6.4 Frames and structures

Although trusses are good, they are not good enough for all purposes. Nor are they even always good-enough models of very truss-looking structures. *Frames* are structures that are more general than trusses. In a truss, every bar is a two-force body. In a more general structure or frame, one or more components is not a two-force body. The analysis of non-truss frames is generally less formulaic, and thus more subtle, than is the method of joints for trusses.

Example: A-frame ladder.
The two-diagonal parts of an A-frame ladder are not two-force bodies and thus the ladder, triangular as it looks, is not a truss. And truss analysis is not appropriate.

A 2D frame is a collection of rigid objects connected by hinges. A special case is a truss, where each bar has exactly two hinges.

Figure 6.52: A ladder is well-modeled as a collection of rigid pieces connected by pin joints ...

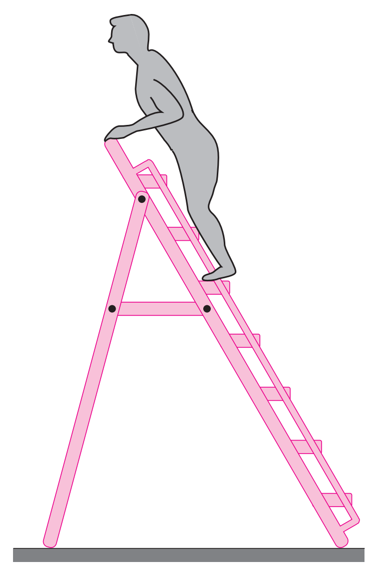


Figure 6.53: ... but A-frame ladder is not a truss because the bars are not all two-force members. The piece the person is standing on has forces at more than two places: the hands, the top pin, the middle brace, the feet and the ground. The left diagonal leg also has three ($3 > 2$) forces acting on it.

One could generalize this definition slightly to include also sliding joints. And one could narrow the definition, to exclude mechanisms, by insisting that the structure be rigid. Being precise about these cases is not important because the analysis methods are the same in all cases.

The overall mechanics recipe applies to frames, of course: a) draw free-body diagrams, b) apply the laws of mechanics to each free-body diagram, and c) solve the mechanics equations for unknowns of interest. For trusses, the free-body diagrams of each bar, with the 3 equilibrium equations (6 in 3D) just yield the "two-force" body result that the bar has equal tensions at the two ends. Because there was no more to learn from the bar free-body diagrams we didn't even draw them. Instead we used the bar tensions as forces on free-body diagrams of the joints. It's as if the bars were just a means to mediate action-reaction pairs between joints.

For more general frameworks we have to pay full respect to the free-body diagrams of all of the parts, not just the pins. At least for all of the parts that are not two-force bodies. Here is the analysis of frameworks recipe:

- Draw free-body diagrams of
  - the whole structure; and
  - the separate parts of the structure; and
  - collections of parts of the structure if such seems likely to be fruitful;
  - Use the principle of action and reaction in the free-body diagrams so that one action-reaction pair has only one unknown;

- For each free-body diagram, write equilibrium conditions (force and moment balance).
- Solve the equilibrium equations for desired unknowns.

For the parts that are not two-force bodies, we will not know the directions of the interaction forces *a priori*, that's why the method of joints is not used for frames ①. Naturally one can be on the look out for shortcuts:

① Conversely, we could have analyzed trusses the way we are now going to analyze frames. This seldom-used approach to trusses, the 'method of bars and pins', is discussed in box 6.1 on page 4.

- for any two-force bodies assign an equal-valued tension to each end (thus eliminating any need or use for equilibrium equations for that object)
- consider each pin as part of one of the bodies to which it is connected (*i.e.,* there is no need to draw a separate FBD of the pin).
- To minimize calculation, look for a subset of the equilibrium equations that
  - contains your unknowns of interest, and
  - has as many unknowns as scalar equations, and
  - contains as few equations as possible.

Our general goal here is to find the reaction forces, the interaction forces and the 'internal' forces in the components of a statically determinate structure.

Example: An X structure
Two bars are joined in an 'X' by a pin at J. Neither of the bars is a two-force body so a free-body diagram of the 'joint' at J, made by cutting and leaving stubs as we did with trusses, has 12 unknown force and moment components.

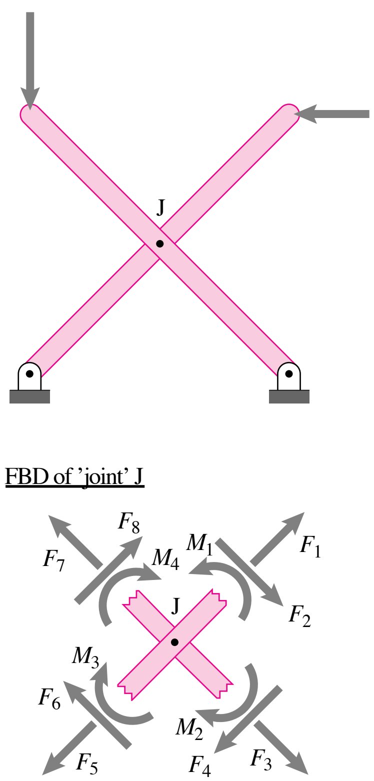


Figure 6.54: A simple frame and a free-body diagram of a joint. Unlike the case for trusses, free-body diagrams of joints (pins) are nearly useless in the study of more general frames. There are too many unknown forces and moments acting at the pin.

Instead of drawing free-body diagrams of the connections, our approach here is to draw free-body diagrams of each of the structure's or machine's parts. Sometimes, as was the case with trusses, it is also useful to draw a free-body diagram of a whole structure or of some multi-piece part of the structure

Static determinacy

A statically determinate structure has

- a solution for all possible applied loads, and
- only one solution, and
- this solution can be found by using equilibrium equations applied to each of the pieces.

Not all practical structures are statically determinate. Some structures are rigid but redundant, thus precluding finding all unknowns from statics. Some structures cannot carry all loads, but can carry the loads of interest (e.g., a vertical cable that can usefully carry a weight but cannot carry a side load). Nonetheless, for starters here we emphasize determinate structures. The basic counting formula

$$\text{number of equations} = \text{number of unknowns}$$

is necessary for determinacy but does not guarantee determinacy. How to count? Frameworks in 2D have three equilibrium equations for each object that isn't just a point (in which case there are only two equilibrium equations). There are two unknown force components for every pin connection, whether to the ground or to another piece. And there is one unknown force component for every roller connection whether to the ground or between objects. Applied forces do not count in this determinacy check, even if they are unknown.

Example: 'X' structure counting
In the 'X' structure above we can count as follows.

$$\begin{aligned}
\text{number of equations} &\stackrel{?}{=} \text{number of unknowns}\\
\text{(3 eqs per bar)} \cdot \text{(2 bars)} &\stackrel{?}{=} \text{(2 unknown force components per pin)} \cdot \text{(3 pins)}\\
\text{6 eqs} &\stackrel{\surd}{=} \text{6 unknown force components}
\end{aligned}$$

So the 'X' structure passes the counting test for static determinacy.

Redundant structures

A redundant structure can carry whatever loads it can carry in more than one way. If not also indeterminate, a redundant structure has fewer equilibrium equations than unknown reaction or interaction force components. Finding all the reaction components is only possible if one models the deformation, a topic for more advanced structural mechanics. **Example: Over-braced 'X'**

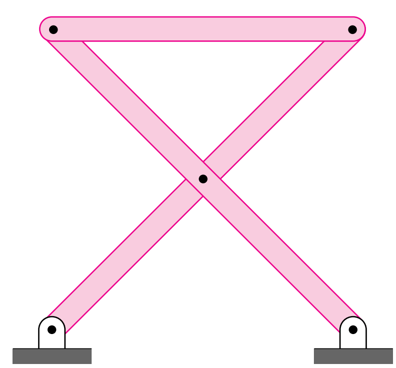


Figure 6.55: Overbraced X. A rigid frame that is not statically determinate.

The structure is evidently redundant. Why? Because it has a bar added to a structure that was already statically determinate. By counting we get

$$\begin{aligned}
\text{number of equations} &\stackrel{?}{=} \text{number of unknowns}\\
3 \cdot \underbrace{\text{(number of bars)}}_{3} &\stackrel{?}{=} 2 \cdot \underbrace{\text{(number of joints)}}_{5}\\
\text{9 eqs} &< \text{10 unknown force components}
\end{aligned}$$

thus demonstrating redundancy.

6.1 The 'method of bars and pins' for trusses

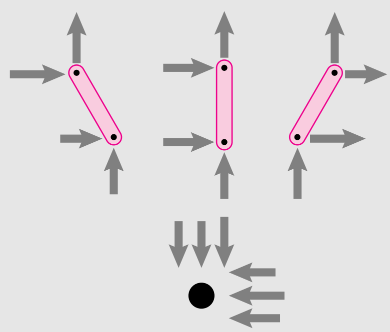
This is an aside for those who wonder why truss analysis seems so different than frame analysis.

Trusses are simple frameworks. So the methods used for more general frameworks should work for trusses. They do. The resulting method, which is essentially never used in such detail, we will call 'the method of bars and pins'.

In the method of bars and pins you treat a truss like any other structure. You draw a free-body diagram of each part.

One approach: treat the pins as parts. One approach is to draw free-body diagrams of each pin also. You use the principle of action and reaction to relate the forces on the different bars and pins. Then you solve the collection of equilibrium equations.

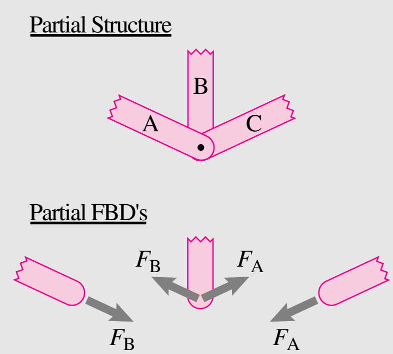
Consider one joint of a truss where three bars meet at a hinge (pin). Below are free-body diagrams of the three bars and of the pin. Assuming a frictionless round pin at the hinge, all the bar forces on the pin pass through its center.



Thus, in 2D, you get two equilibrium equations for each pin and three for each bar. If you apply the three bar equations to a given bar you find that it obeys the two-force body relations. Namely, the reactions on the two bar ends are equal and opposite and along the connecting points. Now application of the pin equilibrium equations is identical to the joint equations we had previously. Thus, the 'method of bars and pins' reduces to the method of joints in the end.

Approach two: draw FBDs of just the bars. Another approach is to associate each pin with one of the bars to which it is attached. Then just think of a truss as bars that are connected with forces and no moments. Draw free-body diagrams of each piece, use the principle of action and reaction, and write the equilibrium equations for each bar. This is the approach that is used in this section for other structures.

If three bars A, B, and C are connected to a pin, consider the pin as part of, say, A. Then consider action-reaction pairs between A and B, and between A and C, but not between B and C. Similarly if there are four or more bars, consider interactions between each bar and the one-bar that has the pin.



Determinate equations. In all cases, if the truss is statically determinate the equilibrium equations generated from the free-body diagrams above will produce a solvable set of linear algebraic equations. But, in all cases above, these will not be the more minimal set of equations we generated in the method of joints in the truss analysis. The methods of this box work, they are just harder to implement.

SAMPLE 6.12 The braced X-frame shown in the figure carries two vertical loads $F_1 = 2\,\text{kN}$ and $F_2 = 3\,\text{kN}$. Points G and H are directly above points A and B respectively. If $d = h = 2\,\text{m}$, find the tension in the brace CD.

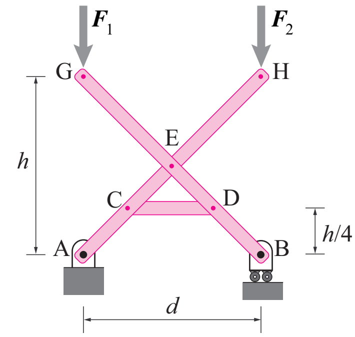


Figure 6.56

Solution The brace CD is pinned to the X-frame at C and D. The only loads acting on the brace are at its ends C and D. Therefore, it is a two-force body. Let us assume that the tension in brace is C_x. We need to find C_x under the given loads.

The free-body diagram of the whole frame is shown in fig. 6.57. Since the frame is supported by a hinge at A and a roller at B, there are three scalar support reactions acting on the frame. We can now determine all the three reactions from the static analysis of the frame:

$$
\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad A_x = 0 \\
\sum M_A = 0 \quad &\Rightarrow \quad B_y d - F_2 d = 0 \\
&\Rightarrow \quad B_y = F_2 \\
\sum F_y = 0 \quad &\Rightarrow \quad A_y = F_1 + F_2 - B_y = F_1.
\end{aligned}
$$

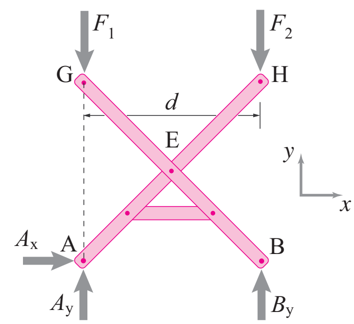


Figure 6.57

Thus all the reactions are known. Now we can analyze either bar AH or bar BG (the analysis is identical) to determine the tension C_x in the brace. The free-body diagram of bar AH is shown in fig. 6.58. Since we are only interested in C_x, we can carry out moment balance about point E ($\sum M_E = 0$) to give

$$
\begin{aligned}
C_x \frac{d}{4} - F_2 \frac{d}{2} - A_y \frac{d}{2} &= 0 \\
\Rightarrow \quad C_x &= 2(F_2 + A_y) \\
&= 2(F_2 + F_1) \\
&= 2(3k\,\text{N} + 2\,\text{kN}) \\
&= 10\,\text{kN}.
\end{aligned}
$$

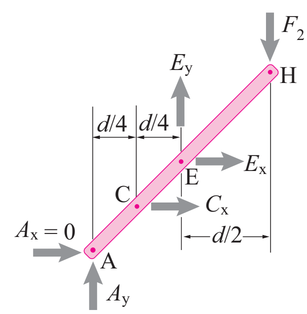


Figure 6.58

Thus the tension in the brace is twice the total load on the structure.

Tension in brace CD = 10 kN

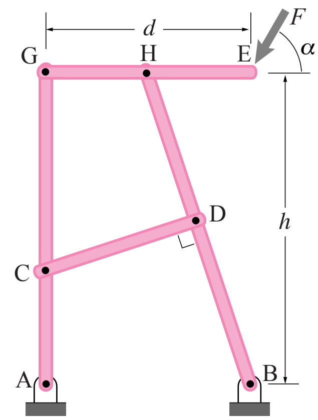


Figure 6.59

SAMPLE 6.13 The frame shown in the figure is supported by hinges at both A and B. Bar GE is as long as the base AB and bar BH is pinned to GE at the mid point H. Brace CD is pinned at D, the mid-point of bar BH, and is orthogonal to bar BH. The load on the structure, $F = 1\,\text{kN}$, is applied at E, at an angle $\alpha = 60°$. Given that $d = 2\,\text{m}$, $h = 3\,\text{m}$, find the forces on the inclined bar BH and the support reactions at A and B.
[Note: Usually, determinate framed structures are made up of overhangs and extensions on a rigid triangle. This structure is an example of a frame that does not contain any rigid triangle.]

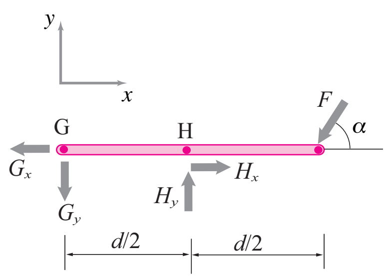


Figure 6.60

Solution The given structure has hinges at both A and B. Therefore, there are four scalar support reactions, two each at A and B. So, from the free-body diagram of the whole structure, we cannot determine all support reactions. In fact, the free-body diagram of each rod will have more than three unknown forces (you can check this mentally). Thus, we are not likely to find all unknown forces on a bar without analyzing other bars. Since bar CD is a two-force member bar, it only contributes one scalar force, the tension in this rod. Now, there are two unknown scalar forces at each pin joint, A, B, G, and H, and one force at C and D (the same force). Thus we have nine unknown scalar forces. We have three bars AG, GE, and BH, each with three independent scalar equations of static equilibrium. Thus we have nine independent equations in nine unknowns. Therefore, we can solve for all the unknown forces.

Consider the free-body diagram of bar GE. The static equilibrium of this bar requires

$$
\begin{aligned}
\sum M_H = 0 \quad &\Rightarrow \quad G_y(d/2) - F\sin\alpha(d/2) = 0 \\
&\Rightarrow \quad G_y = F\sin\alpha \\
\sum F_y = 0 \quad &\Rightarrow \quad -G_y + H_y - F\sin\alpha = 0 \\
&\Rightarrow \quad H_y = F\sin\alpha + G_y = 2F\sin\alpha \\
\sum F_x = 0 \quad &\Rightarrow \quad H_x - G_x - F\cos\alpha = 0. \qquad (6.21)
\end{aligned}
$$

Thus we have found G_y and H_y but only a relationship between G_x and H_x. Since G_x and H_x are colinear, we cannot solve for them from the static analysis of bar GE alone. Now, let us consider bar AG (or bar BH; does not make a difference). The equilibrium analysis of this bar gives

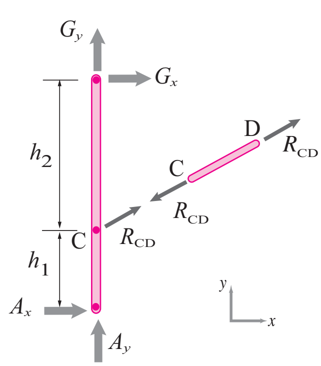


Figure 6.61

$$
\begin{aligned}
\sum F_y = 0 \quad &\Rightarrow \quad A_y + G_y + R_{CD}\sin\theta = 0 \qquad (6.22)\\
\sum M_C = 0 \quad &\Rightarrow \quad A_x h_1 - G_x h_2 = 0 \qquad (6.23)\\
\sum F_x = 0 \quad &\Rightarrow \quad A_x + G_x + R_{CD}\cos\theta = 0. \qquad (6.24)
\end{aligned}
$$

Since none of these equations contains only one unknown, we cannot solve for these forces from the equilibrium equations of bar AG alone. Note that we have written these equations in terms of h_1, h_2, and θ, thus far, undetermined geometric variables. However, we can easily find them from the given geometry. Now let us analyze bar BH.

$$
\begin{aligned}
\sum F_y = 0 \quad &\Rightarrow \quad B_y - H_y - R_{CD}\sin\theta = 0 \qquad (6.25)\\
\sum F_x = 0 \quad &\Rightarrow \quad B_x - H_x - R_{CD}\cos\theta = 0 \qquad (6.26)\\
\sum M_D = 0 \quad &\Rightarrow \quad (H_x + B_x)\frac{h}{2} + (H_y + B_y)\frac{d}{4} = 0 \qquad (6.27)
\end{aligned}
$$

So, now we have seven independent equations, eqns. (6.21)–(6.27), in seven unknowns — $A_x, A_y, B_x, B_y, R_{CD}, G_x$, and H_x (we have already solved for G_y and H_y). We can solve these seven equations on a computer.

Before we go to the computer, let us find the undetermined geometric quantities h_1 and h_2. From fig. 6.63, we see that

$$\begin{aligned} h_1 &= \frac{h}{2} - \Delta \\ h_2 &= \frac{h}{2} + \Delta \end{aligned}$$

where $\Delta = d' \tan\theta$, $d' = 3d/4$, and $\theta = \tan^{-1}(d/2h)$. Now, we are ready to solve the seven equations on a computer.

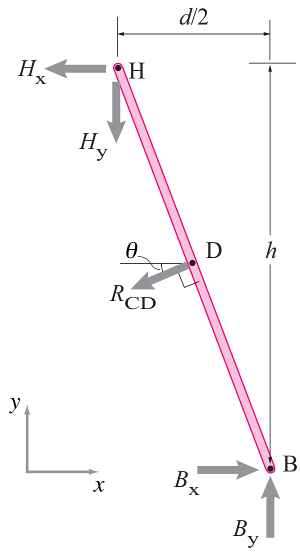


Figure 6.62

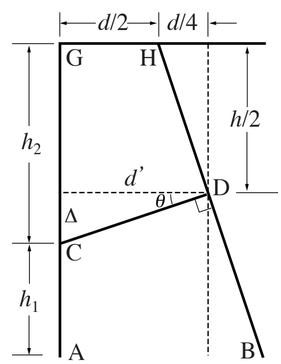


Figure 6.63: The geometry of h_1 and h_2. Since $d' = 3d/4$, $\Delta = d' \tan\theta$ and $\tan\theta = \frac{d/2}{h}$, we get $\Delta = \frac{3d}{4}\frac{d}{2h} = \frac{3d^2}{8h}$ and $h_1 = h/2 - \Delta$, $h_2 = h/2 + \Delta$.

```
% input given quantities
d = 2; h = 3; F = 1; alpha = pi/3;
% Define other used quantities in the equations
Delta = 3*d^2/(8*h);
h1 = h/2 - Delta;  h2 = h/2 + Delta;
theta = arctan(0.5*d/h);
% Input equations
eqset = { Hx - Gx = F*cos(alpha)
          Ay + RCD*sin(theta) = -F*sin(alpha)
          Ax*h1 - Gx*h2 = 0
          Ax + Gx + RCD*cos(theta) = 0
          By - RCD*sin(theta) = 2*F*sin(alpha)
          Bx - Hx - RCD*cos(theta) = 0
          (Hx+Bx)*h/2 + By*d/4 = -F*d/2*sin(alpha) }

solve eqset for Ax, Ay, Bx, By, Gx, Hx, and RCD
```

Including the values of G_y and H_y obtained from the first two equations of equilibrium of bar GE, we get the following values for all unknown forces from the computer solution.

$$\begin{aligned} A_x &= 3.23\,\text{kN} \\ A_y &= 0.75\,\text{kN} \\ B_x &= -2.73\,\text{kN} \\ B_y &= 0.12\,\text{kN} \\ R_{CD} &= -5.11\,\text{kN} \\ G_x &= 1.62\,\text{kN} \\ G_y &= 0.87\,\text{kN} \\ H_x &= 2.12\,\text{kN} \\ H_y &= 1.73\,\text{kN} \end{aligned}$$

$R_{CD} = -5.11\,\text{kN}$

SAMPLE 6.14 An easy-chair uses a curved frame as shown in the small picture in fig. 6.64. To simplify geometry, we can model the chair with straight bars as shown in the figure. Of special significance is the small pin at E that is rigidly attached to bar CDH and slides with negligible friction on bar ABD (see inset). This pin keeps the chair from collapsing and bears a large load. Assume the pin is $\epsilon = 2.5$ cm away from joint D towards B along bar ABD.

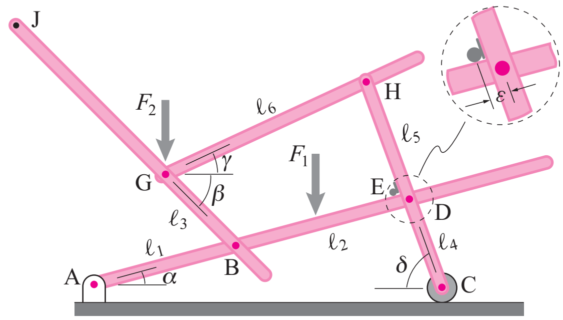


Figure 6.64: Dimensions: $\ell_1 = 45$ cm, $\ell_2 = 60$ cm, $\ell_3 = 30$ cm, $\ell_5 = 40$ cm, $\epsilon = 2.5$ cm, $\alpha = 15°$, $\beta = 45°$ and $\delta = 70°$. Loads: $F_1 = 500$ N and $F_2 = 200$ N. F_1 acts in the middle of bar segment BD and F_2 acts at G.

Solution First, referring to the figures, there is still some geometry needed. We can find γ and ℓ_6 by looking at the position of G relative to H two ways. One is directly from H to G. The other is going from H to E, then E to B, and then B to G. Writing this as a vector equation we get:

$$\vec{r}_{G/H} = \ell_6 \begin{bmatrix} \cos(\gamma) \\ \sin(\gamma) \end{bmatrix} = \begin{bmatrix} \ell_3 \cos(\beta) + \ell_2 \cos(\alpha) - \ell_5 \cos(\delta) \\ -\ell_3 \sin(\beta) + \ell_2 \sin(\alpha) - \ell_5 \sin(\delta) \end{bmatrix}.$$

The right hand side has all given quantities. Look at this vector equation as two scalar equations. Square the first and add it to the square of the second gives $\ell_6 \approx 72.85$ cm. Applying this to either of the two equations then gives $\gamma \approx 25.97°$.

Because the chair is supported by a hinge at A and a roller at B, there are three scalar support reactions. So, we can determine them from the static analysis of the whole chair frame. The free-body diagram of the chair is shown in fig. 6.65. The moment and force equilibrium equations give

$$\begin{aligned} \sum F_x = 0 \quad &\Rightarrow \quad A_x = 0 \\ \sum M_A = 0 \quad &\Rightarrow \quad C_y(d_1 + d_2) - F_1(d_1) - F_2(d_3) = 0 \\ &\Rightarrow \quad C_y = \frac{F_1 d_1 + F_2 d_3}{d_1 + d_2} \\ \sum F_y = 0 \quad &\Rightarrow \quad A_y = F_1 + F_2 - C_y. \end{aligned}$$

From the given geometry,

$$\begin{aligned} d_1 &\approx 72.44 \text{ cm} \\ d_2 &\approx 38.87 \text{ cm} \\ d_3 &\approx 22.25 \text{ cm}. \end{aligned}$$

Substituting these dimensions above with their numerical values, we get

$$C_y = 365\text{ N}, \quad \text{and} \quad A_y = 335\text{ N}.$$

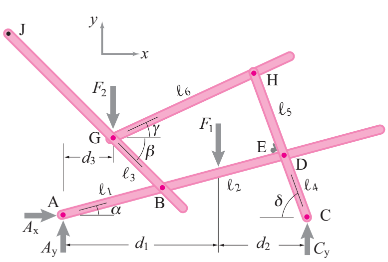


Figure 6.65: Free-body diagram of the whole chair.

Some geometry:

$d_1 = (\ell_1 + \ell_2/2)\cos\alpha$,

$d_2 = (\ell_2/2)\cos(\alpha) + \ell_4 \cos\delta$, and

$d_3 = \ell_1 \cos\alpha - \ell_3 \cos\beta$.

From the law of sines we also have:

$$\ell_4 = \frac{(\ell_1 + \ell_2)\sin(\alpha)}{\sin(\delta)} \approx 28.92\text{ cm}.$$

The support reactions are thus determined. To find the force on the pin E, we can use either bar ABD or bar CDH. In either case however, we have more unknown forces on the bars than we can determine from the equilibrium equations of that bar alone. So, we will have to use equilibrium of some other bar as well. Note that bar GH is a two-force body. Therefore, the tension in this rod can be shown as a single scalar force R_{GH}. Let us now analyze the equilibrium of bar BGJ since it has only three unknown forces on it (see fig. 6.66). The moment and force equilibrium equations give

$$\begin{aligned}
\sum M_G = 0 \quad &\Rightarrow \quad F_2(\ell_3 \cos\beta) - R_{GH}(\ell_3 \sin(\gamma+\beta)) = 0 \\
&\Rightarrow \quad R_{GH} = \frac{F_2(\cos\beta)}{\sin(\gamma+\beta)} = 150\,\text{N}. \\
\sum F_x = 0 \quad &\Rightarrow \quad B_x = R_{GH}\cos\gamma = 134\,\text{N} \\
\sum F_y = 0 \quad &\Rightarrow \quad B_y = -B_x = -134\,\text{N} \quad (\text{Note } \beta = 45°)
\end{aligned}$$

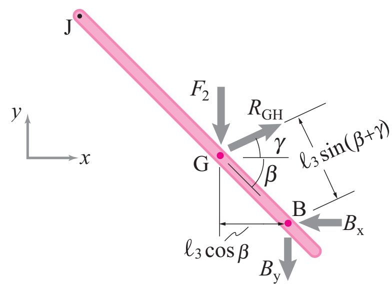


Figure 6.66

Now that we know A_x, A_y, B_x and B_y, we can analyze bar ABD and determine the rest of the unknown forces on it including the force in the pin E, R_E (see the free-body diagram in fig. 6.67):

$$\begin{aligned}
\sum M_D = 0 \quad &\Rightarrow \quad -A_y(d_4+d_5) - B_y d_5 + B_x h + F_1 d_6 + R_E\epsilon = 0 \\
&\Rightarrow \quad R_E = \frac{A_y(d_4+d_5) + B_y d_5 - B_x h - F_1 d_6}{\epsilon} \\
\sum F_x = 0 \quad &\Rightarrow \quad B_x + D_x + R_E \sin\alpha = 0 \\
&\Rightarrow \quad D_x = -B_x - R_E \sin\alpha \\
\sum F_y = 0 \quad &\Rightarrow \quad D_y = R_E\cos\alpha - A_y - B_y + F_y.
\end{aligned}$$

From geometry,

$$\begin{aligned}
d_4 &= \ell_1 \cos\alpha \\
d_5 &= \ell_2 \cos\alpha \\
d_6 &= d_5/2 = (\ell_2/2)\cos\alpha \\
h &= \ell_2 \sin\alpha.
\end{aligned}$$

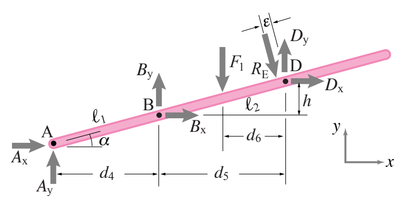


Figure 6.67: Free-body diagram of bar ABD. Note that $d_4 = \ell_1\cos\alpha$, $d_5 = \ell_2\cos\alpha$, $d_6 = d_5/2 = (\ell_2/2)\cos\alpha$, and $h = \ell_2 \sin\alpha$.

Substituting these variables with their numerical values above, we get

$$R_E \approx 3826\,\text{N}, \qquad D_x \approx -1125\,\text{N}, \qquad \text{and} \qquad D_y \approx 3826\,\text{N}.$$

$$A_x = 0, A_y = 335\,\text{N}, C_y = 365\,\text{N}, R_E = 3826\,\text{N}$$

Check: equilibrium of bar CDH

We can check for consistency by seeing if the equilibrium equations for bar CDH are satisfied, and they are (arithmetic not shown).

$$\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad -D_x - R_E\sin(\alpha) - R_{GH}\cos(\gamma) = 0 \quad &(\text{Checks!}) \\
\sum F_y = 0 \quad &\Rightarrow \quad C_y - D_y + R_E\cos(\alpha) - R_{GH}\sin(\gamma) = 0 \quad &(\text{Checks!}) \\
\sum M_D = 0 \quad &\Rightarrow \quad \ell_4\cos(\delta)C_y - \epsilon R_E + \ell_5\sin(\gamma+\beta)R_{GH} = 0 \quad &(\text{Checks!})
\end{aligned}$$

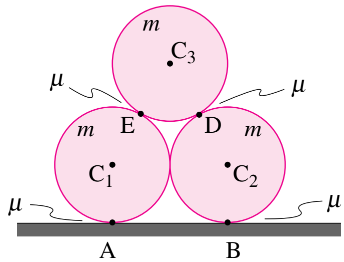


Figure 6.68

SAMPLE 6.15 A stack of three cylinders in static equilibrium? Three identical cylinders, each of mass m and radius R, are stacked such that the top cylinder rests on the lower two cylinders. The two cylinders at the bottom do not touch each other. The coefficient of friction, between the cylinders and between the cylinders and the ground, is μ. Find the minimum value of μ so that the three cylinders are in static equilibrium.

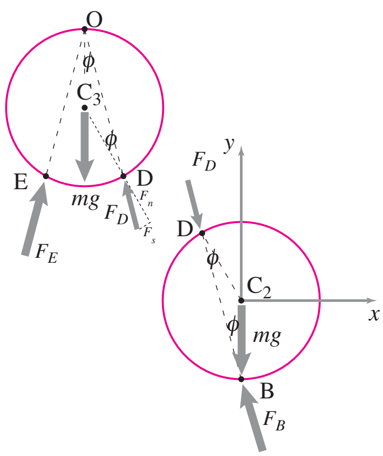


Figure 6.69

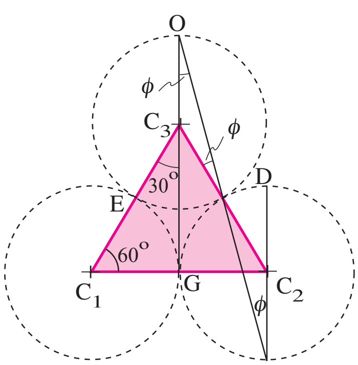


Figure 6.70

Solution We seek the forces for equilibrium. Knowing the forces, we can find the needed friction coefficient μ.

The free-body diagrams of the upper cylinder and the lower right cylinder (why the right cylinder? No particular reason.) are shown in fig. 6.69. The contact forces, $\vec{F}_E$ and $\vec{F}_D$, act on the upper cylinder at points E and D, respectively. Each contact force can be thought of as a force normal N to the contact surface plus a tangential friction force F_s. From the free-body diagrams, we see that each cylinder is a three-force body. Therefore, all the three forces — the two contact forces and the force of gravity — must be concurrent. This requires, for both cylinders, that the two contact forces must intersect on the vertical line passing through the center of the cylinder (the line of action of the force of gravity). Now, if we consider the free-body diagram of the lower right cylinder, we find that force $\vec{F}_D$ has to pass through point B since the other two forces intersect at point B. Thus, we know the direction of force $\vec{F}_D$. Note that ODB is a straight line.

Let ϕ be the angle between the contact force $\vec{F}_D$ and the normal to the cylinder surface at D. Now, from geometry, $\angle C_3DO + \angle C_3OD + OC_3D = 180°$. But, $\phi = \angle C_3DO = \angle C_3OD$. Therefore,

$$\begin{aligned}\phi &= \frac{1}{2}(180° - \angle OC_3D) = \frac{1}{2}(\angle GC_3D)\\ &= \frac{1}{2}30° = 15°\end{aligned}$$

where $\angle GC_3D = 30°$ follows from the fact that $C_1C_2C_3$ is an equilateral triangle and C_3G bisects $\angle\, C_1C_3C_2$.

Note that ϕ is the friction angle between the surface normal and the friction force. Somewhat surprisingly, this same ϕ is the friction angle for both for F_D and for F_B. Now, from fig. 6.70, we see that

$$\tan\phi = \frac{F_{sB}}{F_{nB}} = \frac{F_{sD}}{F_{nD}}.$$

But, the force of friction $F_s \leq \mu N$. Therefore,

$$\mu \geq \tan\phi = \tan 15° = 0.27.$$

The friction coefficient must be at least 0.27 if the three cylinders have to be in static equilibrium.

$\mu \geq 0.27$

6.4 Frames

Preparatory Problems

6.4.1 In what way(s) is/are trusses different from more general frames?

6.4.2 Consider a frame made of 3 pieces connected together. Assume that no free-body diagram cut is within a part.

a) How many different free-body diagrams can you draw?

b) For each free-body diagram how many independent scalar equations can be extracted from the equilibrium relations?

c) In total, from all the free-body diagrams, how many independent scalar equations can be extracted from the various equilibrium conditions?

6.4.3 Consider the two-bar frame shown. Choose appropriate coordinate axes. Find

a) The reaction at D.

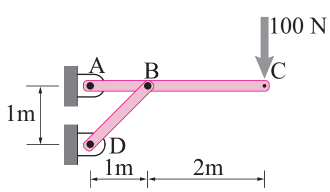
b) The tension in bar DB.

c) The reaction at A.

d) The force of DB on ABC.

e) The moment in ABC

- just (an infinitesimal distance) to the right of A
- just to the left of B
- just to the right of B
- just to the left of C



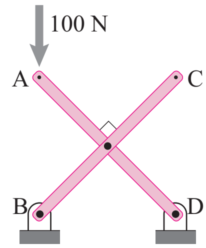
Problem 6.4.3

6.4.4 Two 1m bars are pinned together, at their middles, at right angles. Find

a) the reactions at B and D,

b) the force of BC on AD, &

c) the moment in AD just above the hinge,



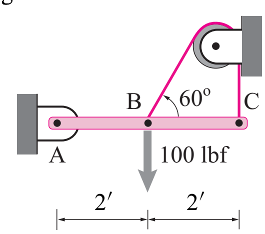
Problem 6.4.4

6.4.5 For the structure shown find

a) the tension in the string

b) the reaction at A

c) the moment in ABC just to the right of B



Problem 6.4.5

More-Involved Problems

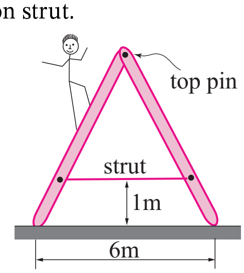
6.4.6 An A-frame aluminum ladder consists of two uniform 5 m, 150 N sections that are pinned at the top and held from splitting by a massless strut 1m above the slippery floor. An 800 N person has climbed halfway up the left side.

a) Find the reactions (the forces of the ground on the two ladder sections).

b) Find the force of the left section on the right at the top pin.

c) Find the tension in the connection strut.

d) Find the moment in the right leg of the ladder just above the tension strut.



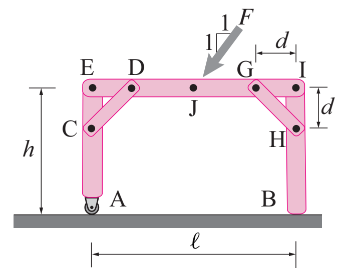
Problem 6.4.6

6.4.7 To make a model of a table statically determinate, we assume that leg EA slides easily on the floor. Assume the other leg does not slip. A force $F = 200\,\text{N}$ acts at 45° at the center of the table. Use $h = 1\,\text{m}$, $\ell = 2\,\text{m}$, and $d = .25\,\text{m}$. Find

a) the reactions at A and B,

b) the tension in GH,

c) the moment in IHB just below H.



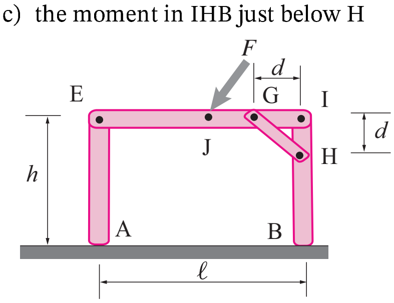
Problem 6.4.7

6.4.8 Another way to make a model of a table statically determinate (see problem 6.4.7) is to assume that one leg is not braced. Now, neither leg can slip. A force $F = 200\,\text{N}$ acts at 45° at the center of the table. Use $F = 200\,\text{N}$, $h = 1\,\text{m}$, $\ell = 2\,\text{m}$, and $d = .25\,\text{m}$. Find

a) the reactions at A and B

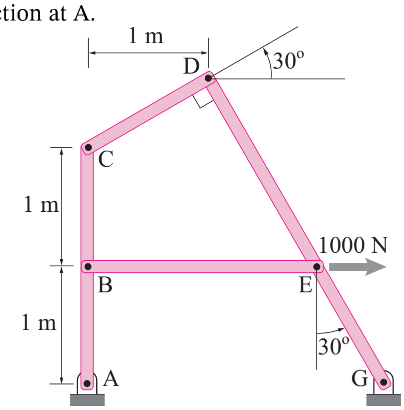
b) the tension in GH

c) the moment in IHB just below H



Problem 6.4.8

6.4.9 For the structure shown find the reaction at A.



Problem 6.4.9

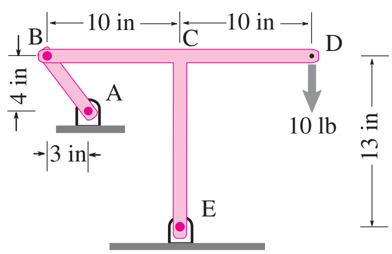
6.4.10 The structure consists of two pieces: bar AB and ‘T’ EBCD. They are connected to each other with a hinge at B. They are connected to the ground with hinges at A and E. The force of gravity is negligible. Find

a) The reaction at A.

b) The reaction at E.

c) The moment in BCED just to the left of C.

d) Why are these forces so big or small? (Your answer should be in words).



Problem 6.4.10