20.4 Mechanics using the moment of inertia matrix

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from the inside cover. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. Linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center-of-mass of the system of interest. One would like something so simple for the calculation of angular momentum.

We are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies, the scalar moment of inertia from Chapter 7 is all we need. The luck is not so great for 3-D rigid bodies but still there is some simplification①.The simplification is to use the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{cm}}]$ from the previous section. One may have to do a sum or integral to find $I \equiv I_{zz}^{cm}$ or $[\boldsymbol{I}^{\mathrm{cm}}]$ if an adequate table is not handy. But once you know $[\boldsymbol{I}^{\mathrm{cm}}]$, say, you need not work with the integrals to evaluate angular momentum and its rate of change. Assuming that you are comfortable calculating and looking-up moments of inertia, we proceed to use it for the purposes of studying mechanics.

①For general motion of *non*-rigid bodies, a topic not covered in this book, the simplification for linear momentum still holds; linear momentum and its rate of change are given by the system mass times the velocity and acceleration of the center-of-mass. But there is no general simplification for the sums needed to evaluate angular momentum and energy or their rates of change.

Let's now consider again the conically swinging rod of fig. 20.18 on page 1081. Method 1 for evaluating $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ was evaluating the sums directly. If we accept the formulae presented for rigid bodies in Table I at the back of the book, we can find all of the motion quantities by setting $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}} = \vec{\mathbf{0}}$.

Method 2 of evaluating $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$: using the xyz moment of inertia matrix about point O

In section 20.1 we examined the conical swinging of a straight uniform rod. Now let's look at that example again using its moment of inertia matrix. For a rigid body in constant rate circular motion about an axis through O,

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}}$$

because the $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ vector rotates with the body. For a rigid body rotating about point O,

$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}}.$$

We assumed at the outset that

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}.$$

So, the only trick is to find $[\boldsymbol{I}^{\mathrm{O}}]$ for a rod in the configuration shown. We recall, look up, or believe for now that

$$[\boldsymbol{I}^{\mathrm{O}}] = \begin{bmatrix} I^O_{xx} & I^O_{xy} & I^O_{xz} \\ I^O_{yx} & I^O_{yy} & I^O_{yz} \\ I^O_{zx} & I^O_{zy} & I^O_{zz} \end{bmatrix}.$$

$$\begin{aligned}
I^O_{xy} &= -\int \underbrace{x}_{0}\, y\, dm = 0 && (x = 0,\text{ all mass in the } yz \text{ plane}) \\
I^O_{xz} &= -\int \underbrace{x}_{0}\, z\, dm = 0 && (x = 0,\text{ all mass in the } yz \text{ plane}) \\
I^O_{yy} &= \int (\underbrace{x^2}_{0} + z^2) dm && (x = 0,\text{ all mass in the } yz \text{ plane}) \\
&= \int_0^{\ell} \underbrace{(-s \cdot \cos\phi)^2}_{z} \rho\, ds \\
&= \cos^2\phi\, \rho \frac{\ell^3}{3} = \cos^2\phi\, m \frac{\ell^2}{3} \\
I^O_{zz} &= \int (\underbrace{x^2}_{0} + y^2) dm && (x = 0,\text{ all mass in the } yz \text{ plane}) \\
&= \int \underbrace{(s \cdot \sin\phi)^2}_{y} \rho\, ds \\
&= \sin^2\phi\, \rho \frac{\ell^3}{3} = \sin^2\phi\, m \frac{\ell^2}{3} \\
I^O_{xx} &= \int (y^2 + z^2) dm && (\text{both integrals have been evaluated above}) \\
&= I^O_{yy} + I^O_{zz} && (\text{perpendicular axis theorem}) \\
&= (\cos^2\phi + \sin^2\phi) m \frac{\ell^2}{3} \\
&= m \frac{\ell^2}{3} \\
I^O_{yz} &= -\int yz\, dm \\
&= -\int (s \cdot \sin\phi)(-s \cdot \cos\phi)\, \rho\, ds \\
&= \sin\phi \cos\phi \cdot \rho \frac{\ell^3}{3} = \sin\phi \cos\phi \cdot m \frac{\ell^2}{3}.
\end{aligned}$$

Putting these terms all together in the matrix, we get

$$[\boldsymbol{I}^{\mathrm{O}}] = m\frac{\ell^2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi & \cos\phi\sin\phi \\ 0 & \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}.$$

Now we can calculate $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ as

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{O}} &= [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}} \\
&= m\frac{\ell^2}{3}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi & \cos\phi\sin\phi \\ 0 & \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}}_{[\boldsymbol{I}^{\mathrm{O}}]}\cdot\underbrace{\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}}_{\vec{\boldsymbol{\omega}}} \\
&= m\frac{\ell^2}{3}\begin{bmatrix} 0 \\ \omega\cos\phi\sin\phi \\ \omega\sin^2\phi \end{bmatrix} \\
&= m\omega\frac{\ell^2}{3}\sin\phi(\cos\phi\,\hat{\boldsymbol{\jmath}} + \sin\phi\,\hat{\boldsymbol{k}}).
\end{aligned}$$

You may notice, by the way, that for this problem, $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ (in the direction of $\cos\phi\,\hat{\boldsymbol{\jmath}} + \sin\phi\,\hat{\boldsymbol{k}}$) is perpendicular to the rod (in the direction of $\sin\phi\,\hat{\boldsymbol{\jmath}} - \cos\phi\,\hat{\boldsymbol{k}}$). So $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is *not* in the direction of $\vec{\boldsymbol{\omega}}$. (2)

(2) $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is only parallel to $\vec{\boldsymbol{\omega}}$ if rotation is about one of the principal axes (eigenvector directions) of the inertia matrix.

Now we calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ as

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{H}}_{/\mathrm{O}} \\
&= (\omega\hat{\boldsymbol{k}})\times\left[\omega m\frac{\ell^2}{3}(\cos\phi\sin\phi\,\hat{\boldsymbol{\jmath}} + \sin^2\phi\,\hat{\boldsymbol{k}})\right] \\
&= -m\frac{\ell^2}{3}\cos\phi\sin\phi\,\omega^2\hat{\boldsymbol{\imath}},
\end{aligned}$$

the same result we got before.

Method 3 of calculating $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$: using $[\boldsymbol{I}^{\mathrm{O}}]$ and a coordinate system lined up with the rod

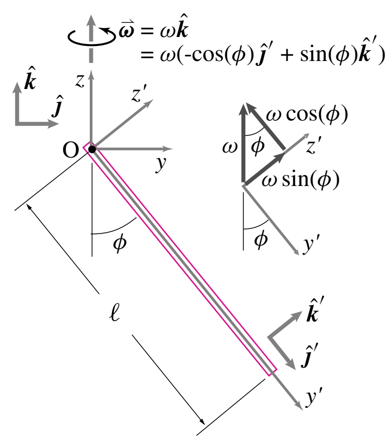


Figure 20.36: The spherical pendulum using $x'y'z'$ axes aligned with the rod.

Here, as suggested above, we redo the problem using a rotated set of axes better aligned with the rod. This method makes calculation of the moment of inertia matrix quite a bit easier — we can even look it up in a table — but makes the determination of $\vec{\boldsymbol{\omega}}$ a little harder. We are stuck finding the components of $\vec{\boldsymbol{\omega}}$ in a rotated coordinate system. Referring to the table of moment of inertias on the inside of the back cover and taking care because different coordinates are used, the moment of inertia matrix about point O for a thin uniform rod, in terms of the rotated coordinates is

$$[\boldsymbol{I}^{\mathrm{O}}]_{x'y'z'} = m\frac{\ell^2}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

and the angular velocity in rotated coordinates is $\vec{\boldsymbol{\omega}} = -\omega\cos\phi\,\hat{\boldsymbol{\jmath}}' + \omega\sin\phi\hat{\boldsymbol{k}}'$, which can be written in component form as

$$[\vec{\boldsymbol{\omega}}]_{x'y'z'} = \omega\begin{bmatrix}0\\ -\cos\phi\\ \sin\phi\end{bmatrix}.$$

We calculate the angular momentum about point O as

$$\overbrace{\vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}}^{\text{vector equation, independent of coordinates}}$$

$$\underbrace{\begin{bmatrix}H_{/\mathrm{O}_{x'}}\\ H_{/\mathrm{O}_{y'}}\\ H_{/\mathrm{O}_{z'}}\end{bmatrix} = m\frac{\ell^2}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}_{x'y'z'}\cdot\,\omega\begin{bmatrix}0\\ -\cos\phi\\ \sin\phi\end{bmatrix}_{x'y'z'}}_{\text{all components in } x'y'z' \text{ coordinate system}}$$

$$\begin{bmatrix}H_{/\mathrm{O}_{x'}}\\ H_{/\mathrm{O}_{y'}}\\ H_{/\mathrm{O}_{z'}}\end{bmatrix} = m\frac{\ell^2}{3}\omega\begin{bmatrix}0\\ 0\\ \sin\phi\end{bmatrix}_{x'y'z'}.$$

Finally, we calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ as

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{H}}_{/\mathrm{O}}\\ &= [\omega(-\cos\phi\,\hat{\boldsymbol{\jmath}}' + \sin\phi\hat{\boldsymbol{k}}'] \times (\omega m\frac{\ell^2}{3}\sin\phi\hat{\boldsymbol{k}}')\\ &= -\sin\phi\cos\phi\; m\omega^2\frac{\ell^2}{3}\hat{\boldsymbol{\imath}}'.\end{aligned}$$

This answer is again the same as what we got before because the x and x' axis are coincident so $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$.

The multitude of ways to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$

We just showed three ways to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ for a conically swinging stick, but there are many more. You can get a sense of the possibilities by studying table I summarizing momenta and energy in the back of the book.

Here are three basic choices.

$$
\begin{aligned}
(1)\ \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm, \\
(2)\ \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}}, \qquad \text{or}
\end{aligned}
$$

$$
(3)\ \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{a}}_{cm} m_{tot} + \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}.
$$

Choice (3) is the safest choice to make if you are in doubt, since it is the only one of the three choices that does not depend on point O being a fixed point (which it was for this example). For option (2) above, we can calculate $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ various ways as

$$
\begin{aligned}
(a)\ \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}}\, dm \qquad \text{or,} \\
(b)\ \vec{\boldsymbol{H}}_{/\mathrm{O}} &= [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}} \qquad \text{or,} \\
(c)\ \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{v}}_{cm} m_{tot} + \vec{\boldsymbol{H}}_{\mathrm{cm}}.
\end{aligned}
$$

For option (3) above, we can calculate $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ as

$$
\begin{aligned}
(d)\ \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}} &= \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{a}}_{/cm} dm \qquad \text{or,} \\
(e)\ \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{cm}}.
\end{aligned}
$$

For (c) and (e) above, we can calculate $\vec{\boldsymbol{H}}_{\mathrm{cm}}$ as

$$
\begin{aligned}
(f)\ \vec{\boldsymbol{H}}_{\mathrm{cm}} &= \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}_{/cm}\, dm \qquad \text{or,} \\
(g)\ \vec{\boldsymbol{H}}_{\mathrm{cm}} &= [\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}.
\end{aligned}
$$

For either of options (b) or (g), we can calculate $[I]$ relative to the xyz axes (as we did for the second method on the previous pages) or relative to some rotated axes better aligned with the rod (as we did for the third method on the previous pages.)

As you can surmise from all the choices above, the list of options for the calculation of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ is too long and boring to show here.

The pros and cons of these methods depend on the problem at hand. If you want to avoid integration you are pretty much stuck using either $[\boldsymbol{I}^{\mathrm{O}}]$ or $[\boldsymbol{I}^{\mathrm{cm}}]$ with (e) and (g) above. Avoiding the integrals depends on your having a table of moments of inertia (like table IV in the back of this book).

Energy of things going in circles at variable rate

The energy only depends on the speeds of the parts of a system, not their accelerations. So, as with constant rate motion,

$$
\begin{aligned}
E_{\mathrm{K}} &= \frac{1}{2}\int \underbrace{v^2}_{\vec{v}\cdot\vec{v}}\, dm \\
&= \frac{1}{2}\int (\vec{\omega}\times\vec{\boldsymbol{r}})\cdot(\vec{\omega}\times\vec{\boldsymbol{r}})\, dm \\
&= \frac{1}{2}\omega^2 \int R^2\, dm
\end{aligned}
$$

R is the distance from the axis to the mass

$$
\begin{aligned}
&= \frac{1}{2}\vec{\omega}\cdot[\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\omega} \\
&= \underbrace{\frac{1}{2}\omega^2 I^{o}_{zz}}
\end{aligned}
$$

For rotation about the z axis the 9 terms in the matrix formula reduce to this one simple term.

The safest bet

The following are the most reliable (least prone to error) formulas for evaluating the motion quantities for a rigid body rotating about a fixed axes (they also apply to arbitrary motion of a rigid body).

$$
\vec{\boldsymbol{H}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{\mathrm{cm/o}}\times m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} + [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\omega} \tag{20.38}
$$

$$
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{\mathrm{cm/o}}\times m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}} + \vec{\omega}\times\underbrace{[\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\omega}}_{\vec{\boldsymbol{H}}_{\mathrm{cm}}} + [\boldsymbol{I}^{\mathrm{cm}}]\cdot\dot{\vec{\omega}} \tag{20.39}
$$

$$
E_{\mathrm{K}} = \frac{1}{2}m_{\mathrm{tot}}v_{cm}^2 + \vec{\omega}\cdot([\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\omega}) \tag{20.40}
$$

Please survey table I at the back of the book which summarizes the ways of evaluating momenta and energy.

The geometry of $\dot{\vec{H}}_{/\mathrm{O}}$

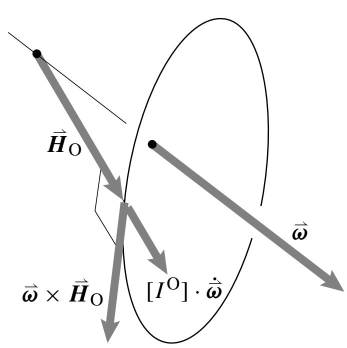


Figure 20.37: The two terms in the rate of change of angular momentum are shown.

It is possible to understand the formula for the change of angular momentum geometrically. Here is one way of looking at it. For this example it is most direct to look at $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ calculated using $[\boldsymbol{I}^{\mathrm{O}}]$, but the same discussion would work with $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ calculated using $[\boldsymbol{I}^{\mathrm{cm}}]$.

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \underbrace{\vec{\omega} \times \vec{\boldsymbol{H}}_{/\mathrm{O}}}_{\text{Contribution from rotation of } \vec{\boldsymbol{H}}_{/\mathrm{O}}} + \underbrace{[\boldsymbol{I}^{\mathrm{O}}] \cdot \dot{\vec{\omega}}}_{\text{Due to the changing length of } \vec{\boldsymbol{H}}_{/\mathrm{O}}}$$

The first term in the equation describes the rotation of the angular momentum vector. The second term describes its rate of change of length.

Since the body is spinning about a fixed axis, the orientation of the axis of rotation is not only fixed relative to the Newtonian frame, the room environment, but also relative to the body. At one instant of time we use a coordinate system to calculate $\vec{\boldsymbol{H}}_{/\mathrm{O}}$. After the body has rotated a little, if we now used a coordinate system that rotated the same amount as the body, a coordinate system 'glued' to the body, we could calculate $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ again.

This new calculation of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ will be almost identical to the calculation before the small rotation, however, because the moment of inertia matrix does not change in time relative to a coordinate system that moves with the body. Also, the coordinates of $\vec{\omega}$ will be unchanged except for possibly a multiplication by a constant because the direction of $\vec{\omega}$ doesn't change. Thus the only change of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ as represented in this rotated coordinate system, is a possible change in its length due to a change in the spinning rate.

But this new coordinate system is a rotated coordinate system. To find the actual change in $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ we need to take this rotation into account. To picture this rotation consider the special case when the rotation rate is constant. Then the vector $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is constant in the rotating coordinate system. That is, the vector $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ rotates with the body.

So the net change in $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is a change due to rotation of the body added to a change due to the change in the rotation rate. For small angular changes, the direction of the first term is the same as the tangent to the circle that is traced by the tip of the angular momentum vector $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ as drawn on the body. To approximate the first term, consider the following reasoning. First, let's denote the first term by $(\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}})_{\mathrm{rot}} = \vec{\omega} \times \vec{\boldsymbol{H}}_{/\mathrm{O}}$, where the subscript 'rot' indicates that this term is the contribution to $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ from the rotation of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$. For small angular changes, $(\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}})_{rot} = (d\vec{\boldsymbol{H}}_{/\mathrm{O}}/dt)_{rot} = \vec{\omega} \times \vec{\boldsymbol{H}}_{/\mathrm{O}} \approx (\Delta\vec{\boldsymbol{H}}_{/\mathrm{O}})_{rot}/\Delta t$. Thus, the term due to rotation is approximately, for small Δt, $(\Delta\vec{\boldsymbol{H}}_{/\mathrm{O}})_{rot} = \Delta t(\vec{\omega} \times \vec{\boldsymbol{H}}_0)$.

The contribution to $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ due to change of length of $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ is $[\boldsymbol{I}^{\mathrm{O}}] \cdot \dot{\vec{\omega}}$. Similarly, this term is approximately, for small Δt, $\Delta t([\boldsymbol{I}^{\mathrm{O}}] \cdot \dot{\vec{\omega}})$.

Rotation of a rigid body about a fixed axis: the general case

Consider a general rigid body of mass m and moment of inertia matrix with respect to the center-of-mass $[\boldsymbol{I}^{\mathrm{cm}}]$ rotating about a fixed axis. Without loss of generality, let the axis of rotation be the $\hat{\boldsymbol{k}}$ axis.

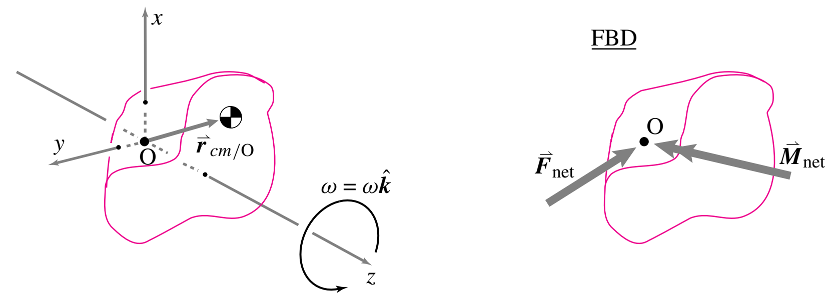


Figure 20.38: Free-body diagram of a general rigid body rotating about the z-axis.

Linear momentum balance

Referring to the free-body diagram of the rigid body, linear momentum balance gives

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= \dot{\vec{\boldsymbol{L}}} \\ \vec{\boldsymbol{F}}_{net} &= m\vec{\boldsymbol{a}}_{cm} \\ &= m[\omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}) + \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}]. \end{aligned} \tag{20.41}$$

The first term on the right hand side of equation 20.41, the centripetal term, is directed from the center-of-mass ( ) through the axis of rotation; that is, it lies in the xy-plane (or has no $\hat{\boldsymbol{k}}$ component). The second term on the right hand side of equation 20.41, the tangential term, is normal to the plane determined by the axis and the center-of-mass. It is tangent to the circle that the center-of-mass travels on. It is zero if the center of mass is on the axis.

Angular momentum balance

Angular momentum balance about point O gives

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \\
\vec{\boldsymbol{M}}_{net} &= \underbrace{\vec{\boldsymbol{r}}_{cm/O} \times m\vec{\boldsymbol{a}}_{cm}}_{\text{term (i)}} \\
&\quad + \underbrace{\omega\hat{\boldsymbol{k}} \times \{[\boldsymbol{I}^{\mathrm{cm}}] \cdot \omega\hat{\boldsymbol{k}}\}}_{\text{term (ii)}} \\
&\quad + \underbrace{[\boldsymbol{I}^{\mathrm{cm}}] \cdot (\dot{\omega}\hat{\boldsymbol{k}})}_{\text{term (iii)}}.
\end{aligned} \tag{20.42}$$

Let's look at each of the three terms on the right hand side in turn.

term (i)

The first term (i) on the right hand side of equation 20.42 is

$$\boxed{\begin{aligned}\vec{\boldsymbol{a}}_{cm} = {} & \omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}) \\ & + \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}\end{aligned}}$$

$$\begin{aligned}
\text{term (i)} &= \vec{\boldsymbol{r}}_{cm/O} \times m\, \overbrace{\vec{\boldsymbol{a}}_{cm}} \\
&= m[\vec{\boldsymbol{r}}_{cm/O} \times (\omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O})) \\
&\quad + \quad \vec{\boldsymbol{r}}_{cm/O} \times (\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O})]
\end{aligned}$$

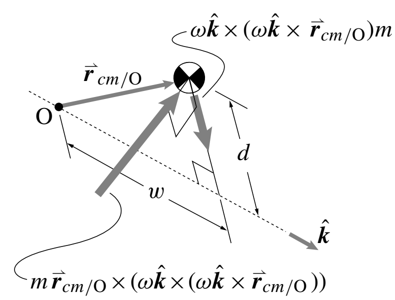


Figure 20.39: The first part of term (i)

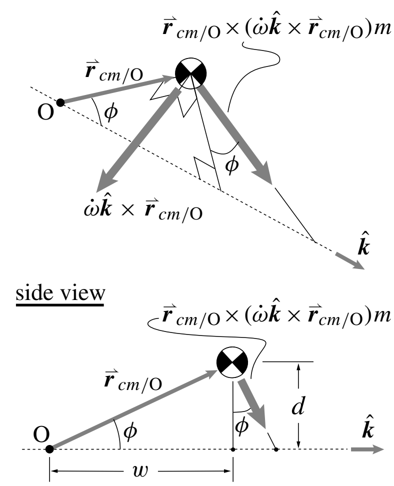


Figure 20.40: The second part of term (i)

Now, let's consider the two parts of term (i) in turn. The first part of term (i) is in the direction of $-\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}$. For example, if the center-of-mass is in the xz plane, this contribution to $\vec{\boldsymbol{M}}_{net}$ is in the $-\hat{\boldsymbol{\jmath}}$ direction and could be accommodated by reaction forces on the axis in the $\hat{\boldsymbol{\imath}}$ and $-\hat{\boldsymbol{\imath}}$ directions.
Now, the second part of term (i) can be decomposed into a part along the $\hat{\boldsymbol{k}}$ direction and a part perpendicular to $\hat{\boldsymbol{k}}$ (along d). The part along $\hat{\boldsymbol{k}}$ is

$$\vec{\boldsymbol{r}}_{cm/O} \times (\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{cm/O}) \cdot \hat{\boldsymbol{k}} = md^2\dot{\omega}\hat{\boldsymbol{k}}.$$

The part of term (i) perpendicular to $\hat{\boldsymbol{k}}$ has magnitude $m\dot{\omega}dw$.

term (ii)

Now, let's look at the second term in equation 20.42, term (ii), and expand it.

$$\begin{aligned}
\text{term (ii)} &= \omega\hat{\boldsymbol{k}} \times [\boldsymbol{I}^{\text{cm}}] \cdot \omega\hat{\boldsymbol{k}} \\
&= \omega\hat{\boldsymbol{k}} \times \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \\
&= \omega\hat{\boldsymbol{k}} \times \left[I_{xz}^{cm}\omega\hat{\boldsymbol{\imath}} + I_{yz}^{cm}\omega\hat{\boldsymbol{\jmath}} + I_{zz}^{cm}\omega\hat{\boldsymbol{k}} \right] \\
&= I_{xz}^{cm}\omega^2\hat{\boldsymbol{\jmath}} - I_{yz}^{cm}\omega^2\hat{\boldsymbol{\imath}}
\end{aligned}$$

So, the second term in equation 20.42, term (ii), has *no* $\hat{\boldsymbol{k}}$ component. The $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components are due to the off-diagonal terms in $[\boldsymbol{I}^{\text{cm}}]$. The off-diagonal terms are responsible for dynamic imbalance.

term (iii)

Now, the third term in equation 20.42, term (iii), is

$$\begin{aligned}
[\boldsymbol{I}^{\text{cm}}] \cdot \dot{\omega}\hat{\boldsymbol{k}} &= \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix} \\
&= I_{xz}^{cm}\dot{\omega}\hat{\boldsymbol{\imath}} + I_{yz}^{cm}\dot{\omega}\hat{\boldsymbol{\jmath}} + I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}}.
\end{aligned}$$

Putting terms (i), (ii), and (iii) back together

Now, let's put the three terms back together into the equation of angular momentum balance about point O, equation 20.42. To help with the interpretation, assume the center-of-mass is in the xz plane. So, we start with

$$\vec{\boldsymbol{M}}_{net} = \dot{\vec{\boldsymbol{H}}}_{/\text{O}}.$$

Breaking this vector equation into components, we get

$$\begin{aligned}
M_{xx} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{\imath}} &= \dot{\vec{\boldsymbol{H}}}_{/\text{O}} \cdot \hat{\boldsymbol{\imath}} \\
M_{yy} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{\jmath}} &= \dot{\vec{\boldsymbol{H}}}_{/\text{O}} \cdot \hat{\boldsymbol{\jmath}} \\
M_{zz} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{k}} &= \dot{\vec{\boldsymbol{H}}}_{/\text{O}} \cdot \hat{\boldsymbol{k}}.
\end{aligned}$$

Now, using all the results so far, we find the torques about the x, y, and z axes. For the z-axis,

$$M_z = \underbrace{[m \overbrace{d^2}^{d\,=\,\text{distance of cm from axis of rotation}} + I_{zz}^{cm}]}_{\text{sometimes called `}[\boldsymbol{I}^{\text{O}}]\text{'}} \dot{\omega}. \tag{20.43}$$

The only torque about the z-axis, the axis of rotation, is due to the acceleration about that axis.

Next, for the x-axis,

$$M_x = -I_{yz}^{cm}\omega^2 + I_{xz}^{cm}\dot{\omega} + m\dot{\omega}dw. \tag{20.44}$$

There is a torque about the x-axis due to off-diagonal terms in $[\boldsymbol{I}^{\mathrm{cm}}]$. One term is the dynamic imbalance term I_{yz}^{cm} and the other is associated with angular acceleration. (Recall, the center-of-mass is on the xz plane.) There is also a term due to the center of mass tangential acceleration since the center-of-mass is on a plane that does not contain point O.

Finally, the torque about the y-axis is

$$M_y = I_{xz}^{cm}\omega^2 + I_{yz}^{cm}\dot{\omega} + (-m\omega^2 dw). \tag{20.45}$$

Here, we have nearly the same types of terms as in equation 20.44. In this case, though, the centripetal acceleration causes the center-of-mass motion to contribute to the torque.

So, if

$$\vec{\omega} = \omega\hat{\boldsymbol{k}} \text{ and } \dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{k}}$$

then

$$\begin{aligned}\vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \quad &= \quad \left(-I_{yz}^{cm}\omega^2 + (I_{xz}^{cm} + mdw)\dot{\omega}\right)\hat{\boldsymbol{\imath}} \\ &\quad\quad +((I_{xz}^{cm} - mdw)\omega^2 + I_{yz}^{cm}\dot{\omega})\hat{\boldsymbol{\jmath}} \\ &\quad\quad +(md^2 + I_{zz}^{cm})\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

SAMPLE 20.11 A scalar times a vector is a vector. A matrix times a vector is a vector. What is the difference? Find the vectors $\vec{\boldsymbol{H}}_1 = I_G\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}}_2 = [\mathbf{I}_G]\vec{\boldsymbol{\omega}}$, if $\vec{\boldsymbol{\omega}} = 2\,\mathrm{rad/s}\hat{\boldsymbol{\imath}} + 3\,\mathrm{rad/s}\hat{\boldsymbol{\jmath}}$, $I_G = 10\,\mathrm{kg\,m^2}$ and $[\mathbf{I}_G] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix}\mathrm{kg\,m^2}$. Draw $\vec{\boldsymbol{\omega}}$, $\vec{\boldsymbol{H}}_1$ and $\vec{\boldsymbol{H}}_2$.

Solution

$$\begin{aligned}\vec{\boldsymbol{H}}_1 &= I_G\vec{\boldsymbol{\omega}} \\ &= 10\,\mathrm{kg\,m^2}(2\,\mathrm{rad/s}\hat{\boldsymbol{\imath}} + 3\,\mathrm{rad/s}\hat{\boldsymbol{\jmath}}) \\ &= (20\hat{\boldsymbol{\imath}} + 30\hat{\boldsymbol{\jmath}})\,\mathrm{kg\,m^2/\,s}.\end{aligned}$$

$$\begin{aligned}\vec{\boldsymbol{H}}_2 &= [\mathbf{I}_G]\cdot\vec{\boldsymbol{\omega}} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix}\mathrm{kg\,m^2}\cdot\begin{Bmatrix} 2 \\ 3 \\ 0 \end{Bmatrix}\mathrm{rad/s} \\ &= (10\hat{\boldsymbol{\imath}} + 15\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}})\,\mathrm{kg\,m^2/\,s}.\end{aligned}$$

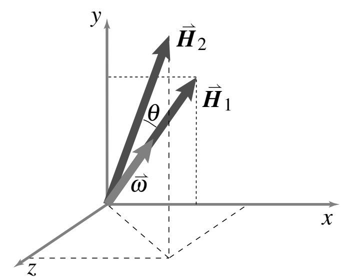


Figure 20.41

These two vectors, $\vec{\boldsymbol{H}}_1$ and $\vec{\boldsymbol{H}}_2$, are shown along with $\vec{\boldsymbol{\omega}}$ in Fig. 20.41. Note that $\vec{\boldsymbol{H}}_1$ has the same direction as $\vec{\boldsymbol{\omega}}$ but $\vec{\boldsymbol{H}}_2$ does not. $\vec{\boldsymbol{H}}_1$ and $\vec{\boldsymbol{\omega}}$ are both in the xy-plane but $\vec{\boldsymbol{H}}_2$ is not; it is in 3-D.

Comments: Multiplying a vector by a scalar does not change the direction of the vector but multiplying by a matrix does change the direction, in general. Find the angles between $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}}_1$ and between $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}}_2$ to convince yourself.

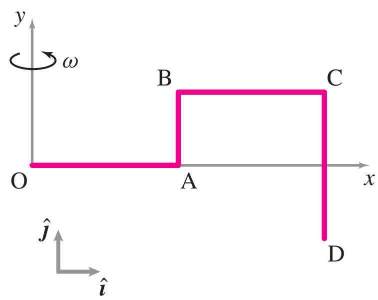


Figure 20.42

SAMPLE 20.12 The composite rod OABCD shown in Figure 20.42 is made up of three identical rods OA, BC, CD of mass 0.5 kg and length 20 cm each, and the rod AB which is half of rod CD. The composite rod goes in circles about the y-axis at a constant rate $\omega = 5\,\mathrm{rad/s}$. Find the angular momentum and the rate of change of angular momentum of the rod at the instant shown (*i.e.*, when the rod is in the xy-plane).

1. Are all the components of $[\boldsymbol{I}^{\mathrm{O}}]$ necessary to compute $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$? Find $[\boldsymbol{I}^{\mathrm{O}}]$ or the necessary components of $[\boldsymbol{I}^{\mathrm{O}}]$.
2. Find the angular momentum $\vec{\boldsymbol{H}}_{/\mathrm{O}}$.
3. Find the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$.
4. If the rod were rotating about the z-axis instead, (*i.e.*, if the motion were in the xy-plane) which components of $[\boldsymbol{I}^{\mathrm{O}}]$ would be required to find $\vec{\boldsymbol{H}}_{/\mathrm{O}}$? What would be the value of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ in that case?

Solution Since the rod rotates about the y-axis, and O is a fixed point on this axis,

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}.$$

1. Since

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{O}} &= \begin{bmatrix} I^{\mathrm{O}}_{xx} & I^{\mathrm{O}}_{xy} & I^{\mathrm{O}}_{xz} \\ I^{\mathrm{O}}_{xy} & I^{\mathrm{O}}_{yy} & I^{\mathrm{O}}_{yz} \\ I^{\mathrm{O}}_{xz} & I^{\mathrm{O}}_{yz} & I^{\mathrm{O}}_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\
&= I^{\mathrm{O}}_{xy}\omega\hat{\boldsymbol{\imath}} + I^{\mathrm{O}}_{yy}\omega\hat{\boldsymbol{\jmath}} + I^{\mathrm{O}}_{yz}\omega\hat{\boldsymbol{k}} \\
&= (I^{\mathrm{O}}_{xy}\hat{\boldsymbol{\imath}} + I^{\mathrm{O}}_{yy}\hat{\boldsymbol{\jmath}} + I^{\mathrm{O}}_{yz}\hat{\boldsymbol{k}})\omega, \qquad (20.46)
\end{aligned}$$

we only need to find three components of $[\boldsymbol{I}^{\mathrm{O}}]$ to compute $\vec{\boldsymbol{H}}_{/\mathrm{O}}$, namely I^{O}_{xy}, I^{O}_{yy} and I^{O}_{yz}.
We can compute these components by considering each rod individually. For any rod, the components can be calculated using the values of the components about the center-of-mass of the rod (see table IV in the back of the book) and then using the parallel axis theorem.

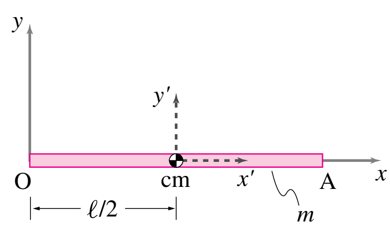


Figure 20.43

Rod OA:

$$\begin{aligned}
I^{\mathrm{O}}_{yy} &= I^{\mathrm{cm}}_{y'y'} + m\frac{l^2}{4} = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 \\
I^{\mathrm{O}}_{xy} &= \underbrace{I^{\mathrm{cm}}_{x'y'}}_{0} + m(-x_{\mathrm{cm/O}}\underbrace{y_{\mathrm{cm/O}}}_{0}) = 0 \\
I^{\mathrm{O}}_{yz} &= \underbrace{I^{\mathrm{cm}}_{y'z'}}_{0} + m(-\underbrace{y_{\mathrm{cm/O}}}_{0}\underbrace{z_{\mathrm{cm/O}}}_{0}) = 0.
\end{aligned}$$

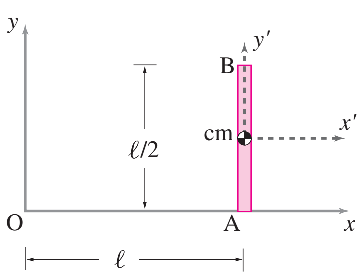


Figure 20.44

Rod AB:

$$\begin{aligned}
I^{\mathrm{O}}_{yy} &= \underbrace{I^{\mathrm{cm}}_{y'y'}}_{0} + \frac{m}{2}(x^2_{\mathrm{cm/O}} + \underbrace{z^2_{\mathrm{cm/O}}}_{0}) = \frac{ml^2}{2} \\
I^{\mathrm{O}}_{xy} &= \underbrace{I^{\mathrm{cm}}_{x'y'}}_{0} + \frac{m}{2}(-\underbrace{x_{\mathrm{cm/O}}}_{l}\underbrace{y_{\mathrm{cm/O}}}_{\frac{l}{4}}) = -\frac{ml^2}{8} \\
I^{\mathrm{O}}_{yz} &= \underbrace{I^{\mathrm{O}}_{y'z'}}_{0} + m(-y_{\mathrm{cm/O}}\underbrace{z_{\mathrm{cm/O}}}_{0}) = 0.
\end{aligned}$$

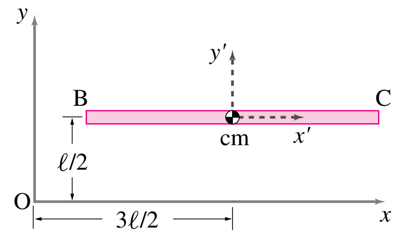


Figure 20.45

$$\underline{\text{Rod BC}}:\quad I^{\text{O}}_{yy} = I^{\text{cm}}_{y'y'} + m(x^2_{\text{cm/O}} + \underbrace{z^2_{\text{cm/O}}}_{0}) = \frac{ml^2}{12} + m\frac{9l^2}{4} = \frac{7}{3}ml^2$$

$$I^{\text{O}}_{xy} = \underbrace{I^{\text{cm}}_{x'y'}}_{0} + m(-x_{\text{cm/O}}y_{\text{cm/O}}) = -m\frac{3l}{2}\cdot\frac{l}{2} = -\frac{3ml^2}{4}$$

$$I^{\text{O}}_{yz} = 0.$$

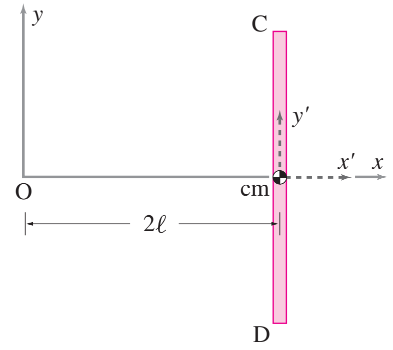


Figure 20.46

$$\underline{\text{Rod CD}}:\quad I^{\text{O}}_{yy} = \underbrace{I^{\text{cm}}_{y'y'}}_{0} + m(x^2_{\text{cm/O}} + \underbrace{z^2_{\text{cm/O}}}_{0}) = m(2l)^2 = 4ml^2$$

$$I^{\text{O}}_{xy} = 0$$

$$I^{\text{O}}_{yz} = 0.$$

Thus for the entire rod,

$$I^{\text{O}}_{yy} = \frac{1}{3}ml^2 + \frac{1}{2}ml^2 + \frac{7}{3}ml^2 + 4ml^2 = \frac{43}{6}ml^2$$

$$I^{\text{O}}_{xy} = -\frac{1}{8}ml^2 - \frac{3}{4}ml^2 = -\frac{7}{8}ml^2$$

$$I^{\text{O}}_{yz} = 0.$$

$$\boxed{I^{\text{O}}_{yy} = \tfrac{43}{6}ml^2,\ I^{\text{O}}_{xy} = -\tfrac{7}{8}ml^2,\ I^{\text{O}}_{yz} = 0.}$$

2.

$$\vec{\boldsymbol{H}}_{/\text{O}} = (I^{\text{O}}_{yy}\hat{\boldsymbol{\imath}} + I^{\text{O}}_{xy}\hat{\boldsymbol{\jmath}} + I^{\text{O}}_{yz}\hat{\boldsymbol{k}})\omega \quad \text{(from Eqn 20.46)}$$

$$= ml^2\omega(\frac{43}{6}\hat{\boldsymbol{\jmath}} - \frac{7}{8}\hat{\boldsymbol{\imath}})$$

$$= 0.5\,\text{kg}(0.2\,\text{m})^2\cdot 5\,\text{rad/s}(7.17\hat{\boldsymbol{\jmath}} - 0.87)\hat{\boldsymbol{\imath}})$$

$$= (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\text{kg}\cdot\text{m}^2/\text{s}.$$

$$\boxed{\vec{\boldsymbol{H}}_{/\text{O}} = (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\text{kg}\cdot\text{m}^2/\text{s}.}$$

3.

$$\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\text{O}} = \omega\hat{\boldsymbol{\jmath}} \times \left([\boldsymbol{I}^{\text{O}}]\cdot\vec{\boldsymbol{\omega}}\right)$$

$$= 5\,\text{rad/s}\,\hat{\boldsymbol{\jmath}} \times (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\text{kg}\cdot\text{m}^2/\text{s}$$

$$= 0.435\,\text{N}\cdot\text{m}\hat{\boldsymbol{k}}.$$

$$\boxed{\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = 0.435\,\text{N}\cdot\text{m}\hat{\boldsymbol{k}}.}$$

4. If the rod were rotating in the xy-plane, it would be planar circular motion. The only component of $[\boldsymbol{I}^{\text{O}}]$ required for the calculation of $\vec{\boldsymbol{H}}_{/\text{O}} = I^{\text{O}}_{zz}\vec{\boldsymbol{\omega}}$ will be I^{O}_{zz}. Also,

$$\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\text{O}} = \vec{\boldsymbol{\omega}} \times I^{\text{O}}_{zz}\vec{\boldsymbol{\omega}} = 0$$

since $\vec{\boldsymbol{\omega}}$ is parallel to $I^{\text{O}}_{zz}\vec{\boldsymbol{\omega}}$.

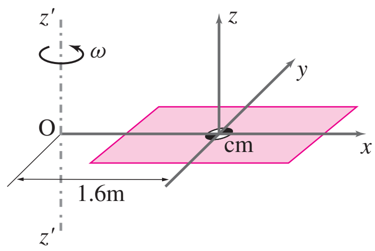


Figure 20.47

SAMPLE 20.13 A 0.5 kg uniform rectangular solar panel rotates about an off-centered axis $z'z'$ with constant angular speed $\omega = 1.5\,\mathrm{rad/s}$. Axis $z'z'$ is parallel to the transverse z-axis of the plate and is 1.6 m away from the center-of-mass of the plate. The in-plane moments of inertia I_{xx}^{cm} and I_{yy}^{cm} of the plate are given: $I_{xx}^{\mathrm{cm}} = 4.0\,\mathrm{kg\cdot m^2}$ and $I_{yy}^{\mathrm{cm}} = 12.4\,\mathrm{kg\cdot m^2}$. At the instant shown in Fig 20.47, calculate

1. the linear momentum of the panel,
2. the angular momentum of the panel, and
3. the kinetic energy of the panel.

Solution Since the plate rotates about the $z'z'$-axis which is parallel to the z-axis,

$$\vec{\omega} = \omega\hat{\boldsymbol{k}} = 1.5\,\mathrm{rad/s}\hat{\boldsymbol{k}}.$$

1. **Linear momentum:**

$$\begin{aligned}\vec{\boldsymbol{L}} &= \mathrm{m_{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} = m(\vec{\omega}\times\vec{\boldsymbol{r}}_{\mathrm{cm/O}}) \\ &= 0.5\,\mathrm{kg}\cdot(1.5\,\mathrm{rad/s}\hat{\boldsymbol{k}}\times 1.6\,\mathrm{m}\hat{\boldsymbol{\imath}}) \\ &= 1.2\,\mathrm{kg\cdot m/s}\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

$$\boxed{\vec{\boldsymbol{L}} = 1.2\,\mathrm{kg\cdot m/s}\,\hat{\boldsymbol{\jmath}}}$$

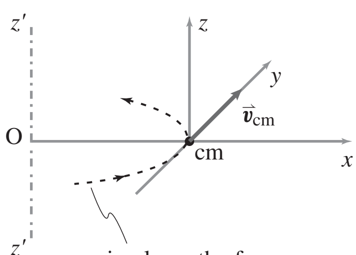


Figure 20.48

We can easily check the direction of $\vec{\boldsymbol{L}}$ since $\vec{\boldsymbol{L}} = m\vec{\boldsymbol{v}}_{\mathrm{cm}}$, it has to be in the same direction as $\vec{\boldsymbol{v}}_{\mathrm{cm}}$. The center-of-mass goes in circles about O, therefore, $\vec{\boldsymbol{v}}_{\mathrm{cm}}$ is tangential to the circular path, i.e. in the y-direction (see Fig 20.48).

2. **Angular momentum:**

$$\vec{\boldsymbol{H}} = I_{z'z'}^{\mathrm{O}}\,\omega\hat{\boldsymbol{k}}$$

Thus to find $\vec{\boldsymbol{H}}$, we need to find $I_{z'z'}^{\mathrm{O}}$. Since $z'z' \parallel zz$, we can use the parallel axis theorem to find $I_{z'z'}^{\mathrm{O}}$ if we know I_{zz}^{cm}. We are given the in-plane moments of inertia I_{xx}^{cm} and I_{yy}^{cm}. Therefore, from the perpendicular axis theorem:

$$I_{zz}^{\mathrm{cm}} = I_{xx}^{\mathrm{cm}} + I_{yy}^{\mathrm{cm}} = (4.0 + 12.4)\,\mathrm{kg\cdot m^2} = 16.4\,\mathrm{kg\cdot m^2}.$$

Now using the parallel axis theorem,

$$\begin{aligned}I_{z'z'}^{\mathrm{O}} &= I_{zz}^{\mathrm{cm}} + Mr_{\mathrm{cm/O}}^2 \\ &= 16.4\,\mathrm{kg\cdot m^2} + 0.5\,\mathrm{kg}(1.6\,\mathrm{m})^2 = 17.68\,\mathrm{kg\cdot m^2}. \\ \text{Therefore,}\quad \vec{\boldsymbol{H}} &= 17.68\,\mathrm{kg\cdot m^2}\cdot(1.5\,\mathrm{rad/s}\hat{\boldsymbol{k}}) \\ &= 26.52\,\mathrm{kg\cdot m^2/s}\hat{\boldsymbol{k}}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}} = 26.52\,\mathrm{kg\cdot m^2/s}\hat{\boldsymbol{k}}.}$$

3. **Kinetic energy:**

$$\begin{aligned}E_{\mathrm{K}} &= \frac{1}{2}I_{z'z'}^{\mathrm{O}}\omega^2 = \frac{1}{2}(17.68\,\mathrm{kg\cdot m^2})(1.5\,\mathrm{rad/s})^2 \\ &= 19.89\frac{\mathrm{kg\cdot m^2}}{\mathrm{s^2}} == 19.89\,\mathrm{N\cdot m} \\ &= 19.89\ \mathrm{J}\end{aligned}$$

$$\boxed{E_{\mathrm{K}} = 19.89\ \mathrm{J}}$$

SAMPLE 20.14 The rotating crooked plate again. For the crooked plate considered in Sample 20.7 and shown again in Fig. 20.49,

1. compute the moment of inertia matrix $[\boldsymbol{I}]$ in the $x'y'z'$ coordinate system,
2. compute the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ using the moment of inertia $[\boldsymbol{I}]_{x'y'z'}$ computed above,
3. express $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ as a vector in the xyz coordinate system.

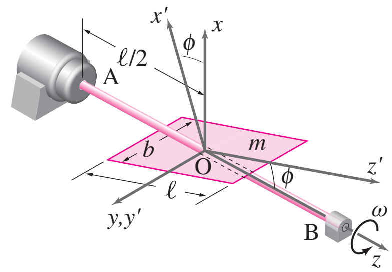


Figure 20.49: A rectangular plate of mass m rotates with shaft AB at a constant speed $\vec{\omega} = \omega\hat{\boldsymbol{k}}$. A coordinate system $x'y'z'$ is aligned with the *principle axes* of the plate.

Solution A line sketch of the plate and the two coordinate systems attached to it are shown in Fig. 20.50. The set of basis vectors $(\hat{\imath}, \hat{\jmath}, \hat{\boldsymbol{k}})$ and $(\hat{\imath}', \hat{\jmath}', \hat{\boldsymbol{k}}')$ associated with coordinate systems xyz and $x'y'z'$ respectively are shown separately for the sake of clarity. From the diagram of the two basis vector sets, we may write

$$\left.\begin{aligned}\hat{\imath}' &= \cos\phi\,\hat{\imath} - \sin\phi\,\hat{\boldsymbol{k}}\\ \hat{\jmath}' &= \hat{\jmath}\\ \hat{\boldsymbol{k}}' &= \sin\phi\,\hat{\imath} + \cos\phi\,\hat{\boldsymbol{k}}\end{aligned}\right\}. \tag{20.47}$$

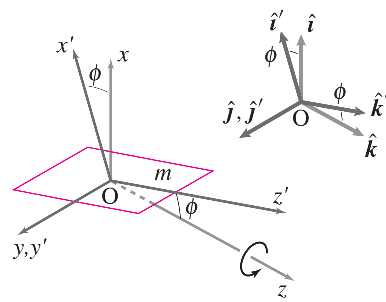


Figure 20.50: A line sketch of the rotating plate along with the two coordinate systems $x'y'z'$ and xyz. The basis vectors associated with the two systems are also shown.

1. **Calculation of $[\boldsymbol{I}]_{x'y'z'}$:**

$$[\boldsymbol{I}]_{x'y'z'} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'}\\ I_{x'y'} & I_{y'y'} & I_{y'z'}\\ I_{x'z'} & I_{y'z'} & I_{z'z'}\end{bmatrix}.$$

Let us first consider the off diagonal terms of $[\boldsymbol{I}]_{x'y'z'}$.
Since $x' = 0$ on the entire plate (the plate is in the $y'z'$ plane and the origin is on the plate.),

$$I_{x'y'} = \int_m \underbrace{x'}_{0}\, y'\,dm = 0, \quad I_{x'z'} = \int_m \underbrace{x'}_{0}\, z'\,dm = 0.$$

How about $I_{y'z'}$? Well, you can calculate it two ways: (a) carry out the integration $I_{y'z'} = \int_m y'z'\,dm$ over the entire plate mass and find that $I_{y'z'} = 0$, or (b) realize that for every mass element $dm\,(=\frac{m}{\ell b}dA)$ with coordinates $(+y'z')$, there exists another element dm at $(-y'z')$ such that the sum of their contributions to the integral is zero. Therefore, $I_{y'z'} = \int_m y'z'\,dm = 0$ ① Now the other terms:

$$\begin{aligned} I_{z'z'} &= \int_m(\underbrace{x'^2}_{0} + y'^2)\,dm = \int_{-\ell/2}^{\ell/2}\int_{-b/2}^{b/2} y'^2\cdot\frac{m}{\ell b}dy'\,dz'\\ &= \frac{m}{\ell b}\int_{-\ell/2}^{\ell/2}\left(\frac{y'^3}{3}\Big|_{-b/2}^{b/2}\right)dz' = \frac{m}{\ell b}\cdot\frac{b^3}{12}\int_{-\ell/2}^{\ell/2}dz'\\ &= \frac{m}{\ell b}\cdot\frac{b^3}{12}\cdot\ell = \frac{mb^2}{12},\end{aligned}$$

and similarly,

$$\begin{aligned} I_{y'y'} &= \int_m(\underbrace{x'^2}_{0} + z'^2)\,dm = \frac{m\ell^2}{12},\\ I_{x'x'} &= \int_m(y'^2 + z'^2)\,dm = I_{y'y'} + I_{z'z'} = \frac{m}{12}(\ell^2 + b^2).\end{aligned}$$

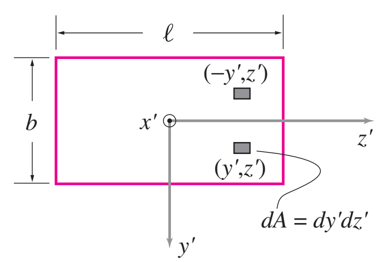


Figure 20.51: Symmetry about the z' axis implies that $I_{y'z'} = 0$. Same result holds if we consider the symmetry about the y' axis.

① You can use a similar argument to find the off-diagonal terms in $[\boldsymbol{I}]$ whenever there is such symmetry with respect to the coordinate axes.

Thus,

$$[\boldsymbol{I}]_{x'y'z'} = \frac{m}{12}\begin{bmatrix} \ell^2+b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}.$$

2. **Calculation of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$:**

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}} \quad \text{and} \quad \vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}]_{x'y'z'}\{\vec{\boldsymbol{\omega}}\}_{x'y'z'}.$$

The subscripts $x'y'z'$ in $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ have been used to denote that both $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ are expressed in $x'y'z'$ coordinate system. 2 We have calculated $[I]_{x'y'z'}$ above. Now we need to find $\{\vec{\boldsymbol{\omega}}\}_{x'y'z'}$.

2 We can express $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ in any coordinate system of our choice but both of them must be in the same system for their product to be valid.

Let $\vec{\boldsymbol{\omega}} = \omega_{x'}\hat{\boldsymbol{\imath}}' + \omega_{y'}\hat{\boldsymbol{\jmath}}' + \omega_{z'}\hat{\boldsymbol{k}}'$. But, we can also write, $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. So,

$$\omega\hat{\boldsymbol{k}} = \omega_{x'}\hat{\boldsymbol{\imath}}' + \omega_{y'}\hat{\boldsymbol{\jmath}}' + \omega_{z'}\hat{\boldsymbol{k}}' \tag{20.48}$$

Dotting both sides of Eqn. (20.48) with $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, and $\hat{\boldsymbol{k}}'$ and using the relationships in (20.47) we get

$$\omega_{x'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\imath}}'}^{-\sin\phi}) = -\omega\sin\phi, \omega_{y'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\jmath}}'}^{0}) = 0, \omega_{k'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'}^{\cos\phi}) = \omega\cos\phi.$$

Thus,

$$\{\vec{\boldsymbol{\omega}}\}_{x'y'z'} = -\omega\sin\phi\hat{\boldsymbol{\imath}}' + \omega\cos\phi\hat{\boldsymbol{k}}'$$

So,

$$\begin{aligned} \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \frac{m}{12}\begin{bmatrix} \ell^2+b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}\begin{Bmatrix} -\omega\sin\phi \\ 0 \\ \omega\cos\phi \end{Bmatrix} \\ &= \begin{Bmatrix} -\frac{m}{12}(\ell^2+b^2)\omega\sin\phi \\ 0 \\ \frac{m}{12}b^2\omega\cos\phi \end{Bmatrix} \\ \text{or} \qquad \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \frac{m\omega}{12}[-(\ell^2+b^2)\sin\phi\hat{\boldsymbol{\imath}}' + b^2\cos\phi\hat{\boldsymbol{k}}']. \end{aligned}$$

Now we can easily compute $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ as follows:

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}} \\ &= (-\omega\sin\phi\hat{\boldsymbol{\imath}}' + \omega\cos\phi\hat{\boldsymbol{k}}') \times \frac{m\omega}{12}[-(\ell^2+b^2)\sin\phi\hat{\boldsymbol{\imath}}' + b^2\cos\phi\hat{\boldsymbol{k}}'] \\ &= \frac{m\omega^2}{12}b^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}' - \frac{m\omega^2}{12}(\ell^2+b^2)\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}' \\ &= -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}'. \end{aligned}$$

3. **Now, back to the xyz coordinate system:** Since $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$ [Eqn (20.47)], we have

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}$$

which is the same result as obtained in Sample 20.7.

SAMPLE 20.15 The calculation of $\dot{\vec{H}}$ using the moment of inertia matrix. For the crooked plate considered in Sample 20.7, find the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$, using the moment of inertia $[\boldsymbol{I}^{\mathrm{O}}]$, calculated in the xyz coordinate system

Solution We calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ using the following formula:

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}]\cdot\underbrace{\dot{\vec{\omega}}}_{\vec{0}} + \vec{\omega}\times\overbrace{([\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\omega})}^{\vec{\boldsymbol{H}}_{/\mathrm{O}}}.$$

Note that point O is the center-of-mass of the plate. Let us write the expression for the angular momentum $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ in matrix form.

$$\begin{Bmatrix} H_{x/O} \\ H_{y/O} \\ H_{z/O} \end{Bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix}.$$

It is clear from the components of $\vec{\omega}$ that we only need the last column of $[\boldsymbol{I}]$ matrix since other elements of $[\boldsymbol{I}]$ will multiply with zeros of $\vec{\omega}$ vector. Carrying out the multiplication we get

$$\begin{Bmatrix} H_{0x} \\ H_{0y} \\ H_{0z} \end{Bmatrix} = \begin{Bmatrix} I_{xz}\omega \\ I_{yz}\omega \\ I_{zz}\omega \end{Bmatrix}$$

or, in vector form

$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = I_{xz}\omega\hat{\boldsymbol{\imath}} + I_{yz}\omega\hat{\boldsymbol{\jmath}} + I_{zz}\omega\hat{\boldsymbol{k}}.$$

Therefore,

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \omega\hat{\boldsymbol{k}}\times\omega(I_{xz}\hat{\boldsymbol{\imath}} + I_{yz}\hat{\boldsymbol{\jmath}} + I_{zz}\hat{\boldsymbol{k}}) \\ &= -\omega^2 I_{yz}\hat{\boldsymbol{\imath}} + \omega^2 I_{xz}\hat{\boldsymbol{\jmath}}. \end{aligned}$$

But,

$$I_{xz} = -\int_m xz\,dm \quad \text{and} \quad I_{yz} = -\int_m yz\,dm$$

and $x = w\sin\phi,\ y = y,\ z = w\cos\phi$ (see Fig. 20.28), hence

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= -\omega^2\int_m(-wy\cos\phi\hat{\boldsymbol{\imath}} + w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}})\,dm \\ &= \int_{-b/2}^{b/2}\int_{-\ell/2}^{\ell/2}\omega^2(-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}})\frac{m}{\ell b}dw\,dy \end{aligned}$$

which is the same integral as obtained in Sample 20.7 for $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$. Therefore, the result is also the same:

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}.$$

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}$$

SAMPLE 20.16 **Direct application of formula:** A rectangular plate is mounted on a massless shaft with the center-of-mass of the plate on the shaft axis. The shaft rotates about its axis with angular acceleration $\dot{\vec{\omega}} = 0.5\,\text{rpm}/\,\text{s}(\hat{\imath} + \hat{\jmath})$. At the instant of interest, the angular velocity is $\vec{\omega} = 100\,\text{rpm}(\hat{\imath} + \hat{\jmath})$ and the components of the moment of inertia matrix of the plate are $I_{xx} = 2\,\text{kg}{\cdot}\text{m}^2$, $I_{yy} = 4\,\text{kg}{\cdot}\text{m}^2$, $I_{zz} = 6\,\text{kg}{\cdot}\text{m}^2$ and $I_{xy} = I_{yz} = I_{xz} = 0$.

1. Find the angular momentum of the plate about its mass-center and show that it is not in the same direction as the angular velocity.
2. Find the net moment acting on the plate.

Solution We are given the angular velocity, the angular acceleration, and the moment of inertia matrix of the plate:

$$\begin{aligned}
\vec{\omega} &= 100\,\text{rpm}(\hat{\imath} + \hat{\jmath}) = 10.47\,\text{rad/s}(\hat{\imath} + \hat{\jmath}) \\
\dot{\vec{\omega}} &= 5\,\text{rpm}/\,\text{s}(\hat{\imath} + \hat{\jmath}) = 0.52\,\text{rad/s}^2(\hat{\imath} + \hat{\jmath}) \\
[\boldsymbol{I}^{\text{cm}}] &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{kg}{\cdot}\text{m}^2.
\end{aligned}$$

1. **The angular momentum:** The angular momentum of the plate is

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\text{cm}} &= [\boldsymbol{I}^{\text{cm}}] \cdot \vec{\omega} \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{kg}{\cdot}\text{m}^2 \cdot \begin{Bmatrix} 10.47 \\ 10.47 \\ 0 \end{Bmatrix} \text{rad/s} \\
&= (20.94\hat{\imath} + 41.88\hat{\jmath})\, \underbrace{\text{kg}{\cdot}\text{m}^2 \cdot \text{s}^{-1}}_{\text{N}\cdot\text{m}\cdot\text{s}} \\
&= (20.94\hat{\imath} + 41.88\hat{\jmath})\,\text{N}{\cdot}\text{m}\cdot\text{s}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}}_{\text{cm}} = (20.94\hat{\imath} + 41.88\hat{\jmath})\,\text{N}{\cdot}\text{m}\cdot\text{s}}$$

There are many ways of showing that $\vec{\boldsymbol{H}}_{\text{cm}}$ is not parallel to $\vec{\omega}$. We can simply draw the two vectors and show that they are not parallel We can, alternatively, take the cross product of the two vectors:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\text{cm}} \times \vec{\omega} &= (20.94\hat{\imath} + 41.88\hat{\jmath})\,\text{N}{\cdot}\text{m}\cdot\text{s} \times 10.47\,\text{rad/s}(\hat{\imath} + \hat{\jmath}) \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 20.94 & 41.88 & 0 \\ 10.47 & 10.47 & 0 \end{vmatrix} \text{N}{\cdot}\text{m} \\
&= \hat{\boldsymbol{k}}(219.24 - 438.48)\,\text{N}{\cdot}\text{m} = -20.94\,\text{N}{\cdot}\text{m}\hat{\boldsymbol{k}}
\end{aligned}$$

which is not zero, implying that the two vectors are not parallel.

2. **The net moment:** The net moment on the plate can be found by applying angular momentum balance:

$$\begin{aligned}
\vec{\boldsymbol{M}}_{/\text{cm}} &= \dot{\vec{\boldsymbol{H}}}_{\text{cm}} = \vec{\omega} \times \underbrace{[\boldsymbol{I}^{\text{cm}}] \cdot \vec{\omega}}_{\vec{H}_{\text{cm}}} + [\boldsymbol{I}^{\text{cm}}] \cdot \dot{\vec{\omega}} \\
&= \underbrace{20.94\,\text{N}{\cdot}\text{m}\hat{\boldsymbol{k}}}_{\vec{\omega}\times\vec{H}_{\text{cm}}} + \underbrace{1.04\,\text{N}{\cdot}\text{m}\hat{\imath} + 2.08\,\text{N}{\cdot}\text{m}\hat{\jmath}}_{[I^{\text{cm}}]\cdot\dot{\vec{\omega}}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{M}}_{/\text{cm}} = (1.04\hat{\imath} + 2.08\hat{\jmath} + 20.94\hat{\boldsymbol{k}})\,\text{N}{\cdot}\text{m}}$$

SAMPLE 20.17 **A rod swings around a tipped axis in 3-D.** A rigid shaft of negligible mass is connected to frictionless hinges at A and B. The shaft is tipped from the horizontal plane (xy-plane) such that the shaft axis makes an angle γ with the vertical axis. A uniform rod OC of mass m and length ℓ is welded to the shaft. At time $t = 0$, the rod is tipped by a small angle ϕ from its position OC' in the yz (or $y'z'$) plane. Find the equation of motion of the rod.

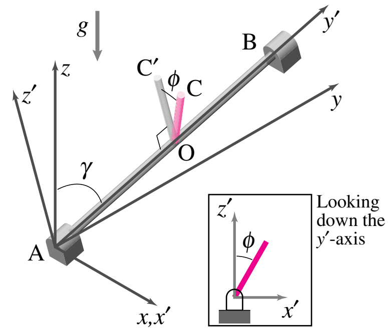


Figure 20.52: A rod, welded to a tipped shaft AB, swings around the shaft axis if it is tipped slightly from the vertical plane yz.

Solution Let $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ be the basis vectors associated with the $x'y'z'$ coordinate system. Since $y'z'$ axes can be obtained by rotating yz axes counterclockwise about the x-axis by an angle $90^o - \gamma$, we can relate the basis vectors of the two coordinate systems with the help of Fig. 20.53:

$$\begin{aligned} \hat{\imath}' &= \hat{\imath}, \\ \hat{\jmath}' &= \sin\gamma\,\hat{\jmath} + \cos\gamma\,\hat{k}, \\ \hat{k}' &= -\cos\gamma\,\hat{\jmath} + \sin\gamma\,\hat{k}. \end{aligned} \tag{20.49}$$

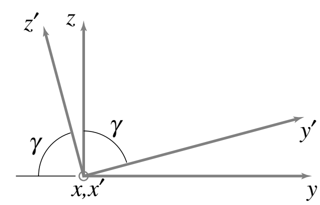


Figure 20.53: $y'z'$ axes are obtained by rotating yz axes counterclockwise about the x-axis by an angle $90^o - \gamma$.

The free-body diagram of the shaft with rod OC is shown in Fig. 20.54. We can write angular momentum balance for this system about any point on the axis of rotation AB. However, rather than writing angular momentum balance about a point, let us write angular momentum balance about axis AB. This 'trick' will eliminate reactions $\vec{R}_A$ and $\vec{R}_B$ from our equations. Angular Momentum Balance about axis AB is:

$$\begin{aligned} \hat{\lambda}_{AB}\cdot[\sum \vec{M}_O &= \dot{\vec{H}}_{/O}] \\ \text{or} \qquad \hat{\jmath}'\cdot\sum \vec{M}_O &= \hat{\jmath}'\cdot\dot{\vec{H}}_{/O}. \end{aligned}$$

Calculation of ($\hat{\jmath}'\cdot\sum\vec{M}_O$): Since $\vec{R}_A$ and $\vec{R}_B$ pass through axis AB, they do not produce any moment about this axis. Therefore,

$$\begin{aligned} \hat{\jmath}'\cdot\sum\vec{M}_O &= \hat{\jmath}'\cdot[\vec{r}_{G/O}\times mg(-\hat{k})] \\ &= \hat{\jmath}'\cdot\left[\frac{L}{2}(\sin\phi\,\hat{\imath}' + \cos\phi\,\hat{k}')\times mg(-\hat{k})\right] \\ &= \overbrace{(\sin\gamma\,\hat{\jmath} + \cos\gamma\,\hat{k})}^{\hat{\jmath}'}\cdot[\frac{L}{2}mg\overbrace{(\sin\phi\,\hat{\jmath} + \cos\phi\,\cos\gamma\,\hat{\imath})}^{\text{using Eqn.(20.49)}}] \\ &= \frac{\ell}{2}mg\sin\phi\,\sin\gamma \end{aligned}$$

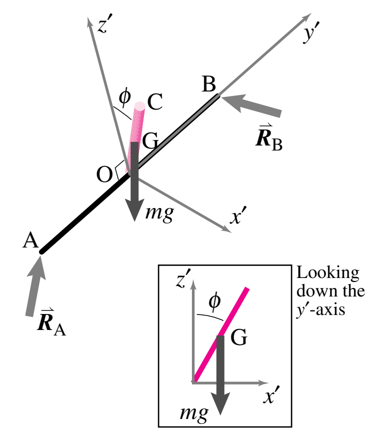


Figure 20.54: Free-body diagram of the bar and shaft system. In the inset, only the weight of the bar, mg, is shown for clarity of geometry.

Calculation of $\hat{\jmath}'\cdot\dot{\vec{H}}_{/O}$: Since the rod rotates about axis AB, we may write

$$\begin{aligned} \vec{\omega} &= \dot{\phi}\,\hat{\jmath}', \\ \dot{\vec{\omega}} &= \ddot{\phi}\,\hat{\jmath}'. \end{aligned}$$

Now,

$$\dot{\vec{H}}_{/O} = [I^O]\{\dot{\vec{\omega}}\} + \vec{\omega}\times[I^O]\{\vec{\omega}\}$$

where

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]\{\dot{\vec{\omega}}\} &= [\boldsymbol{I}^{\mathrm{O}}]_{x'y'z'}\,\{\dot{\vec{\omega}}\}_{x'y'z'} \\
&= \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{x'y'} & I_{y'y'} & I_{y'z'} \\ I_{x'z'} & I_{y'z'} & I_{z'z'} \end{bmatrix} \begin{Bmatrix} 0 \\ \ddot{\phi} \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} I_{x'y'}\ddot{\phi} \\ I_{y'y'}\ddot{\phi} \\ I_{y'z'}\ddot{\phi} \end{Bmatrix}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]\{\vec{\omega}\} &= \begin{Bmatrix} I_{x'y'}\dot{\phi} \\ I_{y'y'}\dot{\phi} \\ I_{y'z'}\dot{\phi} \end{Bmatrix}, \\
\Rightarrow \qquad \vec{\omega}\times[\boldsymbol{I}^{\mathrm{O}}]\{\vec{\omega}\} &= \dot{\phi}\hat{\boldsymbol{\jmath}}'\times[I_{x'y'}\dot{\phi}\hat{\boldsymbol{\imath}}' + I_{y'y'}\dot{\phi}\hat{\boldsymbol{\jmath}}' + I_{y'z'}\dot{\phi}\hat{\boldsymbol{k}}'] \\
&= -I_{x'y'}\dot{\phi}^2\hat{\boldsymbol{k}}' + I_{y'z'}\dot{\phi}^2\hat{\boldsymbol{\imath}}'.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\boldsymbol{\jmath}}'\cdot\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \hat{\boldsymbol{\jmath}}'\cdot[(I_{x'y'}\ddot{\phi}\hat{\boldsymbol{\imath}}' + I_{y'y'}\ddot{\phi}\hat{\boldsymbol{\jmath}}' + I_{y'z'}\ddot{\phi}\hat{\boldsymbol{k}}') + (-I_{x'y'}\dot{\phi}^2\hat{\boldsymbol{k}}' + I_{y'z'}\dot{\phi}^2\hat{\boldsymbol{\imath}}')] \\
&= I_{y'y'}\ddot{\phi}.
\end{aligned}$$

Now, setting $\hat{\boldsymbol{\jmath}}'\cdot\sum\vec{\boldsymbol{M}}_{\mathrm{O}} = \hat{\boldsymbol{\jmath}}'\cdot\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$, we get

$$\begin{aligned}
\frac{L}{2}mg\sin\phi\,\sin\gamma &= I_{y'y'}\ddot{\phi} \\
\text{or} \qquad \ddot{\phi} &= \frac{mg\,(L/2)\,\sin\gamma}{I_{y'y'}}\sin\phi \\
\text{or} \qquad \ddot{\phi} &= C\sin\phi
\end{aligned}$$

where

$$C = \frac{mg\,(L/2)\,\sin\gamma}{I_{y'y'}}.$$

For rod OC,

$$I_{y'y'} = \int_m (x'^2 + z'^2)\,dm = \int_0^L l^2\frac{m}{L}\,dl = \frac{1}{3}mL^2.$$

Therefore, the equation of motion of the rod is

$$\begin{aligned}
\ddot{\phi} - \frac{mg\,(L/2)\,\sin\gamma}{(1/3)mL^2}\sin\phi &= 0 \\
\ddot{\phi} - \frac{3g\,\sin\gamma}{2L}\sin\phi &= 0.
\end{aligned}$$

$$\ddot{\phi} - \frac{3g\,\sin\gamma}{2L}\sin\phi = 0$$

SAMPLE 20.18 **Kinetic energy in 3-D rotation.** A thin rod of mass 2 kg and length $L = \frac{1}{2}$ m is welded to a massless shaft at an angle $\theta = 45^\circ$. The shaft rotates about its longitudinal axis (y-axis) at 100 rpm. The moment of inertia matrix of the rod about the weld point O is

$$[\boldsymbol{I}^{\mathrm{O}}] = 0.08\,\mathrm{kg\cdot m^2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find the kinetic energy of the rod.

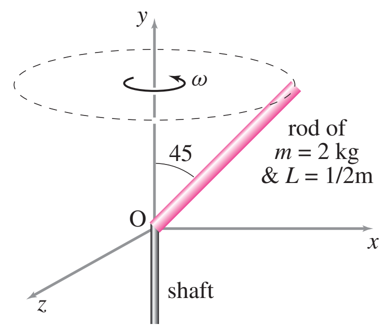


Figure 20.55

Solution The rod rotates about the fixed point O with angular velocity

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}} \quad \text{where} \quad \omega = 100\,\mathrm{rpm} = 10.47\,\mathrm{rad/s}.$$

The kinetic energy of a rigid body rotating about a fixed point O is given by

$$E_{\mathrm{K}} = \frac{1}{2}\vec{\boldsymbol{\omega}}\cdot[[\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}]$$

For the given problem, let us write the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$ of the rod as ③

③ This step is simply to facilitate computation. We carry out the multiplication and at the end substitute the value of K_0.

$$[\boldsymbol{I}^{\mathrm{O}}] = K_0\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where $K_0 = 0.08\,\mathrm{kg\cdot m^2}$. Now

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}} = \{0 \quad \omega \quad 0\}^{\mathrm{T}}.$$

Therefore,

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}} &= K_0\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\
&= K_0\begin{Bmatrix} -\omega \\ \omega \\ 0 \end{Bmatrix} \\
&= -K_0\omega\hat{\boldsymbol{\imath}} + K_0\omega\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

Substituting the expression in the formula for E_{K} we get

$$\begin{aligned}
E_{\mathrm{K}} &= \frac{1}{2}\omega\hat{\boldsymbol{\jmath}}\cdot(-K_0\omega\hat{\boldsymbol{\imath}} + K_0\omega\hat{\boldsymbol{\jmath}}) \\
&= \frac{1}{2}K_0\omega^2 \\
&= \frac{1}{2}\cdot 0.08\,\mathrm{kg\cdot m^2}\cdot(10.47\,\mathrm{rad/s})^2 \\
&= 4.38\,\mathrm{N\cdot m} = 4.38\mathrm{J}
\end{aligned}$$

$E_{\mathrm{K}} = 4.38\mathrm{J}$

20.4 Mechanics using $[\boldsymbol{I}^{\text{cm}}]$ and $[\boldsymbol{I}^{\text{O}}]$

20.4.1 Find the angle between $\vec{\omega}$ and $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\text{cm}}]\vec{\omega}$, if $\vec{\omega} = 3\,\text{rad/s}\,\hat{\boldsymbol{k}}$ and

$$[\boldsymbol{I}^{\text{cm}}] = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{kg}\cdot\text{m}^2.$$

20.4.2 Find the projection of $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\text{cm}}]\vec{\omega}$ in the direction of $\vec{\omega}$, if $\vec{\omega} = (2\hat{\imath} - 3\hat{\jmath})\,\text{rad/s}$ and

$$[\boldsymbol{I}^{\text{cm}}] = \begin{bmatrix} 5.4 & 0 & 0 \\ 0 & 5.4 & 0 \\ 0 & 0 & 10.8 \end{bmatrix} \text{lbm ft}^2.$$

20.4.3 Compute $\vec{\omega} \times \left([\boldsymbol{I}^{\text{cm}}] \cdot \vec{\omega}\right)$ for the following two cases. In each case $\vec{\omega} = 2.5\,\text{rad/s}\,\hat{\jmath}$.

a) $[\boldsymbol{I}^{\text{cm}}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 8 \end{bmatrix} \text{kg m}^2$, *

b) $[\boldsymbol{I}^{\text{cm}}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & -4 \\ 0 & -4 & 8 \end{bmatrix} \text{kg m}^2$.

*

20.4.4 Find the z component of $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\text{cm}}]\vec{\omega}$ for $\vec{\omega} = 2\,\text{rad/s}\,\hat{\boldsymbol{k}}$ and

$$[\boldsymbol{I}^{\text{cm}}] = \begin{bmatrix} 10.20 & -2.56 & 0 \\ -2.56 & 10.20 & 0 \\ 0 & 0 & 14.00 \end{bmatrix} \text{lbm ft}^2.$$

20.4.5 For $\vec{\omega} = 5\,\text{rad/s}\,\hat{\boldsymbol{\lambda}}$, find $\hat{\boldsymbol{\lambda}}$ such that the angle between $\vec{\omega}$ and $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\text{cm}}]\cdot\vec{\omega}$ is zero if $[\boldsymbol{I}^{\text{cm}}] = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 6 & 0 \\ 2 & 0 & 3 \end{bmatrix} \text{kg m}^2$.

20.4.6 Calculation of $\vec{H}_{/\text{O}}$ and $\dot{\vec{H}}_{/\text{O}}$. You will consider a disc spinning about various axes through the center of the disc O. For the most part, you are concerned with calculating $\dot{\vec{H}}_{/\text{O}}$. But, you should keep in mind that the purpose of the calculation might be to calculate reaction forces and moments. Every point on the disk moves in circles about the same axis at the same number of radians per second. These problems can be done at least three different ways: using the$[I]$'s from problem 20.3.6, direct integration of the definition of $\dot{\vec{H}}_{/\text{O}}$, and using the equivalent dumbbell (with the definition of $\dot{\vec{H}}_{/\text{O}}$). You should do them all three ways, first using the way you are most comfortable with.

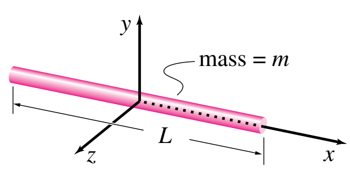
a) Assume that the configuration is as in problem 20.3.6(a) and that the disk is spinning about the z axis at the constant rate of $\omega\mathbf{k}$. What are $\vec{H}_{/\text{O}}$ and $\dot{\vec{H}}_{/\text{O}}$?

b) Assume that the configuration is as in problem 20.3.6(b) and that the disk is spinning about the z axis at the constant rate of $\omega\mathbf{k}$. What are $\vec{H}_{/\text{O}}$ and $\dot{\vec{H}}_{/\text{O}}$? (In this problem there are two ways to use [I]: one is in the xyz coordinate system, the other is in a coordinate system aligned nicely with the disk.)

20.4.7 What is the angular momentum $\vec{\boldsymbol{H}}$ for:

a) A uniform sphere rotating with angular velocity $\vec{\omega}$ (radius R, mass m)?

b) A rod shown in the figure rotating about its center of mass with these angular velocities: $\vec{\omega} = \omega\hat{\imath}$, $\vec{\omega} = \omega\hat{\jmath}$, and $\vec{\omega} = \omega(\hat{\imath} + \hat{\jmath})/\sqrt{2}$?

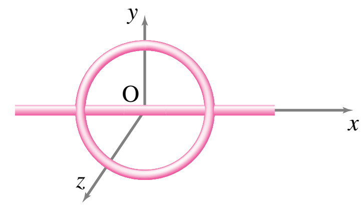


Problem 20.4.7

20.4.8 A hoop is welded to a long rigid massless shaft which lies on a diameter of the hoop. The shaft is parallel to the x-axis. At the instant of interest the hoop is in the xy-plane of a coordinate system which has its origin O at the center of the hoop. The hoop spins about the x-axis at 1 rad/s. There is no gravity.

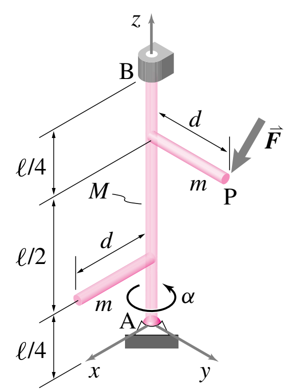
a) What is $[\boldsymbol{I}^{\text{cm}}]$ of the hoop?

b) What is $\vec{\boldsymbol{H}}_{/\text{O}}$ of the hoop?



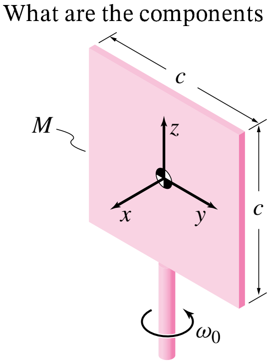
Problem 20.4.8

20.4.9 Two thin rods, each of mass m and length d, are welded perpendicularly to an axle of mass M and length ℓ, which is supported by a ball-and-socket joint at A and a journal bearing at B. A force $\vec{\boldsymbol{F}} = -F\hat{\imath} + G\hat{\boldsymbol{k}}$ acts at the point P as shown in the figure. Determine the initial angular acceleration α and initial reactions at point B.



Problem 20.4.9

20.4.10 The thin square plate of mass M and side c is mounted vertically on a vertical shaft which rotates with angular speed ω_0. In the fixed x, y, z coordinate system with origin at the center of mass (with which the plate is currently aligned) what are the vector components of $\vec{\omega}$? What are the components of $\vec{\boldsymbol{H}}_{cm}$?



Problem 20.4.10

20.4.11 If you take any rigid body (with weight), skewer it with an axle, hold the axle with hinges that cause no moment about the axle, and hold it so that it cannot slide along the axle, it will swing like a pendulum. That is, it will obey the equations:

$$\begin{aligned} \ddot{\theta} &= -C \sin\theta, \quad \text{or} \quad (a) \\ \ddot{\phi} &= +C \sin\phi, \qquad (b) \end{aligned}$$

where

θ is the amount of rotation about the axis measuring zero when the center of mass is directly below the z'-axis; i. e., when the center of mass is in a plane containing the z'-axis and the gravity vector,

ϕ is the amount of rotation about the axis measuring zero when the center of mass is directly above the z'-axis,

$C = M_{total}\, gd \sin\gamma / I_{z'z'}$,

M_{total} is the total mass of the body,

g is the gravitational constant,

d is the perpendicular distance from the axis to the center of mass of the object,

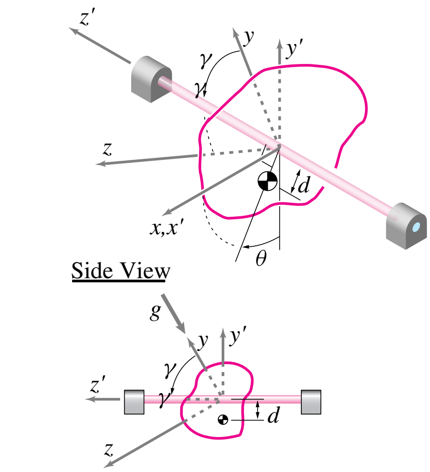
γ is the angle of tip of the axis from the vertical,

$I_{z'z'}$ is the moment of inertia of the object about the skewer axis.

a) How many examples of such pendula can you think of and what would be the values of the constants? (e.g. a simple pendulum?, a broom upside down on your hand?, a door?, a box tipping on one edge?, a bicycler stopped at a red light and stuck in her new toe clips?, a person who got *rigor mortis* and whose ankles went totally limp at the same time?,...)

b) A door with frictionless hinges is misaligned by half an inch (the top hinge is half an inch away from the vertical line above the other hinge). Estimate the period with which it will swing back and forth. Use reasonable magnitudes of any quantities you need.

c) Can you verify the formula (a simple pendulum and a swinging stick are special cases)?



Problem 20.4.11: **-3-D pendulum on a hinge.**

20.4.12 A hanging rod going in circles. A uniform rod with length ℓ and mass m hangs from a well greased ball and socket joint at O. It swings in circles at constant rate $\vec{\omega} = \omega \mathbf{k}$ while the rod makes an angle ϕ with the vertical (so it sweeps a cone).

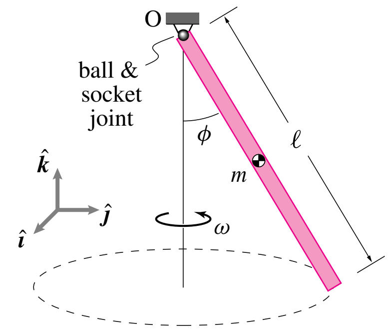
a) Calculate $\vec{H}_{/O}$ and $\dot{\vec{H}}_{/O}$.

b) Using $\dot{\vec{H}}_{/O}$ and angular momentum balance, find the rotation rate ω in terms of ϕ, g, ℓ, and m. How long does it take to make one revolution? *

c) If another stick rotates twice as fast while making the same angle ϕ, what must be its length compared to ℓ?

d) Is there a simple way to describe what two uniform sticks of different length and different cone angle ϕ both have in common if their rate of rotation ω is the same? (If you accurately draw two such sticks, you should see the relation.)

e) Find the reaction force at the ball and socket joint. Is this force parallel to the stick?



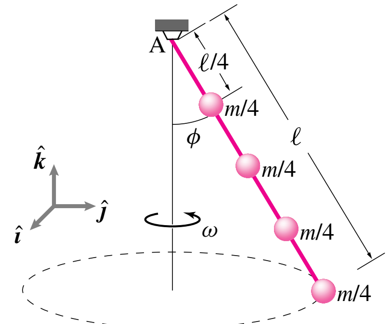
Problem 20.4.12

20.4.13 From discrete to continuous. n equal beads of mass m/n each are glued to a rigid massless rod at equal distances $l = \ell/n$ from each other. For $n = 4$, the system is shown in the figure, but you are to consider the general problem with n masses. The rod swings around the vertical axis maintaining angle ϕ.

a) Find the rate of rotation of the system as a function of g, ℓ, n, and ϕ for constant rate circular motion. [Hint: You may need these series sums: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ and $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.] *

b) Set $n = 1$ in the expression derived in (a) and check the value with the rate of rotation of a *conical pendulum* of mass m and length ℓ.

c) Now, put $n \to \infty$ in the expression obtained in part (a). Does this rate of rotation make sense to you? [Hint: How does it compare with the rate of rotation for the uniform stick of Problem 20.4.12?]



Problem 20.4.13

20.4.14 Spinning dumbbell. Two point masses are connected by a rigid rod of negligible mass and length $2\sqrt{2}\ell$. The rod is welded to a shaft that is along the y axis. The shaft is spinning at a constant rate ω driven by a motor at A (Not shown). At the time of interest the dumbbell is in the yz-plane. Neglect gravity.

a) What is the moment of inertia matrix about point O for the dumbbell at the instant shown? *

b) What is the angular momentum of the dumbbell at the instant shown about point O. *

c) What is the rate of change of angular momentum of the dumbbell at the instant shown? Calculate the rate of change of angular momentum two different ways:

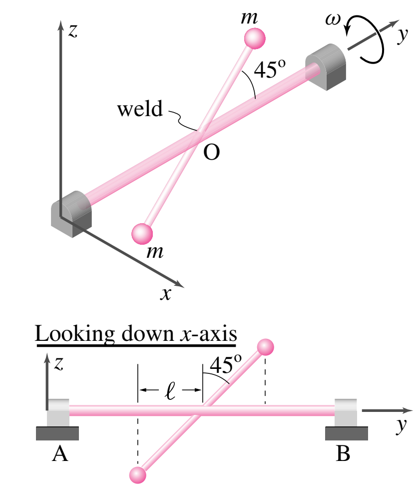
  (i) by adding up the contribution of both of the masses to the rate of change of angular momentum (i.e. $\dot{\vec{H}} = \sum \vec{r}_i \times (m_i \vec{a}_i)$) and

  (ii) by calculating I_{xy} and I_{yz}, calculating $\vec{H}$, and then calculating $\dot{\vec{H}}$ from $\vec{\omega} \times \vec{H}$. *

d) What is the total force and moment (at the CM of the dumbbell) required to keep this motion going? Do the calculation in the following ways:

  (i) Using angular momentum balance (about any point of your choice) for the shaft dumbbell system.

  (ii) By drawing free-body diagrams of the masses alone and calculating the forces on the masses. Then use action and reaction and draw a free-body diagram of the remaining (massless) part of the shaft-dumbbell system. Then use moment balance for this system. *



Problem 20.4.14: Crooked spinning dumbbell.

20.4.15 A uniform rectangular plate of mass m, height h, and width b spins about its diagonal at a constant angular speed ω, as shown in the figure. The plate is supported by frictionless bearings at O and C. The plate lies in the YZ-plane at the instant shown.

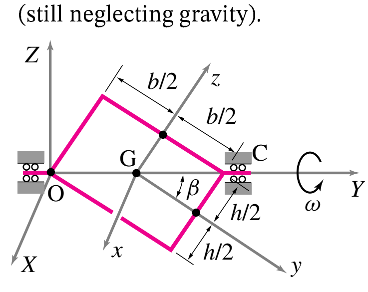
[Note: $\cos(\beta) = b/\sqrt{(h^2 + b^2)}$, and $\sin(\beta) = h/\sqrt{(h^2 + b^2)}$.]

a) Compute the angular momentum of the plate about G (with respect to the body axis $Gxyz$ shown).

b) Ignoring gravity determine the net torque (moment) that the supports impose on the body at the instant shown. (Express your solution using base vectors and components associated with the inertial XYZ frame.)

c) Calculate the reactions at O or C for the configuration shown. *

d) In the case when $h = b = 1\,\mathrm{m}$ what is the vertical reaction at C (still neglecting gravity).



$OXYZ$ - Fixed Frame
(base vectors, $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$)

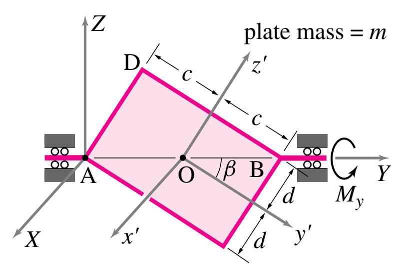
$Gxyz$ - Moving "Body" Frame
(base vectors, $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$)

Problem 20.4.15

20.4.16 A uniform rectangular plate spins about a fixed axis. The constant torque M_y is applied about the Y axis. The plate is held by bearings at the corners of the plate at A and B. At the instant shown the plate and the $y'z'$ axis attached to the plate are in the YZ plane. Neglect gravity. At $t = 0$ the plate is at rest.

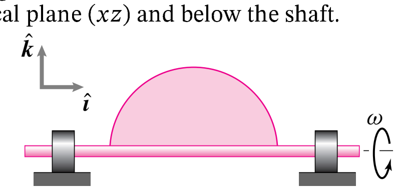
a) At $t = 0^+$ (i.e., *just* after the start), what is the acceleration of point D? Give your answer in terms of any or all of $c, d, m, M_y, \hat{\imath}, \hat{\jmath}$, and $\hat{k}$ (the base vectors aligned with the fixed XYZ axes), and I_{YY}^{cm} [You may use I_{YY}^{cm} in your answer. Or you could use this result from the change of coordinates formulae in linear algebra: $I_{YY}^{cm} = (\cos^2\beta) I_{y'y'}^{cm} + (\sin^2\beta) I_{z'z'}^{cm}$ to get an answer in terms of c, d, m, and M_y.] *

b) At $t = 0^+$, what is the reaction at B? *



Problem 20.4.16

20.4.17 A uniform semi-circular disk of radius $r = 0.2\,\mathrm{m}$ and mass $m = 1.5\,\mathrm{kg}$ is attached to a shaft along its straight edge. The shaft is tipped very slightly from a vertical upright position. Find the dynamic reactions on the bearings supporting the shaft when the disk is in the vertical plane (xz) and below the shaft.



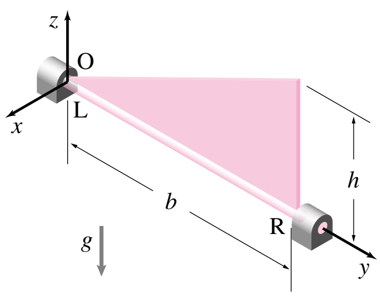
Problem 20.4.17

20.4.18 A thin triangular plate of base b, height h, and mass m is attached to a massless shaft which passes through the bearings L and R. The plate is initially vertical and then, provoked by a teensy disturbance, falls, pivoting around the shaft. The bearings are frictionless. Given,

$$\begin{aligned}
\vec{\boldsymbol{r}}_{cm} &= (0, \tfrac{2}{3}b, \tfrac{1}{3}h),\\
I^O_{xx} &= \frac{m}{2}(b^2 + h^2/3),\\
I^O_{yy} &= mh^2/6,\\
I^O_{zz} &= mb^2/2,\\
I^{cm}_{yy} &= mh^2/18,\\
I^O_{zy} &= -mbh/4,
\end{aligned}$$

a) what is the plate's angular velocity as it passes through the downward vertical position?

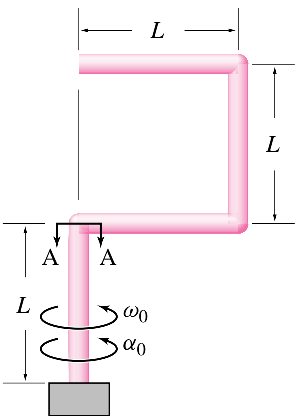
b) Compute the bearing reactions when the plate swings through the downward vertical position.



Problem 20.4.18

20.4.19 A welded structural tubing framework is proposed to support a heavy emergency searchlight. To select properly the tubing dimensions it is necessary to know the forces and moments at the critical section AA, when the framework is moving. The frame moves as shown.

a) Determine the magnitude and direction of the dynamic moments (bending and twisting) and forces (axial and shear) at section AA of the frame if, at the instant shown, the frame has angular velocity ω_o and angular acceleration α_o. Assume that the mass per unit length of the pipe is ρ.



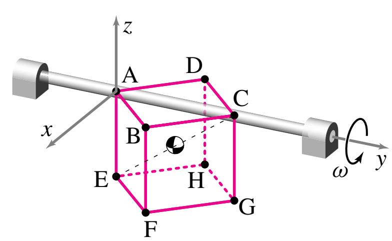
Problem 20.4.19

20.4.20 A uniform cube with side 0.25 m is glued to a pipe along the diagonal of one of its faces $ABCD$. The pipe rotates about the y-axis at a constant rate ω = 30 rpm. At the instant of interest, the face $ABCD$ is in the xy-plane.

a) Find the velocity and acceleration of point F on the cube.

b) Find the position of the center of mass of the cube and compute its acceleration $\vec{\boldsymbol{a}}_{cm}$.

c) Suppose that the entire mass of the cube is concentrated at its center of mass. What is the angular momentum of the cube about point A?

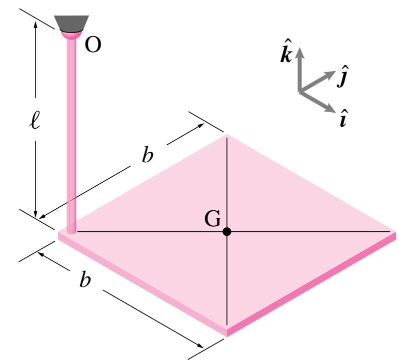


Problem 20.4.20

20.4.21 A uniform square plate of mass m and side length b is welded to a narrow shaft at one corner. The shaft is perpendicular to the plate and has length ℓ. The upper end of the shaft is connected to a ball and socket joint at O. The assembly is rotating about the z axis through O at the fixed rate $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ which happens to be just right to make the plate horizontal and the shaft vertical, as shown. The gravitational constant is g.

a) What is ω in terms of m, ℓ, b, and g?

b) Is the reaction at O on the rod-plate assembly in the direction of the line from G to O? (Why?)



Problem 20.4.21

20.4.22 The uniform cube shown is rotating at constant angular velocity about the fixed axis OA with angular speed 10 rad/s counterclockwise looking towards O from A. Find:

a) $\vec{\boldsymbol{\omega}}$

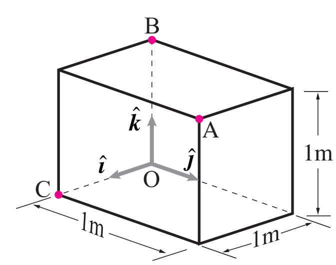
b) $\vec{\boldsymbol{v}}_{\mathrm{B}}$

c) $\vec{\boldsymbol{a}}_{\mathrm{C}}$

d) $\vec{\boldsymbol{H}}_{/\mathrm{O}}$

e) $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$

f) The total torque applied to the block in the configuration shown during this motion.



Problem 20.4.22

20.4.23 A spinning box. A solid box $\mathcal{B}$ with dimensions a, a, and $2a$ (where a = .1 m) and mass = 1 kg. It is supported with ball and socket joints at opposite diagonal corners at A and B. It is spinning at constant rate of $(50/\pi)$ rev/sec and at the instant of interest is aligned with the fixed axis $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\boldsymbol{k}}$. Neglect gravity.

a) What is $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$?

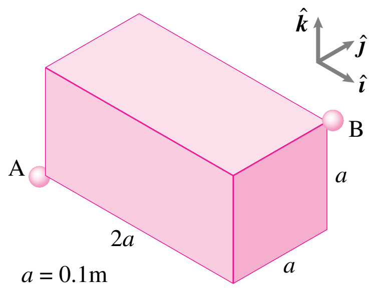
b) What is $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}}$?

c) What is $\vec{\boldsymbol{H}}_{\mathcal{B}/cm}$?

d) What is $\dot{\vec{H}}_{\mathcal{B}/cm}$?

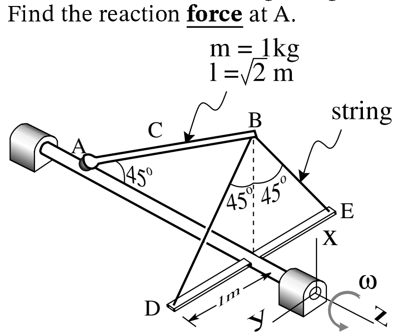
e) What is $\vec{r}_{B/A} \times \vec{F}_B$ (where $\vec{F}_B$ is the reaction force on $\mathcal{B}$ at B)? A dimensional vector is desired.

f) If the hinges were suddenly cut off when $\mathcal{B}$ was in the configuration shown, what would be $\dot{\vec{\omega}}_{\mathcal{B}}$ immediately afterwards?



Problem 20.4.23

20.4.24 Uniform rod attached to a shaft. A uniform 1 kg, $\sqrt{2}$ m long rod is attached to a shaft which spins at a constant rate of 1 rad/s. One end of the rod is connected to the shaft by a ball-and-socket joint. The other end is held by two strings (or pin jointed rods) that are connected to a rigid cross bar. The cross bar is welded to the main shaft. Ignore gravity. Find the reaction **force** at A.



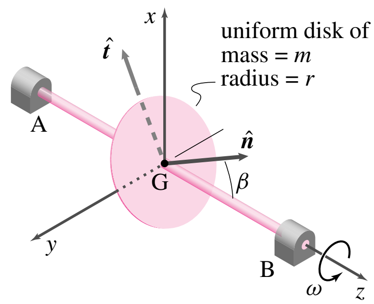
Problem 20.4.24

20.4.25 A round disk spins crookedly on a shaft. A thin homogeneous disk of mass m and radius r is welded to the horizontal axle AB. The normal to the plane of the disk forms an angle β with the axle. The shaft rotates at the constant rate ω.

a) What is the x component of the angular momentum of the disk about point G the center of mass of the disk?

b) What is the x component of the rate of change of angular momentum of the disk about G?

c) At the given instant, what is the total force and moment (about G) required to keep the disk spinning at constant rate?



Problem 20.4.25: A round disk spins crookedly on a shaft.

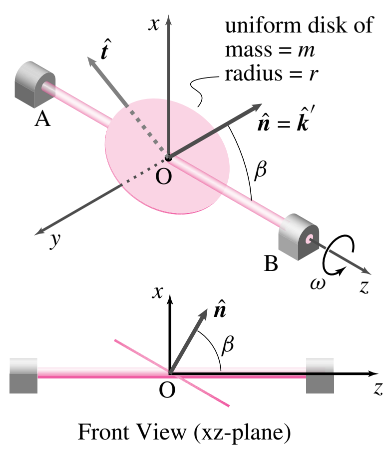
20.4.26 A uniform disk of mass $m = 2$ kg and radius $r = 0.2$ m is mounted rigidly on a massless axle. The normal to the disk makes an angle $\beta = 60°$ with the axle. The axle rotates at a constant rate $\omega = 60$ rpm. At the instant shown in the figure,

a) Find the angular momentum $\vec{H}_{/O}$ of the disk about point O. *

b) Draw the angular momentum vector indicating its magnitude and direction. *

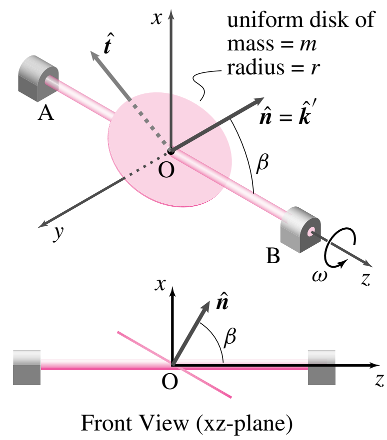
c) As the disk rotates (with the axle), the angular momentum vector changes. Does it change in magnitude or direction, or both? *

d) Find the direction of the net torque on the system at the instant shown. Note, only the *direction* is asked for. You can answer this question based on parts (b) and (c). *



Problem 20.4.26

20.4.27 A uniform disk spins at constant angular velocity. The uniform disk of mass m and radius r spins about the z axis at the constant rate ω. Its normal $\hat{n}$ is inclined from the z axis by the angle β and is in the xz plane at the instant of interest. What is the total force and moment (relative to O) needed to keep this disk spinning at this rate? Answer in terms of m, r, ω, β and any base vectors you need.

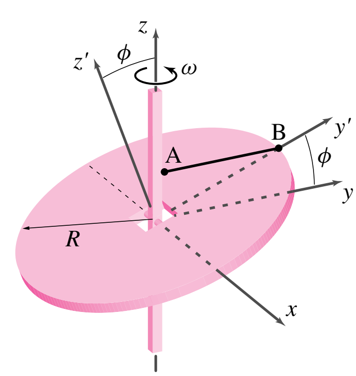


Problem 20.4.27

20.4.28 Spinning crooked disk. A rigid uniform disk of mass m and radius R is mounted to a shaft with a hinge. An apparatus not shown keeps the shaft rotating around the z axis at a constant ω. At the instant shown the hinge axis is in the x direction. A massless wire, perpendicular to the shaft, from the shaft at A to the edge of the disk at B keeps the normal to the plate (the z' direction) at an angle ϕ with the shaft.

a) What is the tension in the wire? Answer in terms of some or all of R, m, ϕ and ω.

b) What is the reaction force at the hinge at O?



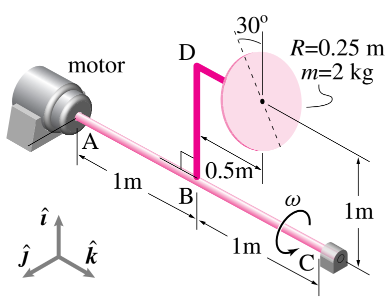
Problem 20.4.28

20.4.29 A disoriented disk rotating with a shaft. A uniform thin circular disk with mass $m = 2\,\text{kg}$ and radius $R = 0.25\,\text{m}$ is mounted on a massless shaft as shown in the figure. The shaft spins at a constant speed $\omega = 3\,\text{rad/s}$. At the instant shown, the disk is slanted at 30° with the xy-plane. Ignore gravity.

a) Find the rate of change of angular momentum of the disk about its center of mass. *

b) Find the rate of change of angular momentum about point A. *

c) Find the total force and moment required at the supports to keep the motion going. *



Problem 20.4.29