Today: e^{At}, damping and forcing

Calculus Review: consider scalar equation $\dot{x} = ax$ with $x(0) = x_0$

Solution is $x(t)$

$\dot{x} = ax$

$\ddot{x} = \frac{d}{dt} ax = a\dot{x} = a(ax) = a^2 x$

$\dddot{x} = \frac{d}{dt}(\ddot{x}) = a^2 \dot{x} = a^2(ax) = a^3 x$

etc.

given first order ODE and its solution at one t

- taylor series for all times

Taylor Series at $x = 0$

$x(t) = x_0 + \dot{x}t + \frac{1}{2}\ddot{x}t^2 + \frac{1}{3!}\dddot{x}\ t^3 + \ldots$

$x(t) = a_0 + ax_0 t + \frac{a^2 x_0}{2} t^2 + \frac{a^3 x_0 t^3}{3!} + \ldots$

$= x_0 (1 + at + \frac{a^2 t^2}{2} + \frac{a^3 t^3}{3!} + \ldots$

Check does $x(t) = e^{at}$ $\dot{x} = ax$

$\frac{d}{dt} x(t) = 0 + a + a^2 t + \frac{3a^3 t^3}{2 \cdot 3} + \ldots$

$= a\, x(t)$ ✓

$\dot{x} = ax$, $x(0) = x_0$ has solution $x = x_0 e^{at}$

Back to multivariable world : $M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0}$

$\dot{\vec{z}} = A\vec{z}$ with $\vec{z}(0) = \vec{z}_0$

$\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix}$, $A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}C & -M^{-1}K \end{array}\right]$

Guess Taylor Series Solution:

$\vec{z}(t) = \vec{z}_0\left(1 + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \dots\right)$

Check: does $\dot{\vec{z}} = A\vec{z}$

$\vec{z}_0\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right)$

$\frac{d}{dt}\left(\vec{z}_0\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right)\right)$

define $e^{At} = \left(I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \dots\right)$ Series converges no matter the value of A

Solution of $\dot{\vec{z}} = A\vec{z}$ with $\vec{z}(0) = \vec{z}_0$

$$\boxed{\vec{z} = e^{At}\vec{z}_0}$$

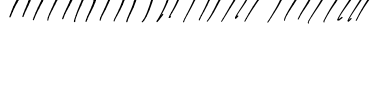
Why is this better than $\vec{z} = c_1e^{\lambda_1 t}\bar{z}_1 + c_2e^{\lambda_2 t}\bar{z}_2 + \dots$

$\bar{z}_i, \lambda_i$ = e - things of A

1.) Tidy
2.) Don't have problem of missing eigenvectors

Simplest vibration problem there is:

$K=0$, $C=0$



1.) old method: $m\ddot{x} + 0\dot{x} + 0x = 0$

guess: $\vec{z} = \begin{bmatrix} x \\ v \end{bmatrix} = \bar{z}e^{\lambda t}$

$\dot{\vec{z}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A}\vec{z}$

$\det(A - \lambda I) = 0$

$\det \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = \lambda^2 = 0 \qquad \lambda = 0, 0$

Look for an eigenvector that goes with this : $\lambda_1 = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \bar{z}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{z}(t) = C_1 e^{0t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + ?$

$\vec{z}(t) = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + ?$

trick $\longrightarrow$ Secular terms $\qquad \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} + \begin{bmatrix} v_0 t \\ 0 \end{bmatrix}$

All vectors on the right!

Try matrix exponential solution

$$\vec{z} = e^{At} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \left(I + At + \frac{A^2 t^2}{2} + \dots\right) \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

$$= I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + 0 + 0 \dots) \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} x_0 + v_0 t \\ v_0 \end{bmatrix}$$

given: $\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{F}(t)$

e^{At} gives analytic solution to homogenous equations (transient solution)

What about particular solution : $M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \bar{F} e^{i w t}$

$M, C, K, \bar{F}, iw$ given

Find $\vec{x}(t) = \vec{X} = \bar{X} e^{iwt}$

good guess unless $C = [0]$ and $w = w_i$ (resonance)

Put the guess in: $-w^2 M\bar{X} + iwC\bar{X} + K\bar{X} = \bar{F}$

$$\rightarrow \left(-\omega^2 M + i\omega C + K\right)\bar{X} = F$$

Matlab : $\boxed{\bar{X} = \left[-\omega^2 M + i\omega C + K\right] \backslash \bar{F}}$, we are told the frequency