5.3 Equilibrium with frictional contact

Contacting objects are prevented from passing through each other by pressing against each other. Generally there is also some frictional resistance to relative slip. We have neglected friction so far for simplicity and because the neglect of friction is a reasonable approximation for some lubricated contact problems. On the other extreme, in some situations we have assumed that friction so well resists slip that we assumed 'no slip' and that frictional contact acts like a hinge or weld. Either way, with friction negligibly small, or reliably large, we have not worried about it.

However, for some purposes friction forces are not reasonably neglected during slip. Or, when there is no slip, sometimes we have to worry about whether the frictional bond is strong enough to prevent slip.

Although slip means motion and motion sounds like dynamics (contradicting the premise of statics), there are many situations where there is enough motion for friction to be important but not so much acceleration that inertial terms ($m\vec{\boldsymbol{a}}$) are important.

How friction forces are represented on free-body diagrams was discussed in Section 2.3 which you should review before proceeding further here. We will now consider friction forces in equilibrium conditions.

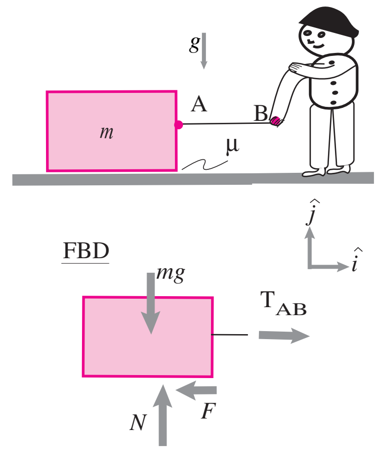


Figure 5.43: Pulling a block on a frictional ground, with related FBD. The complicated distribution of normal forces and friction forces on the block from the ramp have been replaced with the equivalent pair N and F. For particle mechanics, we don't worry about where N and F are applied.

For simplicity, and because of the relatively high accuracy to complexity ratio, we consider only Coulomb friction with a single coefficient of friction μ①.

①If you have studied friction before, our approximation is that $\mu = \mu_{static} = \mu_{dynamic}$ adequately captures the complex and hard to quantify reality of frictional forces

Example: Drag a block with friction.
Consider the block with friction (fig. 5.43). You want to pull it slowly to the right with rod AB. Say $m = 100\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, and $\mu = 0.3$.
Force balance, using the forces on the free-body diagram gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{AB}\hat{\imath} + N\hat{\jmath} - F\hat{\imath} = \vec{0} \tag{5.19}$$

These, with the friction relation $F = \mu N$, are 3 scalar equations in T_{AB}, F and N with solution $N = mg = 1000\,\mathrm{N}, F = \mu mg = 300\,\mathrm{N}$, and $T_{AB} = \mu mg = 300\,\mathrm{N}$.

Example: Drag a block on a ramp with friction.
Consider the block with friction on a slope (fig. 5.44). You want to hold it with rod AB. Maybe you want to (i) slide it up slowly, or (ii) down slowly or (iii) hold it still. Say $m = 100\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, $\theta = 45^\circ$, and $\mu = 0.3$.
Force balance, using the forces on the free-body diagram, gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{AB}\hat{\boldsymbol{e}}_1 + N\hat{\boldsymbol{e}}_2 - F\hat{\boldsymbol{e}}_1 = \vec{0} \tag{5.20}$$

These, with the friction relation, are 3 scalar equations in T_{AB}, F and N.

Summing forces in the rope direction and normal to the plane we get:

$$\begin{aligned} \{(\text{Eqn. 5.20})\}\cdot\hat{\boldsymbol{e}}_1 \quad &\Rightarrow \quad -mg\sin\theta + T_{AB} - F = 0 \\ \{(\text{Eqn. 5.20})\}\cdot\hat{\boldsymbol{e}}_2 \quad &\Rightarrow \quad mg\cos\theta + N = 0 \end{aligned} \tag{5.21}$$

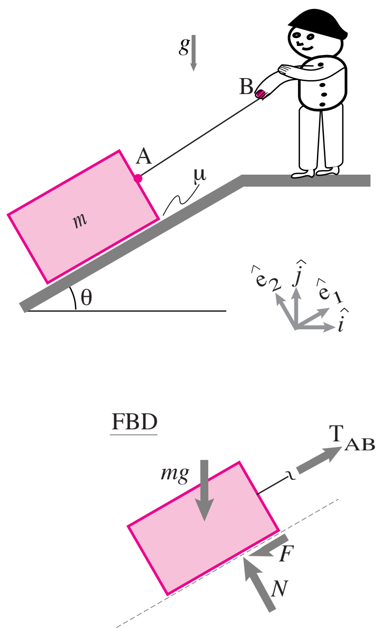


Figure 5.44: Pulling a block up a frictional slope with related FBD. As in the previous example, the pair N and F represent the net normal and frictional forces.

②**Caution:** A common beginner's mistake is to assume the equation $F = \mu N$ applies when there is friction. Rather, if the friction is preventing slip, F could be anything so long as $|F| \le \mu N$. And if the slip is opposite in direction from that implicitly assumed in the free-body diagram, then $F = -\mu N$ (see case (ii) in the example above).

③ We could be tricky and get a single equation for the scalar T_{AB} by dotting both sides of eqn. (5.20) with a vector orthogonal to the resultant of $N\hat{\boldsymbol{e}}_2 - F\hat{\boldsymbol{e}}_1$. For the case of uphill sliding such a vector would be $\hat{\boldsymbol{e}}_1 + \mu\hat{\boldsymbol{e}}_2$.

or, for the quantities given $N = (100\,\text{kg})(10\,\text{m/s}^2)(\cos 45^\circ) = 707\,\text{N}$. We assume that F and N are related by friction described with the standard Coulomb's friction model②:

i) $F = \mu N$ if the block is sliding up;

ii) $F = -\mu N$ if the block is sliding down; or

iii) $-\mu N \le F \le \mu N$ if the block is not sliding.

Solving eqn. (5.20) with the friction relations gives③.

i) $T_{\text{AB}} = mg\,(\mu\cos\theta + \sin\theta)$ if the block is sliding up;

ii) $T_{\text{AB}} = mg\,(-\mu\cos\theta + \sin\theta)$ if the block is sliding down; Note that if $\tan\theta < \mu$ then $T_{\text{AB}} < 0$ and it then takes a push to slide down; or

iii) $mg\,(-\mu\cos\theta + \sin\theta) \le T_{\text{AB}} \le mg\,(\mu\cos\theta + \sin\theta)$ if the block is not sliding. If $\tan\theta < \mu$ then $T_{\text{AB}} = 0$ is amongst the solutions for, so no sliding and the block can sit still on the slope with no pull on the rope.

Note that the tension T_{AB} scales with mg. So doubling m or g doubles all the forces in all of these answers, as you might guess from dimensional considerations. The mathematically-abstract-sounding issues of existence and uniqueness often show up in friction problems. For example, sometimes there is no statics solution (non-existence).

Example: Block on ramp.
A statics problem without a solution. A block with coefficient of friction $\mu = 0.5$ is in static equilibrium sliding steadily down a 45° ramp (fig. 5.45). *Not!* If there is constant velocity motion, then statics would apply. But the forces in the free-body diagram cannot add to zero (because the resultant of the friction and normal force is tipped up and to the left and thus cannot be parallel to the vertical gravity force). The assumptions are not consistent with statics (actually this is a dynamics problem, the block accelerates down the ramp). If you saw a block just sitting there on a ramp, then you can be sure that the slope and friction coefficient are not those given above.

Friction problems might be studied with a particle model, as above, or also with moment balance.

Example: Dragged block as an extended body.
This is a repeat of the first example on page 1. One might wonder if the dragging causes an uneven distribution of force up on the block. Does the block dragging back, for example, cause a bigger pressure on the back? As a simple model, assume all the ground force is at the front and back edge of the block. Force balance gives basically the same information as for the particle model, namely that:

$$N_C + N_D = W \quad \text{and} \quad \underbrace{F_C}_{\mu N_C} + \underbrace{F_D}_{\mu N_D} = T_{AB} \quad \Rightarrow \quad T_{AB} = \mu W.$$

One can find more with moment balance about any point you like, say C, with force balance giving

$$\sum M_{/C} = 0 \quad \Rightarrow \quad N_D = \frac{W}{2} + \frac{\mu h W}{2\ell} \quad \text{and} \quad N_C = \frac{W}{2} - \frac{\mu h W}{2\ell}$$

So there is more pressure on the front than the back. This difference goes away if either the friction or the height of the string attachment vanishes.

Conditional contact, consistency, and contradictions

There is a natural hope that a subject will reduce to the solution of some well defined equations. For better and worse, things are not always this simple. For better, because it means that the recipes are still not so well defined that computers can easily steal the subject of mechanics from people. For worse, because it means you have to think hard to do some mechanics problems.

One source of these difficulties is the conditional nature of the equations that govern contact. For example:

- The ground pushes up on something to prevent interpenetration *if* the pushing is positive, *otherwise* the ground does not push up.
- The force of friction opposes motion and has magnitude μN *if* there is slip, *otherwise* the force of friction is something less than μN in magnitude.
- The distance between two points is kept from increasing by the tension in the string between them *if* the tension is positive, *otherwise* the tension is zero.

These conditions are, implicitly or explicitly, in the equations that govern these interactions. One does not always know which of the alternative contact conditions, if either, apply when one starts a problem. Sometimes multiple possibilities need to be checked[4].

On a FBD at every point of frictional contact

- *If* the direction of slip or impending slip is known, *either*
  - Draw a normal force N and a friction force $F = \mu N$ opposing the relative slip, *or*
  - Draw a single force R at an angle ϕ from the normal of the contact in the direction which resists slip (with $\tan\phi = \mu$)
- *If* there is no slip, *either*
  - Draw a normal force N and tangential force F *or*
  - Draw a single force vector $\vec{\boldsymbol{R}}$ with unknown components
- *If* you don't know whether or not there is slip, *first*
  - Guess that there is no slip, *then*
  - Solve the equilibrium equations, *then*
    * *If* $F \leq \mu N$: you guessed right and have found a solution to both the equilibrium and friction equations.
    * *If* $F > \mu N$: you guessed wrong and have to guess that there is slip in one direction (guess which), *then*

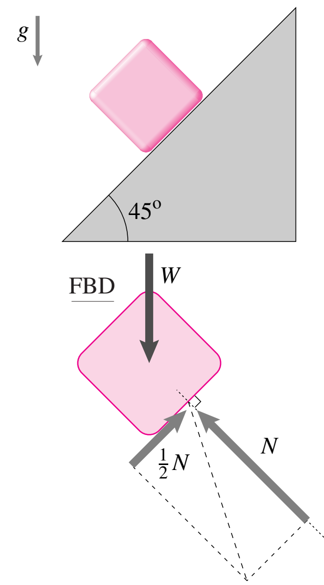


Figure 5.45: Block on steep ramp and related FBD.

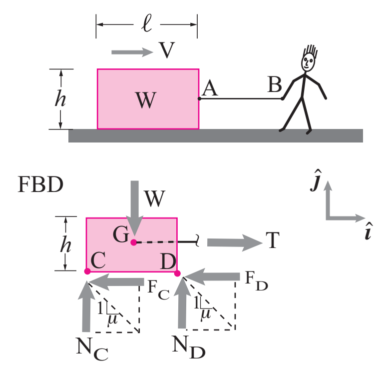
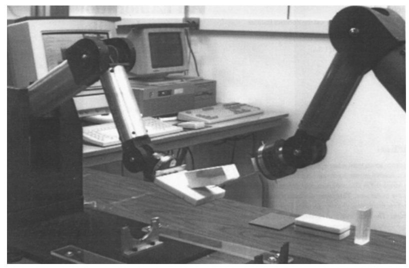


Figure 5.46: Dragging a block, taking account (in a simple way) the distribution of contact forces from the ground. Assume slip to the right is occurring.

4 All of the combinatorics in this recipe follows from the inability of our mathematics to deal easily with relations between variables with the step shape of Coulomb friction (fig. 2.66 on page 188).

- see if you can solve the equilibrium equations, if not *then*
- assume slip in the opposite direction and try to solve the equilibrium equations, if you can't, *then*
- the problem has no solution

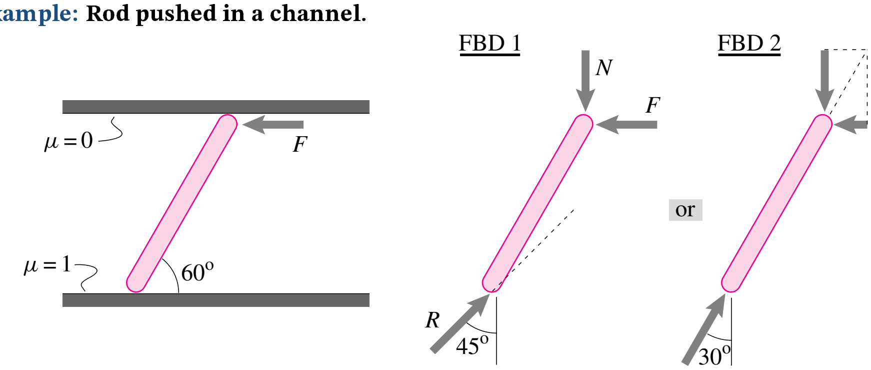
Example: Robot hand



Roboticist Michael Erdmann has designed a palm-like robot 'hand' that manipulates objects without squeezing them. The flat robot palms just move around and the object consequently slides. Determining whether the object slides on one or the other, or possibly on both hands in a given movement is a matter of case study. The computer checks to see if the equilibrium equations can be solved with the assumption of sticking or slipping at one or the other contact.

Once you find a solution to a problem with friction there remains the possibility of multiple solutions, in this case for different reasons than the usual static indeterminacy. The following problem shows a case where a statics problem has multiple solutions due to frictional effects.

Example: Rod pushed in a channel.



A light rod is just long enough to make a 60° angle with the walls of a channel. One channel wall is frictionless and the other has $\mu = 1$. What is the force needed to keep it in equilibrium in the position shown? If we assume it is sliding we get the first free-body diagram. The forces shown can only be in equilibrium if all forces are zero. So a solution is that the rod slides in equilibrium with no force. If we assume that the rod is not sliding, the friction force on the lower wall can be at any angle between $\pm 45^\circ$. Thus we have equilibrium with the second FBD for arbitrary positive F. This is a second set of solutions. A rod like this is said to be *self locking* in that it can hold an arbitrary large force F without slipping. That we have found freely slipping solutions, with no force, and jammed solutions, with arbitrary force, corresponds physically to one being able to easily slide a rod like this down a slot and then also at a different instant, have the rod totally jamb. Some rock-climbing equipment depends on such self-locking and easy release.

Statically indeterminate problems

When there are two or more points of frictional contact and there is no slip nor impending slip, then static indeterminacy is likely.

Example: Chair with friction

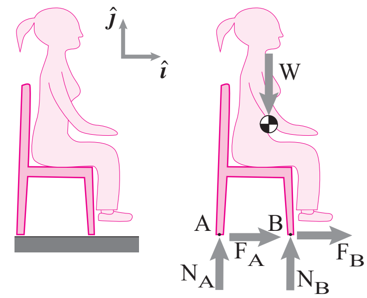


Figure 5.47: A chair with friction at the feet.

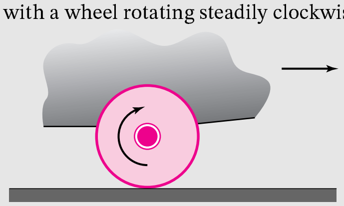
5.7 Undriven wheels and two-force bodies

One often hears whimsical reverence for the "invention of the wheel." Now, using elementary mechanics, we can gain some appreciation for this revolutionary way of sliding things.

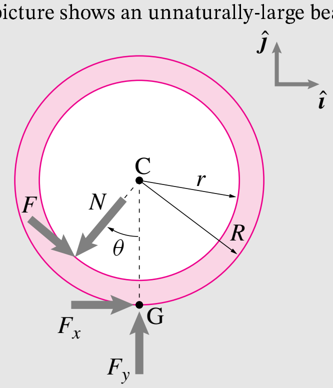
Without a wheel, the force it takes to drag something is about μW. Since μ ranges between about .1 for teflon, to about .6 for stone on ground, to about 1 for rubber on pavement, you need to pull with a force that is on the order of a half of the full weight of the thing you are dragging.

You have seen how rolling on round logs cleverly take advantage of the properties of two-force bodies (page 259). But that good idea has the major deficiency of requiring that logs be repeatedly picked up from behind and placed in front again. So we invent the wheel.

The simplest wheel design uses a dry "journal" bearing consisting of a non-rotating shaft protruding through a near close fitting hole in the wheel. Here is shown part of a cart rolling to the right with a wheel rotating steadily clockwise.



To figure out the forces involved we draw a free-body diagram of the wheel. We neglect the wheel's weight because it is generally much smaller than the forces it mediates. To make things more clear, the picture shows an unnaturally-large bearing hole r.



The force of the axle on the wheel has a normal component N and a frictional component F. The force of the ground on the wheel has a part holding the cart up F_y and a part along the ground F_x which will surely turn out to be negative for a cart moving to the right. If we take the wheel dimensions to be known, and also the vertical part of the ground reaction force F_y (the weight born), we have as unknowns N, F, θ and F_x. To find these we could use the friction equation for the sliding bearing contact

$$F = \mu N;$$

force balance

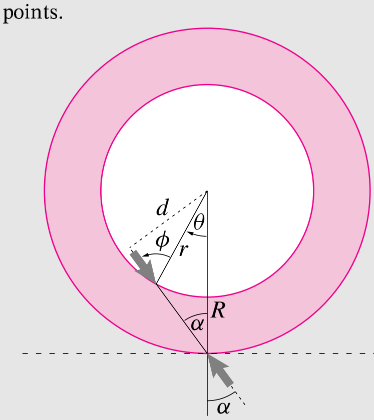
$$F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + N(-\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) + F(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}) = \vec{\mathbf{0}},$$

which could be reduced to 2 scalar equations by taking components or dot products; and moment balance about C, which we calculate with forces and perpendicular distances as

$$Fr + F_x R = 0.$$

Of key interest is finding the force resisting motion F_x. With some mathematical manipulation we could solve the 4 scalar equations above for any of F_x, N, F, and θ in terms of r, R, F_y, and μ. We follow a more intuitive approach instead.

As modeled, the wheel is a two-force body so the free-body diagram shows equal and opposite collinear forces at the two contact points.



(continued...)

If we assume Coulomb friction at the chair feet, we know that

$$|F_A| \leq \mu N_A \qquad \text{and} \qquad |F_B| \leq \mu N_B$$

The equilibrium equations tell us (assuming for simplicity that W acts in the middle of the chair):

$$F_A + F_B = 0, N_A = N_B = W/2.$$

Putting these equations together we find that

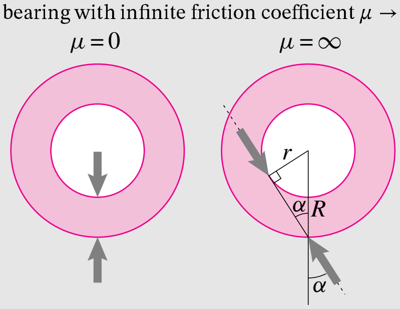
$$-W/2 \leq F_A \leq W/2 \qquad \text{and} \qquad F_B = -F_A$$

and no more. That is, all we can tell is that both are within the friction limits and that the horizontal forces cancel each other.

5.7 Undriven wheels and two-force bodies (continued)

The friction angle ϕ describes the friction between the axle and wheel (with $\tan\phi = \mu$). The angle α describes the *effective* friction of the wheel. This is not the friction angle for sliding between the wheel and ground which is assumed to be larger (if not, the wheel would skid and not roll), probably much larger. The *specific resistance* or the *coefficient of rolling resistance* or the *specific cost of transport* is $\mu_{\text{eff}} = \tan\alpha$. (If there was no wheel, and the cart or whatever was just dragged, the specific resistance would be the friction between the cart and ground $\mu_{\text{eff}} = \mu$.)

Although we can solve for α in terms of μ or ϕ let's first consider two extreme cases: one is a frictionless bearing and the other is a bearing with infinite friction coefficient $\mu \to \infty$ and $\phi \to 90^\circ$.



In the case that the wheel bearing has no friction we satisfyingly see clearly that there is no ground resistance to motion. The case of infinite friction is perhaps surprising. Even with infinite friction we have that

$$\sin\alpha = \frac{r}{R}.$$

Thus if the axle has a diameter of 10 cm and the wheel of 1 m then $\sin\alpha$ is less than .1 no matter how bad the bearing material. For such small values we can make the approximation $\mu_{\text{eff}} = \tan\alpha \approx \sin\alpha$ so that the effective coefficient of friction is .1 or less no matter what the bearing friction.

> The genius of the wheel design is that it makes the effective friction less than r/R no matter how bad the bearing friction.

Going back to the two-force body free-body diagram we can see that

$$\begin{aligned} \Rightarrow \quad \overbrace{r\sin\phi}^{d} &= \overbrace{R\sin\alpha}^{d} \\ \Rightarrow \quad \sin\alpha &= \frac{r}{R}\sin\phi. \qquad (*) \end{aligned}$$

From this formula we can extract the limiting cases discussed previously ($\phi = 0$ and $\phi = 90^\circ$). We can also plug in the small angle approximations ($\sin\alpha \approx \tan\alpha$ and $\sin\phi \approx \tan\phi$) if the friction coefficient is low to get

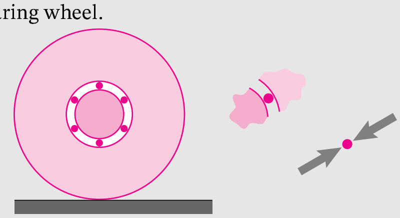
$$\mu_{\text{eff}} \approx \mu\frac{r}{R}.$$

The effective friction is the bearing friction attenuated by the radius ratio. Or, we can use the trig identity $\sin = \sqrt{1+\tan^{-2}}^{\,-1}$ to solve the exact equation (*) for

$$\mu_{\text{eff}} = \mu\frac{r}{R}\left(\frac{1}{\sqrt{1+\mu^2(1-r^2/R^2)}}\right),$$

where the term in parenthesis is always less than one and close to one if the sliding coefficient in the bearing is low.

Finally we combine the genius of the wheel with the genius of the rolling log and invent a wheel with rolling logs inside, a ball bearing wheel.



Each ball is a two-force body and thus only transmits radial loads. It's as if there were no friction on the bearing and we get a specific resistance of zero, $\mu_{\text{eff}} = 0$. Of course real ball bearings are not perfectly smooth or perfectly rigid, so it's good to keep r/R small as a back up plan even with ball bearings.

By this means some wheels have effective friction coefficients as low as about .003. The force it takes to drag something on wheels can be as little as one three hundredth the weight.

If a free-body diagram shows two forces with a common line of action, like the friction forces F_A and F_B on the chair above, the laws of statics might only find their sum, but otherwise can't untangle them.

Only if there is independent information, as would be the case if we knew the chair was sliding to the right (which it clearly isn't in this static example), could we find the friction forces.

