8.1 Free body cuts at arbitrary locations

Tension, shear force, and bending moment diagrams

Engineers often want to know how the internal forces vary from point to point in a structure. How do you find these internal forces? You could draw a variety of free-body diagrams with cuts at the points of interest. Another approach, which we present now, is to leave the position, say x, of the free-body-diagram cut a variable, and then calculate the internal forces in terms of x.

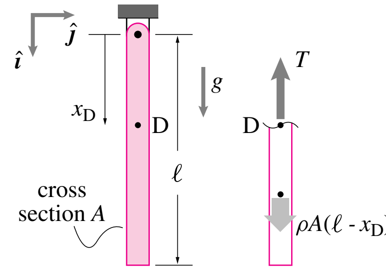


Figure 8.1: a) Rod hanging with gravity. b) free-body diagram with cut at x_D.

Example: Tension in a two-force body
Recall that in the first example of sec. 5.4 we found T without ever using information about the location of the free-body diagram cut. So the location does not affect the tension. For a two-force body the tension is a constant along the length.

Example: Tension in a rod from its own weight.
The uniform $1\,\text{cm}^2$ steel square rod with density $\rho = 7.7\,\text{gm}/\text{cm}^3$ and length $\ell = 100\,\text{m}$ has total weight $W = mg = \rho\ell Ag$ (see fig. 8.1). What is the tension a distance x_D from the top? Using the free-body diagram with cut at x_D we get:

$$
\begin{aligned}
&\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \\
\Rightarrow \quad T \quad &= \quad \rho A g(\ell - x_D) \\
&= \quad (7.7\,\text{gm}/\text{cm}^3)(1\,\text{cm}^2)(9.8\,\text{N}/\text{kg})(100\,\text{m} - x_D) \\
&= \quad 7.7 \cdot 9.8 \frac{\text{gm}\,\text{N}\,\text{m}}{\text{cm}\,\text{kg}} \left(100 - \frac{x_D}{\text{m}}\right) \underbrace{\left(\frac{1\,\text{kg}}{1000\,\text{gm}}\right)}_{1} \underbrace{\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)}_{1} \\
&= \quad 7.5\left(100 - \frac{x_D}{\text{m}}\right)\,\text{N}.
\end{aligned}
$$

So, at the bottom end at $x_D = 100\,\text{m}$ we get $T = 0$ and at the top end where $x_D = 0\,\text{m}$ we get $T = 750\,\text{N}$ and in the middle at $x_D = 50\,\text{m}$ we get $T = 375\,\text{N}$.

Because the free-body diagram cut location is variable, we can plot the internal forces as a function of position. This is most useful in civil engineering where an engineer wants to know the internal forces in a horizontal beam carrying vertical loads. Common examples include bridge platforms and floor joists.

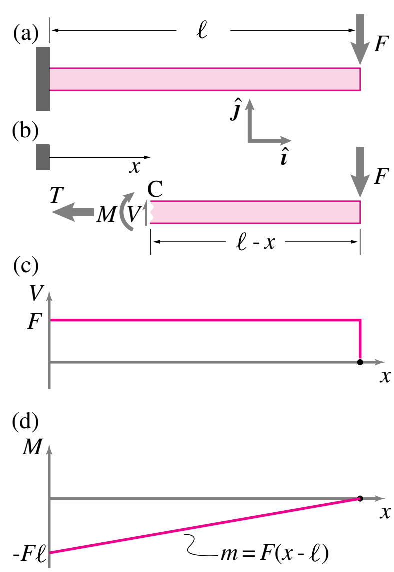


Figure 8.2: a) Cantilever beam, b) free-body diagram, c) Shear force diagram, & d) Bending moment diagram.

Example: Cantilever M and V diagram
A *cantilever* beam is mounted firmly at one end and has various loads orthogonal to its length, in this case a downwards load F at the end (fig. 8.2a). By drawing a free-body diagram with a cut at the arbitrary point C (fig. 8.2b) we can find the internal forces as functions of the position of C.

$$
\begin{aligned}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} \quad &\Rightarrow \quad V &= \quad &F \\
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \quad &\Rightarrow \quad T &= \quad &0 \\
\left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad M(x) &= \quad &F(x - \ell).
\end{aligned}
$$

That the axial tension is zero in these problems is so well known that the tension is often not drawn on the free-body diagram, and isn't calculated. We can now plot $V(x)$ and $M(x)$ as in figs. 8.2c and 8.2d. In this case the shear force is a constant and the bending moment varies from its maximum magnitude at the wall ($M = -F\ell$) to 0 at the end. It is the big value of $|M|$ at the fixed support, the maximum over the length, that makes cantilever beams typically break at the ends.

Often one is interested in distributed loads from gravity on the structure itself or from a distribution (say of people on a floor). The method is the same.

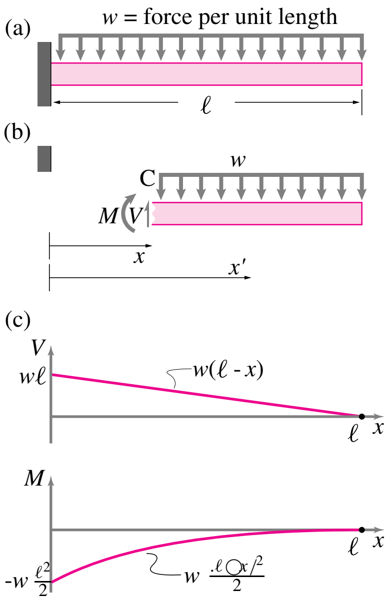


Figure 8.3: a) Cantilever beam with distributed load, b) free-body diagram, c) Shear force diagram, & d) Bending moment diagram.

Example: Distributed load

A cantilever beam has a downwards uniformly distributed load of w per unit length, *e.g.*, the weight of the beam, (fig. 8.3a). Using the free-body diagram shown (fig. 8.3b) we can find:

$$
\begin{aligned}
\{\textstyle\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}}\} \cdot \hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad \{V(x)\hat{\boldsymbol{\jmath}} + \textstyle\int d\vec{\boldsymbol{F}}\} \cdot \hat{\boldsymbol{\jmath}} = 0 \\
&\Rightarrow \quad V(x) = \textstyle\int_x^\ell w\, dx' \\
&\qquad\qquad\;\; = w \cdot (\ell - x)
\end{aligned}
$$

$$
\begin{aligned}
\{\textstyle\sum \vec{\boldsymbol{M}}_C = \vec{\boldsymbol{0}}\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad \{M(x)(-\hat{\boldsymbol{k}}) + \textstyle\int \vec{\boldsymbol{r}}_{/C} \times d\vec{\boldsymbol{F}}\Big\} \cdot \hat{\boldsymbol{k}} = 0 \\
&\Rightarrow \quad M(x) = \textstyle\int_x^\ell (x' - x) w\, dx' \\
&\qquad\qquad\;\; = w \cdot (x'^2/2 - x'x)\Big|_x^\ell \\
&\qquad\qquad\;\; = (\ell^2/2 - \ell x) - (x^2/2 - x^2) \\
&\qquad\qquad\;\; = -w \cdot (\ell - x)^2/2.
\end{aligned}
$$

The integrals were used because of their general applicability for distributed loads. For this problem we could have avoided the integrals by using an equivalent downwards force $w \cdot (\ell - x)$ applied a distance $(\ell - x)/2$ to the right of the cut. Shear and bending moment diagrams are shown in figs. 8.3a and 8.3b.

As for all problems based on the equilibrium equations and a given geometry, the principle of superposition applies.

Example: Superposition

Consider a cantilever beam that simultaneously has both of the loads from the previous two examples. By the principle of superposition:

$$
\begin{aligned}
V &= F + w(\ell - x) \\
M(x) &= F(x - \ell) + -w(\ell - x)^2/2.
\end{aligned}
$$

The shear force at every point is the sum of the shear forces from the previous examples. The bending moment at every point is the sum of the bending moments.

If there are concentrated loads in the middle of the region of interest the calculation gets more elaborate; the concentrated force may or may not show up on the free-body diagram of the cut bar, depending on the location of the cut.

Example: Simply supported beam with point load in the middle

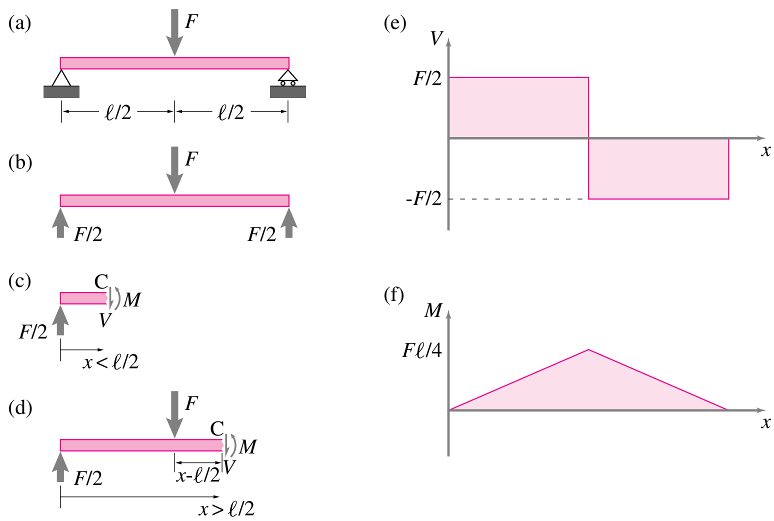


Figure 8.4: a) Simply supported beam, b) free-body diagram of whole beam, c) free-body diagram with cut to the left of the applied force, d) free-body diagram with cut to the right of the applied force e) Shear force diagram, f) Bending moment diagram

A *simply supported* beam is mounted with pivots at both ends (fig. 8.4a). First we draw a free-body diagram of the whole beam (fig. 8.4a) and then two more, one with a cut to the left of the applied force and one with a cut to the right of the applied force (figs. 8.4c and 8.4d). With the free-body diagram 8.4c we can find $V(x)$ and $M(x)$ for $x < \ell/2$ and with the free-body diagram 8.4d we can find $V(x)$ and $M(x)$ for $x > \ell/2$.

$$
\begin{aligned}
\{\textstyle\sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \vec{\mathbf{0}}\} \cdot \hat{\boldsymbol{\jmath}} \quad &\Rightarrow & V &= F/2 && \text{for } x < \ell/2 \\
& & &= -F/2 && \text{for } x > \ell/2 \\
\{\textstyle\sum \vec{\boldsymbol{M}}_{\mathrm{C}} = \vec{\mathbf{0}}\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow & M(x) &= Fx/2 && \text{for } x < \ell/2 \\
& & &= F(\ell - x)/2 && \text{for } x > \ell/2
\end{aligned}
$$

These relations can be plotted as in figs. 8.4e and 8.4f. Some observations: For this beam the biggest bending moment is in the middle, the place where simply supported beams often break. Instead of the free-body diagrams shown in (c) and (d) we could have drawn a free-body diagram of the bar to the right of the cut and would have gotten the same $V(x)$ and $M(x)$. We avoided drawing a free-body diagram cut *at* the applied load where $V(x)$ has a discontinuity.

How to find T, V, and M

Here are some guidelines for finding internal forces and drawing shear and bending moment diagrams.

- Draw a free-body diagram of the whole bar.
- Using the free-body diagram above, find the reaction forces.
- Draw a free-body diagram(s) of the cut bar.
  - For each region between concentrated loads, draw one free-body diagram.

  - Show the piece from the cut to one or the other end (So that all but the internal forces are known).
  - Don't make cuts *at* intermediate points of connection or load application.
- Use the equilibrium equations to find T, V, or M (Moment balance about a point at the cut is a good way to find M.)
- Use the results above to plot $V(x)$ and $M(x)$. $T(x)$ is rarely plotted.
  - Use the same x scale for this plot as for the free-body diagram of the whole bar.
  - Put the plots directly under the free-body diagram of the bar (so you can most easily relate features of the loads to features of the V and M diagrams).

Stress is force per unit area

For a given load, if you replace one bar in tension with two bars side-by-side, you would imagine the tension in each bar would go down by a factor of 2. Thus the pair of bars should be twice as strong as a single bar. If you glued these side-by-side bars together you would again have one bar, but it would be twice as strong as the original bar. Why? Because it has twice the cross sectional area.

What makes a solid break is the force per unit area carried by the material. For an applied tension load T, the force per unit area on an interior free-body diagram cut is T/A. The force per unit area, that is normal to an internal free-body diagram cut, is called *tension stress* and denoted σ (lower case 'sigma', the Greek letter s).

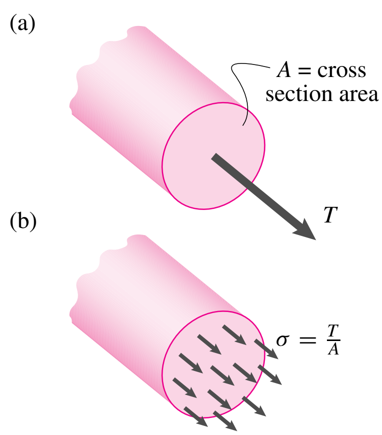


Figure 8.5: a) Tension on a free-body diagram cut is equivalent to b) uniform tension stress.

$$\sigma = T/A$$

Example: Stress in a hanging bar
Look at the hanging bar in the example on page 1. We can find the tension stress in this bar as a function of position along the bar as:

$$\sigma = \frac{T}{A} = \frac{\rho g A(\ell - x)}{A} = \rho g(\ell - x).$$

Note that the stress for this bar doesn't depend on the cross sectional area. The bigger the area the bigger the volume and hence the load. But also, the bigger the area on which to carry it.

For reasons that are beyond this book, the tension stress tends to be uniform in homogeneous (all one material) bars, no matter what their cross sectional shape, so that the average tension stress $\frac{T}{A}$ is actually the tension stress all across the cross section.

We can similarly define the average *shear stress* τ_{ave} ('tau') on a free body diagram cut as the average force per unit area tangent to the cut,

$$\tau_{\text{ave}} = \frac{V}{A}.$$

For reasons you may learn in a strength of materials class, shear stress is not so uniformly distributed across the cross section. But the average shear stress τ_{ave} does give an indication of the actual shear stress in the bar (*e.g.,* for a rectangular elastic bar, the peak shear stress is 50% larger than τ_{ave}).

The biggest stresses typically come from the bending moment. Motivating formulas for these stresses here is too big a digression. The formulas for the stresses due to bending moment are a key part of elementary strength of materials. But just knowing that these stresses tend to be larger, gives you the important notion that bending moment is a common cause of structural failure.

Internal force summary

'Internal forces' are the scalars that describe the force and moment on potential internal free-body diagram cuts. They are found by applying the equilibrium equations to free-body diagrams that have cuts at the points of interest. The internal forces are intimately associated with the internal stresses (force per unit area) and thus are important for determining the strength of structures.

SAMPLE 8.1 Support reactions on a simply supported beam: A uniform beam of length 3 m is simply supported at A and B as shown in the figure. A uniformly distributed vertical load $q = 100\,\mathrm{N/m}$ acts over the entire length of the beam. In addition, a concentrated load $P = 150\,\mathrm{N}$ acts at a distance $d = 1\,\mathrm{m}$ from the left end. Find the support reactions.

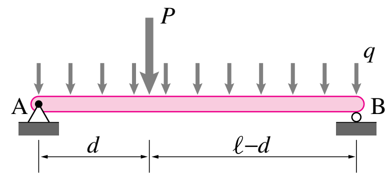


Figure 8.6

Solution Because the beam is supported at A on a pin joint and at B on a roller, the unknown reactions are

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}, \qquad \vec{B} = B_y\hat{\jmath}.$$

The uniformly distributed load q can be replaced by an equivalent concentrated load $W = q\ell$ acting at the center of the beam span. The free-body diagram of the beam, with the concentrated load replaced by the equivalent concentrated load is shown in Fig. 8.7. The moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, gives

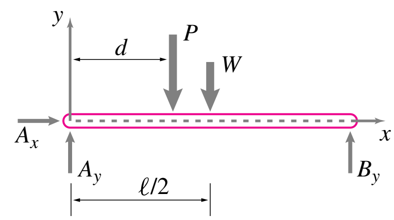


Figure 8.7

$$(-Pd - W\frac{\ell}{2} + B_y\ell)\hat{k} = \vec{0}$$

$$\begin{aligned}\Rightarrow \quad B_y &= P\frac{d}{\ell} + \frac{1}{2}W \\ &= 150\,\mathrm{N}\cdot\frac{1}{3} + \frac{1}{2}\cdot 300\,\mathrm{N} = 200\,\mathrm{N}.\end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\vec{A} + B_y\hat{\jmath} - P\hat{\jmath} - W\hat{\jmath} = \vec{0}$$

$$\begin{aligned}\Rightarrow \quad \vec{A} &= (-B_y + P + W)\hat{\jmath} \\ &= (-200\,\mathrm{N} + 150\,\mathrm{N} + 300\,\mathrm{N})\hat{\jmath} = 250\,\mathrm{N}\,\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{A} = 250\,\mathrm{N}\,\hat{\jmath}, \quad \vec{B} = 200\,\mathrm{N}\,\hat{\jmath}}$$

SAMPLE 8.2 Support reactions on a cantilever beam: A 2 kN horizontal force acts at the tip of an 'L'-shaped cantilever beam as shown in the figure. Find the support reactions at A.

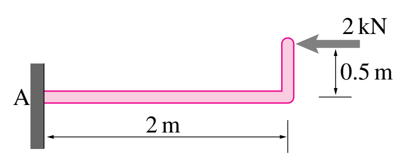


Figure 8.8

Solution The free-body diagram of the beam is shown in Fig. 8.9. The reaction force at A is $\vec{A}$ and the reaction moment is $\vec{M} = M\hat{k}$. Writing the moment-balance equation about point A, $\sum \vec{M}_A = \vec{0}$, we get

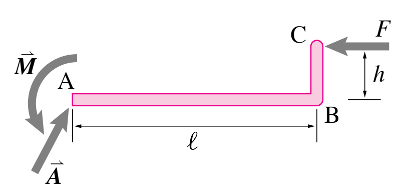


Figure 8.9

$$\begin{aligned}\vec{M} + \vec{r}_{C/A} \times \vec{F} &= \vec{0} \\ \vec{M} + (\ell\hat{\imath} + h\hat{\jmath}) \times (-F\hat{\imath}) &= \vec{0} \\ \Rightarrow \quad \vec{M} &= -Fh\,\hat{k} \\ &= -2\,\mathrm{kN}\cdot 0.5\,\mathrm{m}\,\hat{k} \\ &= -1\,\mathrm{kN\cdot m}\,\hat{k}.\end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned}\vec{A} + \vec{F} &= \vec{0} \\ \Rightarrow \quad \vec{A} &= -\vec{F} = -(-2\,\mathrm{kN}\,\hat{\imath}) = 2\,\mathrm{kN}\,\hat{\imath}.\end{aligned}$$

$$\boxed{\vec{A} = 2\,\mathrm{kN}\,\hat{\imath}, \quad \vec{M} = -1\,\mathrm{kN\cdot m}\,\hat{k}}$$

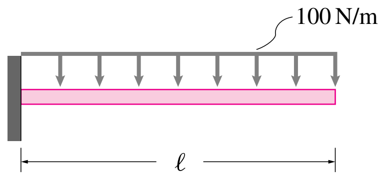


Figure 8.10

SAMPLE 8.3 Net force of a uniformly distributed system: A uniformly distributed vertical load of intensity 100 N/macts on a cantilever beam of length $\ell = 2\,\mathrm{m}$ as shown in the figure.

1. Find the net force acting on the beam.
2. Find an equivalent force-couple system at the mid-point of the beam.
3. Find an equivalent force-couple system at the right end of the beam.

Solution

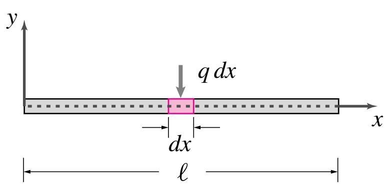


Figure 8.11

1. **The net force:** Because the load is uniformly distributed along the length, we can find the total (or net) load by calculating the load on an infinitesimal segment of length dx of the beam and then integrating over the entire length of the beam. Let the load intensity (load per unit length) be q ($q = 100\,\mathrm{N/m}$, as given). Then the vertical load on segment dx is (see Fig. 8.11),

$$d\vec{\boldsymbol{F}} \quad = \quad q\,dx(-\hat{\boldsymbol{\jmath}}).$$

Therefore, the net force is,

$$\vec{\boldsymbol{F}}_{\mathrm{net}} \quad = \quad \int_0^\ell q\,dx(-\hat{\boldsymbol{\jmath}}) = q\,\ell\,\hat{\boldsymbol{\jmath}} = -100\,\mathrm{N/m}\cdot 2\,\mathrm{m}\,\hat{\boldsymbol{\jmath}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}.$$

$$\boxed{\vec{\boldsymbol{F}}_{\mathrm{net}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}}$$

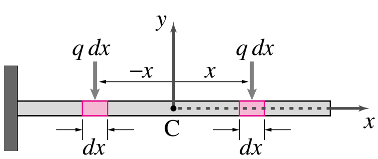


Figure 8.12

2. **The equivalent system at the mid-point:** We have already calculated the net force that can replace the uniformly distributed load. Now we need to calculate the couple at the mid-point of the beam to get the equivalent force-couple system. Again, consider a small segment of the beam of length dx located at distance x from the mid-point C (see Fig. 8.12). The moment about point C due to the load on dx is $(q\,dx)x(-\hat{\boldsymbol{k}})$. But, we can find a similar segment on the other side of C with exactly the same length dx, at exactly the same distance x, that produces a moment of $(q\,dx)x(+\hat{\boldsymbol{k}})$. The two contributions cancel each other and we have a net zero moment about C. Now, you can imagine the whole beam made up of these pairs that contribute equal and opposite moment about C and thus the net moment about the mid-point is zero. You can also find the same result by straight integration:

$$\vec{\boldsymbol{M}}_{\mathrm{C}} = \int_{-\ell/2}^{+\ell/2} qx\,dx(-\hat{\boldsymbol{k}}) = \left.\frac{qx^2}{2}\right|_{-\ell/2}^{+\ell/2}(-\hat{\boldsymbol{k}}) = \vec{\boldsymbol{0}}.$$

$$\boxed{\vec{\boldsymbol{F}}_{\mathrm{net}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}},\text{ and } \vec{\boldsymbol{M}}_{\mathrm{C}} = \vec{\boldsymbol{0}}}$$

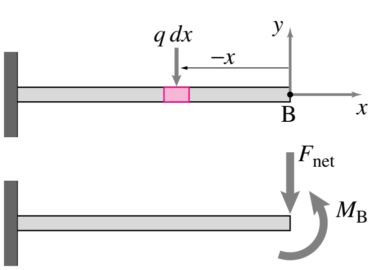


Figure 8.13

3. **The equivalent system at the end:** The net force remains the same as above. We compute the net moment about the end point B, referring to Fig. 8.13, as follows.

$$\begin{aligned}\vec{\boldsymbol{M}}_{\mathrm{B}} \quad &= \quad \int_0^\ell (-x\hat{\boldsymbol{\imath}})\times(-q\,dx\,\hat{\boldsymbol{\jmath}}) = -q\int_0^\ell x\,dx\hat{\boldsymbol{k}} \\ &= \quad -\frac{q\ell^2}{2}\hat{\boldsymbol{k}} = -\frac{100\,\mathrm{N/m}\cdot 4\,\mathrm{m}^2}{2}\hat{\boldsymbol{k}} = -200\,\mathrm{N{\cdot}m}\hat{\boldsymbol{k}}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{F}}_{\mathrm{net}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}\text{ and } \vec{\boldsymbol{M}}_{\mathrm{B}} = -200\,\mathrm{N{\cdot}m}\hat{\boldsymbol{k}}}$$

SAMPLE 8.4 For the uniformly loaded, simply supported beam shown in the figure, find the shear force and the bending moment at the mid-section c-c of the beam.

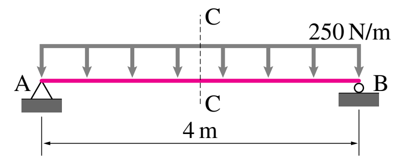


Figure 8.14

Solution To determine the shear force V and the bending moment M at the mid-section c-c, we cut the beam at c-c and draw its free-body diagram as shown in fig. 8.15. For writing force- and moment-balance equations we use the second figure where we have replaced the distributed load with an equivalent single load $F = (q\ell)/2$ acting vertically downward at distance $\ell/4$ from end A.

The force balance, $\sum \vec{F} = \vec{0}$, implies that

$$A_x \hat{\imath} + A_y \hat{\jmath} - V \hat{\jmath} - F \hat{\jmath} = \vec{0}.$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} A_x &= 0 \\ V &= A_y - F && (8.1) \\ &= A_y - \frac{q\ell}{2}. && (8.2) \end{aligned}$$

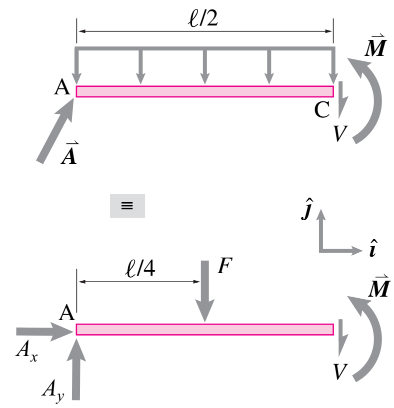


Figure 8.15

From the moment equilibrium about point A, $\sum \vec{M}_{\mathrm{A}} = \vec{0}$, we get

$$\begin{aligned} &M\hat{k} - \left(\frac{q\ell}{2} \cdot \frac{\ell}{4}\right)\hat{k} - V\frac{\ell}{2}\hat{k} = 0 \\ \Rightarrow \quad &M = \frac{q\ell^2 + 4V\ell}{8}. && (8.3) \end{aligned}$$

Thus, to find V and M we need to know the support reaction $\vec{A}$. From the free-body diagram of the beam in fig. 8.16 and the moment equilibrium equation about point B, $\sum \vec{M}_{\mathrm{B}} = \vec{0}$, we get

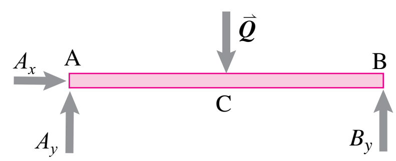


Figure 8.16

$$\begin{aligned} \vec{r}_{\mathrm{A/B}} \times \vec{A} + \vec{r}_{\mathrm{C/B}} \times \vec{Q} &= \vec{0} \\ (-A_y\ell + q\ell\frac{\ell}{2})\hat{k} &= \vec{0} \\ \Rightarrow \quad A_y = \frac{q\ell}{2} &= 500\,\mathrm{N}. \end{aligned}$$

Thus $\vec{A} = 500\,\mathrm{N}\,\hat{\jmath}$. Substituting $\vec{A}$ in eqns. (8.2) and (8.3), we get

$$\begin{aligned} V &= 500\,\mathrm{N} - 500\,\mathrm{N} = 0 \\ M &= \frac{(250\,\mathrm{N} \cdot 4\,\mathrm{m})^2 + 0}{8} \\ &= 500\,\mathrm{N{\cdot}m}. \end{aligned}$$

$V = 0, \quad M = 500\,\mathrm{N{\cdot}m}$

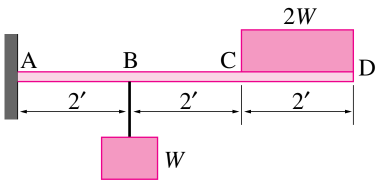


Figure 8.17

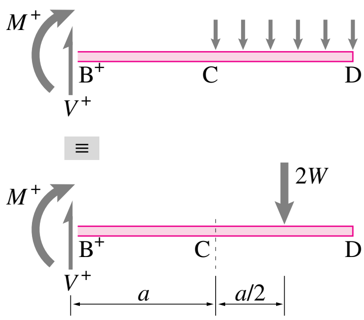


Figure 8.18

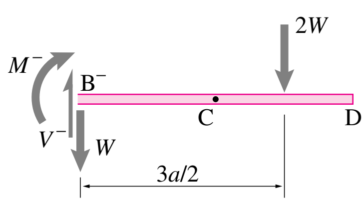


Figure 8.19

SAMPLE 8.5 The cantilever beam AD is loaded as shown in the figure where $W = 200\,\text{lbf}$. Find the shear force and bending moment on a section just left of point B and another section just right of point B.

Solution To find the desired internal forces, we need to make a cut at a section just to the left of B and one just to the right of B. We first take the one that is to the right of point B. The free-body diagram of the right part of the cut beam is shown in fig. 8.18. Note that if we selected the left part of the beam, we would need to determine support reactions at A. The uniformly distributed load $2W$ of the block sitting on the beam can be replaced by an equivalent concentrated load $2W$ acting at point E, at distance $a/2$ from the end D of the beam.

Let us denote the shear force by V^+ and the bending moment by M^+ at the section of our interest. Now, from the force equilibrium of the part-beam BD we get

$$\begin{aligned} V^+\hat{\boldsymbol{\jmath}} - 2W\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad V^+ &= 2W \\ &= 400\,\text{lbf}. \end{aligned}$$

The moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_{\text{B}} = \vec{\mathbf{0}}$, gives

$$\begin{aligned} -M^+\hat{\boldsymbol{k}} - 2W \cdot \frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad M^+ &= -3Wa \\ &= -1200\,\text{lb}\cdot\text{ft}. \end{aligned}$$

Now, we determine the internal forces at a section just to the left of point B. Let the shear and bending moment at this section be V^- and M^-, respectively, as shown in the free-body diagram (fig. 8.19). Note that load W acting at B is now included in the free-body diagram since the beam is now cut just a teeny bit left of this load.

From the force equilibrium of the part-beam, we have

$$\begin{aligned} V^-\hat{\boldsymbol{\jmath}} - W\hat{\boldsymbol{\jmath}} - 2W\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad V^- &= 3W \\ &= 600\,\text{lbf} \end{aligned}$$

and, from moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_{\text{B}} = \vec{\mathbf{0}}$, we get

$$\begin{aligned} -M^-\hat{\boldsymbol{k}} - 2W \cdot \frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad M^- &= -3Wa \\ &= -1200\,\text{lb}\cdot\text{ft}. \end{aligned}$$

$$M^+ = M^- = -1200\,\text{lb}\cdot\text{ft}, \quad V^+ = 400\,\text{lbf}, \quad V^- = 600\,\text{lbf}$$

Note that the bending moment remains the same on either side of point B but the shear force jumps by $V^+ - V^- = 200\,\text{lbf} = W$ as we go from right to the left. This jump is expected because a concentrated load W acts at B, in between the two sections we consider. Concentrated external forces cause a jump in shear, and concentrated external moments cause a jump in the bending moment.

SAMPLE 8.6 Tension in a bar: A T-shaped bar is fixed in a wall at one end and is acted on by three forces as shown in the figure. Find the tension in the rod at

1. section a-a, and
2. section b-b.

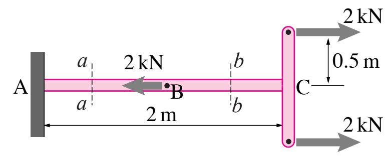


Figure 8.20

Solution

1. Let us cut the bar at section a-a and consider the part of the bar to the right of the cut-section. The free-body diagram of this part of the bar is shown in fig. 8.21. The scalar force balance in the horizontal direction gives

$$\begin{aligned} -T - F + 2F &= 0 \\ \Rightarrow \quad T &= F \\ &= 2\,\text{kN}. \end{aligned}$$

At section a-a:$T = 2\,\text{kN}$

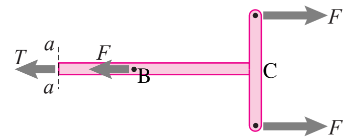


Figure 8.21

2. Now, we cut the bar at section b-b and again consider the section of the bar to the right of the cut-section. The free-body diagram of this part of the bar is shown in fig. 8.22. Again, the force balance in the horizontal direction gives

$$\begin{aligned} -T + 2F &= 0 \\ \Rightarrow \quad T &= 2F \\ &= 4\,\text{kN}. \end{aligned}$$

At section b-b:$T = 4\,\text{kN}$

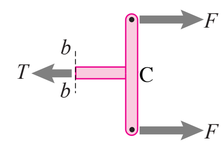


Figure 8.22

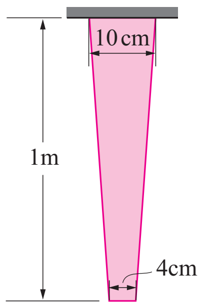


Figure 8.23

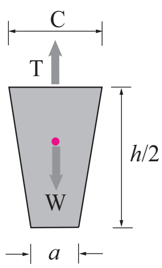


Figure 8.24

SAMPLE 8.7 Tension in a tapered bar due to self weight: A tapered bar of height 1 m, top width 10 cm, bottom width 4 cm and uniform thickness 4 cm hangs upside down from a ceiling. If the density of the material is $7500\,\mathrm{kg/m^3}$, find the tension in the rod halfway from the top. Use $g \approx 10\,\mathrm{m/s^2}$.

Solution Let us cut the bar at a section halfway from the top. The free-body diagram of the bar below the cut is shown in fig. 8.24. From the scalar force balance in the vertical direction, we have

$$T = W$$

where W is the weight of the lower part of bar below the cut section. Now, $W = \rho A t g$ where A is the frontal area, t is the thickness, and $\rho = 7500\,\mathrm{kg/m^3}$ is the density of the rod material. We need to compute W.

The width of the bar at the cut section is $c = (a+b)/2$ where $a = 4\,\mathrm{cm}$ and $b = 10\,\mathrm{cm}$. The frontal area of the bar-part is $A = (a+c)/2 \cdot (h/2)$ where $h = 1\,\mathrm{m}$. Thus,

$$\begin{aligned} W &= \rho\left(\frac{a+c}{2}\cdot\frac{h}{2}\right)tg \\ &= 7500\,\mathrm{kg/m^3}\left(\frac{0.04\,\mathrm{m}+0.07\,\mathrm{m}}{2}\cdot\frac{1\,\mathrm{m}}{2}\cdot(0.04\,\mathrm{m})\right)10\,\mathrm{m/s^2} \\ &= 82.5\,\mathrm{N}. \end{aligned}$$

Thus, $T = 82.5\,\mathrm{N}$.

$T = 82.5\,\mathrm{N}$

SAMPLE 8.8 **A simple frame:** A 2 m high and 1.5 m wide rectangular frame ABCD is loaded with a 1.5 kN horizontal force at B and a 2 kN vertical force at C. Find the internal forces and moments at the mid-section e-e of the vertical leg AB.

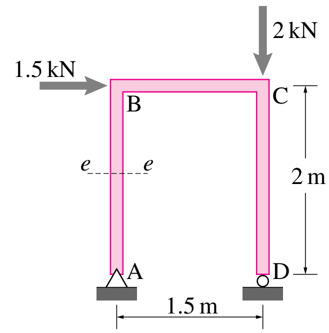


Figure 8.25

Solution To find the internal forces and moments, we need to cut the frame at the specified section e-e and consider the free-body diagram of either AE or EBCD. No matter which of the two we select, we will need the support reactions at A or D to determine the internal forces. Therefore, let us first find the support reactions at A and D by considering the free-body diagram of the whole frame (fig. 8.26). The moment balance about point A, $\sum \vec{\boldsymbol{M}}_{\mathrm{A}} = \vec{\boldsymbol{0}}$, gives

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{B}} \times \vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{r}}_{\mathrm{C}} \times \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{r}}_{\mathrm{D}} \times \vec{\boldsymbol{D}} &= \vec{\boldsymbol{0}} \\
h\hat{\boldsymbol{\jmath}} \times F_1\hat{\boldsymbol{\imath}} + (h\hat{\boldsymbol{\jmath}} + \ell\hat{\boldsymbol{\imath}}) \times (-F_2\hat{\boldsymbol{\jmath}}) + \ell\hat{\boldsymbol{\imath}} \times D\hat{\boldsymbol{\jmath}} &= \vec{\boldsymbol{0}} \\
-F_1 h\hat{\boldsymbol{k}} - F_2\ell\hat{\boldsymbol{k}} + D\ell\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \qquad D &= F_1\frac{h}{\ell} + F_2 \\
&= 1.5\,\mathrm{kN} \cdot \frac{2}{1.5} + 2\,\mathrm{kN} \\
&= 4\,\mathrm{kN}.
\end{aligned}
$$

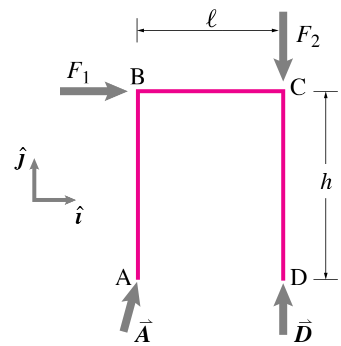


Figure 8.26

From force equilibrium, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, we have

$$
\begin{aligned}
\vec{\boldsymbol{A}} &= -\vec{\boldsymbol{F}}_1 - \vec{\boldsymbol{F}}_2 - \vec{\boldsymbol{D}} \\
&= -F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}} - D\hat{\boldsymbol{\jmath}} \\
&= -1.5\,\mathrm{kN}\hat{\boldsymbol{\imath}} - 2\,\mathrm{kN}\hat{\boldsymbol{\jmath}}.
\end{aligned}
$$

Now we draw the free-body diagram of AE to find the shear force V, axial (tensile) force T, and the bending moment M at section e-e.

From the force equilibrium of part AE, we get

$$
\begin{aligned}
\vec{\boldsymbol{A}} - V\hat{\boldsymbol{\imath}} + T\hat{\boldsymbol{\jmath}} &= \vec{\boldsymbol{0}} \\
(A_x - V)\hat{\boldsymbol{\imath}} + (A_y + T)\hat{\boldsymbol{\jmath}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad V = A_x &= -1.5\,\mathrm{kN} \\
T = -A_y &= 2\,\mathrm{kN}.
\end{aligned}
$$

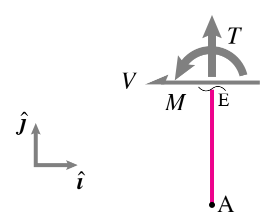


Figure 8.27

From the moment equilibrium about point A, $\sum \vec{\boldsymbol{M}}_{\mathrm{A}} = \vec{\boldsymbol{0}}$, we have

$$
\begin{aligned}
M\hat{\boldsymbol{k}} + \frac{h}{2}\hat{\boldsymbol{\jmath}} \times (-V\hat{\boldsymbol{\imath}}) &= \vec{\boldsymbol{0}} \\
M\hat{\boldsymbol{k}} + V\frac{h}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad M &= -V\frac{h}{2} \\
&= -(-1.5\,\mathrm{kN}) \cdot \frac{2\,\mathrm{m}}{2} \\
&= 1.5\,\mathrm{k\,N{\cdot}m}.
\end{aligned}
$$

$$\boxed{V = 1.5\,\mathrm{kN}, \quad T = 2\,\mathrm{kN}, \quad M = 1.5\,\mathrm{k\,N{\cdot}m}}$$

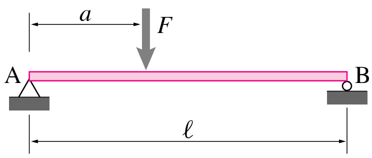


Figure 8.28

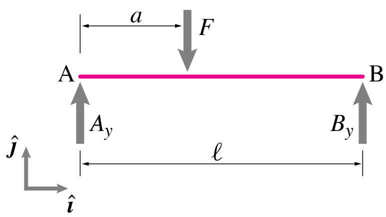


Figure 8.29

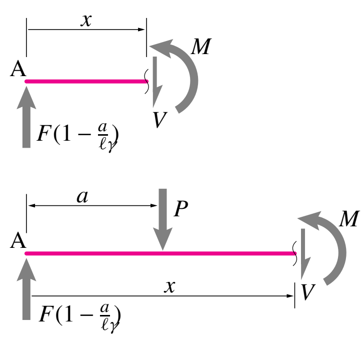


Figure 8.30

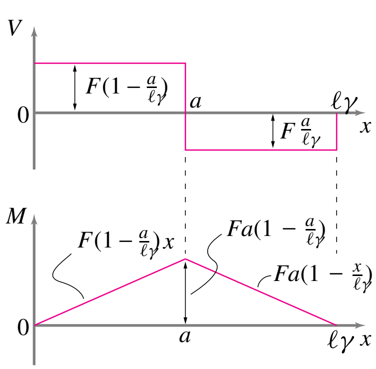


Figure 8.31

SAMPLE 8.9 Shear force and bending moment diagrams: A simply supported beam of length $\ell = 2\,\mathrm{m}$ carries a concentrated vertical load $F = 100\,\mathrm{N}$ at a distance a from its left end. Find and plot the shear force and the bending moment along the length of the beam for $a = \ell/4$.

Solution We first find the support reactions by considering the free-body diagram of the whole beam shown in fig. 8.29. By now, we have developed enough intuition to know that the reaction at A will have no horizontal component since there is no external force in the horizontal direction. Therefore, we take the reactions at A and B to be only vertical. Now, from the moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_{\mathrm{B}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned} F(\ell - a)\hat{\boldsymbol{k}} - A_y \ell \hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad A_y &= \frac{F(\ell - a)}{\ell} \\ &= F\left(1 - \frac{a}{\ell}\right) \end{aligned}$$

and from the force equilibrium in the vertical direction, $(\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}) \cdot \hat{\boldsymbol{\jmath}}$, we get

$$B_y = F - A_y = F\frac{a}{\ell}.$$

Now we make a cut at an arbitrary (variable) distance x from A where $x < a$ (see fig. 8.30). Carrying out the force balance and the moment balance about point A, we get, for $0 \le x < a$,

$$V = A_y = F\left(1 - \frac{a}{\ell}\right) \tag{8.4}$$

$$M = Vx = F\left(1 - \frac{a}{\ell}\right)x \tag{8.5}$$

Thus V is constant for all $x < a$ but M varies linearly with x.

Now we make a cut at an arbitrary x to the right of load F, *i.e.*, $a < x \le \ell$. Again, from the force balance in the vertical direction, we get

$$V = -F + F\left(1 - \frac{a}{\ell}\right) = -F\frac{a}{\ell} \tag{8.6}$$

and from the moment balance about point A,

$$\begin{aligned} M &= Fa + Vx \\ &= Fa - F\frac{a}{\ell}x \\ &= Fa\left(1 - \frac{x}{\ell}\right). \end{aligned} \tag{8.7}$$

Although eqn. (8.5) is strictly valid for $x < a$ and eqn. (8.7) is strictly valid for $x > a$, substituting $x = a$ in these two equations gives the same value for $M(= Fa(1 - a/\ell))$ as it must because there is no reason to have a jump in the bending moment at any point along the length of the beam. The shear force V, however, does jump because of the concentrated load F at $x = a$.

Now, we plug in $a = \ell/4 = 0.5\,\mathrm{m}$, and $F = 100\,\mathrm{N}$, in eqns. (8.4)–(8.7) and plot V and M along the length of the beam by varying x. The plots of $V(x)$ and $M(x)$ are shown in fig. 8.31.

SAMPLE 8.10 Shear force and bending moment diagrams by superposition: For the cantilever beam and the loading shown in the figure, draw the shear force and the bending moment diagrams by

1. considering all the loads together, and
2. considering each load (of one type) at a time and using superposition.

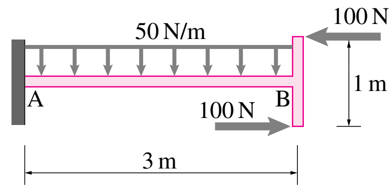


Figure 8.32

Solution

1. *$V(x)$ and $M(x)$ with all forces considered together:* The horizontal forces acting at the end of the cantilever are equal and opposite and, therefore, produce a couple. So, we first replace these forces by an equivalent couple $M_{\text{applied}} = 100\,\text{N} \cdot 1\,\text{m} = 100\,\text{N·m}$. Since we have a cantilever beam, we can consider the right hand side of the beam after making a cut anywhere for finding V and M without first finding the support reactions.

   Let us cut the beam at an arbitrary distance x from the right hand side. The free-body diagram of the right segment of the beam is shown in fig. 8.33. From the force balance, $\sum \vec{\boldsymbol{F}} = \vec{\mathbf{0}}$, we find that

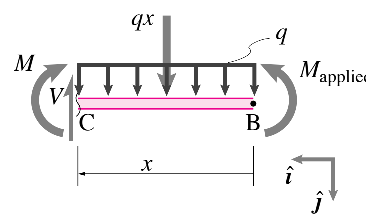


Figure 8.33

$$\begin{aligned} -V\,\hat{\boldsymbol{\jmath}} + qx\,\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad V &= qx \qquad (8.8) \\ &= (50\,\text{N/m})x. \end{aligned}$$

   Thus the shear force varies linearly along the length of the beam with

$$\begin{aligned} V(x=0) &= 0, \\ \text{and} \qquad V(x=3\,\text{m}) &= 150\,\text{N}. \end{aligned}$$

   The moment balance about point C, $\sum \vec{\boldsymbol{M}}_{\text{C}} = \vec{\mathbf{0}}$, gives

$$-M\hat{\boldsymbol{k}} - qx \cdot \frac{x}{2}\,\hat{\boldsymbol{k}} + M_{\text{applied}}\hat{\boldsymbol{k}} = \vec{\mathbf{0}}$$

   where the moment due to the distributed load is most easily computed by considering an equivalent concentrated load qx acting at $x/2$ from the end B. Thus,

$$\begin{aligned} \Rightarrow \quad M &= M_{\text{applied}} - q\frac{x^2}{2} \qquad (8.9) \\ &= 100\,\text{N·m} - 50\,\text{N/m} \cdot \frac{x^2}{2}. \qquad (8.10) \end{aligned}$$

   Thus, the bending moment varies quadratically with x along the length of the beam. In particular, the values at the ends are

$$\begin{aligned} M(x=0) &= 100\,\text{N·m} \\ \text{and} \qquad M(x=3\,\text{m}) &= -125\,\text{N·m}. \end{aligned}$$

   The shear force and the bending moment diagrams obtained from eqns. (8.8) and (8.9) are shown in fig. 8.34. Note that $M = 0$ at $x = 2\,\text{m}$ as given by eqn. (8.9).

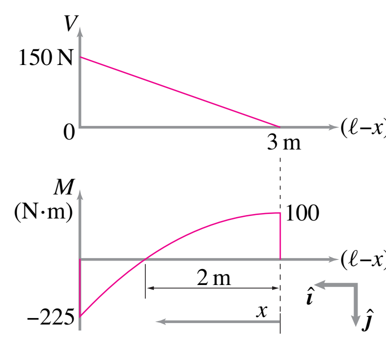


Figure 8.34

2. *$V(x)$ and $M(x)$ by superposition:* Now we consider the cantilever beam with only one type of load at a time. That is, we first consider the beam only with the uniformly distributed load and then only with the end couple. We draw the shear force and the bending moment diagrams for each case separately and then just *add them up*. That is superposition.

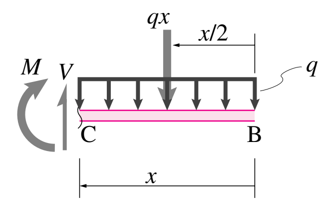


Figure 8.35

So, first let us consider the beam with the uniformly distributed load. The free-body diagram of a segment CB, obtained by cutting the beam at a distance x from the end B, is shown in fig. 8.35. Once again, from force balance, we get

$$V = qx \qquad \text{for } 0 \le x \le \ell \tag{8.11}$$

and from the moment balance about point C, $\sum \vec{\boldsymbol{M}}_C = \vec{\boldsymbol{0}}$, we get

$$M = -qx \cdot \frac{x}{2} = -q\frac{x^2}{2} \qquad \text{for } 0 \le x \le \ell. \tag{8.12}$$

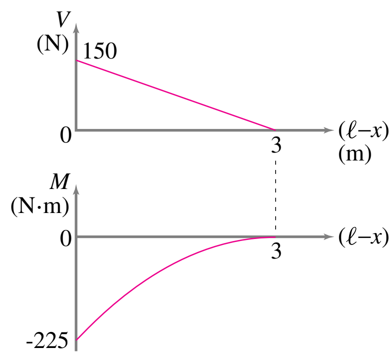


Figure 8.36

Figure 8.36 shows the plots of V and M obtained from eqns. (8.11) and (8.12), respectively, with the values computed from $x = 0$ to $x = 3\,\text{m}$ with $q = 50\,\text{N/m}$ as given.

Now we take the beam with only the end couple and repeat our analysis. A cut section of the beam is shown in fig. 8.37. In this case, it should be obvious that from force balance and moment balance about any point, we get

$$\begin{aligned} V &= 0 \\ \text{and} \qquad M &= M_{\text{applied}}. \end{aligned}$$

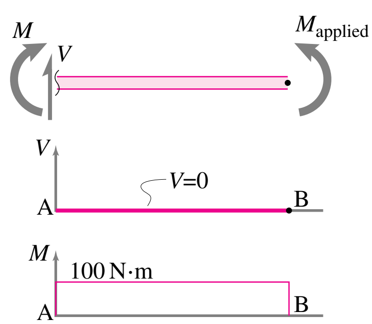


Figure 8.37

Thus, both the shear force and the bending moment are constant along the length of the beam as shown in fig. 8.37.

Now superimposing (adding) the shear force diagrams from Figs. 8.36 and 8.37, and similarly, the bending moment diagrams from Figs. 8.36 and 8.37, we get the same diagrams as in fig. 8.38.

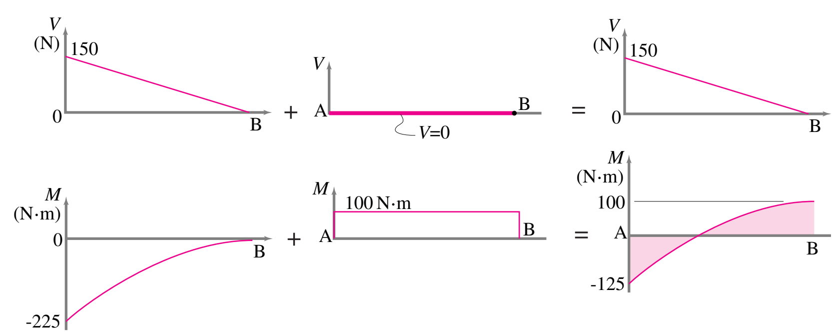
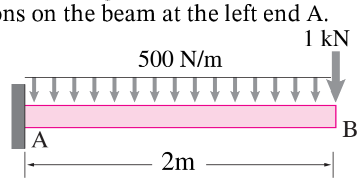


Figure 8.38:

8.1 Shear force, bending moment and tension diagrams

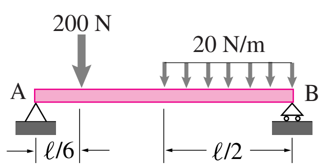
Preparatory Problems

8.1.1 A cantilever beam AB is loaded as shown in the figure. Find the support reactions on the beam at the left end A.



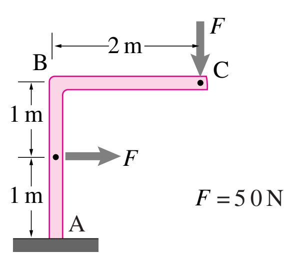
Problem 8.1.1

8.1.2 A simply supported beam AB of length $\ell = 6\,\mathrm{m}$ is partly loaded with a uniformly distributed load as shown in the figure. In addition, there is a concentrated load acting at $\ell/6$ from the left end A. Find the support reactions on the beam.



Problem 8.1.2

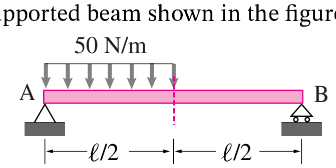
8.1.3 An (inverted) L-shaped frame is loaded with two equal concentrated forces of magnitude 50 N each as shown in the figure. Find the support reactions at A.



Problem 8.1.3

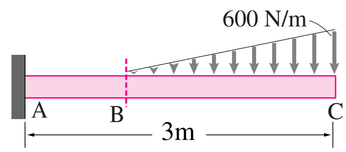
More-Involved Problems

8.1.4 Find the shear force and the bending moment at the mid section of the simply supported beam shown in the figure.



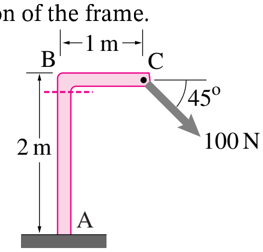
Problem 8.1.4

8.1.5 A cantilever beam ABC is loaded with a linearly variable distributed load along two thirds of its span. The intensity of the load at the right end is $600\,\mathrm{N}/m$. Find the shear force and the bending moment at section B of the beam.



Problem 8.1.5

8.1.6 Analyze the frame shown in the figure and find the shear force and the bending moment at the end of the vertical section of the frame.

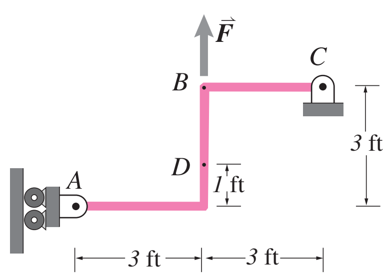


Problem 8.1.6

8.1.7 A force $F = 100\,\mathrm{lbf}$ is applied to the bent rod shown. Before doing any calculations, try to figure out the tension at D in your head.

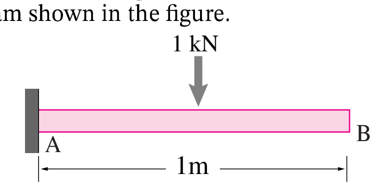
a) Find the reactions at A and C.

b) Find the tension, shear and bending moment at the section D. Check your answer against what you figured out in your head.



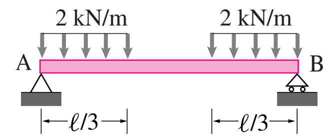
Problem 8.1.7

8.1.8 Draw the shear force and the bending moment diagram for the cantilever beam shown in the figure.



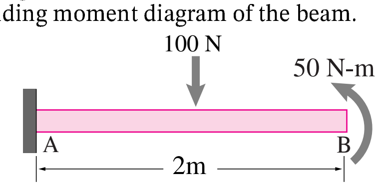
Problem 8.1.8

8.1.9 A simply supported beam AB is loaded along one thirds of its span from both ends by a uniformly distributed load of intensity $2\,\mathrm{kN/m}$. Draw the shear force and the bending moment diagram of the beam.



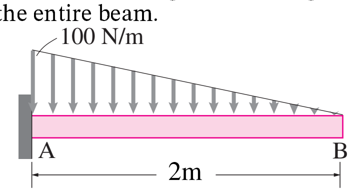
Problem 8.1.9

8.1.10 The cantilever beam shown in the figure is loaded with a concentrated load and a concentrated moment as shown in the figure. Draw the shear force and the bending moment diagram of the beam.



Problem 8.1.10

8.1.11 A cantilever beam AB is loaded with a triangular shaped distributed load as shown in the figure. Draw the shear force and the bending moment diagrams for the entire beam.

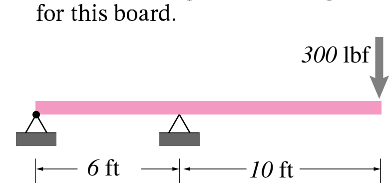


Problem 8.1.11

8.1.12 A regulation 16 ft diving board is supported as shown.

a) Where is the bending moment the greatest and how big is it there?

b) Draw a bending moment diagram for this board.



Problem 8.1.12

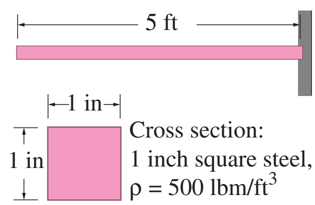
8.1.13 The cantilever steel beam is loaded by its own weight.

a) Find the bending moment and shear force at the free and at the clamped end.

b) Draw a shear force diagram

c) Draw a bending moment diagram

d) The tension stress σ in the beam at the top edge where it is biggest is given by $\sigma = 12M/h^3$ where $h = 1 \in$ for this beam. The strength (the maximum tension stress the material can bear) of soft steel is about $\sigma_{\max} = 30,000\,\mathrm{lbf}/\in^2$. What is the longest a beam with this cross section can be made and still not fail?

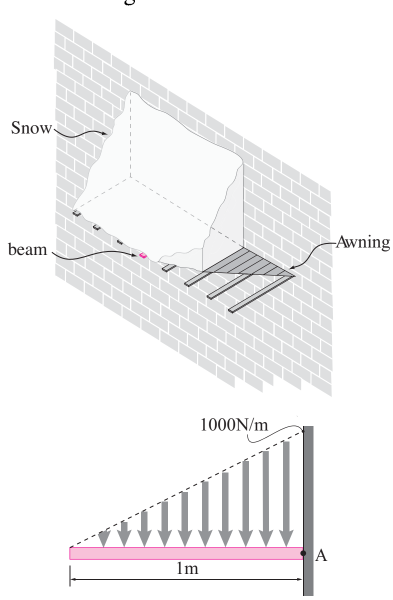


Problem 8.1.13

8.1.14 A snow loaded bus-stop awning (shown partially cut away) on the side of a building is supported by horizontal, cantilevered, beams. The loading that is carried by one beam is as shown below.

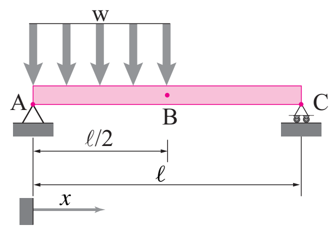
a) Find the reaction force and couple at the wall at A (the force and moment acting on one beam from the wall). *

b) Draw shear force and bending moment diagrams for the beam.



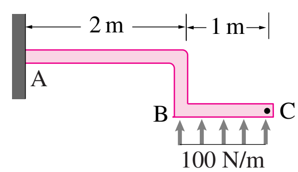
Problem 8.1.14

8.1.15 Draw shear and bending moment diagrams of the beam shown. Clearly label the values of the heights of the curves at jumps, kinks and local maxima (if and where they exist). *



Problem 8.1.15

8.1.16 A frame ABC is much like a cantilever beam with a short bent section of length 0.5 m. The frame is loaded as shown in the figure. Draw the shear force and the bending moment diagrams of the entire frame indicating how it differs from an ordinary cantilever beam.

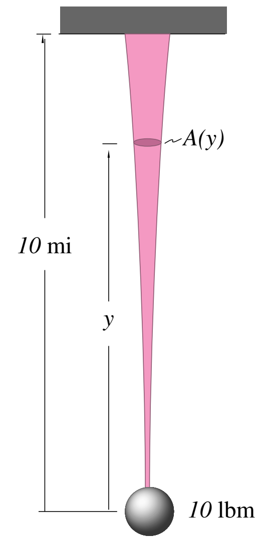


Problem 8.1.16

8.1.17 A 10 pound ball is suspended by a long steel wire. The wire has a density of about $500\,\mathrm{lbm}/\mathrm{ft}^3$. The strength of the wire (the maximum force per unit area it can carry) is about $\sigma_{\max} = 60,000\,\mathrm{lbf}/\in^2$.

a) First, neglecting the weight of the wire in the calculation of stress, what is the weight of wire needed to hold the weight?

b) Taking into account the weight of the wire in the load calculation, what is the weight of wire needed to hold the weight? *



Problem 8.1.17