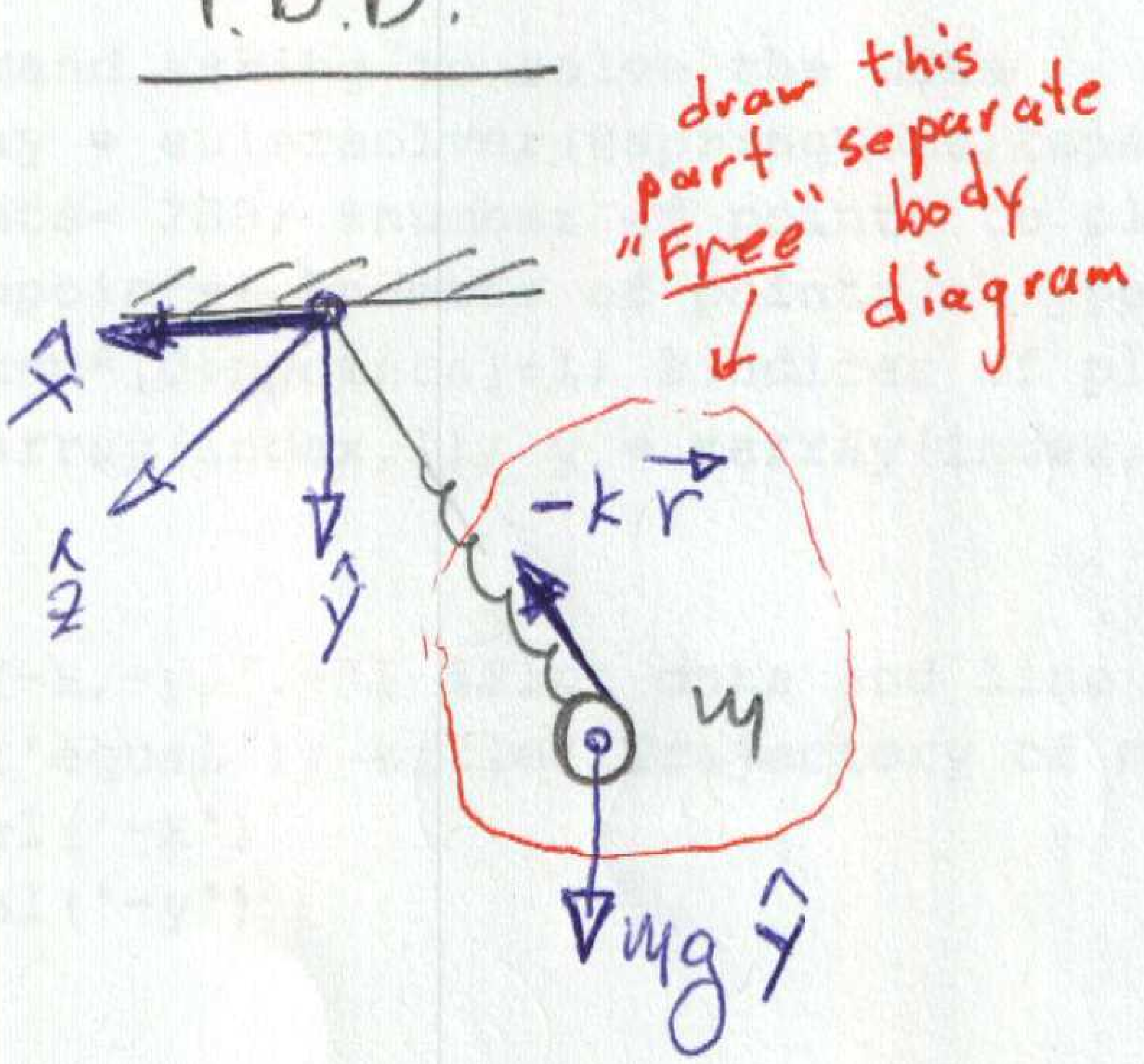
Problem 3 (from fall 2012)

- mass hanging from zero-rest-length linear spring
- const. grav. field

a) set up equations

F.B.D.



L.M.B.

$$m\ddot{\vec{r}} = mg\hat{y} - k\vec{r} \Rightarrow$$

$$\ddot{\vec{r}} = g\hat{y} - \frac{k}{m}\vec{r} \Rightarrow$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = g \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{k}{m} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ ✓

b) From the plots of the trajectory we can see that the pattern of its motion consists of ellipses, which change shape depending on the initial conditions. [MATLAB + PLOT]
The Euler method was applied.

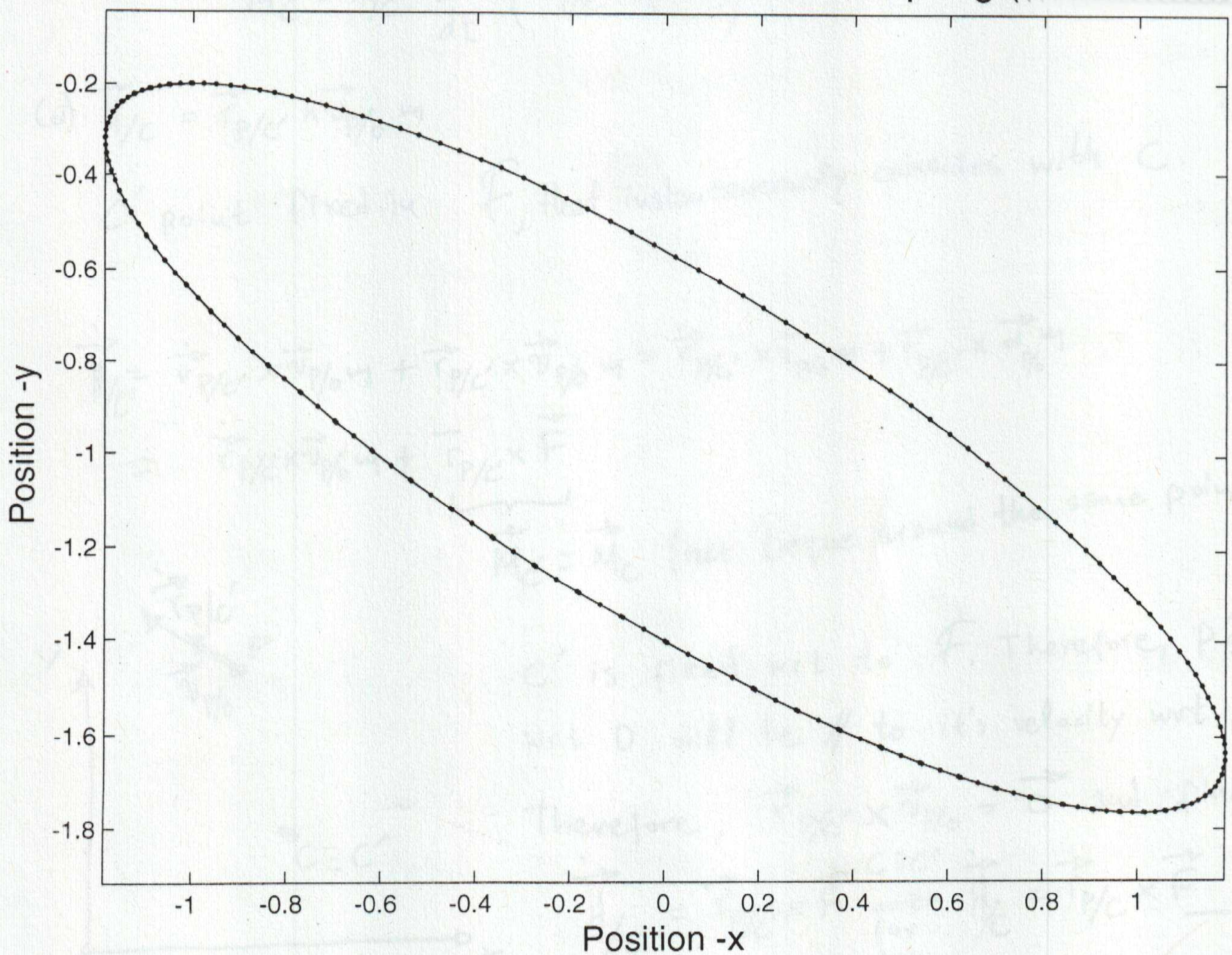
c) We know that the solutions of the previous equations are in the form $x = x_0\cos(\lambda t)$ and $y = y_0 \sin(\lambda t)$ for example in the plane (x,y).
Therefore we can have the system: $\left.\begin{array}{l} x = x_0\cos(\lambda t) \\ y = y_0\sin(\lambda t) \end{array}\right\} \xrightarrow{\wedge 2} \left.\begin{array}{l} x^2 = x_0^2\cos^2(\lambda t) \\ y^2 = y_0^2\sin^2(\lambda t) \end{array}\right\}$

$$\Rightarrow \left.\begin{array}{l} \dfrac{x^2}{x_0^2} = \cos^2(\lambda t) \\ \dfrac{y^2}{y_0^2} = \sin^2(\lambda t) \end{array}\right\} \xrightarrow{(+)} \frac{x^2}{x_0^2} + \frac{y^2}{y_0^2} = 1 \Rightarrow$$ Equation of the ellipse in plane (x,y)! ✓

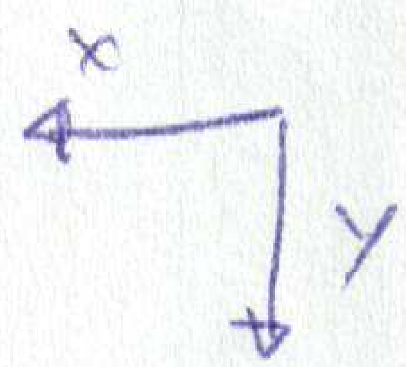
```
function spring()
%Problem set up
clear
close all
p.m = 1; p.k= 100; p.g= 9.81; % parameters are in struct p
n=100000; % number of steps in integration
tmax = 1.3*2*pi/sqrt(p.k/p.m); % duration of integration (tried a lot of tmax's)
tspan= linspace(0,tmax,n+1);
x0=0; y0=0; z0=0; vx0=0.09; vy0=0.01; vz0=0.03;
r0=[x0 y0 z0]'; v0 = [vx0 vy0 vz0]';
z0 = [r0;v0]; %Initial condition

%Command asking to solve the ODEs
zarray = eulersolver(@springODEs,tspan, z0,p);
npoints= 200; %number of points to plot
m=n/npoints; %number of points skipped in each plot point
index=m*[0:npoints]+1; %indices of plotted points
x= zarray(index,1); y = zarray(index,2); z = zarray(index,3);


plot(-x,-y,'.-') %Plot dots and line
axis('equal'); title('Trajectory of mass attached to Linear Spring (C. Mavrogiannis)')
xlabel('-x')
ylabel('-y')
shg

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zdot = springODEs(t,z,p)
%The ODEs for zero-rest-length linear spring
r=z(1:3); v=z(4:6); % position and velocity
rdot = v;
F = p.m*(p.g*[0 1 0]'-p.k/p.m*r);
a = F/p.m;
vdot = a;
zdot = [rdot; vdot];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zarray = eulersolver(rhs,tarray, z0,p)
% General ODE solver using Euler's method
% Soln will be in zarray. One row for each instant in time.
```

```
nrows=length(tarray); ncols= length(z0);
zarray=zeros(nrows,ncols); %initialize the matrix of solutions
zarray(1,:) = z0; % first row is init. conds.
for i=1:nrows-1
h = tarray(i+1) - tarray(i); %time step
zdot = feval(rhs,tarray(i), zarray(i,:)',p);
zarray(i+1,:) = zarray(i,:) + zdot'*h; %Euler's method
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Problem 3-Trajectory of mass attached to Linear Spring [redacted]



plot $(-x, -y)$ to make plot physically represent the reality. Because I set (x, y) to be



2012 # 4

For which of the following definitions of $\vec{H}_{/c}$ is the equation $(\vec{M}_c = \dot{\vec{H}}_{/c})$ of motion true?

(a) $\vec{H}_{/c} = \vec{r}_{p/c'} \times \vec{V}_{p/o}\, m$

$$\dot{H}_{/c} = \dot{\vec{r}}_{p/c'} \times \vec{V}_{p/o}\, m + \vec{r}_{p/c'} \times \dot{\vec{V}}_{p/o}\, m$$
$$= m\left(\dot{\vec{r}}_{p/c'} \times \vec{V}_{p/o} + \vec{r}_{p/c'} \times \dot{\vec{V}}_{p/o}\right)$$
$$= m\left(\vec{V}_{p/c'} \times \vec{V}_{p/o} + \vec{r}_{p/c'} \times \vec{a}_{p/o}\right)$$

c' is the position of point c in its own reference frame, which is fixed. The origin, O, is fixed as well. Therefore

$$\vec{V}_{p/c'} = \vec{V}_{p/o}$$

There cross product is 0:

$$\vec{V}_{p/c'} \times \vec{V}_{p/o} = 0$$

The equation reduces to:

$$\dot{\vec{H}}_{/c} = \vec{r}_{p/c'} \times \vec{a}_{p/o}\, m$$

This holds true at one instant in time, when the position of c instantaneously $= c'$

$$\boxed{\text{For only one instant: } \dot{\vec{H}}_{/c} = \vec{r}_{p/c} \times \vec{a}_{p/o}\, m}$$

(b) $\vec{H}_{/c} = \vec{r}_{p/c} \times \vec{v}_{p/o}\, m$

$$\dot{\vec{H}} = \dot{\vec{r}}_{p/c} \times \vec{V}_{p/o}\, m + \vec{r}_{p/c} \times \dot{\vec{V}}_{p/o}\, m$$

$$= m\left(\dot{\vec{r}}_{p/c} \times \vec{V}_{p/o} + \vec{r}_{p/c} \times \dot{\vec{V}}_{p/o}\right)$$

$$= m\left(\vec{V}_{p/c} \times \vec{V}_{p/o} + \vec{r}_{p/c} \times \vec{a}_{p/o}\right)$$

$$= m\left(\vec{V}_{p/c} \times \left(\vec{V}_{p/c} + \vec{V}_{c/o}\right) + \vec{r}_{p/c} \times \vec{a}_{p/o}\right)$$

$$= m\left(\cancel{\vec{V}_{p/c} \times \vec{V}_{p/c}} + \vec{V}_{p/c} \times \vec{V}_{c/o} + \vec{r}_{p/c} \times \vec{a}_{p/o}\right)$$

$$\dot{\vec{H}} = m\left(\vec{V}_{p/c} \times \vec{V}_{c/o} + \vec{r}_{p/c} \times \vec{a}_{p/o}\right)$$

This holds true for the following special cases:

1. $\vec{V}_{p/c} = 0$: If particle p and point C are moving at the same velocity, $\vec{V}_{p/c} = 0$

2. $\vec{V}_{c/o} = 0$: If point C is instantaneously stable with respect to O.

3. $\vec{V}_{p/c} \times \vec{V}_{c/o} = 0$: If the two velocities are parallel or antiparallel

(c) $\vec{H}_{/c} = \vec{r}_{p/c} \times \vec{V}_{p/c}\, m$

$$\dot{\vec{H}}_{/c} = \dot{\vec{r}}_{p/c} \times \vec{V}_{p/c}\, m + \vec{r}_{p/c} \times \dot{\vec{V}}_{p/c}\, m$$

$$= \vec{V}_{p/c} \times \vec{V}_{p/c}\, m + \vec{r}_{p/c} \times \vec{a}_{p/c}\, m$$

$$= 0 + \vec{r}_{p/c} \times (\vec{a}_{p/o} - \vec{a}_{c/o})$$

$$\dot{\vec{H}} = (\vec{r}_{p/c} \times \vec{a}_{p/o} - \vec{r}_{p/c} \times \vec{a}_{c/o})\, m$$

This holds true for the following special cases:

① $\vec{r}_{p/c} = 0$: if particle c and particle p are at the same position

② $\vec{a}_{c/o} = 0$: if point c reverses direction, $\vec{a}_{c/o}$ will be 0.

③ $\vec{r}_{p/c} \times \vec{a}_{c/o} = 0$: Both can be nonzero, but if the position and acceleration are parallel or antiparallel

Problem 3 (#5 from Fall 2012PDF)

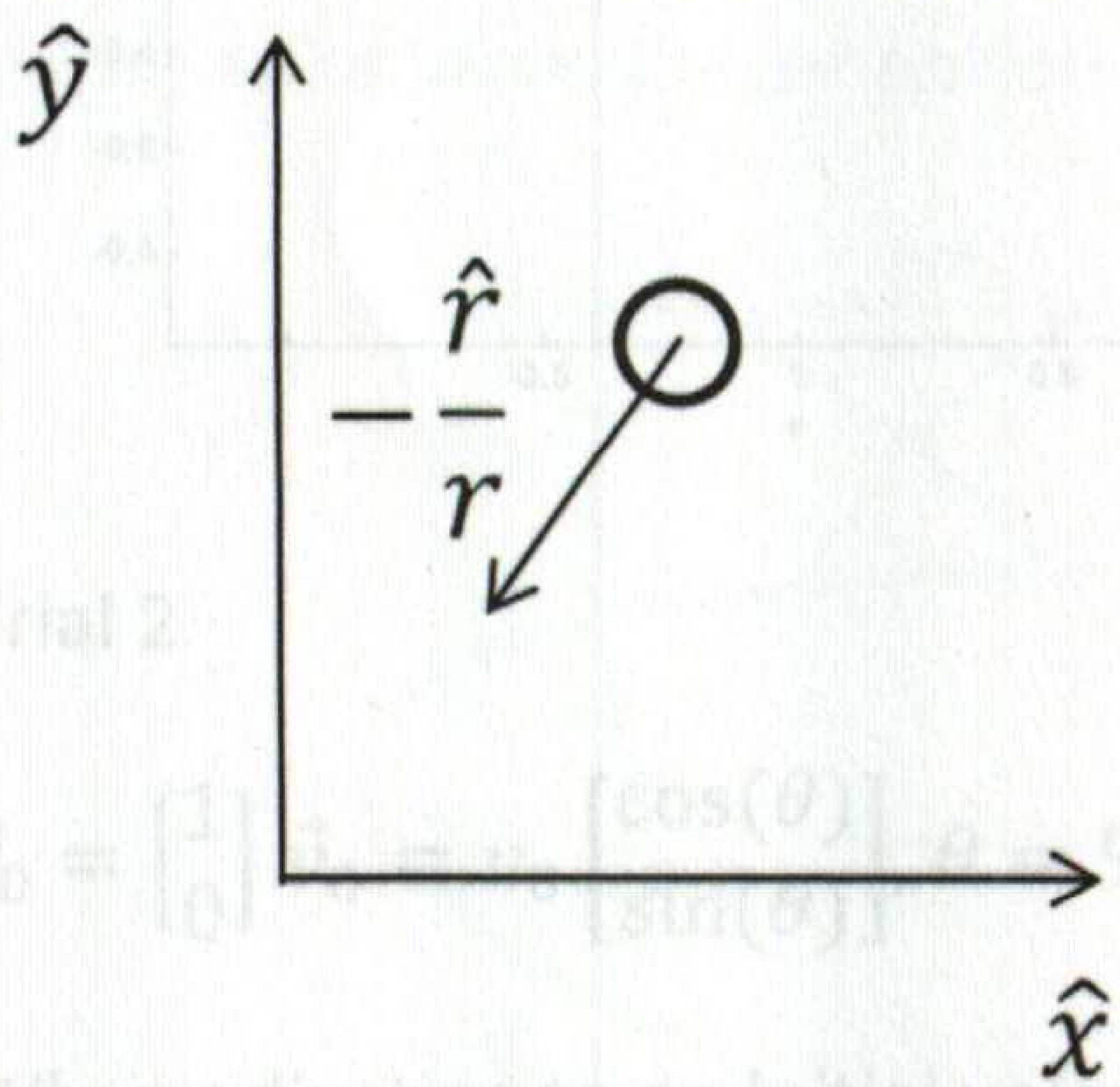
*How to complete this problem was discussed with Professor Ruina in office hours

Investigate trajectories of a particle experiencing a central force that is not of the form $F(r) = -kr$ or $F(r) = -\frac{k}{r^2}$. Try to find trajectories that are not straight lines or ellipses. Try to find initial conditions that will make irregular orbits periodic.

Solution:

The trajectory of a particle experiencing $F(r) = -\frac{1}{r}$ will be investigated in this problem.

FBD



$$\vec{F} = m\vec{a}$$

$$m\ddot{\vec{r}} = -\frac{\hat{r}}{r}$$

Breaking into Two Sets of First Order ODES

$$\dot{\vec{r}} = \vec{v}$$

$$\dot{\vec{v}} = -\frac{\hat{r}}{r}$$

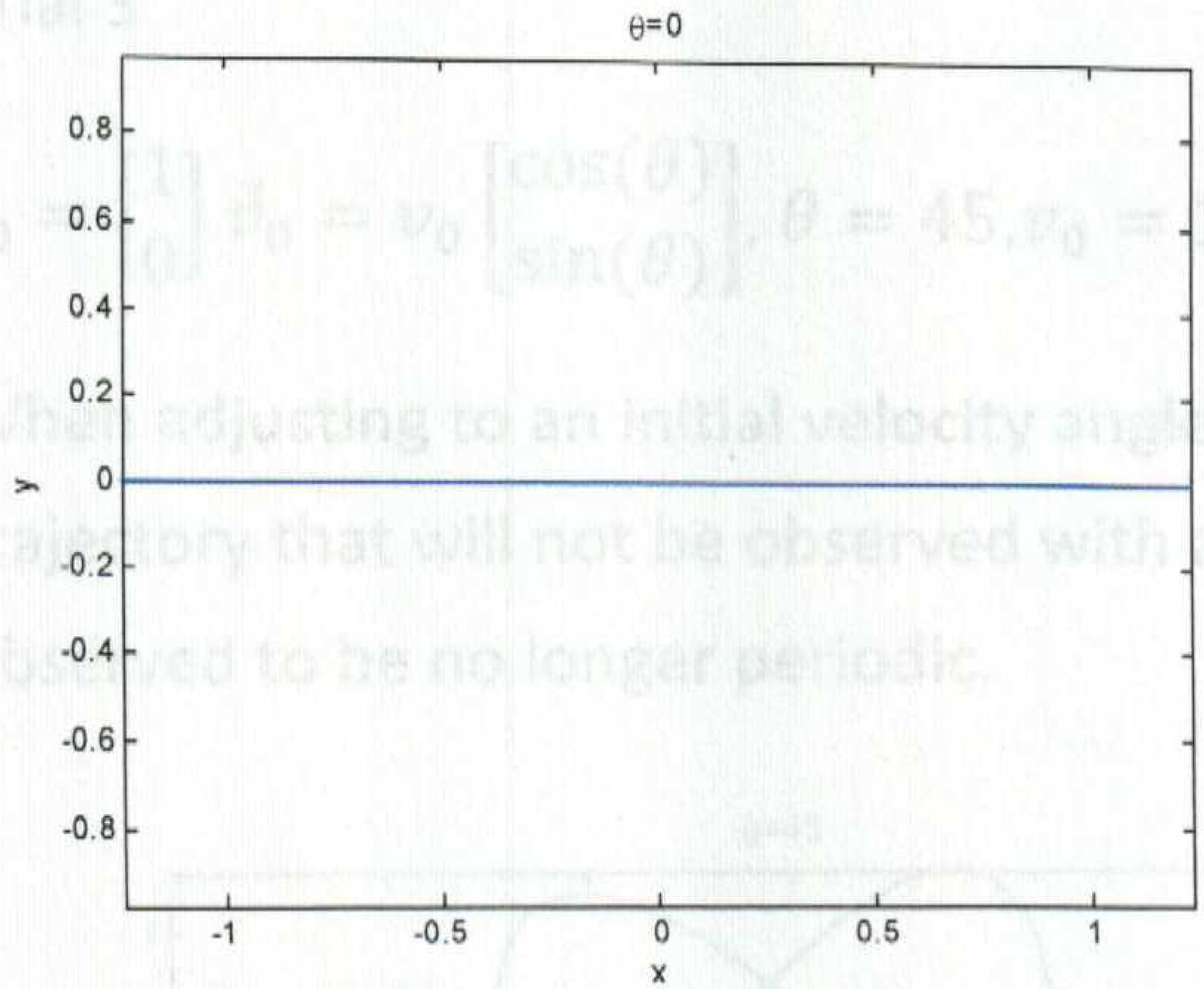
These ODES are evaluated in Matlab using ODE45(Code Included)

Effects of Adjusting Initial Velocity Angle From Horizontal:

Trial 1:

$$\vec{r}_0 = \begin{bmatrix}1\\0\end{bmatrix}\ \vec{v}_0 = v_0\begin{bmatrix}\cos(\theta)\\\sin(\theta)\end{bmatrix}, \theta = 0, v_0 = 1$$

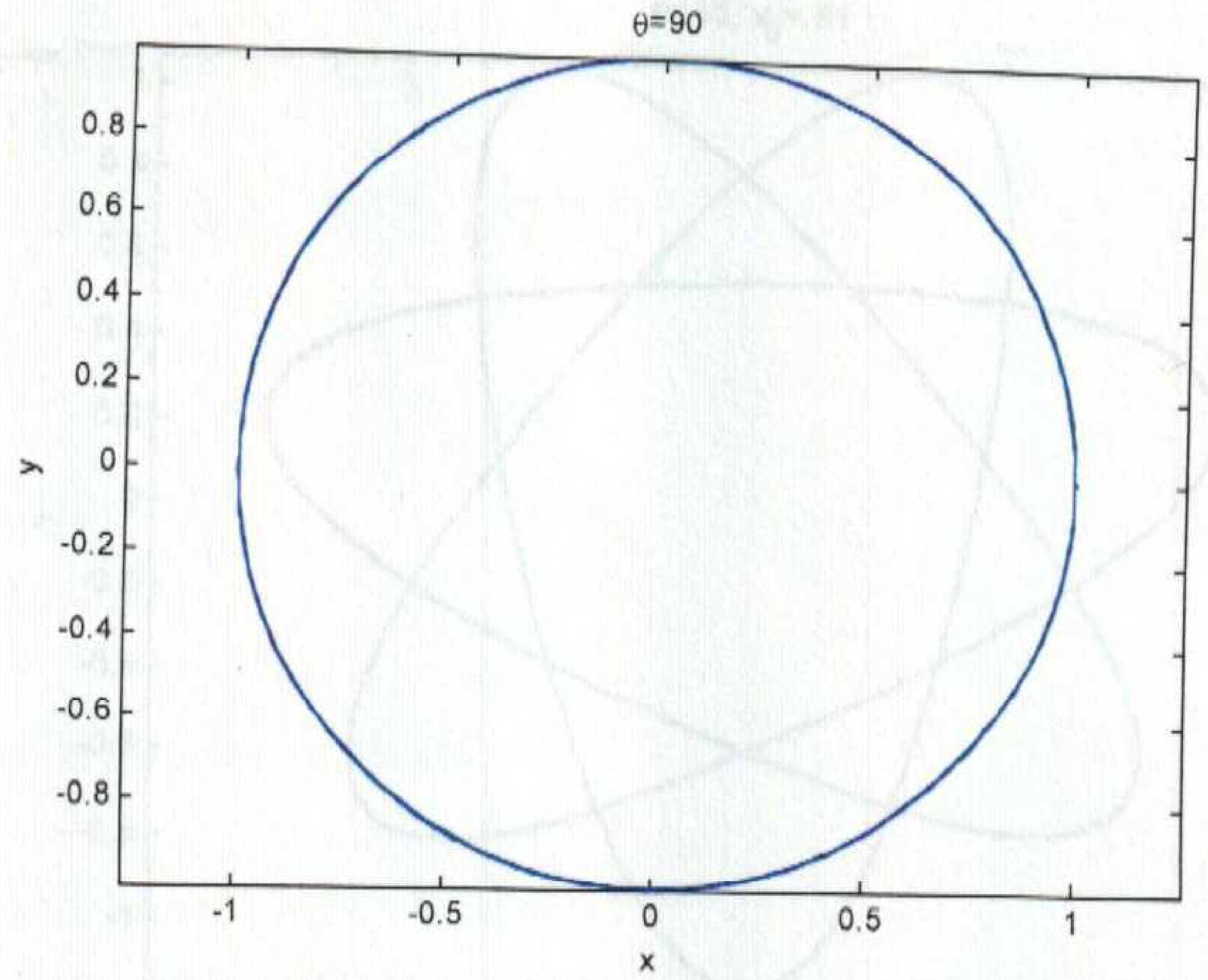
With these initial conditions, the particle oscillates along the x-axis. So no interesting behavior is observed.



Trial 2

$$\vec{r}_0 = \begin{bmatrix}1\\0\end{bmatrix}\ \vec{v}_0 = v_0\begin{bmatrix}\cos(\theta)\\\sin(\theta)\end{bmatrix}, \theta = 90, v_0 = 1$$

When adjusting to an initial velocity angle of 90°, the particle moves periodically in a circle around the origin. So again, no interesting behavior is observed.

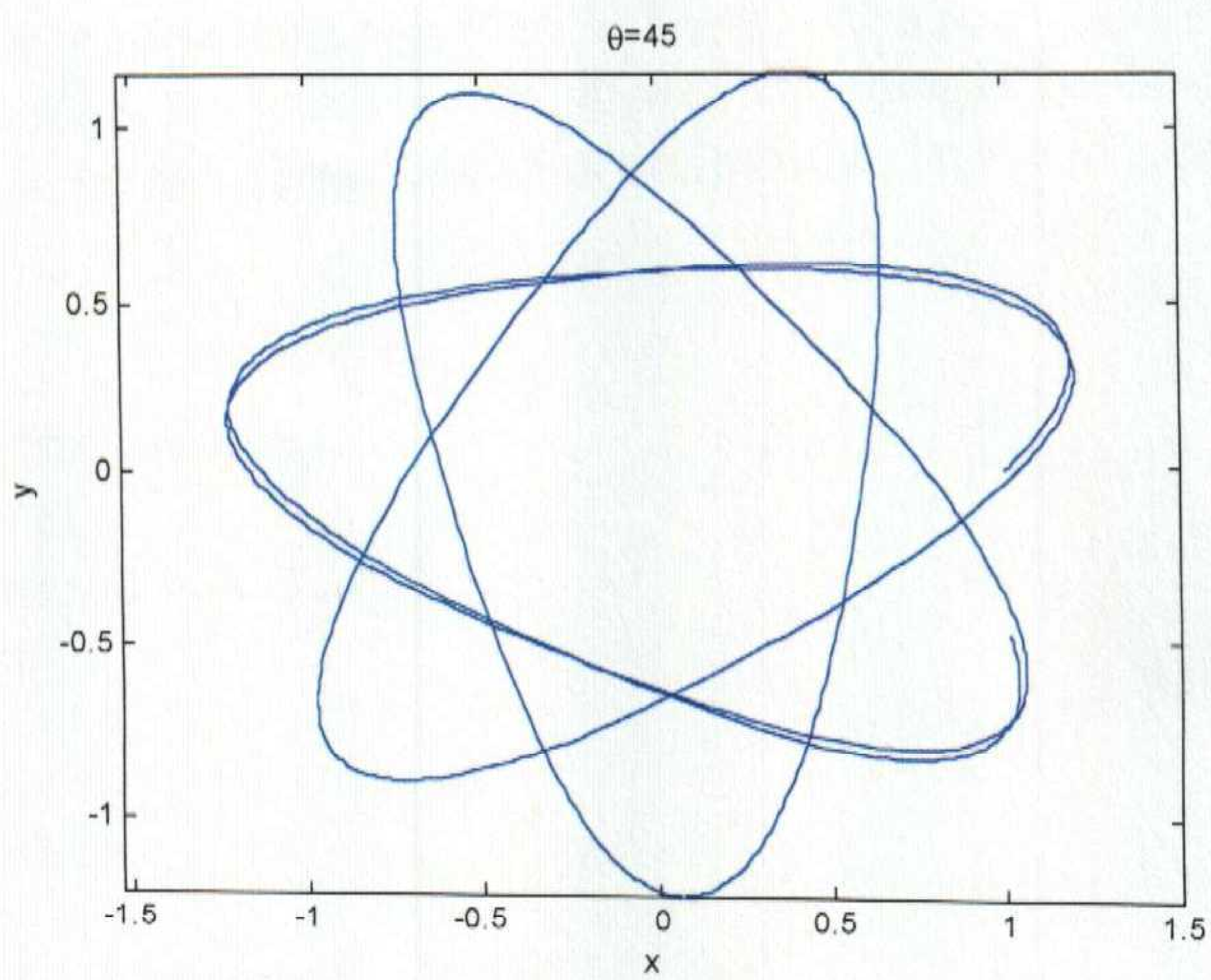


Nice

Trial 3

$$\vec{r}_0 = \begin{bmatrix}1\\0\end{bmatrix} \vec{v}_0 = v_0 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \theta = 45, v_0 = 1$$

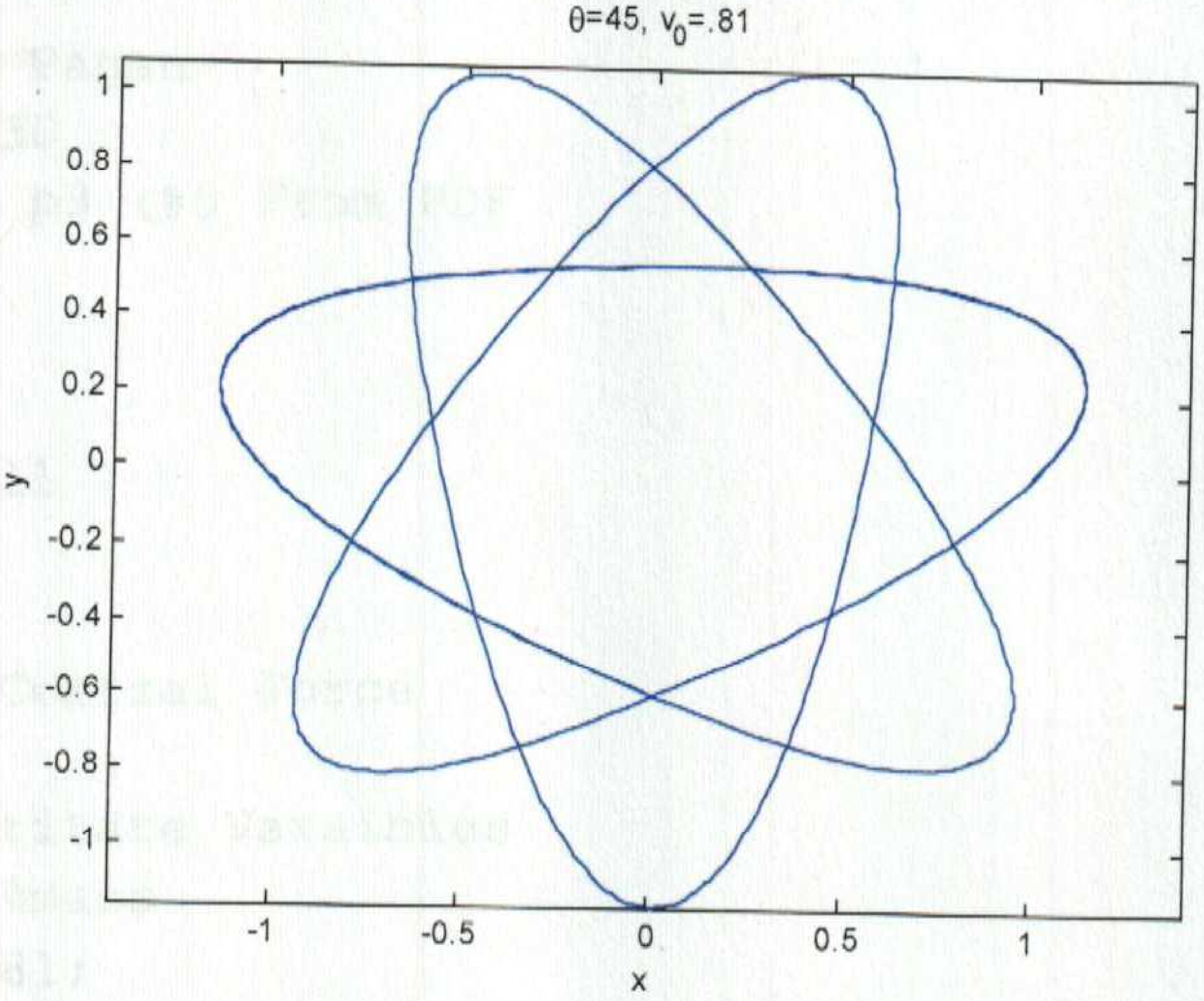
When adjusting to an initial velocity angle of 45°, the particle begins to move in an interesting trajectory that will not be observed with a –kr or $-\frac{k}{r^2}$ central force. However, the trajectory is observed to be no longer periodic.



Trial 4

$$\vec{r}_0 = \begin{bmatrix}1\\0\end{bmatrix} \vec{v}_0 = v_0 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \theta = 45, v_0 = .81$$

With adjustments to the initial speed (v_0), the trajectory can be adjusted to be periodic. This initial velocity was found by trial and error.



Nice

3.27) PROBLEM

A planet is orbiting its sun.

a) show that $\ell = mr^2\omega$

b) show $dA/dt = \frac{1}{2}r^2\omega = \ell/2m$

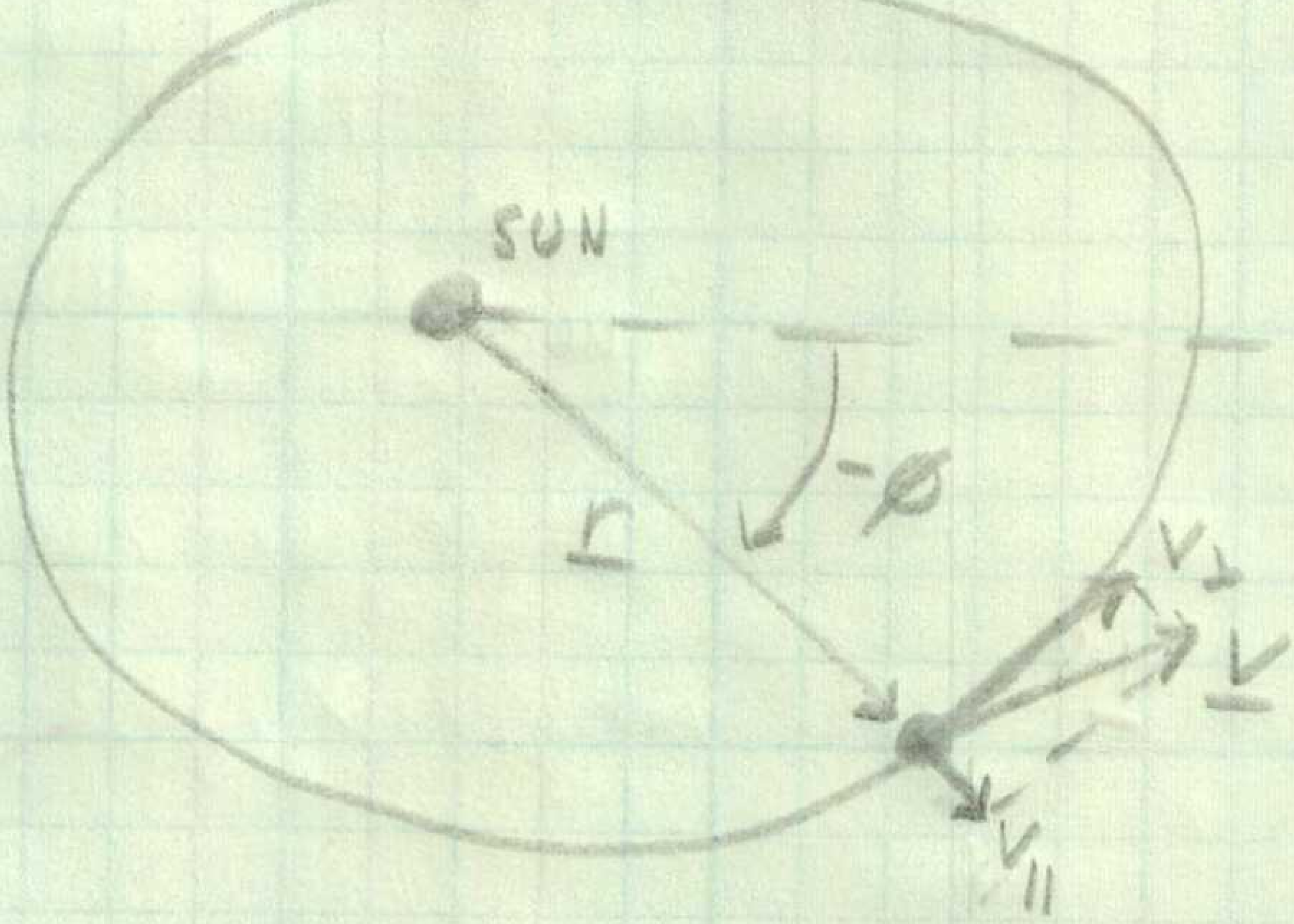
SOLUTION

a) $\ell = m(\underline{r} \times \underline{v})$

$= m(r v_\perp)$

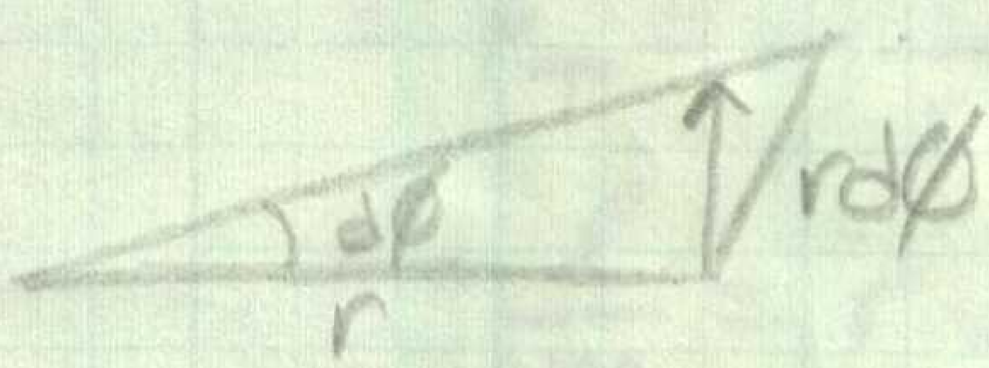
$= m(r(r\omega))$

$= \boxed{mr^2\omega}$ ✓



Sidenote!
Governing ODE
$\underline{F} = m\underline{a}$
$= -G\frac{Mm}{|r|^2}\frac{\underline{r}}{|r|}$

b)



$dA = \frac{1}{2}bh$

$= \frac{1}{2}r(rd\phi)$

$= \frac{1}{2}r^2 d\phi$

$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\phi}{dt}$

$= \boxed{\frac{1}{2}r^2\omega}$ ✓

$= \frac{\ell}{2m}$ = constant because $\ell = cst$, $m = cst$ ✓

5) 4.4

A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. Initially the particle is moving in a circle of radius r_0 with angular velocity ω_0, but I now pull the string down through the hole until a length r remains between the hole and the particle.

a. What is the particles angular velocity now?

cons. of angular momentum

$$\ell = \ell$$

$$m r_0^2 \omega_0 = m r^2 \omega$$

$$r^2 \omega = r_0^2 \omega_0$$

$$\omega = \frac{r_0^2 \omega_0}{r^2}$$

$$\boxed{\omega = \left(\frac{r_0}{r}\right)^2 \omega_0}$$

b. $W = \int_{r_1}^{r_2} \vec{F} \cdot dr = \Delta E_K$ $\quad v = \omega r$

$$\Delta E_K = E_f - E_i = \frac{1}{2} m (v_f^2 - v_0^2) = \frac{1}{2} m \left[(\omega r)^2 - (\omega_0 r_0)^2 \right] = \frac{1}{2} m \left[\left(\frac{r_0^2}{r^2} \omega_0 r \right)^2 - (\omega_0 r_0)^2 \right] = \frac{1}{2} m \left[\left(\frac{r_0^2 \omega_0}{r} \right)^2 - \omega_0^2 r_0^2 \right]$$

$$= \frac{m}{2} \left[\frac{r_0^4 \omega_0^2}{r^2} - \omega_0^2 r_0^2 \right] = \frac{m}{2} \left[\frac{r_0^4 \omega_0^2}{r^2} - \frac{\omega_0^2 r_0^2 r^2}{r^2} \right] = \frac{m}{2} \left[\frac{\omega_0^2}{r^2} \left(r_0^4 - r_0^2 r^2 \right) \right]$$

$$\Delta E_K = \frac{m (r_0 \omega_0)^2}{2} \left(\frac{r_0^2}{r^2} - 1 \right)$$

$$W = \int_{r_1}^{r_2} \vec{F}(r) \cdot dr$$

Tension in the String:

$\vec{F}(r) = m \underbrace{\frac{v^2}{r}}_{\text{radial acceleration}}$ $\quad$ remember: $v = \omega r$ $\quad \therefore \vec{F}(r) = m \frac{(\omega r)^2}{r} = m \omega^2 r$

$$\int_{r_0}^{r} (m \omega_1^2 r)\, dr = m \left[\omega_1^2 \frac{1}{2} r^2 \right]_{r_0, \omega_0}^{r, \omega} = m \left[\frac{\omega^2 r^2}{2} - \frac{\omega_0^2 r_0^2}{2} \right] = m \left[\frac{\left(\frac{r_0^2}{r^2} \omega_0 \right)^2 r^2}{2} - \frac{\omega_0^2 r_0^2}{2} \right]$$

$$= \frac{m}{2} \left[\frac{r_0^4 \omega_0^2}{r^2} - \frac{\omega_0^2 r_0^2}{\cancel{2}} \right] = \frac{m}{2} (r_0 \omega_0)^2 \left(\frac{r_0^2}{r^2} - 1 \right)$$

$$\boxed{W = \frac{m (r_0 \omega_0)^2}{2} \left(\frac{r_0^2}{r^2} - 1 \right)}$$

$\boxed{W = \Delta E_K}$ Work is equal to the change in kinetic Energy!

Problem 6 4.23 (Taylor)

Which forces are conservative? If conservative, find potential energy U and verify by direct differentiation that $F = -\nabla U$

NOTE: F is conservative if ① F depends only on position
② $\nabla \times F = 0$

a) $F = k(x, 2y, 3z)$

$(\nabla \times F)_x = \frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y = 0 - 0 = 0$

$(\nabla \times F)_y = \frac{\partial}{\partial z} F_x - \frac{\partial}{\partial x} F_z = 0 - 0 = 0$

$(\nabla \times F)_z = \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x = 0 - 0 = 0$

$\boxed{\text{conservative}}$ ✓

$\frac{\partial U}{\partial x} = -F_x = -xk$

$\frac{\partial U}{\partial y} = -F_y = -k2y$

$\frac{\partial U}{\partial z} = -F_z = -k3z$

FIND potential energy, U

$\int \frac{\partial U}{\partial x} = \int -kx$

$U(x, y, z) = \frac{-kx^2}{2} + f(y, z)$

$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-kx^2}{2} + f(y, z)\right)$

$\frac{\partial U}{\partial y} = \int \frac{df}{\partial y} = \int -k2y$

$f(y, z) = \frac{-k2y^2}{2} + g(z)$

$U(x, y, z) = \frac{-kx^2}{2} + -ky^2 + g(z)$

$\frac{dU}{\partial z} = \int \frac{dg}{\partial z} = \int -k\frac{3z^2}{2}$

$g(z) = \frac{-k3z^2}{2} + C$

$U(x, y, z) = -\frac{1}{2}(x^2 + 2y^2 + 3z^2)k + C$

✓ $\boxed{U = -\frac{1}{2}(x^2 + 2y^2 + 3z^2)k} + C$

Prove $F = -\nabla U$

$-\nabla U = -\frac{\partial}{\partial x}\hat{x} - \frac{\partial}{\partial y}\hat{y} - \frac{\partial}{\partial z}\hat{z}$

$\boxed{-\nabla U = -kx\hat{x} - k2y\hat{y} - k3z\hat{z} = F = k(x, 2y, 3z)}$

Problem 6 (4.23) (con't)

b) $F = k(y, x, 0)$

$$(\nabla \times F)_x = \frac{\partial}{\partial y} F - \frac{\partial}{\partial z} F_y = 0 - 0 = 0$$

$$(\nabla \times F)_y = \frac{\partial}{\partial z} F_x - \frac{\partial}{\partial x} F_z = 0 - 0 = 0$$

$$(\nabla \times F)_z = \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x = 1 - 1 = 0$$

CONSERVATIVE ✓

FIND potential energy, u

$$\frac{\partial u}{\partial x} = -F_x = -y$$

$$\frac{\partial u}{\partial y} = -F_y = -x$$

$$\frac{\partial u}{\partial z} = -F_z = 0$$

$$\int \frac{\partial u}{\partial x} = \int -ky$$

$$u(x,y) = -y\,x\,k + f(y)$$

$$\frac{\partial u}{\partial y} = -x + \cancel{f'(y)} = -x$$

$$\boxed{u = -kxy} + C$$ ✓

Prove $F = -\nabla u$

$$-\nabla u = \frac{-\partial}{\partial x}\hat{x} - \frac{\partial}{\partial y}\hat{y} - \frac{\partial}{\partial z}\hat{z}$$

$$\boxed{-\nabla u = ky\,\hat{x} + kx\,\hat{y} - 0\,\hat{z} = F = k(x, y, 0)}$$

c) $F = k(-y, x, 0)$

$$(\nabla \times F)_x = \frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y = 0 - 0 = 0$$

$$(\nabla \times F)_y = \frac{\partial}{\partial z} F_x - \frac{\partial}{\partial x} F_z = 0 - 0 = 0$$

$$(\nabla \times F)_z = \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x = 1 - (-1) = 2 \neq 0$$

NOT CONSERVATIVE ✓