5.3 Equilibrium with frictional contact

Contacting objects are prevented from passing through each other by pressing against each other. Generally there is also some frictional resistance to relative slip. We have neglected friction so far for simplicity and because the neglect of friction is a reasonable approximation for some lubricated contact problems. On the other extreme, in some situations we have assumed that friction so well resists slip that we assumed 'no slip' and that frictional contact acts like a hinge or weld. Either way, with friction negligibly small, or reliably large, we have not worried about it.

However, for some purposes friction forces are not reasonably neglected during slip. Or, when there is no slip, sometimes we have to worry about whether the frictional bond is strong enough to prevent slip.

Although slip means motion and motion sounds like dynamics (contradicting the premise of statics), there are many situations where there is enough motion for friction to be important but not so much acceleration that inertial terms ($m\vec{a}$) are important.

How friction forces are represented on free-body diagrams was discussed in Section 2.3 which you should review before proceeding further here. We will now consider friction forces in equilibrium conditions.

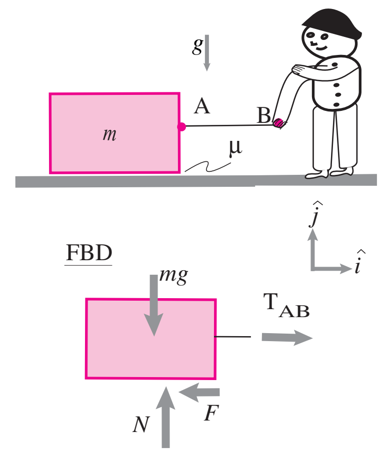


Figure 5.43: Pulling a block on a frictional ground, with related FBD. The complicated distribution of normal forces and friction forces on the block from the ramp have been replaced with the equivalent pair N and F. For particle mechanics, we don't worry about where N and F are applied.

For simplicity, and because of the relatively high accuracy to complexity ratio, we consider only Coulomb friction with a single coefficient of friction μ①.

①If you have studied friction before, our approximation is that $\mu = \mu_{static} = \mu_{dynamic}$ adequately captures the complex and hard to quantify reality of frictional forces

Example: Drag a block with friction.
Consider the block with friction (fig. 5.43). You want to pull it slowly to the right with rod AB. Say $m = 100\,\text{kg}$, $g = 10\,\text{m/s}^2$, and $\mu = 0.3$.
Force balance, using the forces on the free-body diagram gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{AB}\hat{\imath} + N\hat{\jmath} - F\hat{\imath} = \vec{0} \tag{5.19}$$

These, with the friction relation $F = \mu N$, are 3 scalar equations in T_{AB}, F and N with solution $N = mg = 1000\,\text{N}, F = \mu mg = 300\,\text{N}$, and $T_{AB} = \mu mg = 300\,\text{N}$.

Example: Drag a block on a ramp with friction.
Consider the block with friction on a slope (fig. 5.44). You want to hold it with rod AB. Maybe you want to (i) slide it up slowly, or (ii) down slowly or (iii) hold it still. Say $m = 100\,\text{kg}$, $g = 10\,\text{m/s}^2$, $\theta = 45^\circ$, and $\mu = 0.3$.
Force balance, using the forces on the free-body diagram, gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{AB}\hat{e}_1 + N\hat{e}_2 - F\hat{e}_1 = \vec{0} \tag{5.20}$$

These, with the friction relation, are 3 scalar equations in T_{AB}, F and N.

Summing forces in the rope direction and normal to the plane we get:

$$\begin{aligned} \{(\text{Eqn. } 5.20)\} \cdot \hat{e}_1 \quad &\Rightarrow \quad -mg\sin\theta + T_{AB} - F = 0 \\ \{(\text{Eqn. } 5.20)\} \cdot \hat{e}_2 \quad &\Rightarrow \quad mg\cos\theta + N = 0 \end{aligned} \tag{5.21}$$

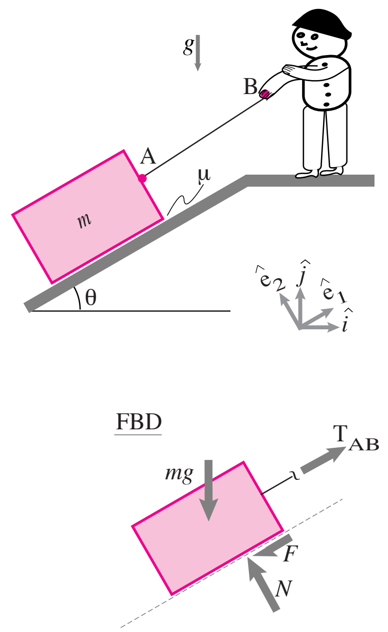


Figure 5.44: Pulling a block up a frictional slope with related FBD. As in the previous example, the pair N and F represent the net normal and frictional forces.

or, for the quantities given $N = (100\,\mathrm{kg})(10\,\mathrm{m/s^2})(\cos 45^\circ) = 707\,\mathrm{N}$. We assume that F and N are related by friction described with the standard Coulomb's friction model②:

i) $F = \mu N$ if the block is sliding up;

ii) $F = -\mu N$ if the block is sliding down; or

iii) $-\mu N \leq F \leq \mu N$ if the block is not sliding.

Solving eqn. (5.20) with the friction relations gives③.

i) $T_{AB} = mg\,(\mu\cos\theta + \sin\theta)$ if the block is sliding up;

ii) $T_{AB} = mg\,(-\mu\cos\theta + \sin\theta)$ if the block is sliding down;
Note that if $\tan\theta < \mu$ then $T_{AB} < 0$ and it then takes a push to slide down; or

iii) $mg\,(-\mu\cos\theta + \sin\theta) \leq T_{AB} \leq mg\,(\mu\cos\theta + \sin\theta)$
if the block is not sliding.
If $\tan\theta < \mu$ then $T_{AB} = 0$ is amongst the solutions for, so no sliding and the block can sit still on the slope with no pull on the rope.

Note that the tension T_{AB} scales with mg. So doubling m or g doubles all the forces in all of these answers, as you might guess from dimensional considerations. The mathematically-abstract-sounding issues of existence and uniqueness often show up in friction problems. For example, sometimes there is no statics solution (non-existence).

Example: Block on ramp.
A statics problem without a solution. A block with coefficient of friction $\mu = 0.5$ is in static equilibrium sliding steadily down a 45° ramp (fig. 5.45). *Not!* If there is constant velocity motion, then statics would apply. But the forces in the free-body diagram cannot add to zero (because the resultant of the friction and normal force is tipped up and to the left and thus cannot be parallel to the vertical gravity force). The assumptions are not consistent with statics (actually this is a dynamics problem, the block accelerates down the ramp). If you saw a block just sitting there on a ramp, then you can be sure that the slope and friction coefficient are not those given above.

Friction problems might be studied with a particle model, as above, or also with moment balance.

Example: Dragged block as an extended body.
This is a repeat of the first example on page 1. One might wonder if the dragging causes an uneven distribution of force up on the block. Does the block dragging back, for example, cause a bigger pressure on the back? As a simple model, assume all the ground force is at the front and back edge of the block. Force balance gives basically the same information as for the particle model, namely that:

$$N_C + N_D = W \quad \text{and} \quad \underbrace{F_C}_{\mu N_C} + \underbrace{F_D}_{\mu N_D} = T_{AB} \quad \Rightarrow \quad T_{AB} = \mu W.$$

One can find more with moment balance about any point you like, say C, with force balance giving

$$\sum M_{/C} = 0 \quad \Rightarrow \quad N_D = \frac{W}{2} + \frac{\mu h W}{2\ell} \quad \text{and} \quad N_C = \frac{W}{2} - \frac{\mu h W}{2\ell}$$

②**Caution:** A common beginner's mistake is to assume the equation $F = \mu N$ applies when there is friction. Rather, if the friction is preventing slip, F could be anything so long as $|F| \leq \mu N$. And if the slip is opposite in direction from that implicitly assumed in the free-body diagram, then $F = -\mu N$ (see case (ii) in the example above).

③ We could be tricky and get a single equation for the scalar T_{AB} by dotting both sides of eqn. (5.20) with a vector orthogonal to the resultant of $N\hat{\boldsymbol{e}}_2 - F\hat{\boldsymbol{e}}_1$. For the case of uphill sliding such a vector would be $\hat{\boldsymbol{e}}_1 + \mu\hat{\boldsymbol{e}}_2$.

So there is more pressure on the front than the back. This difference goes away if either the friction or the height of the string attachment vanishes.

Conditional contact, consistency, and contradictions

There is a natural hope that a subject will reduce to the solution of some well defined equations. For better and worse, things are not always this simple. For better, because it means that the recipes are still not so well defined that computers can easily steal the subject of mechanics from people. For worse, because it means you have to think hard to do some mechanics problems.

One source of these difficulties is the conditional nature of the equations that govern contact. For example:

- The ground pushes up on something to prevent interpenetration *if* the pushing is positive, *otherwise* the ground does not push up.
- The force of friction opposes motion and has magnitude μN *if* there is slip, *otherwise* the force of friction is something less than μN in magnitude.
- The distance between two points is kept from increasing by the tension in the string between them *if* the tension is positive, *otherwise* the tension is zero.

These conditions are, implicitly or explicitly, in the equations that govern these interactions. One does not always know which of the alternative contact conditions, if either, apply when one starts a problem. Sometimes multiple possibilities need to be checked④.

On a FBD at every point of frictional contact

- *If* the direction of slip or impending slip is known, *either*
  - Draw a normal force N and a friction force $F = \mu N$ opposing the relative slip, *or*
  - Draw a single force R at an angle ϕ from the normal of the contact in the direction which resists slip (with $\tan\phi = \mu$)
- *If* there is no slip, *either*
  - Draw a normal force N and tangential force F *or*
  - Draw a single force vector $\vec{\boldsymbol{R}}$ with unknown components
- *If* you don't know whether or not there is slip, *first*
  - Guess that there is no slip, *then*
  - Solve the equilibrium equations, *then*
    - ∗ *If* $F \leq \mu N$: you guessed right and have found a solution to both the equilibrium and friction equations.
    - ∗ *If* $F > \mu N$: you guessed wrong and have to guess that there is slip in one direction (guess which), *then*

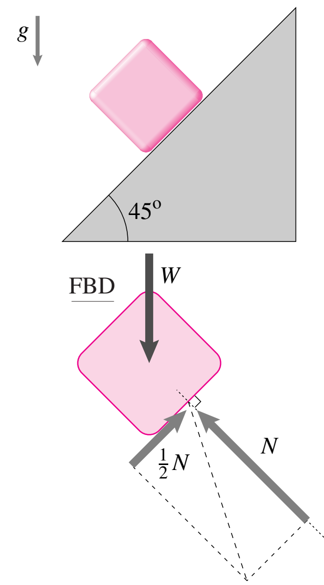


Figure 5.45: Block on steep ramp and related FBD.

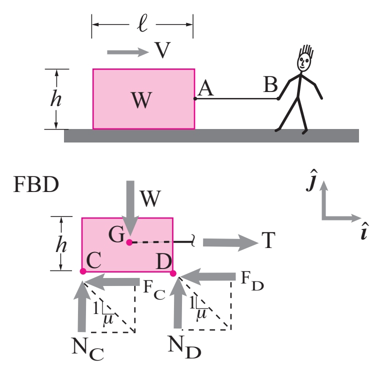
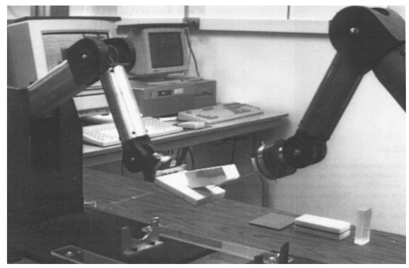


Figure 5.46: Dragging a block, taking account (in a simple way) the distribution of contact forces from the ground. Assume slip to the right is occurring.

④ All of the combinatorics in this recipe follows from the inability of our mathematics to deal easily with relations between variables with the step shape of Coulomb friction (fig. 2.66 on page 188).

- see if you can solve the equilibrium equations, if not *then*
- assume slip in the opposite direction and try to solve the equilibrium equations, if you can't, *then*
- the problem has no solution

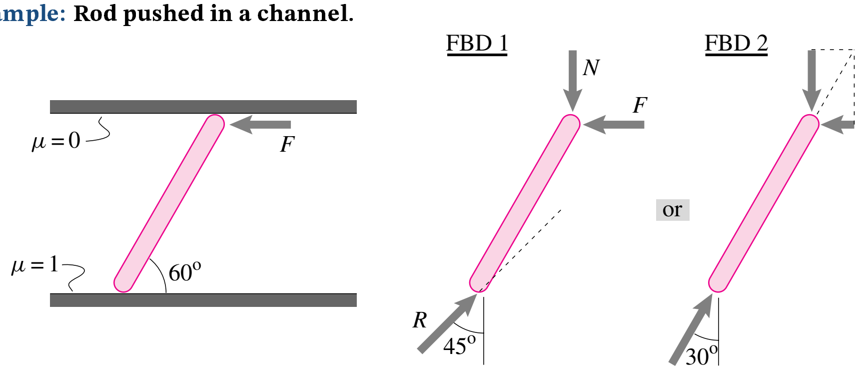
Example: Robot hand



Roboticist Michael Erdmann has designed a palm-like robot 'hand' that manipulates objects without squeezing them. The flat robot palms just move around and the object consequently slides. Determining whether the object slides on one or the other, or possibly on both hands in a given movement is a matter of case study. The computer checks to see if the equilibrium equations can be solved with the assumption of sticking or slipping at one or the other contact.

Once you find a solution to a problem with friction there remains the possibility of multiple solutions, in this case for different reasons than the usual static indeterminacy. The following problem shows a case where a statics problem has multiple solutions due to frictional effects.

Example: Rod pushed in a channel.



A light rod is just long enough to make a 60° angle with the walls of a channel. One channel wall is frictionless and the other has $\mu = 1$. What is the force needed to keep it in equilibrium in the position shown? If we assume it is sliding we get the first free-body diagram. The forces shown can only be in equilibrium if all forces are zero. So a solution is that the rod slides in equilibrium with no force. If we assume that the rod is not sliding, the friction force on the lower wall can be at any angle between $\pm 45^\circ$. Thus we have equilibrium with the second FBD for arbitrary positive F. This is a second set of solutions. A rod like this is said to be *self locking* in that it can hold an arbitrary large force F without slipping. That we have found freely slipping solutions, with no force, and jammed solutions, with arbitrary force, corresponds physically to one being able to easily slide a rod like this down a slot and then also at a different instant, have the rod totally jamb. Some rock-climbing equipment depends on such self-locking and easy release.

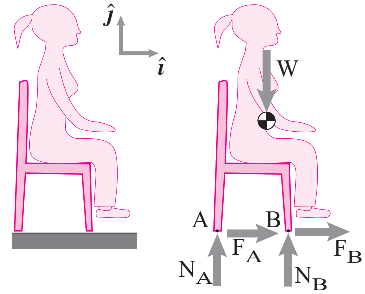


Figure 5.47: A chair with friction at the feet.

Statically indeterminate problems

When there are two or more points of frictional contact and there is no slip nor impending slip, then static indeterminacy is likely.

Example: Chair with friction

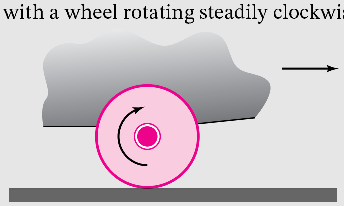
5.7 Undriven wheels and two-force bodies

One often hears whimsical reverence for the "invention of the wheel." Now, using elementary mechanics, we can gain some appreciation for this revolutionary way of sliding things.

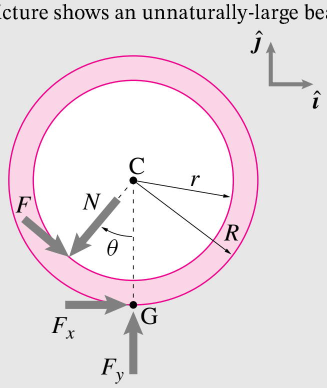
Without a wheel, the force it takes to drag something is about μW. Since μ ranges between about .1 for teflon, to about .6 for stone on ground, to about 1 for rubber on pavement, you need to pull with a force that is on the order of a half of the full weight of the thing you are dragging.

You have seen how rolling on round logs cleverly take advantage of the properties of two-force bodies (page 259). But that good idea has the major deficiency of requiring that logs be repeatedly picked up from behind and placed in front again. So we invent the wheel.

The simplest wheel design uses a dry "journal" bearing consisting of a non-rotating shaft protruding through a near close fitting hole in the wheel. Here is shown part of a cart rolling to the right with a wheel rotating steadily clockwise.



To figure out the forces involved we draw a free-body diagram of the wheel. We neglect the wheel's weight because it is generally much smaller than the forces it mediates. To make things more clear, the picture shows an unnaturally-large bearing hole r.



The force of the axle on the wheel has a normal component N and a frictional component F. The force of the ground on the wheel has a part holding the cart up F_y and a part along the ground F_x which will surely turn out to be negative for a cart moving to the right. If we take the wheel dimensions to be known, and also the vertical part of the ground reaction force F_y (the weight born), we have as unknowns N, F, θ and F_x. To find these we could use the friction equation for the sliding bearing contact

$$F = \mu N;$$

force balance

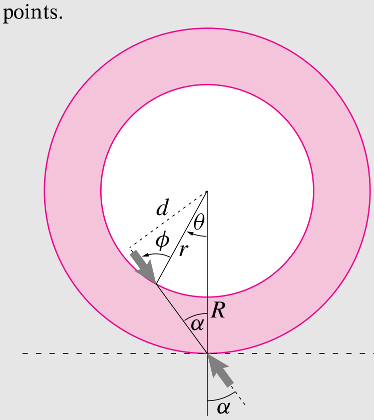
$$F_x \hat{\imath} + F_y \hat{\jmath} + N(-\sin\theta \hat{\imath} - \cos\theta \hat{\jmath}) + F(\cos\theta \hat{\imath} - \sin\theta \hat{\jmath}) = \vec{0},$$

which could be reduced to 2 scalar equations by taking components or dot products; and moment balance about C, which we calculate with forces and perpendicular distances as

$$Fr + F_x R = 0.$$

Of key interest is finding the force resisting motion F_x. With some mathematical manipulation we could solve the 4 scalar equations above for any of $F_x, N, F,$ and θ in terms of $r, R, F_y,$ and μ. We follow a more intuitive approach instead.

As modeled, the wheel is a two-force body so the free-body diagram shows equal and opposite collinear forces at the two contact points.



(continued...)

If we assume Coulomb friction at the chair feet, we know that

$$|F_A| \le \mu N_A \qquad \text{and} \qquad |F_B| \le \mu N_B$$

The equilibrium equations tell us (assuming for simplicity that W acts in the middle of the chair):

$$F_A + F_B = 0, N_A = N_B = W/2.$$

Putting these equations together we find that

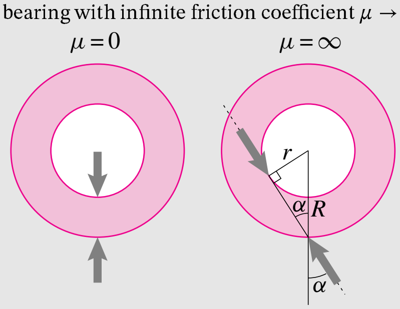
$$-W/2 \le F_A \le W/2 \qquad \text{and} \qquad F_B = -F_A$$

and no more. That is, all we can tell is that both are within the friction limits and that the horizontal forces cancel each other.

5.7 Undriven wheels and two-force bodies (continued)

The friction angle ϕ describes the friction between the axle and wheel (with $\tan\phi = \mu$). The angle α describes the *effective* friction of the wheel. This is not the friction angle for sliding between the wheel and ground which is assumed to be larger (if not, the wheel would skid and not roll), probably much larger. The *specific resistance* or the *coefficient of rolling resistance* or the *specific cost of transport* is $\mu_{\text{eff}} = \tan\alpha$. (If there was no wheel, and the cart or whatever was just dragged, the specific resistance would be the friction between the cart and ground $\mu_{\text{eff}} = \mu$.)

Although we can solve for α in terms of μ or ϕ let's first consider two extreme cases: one is a frictionless bearing and the other is a bearing with infinite friction coefficient $\mu \to \infty$ and $\phi \to 90^\circ$.



In the case that the wheel bearing has no friction we satisfyingly see clearly that there is no ground resistance to motion. The case of infinite friction is perhaps surprising. Even with infinite friction we have that

$$\sin\alpha = \frac{r}{R}.$$

Thus if the axle has a diameter of 10 cm and the wheel of 1 m then $\sin\alpha$ is less than .1 no matter how bad the bearing material. For such small values we can make the approximation $\mu_{\text{eff}} = \tan\alpha \approx \sin\alpha$ so that the effective coefficient of friction is .1 or less no matter what the bearing friction.

> The genius of the wheel design is that it makes the effective friction less than r/R no matter how bad the bearing friction.

Going back to the two-force body free-body diagram we can see that

$$\Rightarrow \quad \overbrace{r\sin\phi}^{d} = \overbrace{R\sin\alpha}^{d}$$

$$\Rightarrow \quad \sin\alpha = \frac{r}{R}\sin\phi. \qquad (*)$$

From this formula we can extract the limiting cases discussed previously ($\phi = 0$ and $\phi = 90^\circ$). We can also plug in the small angle approximations ($\sin\alpha \approx \tan\alpha$ and $\sin\phi \approx \tan\phi$) if the friction coefficient is low to get

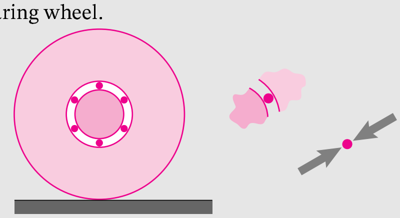
$$\mu_{\text{eff}} \approx \mu\frac{r}{R}.$$

The effective friction is the bearing friction attenuated by the radius ratio. Or, we can use the trig identity $\sin = \sqrt{1+\tan^{-2}}^{\,-1}$ to solve the exact equation (*) for

$$\mu_{\text{eff}} = \mu\frac{r}{R}\left(\frac{1}{\sqrt{1+\mu^2(1-r^2/R^2)}}\right),$$

where the term in parenthesis is always less than one and close to one if the sliding coefficient in the bearing is low.

Finally we combine the genius of the wheel with the genius of the rolling log and invent a wheel with rolling logs inside, a ball bearing wheel.



Each ball is a two-force body and thus only transmits radial loads. It's as if there were no friction on the bearing and we get a specific resistance of zero, $\mu_{\text{eff}} = 0$. Of course real ball bearings are not perfectly smooth or perfectly rigid, so it's good to keep r/R small as a back up plan even with ball bearings.

By this means some wheels have effective friction coefficients as low as about .003. The force it takes to drag something on wheels can be as little as one three hundredth the weight.

If a free-body diagram shows two forces with a common line of action, like the friction forces F_A and F_B on the chair above, the laws of statics might only find their sum, but otherwise can't untangle them.

Only if there is independent information, as would be the case if we knew the chair was sliding to the right (which it clearly isn't in this static example), could we find the friction forces.

SAMPLE 5.12 **A block on a ramp sliding down or up.** Consider a block of mass $m = 10\,\text{kg}$ pushed up by the force F on the ramp as shown in the figure. The coefficient of friction between the ramp and the block is $\mu = 0.7$.

1. Let $\theta = 60^\circ$ and $\alpha = 0^\circ$. Assuming that the block slides steadily downhill, find the tension in the string.
2. Let $\theta = 30^\circ$ and $\alpha = 30^\circ$. If the applied force $F = 20\,\text{N}$, find the force of friction on the block.
3. Let $\theta = 60^\circ$ and $\alpha = 30^\circ$. If the applied force $F = 10\,\text{N}$, find the force of friction on the block.
4. For $\theta = 60^\circ$ and $\alpha = 30^\circ$, what will be the required tension in the string to make the block just about slide up the slope? Express your answer in terms of the weight of the block.

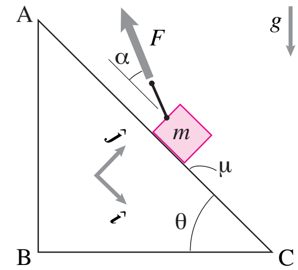


Figure 5.48

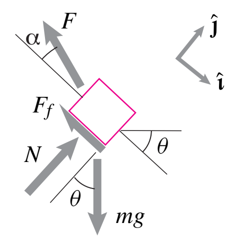


Figure 5.49: Free-body diagram of the block. The friction force $\vec{F}_f$ may be known or unknown depending on whether the block is sliding or not. Under downhill sliding, $F_f = \mu N$, whereas for no sliding $|F_f| \le \mu N$.

Solution The free-body diagram of the block is shown in fig. 5.49. We have assumed that the friction force acts upwards along the inclined plane. The direction of the friction force can be up or down depending on the direction of sliding. We will let the equilibrium equation tell us which way the friction force acts in a particular case. In fig. 5.49, we also use rotated unit vectors $\hat{\imath}$ and $\hat{\jmath}$, parallel and perpendicular to the inclined plane, respectively. This is just to make calculations easier. We can use these basis vectors in any orientation to suit our convenience.

The force balance equation for the static equilibrium of the block gives

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F} + \vec{N} + \vec{F}_f + \vec{W} = \vec{0}. \tag{5.22}$$

$$\Rightarrow \quad F(-\cos\alpha\hat{\imath} + \sin\alpha\,\hat{\jmath}) + N\,\hat{\jmath} - F_f\hat{\imath} + mg(\sin\theta\hat{\imath} - \cos\theta\,\hat{\jmath}) = \vec{0} \tag{5.23}$$

Now depending on what is given and what is unknown, we can manipulate this vector equation to find what we want.

1. **Block sliding down:** If the block slides down steadily or very slowly, we can use the static equilibrium equation written above with $\vec{F}_f = -\mu N\hat{\imath}$ (that is, the friction force is known. This is the case of sliding friction and the friction force is maximum possible). Substituting this value of $\vec{F}_f$ and separating out the $\hat{\imath}$ and $\hat{\jmath}$ components of eqn. (5.23), we get

$$-F\cos\alpha - \mu N + mg\sin\theta = 0 \tag{5.24}$$

$$F\sin\alpha + N - mg\cos\theta = 0. \tag{5.25}$$

Adding μ times eqn. (5.25) to eqn. (5.24) in order to get rid of N, and rearranging terms, we get

$$F(\cos\alpha - \mu\sin\alpha) = mg(\sin\theta - \mu\cos\theta)$$

$$\Rightarrow \quad F = \frac{\sin\theta - \mu\cos\theta}{\cos\alpha - \mu\sin\alpha}mg. \tag{5.26}$$

Substituting $\alpha = 0^\circ$, $\theta = 60^\circ$, and $\mu = 0.7$ in eqn. (5.26), we get

$$F = (\sin 60^\circ - 0.7\cdot\cos 60^\circ)\,mg = 0.52\,mg. = 51\,\text{N}.$$

$$\vec{F} = -(51\,\text{N})\hat{\imath}$$

2. **Block sliding or not sliding – not known:** Now, we are given that $F = 20\,\text{N}$, $\alpha = 30^\circ$, and $\theta = 30^\circ$. We do not know if the block is sliding or not. So, let us assume static equilibrium in the given configuration and solve for the friction force F_f. Then, we will check if it satisfies friction law for static equilibrium ($|F_f| \le \mu N$).

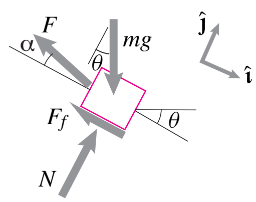


Figure 5.50: Free-body diagram of the block. The friction force $\vec{F}_f$ is not known because we do not know if the block is sliding or not.

Substituting $\vec{F}_f = -F_f\hat{\imath}$ in eqn. (5.23) and separating out the $\hat{\imath}$ and $\hat{\jmath}$ components of the equation, we get

$$\begin{aligned} -F\cos\alpha - F_f + mg\sin\theta &= 0 \\ F\sin\alpha + N - mg\cos\theta &= 0 \end{aligned}$$

which are easily solved for F and N to give

$$\begin{aligned} F_f &= mg\sin\theta - F\cos\alpha \\ N &= mg\cos\theta - F\sin\alpha. \end{aligned}$$

Substituting the given values of F, θ, and α, we get

$$\begin{aligned} F_f &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\sin 30^\circ - 20\,\mathrm{N}\cdot\cos 30^\circ = 31.73\,\mathrm{N} \\ N &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\cos 30^\circ - 20\,\mathrm{N}\cdot\sin 30^\circ = 74.96\,\mathrm{N}. \end{aligned}$$

Now, the maximum possible value of friction force is $\mu N = 0.7\cdot 74.96\,\mathrm{N} = 52.47\,\mathrm{N}$. Thus, $|F_f| < \mu N$, and therefore, our assumption of static equilibrium is valid. This equilibrium requires that $F_f = 31.73\,\mathrm{N}$.

$$\boxed{\vec{F}_f = -(31.73\,\mathrm{N})\hat{\imath}}$$

3. **Block sliding or not sliding – not known, again:** In this case, $F = 10\,\mathrm{N}$, $\alpha = 30^\circ$, and $\theta = 60^\circ$. Again, assuming static equilibrium, we do exactly the same calculations as above (in fact, use the same expressions) and substituting the given values, we get

$$\begin{aligned} F_f &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\sin 60^\circ - 10\,\mathrm{N}\cdot\cos 30^\circ = 76.3\,\mathrm{N} \\ N &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\cos 60^\circ - 10\,\mathrm{N}\cdot\sin 30^\circ = 44\,\mathrm{N}. \end{aligned}$$

Here, $|F_f| > \mu N$. Clearly, F_f is not less than or equal to μN, and therefore, our assumption of static equilibrium is not valid. In fact, the given parameters of the problem will make the block accelerate downhill — a problem of dynamics. However, the friction force remains constant, at its maximum $\vec{F}_f = \mu N = 30.8\,\mathrm{N}$ once the sliding starts, accelerating or not.

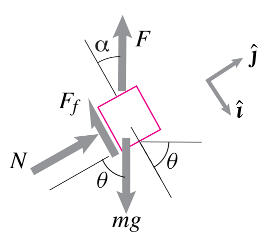


Figure 5.51: Free-body diagram of the block. The friction force $\vec{F}_f$ is not known again because we do not know if the block is sliding or not.

$$\boxed{\vec{F}_f = -30.8\,\mathrm{N}\hat{\imath}}$$

4. **Block just about to slide upwards:** If the block is about to slide upwards, then the friction force must act downwards as shown in fig. 5.52. We also know the magnitude, $F_f = \mu N$ because it is the case of impending slip. Now the force-balance equations in the $\hat{\imath}$ and $\hat{\jmath}$ directions are:

$$\begin{aligned} -F\cos\alpha + \mu N + mg\sin\theta &= 0 \\ F\sin\alpha + N - mg\cos\theta &= 0 \end{aligned}$$

Eliminating N from the two equations, we get F in terms of mg and substituting $\alpha = 30^\circ$, and $\theta = 60^\circ$, we get the desired value:

$$\begin{aligned} F &= \frac{\sin\theta + \mu\cos\theta}{\cos\alpha + \mu\sin\alpha}\, mg \\ &= \frac{1.216}{1.216}\, mg = mg. \end{aligned}$$

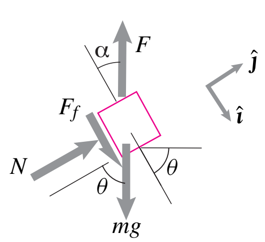


Figure 5.52: Free-body diagram of the block about to slip upwards. The friction force $\vec{F}_f$ is completely known because of impending slip, $\vec{F}_f = \mu N\hat{\imath}$.

$$\boxed{F = mg}$$

Does the answer make sense? Yes, it does. For the given θ and α, the string tension is vertical. If it balances the weight of the block, the normal force goes to zero and so does the friction force. The block is then ready to slide up if the tension increases by any tiny amount.

SAMPLE 5.13 **How much friction does the cylinder need?** A cylinder of mass m sits between an incline and a vertical wall as shown in the figure. There is no friction between the wall and the cylinder but there is friction between the incline and the cylinder. Take the coefficient of friction to be μ and the angle of incline with the horizontal to be θ. Find the force of friction on the cylinder from the incline.

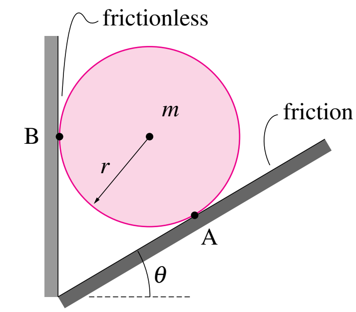


Figure 5.53

Solution The free-body diagram of the cylinder is shown in fig. 5.54. We need to find the force of friction F_s.

Note that the normal reaction of the vertical wall, N, the force of gravity, mg, and the normal reaction of the incline, R, all pass through the center C of the cylinder. So, if we do moment balance about point C, $\sum \vec{\boldsymbol{M}}_C = \vec{\mathbf{0}}$, none of these forces will appear in the equation since their moment about C is zero. Therefore, to find F_s, we should use the moment-balance equation about point C. Noting that F_s acts along the inclined plane, its normal distance (*lever arm*) from point C is simply r, the radius of the cylinder, we have,

$$\sum \vec{\boldsymbol{M}}_C = \vec{\mathbf{0}} \quad \Rightarrow \quad rF_s(-\hat{\boldsymbol{k}}) = \vec{\mathbf{0}}$$
$$\Rightarrow \quad F_s = 0.$$

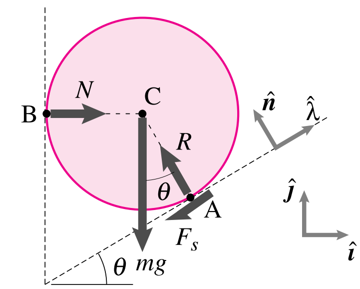


Figure 5.54

Thus the force of friction on the cylinder is zero! Note that F_s is independent of θ, the angle of incline. Thus, irrespective of what the angle of incline is, in the static equilibrium condition, there is no force of friction on the cylinder.

$F_s = 0$

Note: The cylinder here is a three-force body since there are three forces acting on it — two contact forces (at A and B) and one gravity force. Therefore, for equilibrium, all the three forces must intersect at a single point. Now, lines of action of the gravity force and the normal reaction at B intersect at the center C of the cylinder. Therefore, the line of action of the contact force at A also must pass through the center. This is clearly not possible if the contact force is not normal to the incline (see the candidate contact forces marked by the dashed gray arrows in fig. 5.55. If there is any non-zero friction force at A, the contact force (the resultant of the normal reaction and the friction force) at A will be tipped away from the normal, thus making its line of action miss the center of the cylinder and, therefore, violate equilibrium condition.

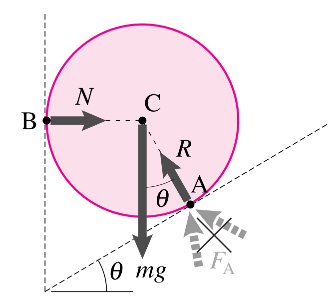


Figure 5.55

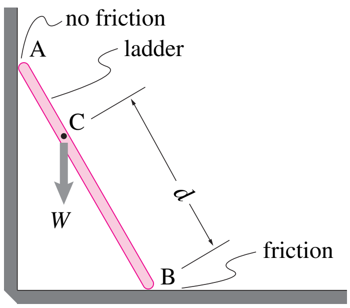


Figure 5.56

SAMPLE 5.14 **Will the ladder slip?** A ladder of length $\ell = 4\,\text{m}$ rests against a wall at $\theta = 60°$. Assume that there is no friction between the ladder and the vertical wall but there is friction between the ground and the ladder with $\mu = 0.5$. A person weighing 700 N starts to climb up the ladder.

1. Can the person make it to the top safely (without the ladder slipping)? If not, then find the distance d along the ladder that the person can climb safely. Ignore the weight of the ladder in comparison to the weight of the person.
2. Does the "no slip" distance d depend on θ? If yes, then find the angle θ which makes it safe for the person to reach the top.

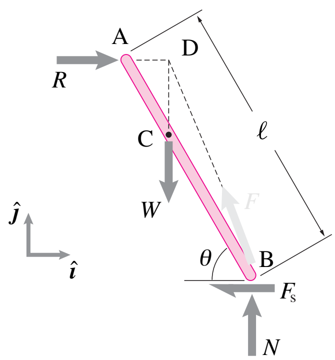


Figure 5.57: The free-body diagram of the ladder indicates that it is a three-force body. Since the direction of the forces acting at points A and C are known (the normal, horizontal reaction at A and the vertical gravity force at C), it is easy to find the direction of the net ground reaction at B — it must pass through point D. The ground reaction F at B (shown by light grey arrow) can be decomposed into a normal reaction N and a horizontal reaction (the force of friction) F_s at B.

Solution

1. The free-body diagram of the ladder is shown in fig. 5.57. There is only a normal reaction $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{\imath}}$ at A since there is no friction between the wall and the ladder. The force of friction at B is $\vec{\boldsymbol{F}}_s = -F_s\hat{\boldsymbol{\imath}}$ where $F_s \le \mu N$. To determine how far the person can climb the ladder without the ladder slipping, we take the critical case of impending slip. In this case, $F_s = \mu N$. Let the person be at point C, a distance d along the ladder from point B. We need to find d and check if $d < \ell$ (cannot make it to point A).

   From moment balance about point B, $\sum \vec{\boldsymbol{M}}_B = \vec{\mathbf{0}}$, we find

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{A/B} \times \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{C/B} \times \vec{\boldsymbol{W}} &= \vec{\mathbf{0}} \\
-R\ell \sin\theta \hat{\boldsymbol{k}} + Wd\cos\theta\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\
\Rightarrow \quad R &= W\frac{d\cos\theta}{\ell\sin\theta}. \qquad (5.27)
\end{aligned}
$$

   From force equilibrium, we get

$$(R - \mu N)\hat{\boldsymbol{\imath}} + (N - W)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}}. \qquad (5.28)$$

   Dotting eqn. (5.28) with $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}}$, respectively, we get

$$
\begin{aligned}
N &= W \\
R &= \mu N = \mu W.
\end{aligned}
$$

   Substituting this value of R in eqn. (5.27) we get

$$
\begin{aligned}
\mu W &= W\frac{d\cos\theta}{\ell\sin\theta} \\
\Rightarrow \quad d &= \mu\ell\tan\theta \qquad (5.29) \\
&= 0.5 \cdot (4\,\text{m}) \cdot \tan 60° = 3.46\,\text{m}.
\end{aligned}
$$

   Thus, the ladder is about to slip when the person is at $d = 3.46\,\text{m}$. But, $d < \ell$, therefore, the person cannot make it to the top of the ladder safely.

   $\boxed{d = 3.46\,\text{m}}$

2. The "no slip" distance d depends on the angle θ via the relationship in eqn. (5.29). The person can climb the ladder safely up to the top if

$$\tan\theta = \frac{1}{\mu} \quad \Rightarrow \quad \theta = \tan^{-1}(\mu^{-1}) = 63.43°.$$

   Thus, any reasonable angle $\theta \ge 64°$ will allow the person to climb up to the top safely.

   $\boxed{\theta \ge 64°}$

SAMPLE 5.15 **Will it tip or will it slide?** Whether or not a box of a given width and height will slide or tip over on an inclined plane depends on the slope of the plane and the coefficient of friction. For a given slope θ, find the relationship between the coefficient of friction μ and the aspect ratio of the box, $\gamma = b/h$ for impending tipping.

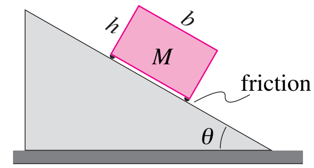


Figure 5.58

Solution Let us imagine that we put the box on a flat surface and then slowly start tilting the surface up with respect to the horizontal. At some slope, the box will either tip over or slide. Just before the instant the box starts to tip over or slide, it is in static equilibrium. The magnitude of the friction force at the contact points is $|F| \leq \mu N$ where N is the magnitude of the normal force at the contact, and the equality holds only in the case of impending slip. That is, if the box is about to slip, then $F = \mu N$ at each contact point.

The free-body diagram of the box is shown in fig. 5.59. Let us first write the equations of static equilibrium assuming there is no impending slip.

The force balance in the $\hat{\imath}$ and $\hat{\jmath}$ directions (see fig. 5.62) gives

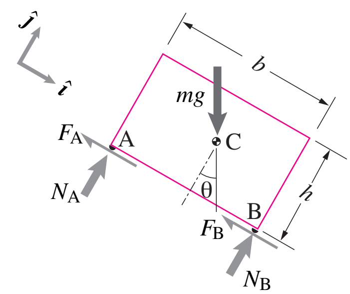


Figure 5.59

$$F_A + F_B = mg\sin\theta \quad (5.30)$$
$$N_A + N_B = mg\cos\theta. \quad (5.31)$$

The moment equilibrium about the center-of-mass, $\sum \vec{M}_C = \vec{0}$, in the $\hat{k}$ direction gives

$$N_B \cdot \frac{b}{2} - N_A\frac{b}{2} - (F_A + F_B)\frac{h}{2} = 0. \quad (5.32)$$

Substituting $F_A + F_B = mg\sin\theta$ from eqn. (5.30) in eqn. (5.32), and solving eqns. (5.31) and (5.32) simultaneously, we get

$$N_A = \frac{1}{2}mg\left(\cos\theta - \frac{h}{b}\sin\theta\right), \text{ and } \quad N_B = \frac{1}{2}mg\left(\cos\theta + \frac{h}{b}\sin\theta\right).$$

If the box were to tip over (about point B), the support forces at A will go to zero (because of loss of contact). Thus, for impending tipping,

$$N_A = 0 \quad \Rightarrow \quad \cos\theta - \frac{h}{b}\sin\theta = 0 \quad \Rightarrow \quad \tan\theta = \frac{b}{h} = \gamma.$$

Thus, the condition for impending tipping is

$$\tan\theta = \gamma. \quad (5.33)$$

This condition, however, does not guarantee that the box *will* tip over. In fact, it may start sliding before it tips over. We need to check if sliding condition is met before eqn. (5.33) is satisfied. In other words, we need to check the value of friction forces and make sure that $|F_A + F_B| \leq \mu(N_A + N_B)$. Thus, for no slipping,

$$F_A + F_B \leq \mu(N_A + N_B) \quad \Rightarrow \quad mg\sin\theta \leq \mu mg\cos\theta \quad \Rightarrow \quad \tan\theta \leq \mu.$$

Using this condition (with equality) in eqn. (5.33), we get the *critical* condition for tipping:

$$\gamma = \mu.$$

$$\gamma = \tan\theta \leq \mu$$

You may know this condition geometrically as the line of action of the weight of the box must pass through B and beyond for tipping over (see fig. 5.60).

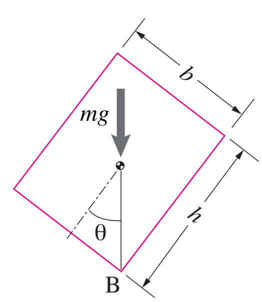


Figure 5.60: The impending tipping condition $\tan\theta = \gamma \equiv b/h$, provided $\tan\theta \leq \mu$.

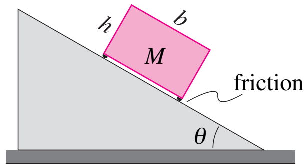


Figure 5.61

SAMPLE 5.16 How big does the friction force get? Consider the box on the inclined plane of Sample 5.15 again. The box has aspect ratio $\gamma = b/h$. The coefficient of friction is μ. Imagine that the angle θ of the inclined plane can be varied. How does the force of friction on the box vary with θ? How does the maximum value of this force depend on μ?

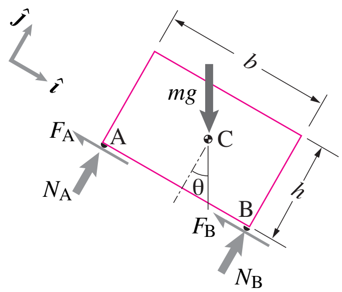


Figure 5.62

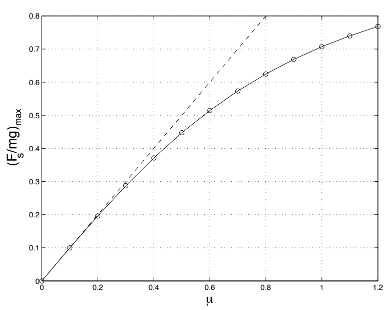


Figure 5.63: The maximum net friction force on the box as a function of the friction coefficient μ. Note that the relationship is almost linear for most practical values of μ.

Solution If we imagine the inclined plane to be not inclined ($\theta = 0$) but horizontal and the box to be just sitting there, the force of friction on the box has to be zero. As we tilt the plane up ($\theta > 0$), the friction force starts increasing. It increases up to the point of impending slip unless the box tips over before that. Assuming that the aspect ratio of the box prevents it from tipping (see Sample 5.15), we can determine the maximum value up to which the friction force rises before the box starts slipping.

From Sample 5.15, we know that the total friction force $F_s = F_A + F_B = mg\sin\theta$. Thus the normalized friction force (as a fraction of the weight of the block), F_s/mg is

$$\frac{F_s}{mg} = \sin\theta.$$

Thus the total friction force varies as sine of the ramp angle. However, this variation is valid only upto the maximum value of the friction force (μN) when the block starts sliding. The critical angle at which this maximum is attained is $\theta_{\text{slip}} = \tan^{-1}\mu \equiv \phi$ (friction angle). Thus,

$$\left.\frac{F_s}{mg}\right|_{\max} = \sin\phi.$$

Figure 5.63 shows how the maximum normalized friction force varies with μ. Note that for lower values of μ (which covers most practical values of μ), the relationship is almost linear. Thus, $|F_s/mg| \approx \mu$ for $\mu \leq 0.5$.

$$\frac{F_s}{mg} = \sin\theta, \quad \left.\frac{F_s}{mg}\right|_{\max} = \sin\phi$$

What happens to the friction force after it attains the maximum value $F_s = mg\sin\phi$? For a given ramp angle, the friction force remains constant and the box slides.

SAMPLE 5.17 A spool of mass $m = 2\,\text{kg}$ rests on an incline as shown in the figure. The inner radius of the spool is $r = 200\,\text{mm}$ and the outer radius is $R = 500\,\text{mm}$. The coefficient of friction between the spool and the incline is $\mu = 0.4$, and the angle of incline $\theta = 60°$.

1. Which way does the force of friction act, up or down the incline?
2. What is the required horizontal pull T to balance the spool on the incline?
3. Is the spool about to slip?

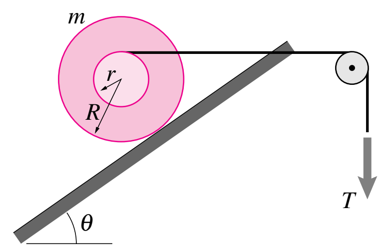


Figure 5.64

Solution

1. The free-body diagram of the spool is shown in fig. 5.65. Note that the spool is a 3-force body. Therefore, in static equilibrium all the three forces — the force of gravity mg, the horizontal pull T, and the incline reaction F — must intersect at a point. Since T and mg intersect at the top of the inner drum (point B), the reaction force $\vec{\boldsymbol{F}}$ of the incline must be along the direction AB. Now the incline reaction $\vec{\boldsymbol{F}}$ is the vector sum of two forces — the normal (to the incline) reaction N and the friction force F_s (along the incline). The normal reaction force N passes though the center C of the spool. Therefore, the force of friction F_s must point up along the incline to make the resultant $\vec{\boldsymbol{F}}$ point along AB.

   Up the incline

2. We need to find the tension T in the string. From the free-body diagram, we see that the force equilibrium will involve $\vec{\boldsymbol{T}}$ along with another unknown force $\vec{\boldsymbol{F}}$, the reaction of the incline. On the other hand, if we do moment balance about point A, we can get rid of $\vec{\boldsymbol{F}}$ and get one scalar equation involving T and mg, giving T in terms of mg. So, writing the moment equilibrium equation about point A,

$$\sum \vec{\boldsymbol{M}}_A = \vec{\boldsymbol{0}},$$

   we get

$$\vec{\boldsymbol{r}}_{C/A} \times (-mg\hat{\boldsymbol{\jmath}}) + \vec{\boldsymbol{r}}_{B/A} \times (T\hat{\boldsymbol{\imath}}) = \vec{\boldsymbol{0}}. \tag{5.34}$$

   These cross products can be easily evaluated by using the scalar form of the moment of a force—the product of force and the lever arm. Thus the moment of mg is $mg \cdot R\sin\theta$ and the moment of T is $-T \cdot (r + R\cos\theta)$ about point A in the $\hat{\boldsymbol{k}}$ direction. Thus the scalar form of the moment-balance equation gives

$$\begin{aligned} mgR\sin\theta &= T(R\cos\theta + r) \\ \Rightarrow \quad T &= mg\frac{\sin\theta}{\cos\theta + r/R} \\ &= 2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2} + \frac{.2\,\text{m}}{.5\,\text{m}}} \\ &= 18.88\,\text{N}. \end{aligned}$$

   $T = 18.88\,\text{N}$

   Alternatively,
   We can also evaluate the net moment on the spool, given by eqn. (5.34), using direct cross products of vectors in the equation. We can use mixed basis vectors ($\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{n}}$) as shown in fig. 5.65. Since,

$$\vec{\boldsymbol{r}}_{C/A} = R\hat{\boldsymbol{n}} \quad \text{and} \quad \vec{\boldsymbol{r}}_{B/A} = R\hat{\boldsymbol{n}} + r\hat{\boldsymbol{\jmath}},$$

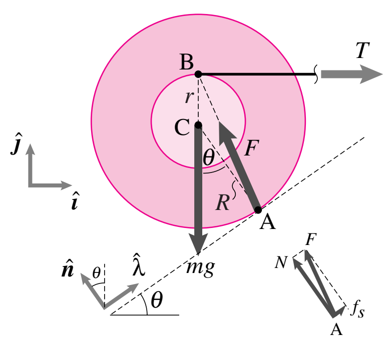


Figure 5.65

we have,

$$\begin{aligned}\vec{r}_{C/A} \times (-mg\hat{\jmath}) &= -mgR(\hat{n} \times \hat{\jmath}) \\ \vec{r}_{B/A} \times T\hat{\imath} &= T[R(\hat{n} \times \hat{\imath}) + r(\hat{\jmath} \times \hat{\imath})].\end{aligned}$$

Now, from the geometry of the basis vectors (see fig. 5.65), we have,

$$\hat{n} \times \hat{\jmath} = -\sin\theta\hat{k} \quad \text{and} \quad \hat{n} \times \hat{\imath} = -\cos\theta\hat{k}.$$

Therefore,

$$\begin{aligned}\vec{r}_{C/A} \times (-mg\hat{\jmath}) &= mgR\sin\theta\hat{k}, \\ \text{and} \qquad \vec{r}_{B/A} \times T\hat{\imath} &= -TR\cos\theta\hat{k} - r\hat{k}.\end{aligned}$$

Hence, eqn. (5.34) becomes

$$mgR\sin\theta\hat{k} - TR\cos\theta\hat{k} - r\hat{k} = \vec{0}.$$

Dotting both sides of this equation with $\hat{k}$, we get the scalar equation

$$mgR\sin\theta = T(R\cos\theta + r)$$

which is the same equation as obtained above using moment lever arms.

3. To find if the spool is about to slip, we need to find the force of friction $|F_s|$ and see if it satisfies the condition of impending slip: $F_s = \mu N$. The force balance on the spool, $\sum\vec{F} = \vec{0}$ gives

$$T\hat{\imath} - mg\hat{\jmath} + F_s\hat{\lambda} + N\hat{n} = \vec{0} \tag{5.35}$$

where $\hat{\lambda}$ and $\hat{n}$ are unit vectors along the incline and normal to the incline, respectively. Dotting eqn. (5.35) with $\hat{\lambda}$ we get

$$\begin{aligned}F_s &= -T(\underbrace{\hat{\imath}\cdot\hat{\lambda}}_{\cos\theta}) + mg(\underbrace{\hat{\jmath}\cdot\hat{\lambda}}_{\sin\theta}) \\ &= -T\cos\theta + mg\sin\theta \\ &= -18.88\,\text{N}(1/2) + 19.62\,\text{N}(\sqrt{3}/2) \\ &= 7.55\,\text{N}.\end{aligned}$$

Similarly, we compute the normal force N by dotting eqn. (5.35) with $\hat{n}$:

$$\begin{aligned}N &= -T(\hat{\imath}\cdot\hat{n}) + mg(\hat{\jmath}\cdot\hat{n}) \\ &= T\sin\theta + mg\cos\theta \\ &= 18.88\,\text{N}(\sqrt{3}/2) + 19.62\,\text{N}(1/2) \\ &= 26.16\,\text{N}.\end{aligned}$$

Now we find that $\mu N = 0.4(26.16\,\text{N}) = 10.46\,\text{N}$ which is greater than $F_s = 7.55\,\text{N}$. Thus $F_s < \mu N$, and therefore, the spool is not about to slip.

Not about to slip

5.3 Friction and equilibrium

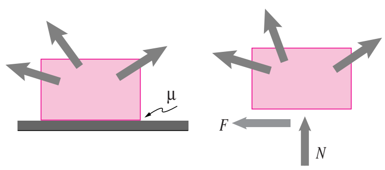
Preparatory Problems

5.3.1 For the block shown in the figure, what do you know about F if

a) the block is sliding to the right

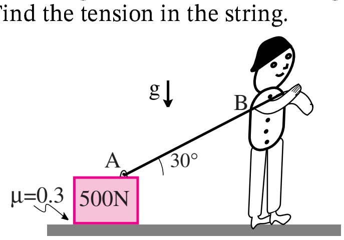
b) the block is sliding to the left

c) the block is not sliding.



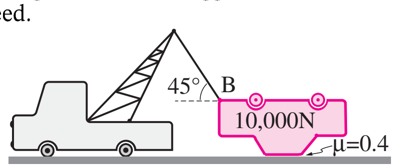
Problem 5.3.1

5.3.2 A block weighing 500 N is dragged slowly on the ground as shown in the figure. Find the tension in the string.



Problem 5.3.2

5.3.3 Find the tension in the cable assuming the car is dragged at constant speed.



Problem 5.3.3

5.3.4 Consider the tow truck dragging the car in Problem 5.3.3 again. In order to ensure safety, you would like to minimize the tension in the rope attached to the car. Assume that the angle shown at point B is θ.

a) What value of θ minimizes the tension in the rope?

b) What is the corresponding value of T?

c) What is the force of the ground on the car?

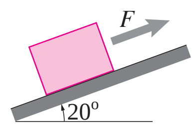
.

More-Involved Problems

5.3.5 A 30,000 N stone cube one meter on a side was dragged up a 20° ramp by 100 of a Pharaoh's slaves by a rope parallel to the slope. The coefficient of friction was $\mu = 0.2$. Assume all the ground contact is at the front and back edges of the cube.

a) Find the dragging force.

b) Find the force on the front and back edges of the cube.

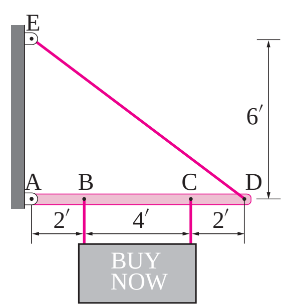


Problem 5.3.5

5.3.6 The 20 lbf uniform rectangular sign is suspended from the strut ABCD by two wires. The strut is supported by cable DE and a pin at A.

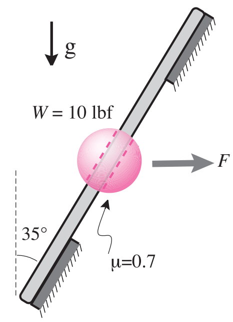
a) Find tension DE.

b) Suppose the workers who hung the sign forgot to pin the strut to the wall at point A. What is the least value of μ between the strut and wall for the system to maintain equilibrium.



Problem 5.3.6

5.3.7 A horizontal force F is applied to slide the bead on the rod shown in the figure. Find the value of F that is required to initiate sliding up the rod. Why is F so big or small?

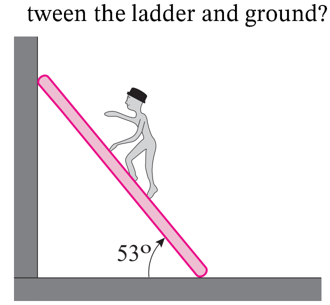


Problem 5.3.7

5.3.8 A 130 pound person climbs a 120 pound ladder that is 30 ft long. The ladder leans against a frictionless wall and makes an angle of 53° with the ground.

a) Find the force of the ground on the ladder when the person is one third of the way up the ladder.

b) When the person gets two thirds of the way up, the bottom of the ladder starts to slip. What is μ between the ladder and ground?



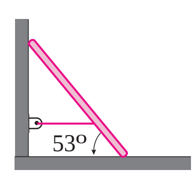
Problem 5.3.8

5.3.9 A uniform 200 N, 10 m ladder leans between a frictionless ground and a wall. It is kept from sliding away from the wall by a horizontal cable 2 m above the ground. Find

a) The tension in the cable.

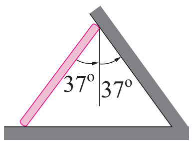
b) The force of the ground on the ladder.

c) The force of the wall on the ladder.



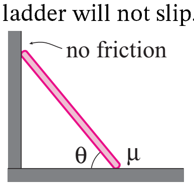
Problem 5.3.9

5.3.10 A uniform ladder of length ℓ and weight W rests against a frictionless slanted wall. What is the minimum μ between ladder and ground that is needed to hold the ladder in position?



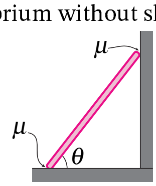
Problem 5.3.10

5.3.11 A uniform ladder with weight W and length ℓ leans against a frictionless vertical wall and makes an angle θ with the ground. In terms of the given quantities, find the values of μ at the ground for which the ladder will not slip.



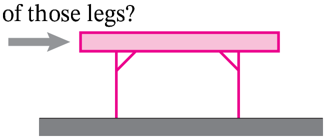
Problem 5.3.11

5.3.12 A uniform ladder with weight W and length ℓ leans against a frictional vertical wall and is supported by the frictional ground. The same coefficient of friction μ applies to the wall and to the ground. In terms of the given quantities, find the values of θ between the ladder and ground for which the ladder can be in equilibrium without slipping. *



Problem 5.3.12

5.3.13 A 2 m square 500 N 4-leg table is pushed across a floor by a horizontal force at its top surface and normal to one edge. Assume the table is 0.8 m high, that its center of mass is 0.6 m high and that all four legs slide on the floor with friction coefficient $\mu = 0.3$. Which legs carry the most load and what is the magnitude of the force from the ground on one of those legs?

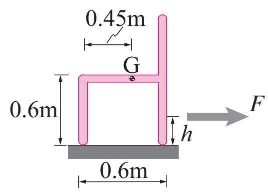


Problem 5.3.13

5.3.14 An 80 N chair is pulled steadily to the right by a rope. The coefficient of friction between the ground and floor is $\mu = 0.25$.

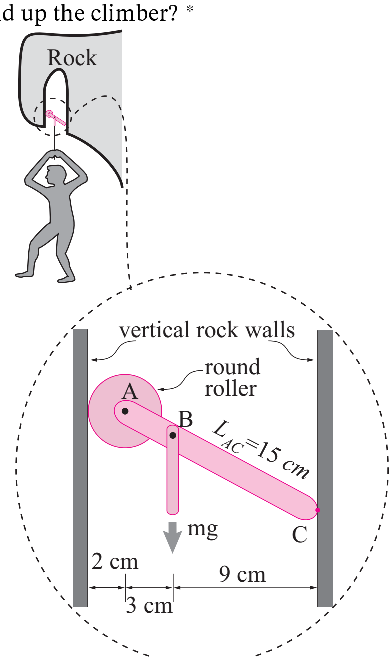
a) What is the force needed to pull the chair?

b) What is the highest point on the chair that the rope can be tied without the chair tipping over?



Problem 5.3.14

5.3.15 A candidate rock-climbing device consists of a roller (radius 2 cm) frictionlessly pinned at A to diagonal-member AC. The length of AC from point A to the wall-contact point at C is $L_{AC} = 15cm$. The climber ($m = 60$ kg) hangs from a rope connected to AC by a pin at B. B is on the line AC and located as shown in the figure. If needed, assume $g = 10$N/kg. What is the minimum coefficient of friction μ at C that is needed to hold up the climber? *



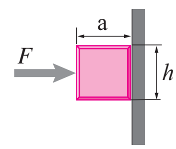
Problem 5.3.15

5.3.16 A uniform $W = 50$ N block with width a and height h is held against a wall with a horizontal force of F acting on the left side half way up the block. The block is prevented from sliding down the wall by friction. There is no glue (no tension between wall and block).

a) Assuming friction is high enough to prevent slip, what is the minimum F to keep the block from tipping away from the wall?

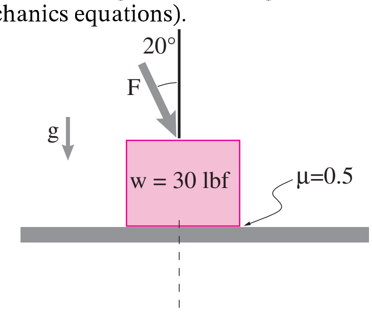
b) For twice that F what is the minimum friction to keep the block from sliding down the wall?

c) For $a = h$ and $F = 3W$ the resultant of all the wall normal and contact forces is a single force that acts on the right side of the block at what position y above the bottom of the block?



Problem 5.3.16

5.3.17 In the figure shown, what is force F required to push the block along the floor? This problem has no solution. Explain why (using free-body diagrams and mechanics equations).



Problem 5.3.17

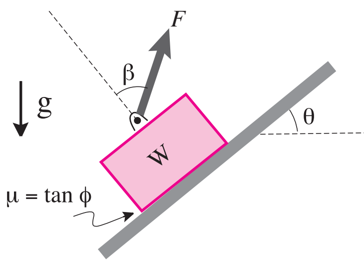
5.3.18 Consider the situation shown in the figure. Give your answers to the following questions in terms of some or all of W, θ, β, g, and ϕ or μ. Assume all values of β and $0 \leq \theta \leq \pi/2$, $0 \leq \mu$, $g > 0$, $W > 0$.

a) Assume the block slides steadily uphill. Find F. For what values of θ, β, and μ does no such F exist (allow $F < 0$)?

b) Assume the block slides downhill. What is F? For what values of θ, β, and μ does no such solution exist?

c) assume the block is not sliding. What are the possible values of F? For what values of θ, β, and μ does such a solution exist?

d) For what values of θ, β, and μ can you have the block slide up, slide down, or lock (that is, no incipient slip) depending on the value of F?

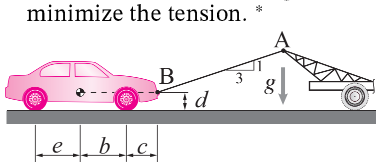


Problem 5.3.18

5.3.19 A car is being towed. Unfortunately all the wheels are locked and skidding with friction coefficient μ. The tow cable AB has a slope of 1/3.

a) In terms of some or all of e, b, c, d, m, g& μ, find the tension in the tow cable AB. *

b) Instead of an angle with slope 1/3, what should the cable angle be to minimize the tension. *



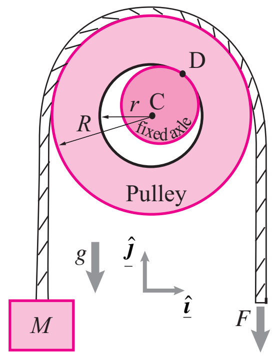
Problem 5.3.19

5.3.20 A weight M is steadily raised by pulling with a force F on a rope going over a negligible-mass pulley on an unlubricated journal bearing (no ball bearings). For an ideal frictionless pulley $F = Mg$. Here, however, we have a friction coefficient between the bearing and its axle which is $\mu = \tan\phi$. [Hint: Finding the location of the contact point D is part of the problem.]

a) Find F in terms of M, g, R, r and μ (or ϕ or $\sin\phi$ or $\cos\phi$ — whichever is most convenient. For example $\cos(\tan^{-1}(\mu))$ is more simply expressed as $\cos\phi$), *and*

b) Evaluate F in the special case that $M = 100\,\text{kg}, g = 10\,\text{m/s}^2, r = 1\,\text{cm}, R = 2\,\text{cm}$, and $\mu = \sqrt{3}/3$ (so $\phi = \pi/6, \sin\phi = 1/2, \cos\phi = \sqrt{3}/2$).

c) Referring back to the general case, for fixed r, R, M, and g what happens to F as $\mu \to \infty$ (does it go to ∞)?



Problem 5.3.20

5.3.21 A reel of mass M and outer radius R is connected by a horizontal string from point P across a pulley to a hanging object of mass m. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle θ. There is no slip between the reel and the slope. There is gravity.

a) Find the ratio of the masses so that the system is at rest. *

b) Find the corresponding tension in the string, in terms of M, g, R, and θ. *

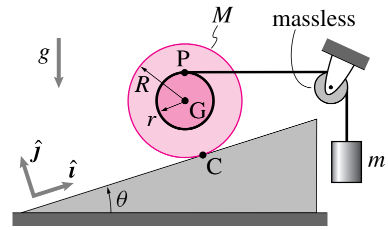
c) Find the corresponding force on the reel at its point of contact with the slope, point C, in terms of M, g, R, and θ. *

Harder Draw a careful sketch and find a point where the lines of action of the gravity force and string tension intersect. For the reel to be in static equilibrium, the line of action of the reaction force at C must pass through this point. Using this information, what must the tangent of the angle ϕ of the reaction force at C be, measured with respect to the normal to the slope? Does this answer agree with that you would obtain from your answer in part(c)? *

d) What is the relationship between the angle ϕ of the reaction at C, measured with respect to the normal to the ground, and the mass ratio required for static equilibrium of the reel? *

e) What is the minimum coefficient of friction μ at C needed to prevent slip.

Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{\boldsymbol{F}}_C = Mg\hat{\boldsymbol{\jmath}}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = 2$ and $\vec{\boldsymbol{F}}_C = Mg(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})$.



Problem 5.3.21

5.3.22 This problem is similar to problem 5.3.21. A reel of mass M and outer radius R is connected by an inextensible string from point P across a pulley to a hanging object of mass m. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle θ. There is no slip between the reel and the slope. There is gravity. In terms of M, g, R, and θ, find:

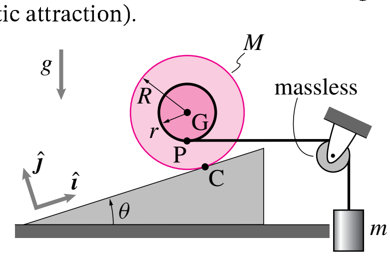
a) the ratio of the masses so that the system is at rest, *

b) the corresponding tension in the string, and *

c) the corresponding force on the reel at its point of contact with the slope, point C. *

d) What is the minimum coefficient of friction μ at C needed to prevent slip.

Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{\boldsymbol{F}}_C = Mg\hat{\boldsymbol{\jmath}}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = -2$ and $\vec{\boldsymbol{F}}_C = Mg(\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}})$.The negative mass ratio is impossible since mass cannot be negative and the negative normal force is impossible unless the wall or the reel or both can 'suck' or they can 'stick' to each other (that is, provide some sort of suction, adhesion, or magnetic attraction).

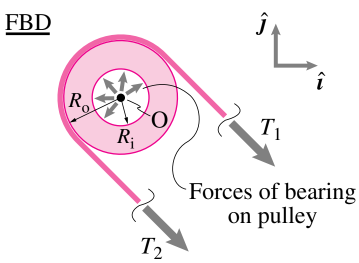


Problem 5.3.22

5.3.23 Assume a massless pulley is round and has outer radius R_2. It slides on a shaft that has radius R_i. Assume there is friction between the shaft and the pulley with coefficient of friction μ, and friction angle ϕ defined by $\mu = \tan(\phi)$. Assume the two ends of the line that are wrapped around the pulley are parallel.

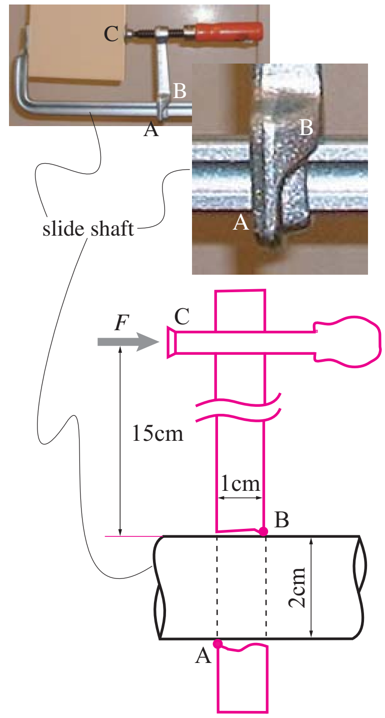
a) What is the relation between the two tensions when the pulley is turning? You may assume that the bearing shaft touches the hole in the pulley at only one point. *.

b) Plug in some reasonable numbers for R_i, R_o and μ (or ϕ) to see one reason why wheels (say pulleys) are such a good idea even when the bearings are not all that well lubricated. *



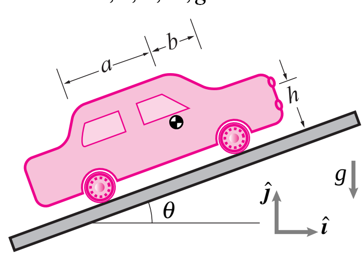
Problem 5.3.23

5.3.24 The so-called *pipe-clamp* has a bracket ABC which loosely fits around the slide-shaft (the'pipe'). When not clamped there is no big force at C and the bracket freely slides on the shaft. However the bracket frictionally locks once the load F at C gets large. Neglecting gravity, find the minimum coefficient of friction μ at A and B for which this clamp holds well (which it does).



Problem 5.3.24

5.3.25 Find the minimum coefficient of friction μ needed for a front wheel drive car to go up hill. Answer in terms of some or all of a, b, h, m, g and θ.



Problem 5.3.25

5.3.26 Solve Problem 5.3.25 for a rear wheel drive car.

5.3.27 Solve Problem 5.3.25 for a four wheel drive car.