# 6.1 Introduction to trusses and the method of joints

Trusses are good.

Trusses are useful in engineering practice, they are easy to analyze, and they provide a good example of more general structural concepts. Your main goal here is to learn ‘truss analysis’, how to find the tensions in the bars of a given truss. But first, what is a truss and why are trusses so common?

You can quickly get a tactile sense of the truss concept. Get 9 short sticks and 9 rubber bands①. Put them together a few different ways and feel the resulting rigidity or lack thereof.

a) **A ‘V’ deforms**. First join two sticks tightly together with a rubber band so that they cannot easily slide along the connection, as in fig. 6.1a. Despite the tight joint connection you can feel that the sticks rotate relative to each other relatively easily; it is easy to open and close the upside-down V.

b) **A triangle is sturdy**. Add a third stick to complete the triangle (fig. 6.1b). The relative rotation of the first two pencils is now almost totally prohibited. Even though each joint on its own made a relatively flexible V, together the 3 joints make a very stiff triangle.

c) **A square deforms**. Now tightly strap four sticks into a square as in fig. 6.1c, making 4 rubber band joints at the corners. Put the square down on a table. The sticks don’t stretch or bend visibly, nor do they slide much along each-other’s lengths, but the connections allow the sticks to rotate relative to each other so the square easily distorts into a parallelogram.

d) **Two triangles are sturdy**. Now add two more sticks to your triangle to make two triangles (fig. 6.1d). So long as you keep this structure flat on the table, it is also sturdy.

Because a triangle is fully determined by the lengths of its sides and the V and quadrilateral are not, the structures made of triangles are *much* harder to distort. A triangle is sturdy even without rigid joints. And a V and a square (and a pentagon, etc) are not. You have just observed the essential inspiration of a truss:

> Triangles make sturdy structures.

## Swiss cheese

A different way to discover a truss is by means of subtraction. Imagine your first initial design for a bridge is to make it from one huge chunk of solid steel. This would be wildly heavy and expensive. So you could cut

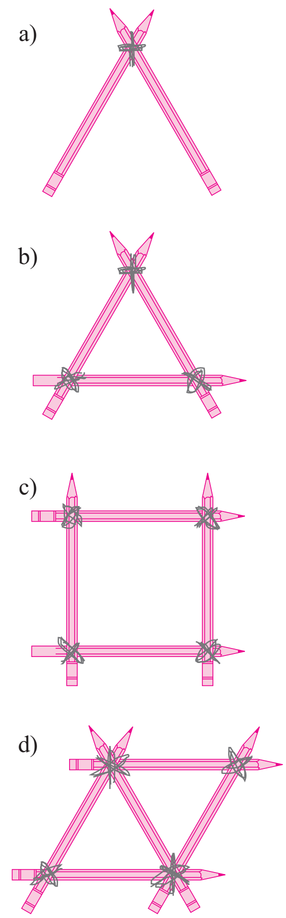



Figure 6.1: a) Two pencils strapped together with a rubber band are not sturdy. b) A triangle made of pencils feels sturdy. c) A square made of 4 pencils easily distorts into a parallelogram. d) A structure made of two triangles feels sturdy (assuming it is kept flat).

①Some useable supplies are surely on hand. For sticks you can use pencils, pens, paper tightly rolled and taped into tubes, chopsticks, popsicle sticks, big wooden matches, knitting needles, barbecue skewers, plastic drinking straws, straw from a broom, Tinkertoy rods, table knives, or strips cut from cardboard. Similarly rubber-bands can be replaced with masking, scotch or duct tape, used chewing gum, lashings using thread or string, hot-melt glue, or (for some of the sticks above) with a paperclip punched through the sticks and bent to make a home-made rivet.

holes out of the chunk here and there, greatly diminishing the weight and amount of material used, but not much reducing the strength. Between these holes you would see other heavy regions of metal from which you might cut more holes leading to a more savings of weight at not much cost in strength. In fact, the reduced weight in the middle decreases the load on the outer parts of the structure possibly making the whole structure stronger rather than weaker②. Eventually you would find yourself with very holey swiss cheese, a structure that looks something like a collection of bars attached from end to end in vaguely triangular patterns; like a microscopic picture of spongy bone (fig. 6.2): As opposed to a solid block, a truss

② Advanced aside: There are three ways that having more material in a structure can make it weaker: 1) the extra material adds to the gravitational load, as for the imagined bridge here, 2) Added material can be wedged in, causing the structure to fight itself (so called *locked-in* or *internal* stresses), and 3) material in the wrong place can cause *stress concentrations* and thus weak spots.

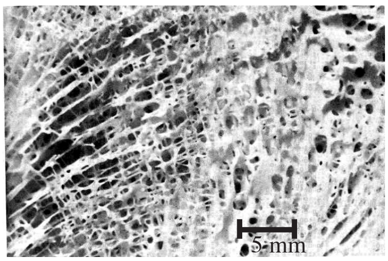



Figure 6.2: The truss inside you. The structure of spongy bone is vaguely truss-like. Shown here is human cancellous bone from the proximal femur, with the marrow removed. (courtesy Rod Lakes).

- Uses less material;
- Puts less gravity load on other parts of the structure;
- Leaves space for other things of interest (*e.g.,* cars, cables, wires, people).

Real trusses are usually not made by removing material from a solid but by joining bars of steel, wood, or bamboo with welds, bolts, rivets, nails, screws, glue, or lashings. Once you are aware, you will notice trusses making up bridges, radio towers, and large-scale construction equipment. Early airplanes were flying trusses (fig. 6.3). Bamboo trusses have been used as scaffoldings for millennia. Birds have had bones whose internal structure is truss-like since they were dinosaurs.

Figure 6.3: Chanute Glider, 1896. The Wright brothers' first planes were near copies of the 4-wing gliders built a few years earlier by Octave Chanute. Chanute was a retired bridge designer. Structurally these early biplanes were essentially flying bridges. The Wright bothers were preoccupied with trusses. Their first inspiration and main airplane patent was about how to control the distortion of a truss ('wing warping'). The Wrights even thought of the threads in their wing fabric as bars in a truss. Take away the outer skin from many small modern planes and you will also find trusses. (National Air and Space Museum Negative 1A-2036084-10697)

Trusses are practical, sturdy, and light structures.

But trusses are also prominent near the front of elementary mechanics books because

- They are perhaps the easiest example of a complex mechanical system that a student can analyze;
- They illustrate a variety of more general structural mechanics issues;
- They help build intuition about structures that are not really trusses; the engineering mind can see an underlying conceptual truss where there is no physical truss.

What is a truss?

A *truss* is a structure made from long narrow bars connected at their ends.

The sturdiness of most trusses comes from the inextensibility of the bars, not the resistance to rotation at the joints (as in the sticks and rubber-band examples at the start of this section). To make the analysis simpler, the relatively small resistance to rotation in the joints (even welded joints) is totally neglected in truss analysis. Thus the interaction of the bars with their neighbors is by forces, with no couples; each bar has one net force acting on each end. So:

An *ideal truss* is an assembly of two-force members.

Or, if you like, an ideal truss is a collection of bars connected at their ends with frictionless pins. Loads are only applied at the pins. In engineering analysis, the word 'truss' refers to an ideal truss even though the object of interest might have, say, welded joint connections. Had we assumed the presence of welding equipment in your study room, the opening paragraph of this section would have described the welding of metal bars instead of the attachment of pencils with rubber bands. Even with welded-together steel you would have found that the triangles would be much more rigid than the V or square.

Figure 6.4: Many bridges, especially old train bridges, are essentially trusses. Here's one that is partially obscured by the truss on a nearby car bridge.

## Bars, joints, loads, and supports

An ideal truss is a collection of *bars* connected at frictionless *joints* at which are applied *loads* as shown in fig. 6.5b (the load at a joint can be $\vec{\mathbf{0}}$ and thus not show on either the sketch of the truss or the free-body diagram of the truss). A truss is held in place with *supports* which are idealized in 2D as either being fixed pins (as for joint E in fig. 6.5a) or as a pin on a roller (as for joint G in fig. 6.5a). *Reaction forces*, the forces on the truss at the supports, show on a FBD of the whole truss (fig. 6.5b) and also on a FBD of any joint at a support. Each bar is a two-force body (fig. 6.5c), with the same magnitude of tension pulling away from each end. A joint can be cut free with a conceptual chain saw, fooling each bar stub with the bar tension, as in the free-body diagram of a joint in fig. 6.5d.

The bar tensions can be negative. A bar with a tension of, say, $T = -5000\,\mathrm{N}$ is said to be in *compression*. A tension of $-5000\,\mathrm{N}$ is a compression of $5000\,\mathrm{N}$.

## Elementary truss analysis

In elementary truss analysis you are given the design of a truss and the loads applied to it. Your goal is to 'solve the truss' which means you are to find the reaction forces and the tensions in the bars (sometimes called the 'bar forces'). As an engineer, this allows you to determine the needed strengths for the bars.

## The 'method of free-body diagrams'

Trusses are always analyzed by the same basic method used in all of mechanics, the 'method of free-body diagrams'.

- Free-body diagrams are drawn of the whole truss and of various parts of the truss.

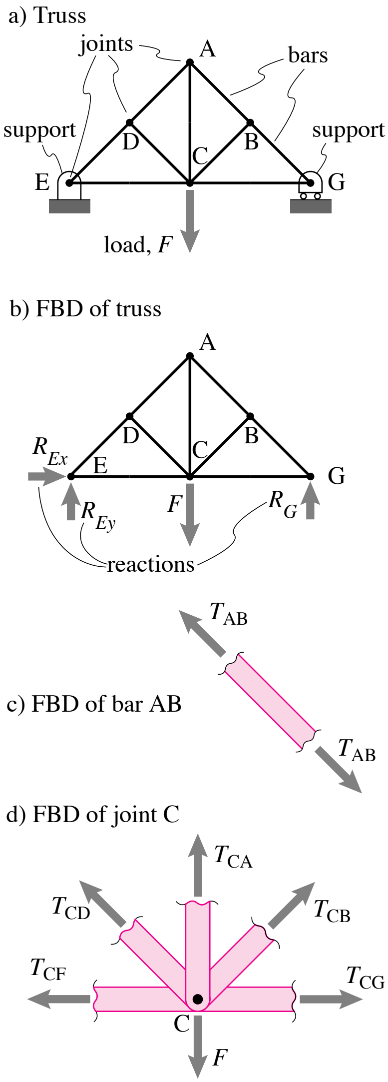



Figure 6.5: a) a truss, b) FBD of truss (It is OK to show $R_{Ex}$ even if it turns out to be zero), c) FBD of a bar, each bar is a two-force body, d) FBD of joint C, a joint is acted on by bar tensions and by applied loads. See Sample 6.1 on page 325 for an analysis of this truss.

- The equilibrium equations are applied to each free-body diagram, and
- The resulting equations are solved for the unknown bar forces and reactions.

The 'method of free-body diagrams' is classically subdivided into two sub-methods.

- In the *method of joints* you draw free-body diagrams of every joint and apply the force-balance equations to each free-body diagram.
- In the *method of sections* you draw a free-body diagrams of one or more parts of the structure, each of which includes 2 or more joints and apply force and moment balance to the part or parts.

These two methods can be used separately or in conjunction. In the rest of this chapter we cover the method of joints, the method of sections, computer solution using the method of joints, and miscellaneous advanced truss topics.

The elementary truss analysis you are about to learn is straightforward and fun. You will learn it without difficulty. However, the analysis of trusses at a more advanced level is mysteriously deep and has occupied great minds from the mid-nineteenth century (*e.g.,* Maxwell and Cauchy) to the present (see, e.g., box 6.2 on page 366).

## Method of joints

Let's start with an example.

**Example: Derrick arm.**
Consider this planar model of the arm of a construction derrick (see fig.6.7). Assume $F$ and $d$ are known. This truss has joints A-S (skipping 'F' to avoid confusion with the load). As is common in truss analysis, we totally neglect the force of gravity on the truss elements [3]. The goal is to find the tensions in the bars (the so-called 'bar forces').

The method of joints is a subset of the more general method of free-body diagrams. Free-body diagrams are drawn of the joints. Here is the method-of-joints recipe:

- Draw a free-body diagram of the whole structure and write 3 independent equilibrium equations (6 in 3D) and solve for unknown reactions if you can. This step is technically superfluous, but is so-often a time-saver that it's best to just do it.
- Draw free-body diagrams of all $n$ joints, 18 such in the example above.
- For each joint free-body diagram you write the force-balance equations, each of which can be broken down into 2 scalar equations (3 in 3D).

- Solve the $2n$ joint equations ($3n$ in 3D) for the unknown bar forces and reactions. In the example above this is $18 * 2 = 36$ equations for 33 unknown tensions and 3 unknown reactions (which you may have found from the FBD of the whole structure, but need not have).

Solving 36 simultaneous equations is generally only feasible with a computer, which is one way to go about things. However, for simple triangulated structures, like the one in fig. 6.7, you can find a sequence of joints for which hand solution is easy. If you solve the equilibrium equations as you go, there are at most two unknown bar forces at each joint. By this means, the joint force-balance equations can be solved, even for some complex structures, without computers.

**Example: Using the FBD of the whole structure**
From the free-body diagram of the whole structure (fig. 6.7) we find that

$$\begin{aligned}
\left\{\sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad R_{Sy} &&= F_{Ay} \\
\left\{\sum \vec{\boldsymbol{M}}_{/\mathrm{S}} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad R_{Rx} &&= 8F_{Ay} - F_{Ax}. \\
\left\{\sum \vec{\boldsymbol{M}}_{/\mathrm{R}} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad R_{Sx} &&= -8F_{Ay}.
\end{aligned}$$

Note, we picked a sign convention for the graphical representation of forces on the Free-Body Diagram (see pages 53 and 169) and let the algebra possibly generate negative numbers: at S the support pushes on the arm with a force of $-8F_{Ay}$ which is pulling (if $F_{Ay} > 0$)

Note that for tension the order of subscripts is not meaningful. The tension $T_{BC}$ is the same scalar as the tension $T_{CB}$. $T_{BC} = T_{CB}$ is the amount of pulling on joint B and also the amount of pulling on joint C. That the two force vectors are negatives of each other is accounted for by the definition of tension as pulling. This unimportance of the order of subscripts is in contrast with the case of position vectors where $\vec{\boldsymbol{r}}_{\mathrm{BC}}$ is the position vector from B to C (also called $\vec{\boldsymbol{r}}_{\mathrm{C/B}}$). For position vectors $\vec{\boldsymbol{r}}_{\mathrm{BC}} = \vec{\boldsymbol{r}}_{\mathrm{C/B}} = -\vec{\boldsymbol{r}}_{\mathrm{B/C}} = -\vec{\boldsymbol{r}}_{\mathrm{CB}}$. Summarizing, the subscript order has meaning for $\vec{\boldsymbol{r}}_{\mathrm{AB}}$ but not for $T_{AB}$.

## FBDs of the joints

In the solve-by-hand method of joints, we first find a joint with at most 2 bars connected. Then we work our way through the structure, one joint at a time, picking joints with at most 2 unknown bar tensions. For each joint we will use

$$\sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \vec{\boldsymbol{0}}.$$

Typically many bars in a truss are parallel to the $x$ or $y$ axis so we often fall into the routine of immediately reducing the above vector equilibrium equation to the component equations

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0.$$

Figure 6.6: The Howrah bridge. Trusses are such a good idea that they are used recursively. The bars in some large trusses, like this one or the Eiffel Tower, are themselves made of trusses (courtesy Bipin Chandra, www.bipinchandra.co.uk).

③ **How to include gravity on the truss elements:** replace the single gravity force at the center of each bar with a pair of equivalent forces at the ends. The gravity loads are then only at the joints, and the truss can still be analyzed in the standard way as a collection of two-force members. Trusses are so efficient, however, that the load they carry is often much greater than their weight. So weights of the truss parts are often neglected when calculating bar forces. If weight is significant, the method in this sidenote will accurately calculate the tensions in the bars, but not the tendency of the bars to bend under their own weight.

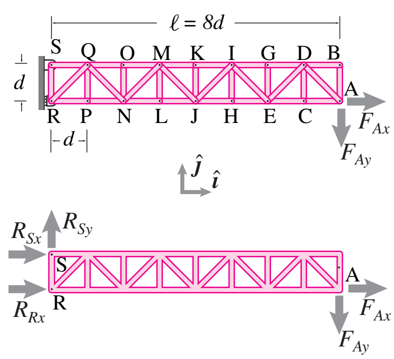



Figure 6.7: A truss with 33 bars, 18 joints and 3 unknown reaction forces. The reactions are sometimes written as, say, $R_{Sx}$ instead of $F_{Sx}$.

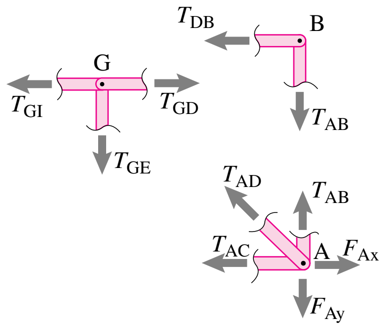



Figure 6.8: Free-body diagram of some joints from the truss in fig. 6.7. Note that bars DB, AB and GE are zero-force members.

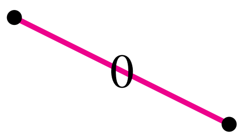



Figure 6.9: A zero-force member is sometimes indicated by writing a zero on top of the bar.

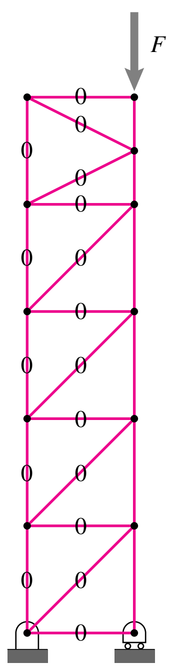



Figure 6.10: A tower with many zero-force members. Although they carry no load they prevent structural collapse. A common use of zero-force members is to brace long bars that are in compression and which would otherwise buckle (pop out to one side).

For the truss in fig. 6.7

- Joint B has only two bars connected (see fig. 6.8). Force balance using FBD 6.8 tells us at a glance that

$$\sum F_x = 0 \Rightarrow T_{DB} = 0 \qquad \text{and} \qquad \sum F_y = 0 \Rightarrow T_{AB} = 0$$

- Now you can draw a free-body diagram of joint A where there are only two unknown tensions (since we just found $T_{AB}$), namely $T_{AD}$ and $T_{AC}$. Force balance gives two scalar equations

$$\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad F_{Ax} - T_{AC} - \sqrt{2}T_{AD}/2 = 0\\
\sum F_y = 0 \quad &\Rightarrow \quad -F_{Ay} + \underbrace{T_{AB}}_{0} + \sqrt{2}T_{AD}/2 = 0
\end{aligned}$$

  which you can solve to find $T_{AD} = \sqrt{2}F_{Ay}$ and $T_{AC} = F_{Ax} - F_{Ay}$.
- Next is joint C. Force balance for joint C will tell you $T_{CD}$ and $T_{CE}$.
- Then you can work your way through the alphabet of joints. Using the bar tensions you have already found you can find, one at a time, joints with only two unknown tensions.

That's it for the method of joints for simple structures.

## Zero-force members

Just by looking at joint B and thinking about the free-body diagram you could probably pick out that bars DB and AB must be *zero-force members*.

Here we explain the unnecessary but useful trick of recognizing such zero-force members even before systematically using the method of joints. *Zero-force members* are bars with $T = 0$, like bars AB, BD and CD in the truss of Fig. 6.7. The basic idea is this:

> If there is any direction for which only one bar contributes a force on a joint, then that bar is a zero-force member.

In particular:

- At any joint where
  - there are no loads, and
  - where there are only two unknown non-parallel bar forces, and
  - where all known bar-tensions are zero,

  then the two new bar tensions are both zero (e.g., joint B in fig. 6.8).
- At any joint where all bars but one are in the same direction as the applied load (if any), the one bar is a zero-force member (see joints C, G, H, K, L, O, and P in fig. 6.7).

In the truss of fig. 6.7 bars AB, BD, CD, EG, IH, JK, ML, NO, and PQ are all zero-force members. Sometimes it is useful to keep track of the zero-force members by marking them with a zero (see fig. 6.9).

### Zero-force members often have a non-zero purpose

Although with the given loading zero-force members have no tension, they are often needed because there are small loads not considered in the basic analysis. These could be from imperfections, or load-induced asymmetries in a structure. This gives the ‘zero-force’ bars a small job to do, a job not noticed by the equilibrium equations in elementary truss analysis, but one that can prevent total structural collapse. Imagine, for example, the tower of fig. 6.10. In a real tower of that design the zero-force members might carry very small loads, say 100 or 1000 times smaller than the tensions (or compressions) calculated for the other bars. But if the zero-force members were removed the tower would collapse. Thus, in practice, you may observe large heavy structures with some very thin bars. Bars which in simple analyzes carry no loads. But bars which prevent structural collapse

## Simple and not-simple trusses

Most elementary texts, like this one, start with structures that yield easily to the method of joints. These are structures where you can totally solve the equilibrium equations for the joints, one joint at a time; each new joint introduces two scalar equations and two unknown bar-tensions.

For more complex trusses, this straightforward approach can fail a few ways:

- Some structures are not designed in a straightforward triangulated manner and cannot be solved 2 equations at a time. Although the method of joints may still yield a solution, it may require simultaneous solution of all of the equilibrium equations④.
- Many structures cannot be solved (that is, the bar tensions can’t be found) by using the laws of statics alone. Such are appropriately called ‘statically indeterminate’ (e.g., see page 387).

For this first truss section we only consider structures that are statically determinate and easily solved. See sec. 6.5 for a detailed discussion of static determinacy.

## Why aren’t trusses everywhere?

Trusses can carry big loads with little use of material and can look nice (See fig. 6.11). They are used in many structures. Why don’t engineers use trusses for all structural designs? Here are some reasons to consider not using a truss:

- Trusses are relatively difficult to build, involving many small parts⑤ and thus requiring much time and effort to assemble.
- Trusses can be sensitive to damage when forces are not applied at the anticipated joints. They are especially sensitive to loads on the middle of the bars.

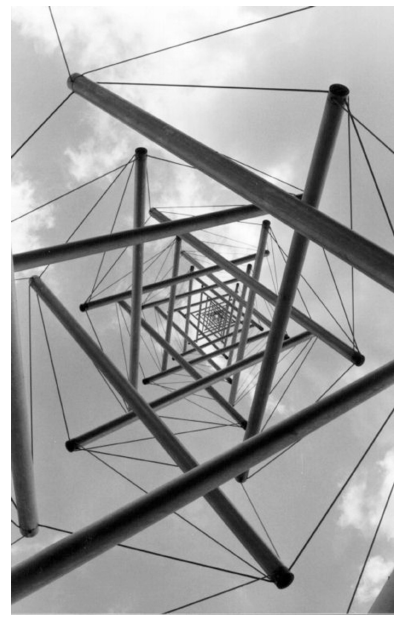


Figure 6.11: Sometimes trusses are used *only* because they look nice. The tensegrity structure ‘Needle Tower’ was designed by artist Kenneth Snelson and is on display in the Hirshhorn Museum in Washington, DC. It doesn’t hold up anything but itself. Here you are looking straight up the middle. (courtesy Christopher Rywalt)

④ In the language of linear algebra: simple structures yield equilibrium equations that are naturally in upper triangular form. More complicated structures do not yield an upper-triangular form.

⑤ If an Indian says to you "Go and count the rivets in the Howrah bridge." she means go away and do something that will take a very long time. The bridge has many rivets (and bars and joints).

- Trusses inevitably depend on the tension strength in some bars. Some common building materials (*e.g.,* concrete, stone, and clay) crack easily when pulled.
- Trusses often have little or no redundancy, so failure in one part can lead to total structural failure.
- The triangulation that trusses require can use space that is needed for other purposes (*e.g.,* doorways, rooms).
- Trusses tend to be stiff, and sometimes more flexibility is desirable (*e.g.,* diving boards, car suspensions).
- In some places, some people consider trusses unaesthetic. (e.g., the Washington Monument is not supposed to look like the Eiffel Tower).

Nonetheless, for situations where you want a stiff, light structure that can carry known loads at pre-defined points, a truss is often the best design choice.

## Three-dimensional trusses

After you have mastered the elementary 2D truss analysis of the previous section, you might wonder

- **Do the ideas generalize to 3D?** Yes, with only minor elaboration.
- **Does at least one of the methods presented always work?** Yes, if you just look at the homework problems for elementary truss analysis. And yes again for many practical structures. But some trusses cannot be analyzed by the simple methods. In this section we classify trusses into types. One type, statically determinate trusses, can be analyzed by simple statics methods; other trusses require study of deformations as well as statics.

The concepts for 3D trusses are basically the same as for 2D trusses with these differences;

- In the method of joints, each joint is associated with 3 scalar force-balance equations instead of 2;
- In the method of sections, and in the free-body diagram of the whole structure one has 6 scalar equations instead of 3;
- To hold the structure in place takes at least 6 reaction components instead of 3;
- The rule-of-thumb check for static determinacy of a grounded structure in 3D is $b + r = 3j$ instead of the 2D relation $b + r = 2j$ (See sec. 6.5 on 361 for discussion of these formulas);
- The rule of thumb for rigidity for a floating truss in 3D is $b + 6 = 3j$ instead of the 2D relation $b + 3 = 2j$.

(Recall: $b$= number of bars; $r$ = number of reaction components; & $j$ = number of joints.) There are various ways to think about the number six in the counts above. Assuming the structure is more than a point, six is the number of ways a rigid structure can move in three-dimensional space (three translations and three rotations), six is the number of equilibrium equations for the whole structure (one 3D vector moment, and one 3D vector force), and six is the number of constraints needed to hold a structure in place.

**Example: A tripod**

A tripod is the simplest rigid 3D structure. With four joints ($j = 4$), three bars ($b = 3$), and nine unknown reaction components ($r = 3 \times 3 = 9$), it exactly satisfies the equation $3j = b + r$, a check for determinacy of rigidity of 3D structures.

A tripod is the 3D equivalent of the two-bar truss shown in fig. 6.73a on page 364.

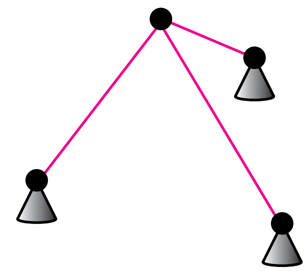


Figure 6.12: A tripod is the simplest rigid 3D truss.

**Example: A tetrahedron**

The simplest rigid floating structure in 3D is a tetrahedron. With four joints ($j = 4$) and six bars ($b = 6$) it exactly satisfies the equation $3j = b + 6$ which is a check for determinacy of rigidity of floating 3D structures.

A tetrahedron is thus, in some sense, the 3D equivalent of a triangle in 2D.

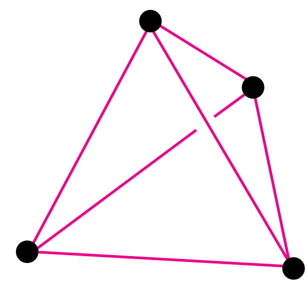


Figure 6.13: A tetrahedron is the simplest rigid truss in 3D that does not depend on grounding (*i.e.,* connection to a rigid external object).

**Example: Geodesic domes**

Any closed polyhedron, with each face a triangle of rods, is a rigid structure. This includes a tetrahedron (above), an octahedron, a cube with a diagonal on each face, an icosahedron, and Buckminster Fuller's geodesic domes.

Well, so Cauchy thought. It turns out that there are some strange non-convex polyhedra that are not rigid. But, for practical purposes, if you see triangles all around the outside of a structure you can assume it's rigid.

**SAMPLE 6.1** The truss shown in the figure carries a load $F = 10\,\text{kN}$ at joint D. The truss is designed with nine rods, six of which (the inclined ones) have the same length $d = 2\,\text{m}$. Rods BC, EC, DE and BD form a square.

1. Find the support reactions at joints A and F.
2. Find the tensions in rods BD and BC.

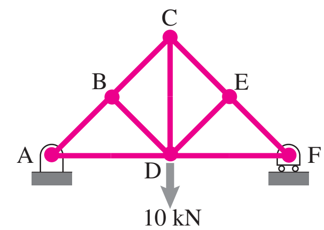



Figure 6.14

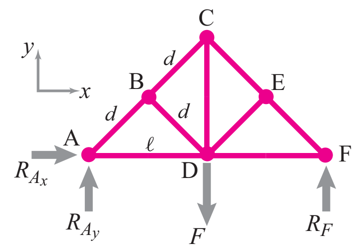



Figure 6.15

### Solution

1. **Support reactions:** To find the support reactions at A and F, we draw the free-body diagram of the entire truss (see fig. 6.15). We are given that $d = 2\,\text{m}$ and that $\angle\text{ABD} = \angle\text{DEF} = \pi/2$. Therefore, $\ell = \sqrt{2}d = 2\sqrt{2}\,\text{m}$.

   The scalar force-balance equation in $x$-direction readily gives $R_{A_x} = 0$. The scalar moment-balance equation about point A gives

   $$2\ell R_F - \ell F = 0 \quad \Rightarrow \quad R_F = \frac{F}{2} = 5\,\text{kN}.$$

   Now, from the scalar force balance in the $y$-direction, we have

   $$R_{A_y} + R_F - F = 0 \quad \Rightarrow \quad R_{A_y} = F - R_F = 5\,\text{kN}.$$

   $$\boxed{R_{A_x} = 0, \quad R_{A_y} = 5\,\text{kN}, \quad R_F = 5\,\text{kN}}$$

2. **Tensions in BD and BC:** We can find the tensions in rods BC and BD by analyzing the equilibrium of joint B. As you can see, joint B has three unknown forces acting on it, namely the tensions of rods AB, BC and BD. Since the joint equilibrium equations (only two scalar equations) can only solve for two unknowns, we need to start at joint A, determine $T_{AB}$ first and then move on to joint B.

   The free-body diagrams of the joints A and B are shown in fig. 6.16. Let us first consider the equilibrium of joint A. From the scalar force-balance equations, we have

   $$\begin{aligned}
   \sum F_y = 0 \quad &\Rightarrow \quad R_{A_y} + T_{AB}\sin\theta = 0 \\
   &\Rightarrow \quad T_{AB} = -R_{A_y}/\sin\theta = -5\,\text{kN}/(1/\sqrt{2}) = -7\,\text{kN}. \\
   \sum F_x = 0 \quad &\Rightarrow \quad T_{AB}\cos\theta + T_{AD} = 0 \\
   &\Rightarrow \quad T_{AD} = -T_{AB}\cos\theta = -7\,\text{kN}(1/\sqrt{2}) = 5\,\text{kN}.
   \end{aligned}$$

   Now, we analyze joint B. From the geometry of forces, it is clear that writing scalar force-balance equations in the $x'$ and $y'$ directions will be advantageous. For example, the force balance in the $x'$ direction immediately gives $T_{BD} = 0$. The force balance in the $y'$ direction gives

   $$-T_{AB} + T_{BC} = 0 \quad \Rightarrow \quad T_{BC} = T_{AB} = -7\,\text{kN}.$$

   $$\boxed{T_{BC} = -7\,\text{kN}, \quad T_{BD} = 0}$$

   Note that it is easy to spot bar BD as a zero-force member since it is perpendicular to rods AB and BC.

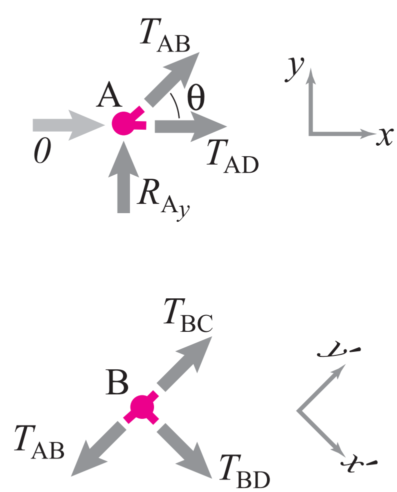



Figure 6.16: Free-body diagrams of joints A and B. From the given geometry, $\theta = pi/4$.

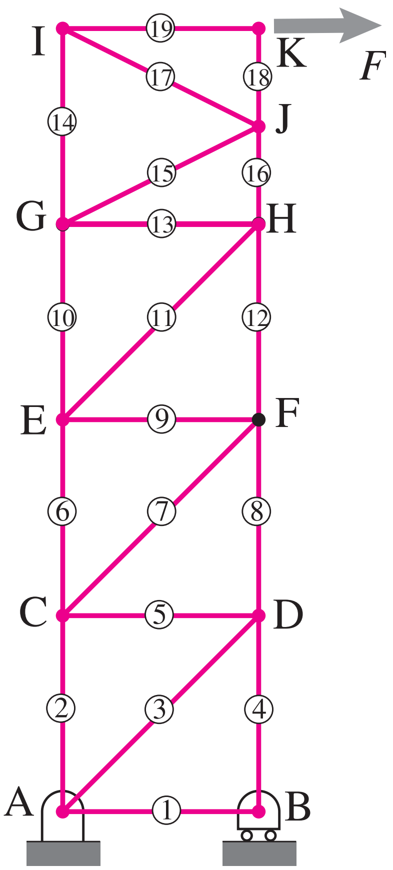



Figure 6.17

**SAMPLE 6.2** For the truss tower shown in the figure, assume all horizontal and vertical rods to be 1 mlong and rods numbered 16 and 18 to be 0.5 mlong. Given that the horizontal load on the truss $F = 500\,\text{N}$, find the tension in rod 15.

**Solution** To find the tension in rod 15, we can use the equilibrium of either joint G or joint K. In either case, the free-body diagram will have four unknown bar tensions (for four bars connected to each of these joints) at the joint. Therefore, we will not be able to solve for them. So, let us start at joint K and work through joint I to joint J. This sequence gets us only two unknown forces at each joint.

The free-body diagrams of the three joints are shown in fig. 6.18. Let us first consider the equilibrium of joint K. A simple inspection (or force balance in the $y$-direction) shows that bar 18 is a zero-force member. The force balance in the horizontal direction then immediately gives $T_{19} = F = 500\,\text{N}$. Thus,

$$T_{19} = 500\,\text{N} \quad \text{and} \quad T_{18} = 0.$$

Next, we consider the equilibrium of joint I. Since $T_{19}$ is already known, there are only two unknown forces, $T_{14}$ and $T_{17}$ at this joint. The force balance in the horizontal direction gives

$$\begin{aligned} T_{19} + T_{17}\cos\theta &= 0 \\ \Rightarrow \quad T_{17} &= -\frac{T_{19}}{\cos\theta} \\ &= -\frac{500\,\text{N}}{\cos(\tan^{-1}(0.5))} \\ &= -559\,\text{N}. \end{aligned}$$

Now we proceed to joint J. Note that we used only one scalar equation (force balance in the $x$-direction) at joint I, since we do not need $T_{14}$. Similarly, to find $T_{15}$, we only need the force balance in the horizontal direction at joint J:

$$\begin{aligned} -T_{17}\cos\theta - T_{15}\cos\theta &= 0 \\ \Rightarrow \quad T_{15} &= -T_{17} = 559\,\text{N}. \end{aligned}$$

$$\boxed{T_{15} = 559\,\text{N}}$$

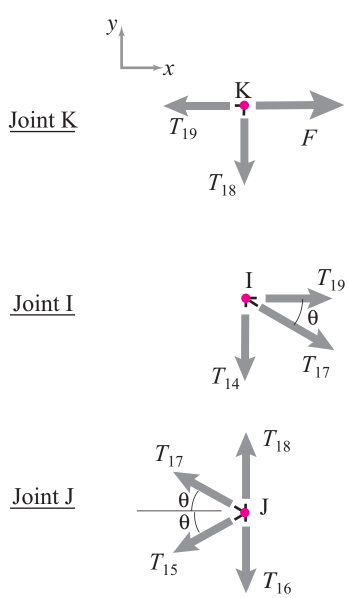



Figure 6.18: Free-body diagrams of joints K, I, and J. From the given geometry, $\theta = \tan^{-1}(\text{KJ}/\text{IK}) = 26.56^{\circ}$.

**Note:** We did not have to find support reactions first in order to proceed to other joints as in the previous sample. As long as you can find a sequence of joints with just two unknown forces at each joint, up to the force that you need to determine, you can easily find the force with hand calculations.

**SAMPLE 6.3** The truss shown in the figure is made up of five horizontal and six inclined rods. All inclined rods are 1 m long and at right angles to each other. The truss carries two vertical loads, $F_1 = 4\,\text{kN}$ and $F_2 = 1\,\text{kN}$ as shown. Find the tensions in rods CE, DE, and DF.

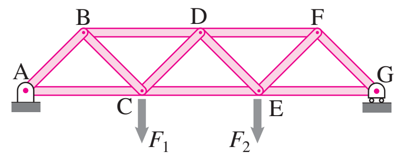



Figure 6.19

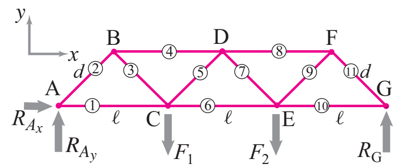



Figure 6.20

**Solution** To find tensions in rods CE, DE and DF, we can either use joints C and D, or joints E and F. However, for either set we need to start from other joints since there are more than two unknown forces at each joint. Let us start from joint G and work our way through joints F and E. To start at joint G, however, we first need to determine the support reaction G.

The free-body diagram of the entire truss is shown in fig. 6.20 where we have numbered the rods for convenience. The scalar moment-balance equation about point A in the $z$-direction gives

$$3\ell R_G - \ell F_1 - 2\ell F_2 = 0 \quad \Rightarrow \quad R_G = \frac{F_1 + 2F_2}{3} = 2\,\text{kN}.$$

The force-balance equations give

$$\sum F_x = 0 \quad \Rightarrow \quad R_{A_x} = 0$$
$$\sum F_y = 0 \quad \Rightarrow \quad R_{A_y} = F_1 + F_2 - R_G = 3\,\text{kN}.$$

Now, we are ready to proceed from joint G. The free-body diagrams of joints G, F, and E are shown in fig. 6.21.

At joint G:

$$\sum F_y = 0 \quad \Rightarrow \quad T_{11}\sin\theta + R_G = 0$$
$$\Rightarrow \quad T_{11} = -\frac{R_G}{\sin\theta} = -\sqrt{2}R_G = -2.83\,\text{kN}.$$
$$\sum F_x = 0 \quad \Rightarrow \quad -T_{11}\cos\theta - T_{10} = 0$$
$$\Rightarrow \quad T_{10} = -T_{11}\cos\theta = 2\,\text{kN}.$$

At joint F:

$$\sum F_y = 0 \quad \Rightarrow \quad -T_{11}\sin\theta - T_9\sin\theta = 0$$
$$\Rightarrow \quad T_9 = -T_{11} = 2.83\,\text{kN}$$
$$\sum F_x = 0 \quad \Rightarrow \quad (T_{11} - T_9)\cos\theta - T_8 = 0$$
$$\Rightarrow \quad T_8 = (T_{11} - T_9)\cos\theta = -4\,\text{kN}.$$

At joint E:

$$\sum F_y = 0 \quad \Rightarrow \quad (T_7 + T_9)\sin\theta - F_2 = 0$$
$$\Rightarrow \quad T_7 = \frac{F_2}{\sin\theta} - T_9 = -1.41\,\text{kN}.$$
$$\sum F_x = 0 \quad \Rightarrow \quad (T_9 - T_7)\cos\theta + T_{10} - T_6 = 0$$
$$\Rightarrow \quad T_6 = (T_9 - T_7\cos\theta + T_{10} = 5\,\text{kN}.$$

$$T_{\text{CE}} = 5\,\text{kN},\ T_{\text{DE}} = -1.41\,\text{kN},\ T_{\text{DF}} = -4\,\text{kN},$$

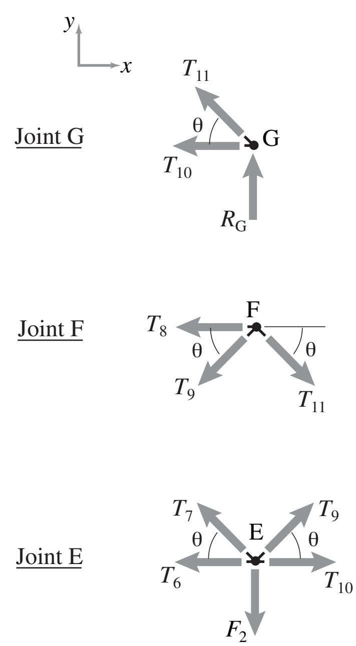



Figure 6.21: Free-body diagrams of joints G, F, and E. From the given geometry, $\theta = 45^\circ$.

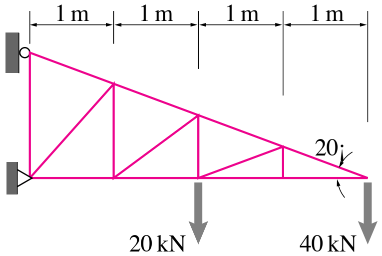



Figure 6.22

**SAMPLE 6.4** The truss shown in the figure has four horizontal bays, each of length 1 m. The top bars make 20° angle with the horizontal. The truss carries two loads of 40 kN and 20 kN as shown. Find the forces in each bar. In particular, find the bars that carry the maximum tensile and compressive forces.

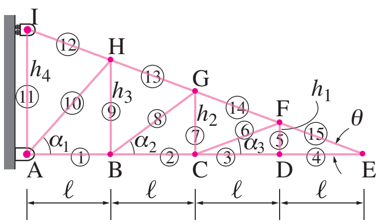



Figure 6.23: To keep track of joints and rods, we first label them. All joints are labeled with letters and rods with numbers.

**Solution** Since we need to find the forces in all the 15 bars, we need to find enough equations to solve for these 15 forces in addition to 3 unknown reactions $A_x, A_y$, and $I_x$. Thus we have a total of 18 unknowns. Note that there are 9 joints and therefore, we can generate 18 scalar equations by writing force equilibrium equations (one vector equation per joint) for each joint.

| | | |
|---|---|---|
| Number of unknowns: | 15 bar forces + 3 reactions | = 18 |
| Number of joints: | (A, B, C, ..., and I) | = 9 |
| Number of equations: | 9 joint × 2 per joint | = 18. |

So, we go joint by joint, draw the free-body diagram of each joint and write the equilibrium equations. After we get all the equations, we can solve them on a computer. All joint equations are just force equilibrium equations, *i.e.*, $\sum \vec{F} = \vec{0}$.

- Joint **A**:
$$(A_x + T_1 + T_{10}\cos\alpha_1)\hat{\imath} + (A_y + T_{11} + T_{10}\sin\alpha_1)\hat{\jmath} = \vec{0}. \tag{6.1}$$
- Joint **B**:
$$(-T_1 + T_2 + T_8\cos\alpha_2)\hat{\imath} + (T_9 + T_8\sin\alpha_2)\hat{\jmath} = \vec{0}. \tag{6.2}$$
- Joint **C**:
$$(-T_2 + T_3 + T_6\cos\alpha_3)\hat{\imath} + (T_7 + T_6\sin\alpha_3)\hat{\jmath} = F_1\hat{\jmath}. \tag{6.3}$$
- Joint **D**:
$$(T_4 - T_3)\hat{\imath} + T_5\hat{\jmath} = \vec{0}. \tag{6.4}$$
- Joint **E**:
$$(-T_4 - T_{15}\cos\theta)\hat{\imath} + T_{15}\sin\theta\,\hat{\jmath} = F_2\hat{\jmath}. \tag{6.5}$$
- Joint **F**:
$$(-T_6\cos\alpha_3 + (T_{15} - T_{14})\cos\theta)\hat{\imath} + (-T_6\sin\alpha_3 + (T_{14} - T_{15})\sin\theta - T_5)\hat{\jmath} = \vec{0}. \tag{6.6}$$
- Joint **G**:
$$(-T_8\cos\alpha_2 + (T_{14} - T_{13})\cos\theta)\hat{\imath} + ((T_{13} - T_{14})\sin\theta - T_8\sin\alpha_2 - T_7)\hat{\jmath} = \vec{0}. \tag{6.7}$$
- Joint **H**:
$$(-T_{10}\cos\alpha_1 + (T_{13} - T_{12})\cos\theta)\hat{\imath} + ((T_{12} - T_{13})\sin\theta - T_{10}\sin\alpha_1 - T_9)\hat{\jmath} = \vec{0}. \tag{6.8}$$
- Joint **I**:
$$(I_x + T_{12}\cos\theta)\hat{\imath} + (-T_{11} - T_{12}\sin\theta)\hat{\jmath} = \vec{0}. \tag{6.9}$$

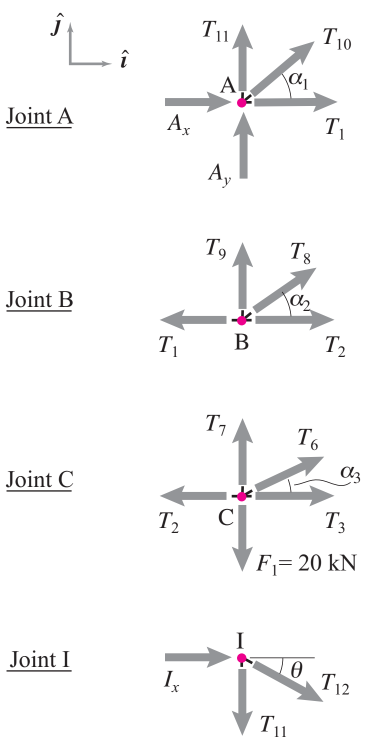



Figure 6.24: Free-body diagrams of some representative joints.

Dotting each equation from (6.1) to (6.9) with $\hat{\imath}$ and $\hat{\jmath}$, we get the required 18 equations. We need to define all the angles that appear in these equations ($\alpha_1, \alpha_2, \alpha_3$, and $\theta$) before we are ready to solve the equations on a computer.

Let $\ell$ be the length of each horizontal bar and let DF $= h_1$, CG $= h_2$, and BH $= h_3$. Then, $h_1/\ell = h_2/2\ell = h_3/3\ell = \tan\theta$. Therefore,

$$\begin{aligned}
\tan\alpha_1 &= \frac{h_3}{\ell} = \frac{3\ell\tan\theta}{\ell} &&\Rightarrow &&\alpha_1 = \tan^{-1}(3\tan\theta)\\
\tan\alpha_2 &= \frac{h_2}{\ell} = 2\tan\theta &&\Rightarrow &&\alpha_2 = \tan^{-1}(2\tan\theta)\\
\tan\alpha_3 &= \frac{h_1}{\ell} = \tan\theta &&\Rightarrow &&\alpha_3 = \tan^{-1}(\tan\theta) = \theta.
\end{aligned}$$

Now, we are ready for a computer solution. You can enter the 18 equations in matrix form or as your favorite software package requires and get the solution by solving for the unknowns. Here is a pseudocode to set up and solve the matrix equation. Let us order the unknown forces in the form

$$x = [T_1 \quad T_2 \quad \ldots \quad T_{15} \quad A_x \quad A_y \quad I_x]^T$$

so that $x_1$ to $x_{15} = T_1$ to $T_{15}$; $x_{16} = A_x$, $x_{17} = A_y$, and $x_{18} = I_x$.

**Entering and solving full matrix equation:**

```
theta = pi/9                      % specify theta in radians
alpha1 = atan(3*tan(theta)        % calculate alpha1
alpha2 = atan(2*tan(theta))       % calculate alpha2 from arctan
alpha3 = theta                    % calculate alpha3 from arctan
C = cos(theta), S = sin(theta)    % compute all sines and cosines
C1 = cos(alpha1), S1 = sin(alpha1)
C2 = .. ..

F1 = 20;                           % input given external loads
F2 = 40;

A = [1 0 0 0 0 0 0 0 0 C1 0 0 0 0 0 1 0 0     % enter matrix A row-
wise
     0 0 0 0 0 0 0 0 0 S1 1 0 0 0 0 0 1 0
     :
     :
     0 0 0 0 0 0 0 0 0 0 -1 -S 0 0 0 0 0 0]
b = [0 0 0 0 0 F1 0 0 0 F2 0 0 0 0 0 0 0 0]'  % enter column vector b

solve A*x = b for x
```

The solution obtained from the computer is

$$\begin{array}{lclllclllcl}
T_1 &=& -128.22\,\text{kN}, & T_2 &=& -109.9\,\text{kN}, & T_3 &=& -109.9\,\text{kN},\\
T_4 &=& -109.9\,\text{kN}, & T_5 &=& 0, & T_6 &=& 0,\\
T_7 &=& 20\,\text{kN}, & T_8 &=& -22.66\,\text{kN}, & T_9 &=& 13.33\,\text{kN},\\
T_{10} &=& -13.56\,\text{kN}, & T_{11} &=& -50\,\text{kN}, & T_{12} &=& 146.19\,\text{kN},\\
T_{13} &=& 136.44\,\text{kN}, & T_{14} &=& 116.95\,\text{kN}, & T_{15} &=& 116.95\,\text{kN},\\
A_x &=& 137.37\,\text{kN}, & A_y &=& 60\,\text{kN}, & I_x &=& -137.37\,\text{kN}.
\end{array}$$

**SAMPLE 6.5 A simple 3-D truss:** The 3-D truss shown in the figure has 12 bars and 6 joints. Nine of the 12 bars that are either horizontal or vertical have length $\ell = 1\,\text{m}$. The truss is supported at A on a ball and socket joint, at B on a linear roller, and at C on a planar roller (all three supports are on the ground). The loads on the truss are $\vec{F}_1 = -50\,\text{N}\hat{\boldsymbol{k}}, \vec{F}_2 = -60\,\text{N}\hat{\boldsymbol{k}}$ , and $\vec{F}_3 = 30\,\text{N}\hat{\boldsymbol{j}}$. Find all support reactions and the tension in bar BC.

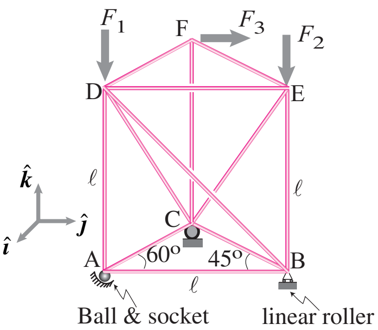



Figure 6.25: This 3-D truss has a triangular base ABC with support points A, B, and C on the ground, and a triangular top DEF. The structure is thus like a prism.

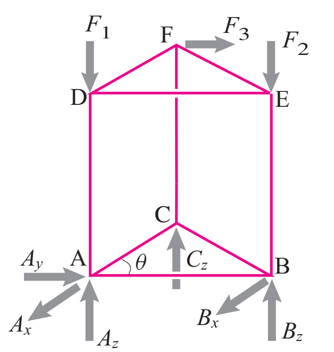



Figure 6.26: The free-body diagram of the structure as a whole. Note that we do not need to show the internal braces DB, DC, or EC here.

**Solution** The free-body diagram of the entire structure is shown in fig. 6.26. Let the support reactions at A, B, and C be $\vec{\boldsymbol{R}}_A = R_{A_x}\hat{\boldsymbol{\imath}} + R_{A_y}\hat{\boldsymbol{j}} + R_{A_z}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{R}}_B = R_{B_x}\hat{\boldsymbol{\imath}} + R_{B_z}\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{R}}_C = R_C\hat{\boldsymbol{k}}$. Then the moment balance about point A, $\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}$, gives

$$\vec{\boldsymbol{r}}_{\text{B/A}} \times \vec{\boldsymbol{R}}_B + \vec{\boldsymbol{r}}_{\text{C/A}} \times \vec{\boldsymbol{R}}_C + \vec{\boldsymbol{r}}_{\text{E/A}} \times \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{r}}_{\text{F/A}} \times \vec{\boldsymbol{F}}_3 = \vec{\boldsymbol{0}}. \tag{6.10}$$

Note that $\vec{\boldsymbol{F}}_1$ passes through A and, therefore, produces no moment about A. Now we compute each term in the equation above.

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{B/A}} \times \vec{\boldsymbol{R}}_B &= \ell\hat{\boldsymbol{j}} \times (R_{B_x}\hat{\boldsymbol{\imath}} + R_{B_z}\hat{\boldsymbol{k}}) &&= -R_{B_x}\ell\hat{\boldsymbol{k}} + R_{B_z}\ell\hat{\boldsymbol{\imath}},\\
\vec{\boldsymbol{r}}_{\text{C/A}} \times \vec{\boldsymbol{R}}_C &= \ell(\cos\theta\,\hat{\boldsymbol{j}} - \sin\theta\,\hat{\boldsymbol{\imath}}) \times R_C\hat{\boldsymbol{k}} &&= R_C\frac{\ell}{2}\hat{\boldsymbol{\imath}} + R_C\frac{\sqrt{3}\ell}{2}\hat{\boldsymbol{j}}\\
\vec{\boldsymbol{r}}_{\text{E/A}} \times \vec{\boldsymbol{F}}_2 &= (\ell\hat{\boldsymbol{j}} + \ell\hat{\boldsymbol{k}}) \times (-F_2\hat{\boldsymbol{k}}) &&= -F_2\ell\hat{\boldsymbol{\imath}},\\
\vec{\boldsymbol{r}}_{\text{F/A}} \times \vec{\boldsymbol{F}}_3 &= [\ell(\cos\theta\,\hat{\boldsymbol{j}} - \sin\theta\,\hat{\boldsymbol{\imath}}) + \ell\hat{\boldsymbol{k}}] \times F_3\hat{\boldsymbol{j}} &&= -F_3\ell\hat{\boldsymbol{\imath}} - F_3\frac{\sqrt{3}\ell}{2}\hat{\boldsymbol{k}}.
\end{aligned}$$

Substituting these products in eqn. (6.10), and dotting the resulting equation with $\hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{\imath}}$, respectively, we get

$$\begin{aligned}
R_C &= 0\\
R_{B_x} &= -\frac{\sqrt{3}}{2}F_3 = -15\sqrt{3}\,\text{N}\\
R_{B_z} &= -\frac{1}{2}R_C + F_2 + F_3 = 90\,\text{N}.
\end{aligned}$$

Thus, $\vec{\boldsymbol{B}} = R_{B_x}\hat{\boldsymbol{\imath}} + R_{B_z}\hat{\boldsymbol{k}} = -15\sqrt{3}\,\text{N}\hat{\boldsymbol{\imath}} + 30\,\text{N}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{R}}_C = \vec{\boldsymbol{0}}$. Now from the force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, we find $\vec{\boldsymbol{R}}_A$ as

$$\begin{aligned}
\vec{\boldsymbol{R}}_A &= -\vec{\boldsymbol{R}}_B - \vec{\boldsymbol{R}}_C - \vec{\boldsymbol{F}}_1 - \vec{\boldsymbol{F}}_2 - \vec{\boldsymbol{F}}_3\\
&= -(-15\sqrt{3}\,\text{N}\hat{\boldsymbol{\imath}} + 90\,\text{N}\hat{\boldsymbol{k}}) - (-50\,\text{N}\hat{\boldsymbol{k}}) - (-60\,\text{N}\hat{\boldsymbol{k}}) - (30\,\text{N}\hat{\boldsymbol{j}})\\
&= 15\sqrt{3}\,\text{N}\hat{\boldsymbol{\imath}} - 30\,\text{N}\hat{\boldsymbol{j}} + 20\,\text{N}\hat{\boldsymbol{k}}.
\end{aligned}$$

To find the force in bar BC, we draw a free-body diagram of joint B (which connects BC) as shown in fig. 6.27. Now, writing the force balance for the joint in the $x$-direction, *i.e.*, $[\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}] \cdot \hat{\boldsymbol{\imath}}$, gives

$$\begin{aligned}
R_{B_x} + \vec{\boldsymbol{T}}_{\text{BC}} \cdot \hat{\boldsymbol{\imath}} &= 0\\
\text{or}\qquad R_{B_x} + T_{BC}\sin\theta &= 0\\
\Rightarrow\qquad T_{BC} &= -\frac{R_{B_x}}{\sin\theta}\\
&= -\frac{-15\sqrt{3}\,\text{N}}{\sqrt{3}/2} = 30\,\text{N}.
\end{aligned}$$

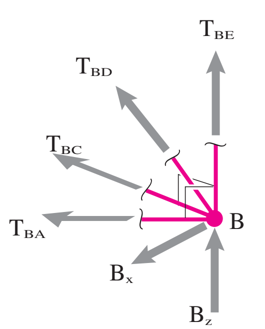



Figure 6.27: The free-body diagram of joint B.

Thus, the force in bar BC is $T_{BC} = 30\,\text{N}$ (tensile force).

$$\vec{\boldsymbol{R}}_A = 15\sqrt{3}\,\text{N}\hat{\boldsymbol{\imath}} - 30\,\text{N}\hat{\boldsymbol{j}} + 20\,\text{N}\hat{\boldsymbol{k}},\ \vec{\boldsymbol{R}}_B = -15\sqrt{3}\,\text{N}\hat{\boldsymbol{\imath}} + 90\,\text{N}\hat{\boldsymbol{k}},\ \vec{\boldsymbol{R}}_C = \vec{\boldsymbol{0}},\ T_{BC} = 30\,\text{N}$$

**SAMPLE 6.6** **A 3-D truss solved on the computer:** The 3-D truss shown in the figure is fabricated with 12 bars. Bars 1–5 are of length $\ell = 1\,\text{m}$, bars 6–9 have length $\ell/\sqrt{2} (\approx 0.71\,\text{m})$, and bars 10–12 are cut to size to fit between the joints they connect. The truss is supported at A on a ball and socket, at B on a linear roller, and at C on a planar roller. A load $F = 2\,\text{kN}$ is applied at D as shown. Write all equations required to solve for all bar forces and support reactions and solve the equations using a computer.

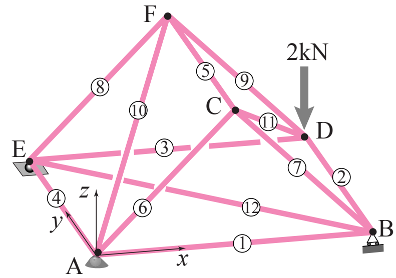



Figure 6.28

**Solution** There are 12 bars and 6 joints in the given truss. The unknowns are 12 bar forces and six support reactions (3 at A $(R_{A_x}, R_{A_y}, R_{A_z})$, 2 at B $(R_{B_y}, R_{B_z})$, and 1 at E $(R_{E_z})$). Therefore, we need 18 independent equations to solve for all the unknowns. Since the force equilibrium at each joint gives one vector equation in 3-D, *i.e.,* three scalar equations, the 6 joints in the truss can generate the required number $(6 \times 3 = 18)$ of equations. Therefore, we go joint by joint, draw the free-body diagram of the joint, write the force equilibrium equation, and extract the 3 scalar equations from each vector equation. We switch from the letters to denote the bars in the force vectors to numbers in its scalar representation $(T_1, T_2, \textit{etc.})$ to facilitate computer solution.

- **Joint A:**

$$T_1\hat{\imath} + \frac{T_6}{\sqrt{2}}(\hat{\imath} + \hat{k}) + \frac{T_{10}}{\sqrt{6}}(\hat{\imath} + 2\hat{\jmath} + \hat{k}) + T_4\hat{\jmath} + R_{A_x}\hat{\imath} + R_{A_y}\hat{\jmath} + R_{A_z}\hat{k} = \vec{0}.$$

- **Joint B:**

$$-T_1\hat{\imath} + \frac{T_7}{\sqrt{2}}(-\hat{\imath} + \hat{k}) + T_2\hat{\jmath} + \frac{T_{12}}{\sqrt{2}}(-\hat{\imath} + \hat{\jmath}) + R_{B_y}\hat{\jmath} + R_{B_z}\hat{k} = \vec{0}.$$

- **Joint C:**

$$-\frac{T_6}{\sqrt{2}}(\hat{\imath} + \hat{k}) - \frac{T_7}{\sqrt{2}}(-\hat{\imath} + \hat{k}) + T_5\hat{\jmath} + \frac{T_{11}}{\sqrt{6}}(\hat{\imath} + 2\hat{\jmath} - \hat{k}) = \vec{0}.$$

- **Joint D:**

$$-T_2\hat{\jmath} - \frac{T_{11}}{\sqrt{6}}(\hat{\imath} + 2\hat{\jmath} - \hat{k}) - T_3\hat{\imath} + \frac{T_9}{\sqrt{2}}(-\hat{\imath} + \hat{k}) - F\hat{k} = \vec{0}.$$

- **Joint E:**

$$-T_4\hat{\jmath} + \frac{T_{12}}{\sqrt{2}}(\hat{\imath} - \hat{\jmath}) + T_3\hat{\imath} + \frac{T_8}{\sqrt{2}}(\hat{\imath} + \hat{k}) + R_{E_z}\hat{k} = \vec{0}.$$

- **Joint F:**

$$-T_5\hat{\jmath} - \frac{T_8}{\sqrt{2}}(\hat{\imath} + \hat{k}) - \frac{T_{10}}{\sqrt{6}}(\hat{\imath} + 2\hat{\jmath} + \hat{k}) - \frac{T_9}{\sqrt{2}}(-\hat{\imath} + \hat{k}) = \vec{0}.$$

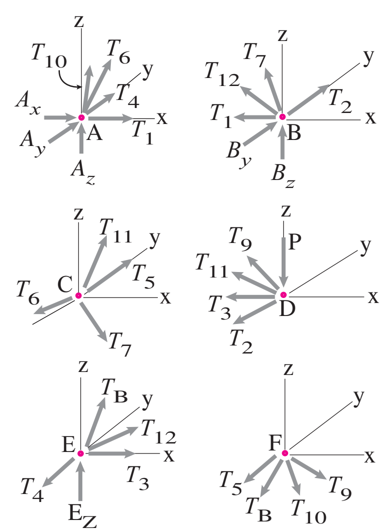


Figure 6.29

Now we can separate out 3 scalar equations from each of the joint vector equations by dotting them with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

| Joint | [Eqn.] $\cdot\,\hat{\imath}$ | [Eqn.] $\cdot\,\hat{\jmath}$ | [Eqn.] $\cdot\,\hat{k}$ |
|---|---|---|---|
| $A$ | $T_1 + \frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + R_{A_x} = 0,$ | $\frac{2}{\sqrt{6}}T_{10} + T_4 + R_{A_y} = 0,$ | $\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + R_{A_z} = 0$ |
| $B$ | $-T_1 - \frac{1}{\sqrt{2}}T_7 - \frac{1}{\sqrt{2}}T_{12} = 0,$ | $T_2 + \frac{1}{\sqrt{2}}T_{12} + R_{B_y} = 0,$ | $\frac{1}{\sqrt{2}}T_7 + R_{B_z} = 0$ |
| $C$ | $-\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0,$ | $T_5 + \frac{2}{\sqrt{6}}T_{11} = 0,$ | $\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0$ |
| $D$ | $-\frac{1}{\sqrt{6}}T_{11} - T_3 - \frac{1}{\sqrt{2}}T_9 = 0,$ | $-T_2 - \frac{2}{\sqrt{6}}T_{11} = 0,$ | $\frac{1}{\sqrt{6}}T_{11} + \frac{1}{\sqrt{2}}T_9 = F$ |
| $E$ | $\frac{1}{\sqrt{2}}T_{12} + T_3 + \frac{1}{\sqrt{2}}T_8 = 0,$ | $-T_4 - \frac{1}{\sqrt{2}}T_{12} = 0,$ | $\frac{1}{\sqrt{2}}T_8 + R_{E_z} = 0$ |
| $F$ | $-\frac{1}{\sqrt{2}}T_8 - \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0,$ | $-T_5 - \frac{2}{\sqrt{6}}T_{10} = 0,$ | $\frac{1}{\sqrt{2}}T_8 + \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0.$ |

Thus, we have 18 required equations for the 18 unknowns. Before we go to the computer, we need to do just one more little thing. We need to order the unknowns in some way in a one-dimensional array. So, let

$$x = [R_{A_x} \quad R_{A_y} \quad R_{A_z} \quad R_{B_x} \quad R_{B_y} \quad R_{E_z} \quad T_1 \ldots T_{12}].$$

Thus $x_1 = R_{A_x}$, $x_2 = R_{A_y}$, …, $x_7 = T_1$, $x_8 = T_2$, …, $x_{18} = T_{12}$. Now we are ready to go to the computer, feed these equations, and get the solution. We enter each equation as part of a matrix [A] and a vector {b} such that `[A]{x} = {b}`. Here is the pseudocode:

```
sq2i = 1/sqrt(2)            % define a constant
sq6i = 1/sqrt(6)            % define another constant
F = 2                       % specify given load
A(1,[1 7 12 16]) = [1 1 sq2i sq6i]
A(2,[2 10 16]) = [1 1 2*sq6i]
.
.
A(18,[14 15 16]) = [sq2i sq2i sq6i]
b(12,1) = F
form A and b setting all other entries to zero
solve A*x = b for x
```

The solution obtained from the computer is the one-dimensional array `x` which after decoding according to our numbering scheme gives the following answer.

$$R_{A_x} = R_{A_y} = 0,\ \ R_{A_z} = -2\,\text{kN},\ \ R_{B_y} = 0,\ \ R_{B_z} = 2\,\text{kN},\ \ R_{E_z} = 2\,\text{kN},$$
$$T_1 = T_3 = -2\,\text{kN},\ \ T_2 = T_4 = T_5 = -4\,\text{kN},\ \ T_6 = 0,$$
$$T_7 = T_8 = -2.83\,\text{kN},\ \ T_9 = 0,\ \ T_{10} = T_{11} = 4.9\,\text{kN},\ \ T_{12} = 5.66\,\text{kN},$$

# 6.1 Method of joints

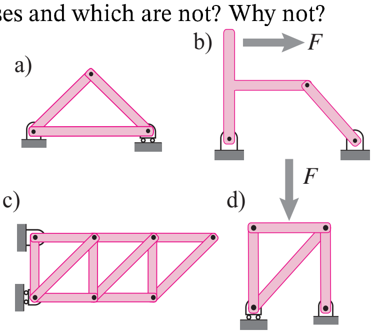
## Preparatory Problems

**6.1.1** Define these terms: a) truss, b) ideal truss, c) bar, d) joint, e) load, f) "bar force", g) bar tension, h) bar compression, i) reaction, j) roller support and k) pin support.

**6.1.2** Name as many positive attributes of trusses as you can.

**6.1.3** Name as many negative attributes of trusses as you can.

**6.1.4** Which of the structures below are trusses and which are not? Why not?



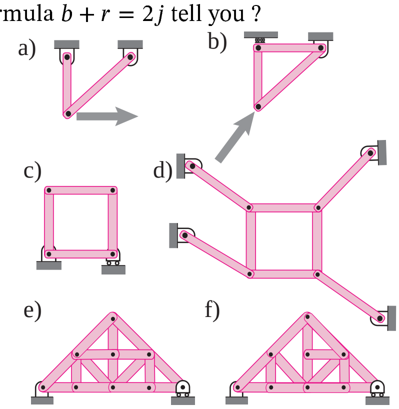

Problem 6.1.4

**6.1.5** Consider this formula

$$b + r = 2j$$

a) What do $b$, $r$, and $j$ stand for?
b) What is the use of this formula?
c) What is the source of this formula?

**6.1.6** For each of the trusses below: **i)** What are $b$, $j$, and $r$? **ii)** What does the formula $b + r = 2j$ tell you ?

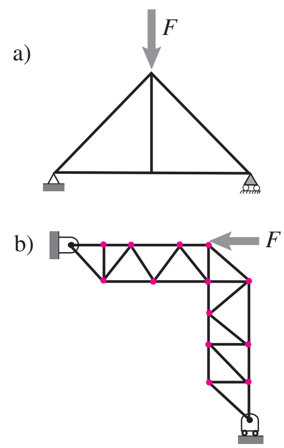



Problem 6.1.6

**6.1.7** For a given truss you are told values for $b$, $j$, and $r$.

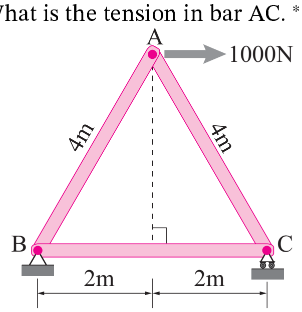
a) When solving the truss how many unknowns are you trying to solve for?
b) How many independent scalar equations do you have from using the method of joints on the whole structure?

**6.1.8** Find the zero-force members in the trusses below.



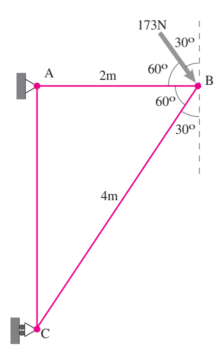

Problem 6.1.8

**6.1.9** What is the tension in bar AC. *



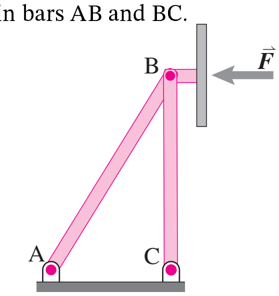

Problem 6.1.9

**6.1.10** The only force acting on the negligible-weight truss ABC is the 173 N force shown. Find the tension in the bar AB. *



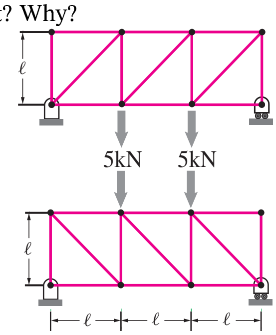

Problem 6.1.10

**6.1.11** A billboard is supported by a two bar truss as shown in the figure. The two bars have pin joints at A, B, and C. Angle ABC is 30°. The total wind load on the board is estimated to be 300 N, find the forces in bars AB and BC.




Problem 6.1.11

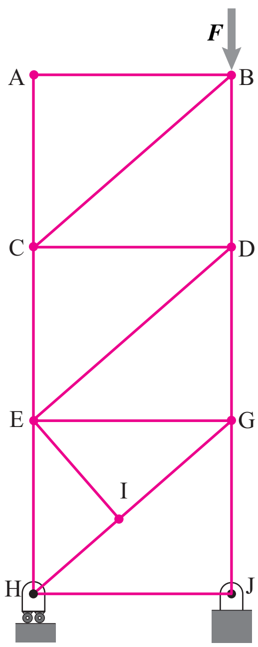
**6.1.12** Find the support reactions for the two trusses without any (written) calculations. Should the support reactions be different? Why?




Problem 6.1.12

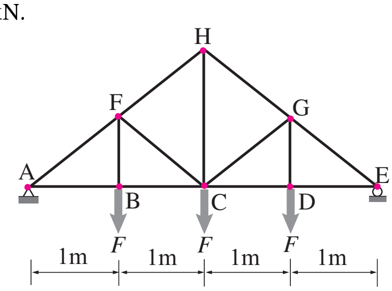
## More-Involved Problems

**6.1.13** Sketch the truss below. Write a big clear zero on top of each of the zero-force members. *



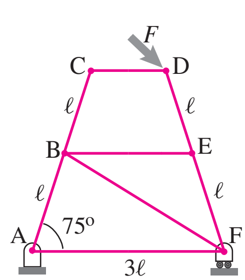

Problem 6.1.13

**6.1.14** Find the support reactions on the truss shown in the figure taking $F = 5\,\text{kN}$.



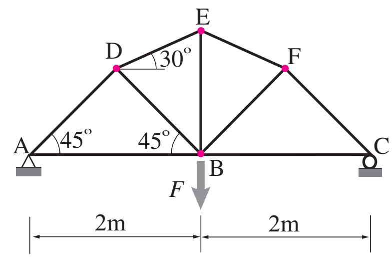

Problem 6.1.14

**6.1.15** Find the support reactions at A and F for a load $F = 3\,\text{kN}$ acting at D at 45° with respect to CD, if $\ell = 1\,\text{m}$ and $\theta = 75°$. How will the support reactions change if bar BF was removed and used to connect joints A and E instead of B and F?



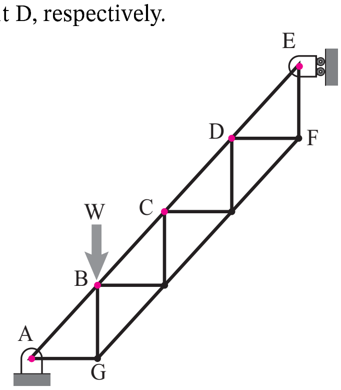

Problem 6.1.15

**6.1.16** How do the support reactions on the truss shown in the figure change if the load at point B is replaced by three equal loads, $F/3$ each, acting at points D, E, and F?



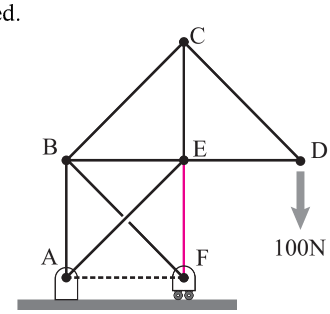

Problem 6.1.16

**6.1.17** The stairstep truss shown in the figure has 500 mm long horizontal and vertical bars. Find the support reactions at A and E when a load $W = 1\,\text{kN}$ is applied at (a) point B, (b) point C, and (c) point D, respectively.




Problem 6.1.17

**6.1.18** In the truss shown in the figure, how does the force in bar EF change if the diagonal bar BF is removed and another bar AF (shown by dotted line) is introduced instead? You can assume any reasonable dimensions for the bars if needed.



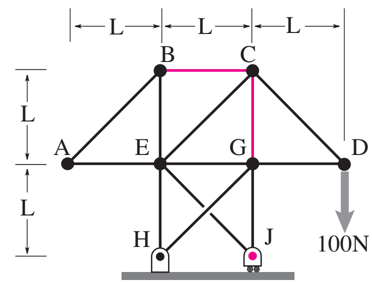

Problem 6.1.18

**6.1.19** For the truss shown, find:

a) The reaction at J.

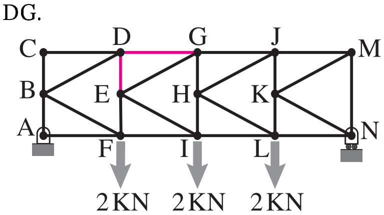
b) The bar force in BC (tension or compression).

c) The force in bar CG (tension or compression).



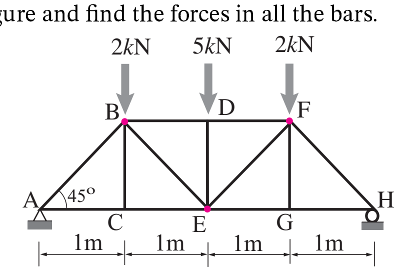

Problem 6.1.19

**6.1.20** The truss shown in the figure consists of 4 square bays of a 'K' structure. Each bay has two 2 m long horizontal, two 1 m long vertical bars, and two diagonal bars. Find the tensions in rods DE and DG.



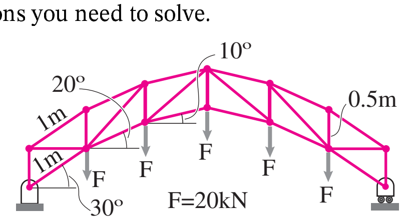

Problem 6.1.20

**6.1.21** Analyze the truss shown in the figure and find the forces in all the bars.




Problem 6.1.21

**6.1.22** Analyze the truss shown in the figure and find out forces in all bars. Use symmetry to reduce the number of equations you need to solve.




Problem 6.1.22