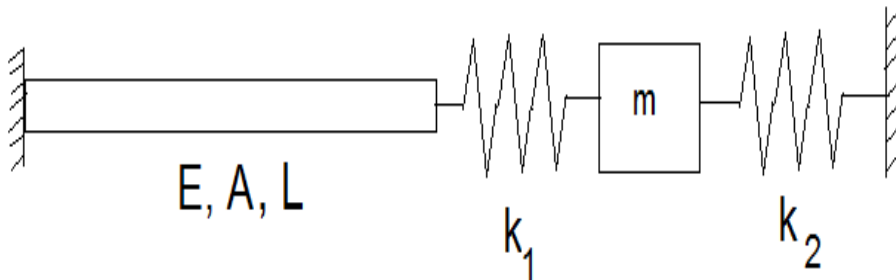


Homework No.1

Due: Monday Feb.6, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. Read Solved Problems 1.9, 1.10, 1.11 on pp.7-10. Then find an expression for the natural frequency of this system:

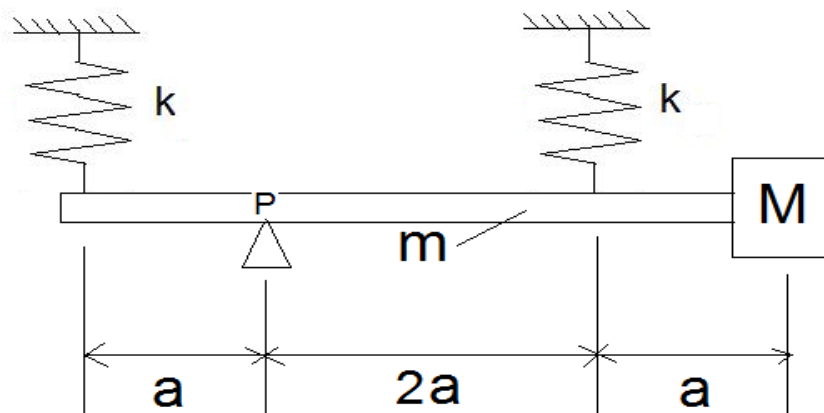


2. A concentrated mass M is attached to the end of a uniform rigid bar of mass m which is pinned at P as shown and restrained by two springs.

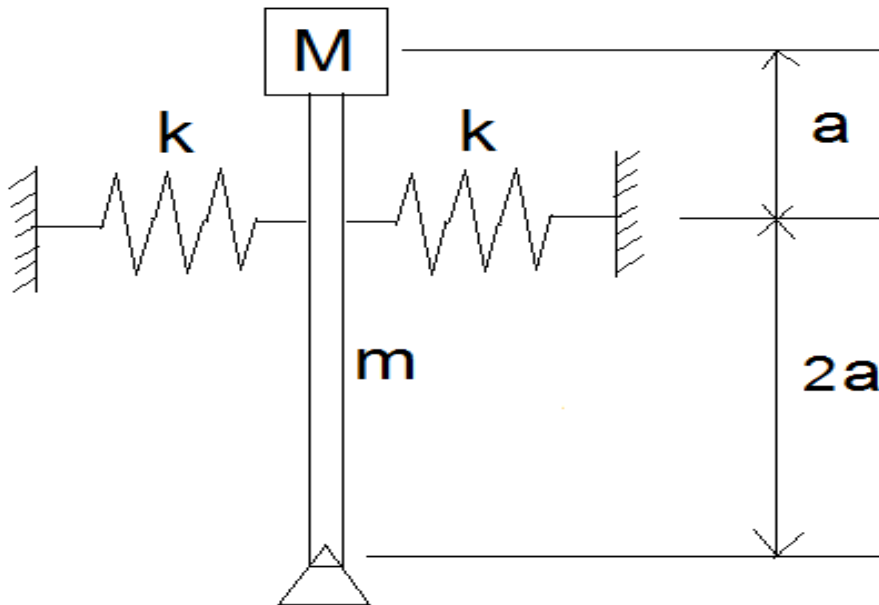
a) Obtain the equation of motion by using free body diagrams. Neglect gravity and friction. Assume small rotation angle.

b) Same thing, but use Lagrange's equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} \quad \text{where} \quad L = T - V$$



3. An upside-down compound pendulum is composed of a concentrated mass M is attached to the end of a uniform rigid bar of mass m . Although normally unstable, its equilibrium position is made stable by the addition of two springs as shown. Find an expression for the minimum value of k for which the equilibrium is stable.



4. The differential equation

$$x'' = 0$$

has, amongst all possible solutions, the solution

$$x = 1 + t$$

The differential equation

$$x'' + w^2 x = 0$$

has the general solution

$$x = c_1 \sin wt + c_2 \cos wt$$

In the limit as $w \rightarrow 0$, the second differential equation approaches the first differential equation. We would therefore expect the second solution to approach the first solution.

But since $\sin wt \rightarrow 0$ and $\cos wt \rightarrow 1$ in the limit as $w \rightarrow 0$, the second solution does not appear to approach the first solution.

How can this be? What is going on here? Explain.

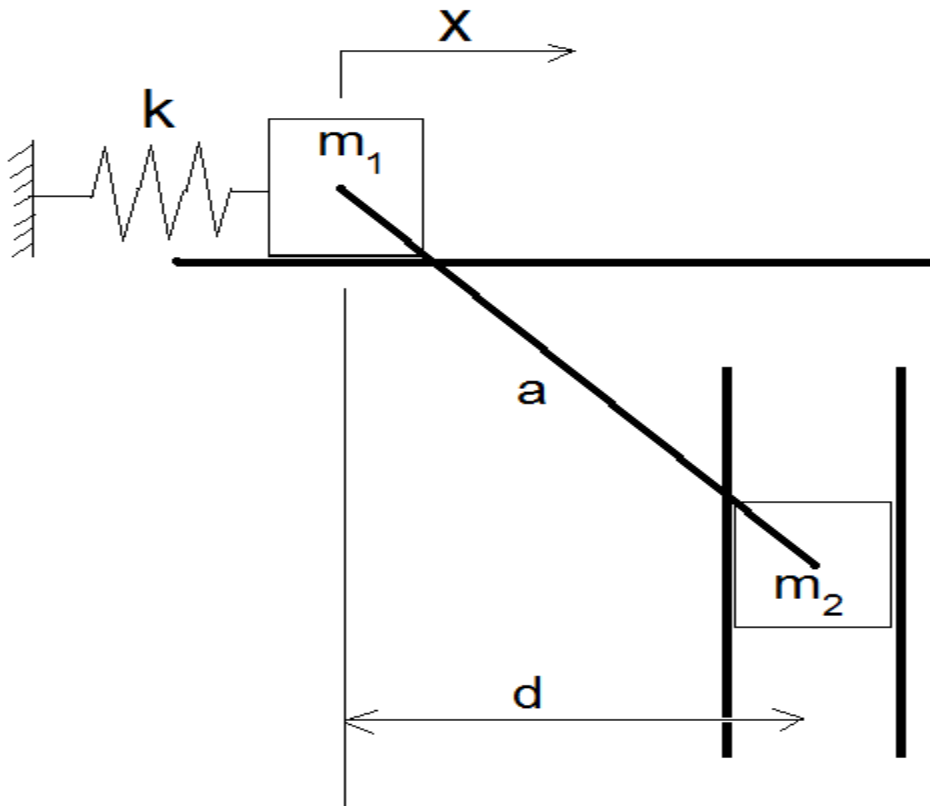
Homework No.2

Due: Monday Feb.13, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. Mass m_1 moves along a horizontal line, while mass m_2 is constrained to move along a vertical channel as shown. They are connected by a massless rigid rod of length a which is pinned to each mass. Mass m_1 is restrained by a linear spring with spring constant k . d is the horizontal distance between the equilibrium position of m_1 and the vertical channel in which m_2 moves.

- a) Use Lagrange's equation to find the equation of motion. Assume that x is measured from equilibrium. Neglect friction and gravity.
- b) What is the natural frequency of small vibrations about equilibrium?



CONTINUED ON NEXT PAGE

2. A point mass m moves without friction on a parabola given by $y = x^2$ where y is vertically upward. Use Lagrange's equation to find the natural frequency of small vibrations about $x = 0$. Include gravity, neglect friction.

3. In class we showed that the maximum steady state response amplitude R of a harmonically forced lightly damped linear system

$$\ddot{x} + 2n\dot{x} + \omega^2 x = \frac{F}{m} \cos \Omega t \quad \Rightarrow \quad x_{steadystate} = R \cos(\Omega t - \phi)$$

occurred at driving frequency

$$\Omega = \sqrt{\omega^2 - 2n^2}$$

At what driving frequency does the maximum amplitude of the steady state *velocity* occur?

CONTINUED ON NEXT PAGE

4. A wheel of radius R and moment of inertia I_0 is mounted at its center O with a frictionless pin. A mass m is constrained to move on a vertical line and is attached to the wheel by an inextensible cable as shown. The system is released from rest and the wheel begins to spin up due to the downward motion of the mass due to gravity. The kinetic energy is given by:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_0\dot{\theta}^2 \quad (1)$$

and the potential energy is given by

$$V = -mgx \quad (2)$$

Neglecting friction, the conservation of energy gives

$$T + V = \frac{1}{2}(mR^2 + I_0)\dot{\theta}^2 - mgR\theta = C \quad (3)$$

where C is a constant and where we have used the relation $x = R\theta$. Since the system is released from rest, the constant C is evaluated using the initial conditions as follows:

$$\theta(0) = 0, \quad \dot{\theta}(0) = 0 \Rightarrow C = 0 \quad (4)$$

Solving eq.(3) for $\dot{\theta}$, we obtain the first order differential equation:

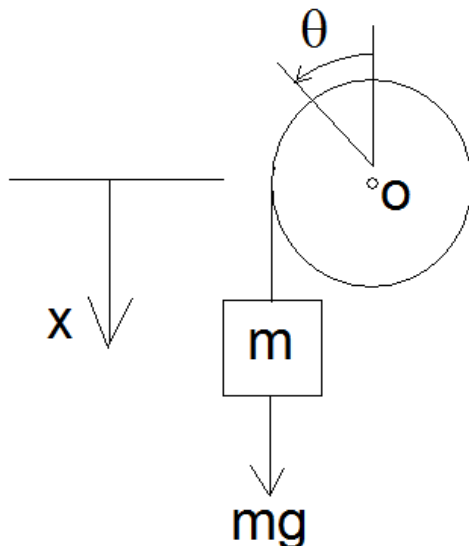
$$\frac{d\theta}{dt} = \sqrt{\frac{2mgR}{mR^2 + I_0}}\sqrt{\theta} \quad (5)$$

where we have taken the positive square root for downward motion.

Now here is the question: Eq.(5) with initial conditions (4) has the exact solution:

$$\theta(t) \equiv 0, \quad \text{for all } t > 0$$

which says that the wheel doesn't rotate. But this is physically unreasonable. Where is the error?

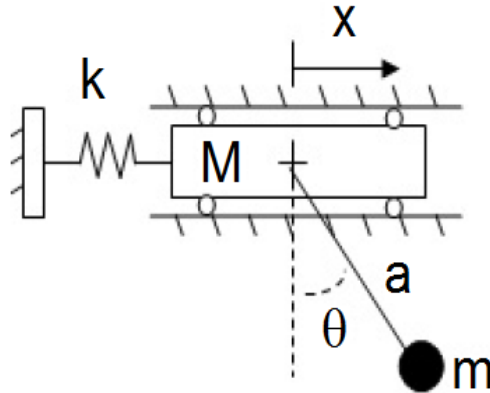


Homework No.3

Due: Monday Feb.20, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. A simple pendulum of mass m and length a has its point of suspension attached to a mass M which is constrained to move on a horizontal line, and which is restrained by a linear spring k :



- Use Lagrange's equations to derive the (full nonlinear) equations of motion for this system. x is measured from the unstretched position of the spring. Neglect friction.
- Linearize the resulting equations about $x = \theta = 0$.
- Write the linearized equations in the matrix form:

$$M\ddot{z} + Kz = 0$$

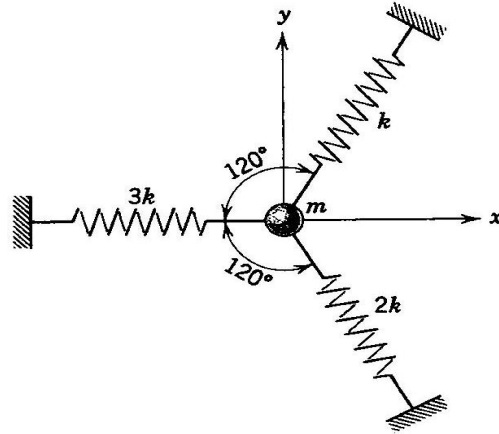
where

$$z = \begin{pmatrix} x \\ \theta \end{pmatrix}$$

and where M and K are *symmetric* matrices.

- Using the numerical values $M = m = a = g = k = 1$, compute the frequencies ω_1, ω_2 and the modal vectors ${}_1Z$ and ${}_2Z$.
- Show by direct computation that the modal vectors are orthogonal with respect to both M and K .
- Defining the matrix R as $R = [{}_1Z \quad {}_2Z]$, show that $R^t M R$ is diagonal.
- Write x and θ as linear combinations of the principal coordinates p_1 and p_2 .
- By substituting the expressions you chose in part g above in $M\ddot{z} + Kz = 0$, obtain differential equations on the principal coordinates p_1 and p_2 .

2. A particle of mass $m = 1$ moves in the $x - y$ plane and is restrained by three springs with spring constants k , $2k$ and $3k$ respectively, where $k = 1$. The springs are unstretched in the configuration shown, and have unstretched length equal to 1. Find the natural frequencies of free vibration.



3. The general motion of the first coordinate of a two degree of freedom system is given by:

$$x_1(t) = R_1 \cos(\omega_1 t - \theta_1) + R_2 \cos(\omega_2 t - \theta_2)$$

Is this a periodic motion? Under what condition will it be periodic?

Homework No.4

Due: Monday Feb.27, 2012

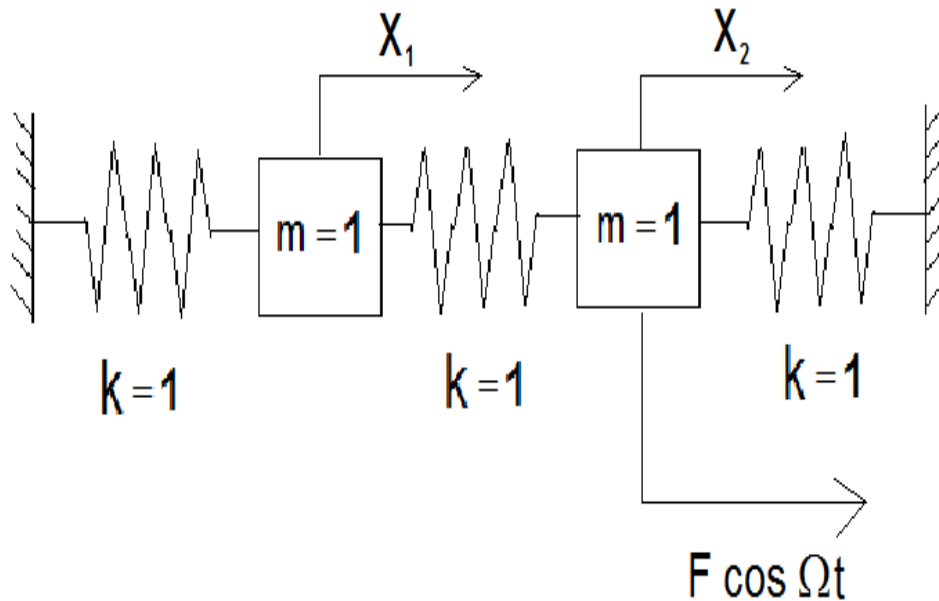
RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. a) Use Lagrange's equations to derive the EOM (equations of motion) for the following system in physical coordinates x_1, x_2 .
- b) Transform to principal coordinates p_1, p_2 , and so write the EOM in the form:

$$\ddot{p}_1 + \omega_1^2 p_1 = g_1(t), \quad \ddot{p}_2 + \omega_2^2 p_2 = g_2(t) \quad (1)$$

Identify $\omega_1, \omega_2, g_1(t), g_2(t)$.

- c) Assuming very small damping, the complementary solution to eqs.(1) will be transient. Ignoring the complementary solution, obtain the steady state solution to eqs.(1).
- d) Transform the steady state solution in p_1, p_2 obtained in c) back to physical variables x_1, x_2 .



2. a) Obtain the steady state response to the EOM derived in 1a) directly (without the use of principal coordinates) by assuming a solution in the form:

$$x_1 = A \cos \Omega t, \quad x_2 = B \cos \Omega t \quad (2)$$

Substituting eq.(2) into the EOM, obtain expressions for A, B .

- b) Show that the solution so obtained agrees with that obtained in 1d).
- c) Plot A, B as functions of Ω . For which value of Ω does B vanish?

3. This question concerns the two foregoing schemes for obtaining the “steady state” response of a very lightly damped linear n degree of freedom vibrating system. We take the equations of motion in the matrix form:

$$M\ddot{x} + Kx = F \cos \Omega t \quad (3)$$

where M and K are symmetric $n \times n$ matrices, and x and F are column vectors. Here we have omitted the damping term from the differential equation, but we assume its presence causes the complementary solution to decay to zero. Thus we look for the steady state in the form:

$$x = X \cos \Omega t \quad (4)$$

We wish to find the amplitude vector X in terms of the forcing vector F .

Method 1: Direct substitution

Substituting (4) into (3) gives:

$$-\Omega^2 MX + KX = F \quad \Rightarrow \quad X = (-\Omega^2 M + K)^{-1} F \quad (5)$$

Method 2: Principal modes

Principal coordinates are defined by

$$x = Rp \quad (6)$$

where R is the $n \times n$ modal matrix which satisfies the equations:

$$R^t MR = D_1, \quad R^t KR = D_2 \quad (7)$$

where D_1 and D_2 are $n \times n$ diagonal matrices. Motivated by eq.(4), we write

$$p = U \cos \Omega t \quad (8)$$

where U , like X and F , is a column vector of constants. Thus eqs.(4),(6) and (8) give:

$$X = RU \quad (9)$$

Substituting (9) into (5), we get

$$-\Omega^2 MRU + KRU = F \quad (10)$$

Next we multiply eq.(10) by R^t

$$-\Omega^2 R^t MRU + R^t KRU = R^t F \quad \Rightarrow \quad -\Omega^2 D_1 U + D_2 U = R^t F \quad (11)$$

where we have used eq.(7). Solving the second eq. in (11) for U , we get:

$$U = (-\Omega^2 D_1 + D_2)^{-1} R^t F \quad (12)$$

Having solved for U , we can now get X from eq.(9):

$$X = RU = R(-\Omega^2 D_1 + D_2)^{-1} R^t F \quad (13)$$

Comparison of the two methods

We now have two distinct expressions for the amplitude vector X in terms of the forcing vector F , namely eqs.(5) and (13).

SHOW THAT THESE TWO EXPRESSIONS FOR X ARE EQUIVALENT.

Homework No.5

Due: Monday March 12, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. As shown in class, the longitudinal vibrations of a rod with free ends are governed by the equations:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

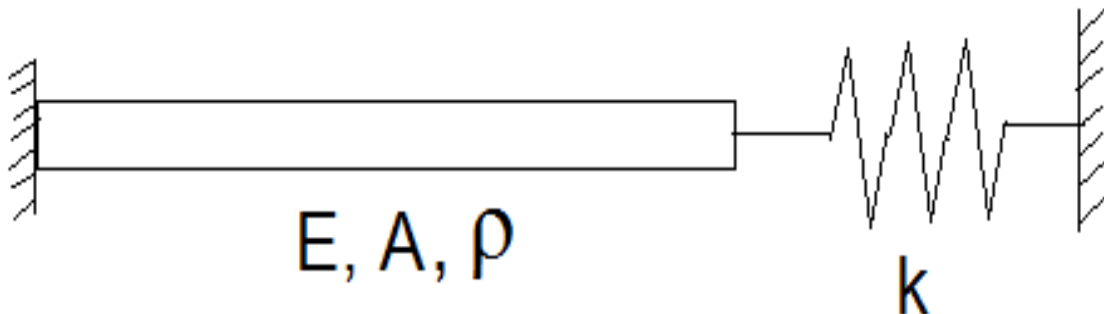
with B.C. $\frac{\partial u}{\partial x} = 0$ at $x = 0, \ell$

- Find the frequencies ω_n and modes of vibration $U_n(x)$.
- Show that the modes $\{U_n(x)\}$ form an orthogonal set.
- Find the general solution.
- Find the solution which corresponds to the initial conditions:

$$\text{I.C. } t = 0, \quad u = \frac{x}{\ell}, \quad \frac{\partial u}{\partial t} = 0$$

- Compute $u(\ell/2, t)$.
- Use matlab or some other software to plot the first three terms of $u(0, t)$ for $c = \ell = 1$.

2. The longitudinal motion of a rod with mass/length ρ , Young's modulus E and cross-sectional area A is restrained by a spring with stiffness k at $x = \ell$, as shown. What is the resulting B.C. at $x = \ell$?



3. Find the B.C. for a rod with mass/length ρ , Young's modulus E and cross-sectional area A which is moving longitudinally and has a particle of mass M attached at its end, $x = \ell$.

Homework No.6: 2nd REVISION

Due: Monday March 26, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. This problem is a forced vibrations version of the free vibrations problem in Homework No.5. The forced longitudinal vibrations of a rod with free ends are governed by the equations:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x^2 \cos t$$

with B.C. $\frac{\partial u}{\partial x} = 0$ at $x = 0, 1$

and I.C. $u = 0$ and $\frac{\partial u}{\partial t} = x$ at $t = 0$.

- a) Find the frequencies ω_n and modes of vibration $U_n(x)$.
(Same as Homework No.5 except that here $c = 1$ and $\ell = 1$.)
- b) Expand the forcing function in a series of $U_n(x)$.
- c) Expand the I.C. in a series of $U_n(x)$.
- d) Write the ODE and B.C. on $p_n(t)$.
- e) Solve d) for $p_n(t)$.
- f) Write out the first 2 terms of the solution, $p_0(t)U_0(x) + p_1(t)U_1(x)$.

2. The free vibrations of a (Bernoulli-Euler) beam are governed by the PDE:

$$\rho \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = 0$$

This problem concerns a beam which is clamped at both ends:

B.C. are $\frac{\partial u}{\partial x} = 0$ and $u = 0$ at $x = 0, \ell$.

NOTE: You may use the "Table of Beam Frequencies" posted on the Blackboard website.

- a) Find the frequency equation on ω_n and an equation for the mode shape $U_n(x)$.
- b) Find the first 4 frequencies $\omega_1, \dots, \omega_4$.
- c) Plot the corresponding 4 mode shapes.

3. A student tries to solve the PDE $\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} - \cos t$ with B.C. $w(0, t) = w(\pi, t) = 0$ by setting $w(x, t) = u(x) \cos t$. This gives the boundary value problem for $u(x)$:

$$\frac{d^2 u}{dx^2} + u = 1 \text{ with the B.C. } u = 0 \text{ at } x = 0, \pi.$$

Try to solve for $u(x)$. Comment on what is going on here.

Homework No.7

Due: Monday April 2, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1a. Using the Table of Beam Frequencies, find the lowest natural frequency of a uniform beam which is clamped at $x = 0$ and pinned at $x = \ell$.

1b. Find the static deflection of a beam as in 1a under a uniform distributed load.

1c. Using $V(x)$ as in 1b, compute Rayleigh's quotient and compare the resulting frequency with your result in 1a.

1d. A clamped-pinned beam as in 1a carries a concentrated mass m at its midspan. Use Rayleigh's method to approximate the beam's lowest natural frequency.

2. Although we have used Rayleigh's Method for beams, it may also be used for approximating the lowest frequency ω_1 in the longitudinal vibrations of rods.

Read section 7.6 on p.204 and then use it to find a bound on ω_1 in a tapered rod of circular cross section and length 2 whose radius is given by

$$r(x) = x + 1$$

with B.C.

$$u(0) = u(2) = 0$$

3. A student tries to solve the PDE $\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^4 w}{\partial x^4} + f(x) \cos t$

with B.C. $w(0, t) = w(\pi, t) = 0$ and $w_{xx}(0, t) = w_{xx}(\pi, t) = 0$ by setting $w(x, t) = u(x) \cos t$. This gives the boundary value problem for $u(x)$:

$$\frac{d^4 u}{dx^4} - u = f(x) \text{ with the B.C. } u = u'' = 0 \text{ at } x = 0, \pi.$$

a) Show by constructing the general solution to this ODE that this system has no solution for $f(x) = 1$.

b) Find a choice for $f(x)$ for which a solution exists.

Homework No.8

Due: Monday April 16, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. A clamped-free beam has a cross-section with constant depth, but whose width varies linearly from a maximum at the fixed end to zero at the free end. Taking the origin of the x -axis at the fixed end, take

$$I = I_0\left(1 - \frac{x}{\ell}\right)$$

$$\rho = \rho_0\left(1 - \frac{x}{\ell}\right)$$

where I_0 and ρ_0 are the moment of inertia and mass per unit length at the fixed end.

Choosing $V(x) = c_1x^2 + c_2x^3$, use the Ritz method to obtain a bound on ω_1 and ω_2 .

2. In class (see lecture notes 10 on the web) we used the Ritz method with two terms to approximate the lowest two natural frequencies of a clamped-free beam. Extend this analysis by using 3 terms:

$$V(x) = c_1x^2 + c_2x^3 + c_3x^4$$

Compare the values you get for ω_1 , ω_2 , ω_3 with the exact values given on the Table of Beam Frequencies.

Homework No.9

Due: Monday April 23, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. The radial vibrations of a fluid contained in a rigid spherical container are governed by the equation $\nabla^2 P = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2}$, where P is the pressure in the fluid. Setting $P = U(r) \cos \omega t$ gives

$$\frac{d^2 U}{dr^2} + \frac{2}{r} \frac{dU}{dr} + \frac{\omega^2}{c^2} U = 0$$

where r is the spherical polar coordinate $r^2 = x^2 + y^2 + z^2$. The associated boundary conditions are:

$$\frac{dU}{dr} = 0 \quad \text{at} \quad r = R$$

and

$$|U| < \infty \quad (\text{that is, } U \text{ is bounded}) \quad \text{at} \quad r = 0$$

where R is the radius of the rigid container.

a) Change variables from r to $\rho = \omega r/c$ and find the new ODE on $U(\rho)$. (The idea of this is to remove the constant $\frac{\omega^2}{c^2}$ from the equation.)

b) Seek a solution the ODE found in a) for $U(\rho)$ in the form of a power series in ρ . Find the first 5 nonzero terms.

$$U(\rho) = a_0 + a_1 \rho + a_2 \rho^2 + a_3 \rho^3 + \dots$$

c) Using the series expression you obtained in b), substitute the B.C. $dU/d\rho = 0$ at $r = R$ (that is, at $\rho = \omega R/c$) and thereby find an approximate value for the lowest natural frequency ω_1 .

2. In studying the axisymmetric vibrations of a circular drumhead (membrane), we have met the Bessel function $J_0(x)$, which satisfies the ODE:

$$x^2 J_0'' + x J_0' + x^2 J_0 = 0.$$

A related function is the Bessel function J_n ($n = 1, 2, 3, \dots$) which comes up in the general (non-axisymmetric) vibrations of a circular drumhead. It satisfies the ODE:

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0.$$

In studying the axisymmetric vibration of a circular plate, we encountered J_1 , where it was claimed that

$$J_1 = -J_0'.$$

Derive this relation by using the ODE's on J_0 and J_1 .

Hint: Show that J_0' satisfies the same ODE as $-J_1$.

Homework No.10

Due: Monday April 30, 2012

RULES: Work alone (no collaboration). Do not copy other students' work. If you use a reference (online or from a book), cite it.

1. Use the method of harmonic balance to determine an approximate relation between the frequency ω and amplitude A of free vibration in the following conservative nonlinear system:

$$\frac{d^2x}{dt^2} + x = \alpha x^5$$

2a. Use the method of harmonic balance to determine an approximate expression for any limit cycles which are exhibited by the following nonconservative system:

$$\frac{d^2x}{dt^2} + x = 0.1 (1 - 2x^2 + bx^4) \frac{dx}{dt}$$

Leave answer in terms of the parameter b .

2b. Find a value for the parameter b such that this equation exhibits a limit cycle fold, that is a limit cycle which is variously known as degenerate, semistable, or a double root, and which represents the merging of two limit cycles into one.

2c. Use the matlab add-on "pplane" to numerically integrate the above ODE. Compare the predictions obtained for limit cycle amplitudes in 2a and 2b above, with those obtained using pplane.

3. This question concerns the following system of 2 first order equations:

$$\begin{aligned} \frac{dx}{dt} &= y + 0.1x^3 + \alpha x \\ \frac{dy}{dt} &= -x + 0.1 \left(\frac{dx}{dt} \right)^3 - \beta \frac{dx}{dt} \end{aligned}$$

3a. Rewrite this system as a single second order ODE on $x(t)$, i.e. in which y is absent.

3b. Apply harmonic balance to the resulting second order ODE and thereby obtain an expression for the amplitude A of any limit cycles which occur.

3c. Find a relation between parameters α and β for a Hopf bifurcation to occur. Identify it as supercritical or subcritical.