HW 5 10/2/13

① Taylor 5.13

Potential energy of a 1-D mass at a distance r is

$U(r) = U_0\left(\frac{r}{R} + \lambda^2 \frac{R}{r}\right)$ for $0 < r < \infty$ with U_0, R, λ all positive const.

FIND r_0 (equilibrium position)

Let x be the distance from equilibrium, and show that for small x the PE has the form $U = \text{const} + \frac{1}{2}kx^2$

FIND angular frequency for small oscillations

$U'(r) = U_0\left(\frac{1}{R} - \lambda^2 \frac{R}{r^2}\right)$ $\quad U''(r) = 2U_0\lambda^2 \frac{R}{r^3}$

$U'(r_0) = 0 = U_0\left(\frac{1}{R} - \lambda^2 \frac{R}{r_0^2}\right)$

$0 = \frac{1}{R} - \lambda^2 \frac{R}{r_0^2}$

$\lambda^2 \frac{R}{r_0^2} = \frac{1}{R}$

$\lambda^2 R^2 = r_0^2$

$\boxed{r_0 = \lambda R}$ ✓

$r = r_0 + x \Rightarrow x = r - r_0$

Use a Taylor series expansion of $U(r) = U(r_0 + x)$

$U(r_0+x) = U(r_0) + \overset{0}{\cancel{U'(r_0)}}x + \frac{1}{2}U''(r_0)x^2 + \ldots$ ~~H.O.T.~~ negligible higher order terms

$U(r_0+x) = U_0\left(\frac{r_0}{R} + \lambda^2\frac{R}{r_0}\right) + \frac{1}{2}\left(2U_0\lambda^2\frac{R}{r_0^3}\right)x^2$

$U(r_0+x) = U_0\left(\frac{\lambda\cancel{R}}{\cancel{R}} + \lambda^{\cancel{2}}\frac{\cancel{R}}{\cancel{\lambda R}}\right) + \frac{1}{2}\left(2U_0\lambda^2\frac{R}{(\lambda R)^3}\right)x^2$

$U(r_0+x) = U_0(2\lambda) + U_0\frac{1}{\lambda R^2}x^2$ ✓

$$\boxed{U(r_0+x) = 2U_0\lambda + \frac{1}{2}\frac{2U_0}{\lambda R^2}x^2 \Rightarrow U(r) = \text{const} + \frac{1}{2}kx^2 \quad \text{where } k = \frac{2U_0}{\lambda R^2}}$$

$\omega = \sqrt{k/m} \Rightarrow \boxed{\omega = \sqrt{\frac{2U_0}{m\lambda R^2}}}$ ✓

Taylor 5.30

The position $X(t)$ of an overdamped oscillator is given by

$$X(t) = C_1 e^{-(\beta-\sqrt{\beta^2-\omega_0^2})t} + C_2 e^{-(\beta+\sqrt{\beta^2-\omega_0^2})t}$$

a.) Find the constants C_1 and C_2 in terms of X_0 and V_0:

The velocity equals:

$$\frac{dx(t)}{dt} = V(t) = C_1\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right) e^{-(\beta-\sqrt{\beta^2-\omega_0^2})t} + C_2\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right) e^{-(\beta+\sqrt{\beta^2-\omega_0^2})t}$$

$$V(0) = V_0 = C_1\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right) + C_2\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)$$

$$X(0) = X_0 = C_1 + C_2$$

So the system of equations is:

$$X_0 = C_1 + C_2$$

$$V_0 = C_1\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right) + C_2\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right) = -\beta(C_1+C_2) + \sqrt{\beta^2-\omega_0^2}\,(C_1-C_2)$$

$$V_0 = -\beta X_0 + C_1\sqrt{\beta^2-\omega_0^2} - \sqrt{\beta^2-\omega_0^2}\,C_2, \quad C_2 = X_0 - C_1$$

$$V_0 = -\beta X_0 + C_1\sqrt{\beta^2-\omega_0^2} - \sqrt{\beta^2-\omega_0^2}\,(X_0 - C_1)$$

$$V_0 = -\beta X_0 + C_1\sqrt{\beta^2-\omega_0^2} + C_1\sqrt{\beta^2-\omega_0^2} - X_0\sqrt{\beta^2-\omega_0^2}$$

$$-2C_1 = \frac{-\beta X_0 - V_0 - X_0\sqrt{\beta^2-\omega_0^2}}{\sqrt{\beta^2-\omega_0^2}} \qquad \boxed{C_1 = \frac{X_0\left(\beta+\sqrt{\beta^2-\omega_0^2}\right) + V_0}{2\sqrt{\beta^2-\omega_0^2}}}$$ ✓

$$C_2 = X_0 - C_1 = X_0 - \frac{X_0\left(\beta+\sqrt{\beta^2-\omega_0^2}\right) + V_0}{2\sqrt{\beta^2-\omega_0^2}} = \boxed{\frac{X_0\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right) - V_0}{2\sqrt{\beta^2-\omega_0^2}}}$$ ✓

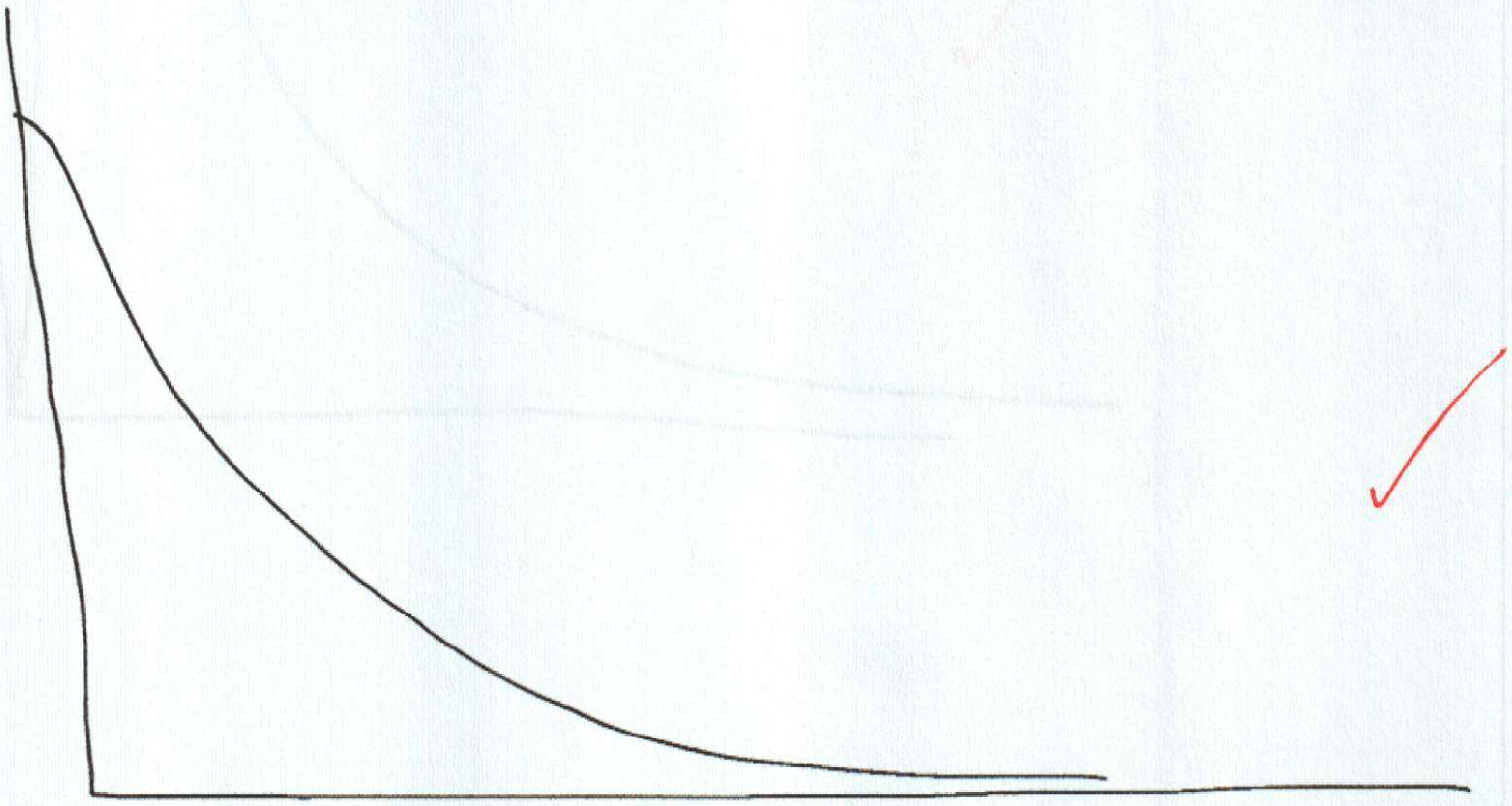
b.) Sketch the behavior of $x(t)$ for the two cases that $V_0=0$ and that $X_0 = 0$

For $V_0=0$

$$C_1 = \frac{X_0(B + \sqrt{B^2-W_0^2}) + 0}{2\sqrt{B^2 W_0^2}} = \frac{X_0(B + \sqrt{B^2-W_0^2}}{2\sqrt{B^2 W_0^2}}$$

$$C_2 = \frac{X_0(-B + \sqrt{B^2-W_0^2}) - 0}{2\sqrt{B^2-W_0^2}} = \frac{X_0(-B + \sqrt{B^2-W_0^2})}{2\sqrt{B^2-W_0^2}}$$

$$x(t) = C_1 e^{-(B-\sqrt{B^2-W_0^2}t} + C_2 e^{-(B+\sqrt{B^2-W_0^2})t}$$



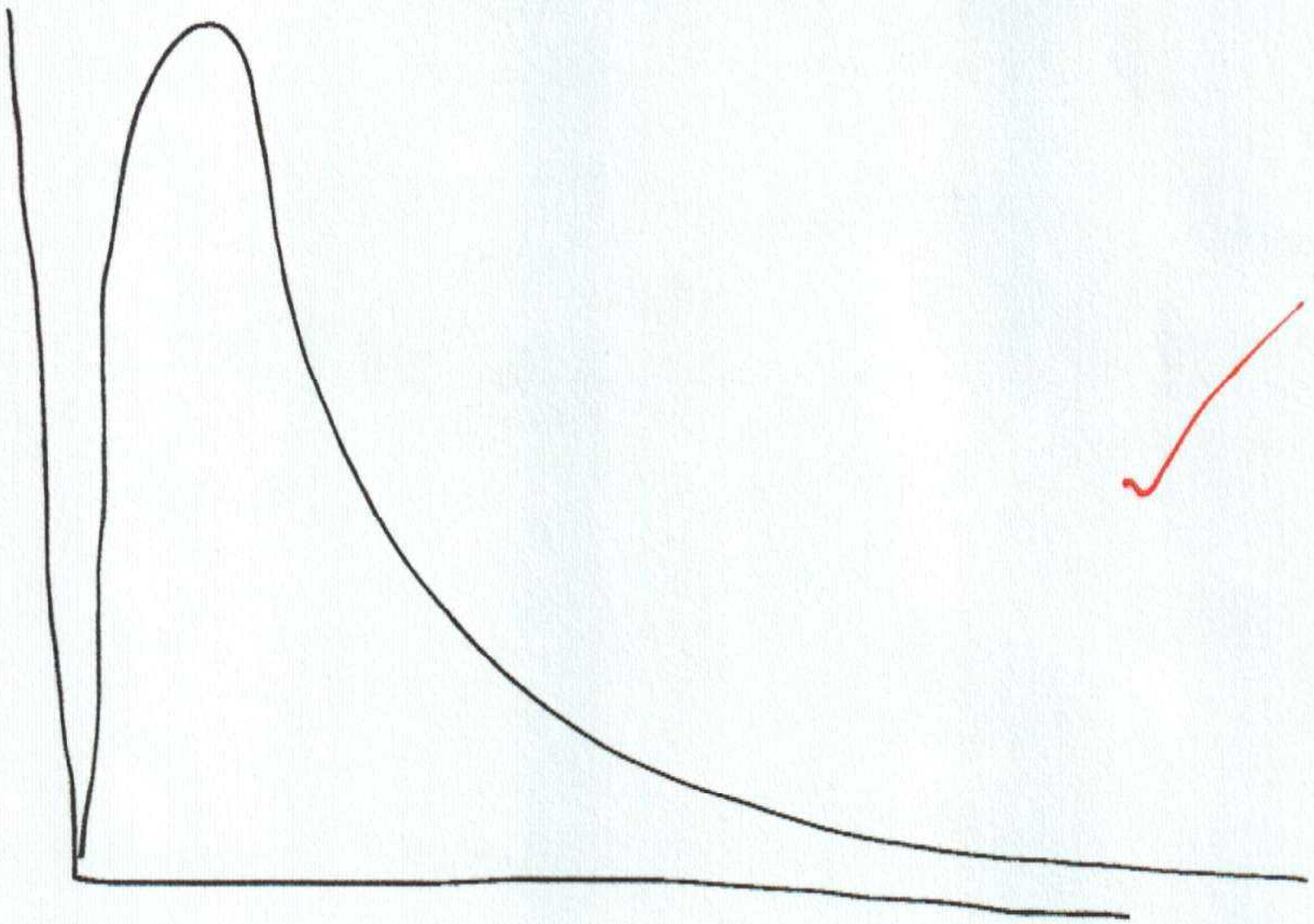
✓

For $X_0 = 0$:

$$C_1 = \frac{0(B + \sqrt{B^2 - W_0^2}) + V_0}{2\sqrt{B^2 - W_0^2}} = \frac{V_0}{2\sqrt{B^2 - W_0^2}}$$

$$C_2 = \frac{0(-B + \sqrt{B^2 - W_0^2}) - V_0}{2\sqrt{B^2 - W_0^2}} = -\frac{V_0}{2\sqrt{B^2 - W_0^2}}$$

$$X(t) = C_1 e^{-(B - \sqrt{B^2 - W_0^2})t} + C_2 e^{-(B + \sqrt{B^2 - W_0^2})t}$$



✓

c.) Set B->0, and see X(t) in part (a) approaches the correct solution for undamped motion:

$$X(t) = C_1 e^{-(\beta - \sqrt{\beta^2 - W_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - W_0^2})t}$$

$$X(t) = C_1 e^{+\sqrt{-W_0^2}\,t} + C_2 e^{-\sqrt{-W_0^2}\,t}$$

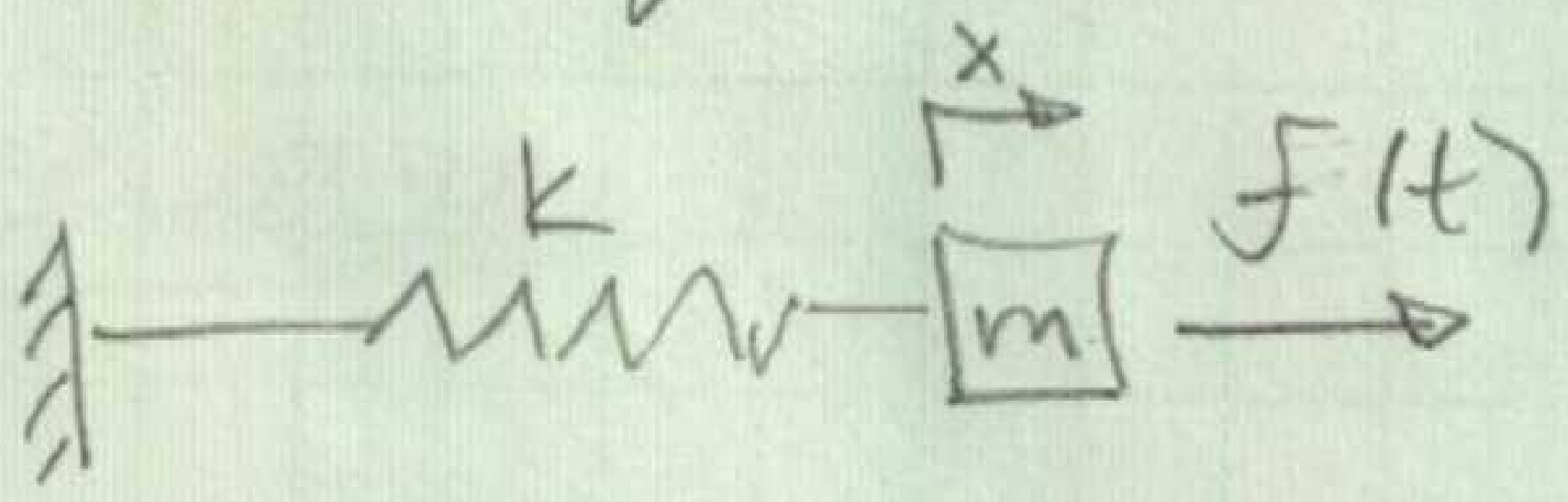
$$\boxed{X(t) = C_1 e^{+iW_0 t} + C_2 e^{-iWt}} \longrightarrow \text{undamped}$$

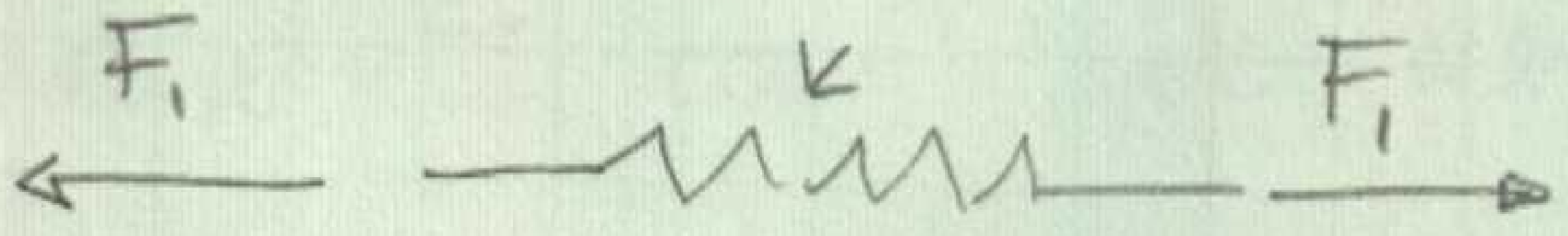
✓

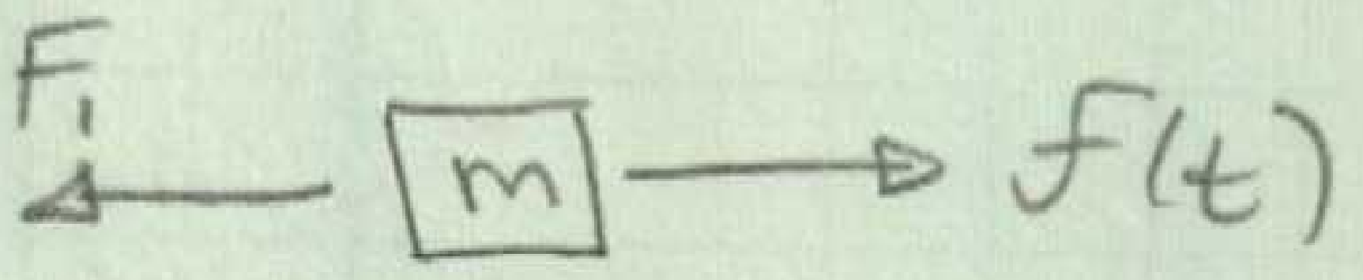
$\boxed{\text{Problem 1.4}}$ (10.1.4)

Find the equation of motion for problems below.

a)





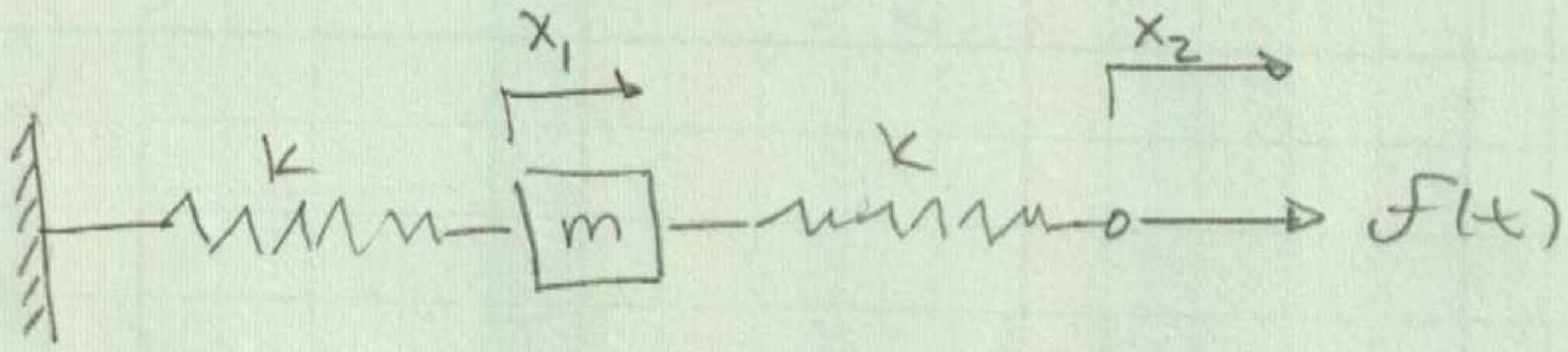


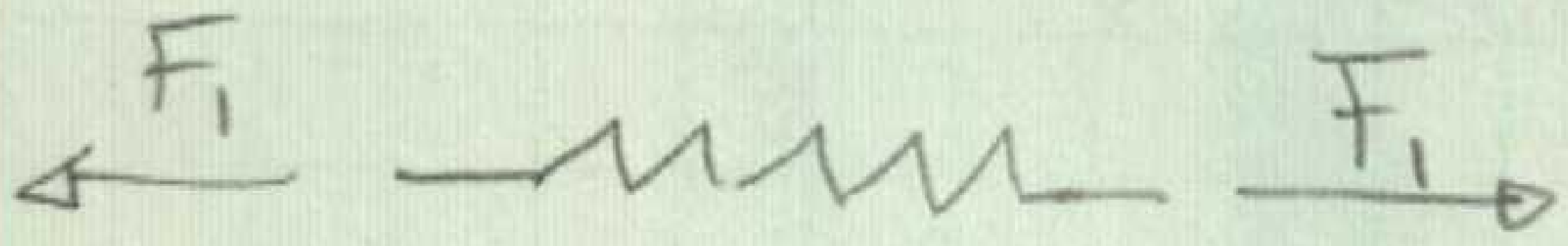
$F_1 = kx$

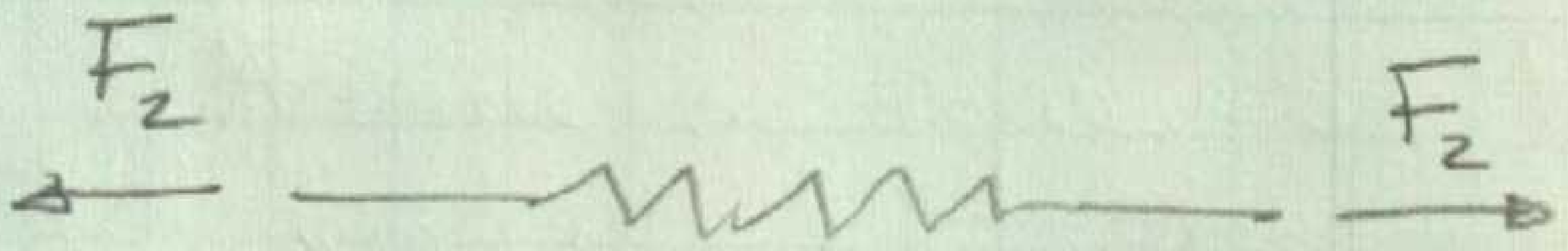
$\xrightarrow{+} \sum F_i = m\ddot{x} = f(t) - F_1$

$$\boxed{m\ddot{x} + kx = f(t)}$$

b)



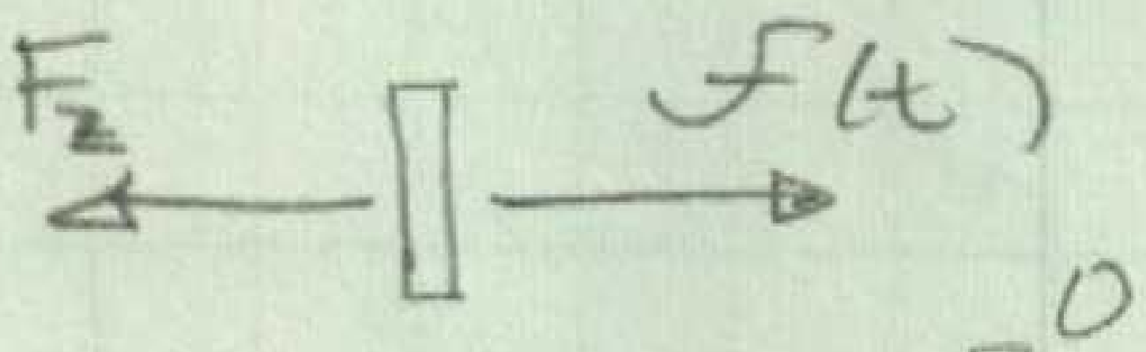




$F_2 = k(x_2 - x_1)$

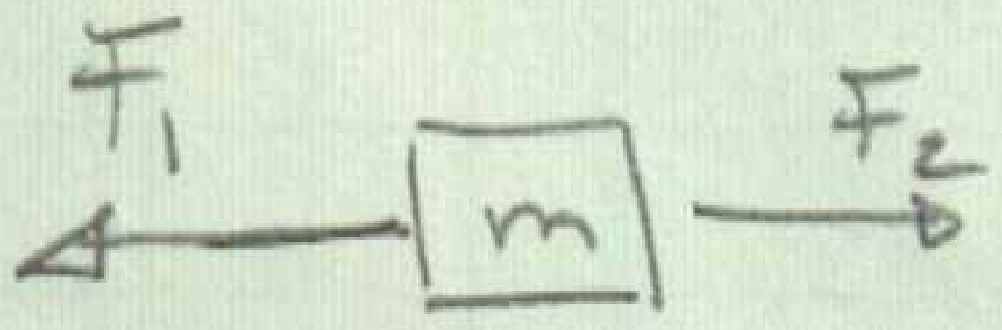
$F = kx_1$

Assume a massless bar at the end



$\xrightarrow{+} \sum F_i = \cancelto{0}{m_b \ddot{x}_2} = f(t) - F_2$

$F_2 = f(t)$

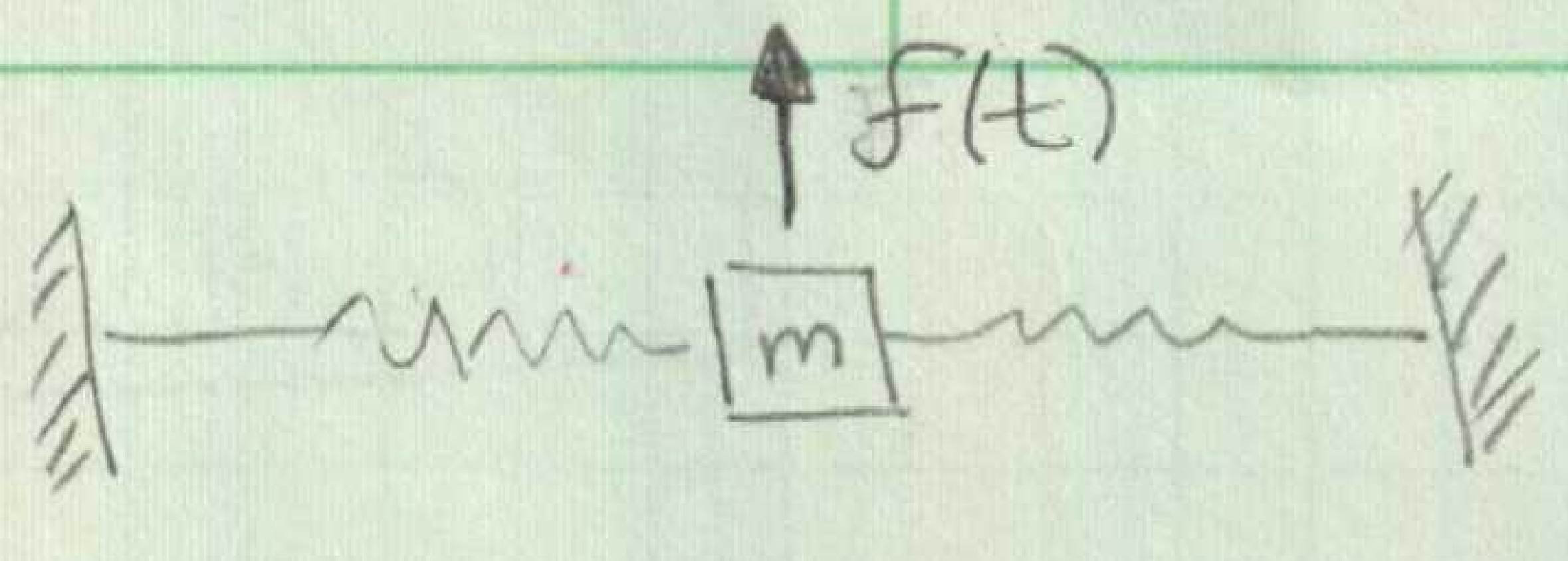


$\xrightarrow{+} \sum F_j = m\ddot{x}_1 = F_2 - F_1$

$m\ddot{x}_1 + F_1 = F_2$

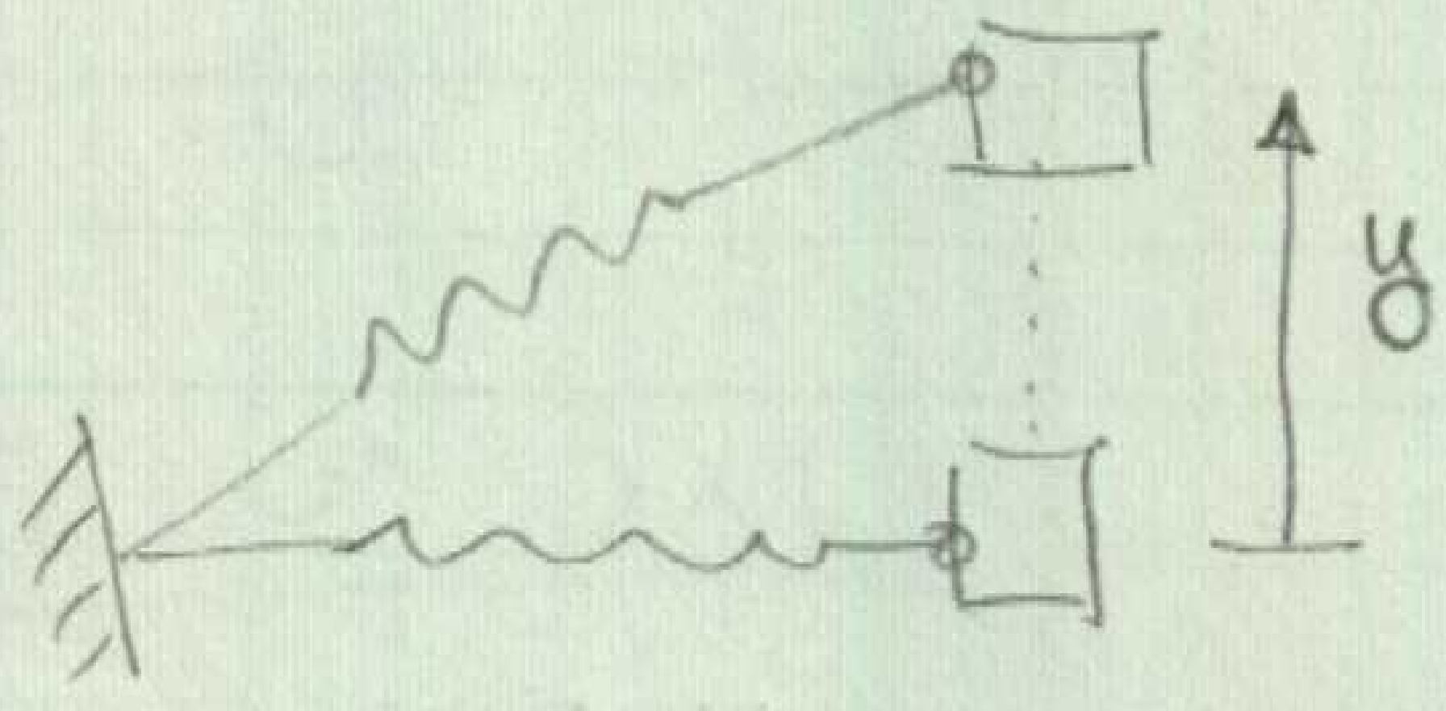
$$\boxed{m\ddot{x} + kx = f(t)}$$

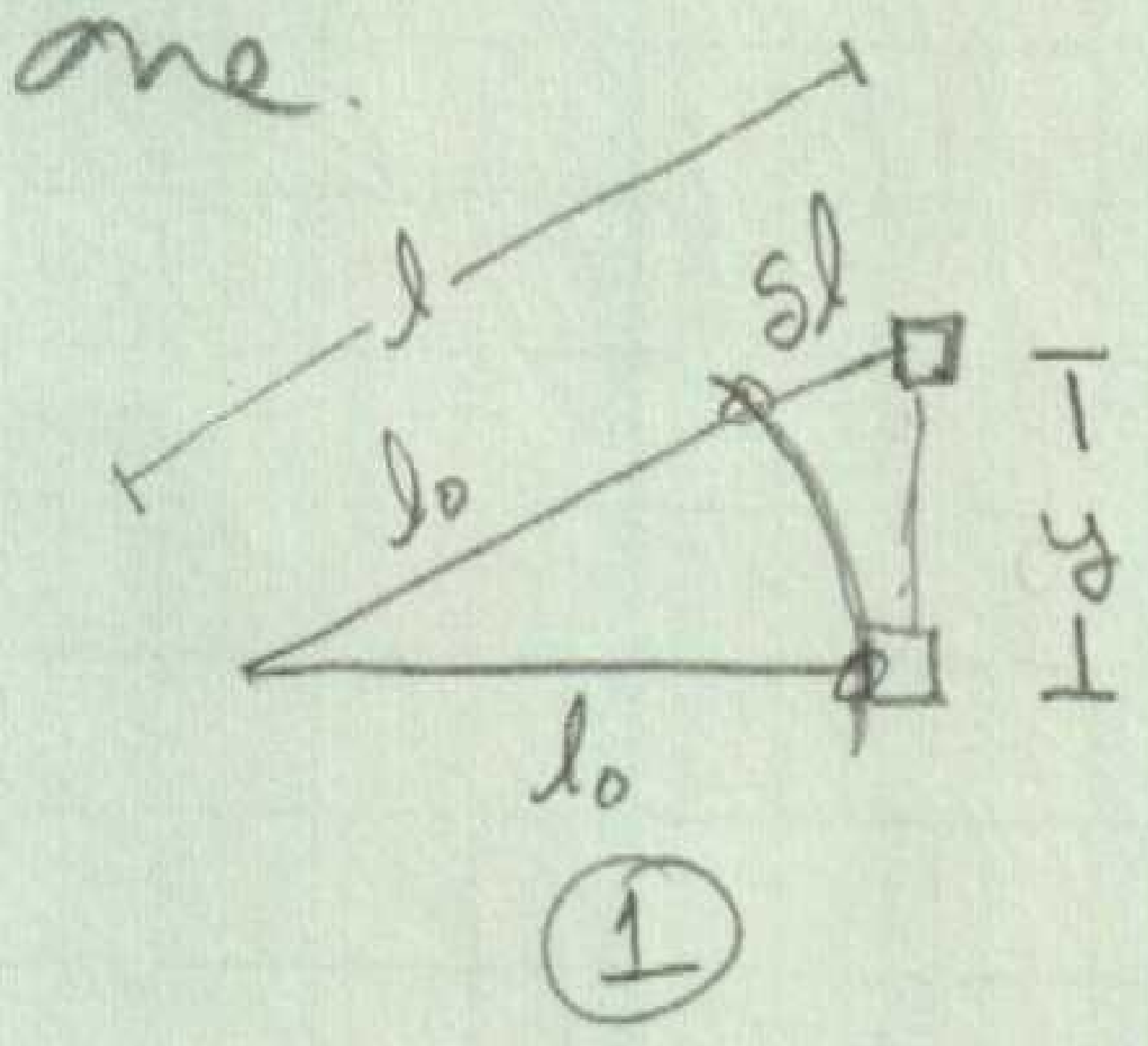
c)

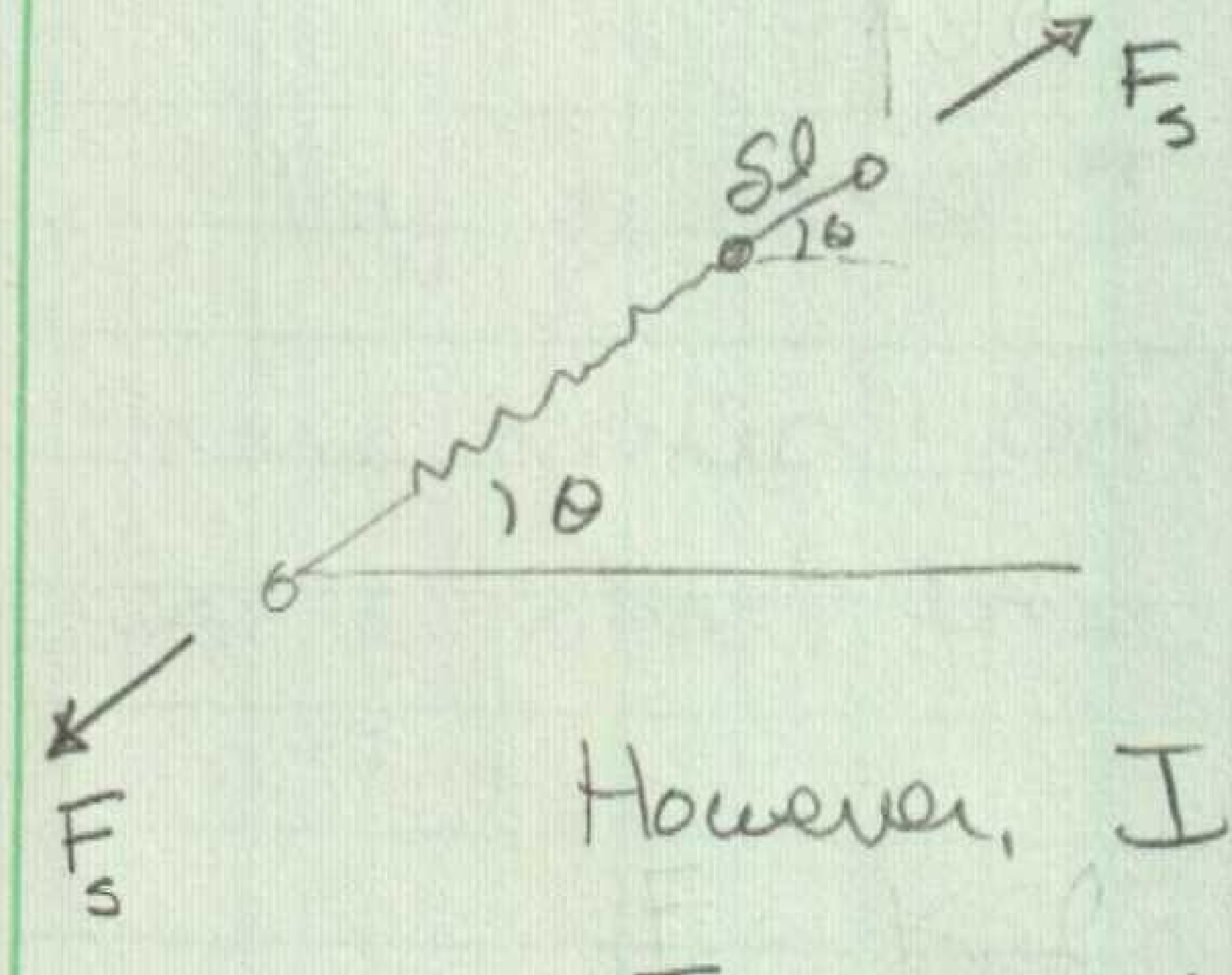


SOLUTION

Since the springs are equal length and spring constant equal value I can look at just one.



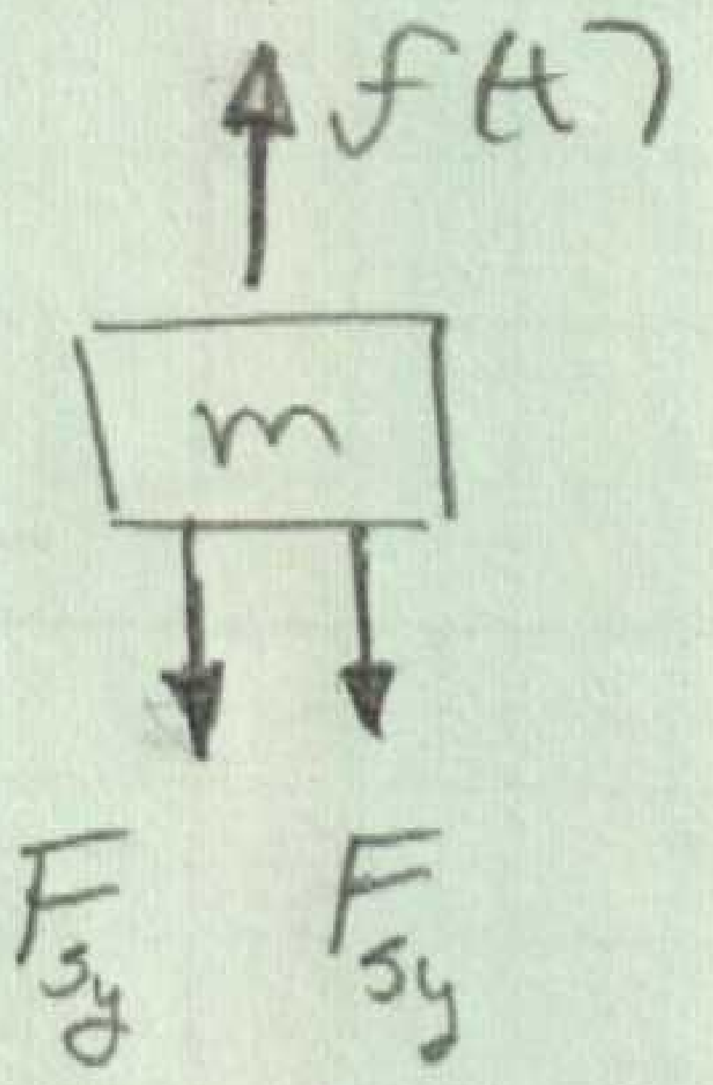




$$F_s = K\delta l$$

However, I'm interested in the vertical load from the spring.

$$\therefore \quad F_{s_y} = K\delta l \sin\theta$$



$$+\hat{\uparrow} \sum_i F_i = m\ddot{y} = f(t) - 2F_{s_y}$$

$$m\ddot{y} + 2K\delta l \sin\theta = f(t)$$

From diagram ① $\quad \sin\theta = \dfrac{y}{l_0 + \delta l} \quad , \quad l_0^2 + y^2 = l^2 = (l_0 + \delta l)^2$

$$\delta l = \sqrt{l_0^2 + y^2} - l_0$$

$$m\ddot{y} + 2K\left(\sqrt{l_0^2+y^2} - l_0\right) \cdot \frac{y}{\sqrt{l_0^2+y^2}} = f(t)$$

$$\boxed{m\ddot{y} + 2K\left(1 - \frac{l_0}{\sqrt{l_0^2+y^2}}\right) y = f(t)} \checkmark$$

4) RP 10.1.13

$m = 1$ kg   mass- spring - dashpot system.

$c = 10$ kg/s

The system returns to its equilibrium position the "quickest" if it approaches the equilibrium the quickest. This will be a critically damped system. To be critically damped ; ✓

$$c^2 = 4mk.$$

$$100 = 4k$$

$$\therefore k = 25 \text{ kgm/s.}$$ ✓

$\therefore$ The stiffness of the spring should be 25 kgm/s.

Tongue

MAE 5730

Hw5 2013.9.2

1.4 Express $2\cos(3t)+4\sin(3t)$ in terms of e^{3it} and e^{-3it}

Solution: Based on the Trignometric Identities that:

$$\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$$

$$\sin\theta=\frac{1}{2i}(e^{i\theta}-e^{-i\theta})$$

Here, we substitue θ with "$3t$"

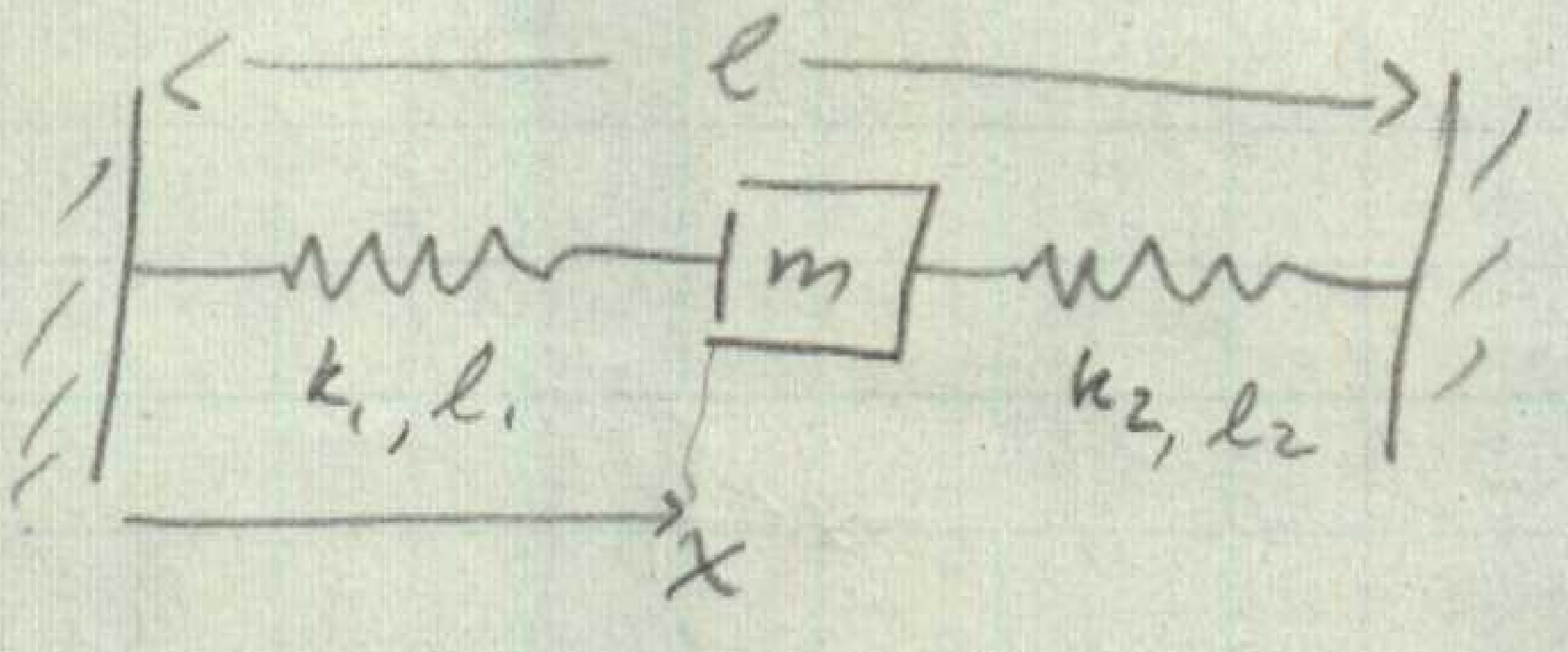
then we can get:

$$2\cos(3t)+4\sin(3t)=2\cdot\left[\frac{1}{2}(e^{3it}+e^{-3it})\right]+4\left[\frac{1}{2i}(e^{3it}-e^{-3it})\right]$$

$$=(1+\frac{2}{i})e^{3it}+(1-\frac{2}{i})e^{-3it}$$

$$=(1-2i)e^{3it}+(1+2i)e^{-3it}$$ ✓

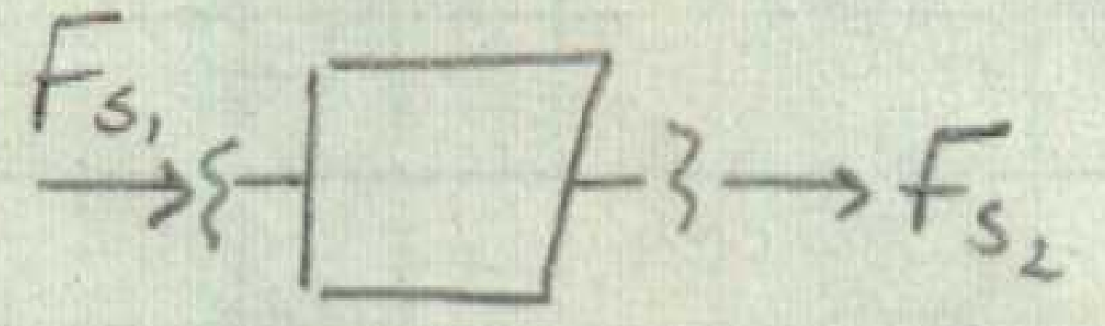
1.31) The unstretched length of k_1 is 0.5m and the unstretched length of k_2 is 0.25 m, $k_1 = 1000$ N/m, $k_2 = 2000$ N/m, $m = 2$ kg, $\ell = 0.5$m. Find the equilibrium position of the mass and determine the natural frequency of the system. Compare this natural freq. to that associated with $\ell = 0.75$ m.



$x + y = \ell$

$y = \ell - x$

FBD:



$F_{s_1} = -K_1(\ell_1 - x)$

$F_{s_2} = K_2 x(\ell - x - \ell_2)$

LMB:

$m\dot{x} = \Sigma F$

$m\ddot{x} = -k_1(\ell_1 - x) + k_2(\ell - x - \ell_2)$

At x_{eq}, $\ddot{x} = 0$

$0 = k_1\ell_1 - k_1 x_{eq} + k_2\ell - k_2 x_{eq} - k_2\ell_2$

$(k_1 + k_2)x_{eq} = k_1\ell_1 + k_2\ell - k_2\ell_2$

$x_{eq} = \dfrac{(k_1\ell_1 + k_2\ell - k_2\ell_2)}{(k_1 + k_2)} = 0.333\text{ m}$ ✓

$m\ddot{x} + (k_1 + k_2)x = k_1\ell_1 + k_2(\ell - \ell_2)$

$\boxed{\omega_0 = \sqrt{\dfrac{(k_1 + k_2)}{m}} = 38.73\text{ rad/sec}}$ ✓

ω_0 for $\ell = 0.5$m or $\ell = 0.75$m will be the same. ✓