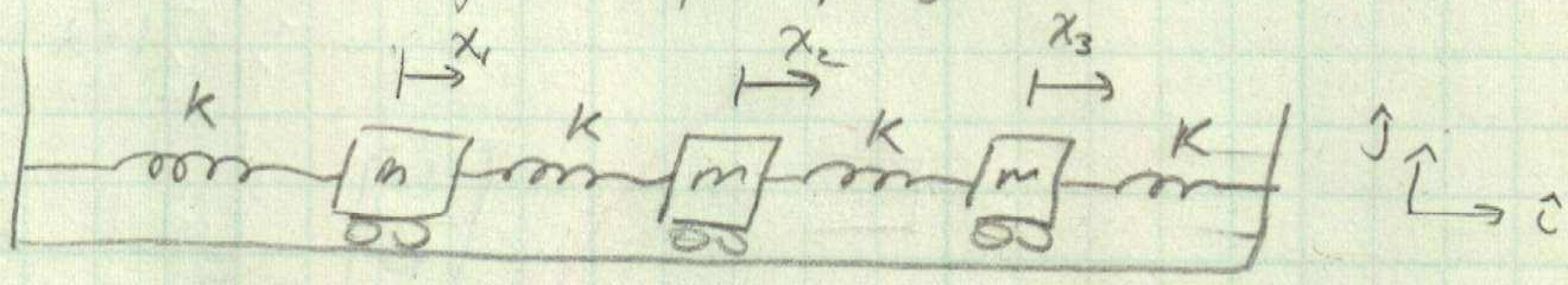
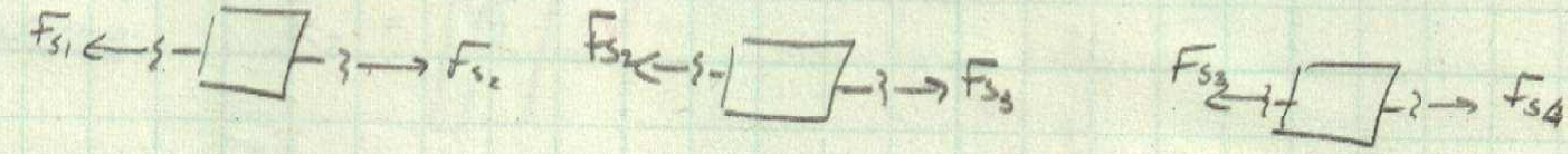
23) Three equal masses are in a line between two rigid walls. They are separated from each other and the walls by four equal springs.



a) Write the equations of motion in matrix form.

FBD



$$m\ddot{x}_1 = F_{s_2} - F_{s_1} = K(x_2 - x_1) - kx_1$$

$$m\ddot{x}_2 = F_{s_3} - F_{s_2} = K(x_3 - x_2) - k(x_2 - x_1)$$

$$m\ddot{x}_3 = F_{s_4} - F_{s_3} = -kx_3 - k(x_3 - x_2)$$

$$\boxed{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

b) By guessing/intuition find one of the normal modes.

$$[-\omega^2 M + k]\vec{x} = 0$$

guess $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\omega_1 = \sqrt{\frac{2k}{m}}$$

23) c) Using the MATLAB eig function find all three normal modes:

using $m=1$
$k=1$

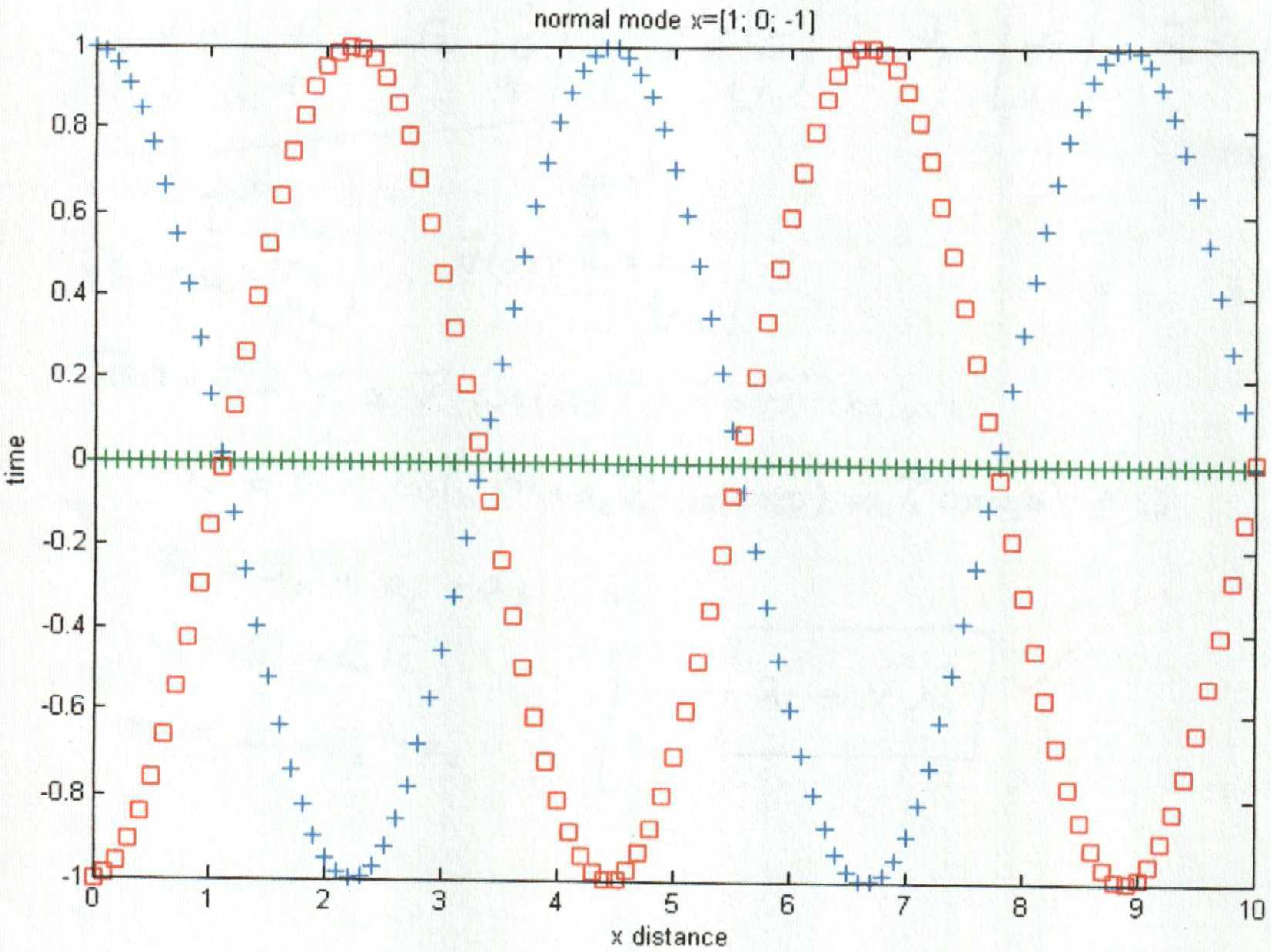
$$\omega_1 = \sqrt{2} \quad , \quad \bar{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\omega_2 = \sqrt{0.5858} \quad , \quad \bar{x}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$\omega_3 = \sqrt{3.4142} \quad , \quad \bar{x}_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

d) Using numerical integration, with masses released from rest with a normal mode shape ($\vec{x}_0 = \vec{u}_i$), show that you get normal mode (synchronous) oscillations.

```
function run_hw7_23()
z0 = [1 0 -1 0 0 0]; %[x1 x2 x3 x1dot x2dot x3dot]
tspan = 0:0.001:10;

[tarray,zarray] = ode45(@fun_hw7_23,tspan,z0);

x1 = zarray(1:100:end,1);
x2 = zarray(1:100:end,2);
x3 = zarray(1:100:end,3);
t = tarray(1:100:end);

plot(t,x1,'+',t,x2,'+',t,x3,'s')

end


function zdot = fun_hw7_23(t,z)
zdot = zeros(6,1);
K = [2 -1 0; -1 2 -1; 0 -1 2];
M = eye(3);

zdot(1:3) = z(4:6);

zdot(4:6) = -M\K*z(1:3);

end
```



H25. For your three mass system (above), find the motion if you are given initial positions and velocities. In some special cases (including at least one normal mode shape), check your motion against direct integration of the ODEs.

Solution: The general solution to the above expression is given by

$$\mathbf{x}(t) \quad = \quad \sum_{i=1}^{3} \mathbf{u}_i (c_{i,c} \cos(\omega_i t) + c_{i,s} \sin(\omega_i t)),$$

where the ω_i are the frequencies of the synchronous normal mode oscillations and $c_{i,c/s}$ denote constants depending on the initial conditions. Thus we must backsolve for the constants given $\mathbf{x}_0 = \mathbf{x}(0)$ and $\mathbf{v}_0 = \mathbf{x}'(0)$. We proceed as follows:

$$\begin{aligned}
\mathbf{x}_0 &= \sum_{i=1}^{3} c_{i,c} \mathbf{u}_i \\
&= \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} \begin{pmatrix} c_{1,c} \\ c_{2,c} \\ c_{3,c} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} c_{1,c} \\ c_{2,c} \\ c_{3,c} \end{pmatrix} &= \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}' \mathbf{x}_0,
\end{aligned}$$

where we have used the fact that the othonormal matrix of eigenvectors has inverse equal to its transpose.

Similarly, we have

$$\begin{aligned}
\mathbf{v}_0 &= \sum_{i=1}^{3} c_{i,s} \omega_i \mathbf{u}_i \\
&= \mathbf{W} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} \begin{pmatrix} c_{1,s} \\ c_{2,s} \\ c_{3,s} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} c_{1,s} \\ c_{2,s} \\ c_{3,s} \end{pmatrix} &= \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}' \mathbf{W}^{-1} \mathbf{x}_0,
\end{aligned}$$

where $\mathbf{W}$ is the diagonal matrix with ω_i on the diagonal.

We now provide code to calculate $\mathbf{x}(t)$ given $(\mathbf{x}_0, \mathbf{v}_0) = \texttt{xinit}(1:6)$:

```
%Actual solution
K=-[-2,1,0;1,-2,1;0,1,-2]
[V,D]=eigs(K); % Calculate omega squared and normal modes
os=sqrt(D);
cc=V'*xinit(1:3); % Cosine constants
cs=(V*os)\ xinit(4:6); % Sine constants
for i=1:3
x1part(i,:)=cc(i)*cos(os(i,i)*t)+cs(i)*sin(os(i,i)*t);
end
for i=1:3
xoft(i,:)=V(i,1)*x1part(1,:)  + V(i,2)*x1part(2,:)  + V(i,3)*x1part(3,:);
% x(t)
end
```

The result of this theory is plotted in comparison with simulation:

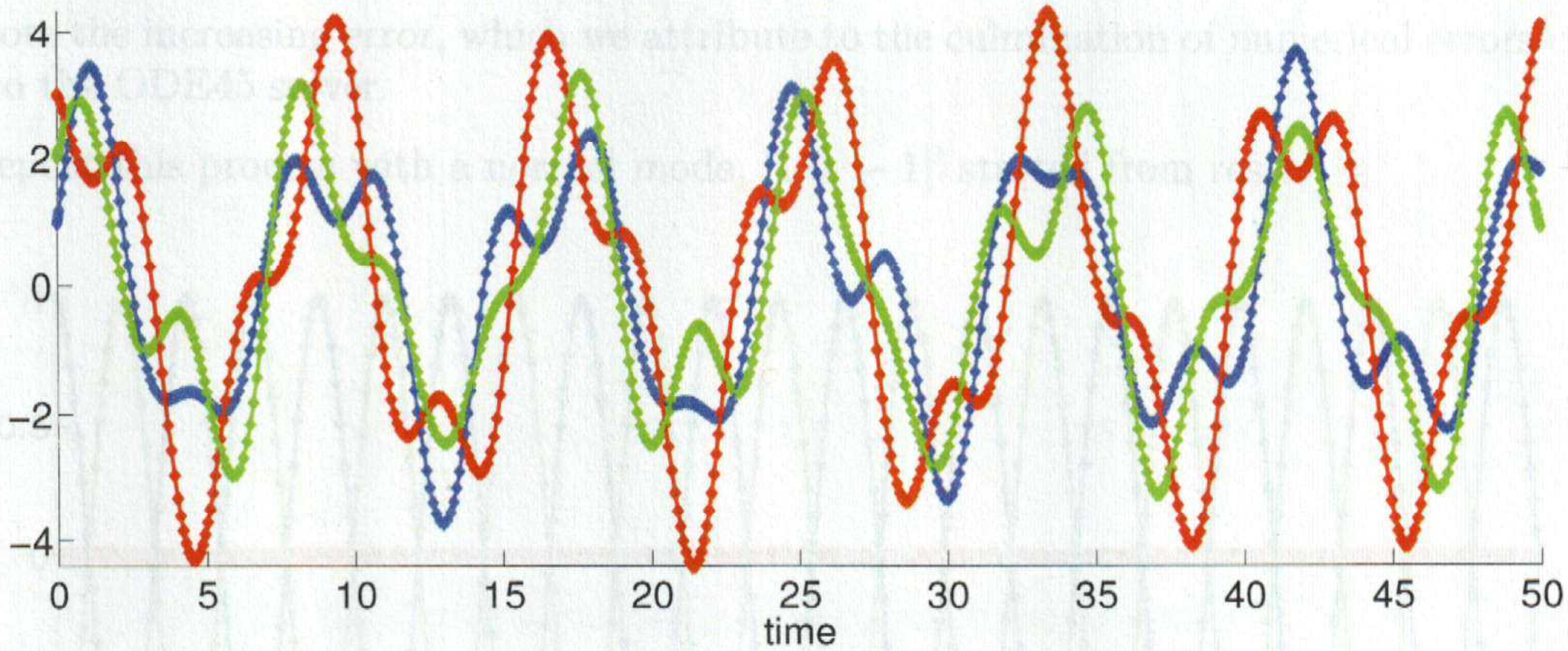


Figure 2: Numerical comparison: diamonds indicate computed output, lines indicate theoretical values. Initial conditions: $x_0 = (1, 3, 2)$, $v_0 = (\pi, -1, 2)$, $k = m = 1$.

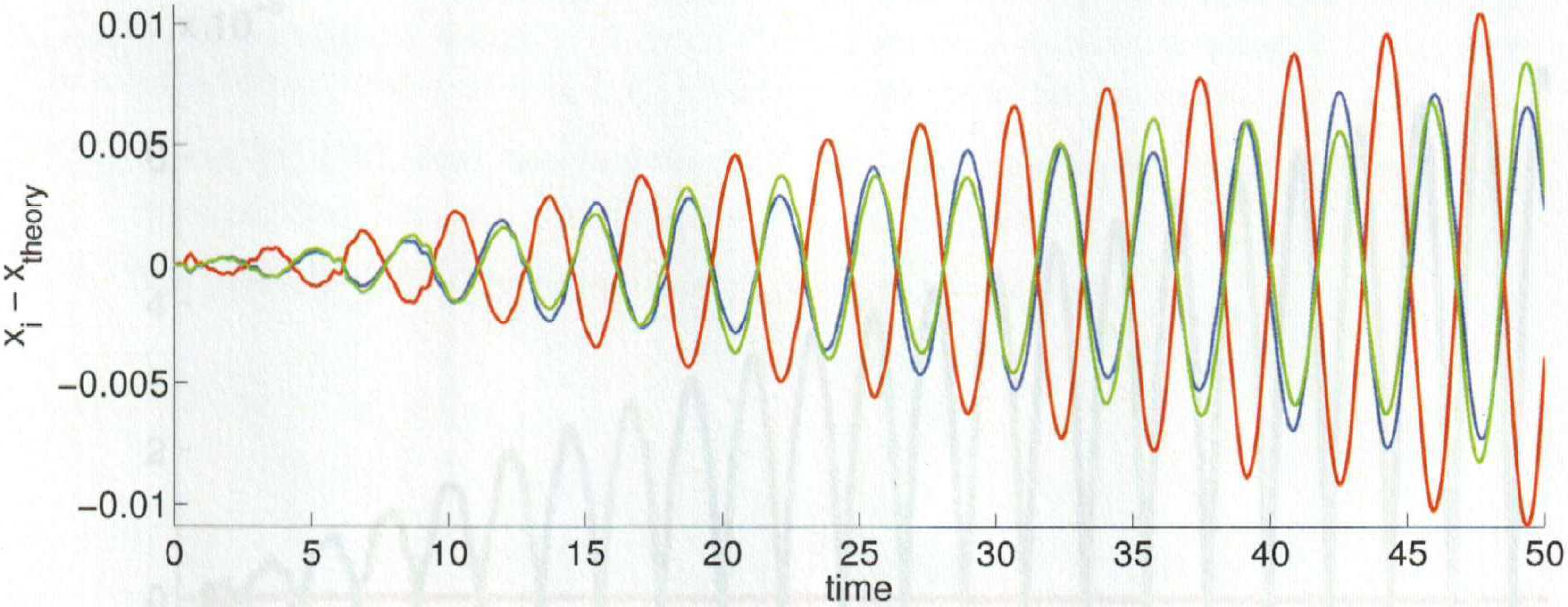


Figure 3: Error plots for each mass: $x_i - x_{i,theory}$. Initial conditions are as above.

We note the increasing error, which we attribute to the culmination of numerical errors due to the ODE45 solver.

We repeat this process with a normal mode, $[1\ 0\ -1]'$ started from rest.

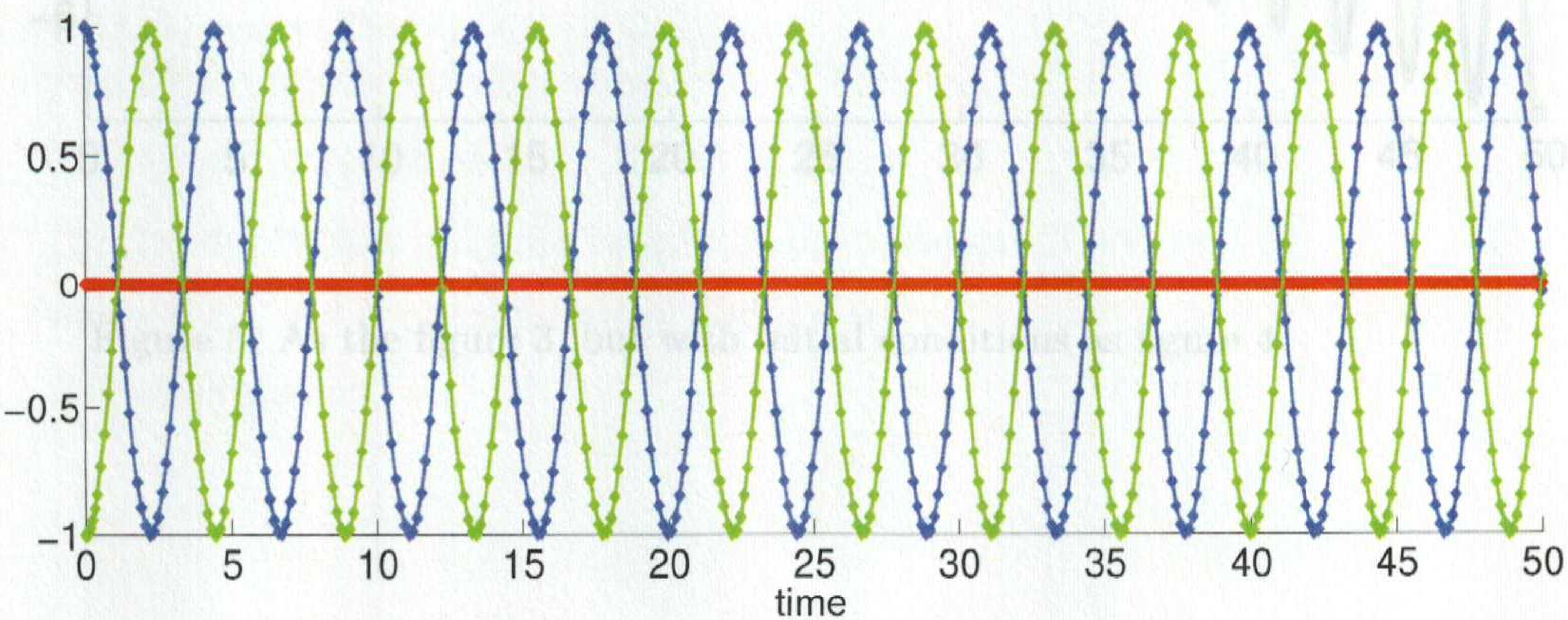


Figure 4: As figure 2, but with initial conditions $\mathbf{x}_0 = [1\ 0\ -1]'$, and $\mathbf{v}_0 = [0\ 0\ 0]$.

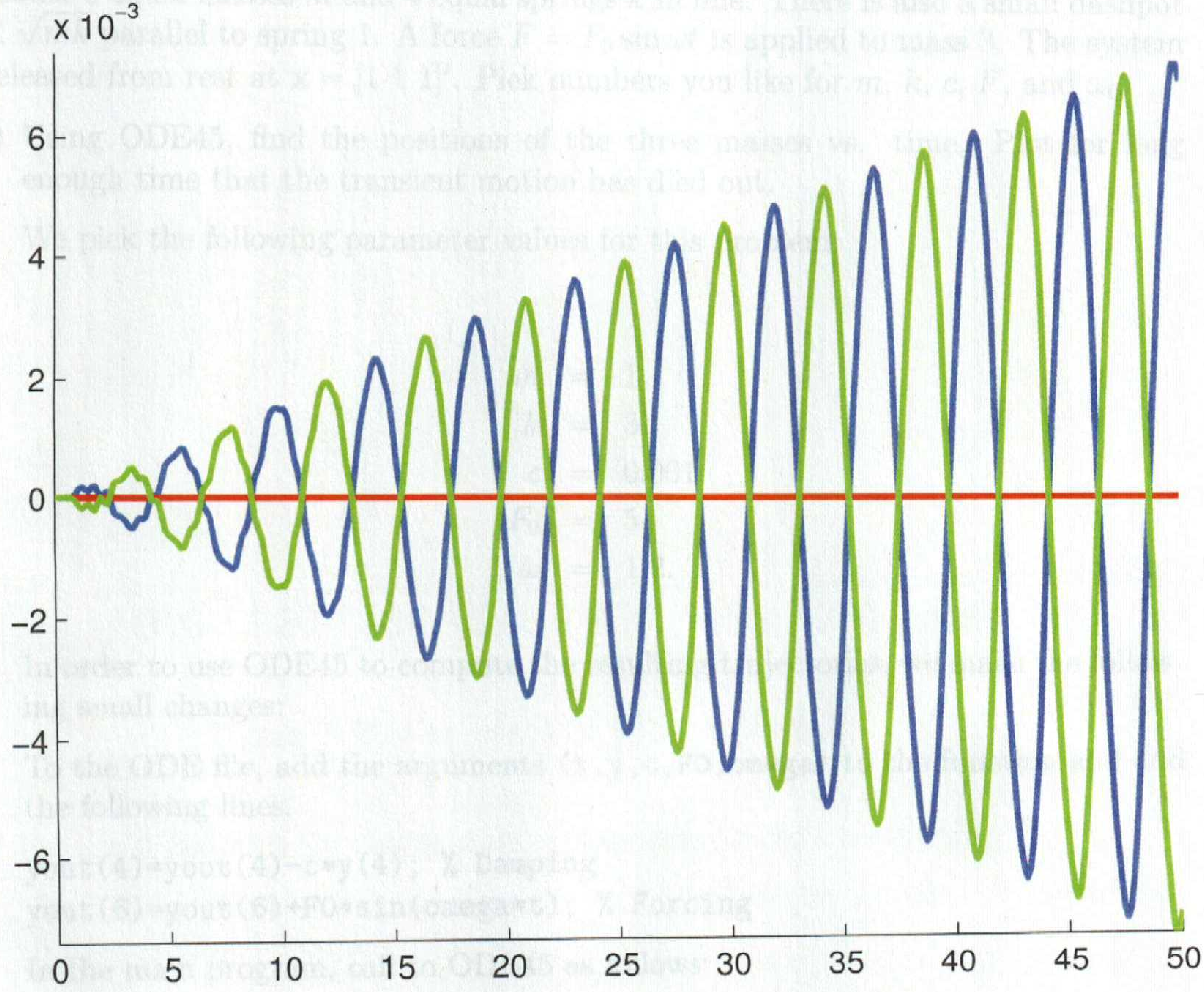
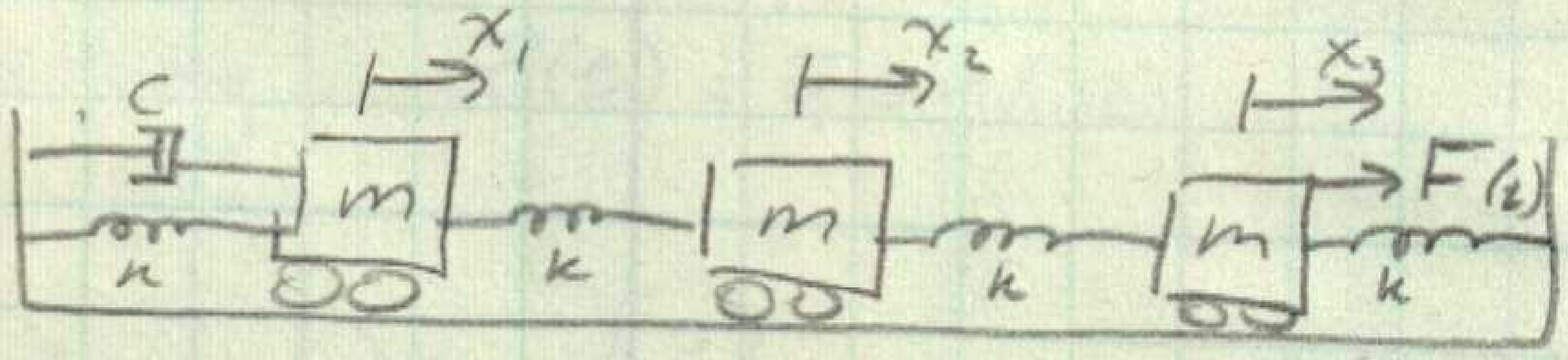


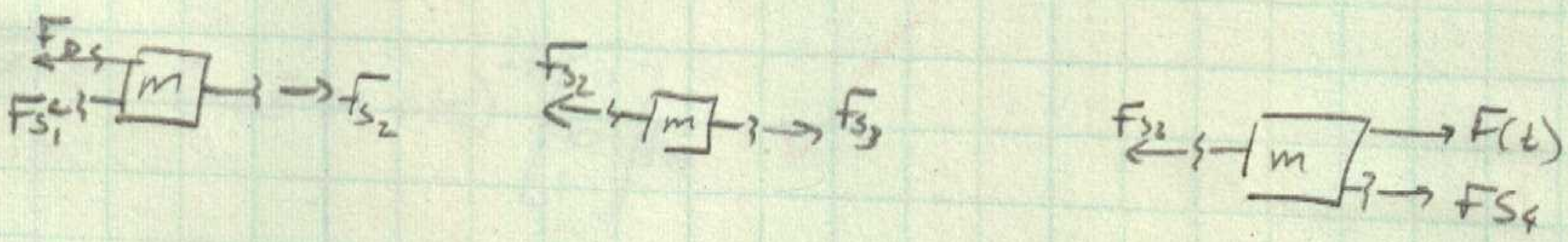
Figure 5: As the figure 3, but with initial conditions as figure 4.

40) Consider 3 equal masses m and 4 equal springs k in line. there is also a small dashpot $c << 2\sqrt{mk}$ parallel to spring 1. A force $F = F_0 \sin(\omega t)$ is applied to mass 3. the system is released from rest at $\vec{x} = [1\ 1\ 1]'$ Pick numbers you like for m, k, c, F_0, and ω.

a) Using ODE45, find the positions of the three masses vs. time.



FBD:



LMB:

$$m_1\ddot{x}_1 = F_{s_2} - F_D - F_{s_1} = k(x_2 - x_1) - kx_1 - c\dot{x}_1$$

$$m\ddot{x}_2 = k(x_3 - x_2) - k(x_2 - x_1)$$

$$m\ddot{x}_3 = -k(x_3) - k(x_3 - x_4) + F(t)$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\ddot{\vec{x}} + \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\vec{x}} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

$m = 1$

$k = 1$

$c = 0.1$

$\omega = \pi$

$F_0 = 1$

40) b) Set $c=0$ and find the steady state motion using matrix method.

$$M\ddot{\vec{x}} + k\vec{x} = \vec{F}$$

guessing $\vec{x}(t) = \bar{x}\sin(\omega t)$

$$(-\omega^2 M + k)\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} F_0$$

$$\bar{x} = (-\omega^2 M + k)^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} F_0$$

$$\vec{x} = \begin{bmatrix} -0.0021 \\ 0.0167 \\ -0.1292 \end{bmatrix}$$

c) Make a plot that shows that the solutions to the two problems above are very close at long times.

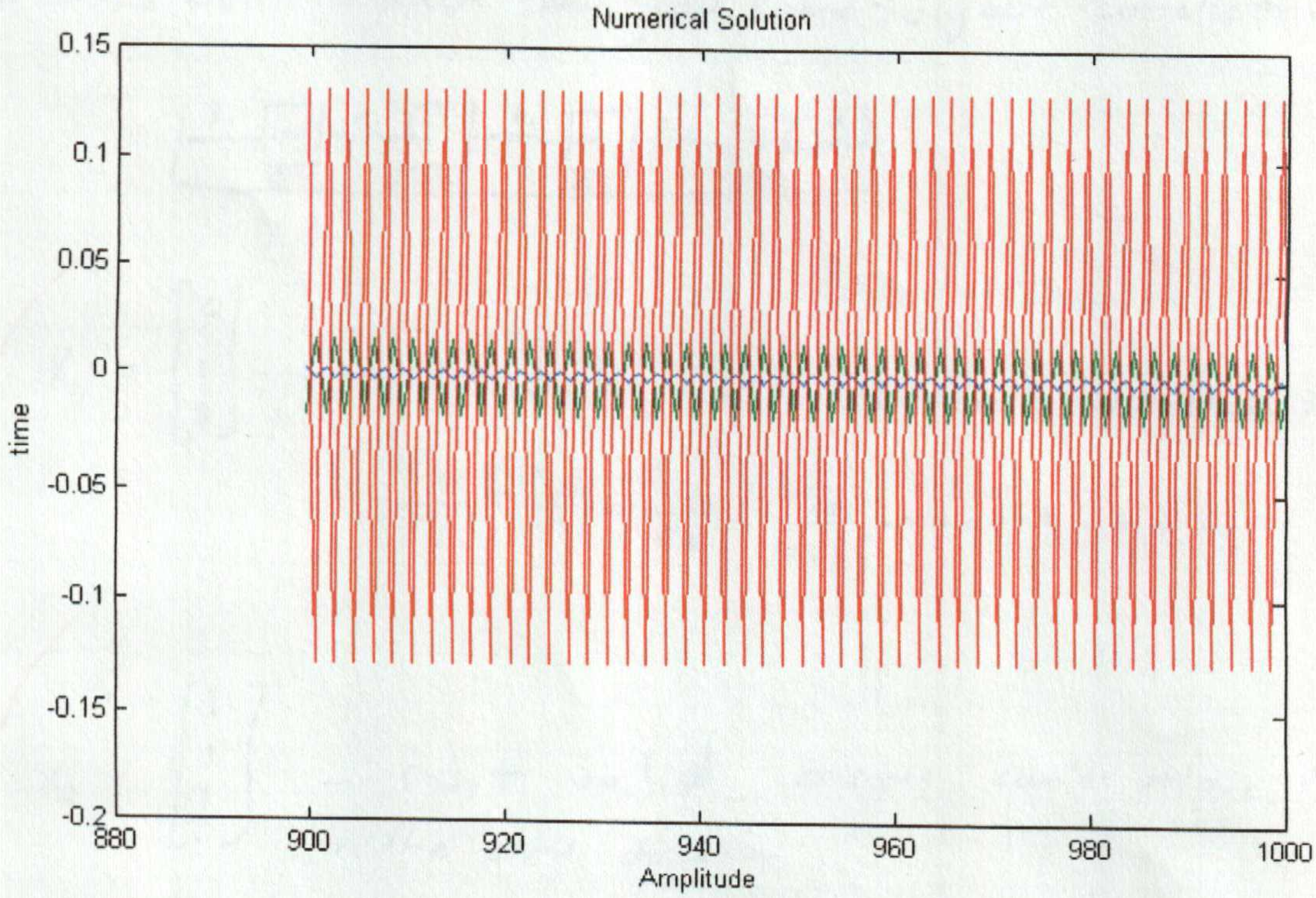
d) Explain.

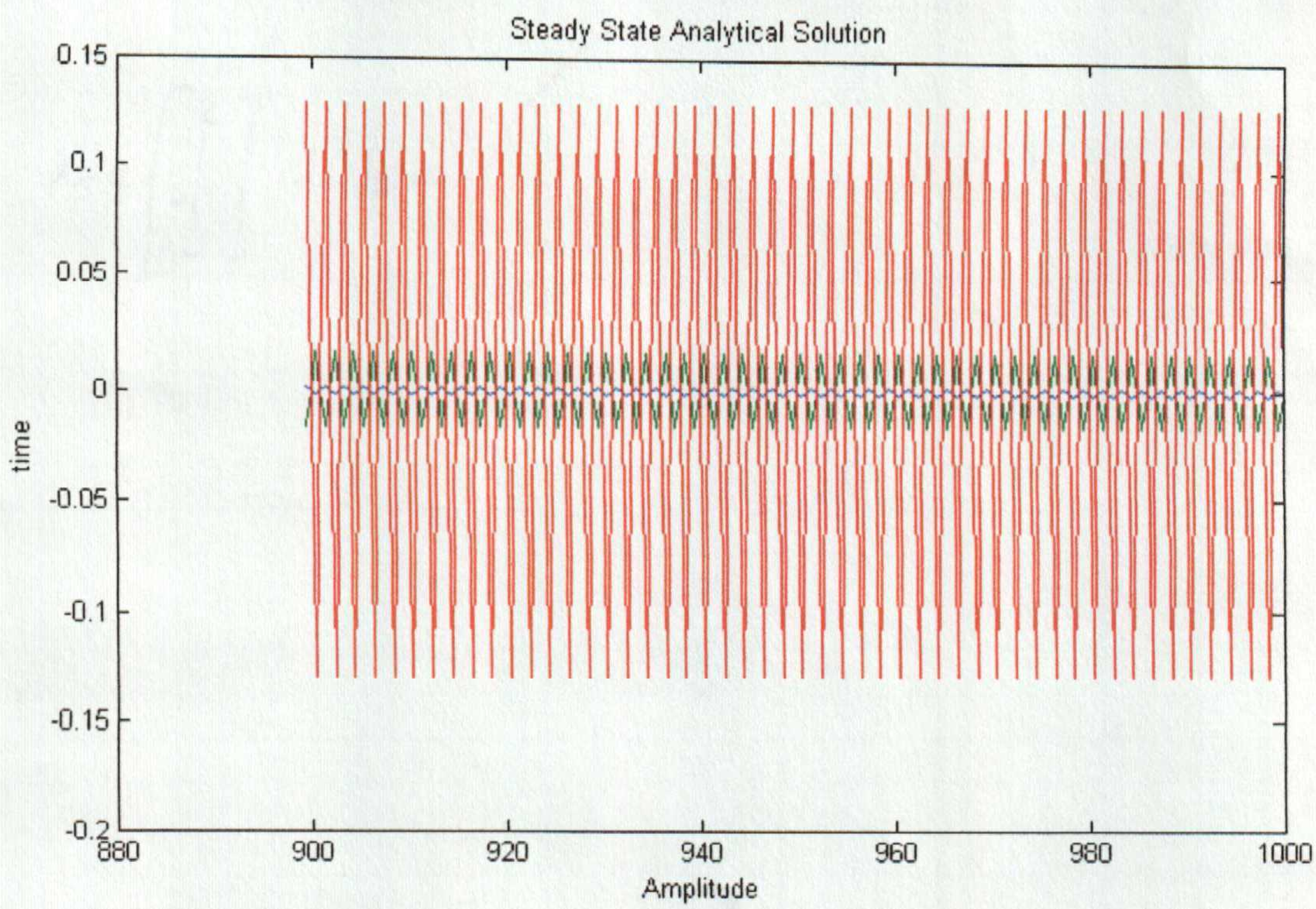
The damper will counter any movement of block 1, so in time, its displacement will go to zero. And since damping depends on velocity, once the block is barely moving, c will become negligible.

So as t grows, c becomes very small, and we can approximate our model as $M\ddot{\vec{x}} + k\vec{x} = \vec{F}$

Hence both graphs are similar for large values of t.

```
function run_hw7_40()

z0 = [1 1 1 0 0 0]; %[x1 x2 x3 x1dot x2dot x3dot]
tspan = 0:0.005:1000;

[tarray,zarray] = ode45(@fun_hw7_40,tspan,z0);

x1 = zarray(1:100:end,1);
x2 = zarray(1:100:end,2);
x3 = zarray(1:100:end,3);
t = tarray(1:100:end);

w = 1*pi;
K = [2 -1 0; -1 2 -1; 0 -1 2];
M = eye(3);
xss = inv(-w^2*M+K)*[0;0;1];

x1b = xss(1)*sin(w*t);
x2b = xss(2)*sin(w*t);
x3b = xss(3)*sin(w*t);

start = floor(9*length(t)/10);
range = start:length(t);
figure
plot(t(range),x1(range),t(range),x2(range),t(range),x3(range))
title('Numerical Solution')
xlabel('Amplitude')
ylabel('time')

figure
plot(t(range),x1b(range),t(range),x2b(range),t(range),x3b(range))
title('Steady State Analytical Solution')
xlabel('Amplitude')
ylabel('time')
end


function zdot = fun_hw7_40(t,z)
zdot = zeros(6,1);
K = [2 -1 0; -1 2 -1; 0 -1 2];
M = eye(3);
C = [0.1 0 0; 0 0 0; 0 0 0];
F = [0;0;1]*1*sin(1*pi*t);

zdot(1:3) = z(4:6);

zdot(4:6) = -M\C*z(4:6)-M\K*z(1:3)+M\F;
end
```

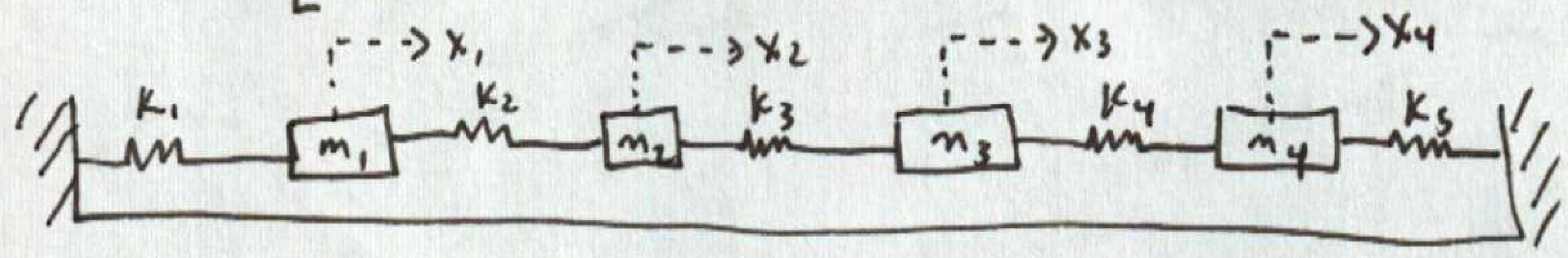

Numerical Solution
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
time
880
900
920
940
960
980
1000
Amplitude


Steady State Analytical Solution
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
time
880
900
920
940
960
980
1000
Amplitude

4

$\boxed{4.18}$ which of those normal modes dont make sense?

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$



EOM

$m_1\ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1$

$m_2\ddot{x}_2 = k_3(x_3 - x_2) - k_2(x_2 - x_1)$

$m_3\ddot{x}_3 = k_4(x_4 - x_3) - k(x_3 - x_2)$

$m_4\ddot{x}_4 = -k_4 x_4 - k_4(x_4 - x_3)$

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} = \begin{bmatrix} (-k_2-k_1) & k_2 & 0 & 0 \\ k_2 & (-k_2-k_3) & k_3 & 0 \\ 0 & k_3 & (-k_3-k_4) & k_4 \\ 0 & 0 & k_4 & (-k_5-k_4) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

knowing that if a periodic solution is assumed of the form $x = ae^{iwt}$, the problem reduces to:

$$-w^2 M\bar{X} = K\bar{X}$$

try $\bar{X}_1$:

$-w^2 m_1 0 = k_2$

$-w^2 m_2 1 = -k_2 \cancel{-k_3} \cancel{+k_3}$

$-w^2 m_3 1 = \cancel{k_3} \cancel{-k_3} - k_4$

$-w^2 m_4 0 = k_4$

$k_2 = 0$

$\sqrt{\frac{k_2}{m_2}} = w$

$\sqrt{\frac{k_4}{m_3}} = w$

$k_4 = 0$

those conditions must be met in order for this to be a normal mode. Since k_2 and k_4 cannot be zero, and since the response cannot have a frequency of 0, this canot be a normal mode!

$\boxed{\text{NOT A NORMAL MODE}}$

try $\bar{X}_2$

$-w^2 m_1 = \cancel{-k_2} - k_1 \cancel{+k_2}$

$-w^2 m_2 = \cancel{k_2} \cancel{-k_2} \cancel{-k_3} \cancel{+k_3}$

$-w^2 m_3 = \cancel{k_3} \cancel{-k_3} \cancel{-k_4} \cancel{+k_4}$

$-w^2 m_4 = \cancel{k_4} - k_5 \cancel{+k_4}$

$w = \sqrt{\frac{k_1}{m_1}}$

$m_2 w = 0$

$m_3 w = 0$

$w = \sqrt{\frac{k_5}{m_4}}$

these conditions must be met in order for this to be a normal mode

m_2 and m_3 camot be equal to zero

$\boxed{\text{NOT A NORMAL MODE}}$

try $\bar{X}_3$

$-w^2 m_1 2 = -2k_2 - 2k_1 + k_2 = -2k_1 - k_2$

$-w^2 m_2 = 2k_2 - k_2 - k_3 - k_3 = k_2 - 2k_3$

$w^2 m_3 = k_3 + k_3 + k_4 - 2k_4 = 2k_3 - k_4$

$2w^2 m_4 = -k_4 + 2k_5 + 2k_4 = 2k_5 + k_4$

$w = \sqrt{\frac{2k_1 + k_2}{2m_1}}$

$w = \sqrt{\frac{2k_3 - k_2}{m_2}}$

$w = \sqrt{\frac{2k_3 - k_4}{m_3}}$

$w = \sqrt{\frac{2k_5 + k_4}{2m_4}}$

these conditions must be met in order for this to be a normal mode

$\boxed{\text{CAN BE A NORMAL MODE}}$

Tongue 4.60

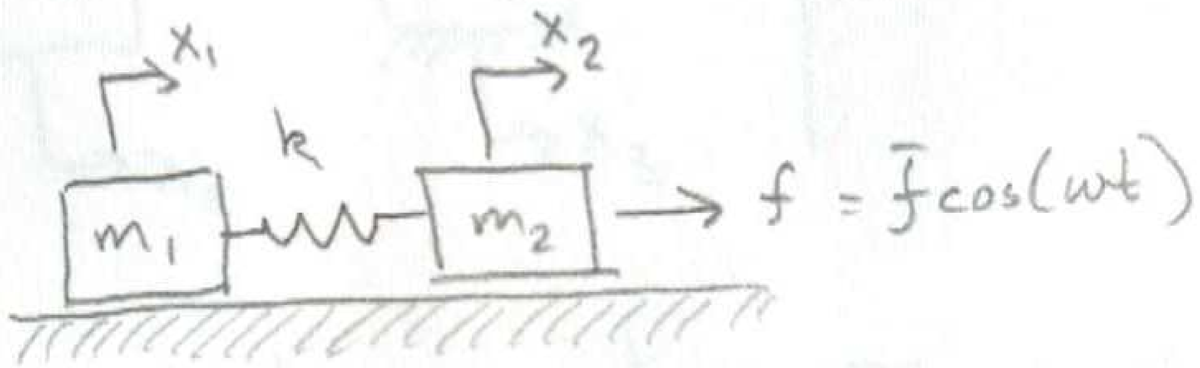
FIND ω such that m_2 will be stationary

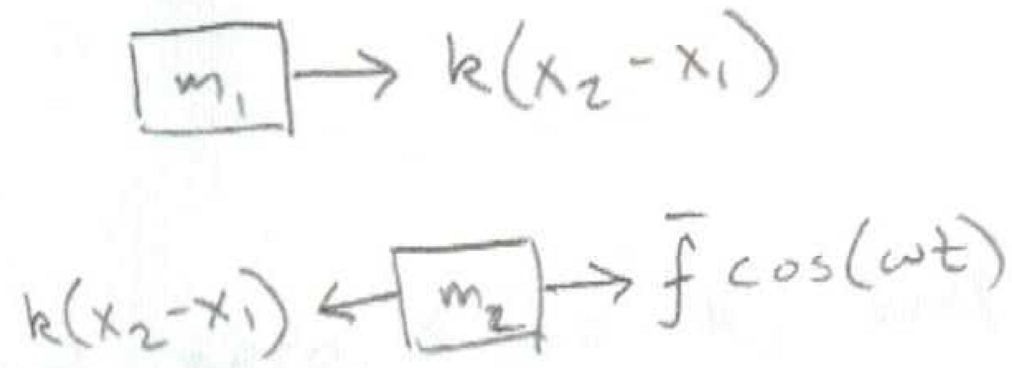
$\bar{f}$ such that $\bar{x}_1 = 3\,mm$

$m_1 = 0.4\,kg$

$m_2 = 0.8\,kg$

$k = 3000\,N/m$





$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

where $x_1(t) = \bar{x}_1 \cos(\omega t)$ $\Rightarrow$ $\ddot{x}_1(t) = -\omega^2 \bar{x}_1 \cos(\omega t)$

$x_2(t) = \bar{x}_2 \cos(\omega t)$ $\quad \ddot{x}_2(t) = -\omega^2 \bar{x}_2 \cos(\omega t)$

$\bar{x}_1, \bar{x}_2$ are scalars representing amplitude of the response

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{f} \end{Bmatrix}$$

$$\begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{f} \end{Bmatrix}$$

solve where $\bar{x}_2 = 0$ (m_2 is stationary)

$$\begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{f} \end{Bmatrix}$$

$(k - \omega^2 m_1)\bar{x}_1 = 0$

$k - \omega^2 m_1 = 0$

$\omega = \sqrt{\frac{k}{m_1}}$

$\omega = \sqrt{\frac{3000\,N/m}{0.4\,kg}}$

$$\boxed{\begin{aligned} \omega &= 86.6\,rad/s \\ f &= 13.8\,Hz \end{aligned}}$$ ✓

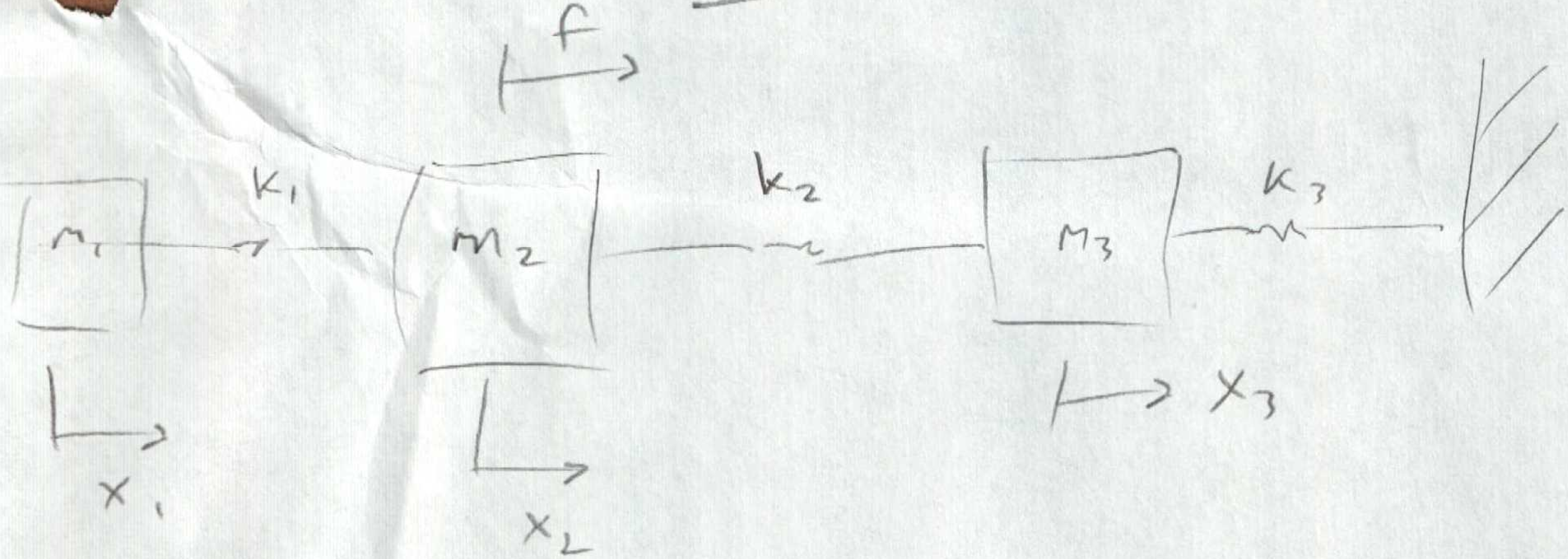
$-k\bar{x}_1 = \bar{f}$

$\bar{f} = -3000\,\frac{N}{m}(0.003\,m)$

✓ $\boxed{\bar{f} = -9\,N}$

negative sign indicates that force is 180° out of phase from motion of m_1

TONGUE 4.68



WHAT FREQUENCY FORCING WILL LET $X_3 = 0$?

ANY MOVEMENT OF M_2 WILL DISPLACE K_2 AND CAUSE A FORCE ON m_3. THEREFORE M_2 MUST ALSO BE STATIONARY. M_1 WILL BE OSCILATING, AND THE FORCING MUST COMPLETELY COUNTERACT THIS FORCE ON K_1. M_1 WILL MOVE @ ITS NATURAL FREQUENCY SO THE FORCING MUST BE AT M_1'S NATURAL FREQUENCY (AND COMPLETELY OUT OF PHASE).

$$\omega_{n1} = \sqrt{\frac{k_1}{m_1}}$$ = OUR FORCING FREQ.

Tongue 4.70

n DOF chain of lumped spring mass elements

FIND - maximum number of frequencies at which a mass can be forced such that the response of one of the masses is zero

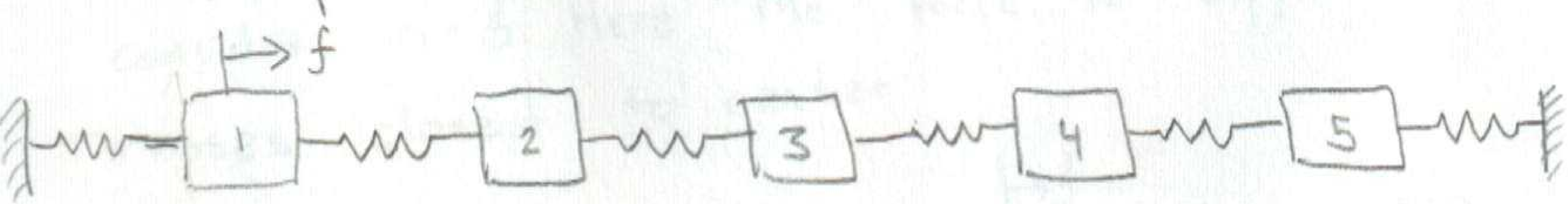
- also find the minimum number and explain the difference

From Section 4.6, p 229

Number of zeros = $n-(s+1)$ where n: number of masses

s: number of masses between the forced mass and the mass where $x=0$

For example, consider $n=5$

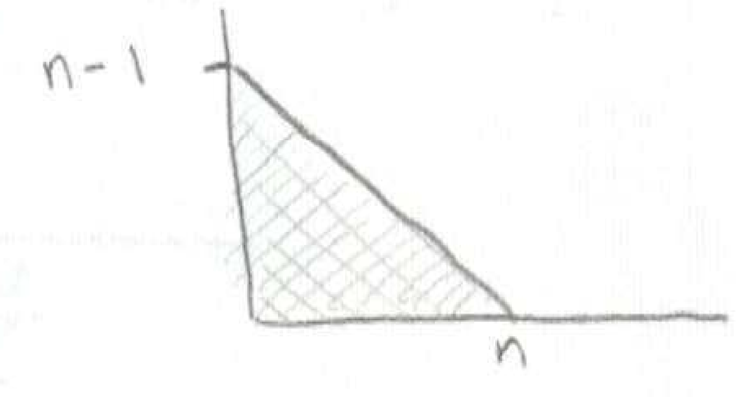


s	0	1	2	3	4
$n-(s+1)$	4	3	2	1	0

For the case where $n=5$, and f is applied to a mass at the end of the chain, we see that the total number of frequencies for which one of the masses is stationary is $4+3+2+1=10$

Written as a summation, we see

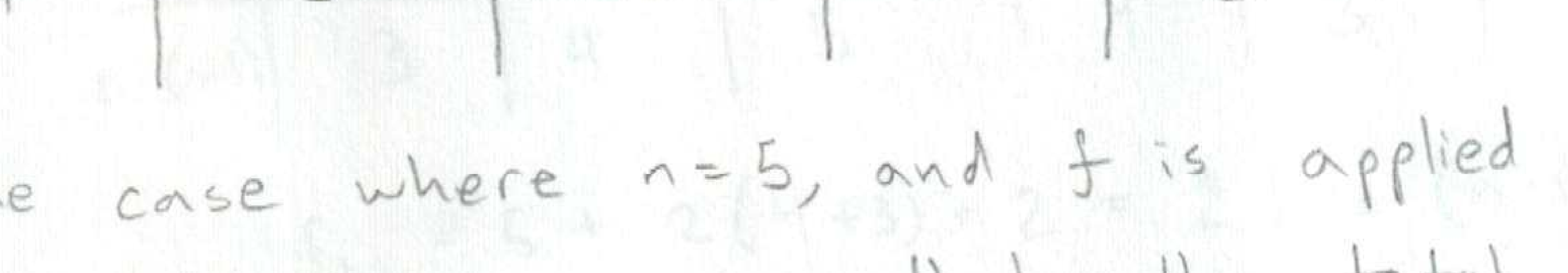
$$(\text{min})\quad f_{total} = \sum_{s=0}^{n-1} n-(s+1)$$



which can be reduced to an algebraic form

$$(\text{min})\quad f_{total} = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$$

This is the minimum number of total frequencies for which one of the masses is stationary. The maximum number occurs when the force is applied to the mass in the center



s	2	1	0	1	2
$n-(s+1)$	2	3	4	3	2

$$f_{total} = 2+3+4+3+2 = 14 = n-1 + \sum_{s=1}^{\frac{n-1}{2}} n-(s+1)$$

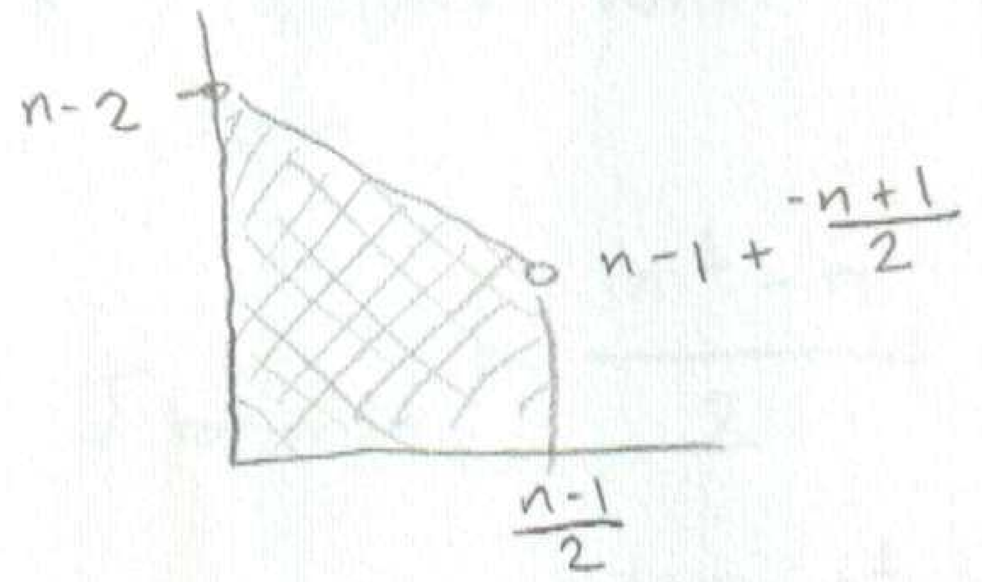
Note: This summation only applies when n is an odd number.

$n-1+\sum_{s=1}^{\frac{n-1}{2}} n-(s+1)$ can be written algebraicly as $\frac{3n^2-4n+1}{4}$

$$n-1+\left(n-2+n-1+\frac{-n+1}{2}\right)\frac{\frac{n-1}{2}}{2}$$

$$n-1+\left(\frac{3n-5}{2}\right)\left(\frac{n-1}{2}\right)$$

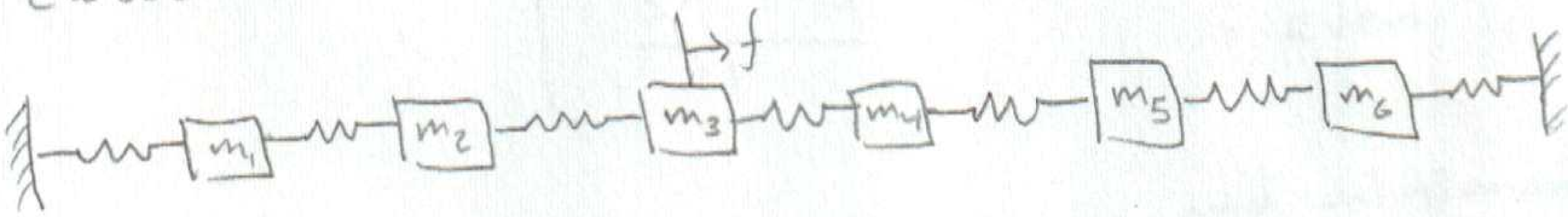
$$\frac{4n-4}{4}+\frac{3n^2-8n+5}{4}=\frac{3n^2-4n+1}{4}$$



(max, n (odd)) $\quad f_{tot}=\frac{3n^2-4n+1}{4}$

A similar expression can be derived for even values of n

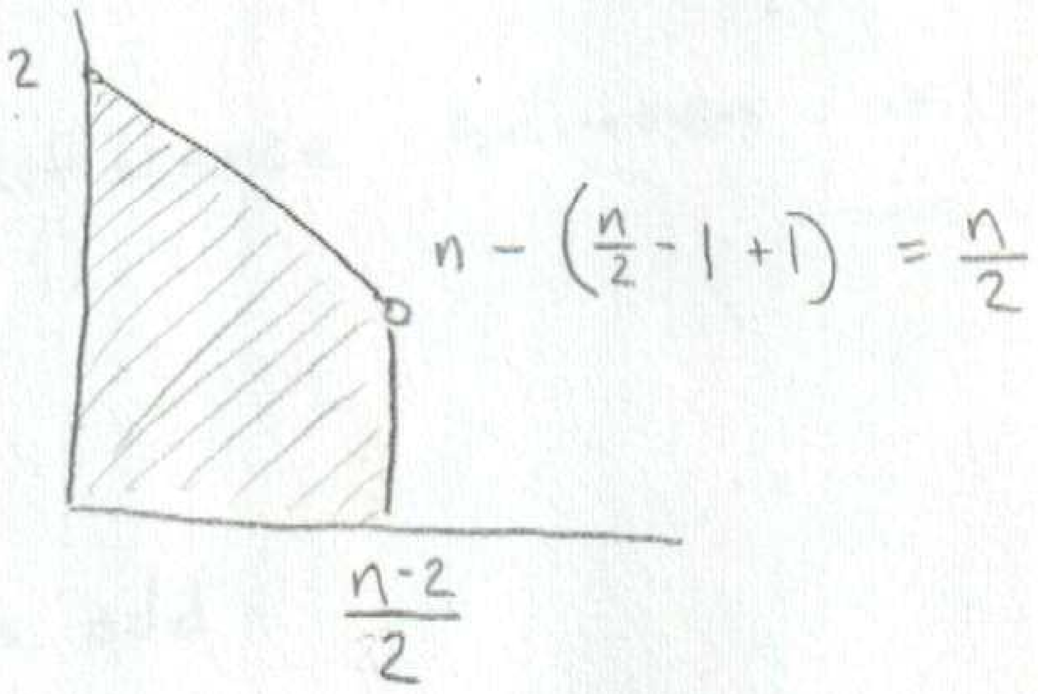
consider $n=6$. Here the force is applied to one of the two masses closest to center



s	2	1	0	1	2	3
$n-(s+1)$	3	4	5	4	3	2

$$f_{tot}=5+2(4+3)+2=21=(n-1)+\left(n-1-\frac{n}{2}\right)+2\sum_{s=1}^{\frac{n}{2}-1} n-(s+1)$$

(even) $\quad f_{tot}=\frac{3n}{2}-2+2\sum_{s=1}^{\frac{n}{2}-1} n-(s+1)$



(even) $\quad f_{tot}=\frac{3n}{2}-2+\frac{(3n^2-10n+8)}{4}$

(even) $\quad f_{tot}=\frac{6n-8}{4}+\frac{3n^2-10n+8}{4}$

$$\frac{n-2}{2}\left(n-2+\frac{n}{2}\right)=\frac{n-2}{2}\left(\frac{3n-4}{2}\right)$$

$$=\frac{(n-2)(3n-4)}{4}$$

(even) $\quad f_{tot}=\frac{3n^2-4n}{4}$

Test: $n=6$

$$\frac{3(36)-24}{4}=21$$

For an n-DOF system,

the minimum number of frequencies to make one of the masses stationary occurs when the force is applied to a force on the end of the chain

$$f_{tot,min} = \frac{n^2 - n}{2}$$

The maximum number of frequencies occurs when the force is applied to the mass closest to the center. The number varies based on rather n is even or odd

$$f_{tot,\,max} = \begin{cases} \dfrac{3n^2 - 4n + 1}{4} & n \text{ is odd} \\[2ex] \dfrac{3n^2 - 4n}{4} & n \text{ is even} \end{cases}$$

Nice!

The difference between maximum and minimum frequency numbers is that masses closer to the applied force have more frequencies for which they are stationary. Thus, when the force is applied at the end of the chain, the average number frequencies/mass is lower because the distance from each mass to the force is on average higher.

We can also find the numerical difference between min and max frequencies for a given n.

$$\Delta f_{max-min} = \begin{cases} \dfrac{n^2 - 2n + 1}{4} & n \text{ is odd} \\[2ex] \dfrac{n^2 - 2n}{4} & n \text{ is even} \end{cases}$$