7.2 Force amplification devices: Levers, wedges, toggles, gears, and pulleys

Simple objects can be connected in various arrangements for various purposes. Here we describe 5 machine fragments that can be used to amplify force. Most machines use these ideas in combination. It might help intuitive understanding of machines to recognize one of these methods in use, although precise categorization of every machine part as one or another of these devices is not possible.

A lever

One of the simplest machines, long understood, and even longer used by humans, is a lever (*e.g.,* , fig. 7.26 and the top of fig. 7.27). Although now we think of statics as a special case of dynamics, statics is, and was, a well-founded subject, independent of dynamics. The statics of a lever was well understood 50 generations before Newton discovered modern dynamics.

> An ideal lever is a rigid body held in place with a frictionless hinge and with two other applied loads.

The free-body diagram fig. 7.27 is the same whether the hinge is at point A, B or C[1].

Lots of things can be viewed as levers including, for example, a wheelbarrow, a hammer extracting a nail, a boat oar, one half of a pair of tweezers, a brake lever, a gear, and, most generally, any three-force body. Using the equilibrium relations on the free-body diagram in fig. 7.27 you get that

$$\frac{F_A}{a} = \frac{F_B}{b} = \frac{F_C}{c}$$

from which you could find the relation between any pair of the forces. In practice it is easier to use moment balance about an appropriate point than to memorize and recall this formula.

An ideal wedge

Wedges are a kind of machine. You see them used in exactly their wedge configuration, to split and separate things (see fig. 7.28). But screws are also, effectively wedges, with torque replacing the downwards force. For an ideal wedge, one neglects friction, effectively replacing sliding contact with rolling contact (see fig. 7.29ab). Although this approximation may not be accurate, it is helpful for building intuition. Because the key idea depends on force balance and not moment balance, for the free-body diagrams of fig. 7.29c, we have not specified the exact location of the contact

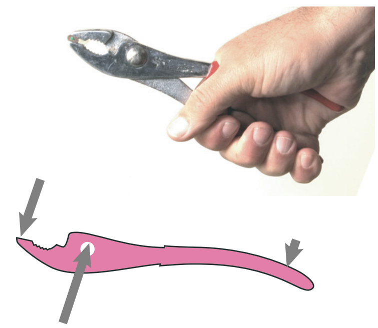

Figure 7.26: One half of a pair of pliers is one of many classic lever examples.

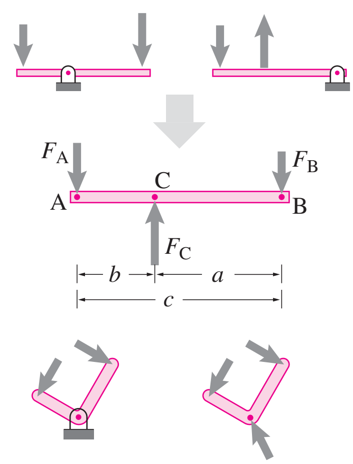

Figure 7.27: A lever can have the pivot in various places. The free-body diagram looks the same whether the pivot is at the left, middle, or right.

[1] People studied levers before mechanics was a quantitative subject. So they have a specialized antiquated vocabulary, classifying them according to which of A, B, or C is the pivot point and which of the other two forces you think of as input and which as the output: A 'class one' lever has the pivot in between; a 'class two' lever has the pivot at one end and the input force at the other; and a 'class three' lever has the pivot at one end and the input force in the middle.

Figure 7.28: A hatchet splitting wood is a classic wedge. In this case it is used dynamically, with the force coming from the deceleration of mass. But other more complex wood splitting devices push the wedge in slowly.

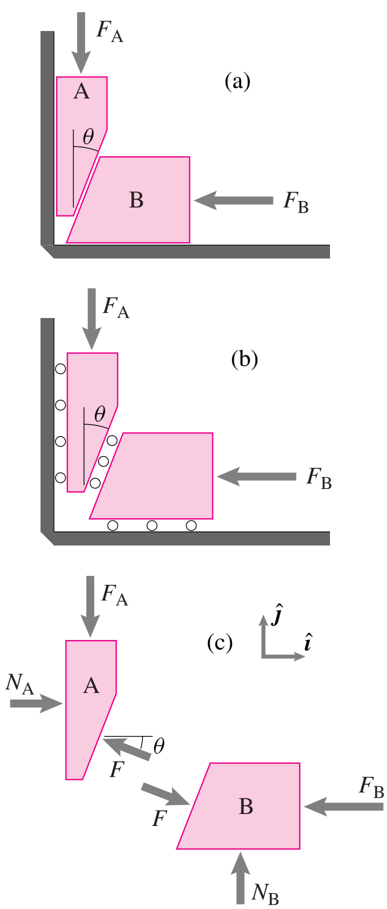

Figure 7.29: a) A wedge, b) treated as frictionless in the ideal case, c) free-body diagrams (assuming no friction)

forces . Neglecting gravity,

$$\begin{aligned}
&\text{For block A,} \left\{\sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \vec{\boldsymbol{0}}\right\}\cdot \hat{\boldsymbol{\jmath}} && \Rightarrow && -F_A + F\sin\theta &&= 0\\
&\text{For block B,} \left\{\sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \vec{\boldsymbol{0}}\right\}\cdot \hat{\boldsymbol{\imath}} && \Rightarrow && -F_B + F\cos\theta &&= 0
\end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_B = \frac{1}{\tan\theta} F_A.$$

Using the small-angle approximation that $\tan\theta \approx \theta$, we have that

> The force amplification of a wedge is roughly the reciprocal of the wedge angle (in radians).

To multiply the force F_A by 10 takes a wedge with a taper of $\theta = \tan^{-1} 0.1 \approx 6^{\circ}$. With this taper, an ideal wedge could also be viewed as a device to attenuate the force F_B by a factor of 10, although wedges are never used for force attenuation in practice, because friction often causes them to bind. A wedge with friction is considered in 7.12 on page 413.

A toggle

The classic *toggle* mechanism for amplifying force tends to have a ‘snap-through’ or bi-stable aspect which is used in the design of some electrical switches. Hence, perhaps, the two dictionary meanings of the word toggle: 1) a force amplifying mechanism, 2) a switch between two states (see fig. 7.32). The simplest version of the toggle mechanism is shown in fig. 7.30. The force amplification is $N/F = 1/\tan\theta$.

Usually the toggle concept is not used with a wall but with a pair of bars (fig. 7.31) Simple truss analysis shows the bar compressions are $-T = N/2\sin\theta$ and $N = F/2\tan\theta$. The toggle-like force amplification occurs for tension as well as compression. But, because of the oft-desirable snap-through and because the amplification increases as the applied force F moves down, the toggle is most often used in compression.

A toggle as lever and wedge. The distinction between toggles and wedges and levers is not precise. On the one hand the toggle is a lever where the lever arm of F is $\ell\cos\theta$ and the lever arm of N is $\ell\sin\theta$. On the other hand the toggle is sort of a rotary wedge with wedge angle θ.

Pulleys and Gears

Here we discuss a few more common machine components which are used to transmit and amplify or attenuate a force or moment.

Gears

One type of transmission is based on gears (fig. 7.34a). If we think of the input and output as the moments on the two gears, we find from the free-body diagram in fig. 7.34b that

$$\begin{aligned}
\text{For gear A,} \left\{\sum \vec{\boldsymbol{M}}_{\mathrm{i/A}} = \vec{\mathbf{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad -R_{\mathrm{A}}F + M_{\mathrm{A}} &= 0\\
\text{For gear B,} \left\{\sum \vec{\boldsymbol{M}}_{\mathrm{i/B}} = \vec{\mathbf{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad -R_{\mathrm{B}}F + M_{\mathrm{B}} &= 0
\end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad M_B = \frac{R_B}{R_A}M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B}M_B$$

depending on which you want to think of input and which as output. The force amplification or attenuation ratio is just the radius ratio, just like for a lever.

Because the spacing of gear teeth for both of a meshed pair of gears is the same, a gear's circumference, and hence its radius is proportional to the number of teeth. And formulas involving radius ratios can just as well be expressed in terms of ratios of numbers of teeth. The tooth ratio is not just used as an approximation to the radius ratio. Averaged over the passage of several teeth, it is exactly the reciprocal ratio of the turning rates of the meshed gears.

Two gears pulled out of a bigger transmission are shown in fig. 7.34c. Gear A has an inner part with radius R_{A_i} welded to an outer part with radius R_{A_o}. Gear B also has an inner part welded to an outer part.

Moment balance about A in the first free-body diagram in fig. 7.34d gives that $R_{A_i}F_A = R_{A_o}F$. You can think of the one gear as a lever (see fig. 7.33). Moment balance about B in the second free-body diagram gives that $R_{B_i}F = R_{B_o}F_B$. Combining, we get

$$F_B = \frac{R_{A_i}}{R_{A_o}}\frac{R_{B_i}}{R_{B_o}}F_A \quad \text{or} \quad F_A = \frac{R_{A_o}}{R_{A_i}}\frac{R_{B_o}}{R_{B_i}}F_B$$

depending on which force you want to find in terms of the other. The transmission attenuates the force if you think of F_A as the input and amplifies the force if you think of F_B as the input. If the inner gears have one tenth the radius of the outer gears then the multiplication or attenuation is a factor of 100.

Trains of gears can build up large net gear ratios. The ratio of the fastest to slowest gear in a common clock or mechanical watch is on the order of 10,000.

In some gear trains, like the example above, large torque amplification comes from a large ratio of concentrically welded gears. A large amplification can also come from differences rather than ratios. The designs based primarily on differences rather than ratios are called 'differentials', 'harmonic drives', or 'planetary gears'.

Example: Planetary gear with a large ratio

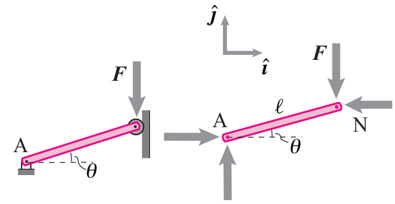

Figure 7.30: **A simple toggle mechanism.** Moment balance about point A gives that $N = F/\tan\theta$ so if θ is small N is large.

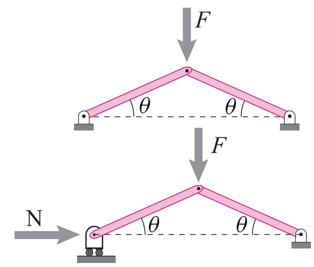

Figure 7.31: Two bars in a toggle mechanism.

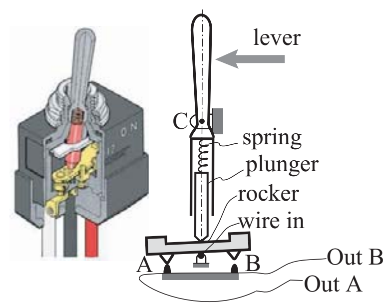

Figure 7.32: Cutaway drawing of a toggle switch and a simplified schematic of the idea. The lever, spring and plunger rotate about C. The spring is pre-stressed so that the plunger presses hard against the rocker. The plunger slides on the rocker (input line) which is pressed against the contact at outputs A or B. The relevant idea here is that the force of the plunger on the rocker can be much bigger than the input force on the lever (cutaway from NKK Switches online catalogue).

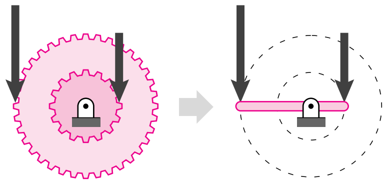

Figure 7.33: One gear may be thought of as a lever.

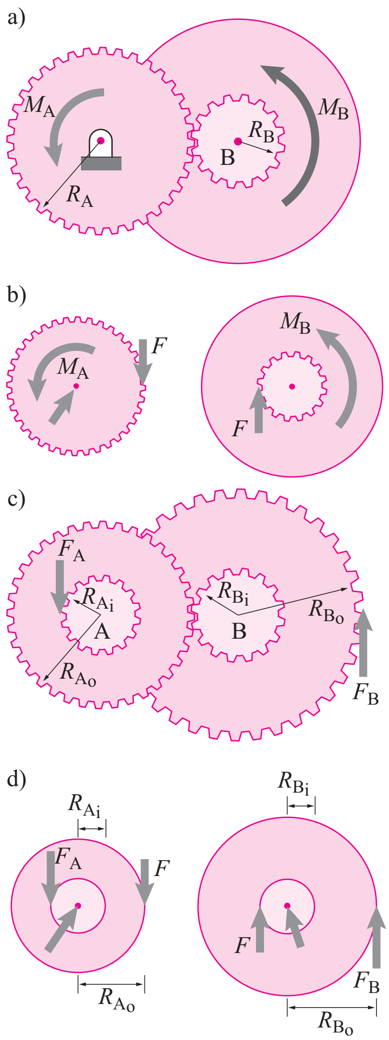

Figure 7.34: a) Two gear pairs pulled out of a transmission with forces on the teeth, b) Free-body diagrams, c) The same gear pair, but loaded with tooth-forces from unseen gears, d) the consequent free-body diagrams.

fig. 7.36 shows a gear design where the ratio of the input torque on the drive gear, to the output torque, on the spider can be huge. In particular, for the design shown, the torque ratio is approximately:

$$\frac{M_{out}}{M_{in}} \approx \frac{2}{R_D/R_R - 1}$$

where R_D is the ratio of the outer drive-gear to inner drive-gear radius and R_R is the ratio of the outer ring-gear to inner ring gear radius. Thus if the inner and outer drive gears have 49 and 50 teeth, respectively, and the inside and outside of the ring gear have 50 and 51 teeth, then the torque multiplication is nearly 5000. (See homework 7.2.17).

Pulleys

We have already studied a pulley as a single object (see page 261). Now we show, as you probably have learned a few times before in school, how to use pulleys to amplify or attenuate force. We assume pulleys are round, massless, and have frictionless bearings.

The key fact for statics analysis is

> For an ideal round pulley with negligible mass (or negligible angular acceleration) the tension on the cable is the same on both sides of the pulley:
>
> $$T_1 = T_2$$

The classic problem is shown in fig. 7.35a where you would like to use a pulley to make the task easier. Figures 7.35b-d show three possible uses of pulleys. If, at a glance, you can't see that these three designs are quite different in their effects you should puzzle them out slowly now.

Because the two tensions in the rope that wraps around the pulley are the same on both sides, the central rope has twice the tension. Design (b) gives no mechanical advantage but does allow one to pull down in order to lift the weight. Design (c) halves the required pulling force. Design (d), which might look superficially similar to (c) doubles the required pulling force, requiring 4 times the force of (c).

By using pulleys in combination, one can get various force attenuations and gains. The 10-pulley design in fig. 7.37 multiplies the force by about 1000.

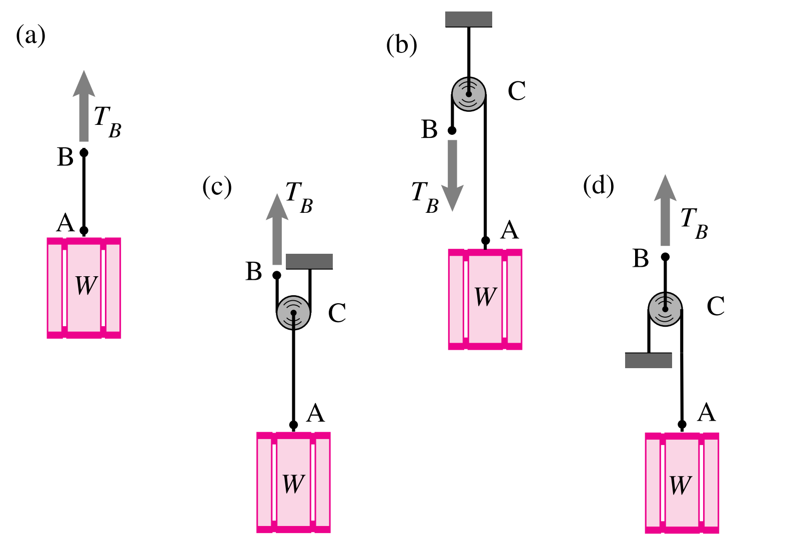

Figure 7.35: **Simple pulley arrangements.** **a)** Lifting a weight with no pulley, **b)** a pulley lets you pull down instead of up, **c)** a pulley halves the needed pull, **d)** a pulley doubles the needed pull.

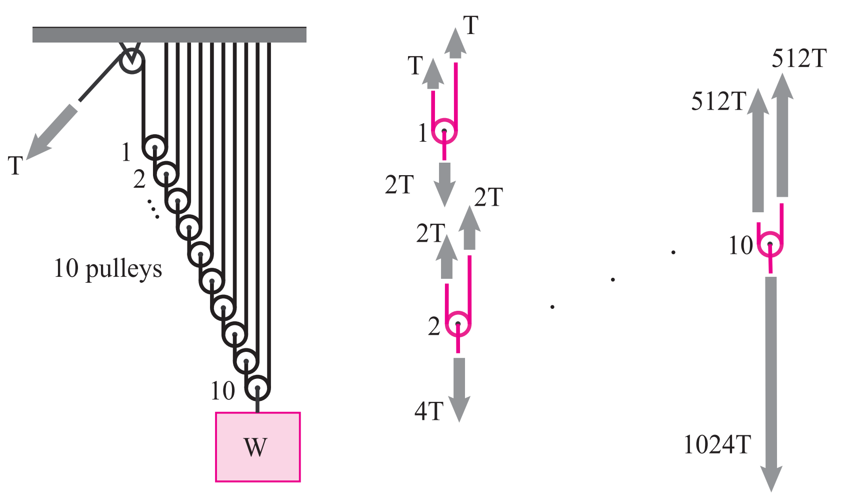

Figure 7.37: **Archimedes pulley.** A pulley arrangement sometimes commonly attributed to Archimedes. With 10 pulleys, he could lift a weight of W with a pull of about $T = 2^{-10}W \approx W/1000$. With 20 pulleys the force amplification is by more than a million. With 67 pulleys he, assuming he was as strong as the average person, could hoist a rock the size of the moon.

General force amplification concepts

If a mechanism generates a large force ratio (output/input), this usually corresponds to a large ratio in some geometric quantities. For a lever we have the ratio of two lever arms. For a wedge, the small wedge angle, and for a toggle also a small angle.

More precisely

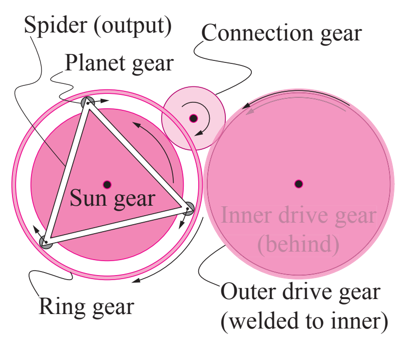

Figure 7.36: **A high-gain planetary gear system,** so named because the 'planet' gears go around the 'sun' gear. Often planetary gears depend on the sun or the spider or the ring not rotating. In this design, all three rotate. A CCW torque is applied to the inner drive gear which is welded to the outer drive gear (they are concentric and have nearly the same radius). The outer drive gear turns the ring gear CW. The inner drive gear turns the connection gear CW which turns the sun gear CCW. The three planet gears spin between the inside of the ring gear and the outside of the sun gear. If the inside of the ring gear moves a little faster down than the outside of the sun gear moves up, then the spider turns CW with the center of the planet gears. As drawn, this motion would be *much* slower than the motion of the drive gears.

For a frictionless transmission, the ratio of the input force to the output force is the reciprocal of the ratio of input motion to output motion.

For a high-gain lever, the handle moves much further than the load. For a narrow wedge, the slip distance is much bigger than the spreading distance. For a toggle the motion of the compressed end is much smaller than that of the applied load. That the force amplification is identical to the motion attenuation follows from energy conservation: the work into the mechanism equals the work out.

So, when Archimedes pulls in a kilometer of rope while lifting a rock, the moon, with 67 pulleys, that huge rock moves 2^{-67} km or about one hundred millionth of a nanometer. A big force, and a small motion. This is as it must be, because Archimedes only supplies so much work. If the transmission greatly amplifies force it must greatly attenuate motion. Similarly with levers. When Archimedes famously said:

> *Give me a lever long enough and a fulcrum on which to place it,*
> *and I shall move the world,*

he was careful not to say how far he would move it. People might have been less impressed if they knew that the distance of movement was far below the resolving power of all past, present and future microscopes.