SAMPLE 1.28 Plain vanilla vector equation in 2-D: Three forces act on a particle as shown in the figure. The equilibrium condition of the particle requires that $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$. It is given that $\vec{W} = -20\,\text{N}\,\hat{\jmath}$. Find the magnitudes of forces $\vec{F}_1$ and $\vec{F}_2$.

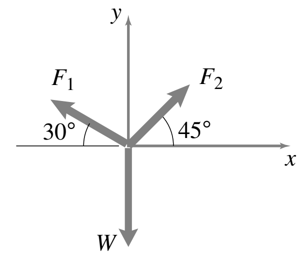

Figure 1.74

Solution We are given a vector equation, $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$, in which one vector $\vec{W}$ is completely known and the directions of the other two vectors are given. We need to find their magnitudes. Let us write the vectors as

$$\vec{F}_1 = F_1\hat{\lambda}_1, \quad \vec{F}_2 = F_2\hat{\lambda}_2, \quad \vec{W} = -W\hat{\jmath},$$

where $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unit vectors along $\vec{F}_1$ and $\vec{F}_2$, respectively (their directions are specified by the given angles in the figure), and $W = 20\,\text{N}$ as given. We can write the unit vectors in component form as

$$\hat{\lambda}_1 = \lambda_{1_x}\hat{\imath} + \lambda_{1_y}\hat{\jmath} \quad \text{and} \quad \hat{\lambda}_2 = \lambda_{2_x}\hat{\imath} + \lambda_{2_y}\hat{\jmath}.$$

Now we can write the given vector equation as

$$F_1(\lambda_{1_x}\hat{\imath} + \lambda_{1_y}\hat{\jmath}) + F_2(\lambda_{2_x}\hat{\imath} + \lambda_{2_y}\hat{\jmath}) = W\hat{\jmath}. \tag{1.46}$$

Dotting both sides of eqn. (1.46) with $\hat{\imath}$ and $\hat{\jmath}$ respectively, we get

$$\lambda_{1_x}F_1 + \lambda_{2_x}F_2 = 0 \tag{1.47}$$

$$\lambda_{1_y}F_1 + \lambda_{2_y}F_2 = W. \tag{1.48}$$

Here, we have two equations in two unknowns (F_1 and F_2). We can solve these equations for the unknowns. Let us solve these two linear equations by first putting them into a matrix form and then solving the matrix equation. The matrix equation is

$$\begin{bmatrix} \lambda_{1_x} & \lambda_{2_x} \\ \lambda_{1_y} & \lambda_{2_y} \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Using Cramer's rule for matrix inversion, we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{\lambda_{1_x}\lambda_{2_y} - \lambda_{2_x}\lambda_{1_y}} \begin{bmatrix} \lambda_{2_y} & -\lambda_{2_x} \\ -\lambda_{1_y} & \lambda_{1_x} \end{bmatrix} \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Substituting the numerical values of $\lambda_{1_x} = -\cos 30^\circ = -\sqrt{3}/2, \lambda_{1_y} = \sin 30^\circ = 1/2$ and similarly, $\lambda_{2_x} = 1/\sqrt{2}, \lambda_{2_y} = 1/\sqrt{2}$, and $W = 20\,\text{N}$, we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 14.64 \\ 17.93 \end{pmatrix} \text{N}.$$

$F_1 = 14.64\,\text{N}, \quad F_2 = 17.93\,\text{N}$

Check: We can easily check if the values we have got are correct. For example, substituting the numerical values in eqn. (1.47), we get

$$14.64\,\text{N} \cdot \left(-\frac{\sqrt{3}}{2}\right) + 17.93\,\text{N} \cdot \frac{1}{\sqrt{2}} \stackrel{\surd}{=} 0.$$

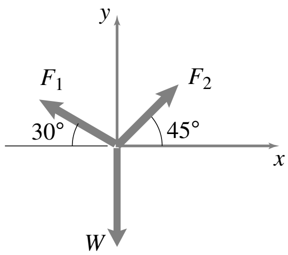

Figure 1.75

SAMPLE 1.29 Solving for a single unknown from a 2-D vector equation: Consider the same problem as in Sample 1.28. That is, you are given that $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$ where $\vec{W} = -20\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}$ and $\vec{F}_1$ and $\vec{F}_2$ act along the directions shown in the figure. Find the magnitude of $\vec{F}_2$.

Solution Once again, we write the given vector equation as

$$F_1\hat{\boldsymbol{\lambda}}_1 + F_2\hat{\boldsymbol{\lambda}}_2 = W\hat{\boldsymbol{\jmath}},$$

where

$$\begin{aligned} W &= 20\,\mathrm{N}, \\ \hat{\boldsymbol{\lambda}}_1 &= \lambda_{1_x}\hat{\boldsymbol{\imath}} + \lambda_{1_y}\hat{\boldsymbol{\jmath}} \\ &= -\sqrt{3}/2\hat{\boldsymbol{\imath}} + 1/2\hat{\boldsymbol{\jmath}}, \quad \text{and} \\ \hat{\boldsymbol{\lambda}}_2 &= \lambda_{2_x}\hat{\boldsymbol{\imath}} + \lambda_{2_y}\hat{\boldsymbol{\jmath}} \\ &= 1/\sqrt{2}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}). \end{aligned}$$

We are interested in finding F_2 only. So, let us take a dot product of this equation with a vector that gets rid of the F_1 term. Any such vector would have to be perpendicular to $\hat{\boldsymbol{\lambda}}_1$. One such vector is $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_1$. Let us call this vector $\hat{\boldsymbol{n}}_1$, that is,

$$\hat{\boldsymbol{n}}_1 = \hat{\boldsymbol{k}} \times (\lambda_{1_x}\hat{\boldsymbol{\imath}} + \lambda_{1_y}\hat{\boldsymbol{\jmath}}) = \lambda_{1_x}\hat{\boldsymbol{\jmath}} - \lambda_{1_y}\hat{\boldsymbol{\imath}}.$$

Now, dotting the given vector equation with $\hat{\boldsymbol{n}}_1$, we get

$$\begin{aligned} F_1 \overbrace{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_1)}^{0} + F_2(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2) &= W(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\jmath}}) \\ \Rightarrow \quad F_2 &= W\frac{\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\jmath}}}{\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2} \\ &= W\frac{(\lambda_{1_x}\hat{\boldsymbol{\jmath}} - \lambda_{1_y}\hat{\boldsymbol{\imath}}) \cdot \hat{\boldsymbol{\jmath}}}{(\lambda_{1_x}\hat{\boldsymbol{\jmath}} - \lambda_{1_y}\hat{\boldsymbol{\imath}}) \cdot (\lambda_{2_x}\hat{\boldsymbol{\imath}} + \lambda_{2_y}\hat{\boldsymbol{\jmath}})} \\ &= W\frac{\lambda_{1_x}}{\lambda_{1_x}\lambda_{2_y} - \lambda_{1_y}\lambda_{2_x}} \\ &= 20\,\mathrm{N}\,\frac{-\sqrt{3}/2}{-\sqrt{3}/2 \cdot 1/\sqrt{2} - 1/2 \cdot 1/\sqrt{2}} \\ &= 20\,\mathrm{N}\,\frac{\sqrt{6}}{\sqrt{3}+1} = 17.93\,\mathrm{N} \end{aligned}$$

which, of course, is the same value we got in Sample 1.28. Note that here we obtained one scalar equation in one unknown by dotting the 2-D vector equation with an appropriate vector to get rid of the other unknown F_1.

$$\boxed{F_2 = 17.93\,\mathrm{N}}$$

SAMPLE 1.30 **Solving a 3-D vector equation on a computer:** Four forces, $\vec{F}_1, \vec{F}_2, \vec{F}_3$ and $\vec{N}$ are in equilibrium, that is, $\vec{F}_1 + \vec{F}_2 + +\vec{F}_3 + \vec{N} = \vec{0}$ where $\vec{N} = -100\,\text{kN}\hat{k}$ is known and the directions of the other three forces are known. $\vec{F}_1$ is directed from (0,0,0) to (1,-1,1), $\vec{F}_2$ from (0,0,0) to (-1,-1,1), and $\vec{F}_3$ from (0,0,0) to (0,1,1). Find the magnitudes of $\vec{F}_1, \vec{F}_2$, and $\vec{F}_3$.

Solution Let $\vec{F}_1 = F_1\hat{\lambda}_1, \vec{F}_2 = F_2\hat{\lambda}_2$, and $\vec{F}_1 = F_3\hat{\lambda}_3$, where $\hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_3$ are unit vectors in the directions of $\vec{F}_1, \vec{F}_2$, and $\vec{F}_3$, respectively. Then the given vector equation can be written as

$$F_1\hat{\lambda}_1 + F_2\hat{\lambda}_2 + F_3\hat{\lambda}_3 = -\vec{N} = -N\hat{k}$$

where $N = -100\,\text{kN}$. Dotting this equation with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively, and realizing that $\hat{\imath} \cdot \hat{\lambda}_1 = \lambda_{1_x}$, $\hat{\jmath} \cdot \hat{\lambda}_1 = \lambda_{1_y}$, etc., we get the following three scalar equations.

$$\begin{aligned}
\lambda_{1_x}F_1 + \lambda_{2_x}F_2 + \lambda_{3_x}F_3 &= 0\\
\lambda_{1_y}F_1 + \lambda_{2_y}F_2 + \lambda_{3_y}F_3 &= 0\\
\lambda_{1_z}F_1 + \lambda_{2_z}F_2 + \lambda_{3_z}F_3 &= -N.
\end{aligned}$$

Thus we get a system of three linear equations in three unknowns. To solve for the unknowns, we set up these equations as a matrix equation and then use a computer to solve it. In matrix form these equations are

$$\begin{bmatrix} \lambda_{1_x} & \lambda_{2_x} & \lambda_{3_x} \\ \lambda_{1_y} & \lambda_{2_y} & \lambda_{3_y} \\ \lambda_{1_z} & \lambda_{2_z} & \lambda_{3_z} \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -N \end{pmatrix}.$$

To solve this equation on a computer, we need to input the matrix of unit vector components and the known vector on the right hand side. From the given coordinates for the directions of forces, we have $\hat{\lambda}_1 = (\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, $\hat{\lambda}_2 = (-\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, and $\hat{\lambda}_3 = (\hat{\jmath} + \hat{k})/\sqrt{2}$. ① We are also given that $N = -100\,\text{kN}$. Now, we use the following pseudo-code to find the solution on a computer.

```
Let s2 = sqrt(2),  s3 = sqrt(3)
    A = [ 1/s3  -1/s3  0
         -1/s3  -1/s3  1/s2
          1/s3   1/s3  1/s2 ]
    b = [ 0   0   100]'
    solve A*F = b for F
```

Using this pseudo-code we find the solution to be

```
    F = [ 43.3013
          43.3013
          70.7107 ]
```

That is, $F_1 = F_2 = 43.3\,\text{kN}$ and $F_3 = 70.7\,\text{kN}$.

$$F_1 = 43.3\,\text{kN},\; F_2 = 43.3\,\text{kN},\; F_3 = 70.7\,\text{kN}$$

① These unit vectors are computed by taking a vector from one end point to the other end point (as given) and then dividing by its magnitude. For example, we find $\hat{\lambda}_1$ by first finding $\vec{r}_1 = (1)\hat{\imath}+(-1)\hat{\jmath}+(1)\hat{k}$, a vector from (0,0,0) to (1,-1,1), and then $\hat{\lambda}_1 = \frac{\vec{r}_1}{|\vec{r}_1|}$.

SAMPLE 1.31 Vector operations on a computer: Consider the problem of Sample 1.30 again. That is, you are given the vector equation $\vec{F}_1 + \vec{F}_2 + +\vec{F}_3 + \vec{N} = \vec{0}$ where $\vec{N} = -100\,\text{kN}\hat{k}$ and the directions of $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ are given by the unit vectors $\hat{\lambda}_1 = (\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, $\hat{\lambda}_2 = (-\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, and $\hat{\lambda}_3 = (\hat{\jmath} + \hat{k})/\sqrt{2}$, respectively. Find F_1.

Solution We can, of course, solve the problem as we did in Sample 1.30 and we get the answer as a part of the unknown forces we solved for. However, we would like to show here that we can extract one scalar equation in just one unknown (F_3) from the given 3-D vector equation and solve for the unknown without solving a matrix equation. Although we can carry out all required calculations by hand, we will show how we can use a computer to do these operations.

We can write the given vector equation as

$$F_1\hat{\lambda}_1 + F_2\hat{\lambda}_2 + F_3\hat{\lambda}_3 = -\vec{N}. \tag{1.49}$$

We want to find F_1. Therefore, we should dot this equation with a vector that gets rid of both F_2 and F_3, i.e., with a vector which is perpendicular to both $\vec{F}_2$ and $\vec{F}_3$. One such vector is $\vec{F}_2 \times \vec{F}_3$ or $\hat{\lambda}_2 \times \hat{\lambda}_3$. Let $\hat{n} = \hat{\lambda}_2 \times \hat{\lambda}_3$. Now, dotting both sides of eqn. (1.49) with $\hat{n}$, we get

$$F_1(\hat{\lambda}_1 \cdot \hat{n}) + F_2(\hat{\lambda}_2 \cdot \hat{n}) + F_3(\hat{\lambda}_3 \cdot \hat{n}) = -\vec{N} \cdot \hat{n}$$

Since $\hat{\lambda}_2 \cdot \hat{n} = 0$ and $\hat{\lambda}_3 \cdot \hat{n} = 0$ ($\hat{n}$ is normal to both $\hat{\lambda}_2$ and $\hat{\lambda}_3$), we get

$$\begin{aligned} F_1(\hat{\lambda}_1 \cdot \hat{n}) &= -\vec{N} \cdot \hat{n} \\ \Rightarrow \quad F_1 &= \frac{-\vec{N} \cdot \hat{n}}{\hat{\lambda}_1 \cdot \hat{n}}. \end{aligned}$$

Thus we have found the solution. To compute the expression on the right hand side of the above equation we use the following pseudo-code which assumes that you have written (or have access to) two functions, `dot` and `cross`, that compute the dot and cross product of two given vectors.

```
lambda_1 = 1/sqrt(3)*[1  -1  1]';
lambda_2 = 1/sqrt(3)*[-1  -1  1]';
lambda_3 = 1/sqrt(2)*[0  1  1]';
       N = [0  0  -100]';
       n = cross(lambda_2, lambda_3);
      F1 = - dot(N, n)/dot(lambda_1, n)
```

By following these steps on a computer, we get the output `F1 = 43.3013`, that is, $F_1 = 43.3\,\text{kN}$, which, of course, is the same answer we obtained in Sample 1.30.

$F_1 = 43.3\,\text{kN}$