9.1 Fluid pressure

Besides the basic laws of mechanics that you already know, elementary hydrostatics is based on the following two constitutive assumptions (see page 27):

1) The force of water on a surface is perpendicular to the surface; and
2) The density of water, ρ (pronounced 'row') is a constant (doesn't vary with depth or pressure),

Sometimes we use the weight density $\gamma = g\rho$ (pronounced 'gammuh equals gee row'), the weight per unit volume . The first assumption, that all static water forces are perpendicular to surfaces on which they act, can be restated:

Still water cannot carry any shear stress.

For near-still water, this constitutive assumption is abnormally accurate (compared to most constitutive assumptions for materials), approximately as good as the laws of mechanics.

The assumption of constant density is called *incompressibility* because it corresponds to the idea that water does not change its volume (compress) much under pressure. This assumption is reasonable for most purposes. At the bottom of the deepest oceans, for example, the extreme pressure (about 800 atmospheres) causes water to increase its density only about 4% from that of water at the surface[1]

We also assume that the direction and magnitude of the local gravitational 'constant' is, well, constant. This assumption becomes inaccurate when considering, say, the hydrostatics of whole oceans (the direction of the gravity force changes as you go around the world, this helps keep the Australians in place). When considering the upper atmosphere one might consider that the magnitude of the gravity decays with distance from the center of the earth. And it's more subtle than that if you care about fine measurement (see page 236). However, for most purposes taking gravity as constant in magnitude and direction is good enough.

[1] That fluid density does depend on salinity, temperature and pressure is sometimes important in hydrostatics, in particular for determining which water floats on which other water. This density variation is important in the ecology of lakes, the effects of the oceans on climate, and in air for the stability of the atmosphere, and the mechanics of fireplace chimneys.

Surface area A, outward normal $\hat{\boldsymbol{n}}$, pressure p, and force $\vec{F}$

We will generalize the high-school physics fact

$$\text{force} = \text{pressure} \times \text{area}$$

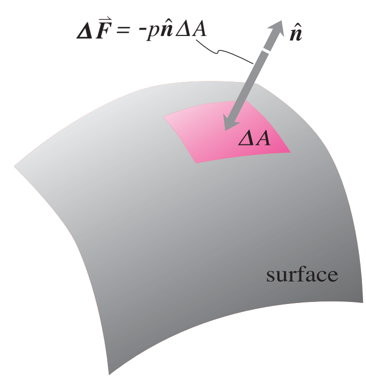


Figure 9.1: A bit of area ΔA on a surface on which pressure p acts. The outward (into the water) normal of the surface is $\hat{\boldsymbol{n}}$ so the force is $\Delta\vec{F} = -p\hat{\boldsymbol{n}}\Delta A$.

to take account that force is a vector, that pressure varies with position, and that not all surfaces are flat. So we need a clear notation and sign convention. The area of a surface is A which we can think of as being the sum of the bits of area ΔA that compose it:

$$A = \sum \Delta A = \int dA.$$

Every bit of surface area has an *outer* normal $\hat{\boldsymbol{n}}$ that points from the surface out into the fluid. The (scalar) force per unit area on the surface is called the pressure p, so that the force on a small bit of surface is

$$\Delta\vec{\boldsymbol{F}} = p\,(-\hat{\boldsymbol{n}})\,(\Delta A)$$

pointing into the surface, assuming positive pressure, and with magnitude proportional to both pressure and area. Thus the total force and moment due to pressure forces on a surface :

$$\begin{aligned} \vec{\boldsymbol{F}} &= \int d\vec{\boldsymbol{F}} &&= -\int_A p\,\hat{\boldsymbol{n}}\ dA \\ \vec{\boldsymbol{M}}_{\mathrm{C}} &= \int_A d\vec{\boldsymbol{M}}_{/\mathrm{C}} &&= -\int_A \vec{\boldsymbol{r}}_{/\mathrm{C}} \times (p\,\hat{\boldsymbol{n}})\ dA \end{aligned} \tag{9.1}$$

Hydrostatics is the evaluation of the (intimidating-at-first-glance) integrals 9.1 and their role in equilibrium equations. In the rest of this section we consider various important special cases.

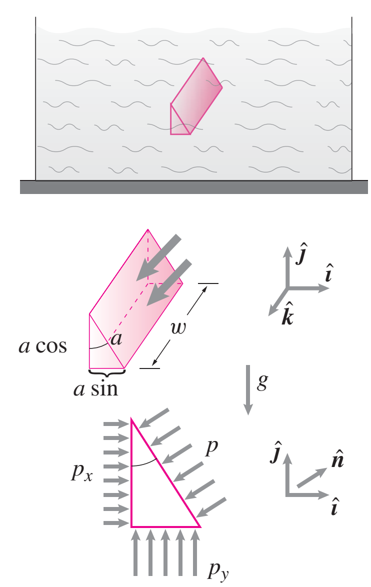


Figure 9.2: A small prism of water in equilibrium is isolated from other water. The free-body diagram does not show the forces in the z direction. Force balance applied to this free-body diagram shows that $p_x = p_y = p$, pressure is the same in all directions.

Water in equilibrium with itself

Before we worry about how water pushes on other things, let's first understand what it means for water to be in static equilibrium. These first important facts about hydrostatics follow from drawing free-body diagrams of various chunks of water and assuming static equilibrium (see box 9.2 on page 5).

1. Pressure is the same in every direction, $p_x = p_y = p$.
2. Pressure doesn't vary with side to side position, $p(x, y, z) = p(y)$.
3. Pressure varies linearly with depth, $p = \rho g h = \gamma h$.

The buoyant force of water on water.

Imagine a chunk of water the shape of a sea monster. Where? In a place under water in a calm swimming pool where there is nothing but water.

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

Now draw a free-body diagram of that water. Because your sea monster is in equilibrium, force balance and moment balance must apply. The only forces are the complicated distribution of pressure forces and the weight of water. The pressure forces must exactly cancel the weight of the water and, to satisfy moment balance, must pass through the center-of-mass of the water monster. So, in static equilibrium:

> The pressure forces acting on a surface enclosing a volume of water are equivalent to the negative weight passing through the center-of-mass of the water.

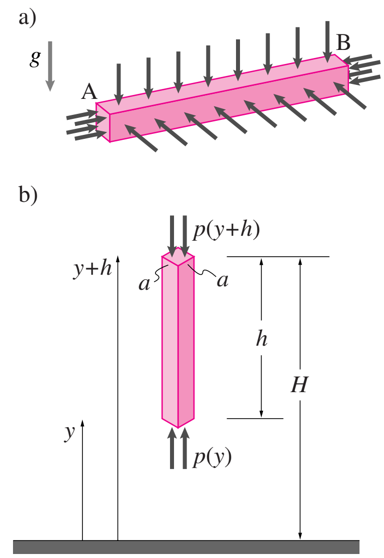


Figure 9.3: Free-body diagrams of aligned boxes of water cut out of a bigger body of water. a) a horizontally aligned box, b) a vertically aligned box. (To avoid visual clutter the self-cancelling horizontal forces are not shown on the second free-body diagram.) Force balance applied to these free-body diagrams shows that $p = p(y)$.

The force of water on submerged and floating objects

The net pressure force and moment on a still object surrounded by still water can be found by a clever argument credited to Archimedes. The pressure at any one point on the outside of the object does not depend on what's inside. The pressure is determined by how far the point of interest is below the surface by eqn. 9.2[2]. So if you can find the resultant force on any object that is the shape of the submerged object, but replacing the submerged object, it tells you what you want to know.

The clever idea is to replace your object with water. In this new system the water is in equilibrium, so the pressure forces exactly balance the weight. We thus obtain Archimedes' Principle:

> The resultant of all pressure forces on a totally submerged object is an upwards force with the same magnitude as the weight of the displaced water. The resultant acts at the centroid of the displaced volume:
> $$\vec{\boldsymbol{F}}_{\text{buoyancy}} = \gamma V \hat{\boldsymbol{\jmath}} \qquad \text{acting at} \qquad \vec{\boldsymbol{r}} = \frac{\int \vec{\boldsymbol{r}}_{/0}\, dV}{V}.$$

The result can also be found by adding the effects of all the pressure forces on the outside surface (see box 9.1 on page 4).

For floating objects, the same argument can be carried out, but since the replaced fluid has to be in equilibrium we cannot replace the whole object with fluid, but only the part which is below the level of the water surface.

[2] If there is no column of water from the point up to the surface it is still true that the pressure is γh, as you can figure out by tracking the pressure changes along on a staircase-like path from the surface to that point.

Displaced fluid

Sometimes people discuss Archimedes' principle in terms of the displaced fluid. A floating object in equilibrium displaces an amount of fluid with the same weight as the object; this is also the amount of volume of the floating object that is below the water level. On the other hand an object

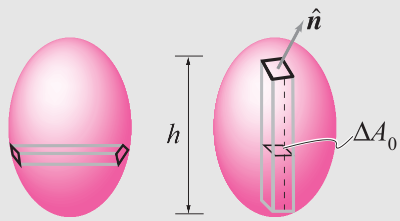
9.1 Adding forces to derive Archimedes' principle

(One can do most hydrostatics calculations, say typical homework problems, without being able to reproduce the derivations here.)

Archimedes' principle follows from adding up all the pressure forces on the outer surfaces of an arbitrarily shaped, submerged solid, say something potato-shaped.

First we find the answer by cutting the potato into french fries. This approach is effectively a derivation of a theorem in vector calculus. After that, for those who have the appropriate math background, we quote the vector calculus directly.

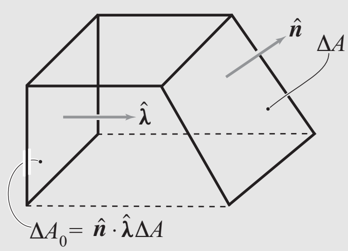
First cut the potato into horizontal french-fries (horizontal prisms) and look at the forces on the end caps (there are no water forces on the sides since those are inside the potato).



The pressure on two ends is the same (because they have the same water depth). The areas on the two ends are probably different because your potato is probably not box shaped. But the area is bigger at one end if the normal to the surface is more oblique compared to the axis of the prism. If the cross sectional area of the prism is ΔA_0 then the area of one of the prism caps is

$$\Delta A = \Delta A_0/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$$

where $\hat{\boldsymbol{\lambda}}$ is along the axis of the prism and $\hat{\boldsymbol{n}}$ is the outer unit normal to the end cap (Note $\Delta A \geq \Delta A_0$ because $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}} \leq 1$).

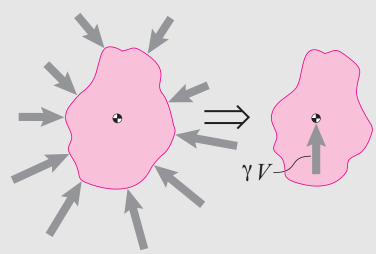


So the net force on the cap is $-p\Delta A_0 \hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$. The component of the force along the prism is $\left[-p\Delta A_0 \hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})\right] \cdot \hat{\boldsymbol{\lambda}}$ which is $-p\Delta A_0$. An identical calculation at the other end of the french fry gives minus the same answer. So the net force of the water pressure along the prism is zero for this and every prism and thus the whole potato. Likewise for prisms with any horizontal orientation. Thus the net sideways force of water on any submerged object is zero.

To find the net vertical force on the potato we cut it into vertical french fries. The net forces on the end caps are calculated just as in the above paragraph but taking account that the pressure on the bottom of the french fry is bigger than at the top. The sum of the forces of the top and bottom caps is an upwards force that is

$$\begin{aligned}
\text{net upwards force on vertical french fry} &= \Delta p \Delta A_0 \\
&= (\gamma h)\Delta A_0 \\
&= \gamma(h\Delta A_0) \\
&= \gamma\,\Delta V_0
\end{aligned}$$

where ΔV_0 is the volume of the french fry. Adding up over all the french fries that make up the potato one gets that the net upwards force is γV. The net result, summarized by the figure below, is that the resultant of the pressure forces on a submerged solid is an upwards force whose magnitude is the weight of the displaced water. The location of the force is the centroid of the displaced volume. (Note that the centroid of the displaced volume is not necessarily at the center of mass of the submerged object.)



. . .

A vector calculus derivation

Here is a derivation of Archimedes' principle, at least the net force part, using multi-variable integral calculus. Only read on if you have taken a math class that covers the divergence theorem. The net pressure force on a submerged object is

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{buoyancy}} &= -\int_A p\,\hat{\boldsymbol{n}}\,dA \\
&= -\int_S p\,\hat{\boldsymbol{n}}\,dS \\
&= -\int_S (H-z)\gamma\,\hat{\boldsymbol{n}}\,dS \\
&= -\int_V \vec{\nabla}\left((H-z)\gamma\right) dV \\
&= -\int_V (-\hat{\boldsymbol{k}})\,\gamma\,dV \\
&= \int_V \gamma\,dV\,\hat{\boldsymbol{k}} \\
&= (\text{weight of displaced water})\,\hat{\boldsymbol{k}}.
\end{aligned}$$

In this derivation we first changed from calling bits of surface area dA to dS because that is a common notation in calculus books. The depth from the surface, of a point with vertical component z from the bottom, is $H - z$. The $\vec{\nabla}$ symbol indicates the gradient and its place in this equation is from the divergence theorem:

$$\int_S (\text{any scalar})\hat{\boldsymbol{n}}\,dS = \int_V \vec{\nabla}(\text{the same scalar})\,dV.$$

The gradient of $(H-z)\gamma$ is $-\hat{\boldsymbol{k}}\gamma$ because H and γ are constants. Note, where we write $\int_S$ some books would write $\iint_S$, and where we write $\int_V$ some books would write $\iiint_V$.

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

that is totally under water, for whatever reason (it is resting on the bottom, or it is being held underwater by a string, etc), displaces as much fluid as the space it occupies. Putting these two ideas together one can remember

9.2 Pressure doesn't depend on direction or horizontal position and increases linearly with depth

We assume that the pressure p does not vary too wildly from point to point, thus if we look at a small enough region we can think of the pressure as constant in that region. If we draw a free-body diagram of a little triangular prism of water the net forces on the prism must add to zero (see fig. 9.2 on page 2). For each surface the magnitude of the force is the pressure times the area of the surface and the direction is minus the outward normal of the surface. We assume, for the time being, that the pressure is different on the differently oriented surfaces. So, for example, because the area of the left surface is $a\cos\theta w$ and the pressure on the surface is p_x, the net force is $a\cos\theta w p_x \hat{\boldsymbol{\imath}}$. Calculating similarly for the other surfaces:

$$\begin{aligned}
\vec{\mathbf{0}} &= \sum \vec{\boldsymbol{F}}_i \\
&= \underbrace{(a\cos\theta) w\, p_x\, \hat{\boldsymbol{\imath}} + (a\sin\theta) w\, p_y\, \hat{\boldsymbol{\jmath}} - a\, w\, p\, \hat{\boldsymbol{n}}}_{\text{pressure terms}} \\
&\qquad - \underbrace{\frac{a^2\cos\theta\sin\theta w}{2}\rho g\, \hat{\boldsymbol{\jmath}}}_{\text{weight}} \\
&= aw\left(\cos\theta\, p_x\, \hat{\boldsymbol{\imath}} + \sin\theta\, p_y\, \hat{\boldsymbol{\jmath}} - p\,(\underbrace{\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}}_{\hat{\boldsymbol{n}}}) - \tfrac{a\cos\theta\sin\theta}{2}\rho g\,\hat{\boldsymbol{\jmath}}\right).
\end{aligned}$$

If a is arbitrarily small, the weight term drops out compared to the pressure terms. Dividing through by aw we get

$$\vec{\mathbf{0}} = \cos\theta p_x \hat{\boldsymbol{\imath}} + \sin\theta\, p_y\, \hat{\boldsymbol{\jmath}} - p\,(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}).$$

Taking the dot product of both sides of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ gives that

$$p = p_x = p_y.$$

Since θ could be anything, force balance for the free-body diagram of a small prism tells us that for a fluid in static equilibrium

> pressure is the same in every direction.

[Other free-body diagrams can be used. That pressure has to be the same in any pair of directions could also be found by drawing a prism with a cross section which is an isosceles triangle. The prism is oriented so that two surfaces of the prism have equal area and have the desired orientations. Force balance along the base of the triangle gives that the pressures on the equal area surfaces are equal. The argument that pressure must not depend on direction in 3D is generally based on equilibrium of a small tetrahedron.]

Pressure doesn't vary with side-to-side position Consider the equilibrium of a horizontally aligned box of water cut out of a bigger body of water (fig. 9.3a on page 3). The forces on the end caps at A and B are the only forces along the box. Therefore they must cancel. Since the areas at the two ends are the same, the pressure must be also. This box could be anywhere and at any length and any horizontal orientation. Thus for a fluid in static equilibrium

> pressure doesn't depend on horizontal position.

If we take the $\hat{\boldsymbol{\jmath}}$ or y direction to be up (fig. 9.3 on page 3), then we have

$$p(x, y, z) = p(y).$$

Pressure increases linearly with depth Consider the vertically aligned box of fig. 9.3b.

$$\begin{aligned}
\left\{\sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \vec{\mathbf{0}}\right\}\cdot\hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad \underbrace{p(y)a^2 - p(y+h)a^2}_{\text{pressure terms}} - \underbrace{\rho g a^2 h}_{\text{weight}} = 0 \\
&\Rightarrow \quad p_{\text{bottom}} - p_{\text{top}} = \rho g h.
\end{aligned}$$

So the pressure increases linearly with depth. If the top of a lake, say, is at atmospheric pressure p_a then we have that

$$p = p_a + \rho g h = p_a + \gamma h = p_a + (H - y)\gamma$$

where h is the distance down from the surface, H is the depth to some reference point underwater and y is the distance up from that reference point (so that $h = H - y$). Neglecting atmospheric pressure at the top surface, we have the useful and easy to remember formula:

$$p = \gamma h. \tag{9.2}$$

Because the pressure at equal depths must be equal and because the pressure at the top surface must be equal to atmospheric pressure, the top surface must be flat and level. Thus waves and the like are a definite sign of static disequilibrium as are any bumps on the water surface even if they don't seem to move (as for a bump in the water where a stream goes steadily over a rock).

© 2026 Andy Ruina & Rudra Pratap

that

> A floating object displaces its weight, a submerged object displaces its volume.

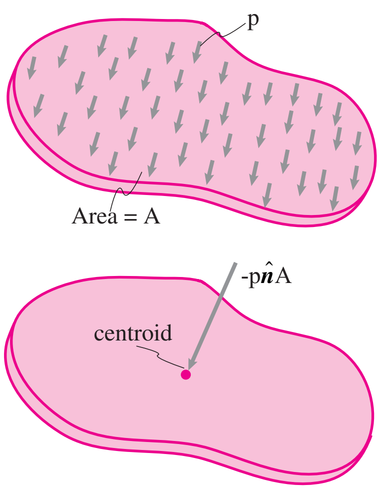


Figure 9.4: The resultant force from a constant pressure p on a flat plate is $\vec{\boldsymbol{F}} = -pA\hat{\boldsymbol{n}}$ acting at the centroid of the plate.

The force of constant pressure on a totally immersed object

When there is no gravity, or gravity is neglected, the pressure in a static fluid is the same everywhere. Exactly the same argument we have just used shows that the resultant of the pressure forces is zero. We could derive this result just by setting $\gamma = 0$ in the formulas above. That is, it is not pressure that causes buoyancy, but the gravity-induced gradient of pressure. If there is no gravity there is no buoyancy.

The force of constant pressure on a flat surface

The net force of constant pressure on a flat surface (not all the way around a submerged volume) is the pressure times the area acting normal to the surface at the centroid of the surface:

$$\begin{aligned} \vec{\boldsymbol{F}}_{\text{net}} &= \textstyle\int_A -p\,\hat{\boldsymbol{n}}\,dA \\ &= \quad -pA\hat{\boldsymbol{n}}. \end{aligned}$$

That this force acts at the centroid can be checked by calculating the moment of the pressure forces relative to the centroid C,

$$\begin{aligned} \vec{\boldsymbol{M}}_{/\text{C,net}} &= \textstyle\int_A \vec{\boldsymbol{r}}_{/\text{C}} \times (-p\,\hat{\boldsymbol{n}}\,dA) \\ &= \underbrace{\left(\int_A \vec{\boldsymbol{r}}_{/\text{C}} dA\right)}_{0} \times (-p\,\hat{\boldsymbol{n}}) \\ &= 0\ , \end{aligned}$$

where the zero is because the position of the center-of-mass, relative to the center-of-mass, is zero.

The force of water on a rectangular plate

Consider a rectangular plate with width (into the page) w and length ℓ. There is water on one side and nothing on the other. Assume the water-side normal to the plate is $\hat{\boldsymbol{n}}$ and that the top edge of the plate is horizontal. Take $\hat{\boldsymbol{\jmath}}$ to be the up direction with y being distance up from the bottom and the total depth of the water is H. Thus the area of the plate is $A = \ell w$. If the bottom and top of the plate are at y_1 and y_2 the net force on the plate

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

can be found as:

$$
\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -\int_A p\hat{\boldsymbol{n}}\,dA \\
&= -\int_A \gamma(H-y)\hat{\boldsymbol{n}}\,dA \\
&= -w\int_0^\ell \gamma(H-y(s))\hat{\boldsymbol{n}}\,ds \\
&= -w\int_0^\ell \gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,s)\right)\hat{\boldsymbol{n}}\,ds \\
&= -w\gamma\left(H\ell-y_1\ell-\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell^2/2\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+(y_2-y_1)/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\left(\gamma(H-y_1)/2+\gamma(H-y_2)/2\right)\hat{\boldsymbol{n}}
\end{aligned}
$$

So

$$
\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -w\ell\,\frac{p_1+p_2}{2}\,\hat{\boldsymbol{n}}. \\
&= -(\text{area})(\text{average pressure})(\text{outwards normal direction}).
\end{aligned}
$$

The net water force is the same as that of the average pressure acting on the whole surface. To find where it acts, it is easiest to think of the pressure distribution as the sum of two different pressure distributions. One is a constant over the plate at the pressure of the top of the plate. The other varies linearly from zero at the top to $\gamma(y_2-y_1)$ at the bottom.

$$
p=\gamma(H-y)=\underbrace{\gamma(H-y_2)}_{\text{Constant pressure, the pressure at the top edge.}}+\underbrace{\gamma(y_2-y)}_{\text{Varies linearly from 0 at the top to }\gamma(y_2-y_1)\text{ at the bottom.}}
$$

The first corresponds to a force of $w\ell\gamma(H-y_2)$ acting at the middle of the plate. The second corresponds to a force of $w\ell\gamma\frac{y_2-y_1}{2}$ acting a third of the way up from the bottom of the plate.

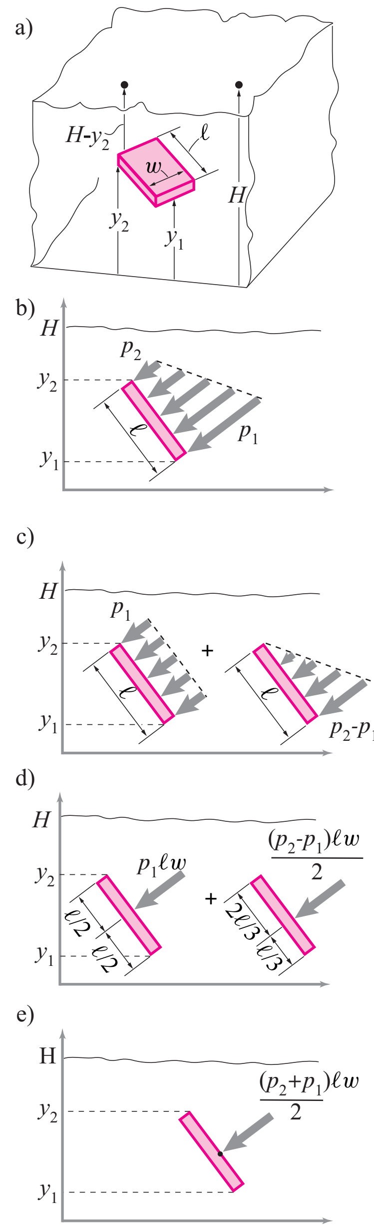


Figure 9.5: The resultant force from a constant depth-increasing pressure on a rectangular plate.

SAMPLE 9.1 A uniform solid cylinder of mass $m = 12\,\text{kg}$, diameter $d = 0.1$ m and height $h = 2$ m floats in water (density $\rho = 1000\,\text{kg/ m}^3$).

1. Assuming the cylinder floats vertically, find the submerged height of the cylinder.
2. If the cylinder floats longitudinally (its longitudinal axis parallel to the water surface), what will be the submerged section of the cylinder?

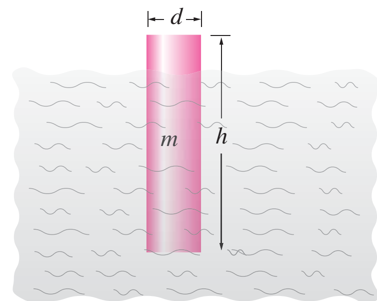


Figure 9.6

Solution

1. **Cylinder floating vertically:** Let h_s be the submerged height of the cylinder and $r = d/2$ be its radius. Then the force of buoyancy F_B is equal to the weight of water replaced by the submerged volume of the cylinder. Thus,

$$\vec{F}_B = \overbrace{\pi r^2 h_s}^{\text{volume}} \gamma\, \hat{\boldsymbol{j}}.$$

From the force balance on the cylinder (see the free-body diagram in fig. 9.7),

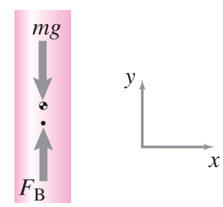


Figure 9.7

$$\begin{aligned} \vec{F}_B - mg\hat{\boldsymbol{j}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad (\pi r^2 h_s \gamma - mg)\hat{\boldsymbol{j}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad h_s &= \frac{mg}{\pi r^2 \gamma} = \frac{m}{\pi r^2 \rho} \\ &= \frac{12\,\text{kg}}{\pi \cdot (0.05\,\text{m})^2 \cdot 1000\,\text{kg/ m}^3} = 1.53\,\text{m}. \end{aligned}$$

$$\boxed{h_s = 1.53\,\text{m}}$$

2. **Cylinder floating horizontally:** No matter how the cylinder floats, the force of buoyancy has to equal the weight of the cylinder. This force is equal to the weight of the displaced water. Thus, the volume of displaced water has to be the same no matter what the orientation of the cylinder is with respect to the water surface. Therefore, the submerged volume of the cylinder while floating longitudinally must equal the volume submerged while floating vertically. That is (see fig. 9.8),

$$\text{area of BCD} \cdot h = \pi r^2 h_s \quad \Rightarrow \quad \text{area of BCD} = \pi r^2 (h_s/h) = 0.006\,\text{m}^2.$$

From the mass and volume we can calculate that the density of the cylinder is .764 that of water. Now we can figure out what d_s should be so that the submerged cross-sectional area is 76% of the total cross-sectional area. This is an exercise in geometry. Since, area of BCD $= \pi r^2 -$ area of ABD, area of ABD $= \pi r^2 -$ area of BCD $= 0.018\,\text{m}^2$. But the area of ABD is the area of the circular sector OBAD ($r^2\theta$) minus the area of triangle OBD ($\frac{1}{2} \cdot r\cos\theta \cdot 2r\sin\theta$). Thus,

$$\begin{aligned} \text{area of ABD} \equiv r^2\theta - \frac{1}{2} r^2 \sin 2\theta &= 0.018\,\text{m}^2 \\ \Rightarrow \quad \theta - \frac{1}{2}\sin 2\theta - 0.738 &= 0. \end{aligned}$$

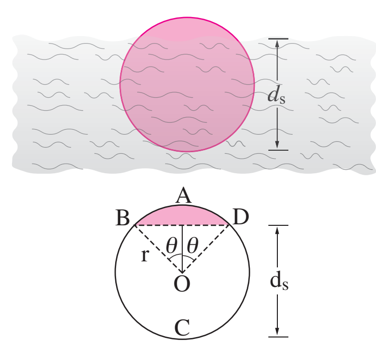


Figure 9.8

We need to solve this nonlinear equation. Using trial and error or root finding on a computer or a graphical method, we find $\theta = 1.126$ rad $= 64.5°$. Using this value, we get, $d_s = r + r\cos\theta = 0.07$ m.

$$\boxed{d_s = 0.07\,\text{m}}$$

© 2026 Andy Ruina & Rudra Pratap

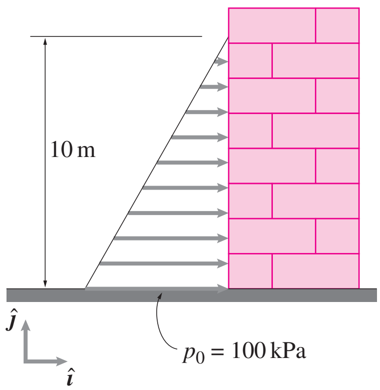
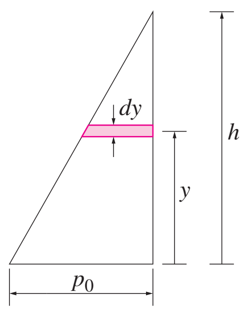


Figure 9.9



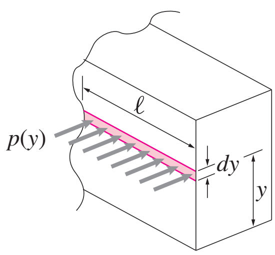


Figure 9.10

SAMPLE 9.2 The force due to varying hydrostatic pressure: The hydrostatic pressure distribution on the face of a wall submerged in water up to a height $h = 10\,\text{m}$ is shown in the figure. Find the net force on the wall from water. Take the length of the wall (into the page) to be 1 m.

Solution Since the pressure varies across the height of the submerged part of the wall, let us take an infinitesimal strip of height dy along the full length ℓ of the wall as shown in fig. 9.10. Since the height of the strip is infinitesimal, we can treat the water pressure on this strip to be essentially constant and equal to $p_0\frac{y}{h}$. Then the force on the strip (of area $\ell\,dy$) due to the constant water pressure $p(y) = p_0 y/h$ is

$$d\vec{F} = (p(y)\cdot\overbrace{\ell\,dy}^{\text{area}})\,\hat{\imath} = p_0\frac{y}{h}\,\ell\,dy\,\hat{\imath}.$$

The net force due to the pressure distribution on the whole wall can now be found by integrating $d\vec{F}$ along the height of the wall.

$$\begin{aligned}\vec{F} &= \int d\vec{F} = \int_0^h p_0\frac{y}{h}\,\ell\,dy\,\hat{\imath}\\ &= \left(p_0\frac{\ell}{h}\int_0^h y\,dy\right)\hat{\imath} = p_0\frac{\ell}{h}\,\frac{h^2}{2}\hat{\imath}\\ &= \frac{1}{2}p_0 h\ell\,\hat{\imath}\\ &= \frac{1}{2}\cdot(100\frac{\text{kN}}{\text{m}^2})\cdot(10\,\text{m})\cdot(1\,\text{m})\hat{\imath}\\ &= (500\,\text{kN})\,\hat{\imath}.\end{aligned}$$

$$\boxed{\vec{F} = 500\,\text{kN}\,\hat{\imath}}$$

Alternatively, the net force can be computed by calculating the area of the pressure triangle and multiplying by the unit length ($\ell = 1\,\text{m}$), *i.e.*,

$$\begin{aligned}\vec{F} &= \overbrace{\frac{1}{2}\cdot h\cdot p_0}^{\text{triangle area}}\ \ell\,\hat{\imath}\\ &= \left(\frac{1}{2}\cdot 10\,\text{m}\cdot 100\frac{\text{kN}}{\text{m}^2}\cdot 1\,\text{m}\right)\hat{\imath}\\ &= 500\,\text{kN}\,\hat{\imath}.\end{aligned}$$

SAMPLE 9.3 **Forces on a submerged sluice gate:** A rectangular plate is used as a gate in a tank to prevent water from draining out. The plate is hinged at A and rests on a frictionless surface at B. Assume the width of the plate to be 1 m. The height of the water surface above point A is h. Ignoring the weight of the plate, find the forces on the hinge at A as a function of h. In particular, find the vertical pull on the hinge for $h = 0$ and $h = 2\,\mathrm{m}$.

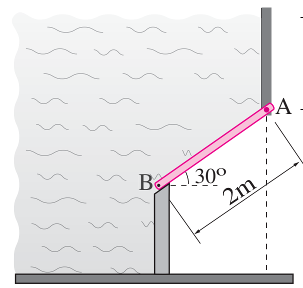


Figure 9.11

Solution Let $\gamma = \rho g$ be the weight density (weight per unit volume) of water. Then the pressure due to water at point A is $p_A = \gamma h$ and at point B is $p_B = \gamma(h + \ell \sin\theta)$. The pressure acts perpendicular to the plate and varies linearly from p_A at A to p_B at B. The free-body diagram of the plate is shown in fig. 9.12. Let $\hat{\boldsymbol{\lambda}}$ be a unit vector along BA and $\hat{\boldsymbol{n}}$ be a unit vector normal to BA. For computing the reaction forces on the plate at points A and B, we first replace the distributed pressure on the plate by two equivalent concentrated forces F_1 and F_2 by dividing the pressure distribution into a rectangular and a triangular region and finding their resultants.

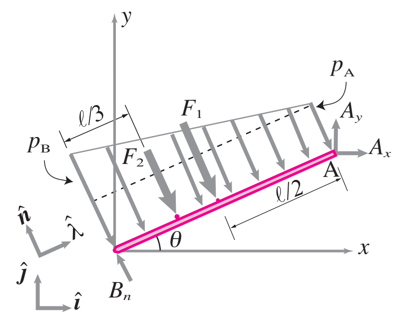


Figure 9.12: The free-body diagram of the gate (plate AB) with distributed water pressure.

$$F_1 = p_A \ell = \gamma h \ell, \quad F_2 = (p_B - p_A)\frac{\ell}{2} = \frac{1}{2}\gamma \ell^2 \sin\theta.$$

Now, we carry out moment balance about point A, $\sum \vec{\boldsymbol{M}}_{\mathrm{A}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{B/A}} \times \vec{\boldsymbol{B}} + \vec{\boldsymbol{r}}_{\mathrm{D/A}} \times \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{r}}_{\mathrm{C/A}} \times \vec{\boldsymbol{F}}_1 &= \vec{\boldsymbol{0}} \\
-\ell\hat{\boldsymbol{\lambda}} \times B_n\hat{\boldsymbol{n}} - \frac{2\ell}{3}\hat{\boldsymbol{\lambda}} \times (-F_1\hat{\boldsymbol{n}}) - \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times (-F_2\hat{\boldsymbol{n}}) &= \vec{\boldsymbol{0}} \\
-B_n\ell\hat{\boldsymbol{k}} + F_1\frac{2\ell}{3}\hat{\boldsymbol{k}} + F_2\frac{\ell}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad B_n = \frac{2F_1}{3} + \frac{F_2}{2} = \gamma\ell\left(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta\right)
\end{aligned}$$

and, from force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned}
\vec{\boldsymbol{A}} &= -B_n\hat{\boldsymbol{n}} + F_1\hat{\boldsymbol{n}} + F_2\hat{\boldsymbol{n}} \\
&= \left(-\gamma\ell\left(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta\right) + \gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} \\
&= \left(\frac{1}{3}\gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} = \gamma\ell\left(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta\right)\hat{\boldsymbol{n}}.
\end{aligned}$$

The force $\vec{\boldsymbol{A}}$ computed above is the force exerted by the hinge at A on the plate. Therefore, the force on the hinge, exerted by the plate, is $-\vec{\boldsymbol{A}}$ as shown in fig. 9.13. From the expression for this force, we see that it varies linearly with h.

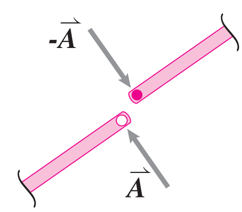


Figure 9.13: The free-body diagram of the pin at A.

Let the vertical pull on the hinge be A_{hinge_y}. Then

$$A_{\mathrm{hinge}_y} = -\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{j}} = -\gamma\ell\left(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta\right)\overbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{j}}}^{\cos\theta} = \frac{1}{4}\gamma\ell\sin 2\theta + \left(\frac{1}{3}\gamma\ell\cos\theta\right)h.$$

Now, substituting $\gamma = 9.81\,\mathrm{kN/m^3}$, $\ell = 2\,\mathrm{m}$, $\theta = 30°$, the two specified values of h, and multiplying the result (which is force per unit length) with the width of the plate (1 m) we get,

$$A_{\mathrm{hinge}_y}\big|_{h=0} = 4.25\,\mathrm{kN}, \quad A_{\mathrm{hinge}_y}\big|_{h=2\,\mathrm{m}} = 15.58\,\mathrm{kN}$$

© 2026 Andy Ruina & Rudra Pratap

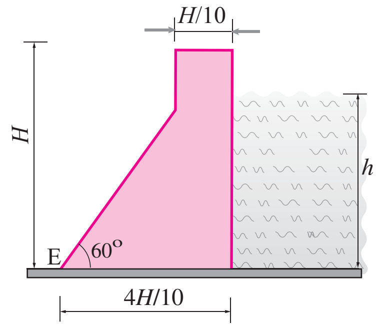


Figure 9.14

SAMPLE 9.4 Tipping of a dam: The cross-section of a concrete dam is shown in the figure. Take the weight-density $\gamma(=\rho g)$ of water to be $10\,\mathrm{kN/m^3}$ and that of concrete to be $25\,\mathrm{kN/m^3}$. For the given design of the cross-section, find the ratio h/H that is safe enough for the dam to not tip over (about the downstream edge E).

Solution Let us imagine the critical situation when the dam is just about to tip over about edge E. In such a situation, the dam bottom would almost lose contact with the ground except along edge E. In that case, there is no force along the bottom of the dam from the ground except at E.① With this assumption, the free-body diagram of the dam is shown in fig. 9.15.

① This assumption is valid only if water does not leak through the lower right of the damn through to the bottom of the dam. If it did, there would be a force on the bottom due to the water pressure. See the following sample where we include the water pressure at the bottom in the analysis.

To compute all the forces acting on the dam, we assume the width w (into the paper) to be unity (*i.e.,* $w = 1$ m). Let γ_w and γ_c denote the weight-densities of water and concrete, respectively. Then the resultant force from the water pressure is

$$F = \frac{1}{2}\gamma_w h \cdot h \cdot w = \frac{1}{2}\gamma_w h^2 w.$$

This is the horizontal force (in the $-\hat{\boldsymbol{\imath}}$ direction) that acts through the centroid of triangle ABC.

To compute the weight of the dam, we divide the cross-section into two sections — the rectangular section CDGH and the triangular section DEF. We compute the weight of these sections separately by computing their respective volumes (we use the shape constant $\alpha = 1/10$):

$$\begin{aligned} W_1 &= \underbrace{\alpha H^2 \cdot w}_{\text{volume}} \cdot \gamma_c = \gamma_c \alpha H^2 w \\ W_2 &= \underbrace{\frac{1}{2}\cdot 3\alpha H \cdot 3\alpha H \tan\theta \cdot w}_{\text{volume}} \cdot \gamma_c = \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta. \end{aligned}$$

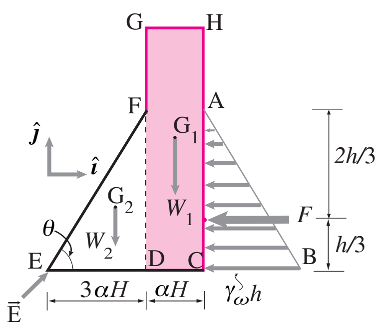


Figure 9.15

Now we apply moment balance about point E, $\sum \vec{\boldsymbol{M}}_{\mathrm{E}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned} \vec{\boldsymbol{r}}_{G_1} \times \vec{\boldsymbol{W}}_1 + \vec{\boldsymbol{r}}_{G_2} \times \vec{\boldsymbol{W}}_2 + \vec{\boldsymbol{r}}_{G_3} \times \vec{\boldsymbol{F}} &= \vec{\boldsymbol{0}} \\ -(3\alpha H + \frac{1}{2}\alpha H)W_1\hat{\boldsymbol{k}} - \frac{2}{3}(3\alpha H)W_2\hat{\boldsymbol{k}} + \frac{h}{3}F\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}}. \end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{k}}$, we get

$$\begin{aligned} \frac{h}{3}F &= (3\alpha H + \frac{1}{2}\alpha H)\cdot\gamma_c\alpha H^2 w + \frac{2}{3}(3\alpha H)\cdot\frac{9}{2}\gamma_c\alpha^2 H^2 w\tan\theta \\ \frac{1}{2}\gamma_w\frac{h^3}{3} &= 9\gamma_c\alpha^3 H^3\tan\theta + \frac{7}{2}\gamma_c\alpha^2 H^3 \\ \Rightarrow \quad \left(\frac{h}{H}\right)^3 &= \frac{\gamma_c}{\gamma_w}(54\alpha^3\tan\theta + 21\alpha^2) \\ &= 2.5(54\cdot 0.1^3\cdot\sqrt{3} + 21\cdot 0.1^2) = 0.7588 \\ \Rightarrow \quad \frac{h}{H} &= 0.91. \end{aligned}$$

Thus, for the dam to not tip over, $h \le 0.91H$ or 91% of H.

$\frac{h}{H} \le 0.91$

© 2026 Andy Ruina & Rudra Pratap

SAMPLE 9.5 **Dam design:** You are to design a dam of rectangular cross-section ($b \times H$), ensuring that the dam does not tip over even when the water level h reaches the top of the dam ($h = H$). Take the specific weight of concrete to be 3. Consider the following two scenarios for your design.

1. The downstream bottom edge of the dam is plugged so that there is no leakage underneath.
2. The lower right edge is not plugged and water leaks under the dam bottom has full pressure across the bottom.

Solution Let γ_c and γ_w denote the weight densities of concrete and water, respectively. We are given that $\gamma_c/\gamma_w = 3$. Also, let $b/H = \alpha$ so that $b = \alpha H$. We consider both scenarios to find a safe cross-sectional shape and size for the dam. In the calculations below, we consider unit length (into the paper) of the dam.

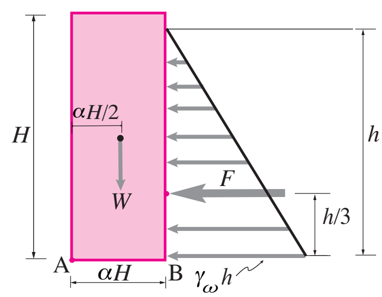


Figure 9.16

1. *No water pressure on the bottom:* When there is no water pressure on the bottom of the dam, then the water pressure acts only on the downstream side of the dam. The free-body diagram of the dam, considering critical tipping (just about to tip), is shown in fig. 9.16 in which F is the resultant force of the triangular water pressure distribution. The known forces acting on the dam are $W = \gamma_c \alpha H^2$, and $F = (1/2)\gamma_w h^2$. The moment balance about point A gives

$$\begin{aligned} F \cdot \frac{h}{3} &= W \cdot \frac{\alpha H}{2} \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= \gamma_c \frac{\alpha^2 H^3}{2} \\ \Rightarrow \quad \alpha^2 &= (1/3)(\gamma_w/\gamma_c)(h/H)^3. \end{aligned}$$

   Considering the case of critical water level up to the height of the dam, *i.e.*, $h/H = 1$, and substituting $\gamma_c/\gamma_w = 3$, we get

$$\alpha^2 = 1/9 \quad \Rightarrow \quad \alpha = 1/3 = 0.333.$$

   Thus the width of the cross-section needs to be at least one-third of the height. For example, if the height of the dam is 9 m then it needs to be at least 3 m wide.

$$\boxed{b/H = 0.33}$$

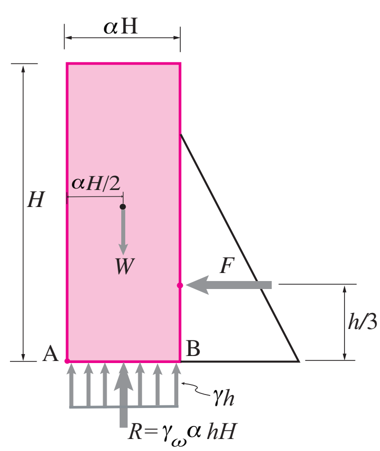


Figure 9.17

2. *Full water pressure on the bottom:* In this case, the water pressure on the bottom is uniformly distributed and its intensity is the same as the lateral pressure at B, *i.e.*, $p = \gamma_w h$. The free-body diagram diagram is shown in fig. 9.17 where the known forces are $W = \gamma_c \alpha H^2$, $F = (1/2)\gamma_w h^2$, and $R = \gamma_w \alpha h H$. Again, we carry out moment balance about point A to get

$$\begin{aligned} F \cdot \frac{h}{3} &= (W - R) \cdot \frac{\alpha h}{2} \\ \gamma_w h^3 &= 3(\gamma_c \alpha H^2 - \gamma_w \alpha h H)\alpha H \\ \alpha^2 &= \frac{(h/H)^3}{3(\gamma_c/\gamma_w - h/H)}. \end{aligned}$$

   Once again, substituting the given values and $h/H = 1$, we get

$$\alpha^2 = 1/6 \quad \Rightarrow \quad \alpha = 0.408.$$

   Thus the width in this case needs to be at least 0.41 times the height H, slightly wider than the previous case.

$$\boxed{b/H \geq 0.41}$$

© 2026 Andy Ruina & Rudra Pratap

9.1 Net force and moments in hydrostatics

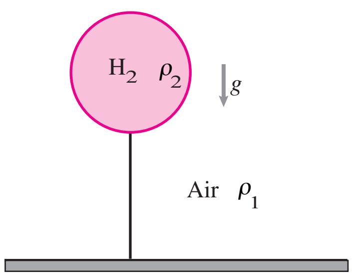
Preparatory Problems

More-Involved Problems

9.1.1 A balloon with volume V, whose membrane has negligible mass, holds a gas with density ρ_2. It is surrounded by a gas with density ρ_1.

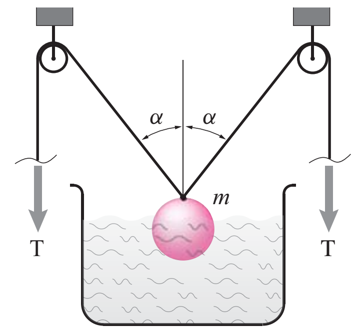
a) In terms of ρ_1, ρ_2, g, and V, find the tension in the string.

b) By some means look up the density of Helium and air at atmospheric temperature and pressure and calculate the volume, in cubic feet and in cubic meters, of a helium balloon that could lift 75 kg.



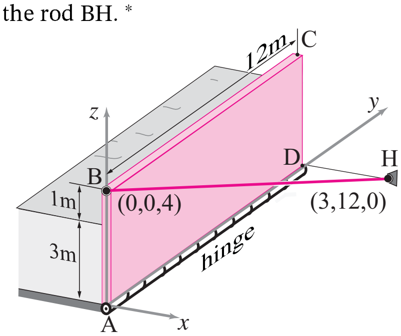
Problem 9.1.1: Balloon

9.1.2 A spherical body of mass $m = 10\,\mathrm{kg}$ and radius $R = 100\,\mathrm{mm}$ hangs from a continuous string as shown in the figure. The body is partially submerged in water and angle $\alpha = 45^\circ$ (fixed). If the force of buoyancy is $\rho V g$ where $\rho = 1000\,\mathrm{kg/m^3}$ = density of water, V is the submerged volume of the body, and g is the usual g; find the tension in the string as a function of the submerged volume V. Find the maximum and the minimum tension corresponding to fully and zero submerged volume of the body respectively.



Problem 9.1.2

9.1.3 A 4-meter-high 'door' holds back a stream ($\gamma = 10{,}000\,\mathrm{N/m^3}$) that is 3m deep and 12m wide. The door is hinged along its bottom and is propped up by a thin rod B that goes from a ball joint at H at (3,12,0) to a ball joint at the upper left corner B of the door at (0,0,4). Neglect the mass of the door. Find the axial-force in the rod BH. *

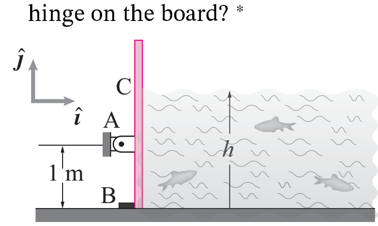


Problem 9.1.3

9.1.4 Water is held in a reservoir by a board with negligible weight that is 5 meters long. It is hinged 1 meter off the bottom at A and kept from leaking by a seal at B. Assume $\rho = 1000\,\mathrm{kg/m^3}$, $g = 10\,\mathrm{N/kg}$.

a) What is h when the board starts to pull away from the stop at B? *

b) At that h what is the force of the hinge on the board? *

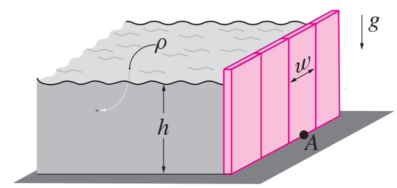


Problem 9.1.4

9.1.5 The side of a pool is made of vertical boards which are stuck in the ground. Assuming that the boards, on average, get no support from their neighbors, and neglect the weight of the board itself,

a) calculate the force and moment from the ground on one board (answer in terms of some or all of w, h, ρ, and g.

b) For a one foot board and 8 foot deep pool, find the size of a force, and its location, so the force is equivalent to the water pressure on the board (answer in lbf).

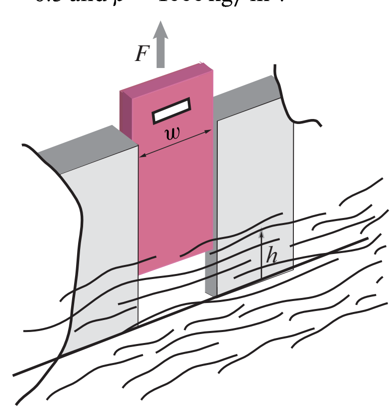


Problem 9.1.5

9.1.6 A sluice gate is a dam that can be opened. Sometimes it is just a board in a slot that is opened by pulling up the board. For water with density ρ and depth h pressing against a board with width w pressing against one face of the slot (the face away from the water) with coefficient of friction μ.

a) find the force F needed to pull up the board in terms of g, ρ, μ, h, and w.

b) Find the force assuming $g = 10\,\mathrm{m/s^2}$, $h = 1\,\mathrm{m}$, $w = 1\,\mathrm{m}$, $\mu = 0.5$ and $\rho = 1000\,\mathrm{kg/m^3}$.



Problem 9.1.6

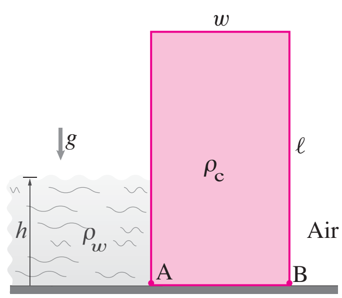
9.1.7 A concrete (density = ρ_c) wall with height ℓ, width w and length (into the paper) d rests on a flat rigid floor and serves as a dam for water with depth h and density ρ_w. Assume the wall only makes contact at edges A and B.

a) Assume there is a seal at A, so no water gets under the dam. What is the coefficient of friction needed to keep the block from sliding?

b) What is the maximum depth of water before the block tips?

c) Assume that there is a seal at B and that water gets under the block. What is the coefficient of friction needed to keep the block from sliding?

d) What is the maximum depth of water before the block tips?

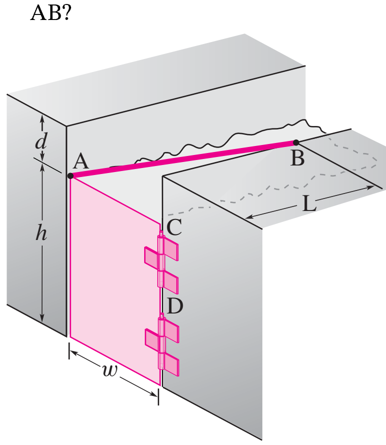


Problem 9.1.7

9.1.8 A door holds back the water at a lock on a canal. The water surface is at the top of the door. The rope AB keeps it from swinging open. The door has hinges at C and D. The height of the door is h, the width w. The point B is a distance d above the top of the door and is set back a distance L. The weight density of the water is γ.

a) What is the total force of the water on the door?

b) What is the tension in the rope AB?

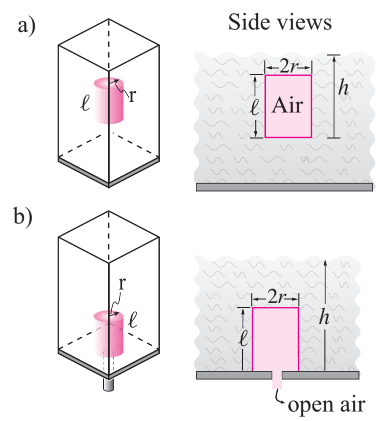


Problem 9.1.8

9.1.9 This problem somewhat explains the workings of some toilet valves. Open the tank of a toilet and look at the rubber piece at the bottom that sits on the bottom but then floats after initially lifted by the turning of the flush lever. The puzzle this problem solves is this: Why does the valve stick to the bottom, but then float when lifted.

a) A hollow cylinder with an open bottom (like an upside down but open can) is filled with air but is under water. What force is required to hold it under water (in terms of ρ, r, h, ℓ and g? *

b) The same can is on the bottom of a tank of water and its edges are sealed. The bottom is open to atmospheric air. How much force is needed to hold the can down now (so there is no force from the bottom of the tank onto the edges of the cylinder)? *



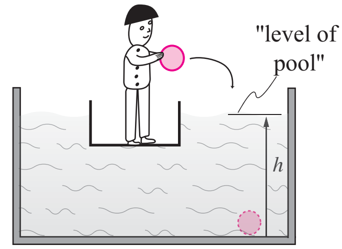
Problem 9.1.9

. . .

Some puzzles. The following puzzles are sometimes presented as brain teasers. You should be able to reason the answers carefully and irrefutably. **9.1.10** A person is in a boat in a pool with surface area A. She is holding a ball with volume V and mass m in a still pool. The ball is then thrown into the pool, no water is splashed out and the pool comes to rest again.

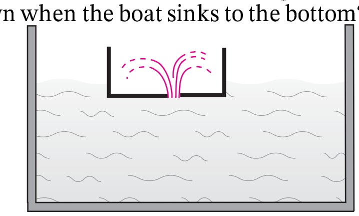
a) Assuming the ball floats, by how much does the pool level go up or down?

b) Assuming the ball sinks to the bottom by how much does the pool level go up or down?



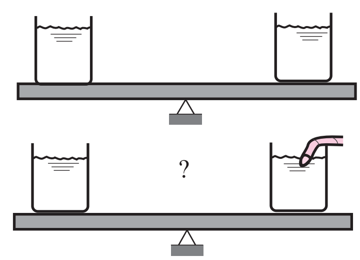
Problem 9.1.10

9.1.11 A steel boat with mass m and density ρ_s is floating in a pool of water with density ρ_w and cross sectional area A. By how much does the pool level go up or down when the boat sinks to the bottom?



Problem 9.1.11

9.1.12 Two cups of water are balanced. You then gently stick your finger into one of them. Does this upset the balance? This experiment can be set up with two cups and a hexagonal-cross-section pencil. The cups need not be identical, they just need to be balanced at the start.



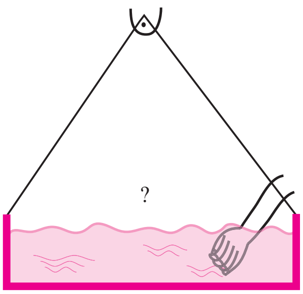
Problem 9.1.12

9.1.13 A tray of water is suspended and level.

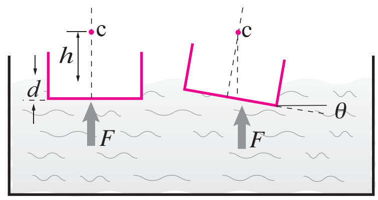
a) A hand is gently placed in the tray but does not touch the edges or bottom. Is the level of the tray upset.

b) *Challenge:* Assuming the tray is massless with width w and water depth h, how high must be the hinge so the equilibrium is stable. That is, imagine the tray is rotated slightly about the hinge, the water pressure should cause a torque which tends to restore the vertical orientation shown.

© 2026 Andy Ruina & Rudra Pratap



Problem 9.1.13



Problem 9.1.14: Ship stability.

9.1.14 *Challenge:* This challenge problem is closely related to the challenge problem above, but is much more famous. It seems to have been first solved by Leonard Euler and Pierre Bouguer in about 1735. This solution seems to be the first mechanics problem in which the significance of the area moment of inertia was appreciated [this is a hint].

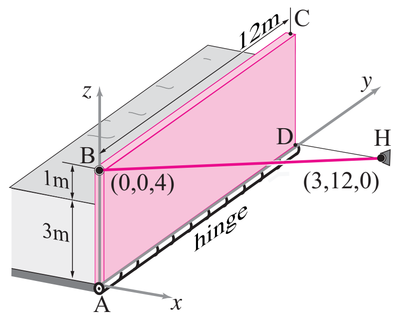
For simplicity assume that a boat is shaped like a box with width h and and length into the paper of b. Assume that the boat floats with its bottom a depth d under water. Now rotate the boat about an axis at the surface of the water and along its length (into the paper). Imagine that giant hands hold the boat in this position. In this rotated position the effect of the water pressure on the boat is a buoyant force and moment. This is equivalent to a force that is displaced slightly sideways.

Your goal is to find the height of the point that the line of action of this force intersects a mast of the boat. For small angles of boat tip the location is independent of the amount of tip.

This point is called the *metacenter* of the hull, and its distance up from the centroid of the boat's submerged volume is the hull's *metacentric height*. The condition of boat stability is that the metacenter be above the center of mass of the boat (thus the moment of the buoyant forces about the center of mass will tend to restore the boat to level).

Euler and Bouguer did the calculation you are asked to do here *after*, *e.g.,* , the launching of the 'great' Swedish ship Vasa, which capsized in the harbor on day 1. This was unfortunate for Sweden at the time, but fortunate now, because the brand-new 375 year old ship is a seeworthy tourist attraction in Stockholm.

9.1.3 A 4-meter-high 'door' holds back a stream ($\gamma = 10{,}000\,\mathrm{N/m^3}$) that is 3m deep and 12m wide. The door is hinged along its bottom and is propped up by a thin rod B that goes from a ball joint at H at (3,12,0) to a ball joint at the upper left corner B of the door at (0,0,4). Neglect the mass of the door. Find the axial-force in the rod BH.



Problem 9.3

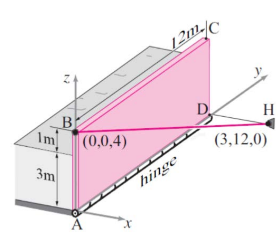
Answer:
$(117\gamma/2)\,\mathrm{m}^3 = 5.85 * 10^5\,\mathrm{N}$

8.1.3

Given, $\gamma = 1000\ \mathrm{N/m^3}$, $l = 12$ m, and $h = 3$ m.

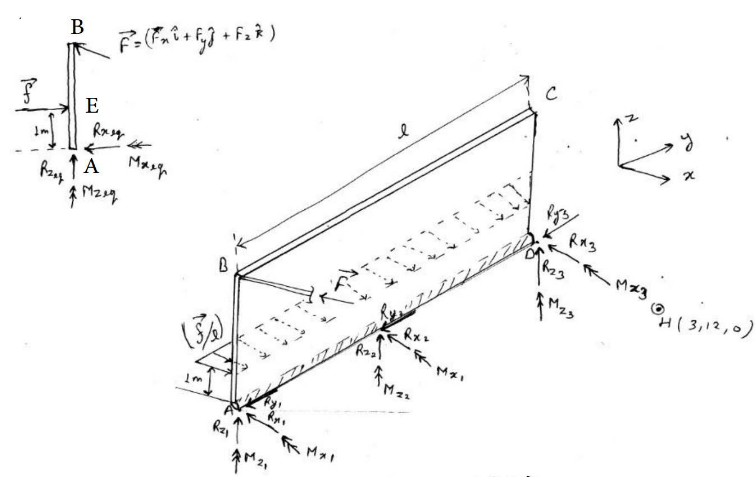
Let the axial force in the rod be $\vec{\boldsymbol{F}} = F\,\hat{\boldsymbol{r}}_{\boldsymbol{HB}}$

$$\Rightarrow \vec{\boldsymbol{F}} = \frac{F}{13}\left(-3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}\right)\ \mathrm{N}$$



Now the pressure force on door is,

$\vec{\boldsymbol{f}} = \frac{1}{2}\gamma h^2 l\,\hat{\boldsymbol{\imath}}$, and this equivalent force of the linear force distribution passes 1/3 up from the edge AD. Hence, $\vec{\boldsymbol{r}}_{AE} = \frac{h}{3}\hat{\boldsymbol{k}} = 1\hat{\boldsymbol{k}}$ m



$\sum\vec{M}_A = \vec{\mathbf{0}}$ on the door ABCD

$\Rightarrow \hat{\boldsymbol{\jmath}}\,.\left(\vec{\boldsymbol{r}}_{AB} \times \vec{\boldsymbol{F}}\right) + \hat{\boldsymbol{\jmath}}\,.\left(\vec{\boldsymbol{r}}_{AE} \times \vec{\boldsymbol{f}}\right) = 0$, Let $\vec{\boldsymbol{r}}_{AB} = b\hat{\boldsymbol{k}} = 4\hat{\boldsymbol{k}}$ m

$$\Rightarrow \left(\frac{-3}{13}\right) F\,b + \frac{1}{2}\gamma\frac{h^3}{3}l = 0$$

$$\Rightarrow F = \left(\frac{13}{18}\right)\frac{\gamma\,h^3 l}{b}$$

$$\Rightarrow \boxed{F = 5.85 \times 10^4\ \mathrm{N}}$$

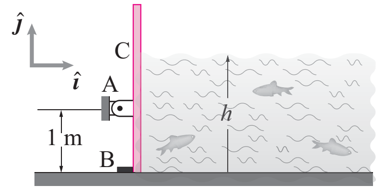
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

Introduction to Mechanics for Engineers • © 2026 Andy Ruina & Rudra Pratap

9.1.4 Water is held in a reservoir by a board with negligible weight that is 5 meters long. It is hinged 1 meter off the bottom at A and kept from leaking by a seal at B. Assume $\rho = 1000\,\text{kg/m}^3$, $g = 10\,\text{N/kg}$.

a) What is h when the board starts to pull away from the stop at B?

b) At that h what is the force of the hinge on the board?



Problem 9.4

Answer:
Water starts to spill at $h = 3r_{AB} = 3\,\text{m}$.
Assuming no friction at B, $\vec{F}_A = 2.25 * 10^5 \hat{\imath}\,\text{N}$

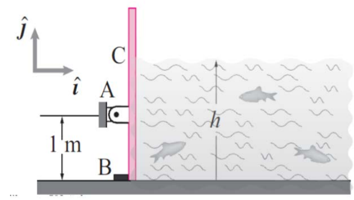
8.1.4

(a) Given, $\rho = 10^3\ \text{kg/m}^3$

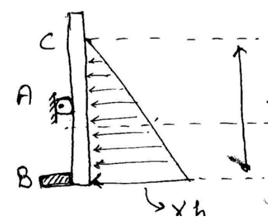
$g = 10\ \text{N/kg}$

$\Rightarrow \gamma = \rho g = 10^4\ \text{N/m}^3$

Length of the board, $l = 5$ m



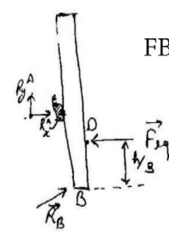
The forces acting on the board are shown in the FBD (I). The equivalent force of the linearly distributed force passes through the point D which is 1/3 up from B. When the board just begins to detach from point B, $R_B = 0$. Hence, from the FBD (II),



$\sum \vec{M}_A = \vec{0}$

$\Rightarrow \vec{r}_{AD} \times \vec{F}_{eq} = \vec{0}$

$\Rightarrow \vec{r}_{AD} \times \left(\int_A p\,\hat{\boldsymbol{n}}\,d\vec{A} \right) = \vec{0}$



FBD (I)

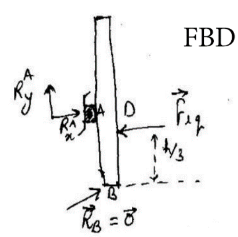
In order for this equation to hold true, the distance from point A to point D (the point of application of equivalent force due to pressure on the board) has to be zero.

$\vec{r}_{AD} = \vec{0}$

$\therefore$ h/3 = AB

so $\boxed{h = 3\ AB = 3\text{m}}$

FBD (II)



(b) $\sum \vec{F} = \vec{0}$

$\Rightarrow \vec{R}_A + \vec{F}_{eq} = \vec{0}$

$\Rightarrow \vec{R}_A - \frac{1}{2}\gamma h^2 l\,\hat{\imath} = \vec{0}$

$\therefore \vec{R}_A = \frac{1}{2}\gamma h^2 l\,\hat{\imath}$

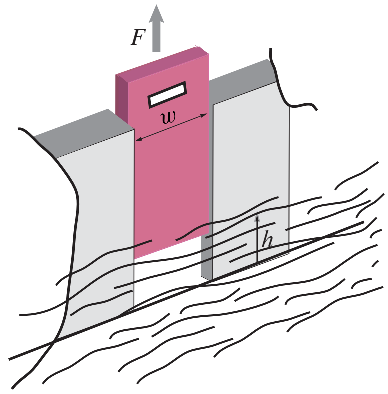
$\boxed{\vec{R}_A = 22.5 \text{ x } 10^4 \hat{\imath}\ \text{N}}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.1.6 A sluice gate is a dam that can be opened. Sometimes it is just a board in a slot that is opened by pulling up the board. For water with density ρ and depth h pressing against a board with width w pressing against one face of the slot (the face away from the water) with coefficient of friction μ.

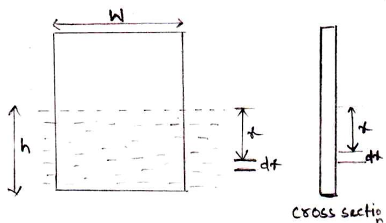
a) find the force F needed to pull up the board in terms of g, ρ, μ, h, and w.

b) Find the force assuming $g = 10\,\mathrm{m/s^2}, h = 1\,\mathrm{m}, w = 1\,\mathrm{m}, \mu = 0.5$ and $\rho = 1000\,\mathrm{kg/m^3}$.



Problem 9.6

8·1·6



Magnitude of friction force depends on the reaction force on the object from the support surface.

Here one face of the slot will apply that reaction force on the board.

The reaction force will be equal to the force exerted by water on the board.

Since force at a particular point due to water depends on depth of water at that point, it will vary with distance from surface.

From the cross sectional figure shown above, pressure due to water at a depth x from the surface is given by

$$P_x = \rho g x$$

Due to this pressure, horizontal force acting on a small element of height dx is given by

$$dF_h = \rho g x\,(w \cdot dx)$$

$w \cdot dx$:- it is the area of the small element under consideration.

Now total force can be found by integrating above

$$\int_0^{F_h} dF_h = \int_0^h \rho g x\, w \cdot dx$$

$$\Rightarrow F_h = \rho g w \int_0^h x\, dx$$

$$\Rightarrow F_h = \frac{\rho g w h^2}{2}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

8.1.6. continued

(a) Force required to pull up the board

$$F = 4F_h$$
$$= \frac{4\rho g w h^2}{2}$$

(b) Substituting the values

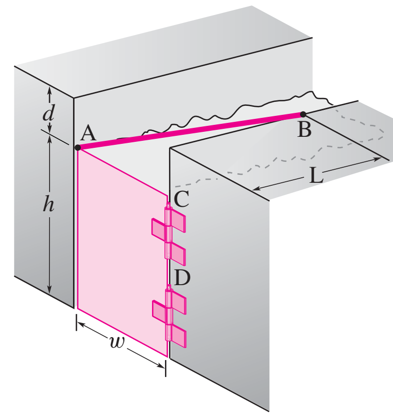
$$F = \frac{0.5 \times 1000\,kg/m^3 \times 10\,m/s^2 \times 1\,m \times (1\,m)^2}{2}$$
$$= 2500\ kg\frac{m}{s^2} = 2500\,N$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.1.8 A door holds back the water at a lock on a canal. The water surface is at the top of the door. The rope AB keeps it from swinging open. The door has hinges at C and D. The height of the door is h, the width w. The point B is a distance d above the top of the door and is set back a distance L. The weight density of the water is γ.

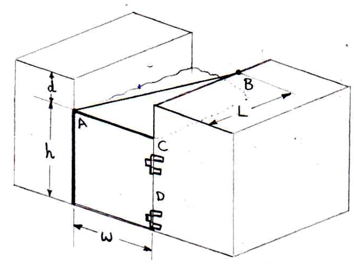
a) What is the total force of the water on the door?

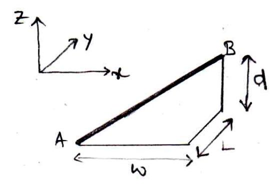
b) What is the tension in the rope AB?



Problem 9.8

8.1.8





Position of B, w.r.t. A

(a) Total force of water on the door :-

Note: calculation of this is similar to that as followed in problem 8.1.8

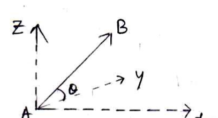
$$F_h = \frac{\rho g w h^2}{2}$$

Weight density water is given as $\gamma = \rho g$

$$\therefore \; F_h = \frac{\gamma w h^2}{2}$$

(b) Moments due to forces exerted by water and that by rope about hinge will be balanced at equillibrium.

Direction of AB is shown in the figure. Angle made by AB with y axis is θ.



Direction of $\vec{AB}$ in the coordinate system

Then we can say that

$$\cos(\theta) = \frac{L}{AB}$$

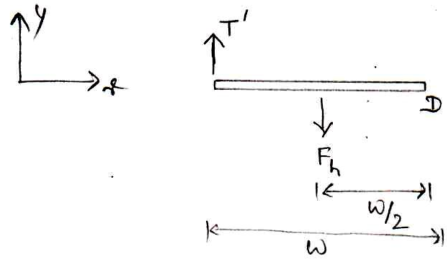
$AB = \sqrt{w^2+L^2+d^2}$ [from the figure ' position of B wrt A]

$$\therefore \cos(\theta) = \frac{L}{\sqrt{w^2+L^2+d^2}}$$

We can see that, Only the component of tension $\vec{T_{AB}}$ in rope AB which is along Y-axis[T'] will exert a moment about CD.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

8.1.8 continued



$T' \to$ component of tension $\vec{T}_{AB}$ along Y-axis.

$F_h \to$ Force exerted by water

F.B.D. of door [Top view]

At equilibrium net moment about point D will be zero

$$\Sigma \vec{M}_{/D} = \vec{0}$$

$$\Rightarrow \vec{M}_{T'/D} + \vec{M}_{F_h/D} = \vec{0}$$

$$\Rightarrow \vec{r}_{T'/D} \times \vec{T}' + \vec{r}_{F_h/D} \times \vec{F}_h = \vec{0}$$

$$\Rightarrow (-w\,\hat{i}) \times (T'\hat{j}) + \left(-\frac{w}{2}\,\hat{i}\right) \times [F_h(-\hat{j})] = \vec{0}$$

$$\Rightarrow -wT'(\hat{k}) + \left(\frac{w}{2} \cdot F_h\right)\hat{k} = \vec{0}$$

$$\Rightarrow \left\{-w \cdot T\cos\theta\,(\hat{k}) + \left(\frac{w}{2} \cdot F_h\right)\hat{k} = \vec{0}\right\} \cdot \hat{k} \quad [\because T' = T \cdot \cos\theta]$$

$$\Rightarrow T = \frac{F_h}{2 \cdot \cos\theta}$$

$$\Rightarrow T = \frac{1}{2} \cdot \frac{\gamma w h^2}{2} \times \frac{\sqrt{w^2 + L^2 + d^2}}{L}$$

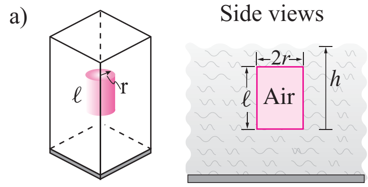
$$\Rightarrow T = \frac{\gamma w h^2}{4L} \sqrt{w^2 + L^2 + d^2}$$

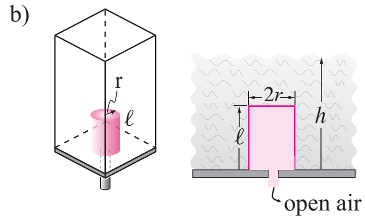
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.1.9 This problem somewhat explains the workings of some toilet valves. Open the tank of a toilet and look at the rubber piece at the bottom that sits on the bottom but then floats after initially lifted by the turning of the flush lever. The puzzle this problem solves is this: Why does the valve stick to the bottom, but then float when lifted.

a) A hollow cylinder with an open bottom (like an upside down but open can) is filled with air but is under water. What force is required to hold it under water (in terms of ρ, r, h, ℓ and g?

b) The same can is on the bottom of a tank of water and its edges are sealed. The bottom is open to atmospheric air. How much force is needed to hold the can down now (so there is no force from the bottom of the tank onto the edges of the cylinder)?





Problem 9.9

Answer:
(a) $\rho g \pi r^2 \ell$
(b) $-\rho g \pi r^2 (h-\ell)$, note the minus sign, it now takes force to lift the can.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

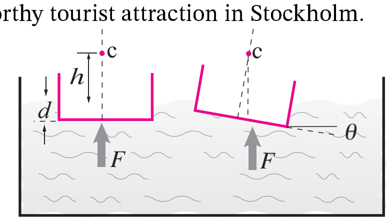
9.1.14 *Challenge:* This challenge problem is closely related to the challenge problem above, but is much more famous. It seems to have been first solved by Leonard Euler and Pierre Bouguer in about 1735. This solution seems to be the first mechanics problem in which the significance of the area moment of inertia was appreciated [this is a hint].

For simplicity assume that a boat is shaped like a box with width h and and length into the paper of b. Assume that the boat floats with its bottom a depth d under water. Now rotate the boat about an axis at the surface of the water and along its length (into the paper). Imagine that giant hands hold the boat in this position. In this rotated position the effect of the water pressure on the boat is a buoyant force and moment. This is equivalent to a force that is displaced slightly sideways.

Your goal is to find the height of the point that the line of action of this force intersects a mast of the boat. For small angles of boat tip the location is independent of the amount of tip.

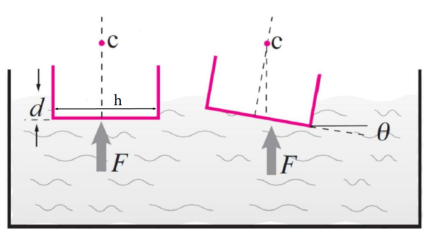
This point is called the *metacenter* of the hull, and its distance up from the centroid of the boat's submerged volume is the hull's *metacentric height*. The condition of boat stability is that the metacenter be above the center of mass of the boat (thus the moment of the buoyant forces about the center of mass will tend to restore the boat to level).

Euler and Bouguer did the calculation you are asked to do here *after*, *e.g.*, , the launching of the 'great' Swedish ship Vasa, which capsized in the harbor on day 1. This was unfortunate for Sweden at the time, but fortunate now, because the brand-new 375 year old ship is a seeworthy tourist attraction in Stockholm.



Problem 9.14: Ship stability.

8.1.14



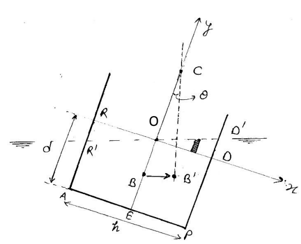
Center of buoyant force is the centroid of the submerged volume of the boat. When the boat tilts, this point shifts from its initial position (say B) to the new position of centroid of the submerged volume (say B'). We need to find this shift in position of the centroid (denoted by $\bar{x}$ and $\bar{y}$), and the metacentric height.

Since the depth (b) of the boat is constant, we consider the area moments in order to find the shift in center of buoyancy. Let the coordinate system be centered at 'O'. The shift in x direction is given as:

$$\bar{x} = \frac{\int_{ARDP} xdA + \int_{D'OD} xdA - \int_{R'OR} xdA}{A_{AR'D'P}}$$

$$\Rightarrow \bar{x} = \frac{0 + \int_0^{h/2} x^2 tan\theta dx - \int_{-h/2}^{0} -x^2 tan\theta dx}{hd}$$

$$\Rightarrow \bar{x} = \frac{tan\theta \int_{-h/2}^{h/2} x^2 dx}{hd} \Rightarrow \bar{x} = \frac{h^2 tan\theta}{12d} \quad (1)$$



The shift in y direction is given as:

$$\bar{y} = \frac{\int_{ARDP} ydA + \int_{D'OD} ydA - \int_{R'OR} ydA}{A_{AR'D'P}}$$

$$\Rightarrow \bar{y} = \frac{-\frac{d}{2}hd + \int_0^{\frac{h}{2}tan\theta} y(\frac{h}{2} - y)cot\theta dy - \int_{-\frac{h}{2}tan\theta}^{0} -y(\frac{h}{2} + y)cot\theta dy}{hd}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

8.1.14

$$\Rightarrow \bar{y} = \frac{-\frac{d}{2}hd + cot\theta\left[\frac{y^2}{2}\frac{h}{2} - \frac{y^3}{3}\right]_0^{\frac{h}{2}tan\theta} + cot\theta\left[\frac{y^2}{2}\frac{h}{2} + \frac{y^3}{3}\right]_{-\frac{h}{2}tan\theta}^{0}}{hd}$$

$$\Rightarrow \bar{y} = -\frac{d}{2} + \frac{h^2tan^2\theta}{24d} \; (2)$$

For small angle (θ), assuming $tan\,\theta \approx \theta$ and $tan^2\,\theta \approx 0$.

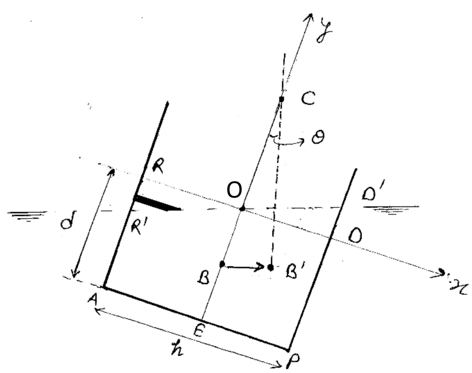
$\therefore\ \bar{x} = \frac{h^2\theta}{12d}$ *and* $\bar{y} = -\frac{d}{2}$

Here $\bar{x}$ and $\bar{y}$ give the approximate position of point B'.

As the center of buoyancy shift in *x* direction is significant, from ΔBB'C

$$tan\theta \approx \frac{BB'}{BC}$$

$$\Rightarrow BC \approx \frac{BB'}{tan\theta} \approx \frac{\bar{x}}{tan\theta} = \frac{h^2\tan\theta}{12d\tan\theta} = \frac{h^2}{12d}$$



When point C (also called the metacenter) lies above the center of gravity of the ship, then the couple applied by gravity and the buoyant force is of restoring nature and the boat stays in stable equilibrium. This metacentric height (BC in triangle BB'C) is independent of tilt for small angle tip and is a feature of the geometric configuration of the ship.