21.2 3D dynamics of a rigid body

Momenta and energy of rigid bodies

The kinematics formulas we have developed allow us to calculate velocities and accelerations of points on a rigid body. We can therefore calculate the motion quantities: the momenta, their rates of change, and kinetic energy (and its rate of change). The three basic laws of mechanics that are at our disposal are:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= \dot{\vec{\boldsymbol{L}}}; && \text{Linear momentum balance} \\ \sum \vec{\boldsymbol{M}}_{\mathrm{C}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}; && \text{Angular momentum balance} \\ \sum(\text{Power in}) &= \frac{d}{dt}(E_{\mathrm{K}}) + \frac{d}{dt}(E_{\mathrm{P}}) && \\ & \quad +\text{Dissipation rate}; && \text{Energy/power balance} \end{aligned}$$

In order to be a master of mechanics, all you need is to be able to effectively use these three equations for a variety of systems. One who knows how to efficiently and accurately evaluate both sides of these equations in terms of knowns and reasonably defined unknowns is an ace mechanic. One who, in addition, knows how to solve these equations is an ace programmer and/or mathematician.

Our emphasis in this book is on systems that can be idealized as a particle or a rigid body or a system of rigid bodies. So, once you know how to get the forces and moments from the free-body diagram and once you know the masses and moments of inertia, then you need to find the accelerations of the centers of mass, the angular velocities, and angular accelerations of the various bodies.

For each body,

$$\begin{aligned} \dot{\vec{\boldsymbol{L}}} &= m\vec{\boldsymbol{a}}_{cm}, \\ \dot{\vec{\boldsymbol{H}}}_{/C} &= \vec{\boldsymbol{r}}_{cm/C} \times \mathrm{m}\vec{\boldsymbol{a}}_{cm} + \overbrace{\vec{\boldsymbol{\omega}} \times \underbrace{\vec{\boldsymbol{H}}_{\mathrm{cm}}} + [\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\alpha}}}^{\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}}, \text{ and} \\ & \qquad \boxed{\vec{\boldsymbol{H}}_{\mathrm{cm}} = [\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}} \\ E_{\mathrm{K}} &= \frac{1}{2} m \vec{\boldsymbol{v}}_{cm} \cdot \vec{\boldsymbol{v}}_{cm} + \frac{1}{2}\vec{\boldsymbol{\omega}} \cdot \left[[\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}\right] \end{aligned}$$

So, as repeatedly mentioned in this chapter, the trick is to evaluate $\vec{\boldsymbol{a}}_{cm}$, $\vec{\boldsymbol{\omega}}$, and $\vec{\boldsymbol{\alpha}}$ in terms of the natural variables in the problem.

The keys to make appropriate use of these formulae are:

$$
\begin{aligned}
\vec{\omega}_{\mathcal{C}/\mathcal{F}} &= \vec{\omega}_{\mathcal{B}/\mathcal{F}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}}, \\
\vec{\alpha}_{\mathcal{C}/\mathcal{F}} &= \vec{\alpha}_{\mathcal{B}/\mathcal{F}} + \vec{\alpha}_{\mathcal{C}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{\omega}_{\mathcal{C}/\mathcal{B}}, \\
\vec{v}_{P/\mathcal{F}} &= \vec{v}_{0'/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}}, \text{ and} \\
\vec{a}_{P/\mathcal{F}} &= \vec{a}_{0'/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times (\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'}) + {}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'} \\
&\quad + \vec{a}_{P/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{v}_{P/\mathcal{B}}.
\end{aligned}
$$

Alternatively, the second, third, and fourth formulae may be effectively derived on an ad-hoc basis by use of the more fundamental formulae

$$ {}^{\mathcal{F}}\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q} \qquad (\text{for any } \vec{Q} \text{ fixed in } \mathcal{B}) $$

on various base vectors.

Now, we look at finding the linear and angular momenta for a rigid body using the 3-D example of the crane in fig. 21.19. The crane is a system of connected bodies moving in well defined ways relative to each other and to the ground.

21.4 $\dot{\vec{H}}$ for a rigid body

We can now derive the formula for rate of change of angular momentum of a rigid body $\mathcal{B}$ about some point C using the '$\dot{\vec{Q}}$ *formula*'

$$
\begin{aligned}
\dot{\vec{H}}_{/C} &= \frac{d}{dt}(\vec{H}_{/C}) \\
&= \frac{d}{dt}\left[\vec{r}_{cm/C} \times m\vec{v}_{cm} + \overbrace{\sum \vec{r}_{i/cm} \times \vec{v}_{i/cm} m_i}^{[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}}\right] \\
&= \underbrace{\dot{\vec{r}}_{cm/C} \times m\vec{v}_{cm}}_{\vec{0}} + \vec{r}_{cm/C} \times m\vec{a}_{cm} + \frac{d}{dt}\left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] \\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} \ldots \\
&\quad \underbrace{+\vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] \quad + {}^{\mathcal{B}}\frac{d}{dt}\left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right]}_{\boxed{\frac{d}{dt}\left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] \text{ using the } \dot{\vec{Q}} \text{ formula}}} \\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} + \vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] \\
&\quad + \underbrace{{}^{\mathcal{B}}\frac{d}{dt}[I^{cm}]}_{\vec{0}\ \leftarrow\ \boxed{[I^{cm}] \text{ is constant in } \mathcal{B}}} + [I^{cm}]\cdot {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}} \\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} + \vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] + [I^{cm}]\cdot {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}} \\
&= \vec{r}_{cm/C} \times m\vec{a}_{cm} + \vec{\omega}_{\mathcal{B}} \times \left[[I^{cm}]\cdot\vec{\omega}_{\mathcal{B}}\right] + [I^{cm}]\cdot \dot{\vec{\omega}}_{\mathcal{B}}
\end{aligned}
$$

In the last step of calculation, we have used the important fact about angular acceleration of a body:

$$
\begin{aligned}
{}^{\mathcal{F}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} &= {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}} + \underbrace{\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{\omega}_{\mathcal{B}/\mathcal{F}}}_{\vec{0}} \\
&= {}^{\mathcal{B}}\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}}.
\end{aligned}
$$

That is, the derivative with respect to time of $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$ is the same in $\mathcal{F}$ as in $\mathcal{B}$.

Example: Mechanics of a rigid body in a mechanism (3-D): A crane, again

We would like to find the rate of change of linear momentum of the boom, $\dot{\vec{\boldsymbol{L}}}_{\mathcal{A}}$, and the rate of change of angular momentum of the boom about point O, $\dot{\vec{\boldsymbol{H}}}_{\mathcal{A}/O}$. Assume the mass $m_{\mathcal{A}}$ and the moment of inertia matrix $[\boldsymbol{I}^{\text{cm}}]_{\mathcal{A}}$ are both known. We have

$$
\begin{aligned}
\dot{\vec{\boldsymbol{L}}}_{\mathcal{A}} &= m_{\mathcal{A}}\vec{\boldsymbol{a}}_G, \text{ and} \\
\dot{\vec{\boldsymbol{H}}}_{\mathcal{A}O} &= \vec{\boldsymbol{r}}_{G/O} \times m_{\mathcal{A}}\vec{\boldsymbol{a}}_G + [\boldsymbol{I}^{\text{cm}}]_{\mathcal{A}} \cdot \vec{\boldsymbol{\alpha}}_{\mathcal{A}} + \vec{\boldsymbol{\omega}}_{\mathcal{A}} \times \left[[\boldsymbol{I}^{\text{cm}}]_{\mathcal{A}} \cdot \vec{\boldsymbol{\omega}}_{\mathcal{A}}\right].
\end{aligned}
$$

So, we must find $\vec{\boldsymbol{r}}_{G/O}$, $\vec{\boldsymbol{a}}_G$, $\vec{\boldsymbol{\omega}}_{\mathcal{A}}$, and $\vec{\boldsymbol{\alpha}}_{\mathcal{A}}$. Here we go. First, we find $\vec{\boldsymbol{r}}_{G/O}$

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{G/O} &= \vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{G/O'} \\
&= \overbrace{d\hat{\boldsymbol{\imath}} + (2s+h)\hat{\boldsymbol{\jmath}}}^{\vec{\boldsymbol{r}}_{O'/O}} + \overbrace{\frac{\ell}{3}(\cos\phi\hat{\boldsymbol{\imath}}' + \sin\phi\hat{\boldsymbol{\jmath}}')}^{\vec{\boldsymbol{r}}_{G/O'}}.
\end{aligned}
$$

Next, we find $\vec{\boldsymbol{\omega}}_{\mathcal{A}}$

$$\vec{\boldsymbol{\omega}}_{\mathcal{A}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{A}/\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{\jmath}}' + \dot{\phi}\hat{\boldsymbol{k}}'.$$

Then, we find $\vec{\boldsymbol{\alpha}}_{\mathcal{A}}$

$$\vec{\boldsymbol{\alpha}}_{\mathcal{A}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} = \ddot{\theta}\hat{\boldsymbol{\jmath}}' + \ddot{\phi}\hat{\boldsymbol{k}}' + \dot{\theta}\dot{\phi}\hat{\boldsymbol{\imath}}'.$$

Finally, we find $\vec{\boldsymbol{a}}_G$

$$\vec{\boldsymbol{a}}_G = \vec{\boldsymbol{a}}_{O'} + \vec{\boldsymbol{\omega}}_{\mathcal{A}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{A}} \times \vec{\boldsymbol{r}}_{G/O'}) + \vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{G/O'} + \vec{\boldsymbol{a}}_{G/\mathcal{B}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{G/\mathcal{B}}.$$

At this point, we see several terms that we need to evaluate to plug into this formula. Here they are:

$$
\begin{aligned}
\vec{\boldsymbol{a}}_{O'} &= \ddot{d}\hat{\boldsymbol{\imath}}, \\
\vec{\boldsymbol{v}}_{G/\mathcal{B}} &= \vec{\boldsymbol{\omega}}_{\mathcal{A}/\mathcal{B}} \times \vec{\boldsymbol{r}}_{G/O'}, \\
&= \frac{\ell}{3}\dot{\phi}(\cos\phi\hat{\boldsymbol{\jmath}}' - \sin\phi\hat{\boldsymbol{\imath}}'), \text{ and} \\
\vec{\boldsymbol{a}}_{G/\mathcal{B}} &= \vec{\boldsymbol{\omega}}_{\mathcal{A}/\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{A}/\mathcal{B}} \times \vec{\boldsymbol{r}}_{G/O'}) + \vec{\boldsymbol{\alpha}}_{\mathcal{A}/\mathcal{B}} \times \vec{\boldsymbol{r}}_{G/O'}, \\
&= \frac{\ell}{3}(-\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi)\hat{\boldsymbol{\imath}}' + \frac{\ell}{3}(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)\hat{\boldsymbol{\jmath}}'.
\end{aligned}
$$

So, we can plug in and find the absolute acceleration of the center-of-mass of body $\mathcal{A}$

$$
\begin{aligned}
\vec{\boldsymbol{a}}_G = \ddot{d}\hat{\boldsymbol{\imath}} + \frac{\ell}{3}\Big[&-(\ddot{\phi}\sin\phi + (\dot{\phi}^2 + \dot{\theta}^2)\cos\phi)\hat{\boldsymbol{\imath}}' \\
&+(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)\hat{\boldsymbol{\jmath}}' + (2\dot{\phi}\dot{\theta}\sin\phi - \ddot{\theta}\cos\phi)\hat{\boldsymbol{k}}'\Big].
\end{aligned}
$$

Thus, for the crane boom, now we have $\vec{\boldsymbol{r}}_{G/O}$, $\vec{\boldsymbol{\omega}}_{\mathcal{A}}$, $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{A}}$, and $\vec{\boldsymbol{a}}_G$ in terms of known dimensions, position vectors, and the angular rates $\dot{\phi}$, and $\dot{\theta}$, and their derivatives $\ddot{\phi}$ and $\ddot{\theta}$. So, now we can evaluate $\dot{\vec{\boldsymbol{L}}}_{\mathcal{A}}$ and $\dot{\vec{\boldsymbol{H}}}_{\mathcal{A}/O}$.

Yes – this situation is a mess! Rigid body calculations in three dimensions lead to big messy equations. There are a few special cases where things simplify. Some of these situations are shown in the sample problems and the homework.

A note of caution: When back substituting with the formulae given, be sure that the same base vectors are used for $[\boldsymbol{I}^{\text{cm}}]$ as for $\vec{\boldsymbol{\omega}}_{\mathcal{A}}$ and $\vec{\boldsymbol{\alpha}}_{\mathcal{A}}$.

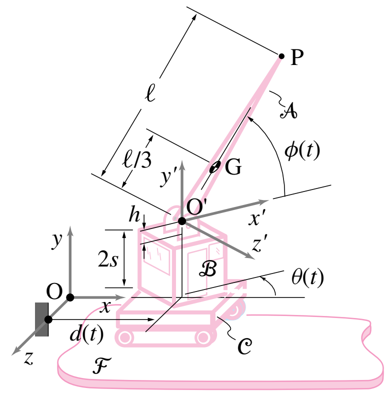

Figure 21.19