$= \frac{1}{4} - x + x^2 + \frac{3}{4} - \sqrt{3}\,y + y^2 + 1$$

$$- 2 + x + \sqrt{3}\,y - \frac{3}{4}x^2 - \frac{1}{4}y^2 + \frac{\sqrt{3}}{2}xy$$

$$= x^2 + y^2 - \frac{3}{4}x^2 - \frac{1}{4}y^2 + \frac{\sqrt{3}}{2}xy$$

$$= \frac{1}{4}x^2 + \frac{3}{4}y^2 + \frac{\sqrt{3}}{2}xy$$

$$\sqrt{(1+x)^2+y^2} = \sqrt{1+2x+x^2+y^2}$$

$$= 1+\frac{1}{2}\left(2x+x^2+y^2\right) - \frac{1}{8}\left(2x+\cancel{x^2}+\cancel{y^2}\right)^2+\cancel{\cdots}$$

$$= 1+x+\frac{x^2}{2}+\frac{y^2}{2}-\frac{x^2}{2}$$

$$= 1+x+\frac{y^2}{2}$$

$$\left(\sqrt{(1+x)^2+y^2}-1\right)^2 = (1+x)^2+y^2+1-2\left(1+x+\frac{y^2}{2}\right)$$

$$= 1+2x+x^2+y^2+1-2-2x-y^2$$

$$= x^2$$

$$V = \frac{1}{2}k\left(\frac{x^2}{4}+\frac{3}{4}y^2+\frac{\sqrt{3}}{2}xy\right)$$

$$+\frac{1}{2}(2k)\left(\frac{x^2}{4}+\frac{3}{4}y^2-\frac{\sqrt{3}}{2}xy\right)$$

$$+\frac{1}{2}(3k)\,x^2$$

$$V = \frac{1}{2}k\left[x^2\left(\frac{1}{4}+\frac{2}{4}+3\right)+y^2\left(\frac{3}{4}+2\cdot\frac{3}{4}\right)+\sqrt{3}\,xy\left(\frac{1}{2}-1\right)\right]$$

$$= \frac{1}{2}k\left[\frac{15}{4}x^2+\frac{9}{4}y^2-\frac{\sqrt{3}}{2}xy\right]$$

$$m\ddot{x} = -\frac{\partial V}{\partial x} = -\frac{15}{4}kx+\frac{\sqrt{3}}{4}yk$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -\frac{9}{4}ky+\frac{\sqrt{3}}{4}xk$$

$$M\ddot{x} + Kx = 0$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \frac{15}{4}k & -\frac{\sqrt{3}}{4}k \\ -\frac{\sqrt{3}}{4}k & \frac{9}{4}k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = A\cos\omega t$$
$$y = B\cos\omega t$$

$$\begin{vmatrix} -\omega^2 m + \frac{15}{4}k & -\frac{\sqrt{3}}{4}k \\ -\frac{\sqrt{3}}{4}k & -\omega^2 m + \frac{9}{4}k \end{vmatrix} = 0$$

$$\left(-\beta^2 + \frac{15}{4}\right)\left(-\beta^2 + \frac{9}{4}\right) - \frac{3}{16} = 0 \quad \text{where } \beta = \omega\sqrt{\frac{m}{k}}$$

$$\beta^4 - 6\beta^2 + \frac{33}{4} = 0$$

$$\beta^2 = \frac{6 \pm \sqrt{36-33}}{2} = 3 \pm \frac{\sqrt{3}}{2}$$

$$\omega = \sqrt{3 \pm \frac{\sqrt{3}}{2}}\sqrt{\frac{k}{m}} = 1.461\sqrt{\frac{k}{m}},\ 1.966\sqrt{\frac{k}{m}}$$

3. The general motion of the first coordinate of a two degree of freedom system is given by:

$$x_1(t) = R_1 \cos(\omega_1 t - \theta_1) + R_2 \cos(\omega_2 t - \theta_2)$$

Is this a periodic motion? Under what condition will it be periodic?

− −

At $t = 0$,

$$x_1(0) = R_1 \cos(\theta_1) + R_2 \cos(\theta_2)$$

At what time t will this happen again?
Suppose that $\omega_2 = \frac{m}{n}\omega_1$, where m and n are whole numbers. Then

$$x_1(t) = R_1 \cos(\omega_1 t - \theta_1) + R_2 \cos(\frac{m}{n}\omega_1 t - \theta_2)$$

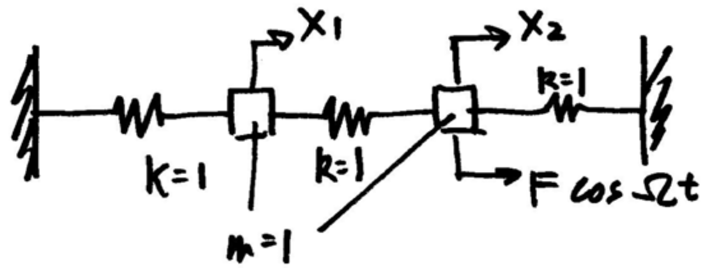
After time $T = \frac{2\pi n}{\omega_1}$, we have

$$x_1(T) = R_1 \cos(2\pi n - \theta_1) + R_2 \cos(2\pi m - \theta_2) = x_1(0)$$

In fact, $x_1(T + t) = x_1(t)$ for all t, not just $t = 0$. Thus in this case the motion is periodic.

However, if the ratio of ω_2 to ω_1 is an irrational number, then $x_1(t)$ will never return to $x_1(0)$ and the motion will not be periodic.

1



$$T = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2)$$

$$V = \frac{1}{2}(x_1^2 + x_2^2 + (x_1 - x_2)^2)$$

$$= x_1^2 + x_2^2 - x_1 x_2$$

$$\delta W_2 = F\cos\Omega t \, \delta x_2 \Rightarrow Q_2 = F\cos\Omega t$$

$$\delta W_1 = 0 \Rightarrow Q_1 = 0$$

$$\ddot{x}_1 + 2x_1 - x_2 = 0$$

$$\ddot{x}_2 - x_1 + 2x_2 = F\cos\Omega t$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ F\cos\Omega t \end{pmatrix}$$

$$M\ddot{x} + Kx = f(t)$$

let $x = X\cos\omega t$ for $f(t) = 0$

$$(-\omega^2 I + K)X = 0$$

$$\begin{vmatrix} -\omega^2+2 & -1 \\ -1 & -\omega^2+2 \end{vmatrix} = 0, \quad (-\omega^2+2)^2 = 1$$

$$-\omega^2 + 2 = \pm 1$$

$$\omega^2 = 2 \mp 1 = 3, 1$$

$$\omega_1 = 1, \quad \begin{bmatrix} -1+2 & -1 \\ -1 & -1+2 \end{bmatrix} {}_1X = 0 \Rightarrow {}_1X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2 = \sqrt{3}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} {}_2X = 0 \Rightarrow {}_2X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Set $x = Rp$, $\quad R = [{}_1\mathbb{X}, {}_2\mathbb{X}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$x_1 = p_1 + p_2$

$x_2 = p_1 - p_2$

$$\underbrace{R^t M R}_{R^t I R} \ddot{p} + \underbrace{R^t K R}\, p = R^t f(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ F \end{bmatrix} \cos \Omega t$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$\ddot{p}_1 + \omega_1^2 p_1 = \frac{F}{2} \cos \Omega t$, $\quad \omega_1 = 1$

$\ddot{p}_2 + \omega_2^2 p_2 = -\frac{F}{2} \cos \Omega t$, $\quad \omega_2 = \sqrt{3}$

$p_1 = K \cos \Omega t$, $\quad (-\Omega^2 + \omega_1^2) K = F$

So $\quad p_1 = \dfrac{F/2}{1-\Omega^2} \cos \Omega t$

Similarly $\quad p_2 = \dfrac{-F/2}{3-\Omega^2} \cos \Omega t$

$$\therefore \quad x_1 = p_1 + p_2 = \left(\frac{1}{1-\Omega^2} - \frac{1}{3-\Omega^2} \right) \frac{F}{2} \cos \Omega t$$

$$= \frac{F}{(1-\Omega^2)(3-\Omega^2)} \cos \Omega t$$

$$x_2 = p_1 - p_2 = \left(\frac{1}{1-\Omega^2} + \frac{1}{3-\Omega^2} \right) \frac{F}{2} \cos \Omega t$$

$$= \frac{(2-\Omega^2) F}{(1-\Omega^2)(3-\Omega^2)} \cos \Omega t$$

(2)

$$\ddot{x}_1 + 2x_1 - x_2 = 0$$

$$\ddot{x}_2 - x_1 + 2x_2 = F\cos\Omega t$$

Set $x_1 = A\cos\Omega t$
$x_2 = B\cos\Omega t$

$$\left.\begin{array}{l} -\Omega^2 A + 2A - B = 0 \\ -\Omega^2 B - A + 2B = F \end{array}\right\} \text{Solve for } A, B$$

$$\Rightarrow \quad A = \frac{F}{(1-\Omega^2)(3-\Omega^2)}, \quad B = \frac{F(2-\Omega^2)}{(1-\Omega^2)(3-\Omega^2)}$$

Agrees with (1)

SOLUTION to question 3:

Multiply the first eq. in (7) by $-\Omega^2$ and add to the second eq. in (7) giving:

$$R^t(-\Omega^2 M + K)R = -\Omega^2 D_1 + D_2$$

Take the inverse of both sides:

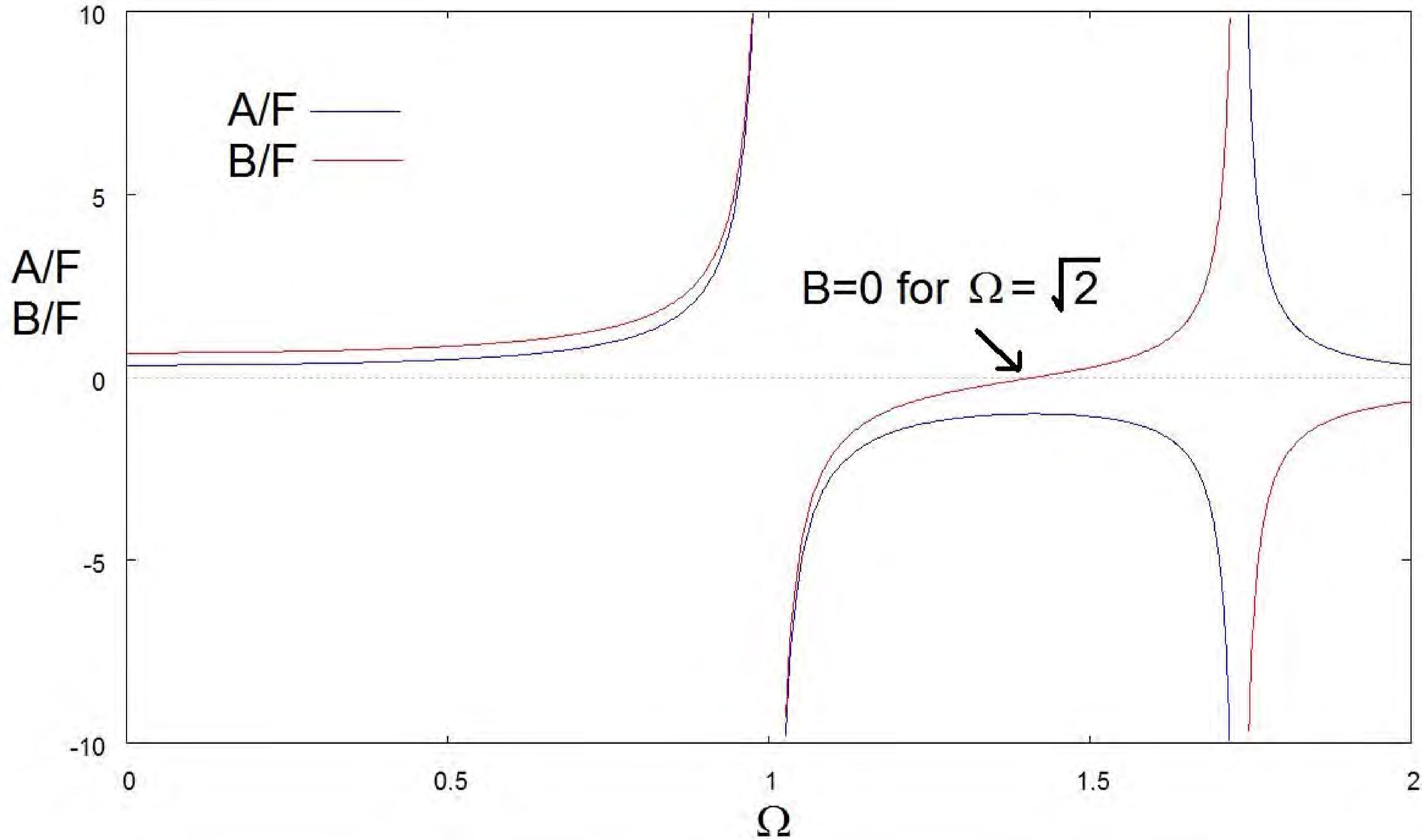
$$(R^t(-\Omega^2 M + K)R)^{-1} = (-\Omega^2 D_1 + D_2)^{-1}$$

$$R^{-1}(-\Omega^2 M + K)^{-1}(R^t)^{-1} = (-\Omega^2 D_1 + D_2)^{-1}$$

Now multiply on the left by R and on the right by R^t, giving

$$(-\Omega^2 M + K)^{-1} = R(-\Omega^2 D_1 + D_2)^{-1}R^t$$

This demonstrates the equivalence of eqs.(5) and (13).



A/F
B/F
A/F
B/F
B=0 for $\Omega = \sqrt{2}$
10
5
0
-5
-10
0
0.5
1
1.5
2
Ω

$$u_{tt} = c^2 u_{xx} \quad , \quad u_x = 0 \text{ at } x = 0, \ell$$

Set $u = U(x) \cos \omega t$

$$-\omega^2 U = c^2 U''$$

$$U(x) = c_1 \sin \frac{\omega}{c} x + c_2 \cos \frac{\omega}{c} x$$

$$U' = \frac{\omega}{c}\left(c_1 \cos \frac{\omega}{c} x - c_2 \sin \frac{\omega}{c} x \right)$$

$$U'(0) = U'(\ell) = 0 \Rightarrow c_1 = 0 \text{ and}$$

$$\sin \frac{\omega \ell}{c} = 0$$

$$\frac{\omega \ell}{c} = n\pi, \quad n = 0, 1, 2, \ldots$$

a) $$\boxed{\omega_n = \frac{n\pi c}{\ell} \ , \quad U_n(x) = \cos \frac{\omega_n}{c} x}$$

b) Show $\{U_n\}$ is orthogonal

$$\int_0^\ell U_n U_m \, dx = 0 \quad , \quad n \neq m$$

$$\int_0^\ell \underbrace{\cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell}}_{\frac{1}{2} \cos\left(\left(\frac{n+m}{\ell}\right)\pi x\right) + \frac{1}{2} \cos\left(\left(\frac{n-m}{\ell}\right)\pi x\right)} dx = 0$$

$$= \frac{\ell}{2\pi} \left(\frac{\sin\left(\frac{n+m}{\ell}\pi x\right)}{n+m} + \frac{\sin\left(\frac{n-m}{\ell}\pi x\right)}{n-m} \Bigg|_0^\ell \right)$$

$$= \frac{\sin (n+m)\pi}{n+m} + \frac{\sin (n-m)\pi}{n-m} = 0$$

c) $$u(x,t) = \begin{cases} \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \cos \frac{\omega_n}{c} x \\ + a_0 + b_0 t \end{cases}$$

$\uparrow$ rigid body mode

d) IC $t=0,\ u_t = 0 \Rightarrow b_n = 0,\ n = 0, 1, 2, \ldots$

$$t=0,\ u = \frac{x}{\ell} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

Mult. by $\cos \frac{m\pi x}{\ell}$ & $\int_0^{\ell} \Rightarrow$

$$\int_0^{\ell} \frac{x}{\ell} \cos \frac{m\pi x}{\ell}\, dx = \int_0^{\ell} a_m \left(\cos \frac{m\pi x}{\ell}\right)^2 dx = \frac{\ell}{2} a_m$$

$(m>0)$

$$a_m = \frac{2}{\ell} \int_0^{\ell} \frac{x}{\ell} \cos \frac{m\pi x}{\ell}\, dx, \quad m > 0$$

$$a_0 = \frac{1}{\ell} \int_0^{\ell} \frac{x}{\ell}\, dx = \frac{1}{\ell^2} \left[\frac{x^2}{2} \Big|_0^{\ell} \right] = \frac{1}{2}$$

$$a_m = \begin{cases} \frac{1}{2}, & m = 0 \\ 0, & m = 2, 4, 6, \ldots \\ \frac{-4}{m^2 \pi^2}, & m = 1, 3, 5, \ldots \end{cases}$$

e) at $x = \frac{\ell}{2}$, $u(\frac{\ell}{2}, t) = \frac{1}{2} + \sum_{n=1,3,5} a_n \cos \omega_n t \cos \frac{n\pi}{2} = \frac{1}{2}$

at $x=0$, $u(0,t) = \frac{1}{2} + \sum_{n=1,3,5}^{\infty} a_n \cos \frac{n\pi c t}{\ell}$

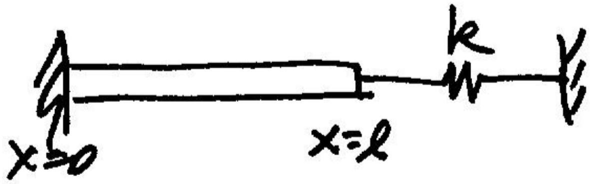
at $x = \ell$, $u(\ell, t) = \frac{1}{2} + \sum_{n=1,3,5}^{\infty} a_n \cos \frac{n\pi c t}{2} \overbrace{\cos n\pi}^{-1}$

$$= \frac{1}{2} - \sum_{n=1,3,5}^{\infty} a_n \cos \frac{n\pi c t}{2}$$

f) $$u(0,t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\frac{\pi c}{\ell}t + \frac{1}{9}\cos\frac{3\pi c}{\ell}t + \frac{1}{25}\cos\frac{5\pi c}{\ell}t + \cdots \right)$$

See figure attached.

2.



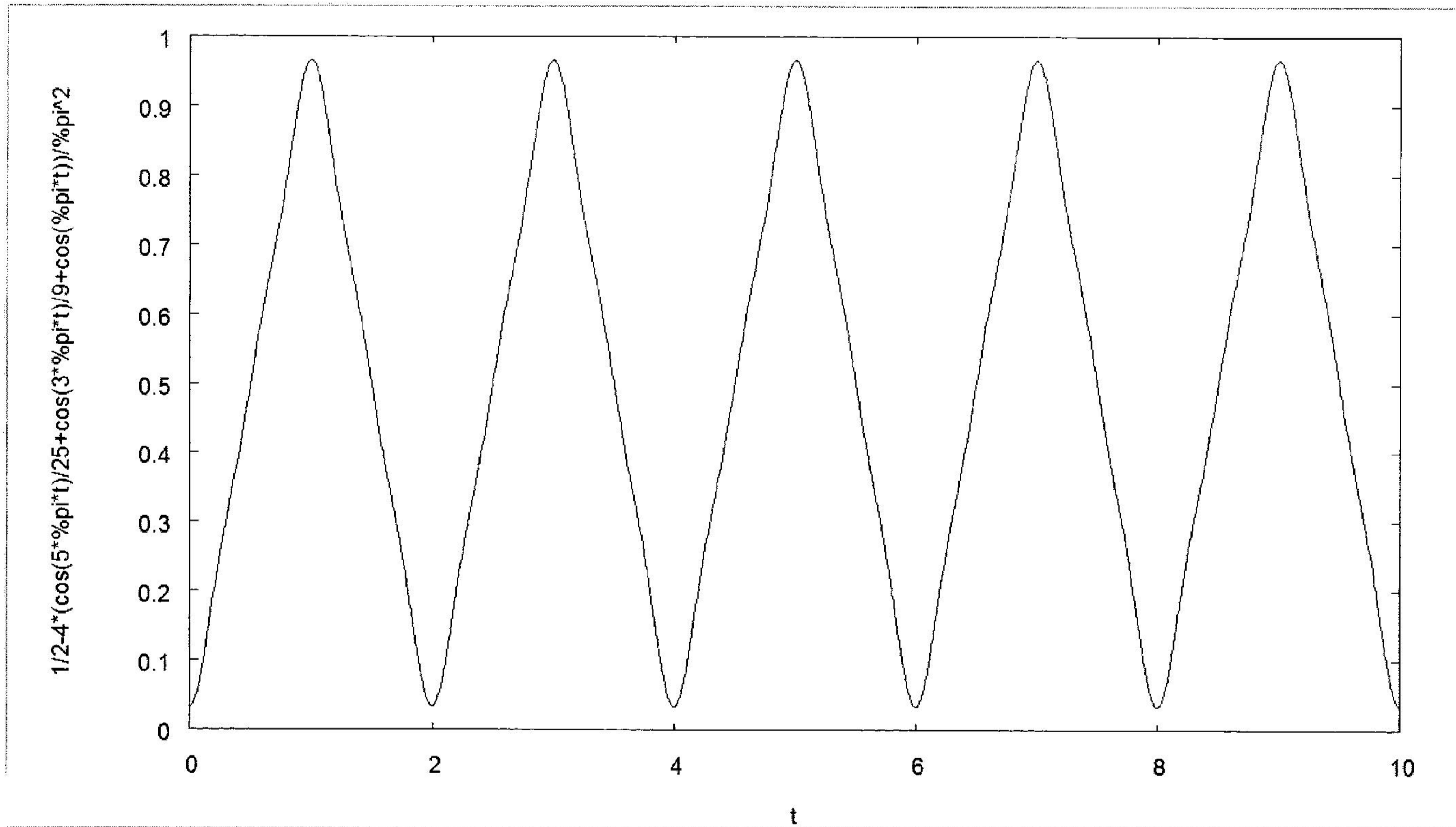
$$EA\frac{\partial u}{\partial x} = -ku \quad \text{at } x=\ell$$

see text, problem 7.8, pp. 210-211

3.

$$EA\frac{\partial u}{\partial x} = -M\frac{\partial^2 u}{\partial t^2} \quad \text{at } x=\ell$$

see text, problem 7.4, p. 207


1/2-4*(cos(5*%pi*t)/25+cos(3*%pi*t)/9+cos(%pi*t))/%pi^2
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
2
4
6
8
10

1a) $\omega_n = n\pi, \quad U_n(x) = \cos n\pi x$

b) $f(x,t) = x^2 \cos t = \sum f_i(t)\, U_i(x)$

$$f_n(t) = \left(\frac{\int_0^1 (\cos n\pi x)\, x^2\, dx}{\int_0^1 (\cos n\pi x)^2\, dx} \right) \cos t$$

$$= \frac{\frac{2}{\pi^2 n^2}(-1)^n}{\frac{1}{2}} \cos t = F_n \cos t$$

where $F_n = \frac{4}{\pi^2 n^2}(-1)^n,\ n>0. \quad F_0 = \frac{1}{3}$

c) $H(x) = u_t(x,0) = x = \sum H_n \cos n\pi x$

$$H_n = \frac{\int_0^1 x \cos n\pi x\, dx}{\int_0^1 (\cos n\pi x)^2 dx} = \frac{\frac{1}{\pi^2 n^2}\left((-1)^n - 1\right)}{\frac{1}{2}}, \ n>0$$

$$H_n = \left\{ \begin{array}{ll} -\frac{4}{\pi^2 n^2} & , n \text{ odd} \\ 0 & , n \text{ even} \end{array} \right\} n>0; \quad H_0 = \int_0^1 x\, dx = \frac{1}{2}$$

$G(x) = u(x,0) = 0 \Rightarrow G_n = 0$

d) $\ddot{p}_n + \omega_n^2 p_n = F_n \cos t$

$t=0,\ p_n = 0,\ \dot{p}_n = H_n$

e) $p_n(t) = A_n \cos n\pi t + B_n \sin n\pi t + \frac{F_n \cos t}{(n\pi)^2 - 1}, \ n>0$

(continued) $\quad p_0(t) = \frac{1}{2} t - \frac{1}{3}\cos t + \frac{1}{3}$

where $A_n = \dfrac{-F_n}{(n\pi)^2-1}$

$B_n = \dfrac{H_n}{n\pi}$

where F_n and H_n are given in b) and c)

f)

$$u(x,t) \approx p_1(t)\,U_1(x) + p_0(t)\,U_0(x)$$

$$\approx \left(A_1 \cos\pi t + B_1 \sin\pi t + \frac{F_1 \cos t}{\pi^2-1}\right)\cos\pi x + p_0(t)$$

$$\approx \left(\frac{-4}{\pi^2(\pi^2-1)}\left(\cos t - \cos\pi t\right) - \frac{4}{\pi^3}\sin\pi t\right)\cos\pi x$$

$$+ \frac{1}{2}t - \frac{1}{3}\cos t + \frac{1}{3}$$

2a) $u = U(x)\cos\omega t$

$$-\omega^2 U + \frac{EI}{\rho}U^{IV} = 0$$

$$U^{IV} - k^4 U = 0, \qquad k^4 = \omega^2\frac{\rho}{EI}$$

$$U = e^{\lambda x} \Rightarrow \quad \lambda^4 - k^4 = 0 \Rightarrow \lambda = k, -k, ik, -ik$$

$$U = c_1\cosh kx + c_2\sinh kx + c_3\cos kx + c_4\sin kx$$

BC $\quad U(0) = U(\ell) = U'(0) = U'(\ell) = 0$

4 homogeneous algebraic eqs.
For nontrivial solution, set determinant $=0$
which gives

$$\cos k\ell \cosh k\ell = 1, \qquad k = \sqrt{\omega}\left(\frac{\rho}{EI}\right)^{1/4}$$

$$\omega = \frac{(k\ell)^2}{\ell^2}\left(\frac{EI}{\rho}\right)^{1/2}$$

Solving the system of 4 homog. eqs, obtain

$$U_n(x) = \cosh(kl)\frac{x}{l} - \cos(kl)\frac{x}{l}$$

$$+ \mu\left(\sinh(kl)\frac{x}{l} - \sin(kl)\frac{x}{l}\right)$$

where $$\mu = -\frac{(\cosh kl - \cos kl)}{(\sinh kl - \sin kl)}$$

b) $kl = 4.73, 7.85, 10.99, 14.13$

c) see plot, attached.

Note: This information may be obtained directly from the "Table of Beam Frequencies" posted on the web.

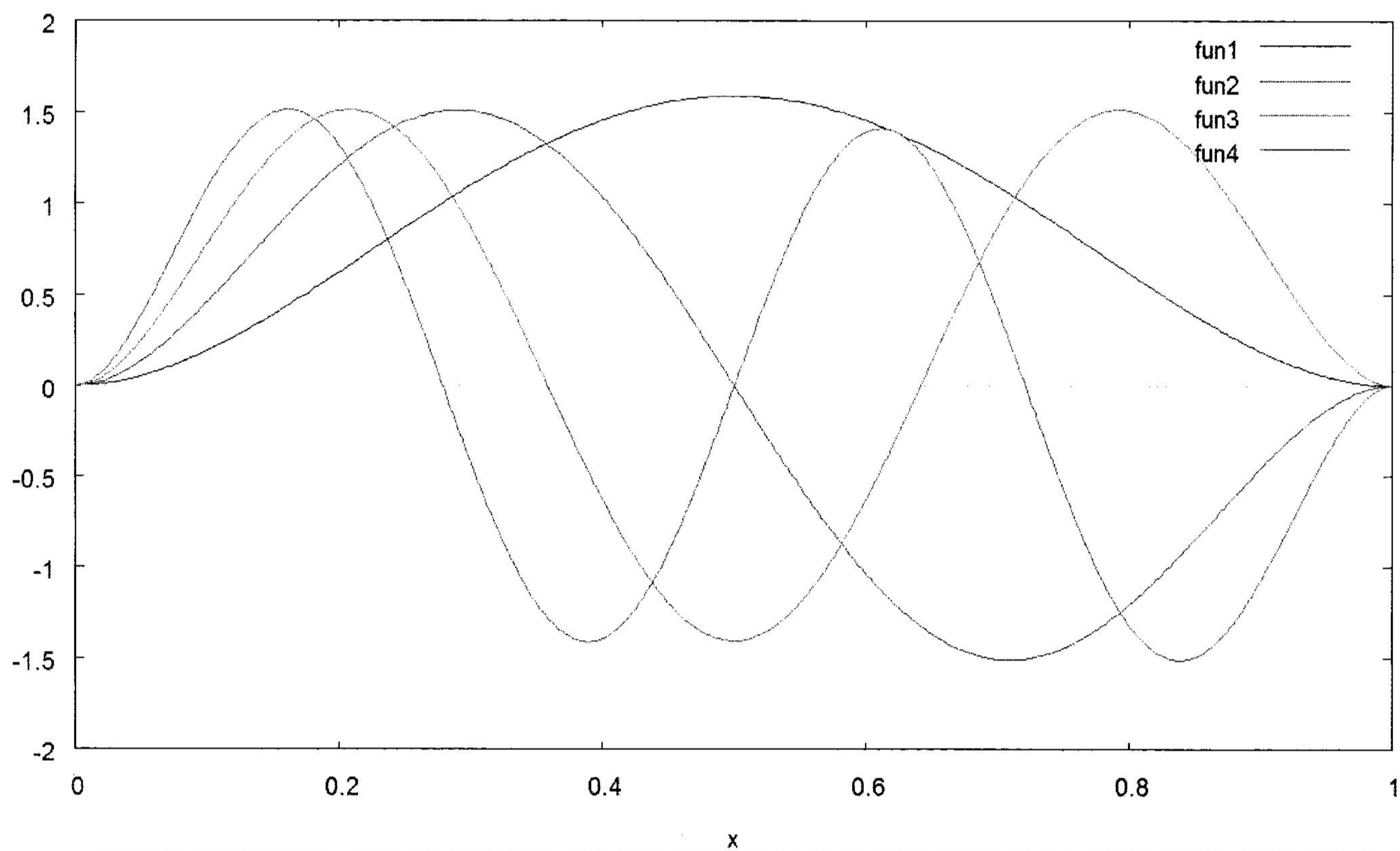
At the bottom of the Table we find:

$$\omega = \frac{\lambda^2}{l^2}\sqrt{\frac{EI}{\rho}}$$ where $\lambda = kl$ in above notation and ρ is listed as μ in Table.

$$J(u) = \cosh u - \cos u$$
$$H(u) = \sinh u - \sin u$$

Using this notation, the Table gives the mode shape as

$$U_n(x) = J\left(\lambda_n \frac{x}{l}\right) - \frac{J(\lambda_n)}{H(\lambda_n)} H\left(\lambda_n \frac{x}{l}\right)$$



④

3. $$\frac{d^2u}{dx^2} + u = 1$$

$$u(0) = 0$$
$$u(\pi) = 0$$

General solution

$$u = A\sin x + B\cos x + 1$$

$$u(0) = B + 1 = 0 \Rightarrow B = -1$$
$$u(\pi) = -B + 1 = 0 \Rightarrow B = 1$$

Since B cannot be equal to both 1 and -1, this problem has no solution.

Note that the above system does have a solution for appropriate choices of the right hand side. For example

$$\frac{d^2u}{dx^2} + u = \cos 3x$$

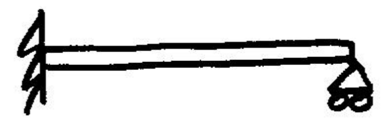
particular solution

$$u = A\sin x + B\cos x - \frac{1}{8}\cos 3x$$

$$\left.\begin{aligned} u(0) &= B - \tfrac{1}{8} = 0 \Rightarrow B = 1/8 \\ u(\pi) &= -B - \tfrac{1}{8}(-1) = 0 \Rightarrow B = 1/8 \end{aligned}\right\} \text{no contradiction}$$

This can be explained in terms of the "Fredholm alternative theorem" which I will go over in class.

1a.



From Table, $\lambda_1 = 3.9266$

$$\omega_1 = \lambda_1^2 \sqrt{\frac{EI}{\rho l^4}} = 15.42 \sqrt{\frac{EI}{\rho l^4}}$$

1b.

$$\frac{d^4 u}{dx^4} = 1$$

$$u = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{x^4}{24}$$

BC: $x=0,\ u=0,\ u'=0 \Rightarrow c_1 = c_2 = 0$

$x = l,\ u=0,\ u''=0 \Rightarrow c_3 = \frac{l^2}{16},\ c_4 = -\frac{5}{48} l$

$$u = \frac{x^2 l^2}{16} - \frac{5}{48} x^3 l + \frac{x^4}{24} \quad (= V(x) \text{ below})$$

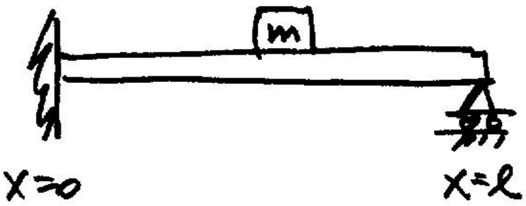
1c.

$$Q = \frac{EI \int_0^l (V'')^2 dx}{\rho \int_0^l V^2 dx} = \frac{EI \frac{l^5}{320}}{\rho \frac{19 l^9}{1451520}}$$

$$\omega_1 < \sqrt{Q} = \sqrt{\frac{4536}{19}} \sqrt{\frac{EI}{\rho l^4}} = 15.45 \sqrt{\frac{EI}{\rho l^4}}$$

Great agreement with 1a!

1d. Again take $V(x) = \frac{x^2 \ell^2}{16} - \frac{5}{48} x^3 \ell + \frac{x^4}{24}$



$$\rho = \rho_0 + m\,\delta\left(x - \frac{\ell}{2}\right)$$

$$Q = \frac{EI \int_0^\ell (V'')^2 dx}{\rho_0 \int_0^\ell V^2 dx + m\,V\left(\frac{\ell}{2}\right)^2}$$

$$V\left(\frac{\ell}{2}\right) = \frac{\ell^4}{192}$$

$$Q = \frac{EI \frac{\ell^5}{320}}{\rho_0 \frac{19\,\ell^9}{1451520} + \frac{m\,\ell^8}{(192)^2}}$$

$$= \frac{EI\left(\frac{4536}{19}\right)}{\rho_0 \ell^4 + \left(\frac{315}{152}\right) m \ell^3}$$

$$\omega_1 < \sqrt{Q} = 15.45 \sqrt{\frac{EI}{\rho_0 \ell^4 + 2.07\, m \ell^3}}$$

2. $r(x) = x+1$ (See 7.22 on p. 225)

$A(x) = \pi(x+1)^2$

$$Q = \frac{\int_0^2 E A(x)(u')^2\,dx}{\int_0^2 \rho A(x)\, u^2\,dx}$$

Choose $u(x) = x(x-2)$

which satisfies the BC $u(0) = u(2) = 0$

Then $u' = 2x - 2$

$$Q = \frac{E\int_0^2 \pi(x+1)^2(2x-2)^2\,dx}{\rho\int_0^2 \pi(x+1)^2 x^2(x-2)^2\,dx} = \frac{\frac{184}{15}E}{\frac{464}{105}\rho}$$

$$Q = \frac{161}{58}\frac{E}{\rho} \Rightarrow \omega_1 \le \sqrt{Q} = 1.67\sqrt{\frac{E}{\rho}}$$

3. $$u^{IV} - u = 1$$

a) $u = c_1 \sin x + c_2 \cos x + c_3 \sinh x + c_4 \cosh x - 1$

$$\left.\begin{aligned} u(0) &= 0 = c_2 + c_4 - 1 \\ u''(0) &= 0 = -c_2 + c_4 \end{aligned}\right\} \quad c_2 = c_4 = \tfrac{1}{2}$$

$u(\pi) = 0 = c_4 \tilde{c} + c_3 \tilde{s} - c_2 - 1$ where $\tilde{c} = \cosh(\pi)$, $\tilde{s} = \sinh(\pi)$

$u''(\pi) = 0 = c_4 \tilde{c} + c_3 \tilde{s} + c_2$

Substituting $c_2 = c_4 = \frac{1}{2}$ from above gives

2 incompatible values for c_3 $\Rightarrow$ no solution

b) The Fredholm Alternative says $f(x)$ must be orthogonal to the null space of the adjoint.

[Note that (i) $u^{IV} - u = 0 \Rightarrow u = \sin x$ satisfies B.C.

i.e. the homogeneous system has a nontrivial soln

and (ii) the operator $Lu = u^{IV} - u$ is self-adjoint.]

Only those $f(x)$ which satisfy $\int_0^\pi f(x) \sin x \, dx = 0$

will give a solution. E.g. $f(x) = \cos 3x$.

Homework 8, Problem 1

Consider a clamped-free beam of constant depth, and a width which varies linearly from a maximum at the fixed end to zero at the free end. Taking the origin of coordinates at the fixed end, $I = I_0(1 - x/l)$ and $\mu = \mu_0(1 - x/l)$, where I_0 and μ_0 are respectively the moment of inertia and mass per unit length at the fixed end. Choose a two-term series,

$$W = A_1x^2 + A_2x^3$$

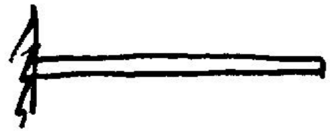
The kinetic and potential energies are:

$$T^* = \tfrac{1}{2}\int_0^l \mu W^2\,dx = \frac{\mu_0}{2}\int_0^l \left(1 - \frac{x}{l}\right)(A_1x^2 + A_2x^3)^2\,dx$$
$$V_{\max} = \tfrac{1}{2}\int_0^l EI(W'')^2\,dx = \frac{EI_0}{2}\int_0^l \left(1 - \frac{x}{l}\right)(2A_1 + 6A_2x)^2\,dx$$

Integrating, taking the partial derivatives, and substituting into (61.127) gives the pair of equations,

$$\begin{aligned}(2 - \beta/30)A_1 + (2 - \beta/42)A_2l &= 0\\ (2 - \beta/42)A_1 + (3 - \beta/56)A_2l &= 0\end{aligned} \qquad (61.128a,b)$$

where $\beta = \mu_0l^4\lambda/EI_0$. The roots of the frequency equation are $\beta_1 = 51.25$ and $\beta_2 = 1377$, and, hence, the first two frequencies are $\omega_1^2 \leq 51.25EI_0/\mu_0l^4$ and $\omega_2^2 \leq 1377EI_0/\mu_0l^4$

2. Ritz on 

w/ 3 terms

$$V = c_1 x^2 + c_2 x^3 + c_3 x^4$$

$$\overline{Q} = Q \frac{\rho \ell^4}{EI}, \quad \overline{c}_2 = c_2 \ell, \quad \overline{c}_3 = c_3 \ell^2$$

$$\overline{Q} = \ell^4 \frac{\int_0^\ell (V'')^2 dx}{\int_0^\ell V^2 dx}$$

$$\int_0^\ell (V'')^2 dx = \ell \Big(\frac{144}{5} \overline{c}_3^2 + 36 \overline{c}_2 \overline{c}_3 + 16 c_1 \overline{c}_3 + 12 \overline{c}_2^2 + 12 c_1 \overline{c}_2 + 4 c_1^2 \Big) = \ell F$$

$$\int_0^\ell V^2 dx = \ell^5 \Big(\frac{\overline{c}_3^2}{9} + \frac{\overline{c}_2 \overline{c}_3}{4} + \frac{2}{7} c_1 \overline{c}_3 + \frac{\overline{c}_2^2}{7} + \frac{c_1 \overline{c}_2}{3} + \frac{c_1^2}{5} \Big) = \ell^5 G$$

$$\overline{Q} = \frac{F}{G}, \qquad G \overline{Q} = F$$

take $\frac{\partial}{\partial c_1}$, $\frac{\partial}{\partial c_2}$ & $\frac{\partial}{\partial c_3}$ of this eq.

& set $\frac{\partial \overline{Q}}{\partial c_i} = 0$

$$\begin{bmatrix} \frac{2\bar{Q}-40}{5} & \frac{\bar{Q}-36}{3} & \frac{2\bar{Q}-112}{7} \\ \frac{\bar{Q}-36}{3} & \frac{2\bar{Q}-168}{7} & \frac{\bar{Q}-144}{4} \\ \frac{2\bar{Q}-112}{7} & \frac{\bar{Q}-144}{4} & \frac{10\bar{Q}-2592}{45} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \bar{0}$$

Set det $= 0 \Rightarrow$

$$5\bar{Q}^3 - 72324\bar{Q}^2 + 35392896\bar{Q} - 426746880 = 0$$

$$\bar{Q} = 12.369,\ 494.322,\ 13958.107$$

EXACT:

$\bar{\omega}_1 \leq \sqrt{\bar{Q}} = 3.517$ $\quad$ (vs. $1.8751^2 = 3.516$)

$\bar{\omega}_2 \leq \sqrt{\bar{Q}} = 22.233$ $\quad$ (vs. $4.6941^2 = 22.03$)

$\bar{\omega}_3 \leq \sqrt{\bar{Q}} = 118.144$ $\quad$ (vs. $7.8548^2 = 61.69$)

where $\omega_i = \bar{\omega}_i \sqrt{\frac{EI}{\rho \ell^4}}$

HW #9 Solution

① a) $$u'' + \frac{2}{\rho} u' + u = 0 \quad , \quad ' = \frac{d}{d\rho}$$

b) $$u = a_0 + a_1 \rho + a_2 \rho^2 + \cdots$$

Substitute, collect terms, set coefficient of $\rho^n = 0$

Find $a_1 = 0$ and all $a_{odd} = 0$

$$a_2 = -\frac{a_0}{6}, \quad a_4 = \frac{a_0}{120} \left(= -\frac{a_2}{20} \right)$$

$$a_6 = -\frac{a_4}{42} = -\frac{a_0}{5040} = -\frac{a_0}{7!} \quad , \quad a_8 = \frac{a_0}{9!}$$

$$u(\rho) = a_0 \left(1 - \frac{\rho^2}{3!} + \frac{\rho^4}{5!} - \frac{\rho^6}{7!} + \frac{\rho^8}{9!} + \cdots \right)$$

$$\left(= a_0 \frac{\sin \rho}{\rho} \right)$$

c) $$\frac{du(\rho)}{d\rho} = 0 = a_0 \left(-\frac{2\rho}{3!} + \frac{4\rho^3}{5!} - \frac{6\rho^5}{7!} + \frac{8\rho^7}{9!} - \cdots \right)$$

$$\rho = 0 \text{ and } -\frac{\rho}{3} + \frac{\rho^3}{30} - \frac{\rho^5}{840} + \frac{\rho^7}{45360} - \cdots = 0$$

a root solver gives $\rho = 4.14$

$$\Rightarrow \quad \rho = \frac{\omega_1 R}{c} = 4.14 \quad , \quad \omega_1 = \frac{4.14\, c}{R}$$

Lord Rayleigh (1872) gives $1.43\pi = 4.49$

(2) $$x^2 J_0'' + x J_0' + x^2 J_0 = 0$$

divide by x^2: $$J_0'' = -\frac{J_0'}{x} - J_0$$

Differentiate $\Rightarrow$ $$J_0''' = -\frac{J_0''}{x} + \frac{J_0'}{x^2} - J_0'$$

Mult by x^2: $$x^2 J_0''' + x J_0'' - (1-x^2) J_0' = 0$$

Let $f = -J_0'$

$$-x^2 f'' - x f' + (1-x^2) f = 0$$

$$\left.\begin{array}{ll} & x^2 f'' + x f' + (x^2-1) f = 0 \\ J_1 \text{ satisfies} & x^2 J_1'' + x J_1' + (x^2-1) J_1 = 0 \end{array}\right\}$$

Comparison of eqs $\Rightarrow$ J_1 and f satisfy the same ODE

Both J_0 and J_1 ODE's admit two linearly independent solutions, one bounded as $x \to 0$, one unbounded.

Since both J_0 & J_1 are bounded, we have that f and J_1 are at least a multiple of one another. Normalization gives $f = J_1$. But $f = -J_0'$

$$\therefore \; -J_0' = J_1$$

HW #10 Solutions

1. $$\frac{d^2x}{dt^2} + x = \alpha x^5$$

Set $x = A\cos\omega t$

$$-A\omega^2\cos\omega t + A\cos\omega t = \alpha A^5\cos^5\omega t$$

Identity (maxima): $\cos^5\theta = \frac{5}{8}\cos\theta + \frac{5}{16}\cos3\theta + \frac{1}{16}\cos5\theta$

So $\alpha A^5\cos^5\omega t = \frac{5}{8}A^5\cos\omega t$ + nonresonant terms

$$-A\omega^2 + A = \frac{5}{8}A^5\alpha$$

$$\Rightarrow \quad \omega^2 = 1 - \frac{5}{8}\alpha A^4$$

2. $$\ddot{x} + x = 0.1(1 - 2x^2 + bx^4)\dot{x}$$

2a. $x = A\cos\omega t$

$$-A\omega^2\cos\omega t + A\cos\omega t = 0.1(1 - 2A^2\cos^2\omega t + bA^4\cos^4\omega t) * -A\omega\sin\omega t$$

Identities (maxima): $\cos^2\theta\sin\theta = \frac{1}{4}\sin\theta + \frac{1}{4}\sin3\theta$

$\cos^4\theta\sin\theta = \frac{1}{8}\sin\theta + \frac{3}{16}\sin3\theta + \frac{1}{16}\sin5\theta$

$$(-\omega^2+1)A\cos\omega t = 0.1(-A\omega)(\sin\omega t)\left(1 - 2A^2\left(\tfrac{1}{4}\right) + bA^4\left(\tfrac{1}{8}\right)\right)$$

+ nonresonant terms

Balancing the harmonics:

$\cos\omega t$: $(-\omega^2+1)A = 0 \Rightarrow \omega = 1$

$\sin\omega t$: $A\omega\left(1 - \frac{A^2}{2} + b\frac{A^4}{8}\right) = 0 \Rightarrow$

$$A^2 = \frac{2}{b}\left(1 \pm \sqrt{1-2b}\right)$$

2b. For $b > \frac{1}{2}$ there are no real roots, no LC's

$b < \frac{1}{2}$ " " 2 real positive roots (= 2 LC's)

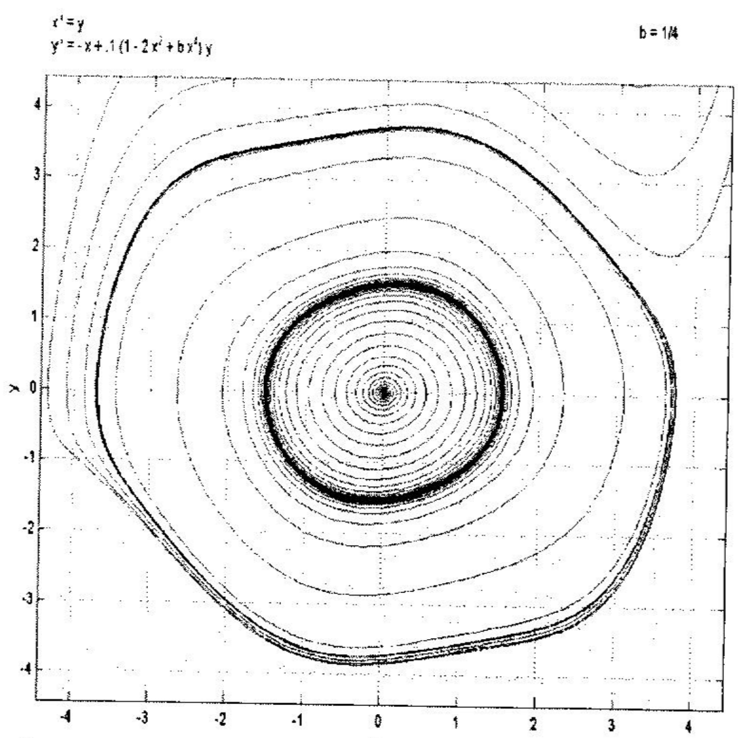
For $\boxed{b = \frac{1}{2}}$ there is one degenerate LC

2c. Check with "pplane".

For $b = \frac{1}{4}$, theory gives $A^2 = 8 \pm 2^{5/2} = 2.34, 13.65$

$\Rightarrow A = 1.53, 3.70$

Agrees w/ pplane plot:



3. $$\dot{x} = y + .1x^3 + \alpha x$$
$$\dot{y} = -x + .1\dot{x}^3 - \beta\dot{x}$$

3a. Differentiate 1st equation:
$$\ddot{x} = \dot{y} + (.1)(3x^2\dot{x}) + \alpha\dot{x}$$
$$= -x + (.1)\dot{x}^3 - \beta\dot{x} + (.1)(3x^2\dot{x}) + \alpha\dot{x}$$

or
$$\ddot{x} + x = (\alpha - \beta)\dot{x} + (.1)(\dot{x}^3 + 3x^2\dot{x})$$

3b. Harmonic balance: $x = A\cos\omega t$
$$-\omega^2 A\cos\omega t + A\cos\omega t = (\alpha-\beta)(-A\omega\sin\omega t) + (.1)(A^3)(-\omega^3\sin^3\omega t - 3\cos^2\omega t\sin\omega t)$$

$$\sin^3\omega t = \frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t$$
$$\cos^2\omega t\sin\omega t = \frac{1}{4}\sin\omega t + \frac{1}{4}\sin 3\omega t$$
$$\cos\omega t:\ (-\omega^2+1)A = 0 \Rightarrow \omega = 1$$
$$\sin\omega t:\ 0 = (\alpha-\beta)(-A\omega) + (.1)A^3\left(-\frac{3}{4}\omega^3 - \frac{3}{4}\right) \Rightarrow$$

(Using $\omega = 1$) $\boxed{A^2 = \frac{20}{3}(\beta - \alpha)}$

For a real solution, $A^2 > 0 \Rightarrow \beta > \alpha$

Example $\beta = .2$, $\alpha = .1$, $A^2 = \frac{2}{3}$, $A \approx .82$

agrees w/ simulation on pplane6

3c. A Hopf bifurcation occurs for $\boxed{\beta = \alpha}$

The LC exists for $\beta > \alpha$.

What is the stability of the origin for $\beta > \alpha$?

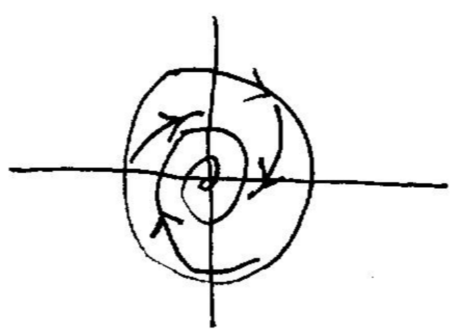
The ODE becomes (linearize near the origin);

$$\ddot{x} + x = (\alpha - \beta)\dot{x} + \text{nonlinear terms}$$

for $\beta > \alpha$ this is a damping term

$\Rightarrow$ the origin is stable

$\Rightarrow$ the LC is unstable:



Motions near the LC move away from it and head towards the origin

$\therefore$ The Hopf bifurcation is SUBCRITICAL

(This agrees with simulation using pplane.
Under "OPTIONS" choose "SOLUTION DIRECTION" as <u>forward</u>. You will see the LC as repelling (i.e. unstable.))