2.4 Contact forces: Sliding, rolling, and collisions

The primary mechanical interaction between intermediate-sized objects, say much smaller than the earth and much larger than an atom, is through contact ①. Contact between two objects restricts their possible motions and causes forces on the objects. Objects cause contact forces on each other when and where they touch. While some contact situations are modeled in standard ways that we have discussed, including contact at a pin-joint, ball-and-socket, hinge, weld, and tied string, here, we consider objects that press against each other in ways not necessarily well-idealized with one of those standard mechanical connections.

① In common engineering, the most-often encountered *non*-contact force is gravity, especially the gravity force from the earth on terrestrial objects like cars, buildings, and rain drops.

Generally, people categorize contact as being one of the three major types: friction, rolling, or collision②. Discussion of collisional free-body diagrams is postponed until the Dynamics book.

② In practice it is not always clear how to make the distinctions between sliding and rolling or between sliding and collision. But, at least for a first pass it is a useful conceptual distinction to think of sliding, rolling, and collisions as three different kinds of contact.

To solve problems, we need rules that will help us find the forces during sliding and rolling contact. There are many candidate descriptions for friction and rolling. They vary in their conceptual simplicity, their ease of use in analytical or numerical calculations, and their accuracy and applicability. Here, we will present the simplest useful rules, describe some of their short-comings, and then give some guidance towards more sophisticated contact rules. A rule for finding the forces of interaction in terms of the objects' positions and velocities is called, as mentioned in Chapter 1, a *constitutive law* or *constitutive relation* (see particularly page 28). This section is about constitutive laws for contact.

Contact laws are all rough approximations

We must emphasize at the outset:

> Constitutive laws for contact are rough approximations. For at least some of the quantities of interest, theory and practice typically differ by at least 10%, and up to a factor of two or more.

That is, equations for contact are of a lower class than the momentum equations. For most engineering purposes the momentum balance equations (the basic mechanics laws) are extremely accurate, with error of well less than a part per billion. Newton's law of gravitational attraction is a similarly accurate law. And the laws of Euclidean (non-Riemannian) geometry and calculus (the kinds you studied) are also extremely accurate (See chapter 0.2 page 33). The laws for springs and dashpots are less accurate. But still, accuracies of one part per thousand are possible for measuring spring stiffness, and perhaps parts per hundred for some dashpot constants.

But the laws for the contact interactions of solids are much less accurate. Not only can't you know the coefficient of friction between any two pieces of steel with any certainty, you also can't even trust the *concept* of a coefficient of friction to be very accurate. Because you will see contact-force equations in books, and be asked to learn them, it is easy to forget their inaccuracy. Once we see an equation in print, we are too-easily tempted into believing it is 'true.' So, a common mistake by beginning engineers is to use contact constitutive equations with confidence, as if accurate. Rather, all contact simple equations are only rough approximations, at best.

Friction

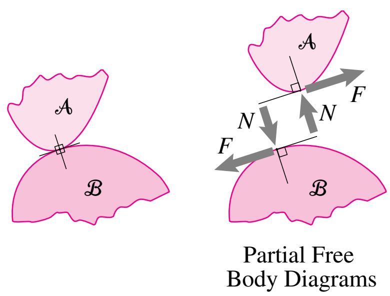


Figure 2.65: Two bodies in contact. The forces between them satisfy the law of action and re-action. It is often convenient to decompose the force of interaction into a part F tangent to the surface of interaction and a part N perpendicular to the surface of interaction.

When two objects are in contact, and one is sliding with respect to the other, we call the force which resists this sliding *friction*. Frictional contact is usually assumed to be either 'lubricated' or 'dry.' When bodies are in lubricated contact they are not in real contact at all, a thin layer of liquid or gas separates them. For example, most of the nominally metal-to-metal contact in a car engine is so lubricated. The contact of the car tires with the road is 'dry' unless the car is 'hydroplaning' on worn-smooth tires on a very wet road. The friction forces in lubricated contact are very small compared to forces of unlubricated contact. There is no quick way to estimate these small lubricated slip forces. The accurate estimation of lubricated friction forces requires use of lubrication theory, a part of fluid mechanics. For many purposes lubricated friction forces are so small that they are negligible. Because lubricated sliding forces are so small and often negligible, and because estimating them is a more advanced topic, we have no more to say about them here.

Dry friction forces are usually not small and thus cannot be sensibly neglected in mechanics problems involving sliding contact. Thus, we need, even if not accurate, friction laws. The simplest model for friction forces (simplest friction law) is called *Coulomb's law of friction* or just *Coulomb friction*. Despite that Coulomb friction is often a high-school physics topic, even this simplest of friction laws, has subtleties.

'Smooth' and 'Rough' are common misnomers for low-friction and high-friction

As a simplification, when we think friction is not important, we sometimes neglect it by setting $\mu = \phi = 0$. In many books this neglect is named "perfectly smooth". Smooth surfaces separated by a little fluid (say water between your feet and the bathroom tile, or oil between pieces of a bearing) do slide easily by each other. And even without a lubricant, sometimes slipping forces can be reduced by smoothing a surface. But, making a surface progressively smoother does not diminish the friction to zero. Rather, extremely smooth surfaces sometimes have anomalously *high* friction (extremely clean flat surfaces can even bond to each other). In general, there is no reliable correlation between smoothness and low friction.

Similarly many books use the phrase “perfectly rough” to mean perfectly high friction ($\mu \to \infty$ and $\phi \to 90^\circ$) and hence that no slip is allowed. This is misleading twice over. First, as just stated, rougher surfaces do not reliably have more friction than smooth ones. Second, even when $\mu \to \infty$ slip can proceed in some situations (see, for example, box 5.7 on page 273).

We use the phrase *frictionless* or *negligible friction* to mean that there is no tangential force component. We use the phrase *no slip* to mean that no tangential motion is allowed and that there is some unknown tangential force. So

> Because they do not correspond to reality, we do not use the words smooth and rough in this book to indicate low and high friction.

Coulomb friction

‘Coulomb’s’ law of friction is also sometimes attributed to Amonton and sometimes to da Vinci. It summarized by the deceptively simple equation:

$$F = \mu N. \tag{2.6}$$

This equation, like many other simple equations, needs some descriptive words to be useful. What is the direction of $\vec{\boldsymbol{F}}$? When does this equation apply, or not?

The direction of the force F on body $\mathcal{A}$ is in the opposite direction of the slip velocity of $\mathcal{A}$ relative to $\mathcal{B}$. By the principle of action and reaction we deduce that the force on body $\mathcal{B}$ is in the opposite direction. This force is also opposite to the relative slip velocity of $\mathcal{B}$ relative to $\mathcal{A}$. That is, F resists relative motion between $\mathcal{A}$ and $\mathcal{B}$.

The friction force F is proportional to the normal force N with the proportionality constant μ. The constant μ is assumed to be independent of the area of contact between bodies $\mathcal{A}$ and $\mathcal{B}$. In the simplest renditions of Coulomb’s law, μ is assumed to be independent of slip distance, slip velocity, time of contact, etc. The friction coefficient is assumed to be a property of the two contacting materials. Thus, one can look up μ for the contact between various pairs of materials. When contacting bodies are not sliding the meaning of the friction equation 2.6 changes. Friction still resists slip, it is the presence of the friction force that prevents slip. But when there is no slip eqn. (2.6) doesn’t describe the force, but rather an upper limit on the possible size of the force. That is, the friction force must be less than or equal to μN in magnitude during non-sliding contact.

$$|F| \leq \mu N \tag{2.7}$$

All of the discussion above can be summarized with the following equations for the friction force

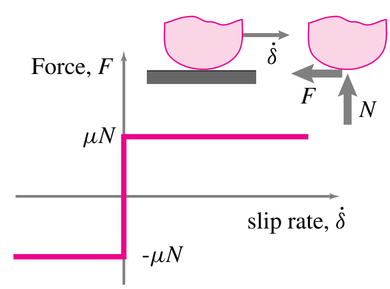


Figure 2.66: **Coulomb friction.** The relation between friction force F and relative slip rate $\dot{\delta}$ is described by the dark line. Since there is a jump from $-\mu N$ to μN in the friction force when the slip rate goes from negative to positive the relation is not a proper mathematical function between F and $\dot{\delta}$. Instead the relation is a curve in the $F, \dot{\delta}$ plane.

$$\overbrace{\vec{F}_{\text{on }\mathcal{A}\text{ from }\mathcal{B}}}^{\text{The friction force}} = \frac{-\mu\,\overbrace{\vec{v}_{\mathcal{A}/\mathcal{B}}}^{\text{Relative slip velocity.}}}{|\vec{v}_{\mathcal{A}/\mathcal{B}}|}N \qquad \text{during slip}$$

$$\underbrace{|\vec{F}_{\text{on }\mathcal{A}\text{ from }\mathcal{B}}|}_{\text{The magnitude of the friction force}} \leq \underbrace{\mu N}_{\text{An upper bound on the friction force}} \qquad \text{during stationary contact}$$

For two-dimensional problems where slip can only be in one direction (or the opposite) this pair of functions describes the dark line in the friction graph of fig. 2.66 in which $\dot{\delta}$ is the speed of relative slip.

The simplest friction law, the one we use in this book, uses a single constant coefficient of friction μ. Almost always $.05 \leq \mu \leq 1.2$ and more commonly $.2 \leq \mu \leq 1$. In this book, we do not distinguish the static coefficient μ_s from the dynamic coefficient μ_d or μ_k. ③ That is $\mu = \mu_s = \mu_k = \mu_d$ for our purposes. We use only this simplest law for a few reasons.

③ How dynamic friction differs from static friction, and how that matters in mechanics, is interesting. If you are wildly curious, here is a random paper on the topic: *Slip instability and state variable friction laws*, A Ruina - JGR, 1983.

- All friction laws used are quite approximate, no matter how complex. Unless the distinction between static and dynamic coefficients of friction is essential to the engineering calculation, using $\mu_s \neq \mu_k$ doesn't add to the calculation's usefulness.
- The concept of a static coefficient of friction that is larger than a dynamic coefficient is, it turns out, not well defined if the contacting objects have more than one point of contact. And, contacting objects most-often do have more than one point of contact (See page 1187.)
- Students learning mechanics are often confused about friction. Because the more complex friction laws are of questionable accuracy and usefulness anyway, it seems time is better spent understanding the simplest friction laws.

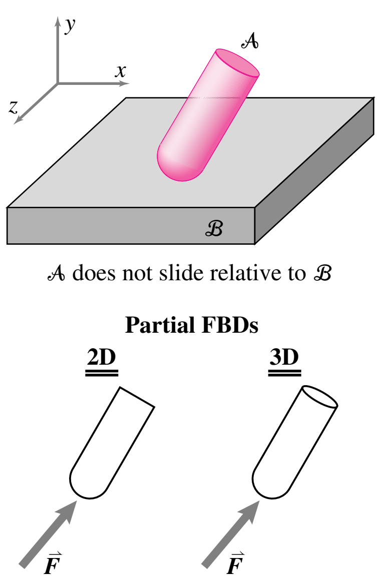


Figure 2.67: Object $\mathcal{A}$ does not slide relative to the plane $\mathcal{B}$.

See page 1189 for more discussion of the pros and cons of the Coulomb-friction approximation.

In summary, the simple model of friction we use is:

Friction resists relative slipping motion. *During slip, the friction force opposes relative motion and has magnitude* $F = \mu N$. When there is no slip the magnitude of the friction force F cannot be determined from the friction law, except that it cannot exceed μN, that is $F \leq \mu N$.

Friction angle

Sometimes people describe the friction coefficient with a *friction angle* ϕ rather than the coefficient of friction (see fig. 2.68). The friction angle is the angle between the net interaction force (normal force plus friction force) and the normal to the sliding surface when slip is occurring. The relation between the friction coefficient μ and the friction angle ϕ is

$$\tan \phi \equiv \mu.$$

The use of ϕ or μ to describe friction is equivalent. Which you use is a matter of taste and convenience. Sometimes analytic formulas in problems come out simpler looking with one or the other of μ and ϕ used to describe the friction.

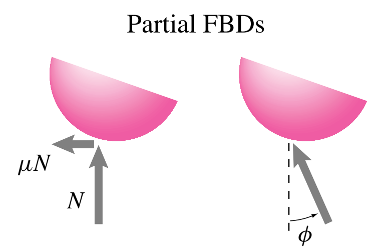


Figure 2.68: Two ways of characterizing friction: the friction coefficient μ and friction angle ϕ.

Rolling contact

An idealization for the non-skidding contact of balls, wheels, and the like is *pure rolling*.

> *Objects $\mathcal{A}$ and $\mathcal{B}$ are in pure rolling contact when their (relatively convex) contacting points have equal velocity. They are not slipping, separating, or interpenetrating.*

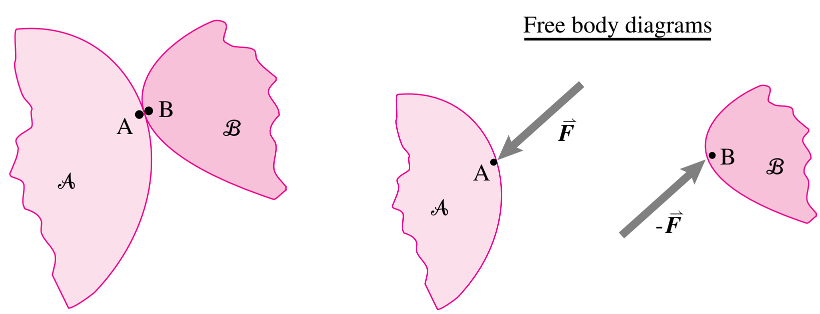


Figure 2.69: Rolling contact: Points of contact on adjoining bodies have the same velocity, $\vec{v}_A = \vec{v}_B$.

Most often, we are interested in cases where the contacting bodies have some non-zero relative angular velocity — a ball sitting *still* on level ground may be technically in rolling contact, but not interestingly so.

The simplest common example is the rolling of a round wheel on a flat surface in two dimensions. See fig. 2.70.

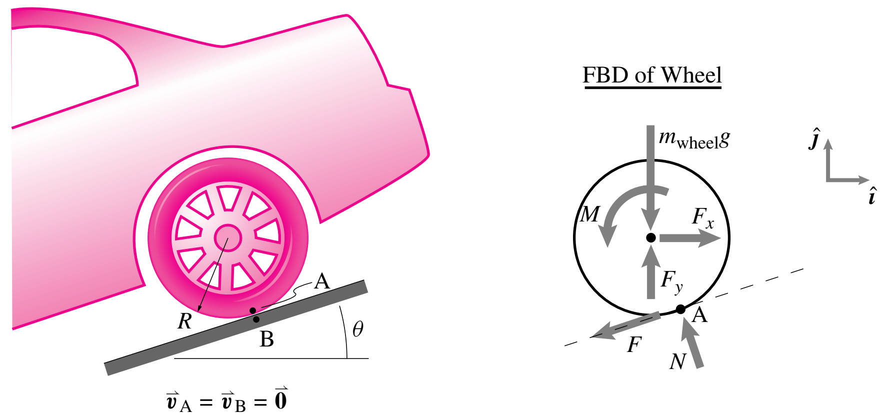


Figure 2.70: Pure rolling of a round wheel on a flat slope in two dimensions.

In practice, there is often confusion about the direction and magnitude of the force F shown in the free-body diagram in fig. 2.70. Here is a recipe:

1.) Draw F as shown in any direction which is tangent to the surface.

2.) Solve the statics or dynamics problem and find the scalar F. (It may turn out to be a negative, which is fine.)

3.) Check that rolling is really possible; that is, that slip would not occur. If the force is greater than the frictional strength, $|F| > \mu N$, the assumption of rolling contact is not appropriate. In this case, you must assume that $F = \mu N$ or $F = -\mu N$ and that slip occurs; then, re-solve the problem.

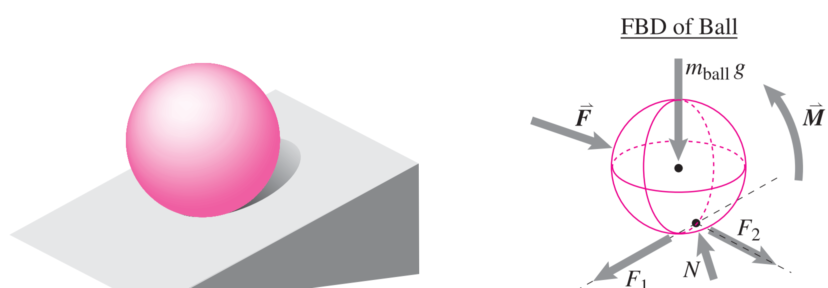


Figure 2.71: **Rolling ball in 3-D.** The force $\vec{F}$ and moment $\vec{M}$ are applied loads from, say, wind, gravity, and any attachments. N is the normal reaction and F_1 and F_2 are the in plane components of the frictional reaction. One must check the no-slip condition, $\mu^2N^2 \geq F_1^2 + F_2^2$.

In three-dimensional rolling contact, we have a free-body diagram that again looks like a free-body diagram for non-slipping frictional contact. Consider, for example, the ball shown in fig. 2.71. For the friction force to be less than the friction coefficient times the normal force, we have the *no slip condition*

$$\sqrt{F_1^2 + F_2^2} \;\leq\; \mu N \quad \text{or} \quad F_1^2 + F_2^2 \;\leq\; \mu^2 N^2$$

Rolling is just a special case of frictional contact. It is the case where bodies contact at a single point (or on a line, as with cylinders) and have relative rotation yet have no relative velocity at their contacting points.

Rolling resistance

Non-ideal rolling contact includes provision for *rolling resistance*. This resistance is simply represented by either moving the location of the point of contact force or by a contact couple. Rolling resistance leads to subtle questions which we skip here④.

④Note that the tangential forces in fig. 2.72 and fig. 2.4 are not rolling resistance.

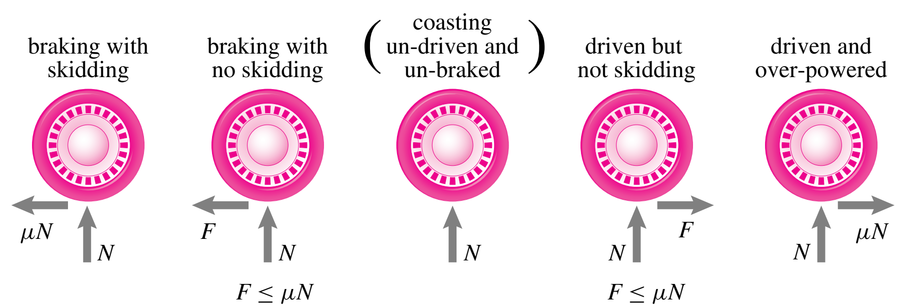


Figure 2.72: Partial free-body diagrams of wheel in a braking or accelerating car that is pointed and moving to the right. The force of the ground on the tire is shown. But, for simplicity, the forces of the axle, gravity, and brakes on the wheel are not shown (that's why it's a partial FBD). An ideal point-contact wheel is assumed. There is no 'rolling resistance' here.

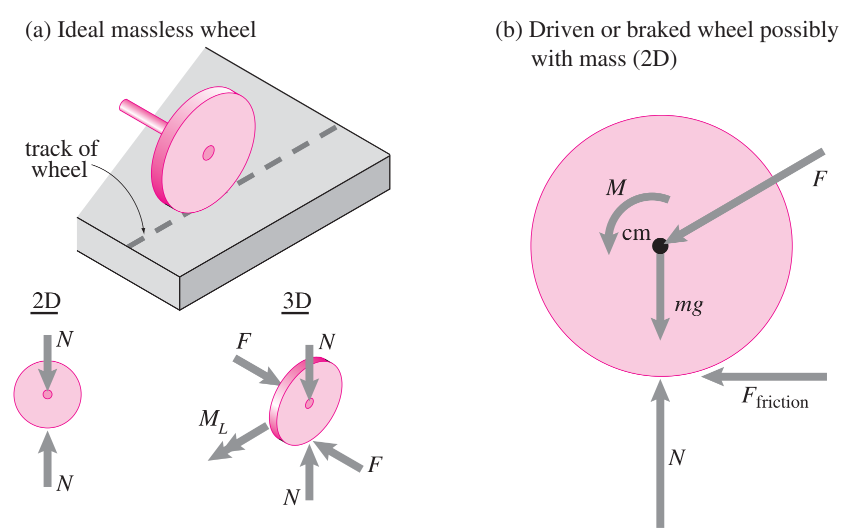


Figure 2.73: An ideal wheel is round, massless, rigid, undriven, and rolls on flat rigid ground with no rolling resistance. Free body diagrams of ideal undriven wheels are shown in two and three dimensions. The force F shown in the three-dimensional picture is perpendicular to the path of the wheel. The lateral moment M_L keeps the wheel from falling over sideways. (b) 2D free-body diagram of a wheel with mass, possibly driven or braked. If the wheel has mass but is not driven or braked the figure is unchanged but for the moment M being zero.

Ideal wheels

An ideal wheel is an approximation of a real wheel. It is a sensible approximation if the mass of the wheel is negligible, bearing friction is negligible, and rolling resistance is negligible. Free-body diagrams of undriven ideal wheels in two and three dimensions are shown in fig. 2.73. This idealization is rationalized in chapter 4 in box 5.7 on page 273. Note that if the wheel is not massless, the 2-D free-body diagram looks more like the one in fig. 2.73b with $F_{\text{friction}} \leq \mu N$.

Extended contact

When things touch each other over an extended region, like the block on the plane of fig. 2.74a, it is not clear what forces to put where on the free-body diagram. On the one hand one imagines reality to be somewhat reflected by millions of small forces as in fig. 2.74b which may or may not be divided into normal (n_i) and frictional (f_i) components. But one generally is not interested in such detail, and even if interested one cannot find it easily.

A simple approach is to replace the detailed force distribution with a single equivalent force, as shown in fig. 2.74c broken into components. The location of this force is not relevant for some problems. ⑤

If one wants to make clear that the contact forces serve to keep the block from rotating, one may replace the contact force distribution with a pair of contacts at the corners as in fig. 2.74d.

Collisional free-body diagrams

As noted earlier, there are special conventions for drawing free-body diagrams of objects that are in the process of colliding. These we treat in the relevant dynamics portions of the book.

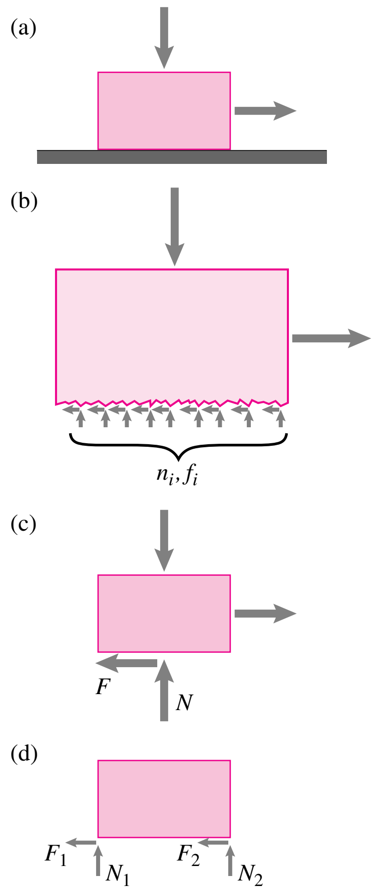


Figure 2.74: The contact forces of a block on a plane can be sensibly modeled in various ways.

⑤In 3D, contact force distributions cannot always be replaced with an equivalent force at an appropriate location (see section 2.1). A couple may be required. Nonetheless, many people often make the approximation that a contact force distribution can be replaced by a force at an appropriate location. For example, this is the "center of pressure" approach used to describe the location of an imagined-equivalent ground force on a robot's foot. This approximation neglects any frictional resistance to twisting about the normal to the contact plane.

SAMPLE 2.19 **Stacked blocks at rest on an inclined plane.** Blocks A and B with masses m and M, respectively, rest on a frictionless inclined surface with the help of force T as shown in Fig. 2.75. There is friction between the two blocks. Draw free-body diagrams of each of the two blocks separately and a free-body diagram of the two blocks as one system.

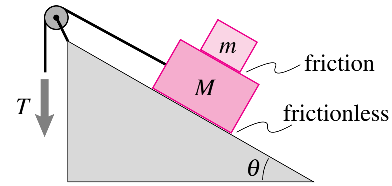


Figure 2.75: Two blocks held in place on an inclined surface

Solution The three free-body diagrams are shown in Fig. 2.76 (a) and (b). Note the action and reaction pairs between the two blocks; the normal force N_A and the friction force F_f between the two bodies A and B. If we consider the two blocks together as a system, then the forces N_A and F_f do not show on the free-body diagram of the system (See Fig. 2.76(b)), because now they are internal to the system.

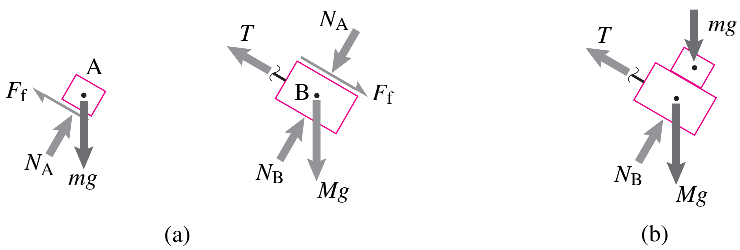


Figure 2.76: Free-body diagrams of (a) block A and block B separately and (b) blocks A and B together.

SAMPLE 2.20 **Two blocks slide down a frictional inclined plane.** Two blocks of identical mass but different material properties are connected by a massless rigid rod. The system slides down an inclined plane which provides different friction to the two blocks. Draw free-body diagrams of the two blocks separately and of the system (two blocks with the rod).

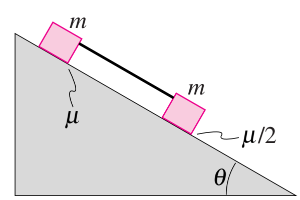


Figure 2.77: Two blocks slide down a frictional inclined plane. The blocks are connected by a light rigid rod.

Solution The Free-body diagrams are shown in Fig. 2.78. Note that the friction forces on the two blocks are different because the coefficients of friction are different for the two blocks. The normal reaction of the plane, however, is the same for each block (why?).

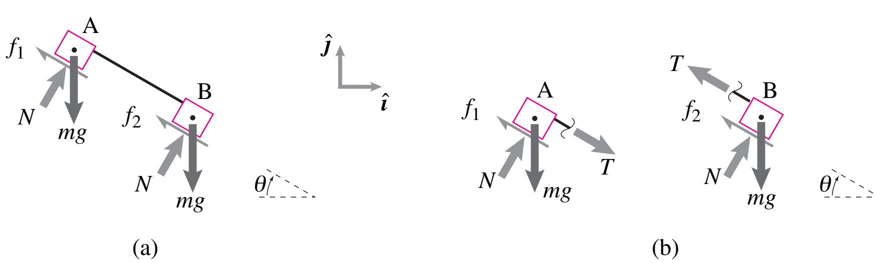


Figure 2.78: Free-body diagrams of (a) the two blocks and the rod as a system and (b) the two blocks separately.

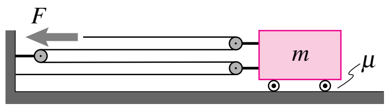


Figure 2.79: A cart with pulleys

SAMPLE 2.21 Massless pulleys. A force F is applied to the pulley arrangement connected to the cart of mass m shown in Fig. 2.79. All the pulleys are massless and frictionless. The wheels of the cart are also massless but there is friction between the wheels and the horizontal surface. Draw a free-body diagram of the cart, its wheels, and the two pulleys attached to the cart, all as one system.

Solution The free-body diagram of the cart system is shown in Fig. 2.80. The force in each part of the string is the same because it is the same string that passes over all the pulleys.

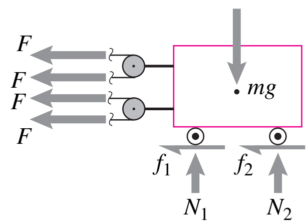


Figure 2.80: Free-body diagram of the cart.

SAMPLE 2.22 A unicyclist in action. A unicyclist weighing 160 lbs exerts a force on the front pedal with a vertical component of 30 lbf at the instant shown in figure 2.81. The rear pedal barely touches the other foot. Assume the wheel and the frame are massless. Draw free-body diagrams of the cyclist and the cycle. Make other reasonable assumptions if required.

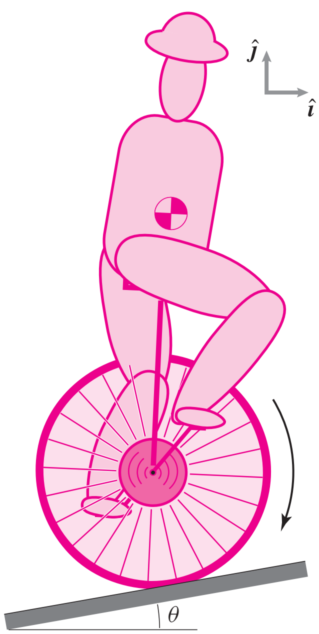


Figure 2.81: The unicyclist

Solution Let us assume, there is friction between the seat and the cyclist and between the pedal and the cyclist's foot. Let's also assume a 2-D analysis. The free-body diagrams of the cyclist and the cycle are shown in Fig. 2.82. We assume no couple interaction at the seat.

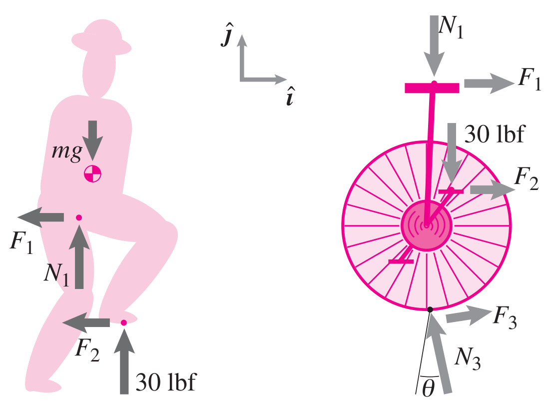
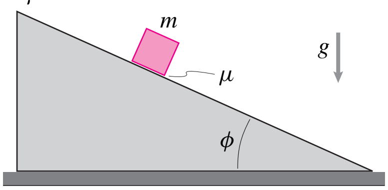


Figure 2.82: Free-body diagram of the cyclist and the cycle.

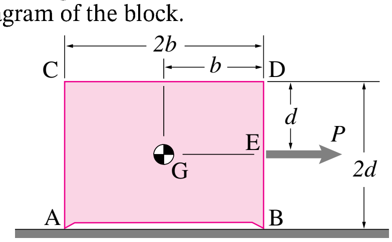
2.4 Contact and friction

2.4.1 A block on an inclined plane. A block of mass m sits on an inclined plane. The coefficient of friction between the block and the plane is μ. Draw the free-body diagram of the block and write the expression for the force(s) applied by the incline on the block in terms of incline angle ϕ.



Problem 2.4.1

2.4.2 A block of mass m sits on a surface supported at points A and B. A horizontal force P acts at point E. There is gravity. The block is sliding to the right. The coefficient of friction between the block and the ground is μ. Draw a free-body diagram of the block.

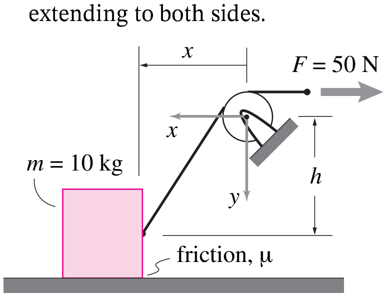


Problem 2.4.2

2.4.3 A block sliding on level ground. A block of mass 10 kg is pulled by an inextensible cable over the pulley.

a) Assuming the block remains on the floor, draw a free-body diagram of the block. (There are various correct answers depending how you model the interaction of the bottom of the block with the ground. See fig. 2.74 on page 193)

b) Draw a free-body diagram of the pulley with a little bit of the cable extending to both sides.

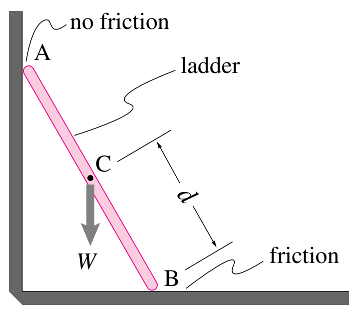


Problem 2.4.3

2.4.4 A ladder standing still. A ladder of mass m rests against a frictionless wall and a floor with more than enough friction to prevent slip. There is gravity.

a) Draw a free-body diagram of the ladder.

b) What is force on the ladder at point B? Find the direction of this force at B assuming the coefficient of friction to be μ and the ladder to be in a state of impending slip. Does the direction of the net force at B depend on the relative positions of A and B (again, assuming impending slip)?

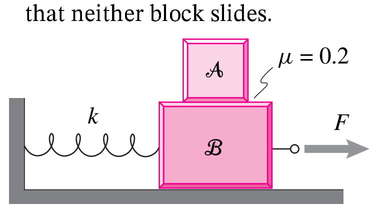


Problem 2.4.4

2.4.5 For the system shown in the figure draw free-body diagrams of each mass separately and of the system of two blocks.

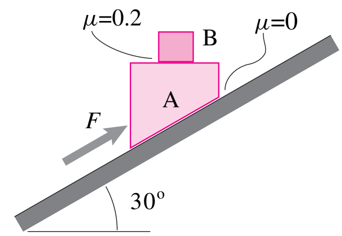
a) Assume there is friction with coefficient μ. At the time of interest block $\mathcal{B}$ is sliding to the right and block $\mathcal{A}$ is sliding to the left relative to $\mathcal{B}$.

b) Assume there is so much friction that neither block slides.



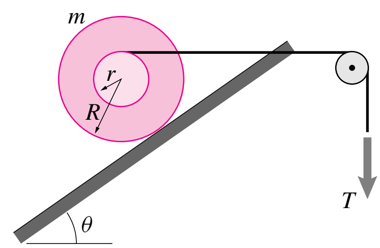
Problem 2.4.5

2.4.6 Two blocks A and B, with mass m_A and m_B respectively, are held on an inclined plane as shown in the figure. Draw a free-body diagram of block A and find the net force acting on the block.



Problem 2.4.6

2.4.7 A spool rolling up an inclined plane. Draw a free-body diagram of the spool shown, including a bit of the rope. Assume the spool does not slip on the ramp.

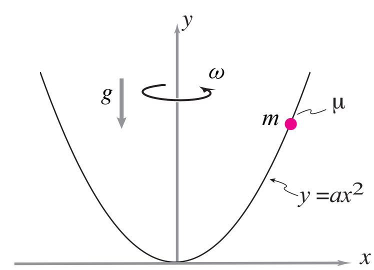


Problem 2.4.7

2.4.8 A bead riding on a rotating wire. A bead of mass m is free to slide on a wire bent in the shape of a parabola. The coefficient of friction between the wire and the bead is μ. The wire rotates about the y-axis. A snapshot during the motion captures the wire in the xy-plane. At this instant, assume the position of the bead to be (x, y).

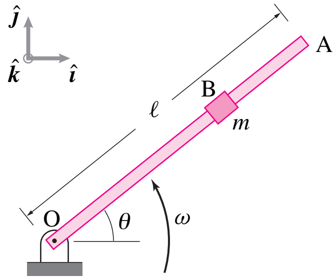
a) Draw a free-body diagram of the bead.

b) Write the vector expression for the force on the bead from the wire. [Hint: you need to find the normal vector to the wire at the position of the bead.]



Problem 2.4.8

2.4.9 A collar sliding up a rotating rod. A uniform rod OA of negligible mass rotates in the plane about point O. A collar B of mass m slides on the rod but faces friction with coefficient $\mu = 0.1$. At the instant shown, draw the free-body diagram of the rod and the collar separately evaluating the force of their interaction as explicitly as possible. Ignore gravity.

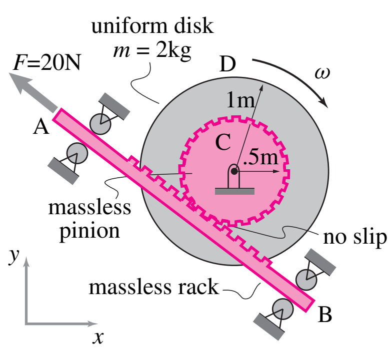


Problem 2.4.9

2.4.10 A rack and a pinion. In the rack and pinion arrangement shown in the figure, the pinion C is welded to the disk D. The rack is pulled up with a force F. As a result the pinion rotates clockwise along with disk D. Assume that the rack and the pinion mesh perfectly together and that there is no slip between them. You can include or ignore gravity.

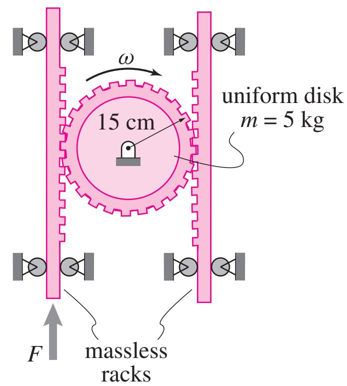
a) Draw a free-body diagram of the rack.

b) Draw a free-body diagram of the pinion along with the disk D.



Problem 2.4.10

2.4.11 Two racks with one pinion A pinion of mass 5 kg and radius 15 cm meshes with two massless racks. The left rack is pushed up with force F making the pinion rotate clockwise. Assuming no slipping between the meshing teeth, draw the free-body diagrams of each rack and the pinion separately. Ignore gravity.



Problem 2.4.11

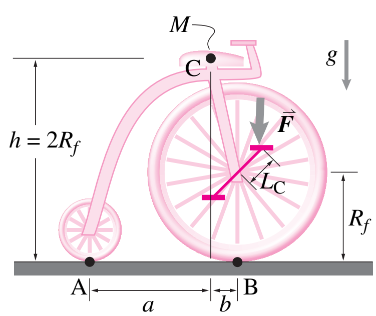
2.4.12 A bicycle with unequal wheels. For the bicycle shown in the figure, assume the mass of the bicycle (and possibly the rider) to be a point mass located at C. A vertical downward force F is applied on the front pedal.

a) Draw a free-body diagram of the front wheel.

b) Draw a free-body diagram of the back wheel.

c) Draw a free-body diagram of the entire bicycle.

d) What assumptions have you made in modeling the interaction force of the ground with the wheels?



Problem 2.4.12