SAMPLE 1.17 Cross product in 2-D: Two vectors $\vec{a}$ and $\vec{b}$ of length 10 ft and 6 ft, respectively, are shown in the figure. The angle between the two vectors is $\theta = 60°$. Find the cross product $\vec{a} \times \vec{b}$.

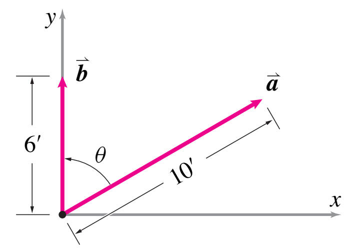

Figure 1.45

Solution Both vectors $\vec{a}$ and $\vec{b}$ are in the xy plane. Therefore, their cross product will point in either $+\hat{k}$ or $-\hat{k}$ direction (normal to the plane formed by $\vec{a}$ and $\vec{b}$). To determine the direction of the normal, we use the right hand rule and curl our fingers from $\vec{a}$ towards $\vec{b}$, tracing angle θ counterclockwise. The thumb then points in the positive $\hat{k}$ direction, indicating $\hat{n} = \hat{k}$. Thus,

$$\begin{aligned}
\vec{a} \times \vec{b} &= |\vec{a}||\vec{b}| \sin\theta \hat{n} \\
&= (10\,\text{ft}) \cdot (6\,\text{ft}) \cdot \sin 60° \hat{k} \\
&= 60\,\text{ft}^2 \cdot \frac{\sqrt{3}}{2} \hat{k} \\
&= 30\sqrt{3}\,\text{ft}^2 \hat{k}.
\end{aligned}$$

$$\boxed{\vec{a} \times \vec{b} = 30\sqrt{3}\,\text{ft}^2 \hat{k}}$$

SAMPLE 1.18 Computing 2-D cross product in different ways: The two vectors shown in the figure are $\vec{a} = 2\hat{\imath} - \hat{\jmath}$ and $\vec{b} = 4\hat{\imath} + 2\hat{\jmath}$. The angle between the two vectors turns out to be $\theta = \sin^{-1}(4/5)$. Find the cross product $\vec{a} \times \vec{b}$

1. using the angle θ, and
2. using the components of the vectors.

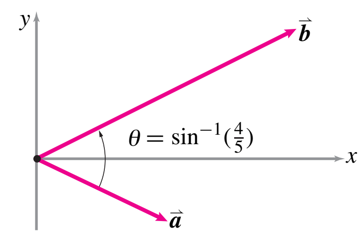

Figure 1.46

Solution

1. Cross product using the angle θ:

$$\begin{aligned}
\vec{a} \times \vec{b} &= |\vec{a}||\vec{b}| \sin\theta \hat{n} \\
&= |2\hat{\imath} - \hat{\jmath}||4\hat{\imath} + 2\hat{\jmath}| \cdot \sin(\sin^{-1} \frac{4}{5}) \hat{k} \\
&= (\sqrt{2^2 + 1^2})(\sqrt{4^2 + 2^2}) \cdot \frac{4}{5} \hat{k} \\
&= \sqrt{5} \cdot \sqrt{20} \cdot \frac{4}{5} \hat{k} = 10 \cdot \frac{4}{5} \hat{k} \\
&= 8\hat{k}.
\end{aligned}$$

2. Cross product using components:

$$\begin{aligned}
\vec{a} \times \vec{b} &= (2\hat{\imath} - \hat{\jmath}) \times (4\hat{\imath} + 2\hat{\jmath}) \\
&= 2\hat{\imath} \times (4\hat{\imath} + 2\hat{\jmath}) - \hat{\jmath} \times (4\hat{\imath} + 2\hat{\jmath}) \\
&= 8 \underbrace{\hat{\imath} \times \hat{\imath}}_{\vec{0}} + 4 \underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{k}} - 4 \underbrace{\hat{\jmath} \times \hat{\imath}}_{-\hat{k}} - 2 \underbrace{\hat{\jmath} \times \hat{\jmath}}_{\vec{0}} \\
&= 4\hat{k} + 4\hat{k} \\
&= 8\hat{k}.
\end{aligned}$$

The answers obtained from the two methods are, of course, the same as they must be.

$$\boxed{\vec{a} \times \vec{b} = 8\hat{k}}$$

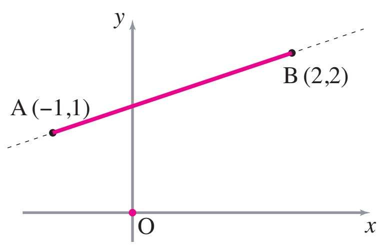

Figure 1.47

SAMPLE 1.19 Find the shortest distance from a point to a line: A straight line passes through two points, A (-1,1) and B (2,2), in the xy plane. Find the shortest distance from the origin to the line.

Solution There are various ways we can find the required distance. In 2-D, the following two methods can be used for quick calculations.

Method-1: The shortest distance d between a point P and a given line AB is obtained by taking the dot product of a vector from P to any point on the line with a unit normal to the given line. Thus, $d = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{r}}_A = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{r}}_B$ where $\hat{\boldsymbol{n}}$ is a unit vector normal to $\vec{\boldsymbol{r}}_{B/A}$. Why?

From geometry we know that the shortest distance from a given point to a given line is the perpendicular distance from the line to the given point. Thus to find this distance, we need to draw a perpendicular from the given point to the line of interest. In fig. 1.48, this distance d is shown as line OC. From the figure, we also see that $d = \mathrm{OC} = \mathrm{OB}\cos\alpha$, that is, d is the projection of vector $\vec{\boldsymbol{r}}_B$ along OC. But OC is normal to AB, hence, we can find a unit vector $\hat{\boldsymbol{n}}$ along OC by finding a normal to vector $\vec{\boldsymbol{r}}_{B/A}$. Thus,

$$d = \vec{\boldsymbol{r}}_B \cdot \hat{\boldsymbol{n}}, \quad \text{where} \quad \hat{\boldsymbol{n}} \perp \vec{\boldsymbol{r}}_{B/A}.$$

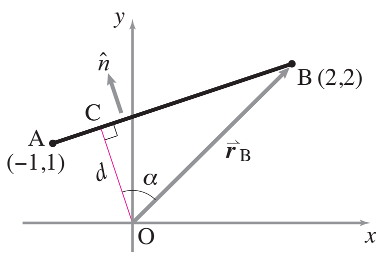

Figure 1.48

We can use cross product to find $\hat{\boldsymbol{n}}$. Since $\vec{\boldsymbol{r}}_{B/A}$ is in the xy plane, $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}$ gives us a vector perpendicular to AB and in the xy plane. Thus,

$$\hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}}{|\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}|}, \quad \text{and hence,} \quad d = \vec{\boldsymbol{r}}_B \cdot \frac{\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}}{|\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}|}.$$

From the given coordinates, $\vec{\boldsymbol{r}}_B = 2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{r}}_{B/A} = \vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A = 3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$, we get

$$\begin{aligned} \hat{\boldsymbol{n}} &= \frac{\hat{\boldsymbol{k}} \times (3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}{|\hat{\boldsymbol{k}} \times (3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})|} = \frac{3\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}}{|3\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}|} = -\frac{1}{\sqrt{10}}\hat{\boldsymbol{\imath}} + \frac{3}{\sqrt{10}}\hat{\boldsymbol{\jmath}} \\ d &= \vec{\boldsymbol{r}}_B \cdot \hat{\boldsymbol{n}} = (2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \cdot \left(-\frac{1}{\sqrt{10}}\hat{\boldsymbol{\imath}} + \frac{3}{\sqrt{10}}\hat{\boldsymbol{\jmath}}\right) = \frac{4}{\sqrt{10}}. \end{aligned}$$

$$\boxed{d = \frac{4}{\sqrt{10}}}$$

You can easily verify that we get the same result even if we use $\vec{\boldsymbol{r}}_A$ in place of $\vec{\boldsymbol{r}}_B$ (or for that matter, any vector from O to a point on line AB). For example, $\vec{\boldsymbol{r}}_A \cdot \hat{\boldsymbol{n}} = (-ui + \hat{\boldsymbol{\jmath}}) \cdot (-\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})/\sqrt{10} = 4/\sqrt{10}$, the same value as obtained above.

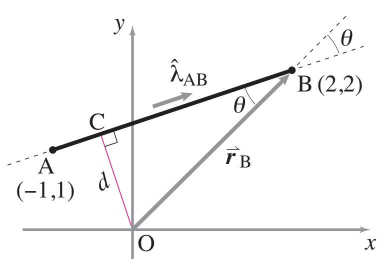

Figure 1.49

Method-2: The shortest distance between a point P and a given line AB can also be found by taking a cross product of a unit vector $\hat{\boldsymbol{\lambda}}_{AB}$ along the line with any vector from P to a point on the line, e.g., $\vec{\boldsymbol{r}}_{B/P}$. For the given case, where P is the origin, we get, $d = \hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B$. Why?

From fig. 1.49, we see that $d \equiv \mathrm{OC} = \mathrm{OB}\sin\theta$ where θ is the angle between $\vec{\boldsymbol{r}}_B$ and $\vec{\boldsymbol{r}}_{B/A}$. Let $\hat{\boldsymbol{\lambda}}_{AB}$ be a unit vector along line AB. Then,

$$\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B = \underbrace{|\hat{\boldsymbol{\lambda}}_{AB}|}_{1} |\vec{\boldsymbol{r}}_B| \sin\theta \hat{\boldsymbol{n}} = |\vec{\boldsymbol{r}}_B| \sin\theta \hat{\boldsymbol{k}}.$$

Thus, $d = OB\sin\theta = |\vec{\boldsymbol{r}}_B|\sin\theta = |\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B|$. We can, therefore, compute d easily by computing the cross product as follows.

$$\begin{aligned} d &= |\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B| = \left|\left(\frac{3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{3^2 + 1^2}}\right) \times (2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\right| \\ &= \left|\frac{4}{\sqrt{10}}\hat{\boldsymbol{k}}\right| = \frac{4}{\sqrt{10}}. \end{aligned}$$

which is the same answer as obtained by Method-1. Once again, you can verify that $\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_A$ also gives the same answer.

$d = 4/\sqrt{10}$

SAMPLE 1.20 **Computing cross product in 3-D:** Compute $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}$, where $\vec{\boldsymbol{a}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{b}} = 3\hat{\boldsymbol{\imath}} + -4\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$.

Solution The calculation of a cross product between two 3-D vectors can be carried out by either using a determinant or the distributive rule. Usually, if the vectors involved have just one or two components, it is easier to use the distributive rule. We show you both methods here and encourage you to learn both. We are given two vectors:

$$\begin{aligned}\vec{\boldsymbol{a}} &= a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}},\\ \vec{\boldsymbol{b}} &= b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}} = 3\hat{\boldsymbol{\imath}} + -4\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}.\end{aligned}$$

- **Calculation using the determinant formula:** In this method, we first write a 3×3 matrix whose first row has the basis vectors as its elements, the second row has the components of the first vector as its elements, and the third row has the components of the second vector as its elements. Thus,

$$\begin{aligned}\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}} &= \begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\ &= \hat{\boldsymbol{\imath}}(a_2b_3 - a_3b_2) + \hat{\boldsymbol{\jmath}}(a_3b_1 - a_1b_3) + \hat{\boldsymbol{k}}(a_1b_2 - b_1a_2)\\ &= \hat{\boldsymbol{\imath}}(1-8) + \hat{\boldsymbol{\jmath}}(-6-1) + \hat{\boldsymbol{k}}(-4-3)\\ &= -7(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}).\end{aligned}$$

- **Calculation using the distributive rule:** In this method, we carry out the cross product by distributing the cross product properly over the three basis vectors. The steps involved are shown below.

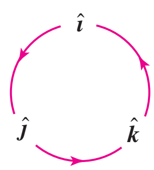

Figure 1.50: The cross product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, e.g. $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$ but $\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$.

$$\begin{aligned}\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}} &= (a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}})\times(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}})\\ &= a_1\hat{\boldsymbol{\imath}}\times(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}) +\\ &\quad\; a_2\hat{\boldsymbol{\jmath}}\times(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}) +\\ &\quad\; a_3\hat{\boldsymbol{k}}\times(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}})\\ &= a_1b_1\overbrace{(\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\imath}})}^{\mathbf{0}} + a_1b_2\overbrace{(\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}})}^{\hat{\boldsymbol{k}}} + a_1b_3\overbrace{(\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{k}})}^{-\hat{\boldsymbol{\jmath}}} +\\ &\quad\; a_2b_1\overbrace{(\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\imath}})}^{-\hat{\boldsymbol{k}}} + a_2b_2\overbrace{(\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\jmath}})}^{\mathbf{0}} + a_2b_3\overbrace{(\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{k}})}^{\hat{\boldsymbol{\imath}}} +\\ &\quad\; a_3b_1\overbrace{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\imath}})}^{\hat{\boldsymbol{\jmath}}} + a_3b_2\overbrace{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\jmath}})}^{-\hat{\boldsymbol{\imath}}} + a_3b_3\overbrace{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}})}^{\mathbf{0}}\\ &= \hat{\boldsymbol{\imath}}(a_2b_3 - a_3b_2) + \hat{\boldsymbol{\jmath}}(a_3b_1 - a_1b_3) + \hat{\boldsymbol{k}}(a_1b_2 - b_1a_2)\\ &= \hat{\boldsymbol{\imath}}(1-8) + \hat{\boldsymbol{\jmath}}(-6-1) + \hat{\boldsymbol{k}}(-4-3)\\ &= -7(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})\end{aligned}$$

which, of course, is the same result as obtained above using the determinant. Making a sketch such as Fig. 1.50 is helpful while calculating cross products this way. The product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, *e.g.*, $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$ but $\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$.

$$\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}} = -7(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$$

SAMPLE 1.21 Finding a vector normal to two given vectors: Find a unit vector perpendicular to the vectors $\vec{\boldsymbol{r}}_A = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{r}}_b = 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$.

Solution The cross product between two vectors gives a vector perpendicular to the plane formed by the two vectors. The sense of direction is determined by the right hand rule.

Let $\vec{\boldsymbol{N}} = N\hat{\boldsymbol{\lambda}}$ be the perpendicular vector.

$$\begin{aligned}\vec{\boldsymbol{N}} &= \vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{r}}_B \\ &= (\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) \times (3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}) \\ &= -7\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}.\end{aligned}$$

This calculation can be done in either of the two ways shown in the previous sample.

Therefore,

$$\begin{aligned}\hat{\boldsymbol{\lambda}} &= \frac{\vec{\boldsymbol{N}}}{N} \\ &= \frac{-7\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}}{\sqrt{7^2 + 2^2 + 3^2}} \\ &= -0.89\hat{\boldsymbol{\imath}} - 0.25\hat{\boldsymbol{\jmath}} + 0.38\hat{\boldsymbol{k}}\end{aligned}$$

$$\hat{\boldsymbol{\lambda}} = -0.89\hat{\boldsymbol{\imath}} - 0.25\hat{\boldsymbol{\jmath}} + 0.38\hat{\boldsymbol{k}}$$

Check:

- $|\hat{\boldsymbol{\lambda}}| = (0.89)^2 + (0.25)^2 + (0.38)^2 \stackrel{\surd}{=} 1.$ (it is a unit vector)
- $\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}_A = 1(-0.89) - 2(-0.25) + 1(0.38) \stackrel{\surd}{=} 0.$ $(\hat{\boldsymbol{\lambda}} \perp \vec{\boldsymbol{r}}_A)$.
- $\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}_B = 3(-0.25) + 2(0.38) \stackrel{\surd}{=} 0.$ $(\hat{\boldsymbol{\lambda}} \perp \vec{\boldsymbol{r}}_B)$.

Comments: If $\hat{\boldsymbol{\lambda}}$ is perpendicular to $\vec{\boldsymbol{r}}_{\mathrm{A}}$ and $\vec{\boldsymbol{r}}_{\mathrm{B}}$, then so is $-\hat{\boldsymbol{\lambda}}$. The perpendicularity does not change by changing the *sense* of direction (from positive to negative) of the vector. In fact, if $\hat{\boldsymbol{\lambda}}$ is perpendicular to a vector $\vec{\boldsymbol{r}}$ then any scalar multiple of $\hat{\boldsymbol{\lambda}}$, *i.e.,* $\alpha\hat{\boldsymbol{\lambda}}$, is also perpendicular to $\vec{\boldsymbol{r}}$. This follows because

$$\alpha\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}} = \alpha(\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}) = \alpha(0) = 0.$$

The case of $-\hat{\boldsymbol{\lambda}}$ is just a particular instance of this rule with $\alpha = -1$.

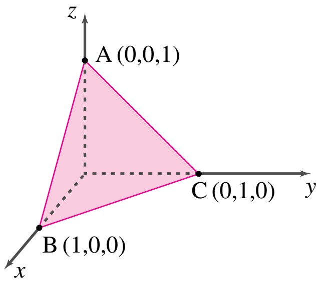

Figure 1.51

SAMPLE 1.22 Finding a vector normal to a plane: Find a unit vector normal to the plane ABC shown in the figure.

Solution A vector normal to the plane ABC would be normal to any vector in that plane. In particular, if we take any two vectors, say $\vec{r}_{AB}$ and $\vec{r}_{AC}$, the normal to the plane would be perpendicular to both $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Since the cross product of two vectors gives a vector perpendicular to both vectors, we can find the desired normal vector by taking the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Thus,

$$\begin{aligned}
\vec{N} &= \vec{r}_{AB} \times \vec{r}_{AC} \\
&= (\hat{\imath} - \hat{k}) \times (\hat{\jmath} - \hat{k}) \\
&= \underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{k}} - \underbrace{\hat{\imath} \times \hat{k}}_{-\hat{\jmath}} - \underbrace{\hat{k} \times \hat{\jmath}}_{-\hat{\imath}} + \underbrace{\hat{k} \times \hat{k}}_{\vec{0}} \\
&= \hat{\imath} + \hat{\jmath} + \hat{k} \\
\Rightarrow \quad \hat{n} &= \frac{\vec{N}}{|\vec{N}|} \\
&= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}).
\end{aligned}$$

$$\boxed{\hat{n} = \tfrac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

Check: Now let us check if $\hat{n}$ is normal to any vector in the plane ABC. It is fairly easy to show that $\hat{n} \cdot \vec{r}_{AB} = \hat{n} \cdot \vec{r}_{AC} = 0$. This is not a surprise because we found $\hat{n}$ from the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Let us check if $\hat{n}$ is normal to $\vec{r}_{BC}$:

$$\begin{aligned}
\hat{n} \cdot \vec{r}_{BC} &= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (-\hat{\imath} + \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-\hat{\imath} \cdot \hat{\imath} + \hat{\jmath} \cdot \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-1 + 1) \stackrel{\checkmark}{=} 0.
\end{aligned}$$

SAMPLE 1.23 **The shortest distance between two lines:** Two lines, AB and CD, in 3-D space are defined by four specified points, A(0,2 m,1 m), B(2 m,1 m,3 m), C(-1 m,0,0), and D(2 m,2 m,2 m) as shown in the figure. Find the shortest distance between the two lines (which are the infinite extensions of the segments shown).

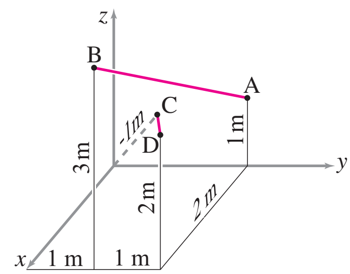

Figure 1.52

Solution The shortest distance between any pair of lines is the length of the line that is perpendicular to both the lines. We can find the shortest distance in three steps:

1. First find a vector that is perpendicular to both the lines. This is easy. Take two vectors $\vec{r}_1$ and $\vec{r}_2$, one along each of the two given lines. Take the cross product of the two vectors and normalize the result to get unit vector $\hat{\boldsymbol{n}}$.
2. Find a vector parallel to $\hat{\boldsymbol{n}}$ that connects the two lines. This is a little tricky. We don't know where to start on any of the two lines. However, we can take any vector from one line to the other and then, take its component along $\hat{\boldsymbol{n}}$.
3. Find the length (magnitude) of the vector just found (in the direction of $\hat{\boldsymbol{n}}$). This is simply the component we find in step (b) devoid of its sign.

Now let us carry out these steps on the given problem.

1. Step-1: Find a unit vector $\hat{\boldsymbol{n}}$ that is perpendicular to both the lines.

$$\begin{aligned}
\vec{r}_{AB} &= 2\,\mathrm{m}\hat{\imath} - 1\,\mathrm{m}\hat{\jmath} + 2\,\mathrm{m}\hat{k} \\
\vec{r}_{CD} &= 3\,\mathrm{m}\hat{\imath} + 2\,\mathrm{m}\hat{\jmath} + 2\,\mathrm{m}\hat{k} \\
\Rightarrow \quad \vec{r}_{AB} \times \vec{r}_{CD} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} \mathrm{m}^2 \\
&= \hat{\imath}(-2-4)\,\mathrm{m}^2 + \hat{\jmath}(6-4)\,\mathrm{m}^2 + \hat{k}(4+3)\,\mathrm{m}^2 \\
&= (-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k})\,\mathrm{m}^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\boldsymbol{n}} &= \frac{\vec{r}_{AB} \times \vec{r}_{CD}}{|\vec{r}_{AB} \times \vec{r}_{CD}|} \\
&= \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}).
\end{aligned}$$

2. Step-2: Find any vector from one line to the other line and find its component along $\hat{\boldsymbol{n}}$.

$$\begin{aligned}
\vec{r}_{AC} &= -1\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\hat{\jmath} - 1\,\mathrm{m}\hat{k} \\
\vec{r}_{AC} \cdot \hat{\boldsymbol{n}} &= -(\hat{\imath} + 2\hat{\jmath} + \hat{k})\,\mathrm{m} \cdot \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}) \\
&= \frac{1}{\sqrt{89}}(6-4-7)\,\mathrm{m} = -\frac{5}{\sqrt{89}}\,\mathrm{m}.
\end{aligned}$$

3. Step-3: Find the required distance d by taking the magnitude of the component along $\hat{\boldsymbol{n}}$.

$$d = \left|\vec{r}_{AC} \cdot \hat{\boldsymbol{n}}\right| = \left|-\frac{5}{\sqrt{89}}\,\mathrm{m}\right| = 0.53\,\mathrm{m}$$

$d = 0.53\,\mathrm{m}$

SAMPLE 1.24 The mixed triple product: Calculate the mixed triple product $\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})$ for $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$, $\vec{\boldsymbol{a}} = 3\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{b}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$.

Solution We compute the given mixed triple product in two ways here:

- **Method-1:** Straight calculation using cross product and dot product.

$$\begin{aligned}
\text{Let} \quad \vec{\boldsymbol{c}} &= \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} \\
&= (3\hat{\boldsymbol{\imath}}) \times (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}) \\
&= 3\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}}_{\vec{\boldsymbol{0}}} + 3\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}}_{\hat{\boldsymbol{k}}} + 9\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}}}_{-\hat{\boldsymbol{\jmath}}} \\
&= -9\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}} \\
\text{So,} \quad \hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) &= \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{c}} \\
&= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \cdot (-9\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}) \\
&= \frac{1}{\sqrt{2}}(-9\underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}}_{0} + 3\underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}}}_{0} - 9\underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}}_{1} + 3\underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}}}_{0}) \\
&= -\frac{9}{\sqrt{2}}.
\end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) = -\tfrac{9}{\sqrt{2}}}$$

- **Method-2:** Using the determinant formula for mixed product.

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) &= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 3 \end{vmatrix} \\
&= \frac{1}{\sqrt{2}}(0-0) + \frac{1}{\sqrt{2}}(0-9) + 0 \\
&= -\frac{9}{\sqrt{2}}.
\end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) = -\tfrac{9}{\sqrt{2}}}$$

