3.1 Basic summary of ODEs for mechanics

This section is *not* an aside. Know it well.

Here are some of the simplest useful ordinary differential equations (ODEs) and their general solutions. Think of u as the distance an object has moved to the right of its 'home' at $u = 0$ in time t. The velocity and acceleration to the right are the derivatives of u.

$$\begin{array}{rr} \text{position} & u \\ \text{velocity} & du/dt = \dot{u} \\ \text{acceleration} & d^2u/dt^2 = \ddot{u}. \end{array} \tag{3.1}$$

If $\dot{u} < 0$ the particle is moving to the left. If $\ddot{u} < 0$ the particle is accelerating to the left.

In all cases below A and B are constants and λ is a positive constant. C_1, C_2, C_3, and C_4 are arbitrary (undetermined) constants in the solutions that get pinned down (determined) by fixing the initial conditions. This section is not about how to derive the solutions, it is about how to *know* them.

Constant position: $\dot{u} = 0 \;\Rightarrow\; u = C_1$

$\dot{u} = 0$ means that the velocity is zero. This equation would arise in dynamics if a particle has no initial velocity and no force is applied to it. The particle doesn't move. Its position must be constant. But it could be anywhere, say at position C_1. Hence the general solution $u = C_1$, as can be found by direct integration.

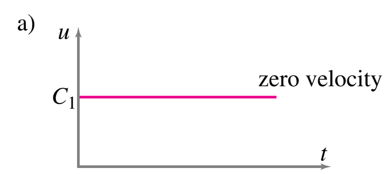

Figure 3.1: Constant position.

Constant velocity: $\dot{u} = A \;\Rightarrow\; u = At + C_1$

$\dot{u} = A$ means the object has constant speed. This equation describes the motion of a particle that starts with speed $v_0 = A$ and because it has no force acting on it continues to move at constant speed. How far does it go in time t? It goes $v_0 t$. Where was it at time $t = 0$? It could have been anywhere then, say C_1. So where is it at time t? It's at its original position plus how far it has moved, $u = v_0 t + C_1$, as can also be found by direct integration.

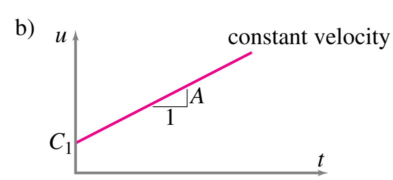

Figure 3.2: Constant velocity.

Zero acceleration: $\ddot{u} = 0 \;\Rightarrow\; u = C_1 + C_2 t$

$\ddot{u} = 0$ means the acceleration is zero. That is, the rate of change of velocity is zero. This constant-velocity motion is the general equation for a particle with no force acting on it. The velocity, if not changing, must be constant. What constant? It could be anything, say C_2. Now we have the same situation as in case (b). So the position as a function of time is anything consistent with an object moving at constant velocity: $u = C_1 + C_2 t$, where the constants C_1 and C_2 depend on the initial position and initial velocity.

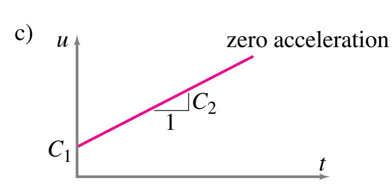

Figure 3.3: Zero acceleration.

If you know that the position at $t = 0$ is u_0 and the velocity at $t = 0$ is v_0, then the position is $u = u_0 + v_0 t$.

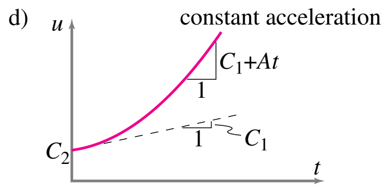

Figure 3.4: Constant acceleration.

Constant acceleration: $\ddot{u} = A \Rightarrow \quad u = At^2/2 + C_1 t + C_2$

This constant acceleration A, constant rate of change of velocity, is the classic (all-too-often studied) case. This situation arises for vertical motion of an object in a constant gravitational field as well as in problems of constant acceleration or deceleration of vehicles. The velocity increases in proportion to the time that passes. The change in velocity in a given time is thus At and the velocity is $v = \dot{u} = v_0 + At$ (given that the velocity was v_0 at $t = 0$). Because the velocity is increasing constantly over time, the average velocity in a trip of length t occurs at $t/2$ and is $v_0 + At/2$. The distance traveled is the average velocity times the time of travel so the distance of travel is $t \cdot (v_0 + At/2) = v_0 t + At^2/2$. The position is the position at $t = 0$, u_0, plus the distance traveled since time zero. So $u = u_0 + v_0 t + At^2/2 = C_2 + C_1 t + At^2/2$. This solution can also be found by direct integration.

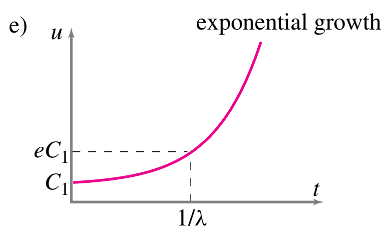

Figure 3.5: Exponential growth.

Exponential growth: $\dot{u} = \lambda u \;\Rightarrow\; u = C_1 e^{\lambda t}$

The displacement u grows in proportion to its present size. This equation describes the initial falling of an inverted pendulum in a thick viscous fluid. The bigger the u, the faster it moves. Such situations are called exponential growth (as in population growth or monetary inflation) for a good mathematical reason. The solution u is an exponential function of time: $u(t) = C_1 e^{\lambda t}$, as can be found by separating variables or guessing.

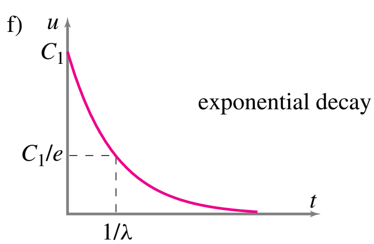

Figure 3.6: Exponential decay.

Exponential decay: $\dot{u} = -\lambda u \;\Rightarrow\; u = C_1 e^{-\lambda t}$

The smaller u is, the more slowly it gets smaller. u gradually tapers towards nothing: u decays exponentially. The solution to the equation is: $u(t) = C_1 e^{-\lambda t}$. This expression is essentially the same equation as in (e) above.

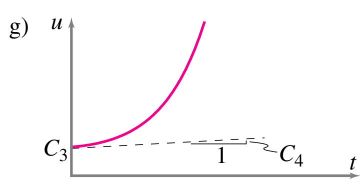

Figure 3.7: Acceleration proportional to position.

Acceleration proportional to position: $\ddot{u} = \lambda^2 u$

$$\Rightarrow u = C_1 e^{\lambda\, t} \; + \; C_2 e^{-\lambda\, t}$$

$$\textbf{or} \Rightarrow u = C_3 \cosh(\lambda\, t) \; + \; C_4 \sinh(\lambda\, t)$$

Note, sinh and cosh are just combinations of exponentials. For $\ddot{u} = \lambda^2 u$, the point accelerates more and more away from the origin in proportion to the distance from the origin. This equation describes the initial falling of a nearly vertical inverted pendulum when there is no friction. Most often, the solution of this equation gives roughly exponential growth. The pendulum accelerates away from being upright. The reason there is also an exponentially decaying solution to this equation is a little more subtle to understand intuitively: if a not quite upright pendulum is given just the right initial velocity it will slowly approach becoming just upright with an exponentially decaying displacement. This decaying solution is not easy to see experimentally because, without the perfect initial condition, the exponentially growing part of the solution eventually dominates and the pendulum accelerates away from being just upright.

Harmonic oscillator: $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$ $\Rightarrow$ $u = C_1 \sin(\lambda\, t) + C_2 \cos(\lambda\, t)$.

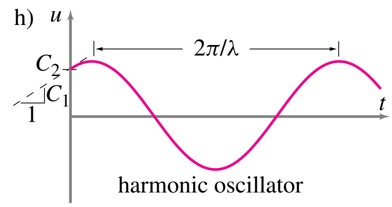

Figure 3.8: Harmonic oscillator.

This equation describes a mass that is restrained by a spring which is relaxed when the mass is at $u = 0$. When u is positive, $\ddot{u}$ is negative. That is, if the particle is on the right side of the origin it accelerates to the left. Similarly, if the particle is on the left it accelerates to the right. In the middle, where $u = 0$, it has no acceleration, so it neither speeds up nor slows down in its motion whether it is moving to the left or the right. So the particle goes back and forth: its position oscillates. A function that correctly describes this oscillation is $u = \sin(\lambda\, t)$, that is, sinusoidal oscillations. The oscillations are faster if λ is bigger. Another solution is $u = \cos(\lambda\, t)$. The general solution is $u = C_1 \sin(\lambda\, t) + C_2 \cos(\lambda\, t)$. A plot of this function reveals a sine wave shape for any value of C_1 or C_2, although the phase depends on the relative values of C_1 and C_2. The equation $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$ is called the 'harmonic oscillator' equation and is important in almost all branches of science. The solution may be found by guessing or other means (which are usually guessing in disguise). In the context of this equation, λ is called the (angular) frequency of oscillation.

More ODEs

Besides the ODEs listed here there are a few others that are often solved by hand rather than with numerical simulation. Most famously there is the damped oscillator equation $A\ddot{u} + B\dot{u} + Cu = 0$ discussed in Box 11.3 on page 563.

With the exception of the damped or forced oscillator, most engineers now-a-days will use numerical integration if they want to solve an ODE not in this appendix.

Learn, memorize, know, whatever

Conversely,

> Anyone competent at dynamics knows all the equations and solutions on these few pages outside, inside, and inside-out.

Whether you have or have not learned these in a calculus course you should learn them now every way possible.

SAMPLE 3.1 A simple first order differential equation: Show that $y(t) = vt$ and $y(t) = C + vt$ are both solutions of the first order differential equation $\dot{y} = v$ where v and C are constants. Which of the two solutions is a *general solution* of the given differential equation and why?

Solution Any function $y(t)$ will be a solution of the given differential equation if it satisfies the differential equation. That is, if we plug the function into the differential equation, the left hand side must equal the right hand side. So, let us check if the given *solutions* satisfy the given differential equation:

(a) $y(t) = vt$: Differentiating $y(t)$ we get, $\dot{y} = \frac{d}{dt}(vt) = v$. Substituting in the given differential equation, we get,

$$\underbrace{v}_{\dot{y}} = v.$$

Thus the left hand side is equal to the right hand side and hence the differential equation is satisfied. Therefore, $y = vt$ is a legitimate solution of the given differential equation.

(b) $y(t) = C + vt$: Differentiating $y(t)$ we get, $\dot{y} = \frac{d}{dt}(C + vt) = 0 + v = v$. Substituting in the given differential equation, $\dot{y} = v$, we get

$$\underbrace{v}_{\dot{y}} = v.$$

Thus again, the left hand side is equal to the right hand side and hence the differential equation is satisfied. Therefore, $y(t) = C + vt$ is a legitimate solution of the given differential equation.

General Solution: The general solution of a differential equation refers to a solution that has enough free parameters (unknown constants) to accommodate any given initial conditions (for *Initial-Value-Problems*) or boundary conditions (for *Boundary-Value-Problems*) and generate a unique solution for the given problem. In dynamics, we have mostly initial-value-problems where initial conditions (*e.g.,* position, displacement, or velocity at $t = 0$) are specified. We can then plug the given initial conditions in the general solution and set $t = 0$ to find the unknown constants. The resulting solution then becomes a unique solution.

Here, $y(t) = C + vt$ is the general solution as it contains one free parameter C that can accommodate any given initial condition $y(t = 0) = y_0$ (say). By substituting this initial condition in the solution ($y(t) = C + vt$), we get,

$$y_0 = C + v\overbrace{t}^{0} \quad \Rightarrow \quad C = y_0.$$

Thus, the solutions to the differential equation, $\dot{y} = v$, $y(0) = y_0$, is

$$y(t) = y_0 + vt.$$

Clearly, the first given solution $y = vt$ is a subset of the general solution, with $C = 0$, and thus it represents the solution of the given differential equation with the initial condition $y(0) = 0$. It is, therefore, a unique solution of the problem $\dot{y} = v$, $y(0) = 0$, and not a general solution of the differential equation $\dot{y} = v$.

General solution: $y(t) = C + vt$

SAMPLE 3.2 Initial conditions and constants of integration: Let $y(t) = C_1 + C_2 t$ be the solution of the differential equation $\ddot{y} = 0$. Find the solutions corresponding to the following initial conditions and sketch $y(t)$ vs t in each case.

1. $y(0) = 1\,\mathrm{m}$ and $\dot{y}(0) = 2\,\mathrm{m/s}$.
2. $y(0) = 0$ and $\dot{y}(0) = 5\,\mathrm{m/s}$.

Solution The general solution has two free constants C_1 and C_2. We can find their values by substituting the given set of initial conditions and thus determine the corresponding solutions. From the general solution, we have,

$$y = C_1 + C_2 t \quad (3.2)$$

$$\dot{y} = C_2. \quad (3.3)$$

1. $y(0) = 1\,\mathrm{m}; \dot{y}(0) = 2\,\mathrm{m/s}$:
   Substituting these values in eqn. (3.2) and eqn. (3.3), we get,

$$\begin{aligned} 1\,\mathrm{m} &= C_1 \\ \text{and} \qquad 2\,\mathrm{m/s} &= C_2. \end{aligned}$$

Thus, the corresponding solution is,

$$y(t) = 1\,\mathrm{m} + (2\,\mathrm{m/s})t.$$

We can now sketch $y(t)$ vs t as shown in fig. 3.9.

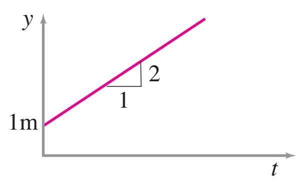

Figure 3.9: Plot of $y(t) = 1\,\mathrm{m} + (2\,\mathrm{m/s})t$ as the solution of $\ddot{y} = 0$; $y(0) = 1\,\mathrm{m}$ and $\dot{y}(0) = 2\,\mathrm{m/s}$.

$$\boxed{y(t) = 1\,\mathrm{m} + (2\,\mathrm{m/s})t}$$

2. $y(0) = 0; \dot{y}(0) = 5\,\mathrm{m/s}$:
   Again substituting the given values in eqn. (3.2) and eqn. (3.3), we get,

$$\begin{aligned} 0 &= C_1 \\ \text{and} \qquad 5\,\mathrm{m/s} &= C_2. \end{aligned}$$

Thus the corresponding solution is

$$y(t) = (5\,\mathrm{m/s})t.$$

This solution is shown in fig. 3.10.

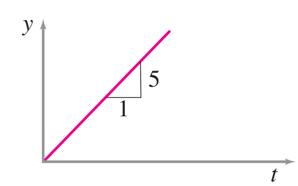

Figure 3.10: Plot of $y(t) = (5\,\mathrm{m/s})t$ as the solution of $\ddot{y} = 0$; $y(0) = 0$ and $\dot{y}(0) = 5\,\mathrm{m/s}$.

$$\boxed{y(t) = (5\,\mathrm{m/s})t}$$

Comments: The differential equation $\ddot{y} = 0$ represents a zero acceleration, or alternatively, a constant speed motion in the y direction. Therefore, the displacement y is a linear function of t as shown in the two graphs. The intercept, $y(t = 0)$, represents the initial displacement and the slope $\frac{dy}{dt}$ represents the constant velocity. In fig. 3.9, the initial displacement is $1\,\mathrm{m}$which is shown as the intercept (at $t = 0$) and the slope is the given velocity $2\,\mathrm{m/s}$. In fig. 3.10, the initial position is $y(0) = 0$ and hence the graph passes through the origin. The slope is the given velocity $\dot{y}(0) = 5\,\mathrm{m/s}$ (since there is no acceleration, the initial velocity remains constant throughout the motion).

SAMPLE 3.3 **Constant speed motion in 1-D again:** A particle moves in the x-direction with constant speed. What is the differential equation that represents this motion? Find the motion of the particle, *i.e.,* find the expression for $x(t)$, given the position of the particle at three instances: $x(0) = x_0$, $x(t_1) = x_1$ and $x(t_2) = x_2$. Do you need $x(t)$ at three instances to determine $x(t)$ completely? What could be alternate information that will help you specify the motion completely?

Solution Let x be the displacement. Then $\dot{x}$ represents speed and $\ddot{x}$ represents acceleration. Since it is motion in 1-D, we will not worry about the vector notation. The given information about *constant speed* can be represented by either of the two differential equations:

1. $\dot{x} = C$ where C is the given constant speed, or
2. $\ddot{x} = 0$, *i.e.,* the acceleration is zero because the speed is constant.

In the first case, the solution of the differential equation is,

$$x(t) = Ct + C_1$$

where C_1 is the constant of integration. We find this solution by direct integration as follows.

$$\dot{x} = C \quad \Rightarrow \quad dx = Cdt \quad \Rightarrow \quad \int dx = \int Cdt \quad \Rightarrow \quad x = Ct + C_1.$$

In the second case, we find the solution by integrating twice:

$$\ddot{x} = 0 \quad \Rightarrow \quad \int d\dot{x} = \int 0\, dt \quad \Rightarrow \quad \dot{x} = C_1$$
$$\dot{x} = C_1 \quad \Rightarrow \quad \int dx = \int C_1 dt \quad \Rightarrow \quad x = C_1 t + C_2.$$

Both solutions above represent the same motion, as they must. The displacement $x(t)$ varies linearly with time t as we would expect in a constant speed motion. This solution is represented by a line in t-x coordinates as shown in fig. 3.11 with C_1 as the slope and C_2 as the intercept. Now, from the given information about values of x at three time instances, it is easy to determine C_1 and C_2. We can do it multiple ways:

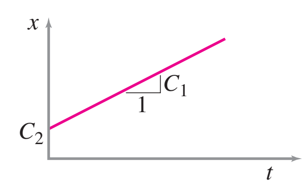

Figure 3.11: The graph of the general solution $x(t) = C_1 t + C_2$

Using the general solution: Substituting the values of x for the three given time instances, $x(t = 0) = x_0, x(t = t_1) = x_1$ and $x(t = t_2) = x_2$, we get

$$\begin{aligned} x_0 &= C_1 \cdot 0 + C_2 \quad \Rightarrow \quad C_2 = x_0 \\ x_1 &= C_1 t_1 + x_0 \\ x_2 &= C_1 t_2 + x_0 \end{aligned}$$

From the last two equations, we get $C_1 = \frac{x_2 - x_1}{t_2 - t_1}$ and thus the solution that satisfies the given conditions is,

$$x(t) = x_0 + \left(\frac{x_2 - x_1}{t_2 - t_1}\right) t.$$

From geometry of the function $x(t)$: We can mark the given values of $x(t)$ for the three specified time instants on the graph of $x(t)$ as shown in fig. 3.12. From the given values, it is obvious that $C_2 = x_0$ and the slope of the line C_1 is $\Delta x / \Delta t = (x_2 - x_1)/(t_2 - t_1)$, and hence the equation of the line becomes,

$$x(t) = x_0 + \left(\frac{x_2 - x_1}{t_2 - t_1}\right) t$$

which is the solution of the differential equation that satisfies all given conditions.

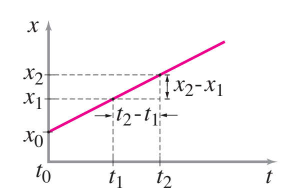

Figure 3.12: Given values of $x(t)$ determine the slope and the intercept

$$\boxed{x(t) = x_0 + \left(\frac{x_2 - x_1}{t_2 - t_1}\right) t}$$

SAMPLE 3.4 Find the solution of the differential equation

$$\ddot{x} + \alpha\dot{x} = 0; \text{ with initial conditions } x(0) = 0 \text{ and } \dot{x}(0) = u_0.$$

Solution The given equation is a second order differential equation that involves derivative terms of x but not x itself. We can solve this equation in two steps. First we solve for $\dot{x}$ and then we solve for x. Let us introduce a new variable y by renaming $\dot{x}$ as y, that is, let $y = \dot{x}$. Then the given equation becomes a first order equation in y:

$$\dot{y} + \alpha y = 0, \text{ or } \dot{y} = -\alpha y.$$

We can solve this equation by integration as follows:

$$\begin{aligned} \frac{dy}{dt} &= -\alpha y \\ \Rightarrow \quad \frac{dy}{y} &= -\alpha dt \\ \Rightarrow \quad \int \frac{1}{y} dy &= -\alpha \int dt \\ \Rightarrow \quad \ln y &= -\alpha t + C \\ \Rightarrow \quad y &= e^{-\alpha t + C} \\ \text{or, } y &= C_1 e^{-\alpha t} \end{aligned}$$

where the last expression is obtained by substituting e^C with another constant C_1 (note that $e^{-\alpha t + C} = e^C e^{-\alpha t}$). So, we have found the solution for $y(t)$. But $y(t) = \dot{x}$, so we now have another differential equation to solve:

$$\dot{x} = C_1 e^{-\alpha t}.$$

We can solve this equation by direct integration as follows:

$$\begin{aligned} \frac{dx}{dt} &= C_1 e^{-\alpha t} \\ \Rightarrow \quad \int dx &= C_1 \int e^{-\alpha t} dt \\ \Rightarrow \quad x &= C_1 \frac{e^{-\alpha t}}{-\alpha} + C_2 \\ \text{or, } x &= C_2 - \frac{C_1}{\alpha} e^{-\alpha t}. \end{aligned}$$

We now have expressions for both $x(t)$ and $\dot{x}(t) \equiv y(t)$. We can substitute the given initial conditions and determine the constants C_1 and C_2:

$$\begin{aligned} \dot{x}(0) &\equiv y(0) = u_0 \\ \Rightarrow \quad u_0 &= C_1 e^{-\alpha 0} = C_1 \end{aligned}$$

and

$$\begin{aligned} x(0) &= 0 \\ \Rightarrow \quad 0 &= C_2 - \frac{u_0}{\alpha} e^{-\alpha 0} \\ \Rightarrow \quad C_2 &= \frac{u_0}{\alpha}. \end{aligned}$$

So, now substituting the values of C_1 and C_2, we get the final solution:

$$x(t) = \frac{u_0}{\alpha}(1 - e^{-\alpha t}).$$

The graph of $x(t)$ is shown in fig. 3.13.

$$\boxed{x(t) = \frac{u_0}{\alpha}(1 - e^{-\alpha t})}$$

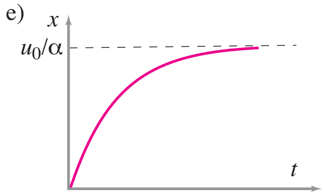

Figure 3.13: Graph of $x(t) = \frac{u_0}{\alpha}(1 - e^{-\alpha t})$. The system reaches the equilibrium value of $x = u_0/\alpha$ asymptotically as $t \to \infty$.

SAMPLE 3.5 **Units of coefficients:** The equation of motion of a mechanical system is given as follows:

$$C_1\ddot{y} + C_2\dot{y} + C_3 y = C_4$$

where y has units of displacement (*e.g.,* meters) and C_1 has units of mass (*e.g.,* kg). What must be the units of C_2, C_3 and C_4 in SI system?

Solution The units of coefficients in any equation, differential or algebraic, have to be such that each term in the summation has the same units. So, in the given differential equation, $C_1\ddot{y}$, $C_2\dot{y}$, $C_3 y$, and C_4 must have the same units. Now, y has the units of displacement (*i.e.,* distance or length) and C_1 has the units of mass. So, in order to find the units of each term, let us first find the units of $\dot{y}$ and $\ddot{y}$:

$$\begin{aligned}
\dot{y} &= \frac{dy}{dt} = \frac{\mathrm{m}}{\mathrm{s}} = \mathrm{m/s} \\
\ddot{y} &= \frac{d\dot{y}}{dt} = \frac{\mathrm{m/s}}{\mathrm{s}} = \mathrm{m/s^2}.
\end{aligned}$$

So, the units of the first term are:

$$C_1\ddot{y} = \mathrm{kg}\cdot\mathrm{m/s^2} = \mathrm{N},$$

that is, unit of force. For the equation to hold good, each term, therefore, must have units of force. So, we can now find the units of all coefficients as follows:

$$\begin{aligned}
C_2\dot{y} &= \mathrm{N} \\
\Rightarrow \quad C_2 &= \frac{\mathrm{N}}{\mathrm{m/s}} = \frac{\mathrm{N\cdot s}}{\mathrm{m}} = \frac{\mathrm{kg}}{\mathrm{s}}, \\
C_3 y &= \mathrm{N} \\
\Rightarrow \quad C_3 &= \frac{\mathrm{N}}{\mathrm{m}}, \text{ and} \\
C_4 &= \mathrm{N}.
\end{aligned}$$

While units of C_4 are clearly that of force, the units of C_2 and C_3 are somewhat unusual and worth thinking about. C_3 has the units of *stiffness*, a quantity that measures the resistance of a body to deformation, and that is why its units are $\mathrm{N/m}$. You may be familiar with this unit as the unit of spring stiffness. The units of C_2 are even more unusual. You can think of these units as $\mathrm{kg/s}$ or $\mathrm{N/(m/s)}$. C_2 has the units of *damping*, a quantity that measures the resistance to motion of a body in a viscous medium in terms of force per unit speed.

C_2: $\mathrm{kg/s}$, C_3: $\mathrm{N/m}$, and C_4: N

SAMPLE 3.6 Initial conditions dictate solutions: Consider the second order differential equation $\ddot{y} + \omega^2 y = 0$. Show that $y(t) = A\sin\omega t$, $y(t) = B\cos\omega t$, and $y(t) = A\sin\omega t + B\cos\omega t$ are all valid solutions of the differential equation. Which of the three solutions will satisfy the following initial conditions?

1. $y(0) = y_0$ and $\dot{y}(0) = 0$.
2. $y(0) = 0$ and $\dot{y}(0) = v_0$.
3. $y(0) = y_0$ and $\dot{y}(0) = v_0$.

Solution In order to show that a given function is a possible solution of a differential equation, we just need to plug the function into the differential equation and see if it satisfies the equation. We can easily do this for the three given functions:

- $y(t) = A\sin\omega t$: We need to find $\ddot{y}$ and plug into the given differential equation.

$$\dot{y} = -A\omega\cos\omega t \quad \Rightarrow \quad \ddot{y} = -A\omega^2\sin\omega t$$

  Therefore,

$$\ddot{y} + \omega^2 y = \underbrace{-A\omega^2\sin\omega t}_{\ddot{y}} + \omega^2\underbrace{A\sin\omega t}_{y} = 0.$$

  Thus $y = A\sin\omega t$ satisfies the given differential equation and hence it is a solution.

- $y(t) = B\cos\omega t$: In this case, $\ddot{y} = -B\omega^2\cos\omega t$, and substituting y and $\ddot{y}$ in the given differential equation, we again find that the equation is satisfied. Hence $y(t) = B\cos\omega t$ is also a solution.
- $y(t) = A\sin\omega t + B\cos\omega t$: Since each of the two terms here satisfies the equation individually, their sum will obviously also satisfy the equation $(0 + 0 = 0)$. Hence this is also a solution of the given differential equation. In fact, this is the most general solution.

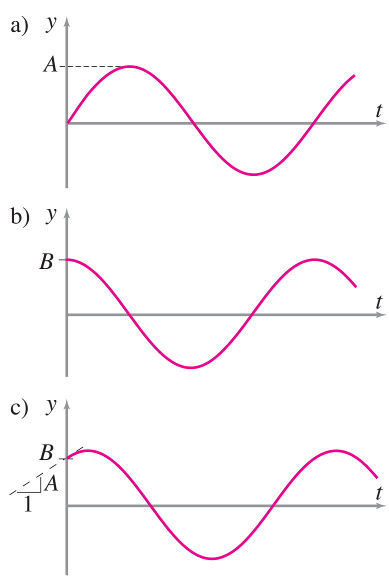

Figure 3.14: Graph of the three solutions: (a) $y(t) = A\sin\omega t$, (b) $y(t) = B\cos\omega t$, and (c) $y(t) = A\sin\omega t + B\cos\omega t$

The three solutions are shown in fig. 3.14. Now let us examine which ones can satisfy which initial conditions.

1. $y(0) = y_0$ and $\dot{y}(0) = 0$: From fig. 3.14(a), it is clear that $y(t) = A\sin\omega t$ cannot support non-zero y at $t = 0$, and hence cannot satisfy $y(0) = y_0$. We also see that this function has a non-zero slope at $t = 0$, *i.e.*, $\dot{y}(0) \neq 0$ and hence it cannot support the second initial condition either. On the other hand, fig. 3.14(b) can easily satisfy both initial conditions if we set $B = y_0$. Similarly, the third solution shown graphically in fig. 3.14(c) can also satisfy the given initial conditions if we set $A = 0$ and $B = y_0$ (then this solution reduces to the second solution).
2. $y(0) = 0$ and $\dot{y}(0) = v_0$: Similar arguments as above lead us to conclude that $y(t) = A\sin\omega t$ satisfies the given initial conditions if we set $A = v_0/\omega$, and $y(t) = A\sin\omega t + B\cos\omega t$ also satisfies the given initial conditions if we set $B = 0$ and $A = v_0/\omega$.
3. $y(0) = y_0$ and $\dot{y}(0) = v_0$: Now we have both initial displacement and initial velocity as non-zero quantities and the graphs in fig. 3.14(a) and (b) make it obvious that neither of these two functions can satisfy these initial conditions. However, the third solution $y(t) = A\sin\omega t + B\cos\omega t$ can certainly satisfy these conditions if we set $A = v_0/\omega$ and $B = y_0$. Thus the solution will be

$$y(t) = y_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t.$$

   Thus the third solution can satisfy any given initial conditions (y_0 can take any value including zero and so can v_0). This is why it is called the *general solution*.

3.1 Basic computation: linear equations and plotting

3.1.1 Write the equations, and general solutions, for as many problems as you can.