Normal Modes continued

Overall goal: Understand $m\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{F}(t)$

What's the relationship to Lagrange? $(E_K(\vec{q},\dot{\vec{q}}), E_p(\vec{q}))$

$$\mathcal{L} = E_K - E_p$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Coordinates

Near equilibrium: $\vec{q} = \vec{0}$

$\ddot{\vec{q}}|_{\vec{q}=0} = 0$

Want it to be stable: $E_p > 0 \rightarrow \vec{q} \neq \vec{0}$

$E_p = 0 \rightarrow \vec{q} = 0$

Kinetic Energy always positive : $E_K = \frac{\sum m_i v_i^2}{2}$

-near equilibrium: $E_K = \frac{1}{2}\underbrace{\left[\sum\sum \frac{\partial E_K}{\partial \dot{q}_i \partial \dot{q}_j}\right]}_{M_{ij}}{}_{q_i=0} \cdot \dot{q}_i \dot{q}_j + \dots$

$M_{ij} \rightarrow$ symmetric positive definite: $M_{ij}\dot{q}_i\dot{q}_j > 0$ for all $\vec{q} \neq \vec{0}$

Side note on taylor series:

Any function of $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $f(\vec{x}) = f(\vec{0}) + \sum \frac{\partial f}{\partial x_i}\Big|_{\vec{x}=\vec{0}} + \frac{1}{2}\sum\sum \frac{\partial^2 f}{\partial x_i x_j} + \sum\sum\sum \dots$

$$\mathcal{L} = E_K - E_p = \frac{1}{2}\sum\sum M_{ij}\dot{q}_i\dot{q}_j + \frac{1}{2}\sum\sum K_{ij} q_i q_j$$

$\rightarrow \boxed{M\ddot{\vec{q}} + K\vec{q} = \vec{0}}$ M/K are both symmetric and positive definite

-K only needs to be semi-definite

How to solve $M\ddot{\vec{x}} + K\vec{X} = \vec{0}$ with $\underline{\text{givens:}}$ $\vec{X}(0) = \vec{X}_0$, $\vec{V}(0) = \vec{V}_0$

<u>Method 1:</u> (Matlab) $z = [x; v]$, $z_0 = [x_0; v_0]$, $\dot{x} = v$; $\dot{v} = -M\backslash(K*x)$;

Method 2.a: guess: $\vec{X} = \bar{X}\cos(\omega t)$

$$\rightarrow \underbrace{(-\omega^2 M + K)}_{A}\bar{X} = \vec{0} \qquad \det(A) = 0$$

solve polynomial, find $\bar{X}$ $\longrightarrow$ add up solutions: $\vec{X} = \sum 3_i \bar{X}_i \cos(\omega_i t)$ + sin terms

constants

Use initial conditions: $M \cdot K \rightarrow \omega_i, X_i$

$IC's \rightarrow 3c_i, 3s_i$

Method 2.b: Shortcut: $[V\ D] = eig(K,M)$ $\rightarrow$ gives ω's and X's

$$V = [\bar{X}_1 | \bar{X}_2 |] \qquad D = \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 & \cdots \\ 0 & \omega_2^2 & 0 & 0 & \cdots \\ 0 & 0 & \omega_3^2 & \cdots & \cdots \end{bmatrix}$$

Method 2.c: $M\ddot{\vec{X}} + K\vec{X} = \vec{0}$

$$M^{-1}[M\ddot{\vec{X}} + K\vec{X}] = 0 \longrightarrow \ddot{\vec{X}} + \underbrace{M^{-1}K}\vec{X} = \vec{0}$$

- nonsingular
- generally not symmetric

$$[V\ D] = eig(M^{-1}K) \qquad D = \begin{bmatrix} \omega_1^2 & 0 & 0 & \cdots \\ 0 & \omega_2^2 & \cdots & \cdots \\ 0 & 0 & \omega_3^2 & \cdots \end{bmatrix}$$

$V \rightarrow [\bar{X}_1 | \bar{X}_2 | \ldots]$

Works but don't have confidence because $M^{-1}K$ is generally not symmetric

insecure: maybe not enough eigenvectors to span set of IC's