5.5 3D statics

The structures and machines we study are most-often adequately modeled as 2D. In those cases a 2D analysis gives about the same answer as a 3D analysis would have given. However, sometimes a 2D model is inadequate, and a 3D analysis is needed. Here we use 3D statics to find various unknown aspects of forces acting on one part. By learning the 3D approach you can get a better sense of when to use a 2D model (which is most of the time for most engineers).

3D statics is conceptually the same as 2D: draw a free-body diagram and use the force and moment-balance equations. However, the geometry can be more of a challenge, the moment balance equation becomes a full vector equation (instead of just having one non-zero component, it has three)[1], and the number of scalar equations from one free-body diagram increases from 3 to 6. In 3D, issues related to static-indeterminacy arise more often and more subtly.

[1]For 2D problems we used the phrase 'moment about a point' to be short for 'moment about an axis in the z direction that passes through the point. In 3D moment about a point is a 3-component vector.

The statics-of-a-3D-object recipe

Our recipe here:

1) **Draw a free-body diagram (FBD)** of the part of interest. Use knowledge of the contact conditions (see Chapter 3) to draw known and unknown aspects of the forces appropriately (see fig. 2.42 on page 169) [hint: use of the form $F\hat{\boldsymbol{\lambda}}$ is often appropriate];
2) **Write equilibrium equations** in terms of the forces (and couples) shown on the FBD;
3) **Solve the equilibrium equations** for unknowns.

The brute-force approach to statically determinate problems

A problem is *statically determinate* when all as-yet-unknown forces can be found using the equilibrium equations. In 3D statics, this generally means that the two vector equilibrium equations

$$\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}} \qquad \text{and} \qquad \sum \vec{\boldsymbol{M}}_{i/C} = \vec{\boldsymbol{0}}$$

(where C is any point that you like) make up 6 independent scalar equations which you can solve for 6 unknown aspects of the applied forces (say the magnitudes of 6 forces whose directions are known *a priori*)[2].

[2]Most elementary text-book problems are statically determinate. Unfortunately most real-world problems, when you first model them, are not statically determinate.

Alternative equation sets

In 2D single-part statics, we noted various alternatives to using vector force balance and moment about one point (see page 257). Similarly, here there are also an infinite number of true equilibrium equations, for example,

- $\left(\sum \vec{F}_{\rm i}\right) \cdot \hat{\boldsymbol{\lambda}} = 0$ where $\hat{\boldsymbol{\lambda}}$ is a vector in any direction you please; and
- $\left(\sum \vec{M}_{/\rm C}\right) \cdot \hat{\boldsymbol{\lambda}} = 0$. This is moment balance about an axis through C in the $\hat{\boldsymbol{\lambda}}$ direction.

From these there are various ways to extract 6 independent scalar equations, including:

- Cartesian components of force balance and moment balance about any point C: $\sum F_x = 0, \sum F_y = 0, \sum F_z = 0, \sum M_{Cx} = 0, \sum M_{Cy} = 0$, and $\sum M_{Cz} = 0$. This always works, although it does not necessarily minimize algebra.
- Force balance in any 3 non-coplanar directions and moment balance about point C resolved in any three non-coplanar directions.
- Moment balance about 6 independent axes. There seems to be no simple description of independent axes but that they give independent equilibrium equations [3]. Practically speaking, six moment-about-an-axis equations are likely to be independent if not too many axes are parallel with each other, not too many are coplanar, and not too many intersect at one point.

③ One can test the sufficiency of the equations by this check: if a force at the origin and a couple are the only forces applied to a system, do the equations demand that they must both be zero? If so, then you have an independent set of equations.

In any case, force balance contributes at most 3 independent equations and moment balance can contribute up to 6 (thus rendering force balance a non-essential tool).

Solving 6 equations in 6 unknowns, or even setting up such for computer solution, is relatively time-consuming and error-prone. Thus one looks for shortcuts when one can, namely:

Useful shortcuts:

- Use moment balance about an axis that intersects, or is parallel to, as many unknown force lines-of-action as possible (thus those forces do not show up in that equilibrium equation);
- Use force balance in a direction orthogonal to as many of the unknown forces as possible (so those forces don't show up in that equation).

Special loadings

Two- and three-force bodies

The concepts of two-force (page 258) and three-force (page 260) bodies are identical in 3D.

- If there are only two forces applied to a body in equilibrium, they must be equal and opposite and acting along the line connecting the points of application. The full set of six equations tells you no more.
- If there are only three forces applied to an object, they must all be in the plane of the points of application and the three forces must have lines of action that intersect at one point. The three equations of force balance are an additional restriction on these three forces.

There are other special loadings where the equilibrium equations offer less than 6 independent equations:

- **2D**. If all of the forces have **lines of action in one plane**, then there are only three independent scalar equations and thus one can solve for 3 unknowns. For example, if all the forces lie in the xy plane, then automatically $\sum F_{iz} = 0$, $\sum M_{ix/C} = 0$, and $\sum M_{iy/C} = 0$.
- **Concurrent forces**. If all the lines of action intersect in one point, say D, then $\sum \vec{\boldsymbol{M}}_{/\mathrm{D}} = \vec{\boldsymbol{0}}$ is automatically satisfied and only the 3 equations of force balance are independent.
- If all the forces are **parallel** in, say, the $\hat{\boldsymbol{k}}$ direction, then force balance in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions as well as moment balance about any axis in the $\hat{\boldsymbol{k}}$ direction, is automatically satisfied and there are only three independent equilibrium equations (say $\sum F_z = 0$, $\sum M_x = 0$ and $\sum M_y = 0$.

What does it mean for a problem to be ‘2D’?

The world we live in is three-dimensional: all the objects we wish to study mechanically are three-dimensional, and if they are in equilibrium, they satisfy the three-dimensional equilibrium equations. How then can an engineer justify doing 2D mechanics? There are a variety of overlapping justifications.

- The 2D equilibrium equations are a subset of the 3D equations. In both 2D and 3D, $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \vec{\boldsymbol{M}}_{/0} \cdot \hat{\boldsymbol{k}} = 0$. So, if when doing 2D mechanics, one just neglects the z component of any applied forces and the x and y components of any applied couples, one is doing correct 3D mechanics, just not all of 3D mechanics. If the forces or conditions of interest to you are contained in the 2D equilibrium equations, then 2D mechanics is really 3D mechanics, ignoring equations you don’t need.
- If the xy plane is a plane of symmetry for the object and of any applied loading, then the 3D equilibrium equations not covered by the

2D equations are automatically satisfied. For a car, say, the assumption of symmetry implies that the forces in the z direction will automatically add to zero, and the moments about the x and y axis will automatically be zero.

- If the object is thin and there are constraint forces holding it near the xy plane, and these constraint forces are not of interest, then 2D statics is also appropriate. This last case is caricatured by all the poor mechanical objects you have drawn so. They are conceptually constrained to lie in your flat paper by invisible slippery glass in front of and behind the paper.

“Internal forces” in 3D

At a free-body-diagram cut on a long narrow structural piece in 2D, we have two force components (tension and shear) and one scalar moment. In 3D, such a cut shows a force $\vec{\boldsymbol{F}}$ and a moment $\vec{\boldsymbol{M}}$ each with three components. If one picks a coordinate system with the x axis aligned with the bar at the cut, the concept of tension remains the same. Tension is the force component along the bar.

$$T = F_x = \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{\imath}}.$$

The two other force components, F_y and F_z, are two components of shear. The net shear force is a vector in the plane orthogonal to $\hat{\boldsymbol{\imath}}$.

The new concept, often called *torsion* is the component of $\vec{\boldsymbol{M}}$ along the axis:

$$\text{torsion} = M_x = \vec{\boldsymbol{M}} \cdot \hat{\boldsymbol{\imath}}$$

Torsion is the part of the moment that twists the shaft.

The remaining part of the $\vec{\boldsymbol{M}}$, in the yz plane, is the bending moment. It has two components M_y and M_z.

The preponderance of statically indeterminate problems

As noted in box 5.1 on page 247, non-uniqueness of solutions is common in the real world. Especially in 3D, real world problems are, at first blush, rarely statically determinate. The statics equations are relevant and provide useful information, but are not sufficient for finding all unknowns of interest. Finding the forces then depends on knowing the deformation properties of the structures as well as details of their initial state.

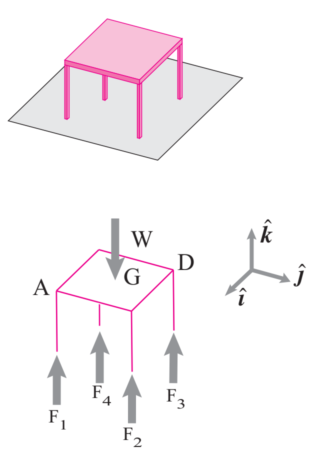


Figure 5.78: A symmetric 4 leg table on a flat floor. Statics is insufficient to find the 4 forces of the ground on the table. The full weight could be carried by either diagonal pair of legs.

Example: Four-leg furniture

Take the table, chair or bed you are now interacting with. It probably has 4 legs. To keep it simple, imagine the legs are on a slippery (negligible-friction) floor and the table is symmetric (left-right and front-back). What are the forces of the floor on the legs? The most we can get from the statics equations (force and moment balance) are that

$$F_1 = F_3, \quad F_2 = F_4, \qquad \text{and} \qquad F_1 + F_2 = W/2.$$

If we insist that there is no glue between the floor and table then $F_1 \geq 0, F_2 \geq 0, F_3 \geq 0, F_4 \geq 0$. But we still can't find the reactions. Here is the variety of solutions:

	$F_1 = F_3 = W/2$	and	$F_2 = F_4 = 0$
or	$F_1 = F_3 = 0$	and	$F_2 = F_4 = W/2$
or	$F_1 = F_3 = W/4$	and	$F_2 = F_4 = W/4$
or	$F_1 = F_3 = C$	and	$F_2 = F_4 = W/2 - C$
			(with C anything in the interval $0 \leq C \leq W/2$).

It takes more than just statics to find the forces. One has to know the exact initial shape of the table and floor, and how the table and floor 'give way' in response to loads.

The lack of static determinacy of a table is not merely an academic curiosity. If you measured the forces of the floor on your table legs, they each could well differ noticeably from $W/4$, with, probably, two bigger and two smaller. Once friction is taken into account, the situation is near hopeless.

Example: Statically determinate stool
Is it even possible to make a stool in 3 dimensions that is statically determinate? Here's one way. Give it three legs. One leg can have a point frictional contact (3 reaction components), one leg can have a wheel (2 reaction components) and one can be frictionless (like with a castered wheel, 1 reaction component). $3 + 2 + 1 = 6$.

In general, it is hard to hold an object in place in three dimensions in a statically determinate manner. Here are some other ways (besides the unusual stool above):

- with six rods that have ball-and-socket joints at both the object-end and at the ground-end. The rods need to have a variety of orientations and attachment points (this idea is used in a 'Stewart Platform').
- With one ball-and-socket joint and three rods.
- A 3-leg stool with three wheels (at the contact points. For each wheel one can draw a line in the plane in the direction normal to rolling. These three lines must not intersect at a point).
- With one hinge and one two-force-member rod.
- With one axially sliding hinge and two rods.
- With a single welded connection.

Given that many things are held in place in a manner that seems statically indeterminate, what can one do in practice? A common approach is to remove reaction components that you think are relatively unimportant. Some examples:

- A door held by two hinges. That's 10 reaction components. Usually one replaces, in the analysis, the hinges with ball-and-socket joints. That makes 6 unknown reaction components but is still statically indeterminate no matter what the loading (the force along the line connecting the joints cannot be decomposed into parts acting at each joint). So one joint is allowed to slide along the nominal hinge axis.

- 4-leg furniture. Counting friction, there are 12 reaction components. If side loads are not an issue, then we can assume-away friction. Thus we have only 4 reaction components for 3 equations (see table example above). We can get a unique solution by assuming the forces share the symmetry of the table (thus $F_1 = F_2$).

Given this complex state of affairs in 3D, it is easy to see why engineers often resort to the more-easily-made determinate 2D world for their models and analyses.

5.8 Statically determinate ways to hold an object in 3D

An advanced aside.

It takes some thought to find ways to hold something in place in 3D that are statically determinate. If you hold it securely enough so that it can't move you will often do it in a way that there can be reaction forces even when there are no loads on the structure. These 'locked-in' forces are sometimes called 'pre-stresses', 'pre-loads' or 'self-stresses'. The values of these locked-in forces will depend on the construction. You can't find them from the equations of statics. Hence the indeterminacy.

Below we show some ways to hold things in a statically determinate way. Static determinacy is great for homework problems. Whether it is good or bad in structural design is not a simple question.

The big tradeoff. Before showing how to achieve determinacy, a warning.

> A statically determinate structure has no redundancy.
> A redundant structure can have locked-in forces.

And locked-in forces can contribute to structural failure. Two parallel rods holding something in place can break because they fight each other, even when there is no external load on the thing.

Counting rules. Counting dominates the discussion. Here's how to do that counting. The rule of thumb for determinacy is:

$$\underbrace{\text{Number of equations}}_{n_{eq}} = \underbrace{\text{number of unknowns}}_{n_{unkn}}$$

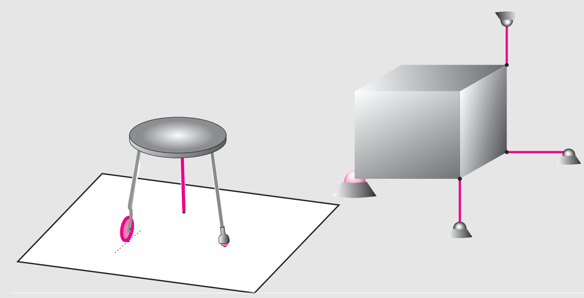
For one object in 3 dimensions we have 6 independent equilibrium equations, for example 3 components of force balance and 3 components of moment balance about a given reference point. Thus $n_{eq} = 6$. To achieve determinacy we thus want $n_{unkn} = 6$. That is, in a free-body diagram of the object we also want 6 unknown reaction components.

Attachment schemes. For each attachment point we can select an attachment type from the table of common connections on page 1219. Here's a compact list with the number (#) of unknown reaction components:

3D Connection	**#**	**what**
weld	6	3 moment and 3 force comps
keyed slot	5	3 moment & 2 force comps
hinge	5	3 moment & 2 force comps
linear bearing	4	2 moment & 2 force comps
ball &socket	3	3 force components
no-slip point contact	3	3 force comps
skate or wheel	2	normal force and side force
sloppy linear bearing	2	2 force components
rod or string	1	tension along 2-force member
frictionless point contact	1	normal force

Mix and match. At least to satisfy the counting rules, you can hold something in place in a statically determinate manner by selecting any number of connections from the table above just make the number of unknown components add to 6. Here are some options:

Scheme	$\sum = 6$
* One weld	6=6
* Hinge+rod	5+1=6
* Ball & socket + sloppy linear bearing + rod	3+2 + 1 =6
* 6 rods	1+1+1+1+1+1=6
* Ball & socket + 3 rods	3 +1+1+1=6
* No-slip point contact + wheel + frictionless point contact	3 + 2+1 =6
* 3 wheels or skates	2+2+2 = 6
* Etc	n + m + ... = 6



A statically determinate stool has one normal leg (with friction), one wheel and one frictionless leg (shown as a ball). **A box** is held in a statically determinate way with a ball & socket and three bars.

Bad luck. Although the rule

$$n_{eq} = n_{unkn}$$

is a good starting point, it is not enough. In the bad cases, two things happen together:

1) The object is not held in place (some loads can not be equilibrated), and
2) The constraints can fight each other.

Then, in the language of linear algebra, the 6×6 matrix for the linear equilibrium equations is singular. So, for some loadings no solution exists (the object is not held in place). And if a solution exists, it is not unique (the homogeneous equations have non-trivial solutions, there can be locked-in forces).

Bad examples. Some ways of holding where 6 reaction forces lead to *indeterminacy* include:

- A hinge + rod with the rod co-planar with the hinge.
- 6 rods where some 4 or more of them have lines of action that intersect in one point.
- A ball & socket with 3 rods where one or more of the rods has a line of action that goes through the ball & socket.
- A ball & socket with two rods that are coplanar with the ball.
- Three wheels on a plane where the normal-lines to the rolling direction all intersect in a single point (the wheels roll on a common circle).