PROBLEM 4-27

Statement: A storage rack is to be designed to hold the paper roll of Problem 4-8 as shown in Figure P4-12. Determine suitable values for dimensions a and b in the figure. Consider bending, shear, and bearing stresses. Assume an allowable tensile/compressive stress of 109 MPa and an allowable shear stress of 50 MPa for both stanchion and mandrel, which are steel. The mandrel is solid and inserts halfway into the paper roll. Balance the design to use all of the material strength. Calculate the deflection at the end of the roll.

Units: \[ N := \text{newton} \quad kN := 10^3 \text{ newton} \quad MPa := 10^6 \text{ Pa} \quad GPa := 10^9 \text{ Pa} \]

Given: Paper roll dimensions \[ OD := 1.50 \text{ m} \quad ID := 0.22 \text{ m} \quad L_{roll} := 3.23 \text{ m} \]

Material properties \[ S'_{y} := 100 \text{ MPa} \quad S_{y} := 50 \text{ MPa} \]

Roll density \[ \rho := 984 \text{ kg/m}^3 \]

Assumptions: 1. The paper roll's weight creates a concentrated load acting at the tip of the mandrel.
2. The mandrel's root in the stanchion experiences a distributed load over its length of engagement

Solution: See Figures 4-27 and Mathcad file P0427.

1. In Problem 3-27, we were concerned only with the portion of the mandrel outside of the stanchion. Therefore, we modeled it as a cantilever beam with a shear and moment reaction at the stanchion. Unfortunately, this tells us nothing about the stress or force distributions in the portion of the mandrel that is inside the stanchion. To do this we need to modify the model by replacing the concentrated moment (and possibly the concentrated shear force) with a force system that will yield information about the stress distribution in the mandrel on that portion that is inside the stanchion. Figure 4-27A shows the FBD used in Problem 3-27. Figure 4-27B is a simple model, but is not representative of a built-in condition. It would be appropriate if the hole in the stanchion did not fit tightly around the mandrel. Figure 4-27C is an improvement that will do for our analysis.

2. Determine the weight of the roll and the length of the mandrel.

\[ W := \frac{\pi}{4} (OD^2 - ID^2) \cdot L_{roll} \rho \quad W = 53.9 \text{ kN} \]

\[ L_m := 0.5L_{roll} \quad L_m = 1.615 \text{ m} \]

3. From inspection of Figure 4-27C, write the load function equation

\[ q(x) = -w < x > + w < x - b > + R < x - b >= W < x - b - L_m > \]

4. Integrate this equation from \(-\infty\) to \(x\) to obtain shear, \(V(x)\)

\[ V(x) = -w < x > + w < x - b > + R < x - b > + W < x - b - L_m > \]

5. Integrate this equation from \(-\infty\) to \(x\) to obtain moment, \(M(x)\)
M(x) = -(w/2)<x^2> + (w/2)<x - b>^2 + R<x - b> + W<x - b> - L_m^2

6. Solve for the reactions by evaluating the shear and moment equations at a point just to the right of x = b + L_m, where both are zero.

At x = (b + L_m)^2, V = M = 0

0 = w(b + L_m) + w(L_m) + R - W

R = W + w - b

0 = \frac{w}{2}(b + L_m)^2 + \frac{w}{2}L_m^2 + RL_m - \frac{w}{2}(b + L_m)^2 + \frac{w}{2}L_m^2 + (W + w - b)L_m

\frac{2WL_m}{b^2}

Note that R is inversely proportional to b and w is inversely proportional to b^2.

7. To see the value of x at which the shear and moment are maximum, let

\[ b := 400 \text{ mm} \]

then \[ w := \frac{2WL_m}{b^2} \]

and \[ R := W + w - b \]

\[ L := b + L_m \]

8. Define the range for x

\[ x := 0 \text{ mm}, 0.002L_L, L \]

9. For a Mathcad solution, define a step function S. This function will have a value of zero when x is less than z, and a value of one when it is greater than or equal to z.

\[ S(x, z) := \text{if}(x \geq z, 1, 0) \]

10. Write the shear and moment equations in Mathcad form, using the function S as a multiplying factor to get the effect of the singularity functions.

\[ V(x) := wS(x, 0 \text{ mm})x + wS(x, b)(x - b) + RS(x, b) - WS(x, L) \]

\[ M(x) := \frac{w}{2}S(x, 0 \text{ mm})x^2 + \frac{w}{2}S(x, b)(x - b)^2 + RS(x, b)(x - b) - WS(x, L)(x - L) \]

11. Plot the shear and moment diagrams.

Shear Diagram

Moment Diagram

\[ \text{FIGURE 4-27D} \]

Shear and Moment Diagram Shapes for Problem 4-27

12. From Figure 4-27D, the maximum internal shear and moment occur at \( x = b \) and are

\[ V_{\text{max}} = \frac{2WL_m}{b} \]

\[ M_{\text{max}} := WL_m \]

\[ M_{\text{max}} = 87.94 \text{kN.m} \]
13. The bending stress will be a maximum at the top or bottom of the mandrel at a section through \( x = b \).

\[
\sigma_{\text{max}} = \frac{M_{\text{max}} a}{2I}
\]

where \( I = \frac{\pi a^4}{64} \) so,

\[
\sigma_{\text{max}} = \frac{32 M_{\text{max}}}{\pi a^3} = S_y \]

Solving for \( a \),

\[
a := \left( \frac{32 W L_m}{\pi S_y} \right)^{\frac{1}{3}}
\]

so,

\[
a = 206.97 \text{ mm}
\]

Round this to \( a := 210 \text{ mm} \)

14. Using this value of \( a \) and equation 4.15c, solve for the shear stress on the neutral axis at \( x = b \).

\[
\tau_{\text{max}} = \frac{4 \cdot V_{\text{max}}}{3A} \cdot \frac{8 W L_m}{3 \left( \frac{\pi a^2}{4} \right) b} = S_{ys}
\]

Solving for \( b \)

\[
b := \frac{8 W L_m}{3 \left( \frac{\pi a^2}{4} \right) S_{ys}}
\]

so,

\[
b = 134.026 \text{ mm}
\]

Round this to \( b := 134 \text{ mm} \)

15. These are minimum values for \( a \) and \( b \). Using them, check the bearing stress.

Magnitude of distributed load

\[
w := \frac{2 \cdot W L_m}{b^2}
\]

so,

\[
w = 9695 \frac{N}{\text{mm}}
\]

Bearing stress

\[
\sigma_{\text{bear}} = \frac{w \cdot b}{a \cdot b} = \sigma_{\text{bear}} = 46.2 \text{ MPa}
\]

Since this is less than \( S_y \), the design is acceptable for \( a = 210 \text{ mm} \) and \( b = 134 \text{ mm} \)

16. Assume a cantilever beam loaded at the tip with load \( W \) and a mandrel diameter equal to \( a \) calculated above.

Moment of inertia

\[
I := \frac{\pi a^4}{64}
\]

so,

\[
I = 9.547 \times 10^7 \text{ mm}^4
\]

Deflection at tip (Appendix D)

\[
y_{\text{max}} := \frac{W L_m^3}{3EI}
\]

so,

\[
y_{\text{max}} = -3.83 \text{ mm}
\]

This can be accommodated by the 220-mm inside diameter of the paper roll.
Find the fluctuating stresses
Find the fatigue safety factor

- First determine the worst case stresses. These will occur at C on top of the arm when $F = 1500 \text{ N}$

\[
\sigma_x = \frac{M_c}{I} = 769.6 \text{ MPa} \quad \sigma_y = 0 \text{ MPa}
\]

\[
\tau_{xy} = \frac{T_c}{J} = 135.8 \text{ MPa}
\]

So:
\[
\sigma_{I \text{ max}} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 793 \text{ MPa}
\]
\[
\sigma_{II \text{ max}} = 0 \text{ MPa}
\]
\[
\sigma_{III \text{ max}} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = -23 \text{ MPa}
\]

Mohr's circle.
Having principal stresses will do you no good in this case. You must determine an "effective" stress or overall stress magnitude. Eq. 5.7c, for von Mises stress is:

\[ \sigma' = \sqrt{\sigma_{1}^{2} - \sigma_{1} \sigma_{III} + \sigma_{III}^{2}} \] (for 2D)

\[ \sigma_{\text{max}} = \sqrt{793^2 - (793)(-23) + (-23)^2} = 804.7 \text{ MPa} \]

\[ \sigma_{\text{min}} = 0 \text{ MPa} \quad (\text{when } F = 0) \]

Now the alternating and mean stress components can be calculated.

\[ \sigma'_{a} = \frac{\sigma'_{\text{max}} - \sigma'_{\text{min}}}{2} = 402.4 \text{ MPa} \]

\[ \sigma'_{M} = \frac{\sigma'_{\text{max}} + \sigma'_{\text{min}}}{2} = 402.4 \text{ MPa} \]
Now handle the fatigue safety factor.

The endurance limit: \( S_e' = 0.5 \times S_{at} = 250 \text{ MPa} \)

The modified endurance limit is:

\[
S_e = C_{load} C_{size} C_{surf} C_{temp} C_{reliab} S_e'
\]

where the \( C_\text{s} \) are modification factors

- \( C_{load} = 1 \) (bending) \([\text{eq. 6.7a}]\)
- \( C_{temp} = 1 \) \( T \approx 450^\circ C \) \([\text{eq. 6.7f}]\) (the rider is not on fire)
- \( C_{size} = 1.189 \text{ deg.} \times 0.976 = 1 \)
  - for \( d = 250 \text{ mm} \) \([\text{eq. 6.7b}]\)
  - \( A_{qs} = \frac{2.354 \text{ in}}{0.766} = 3.1 \) (non-rotating)
  - from Fig. 6-25

\[
C_{surf} = A \times (S_{at})^b
\]

- \( C_{surf} = 0.869 \)

- \( C_{reliab} = 0.753 \)

\[
S_e = 163.56 \text{ MPa}
\]

and:

\[
N_f = \frac{S_e S_{at}}{n_1 + n_2 \times S_e} = 0.31
\]
Using 2000 Series Aluminum, design the cantilever beam sections to survive jumps of 2 in w/ a dynamic safety factor of 2 for a life of 500 cycles.

First assumes: 2024 Al

- \( w = 60 \text{ lb} \)
- \( k = 100 \text{ lb/in} \)
- \( \frac{w}{2} = 30 \text{ lb} \)
- \( \frac{b}{2} = 5 \text{ in} \)
- \( b = 1.5 \text{ in} \)

\[ \Rightarrow \text{weight applied at} \frac{w}{2} \]

To give adequate support

\[ \Rightarrow S_{ut} = 64 \text{ ksi} \text{ for} 2024 \text{ Al} \]

(See Table 8-2, p. 944)

*Note: your values will be different based on your assumptions

- With the above assumptions all that remains is to solve for \( t \)

Begin by finding \( F_{\text{max}} \) as in HW #9 using Energy Method

\[ F_{\text{max}} = ky = k \left[ \frac{mg}{12} + \frac{1}{2} \sqrt{\frac{(gmgh)^2}{k^2} + \frac{8mgh}{k}} \right] = 338 \text{ lb} \]

Recall: \( U_n = U_s \Rightarrow \) from \( E_{\text{max height}} = E_{\text{min height}} \)

\[ (mg\frac{y}{2} + mgh) = \frac{1}{2} ky^2 \]
So: \( P_{\text{max}} = \frac{F_{\text{max}}}{2} = 119 \text{ lbs} \)
\( P_{\text{min}} = 0 \text{ lbs} \)

\( M_{\text{max}} = P_{\text{max}} \left( \frac{L}{2} \right) = 595 \text{ lb-in} \)
\( M_{\text{min}} = 0 \text{ lb-in} \)

- Find the alternating and mean components of the moment.

\[ M_a = \frac{M_{\text{max}} - M_{\text{min}}}{2} = 297.5 \text{ lb-in} \]
\[ M_m = \frac{M_{\text{max}} + M_{\text{min}}}{2} = 297.5 \text{ lb-in} \]

- Find the Endurance limit: \( S_e = 20 \text{ KSI} \) @ 50 cycles

- Find modification factors

  - \( C_{\text{load}} = 1 \) (bending)
  - \( C_{\text{temp}} = 1 \)
  - \( C_{\text{reliab}} = 1 \) (50%, Table 6-4)

\( C_{\text{surf}} = A(S_{\text{uf}})^b \quad A = 2.7 \quad b = -0.265 \)
\( C_{\text{surf}} = 0.897 \)

\[ C_{\text{size}} = 0.869 \left( \frac{\text{deg}}{90} \right) - 0.097 \quad \text{deg} = \sqrt{\frac{0.05(w)(t)}{0.0766}} \]
\[ C_{\text{size}} = 0.87(t) 0.403 \]

\[ S_e = (1)(0.87 + 0.403)(0.897)(1)(1)(20 \text{ KSI}) = 15.61 + 0.403 \]
Find fatigue strength at $N = 10^3$ cycles ($S_m$)

$$S_m = 0.9 \ S_{ut} = 57.6 \text{ Ksi} \quad (eq \ 6.9)$$

Use eq 6.10(a), $S_f = a \ N^b$, to find corrected fatigue strength

where $a(t) = \frac{S_m}{(10^3)^b(t)}$, $b(t) = \frac{1}{Z} \log \frac{S_m}{S_{e(t)}}$

$$N = 5 \times 10^4$$

$$Z = -5.694 \text{ for } 5 \times 10^8$$

$$S_f(t) = \left[ \frac{57.6}{(10^3)^{0.175 \log \left( \frac{3.69}{t^{0.0405}} \right)}} \right]^{0.0175 \log \left( \frac{3.69}{t^{0.0405}} \right)} \text{ from table 6-5}$$

Now: $N_{fd} = 2 = \frac{w t^2}{6} \frac{S_f(t) \ S_{ut}}{M_c \ S_{ut} + M_m \ S_f(t)}$

plug numbers in and solve for $t$

$$t = 0.304 \text{ in}$$