SAMPLE 20.11 A scalar times a vector is a vector. A matrix times a vector is a vector. What is the difference? Find the vectors $\vec{H}_1 = I_G\vec{\omega}$ and $\vec{H}_2 = [\mathbf{I}_G]\vec{\omega}$, if $\vec{\omega} = 2\,\text{rad/s}\hat{\imath} + 3\,\text{rad/s}\hat{\jmath}$, $\quad I_G = 10\,\text{kg}\,\text{m}^2$ and $[\mathbf{I}_G] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix} \text{kg}\,\text{m}^2$. Draw $\vec{\omega}$, $\vec{H}_1$ and $\vec{H}_2$.

Solution

$$\begin{aligned} \vec{H}_1 &= I_G\vec{\omega} \\ &= 10\,\text{kg}\,\text{m}^2(2\,\text{rad/s}\hat{\imath} + 3\,\text{rad/s}\hat{\jmath}) \\ &= (20\hat{\imath} + 30\hat{\jmath})\,\text{kg}\,\text{m}^2/\,\text{s}. \end{aligned}$$

$$\begin{aligned} \vec{H}_2 &= [\mathbf{I}_G]\cdot\vec{\omega} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix} \text{kg}\,\text{m}^2 \cdot \begin{Bmatrix} 2 \\ 3 \\ 0 \end{Bmatrix} \text{rad/s} \\ &= (10\hat{\imath} + 15\hat{\jmath} - 6\hat{k})\,\text{kg}\,\text{m}^2/\,\text{s}. \end{aligned}$$

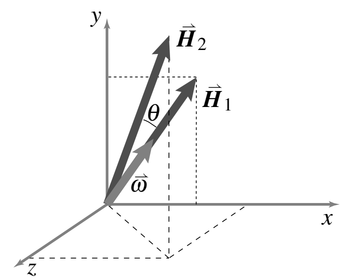

Figure 20.41

These two vectors, $\vec{H}_1$ and $\vec{H}_2$, are shown along with $\vec{\omega}$ in Fig. 20.41. Note that $\vec{H}_1$ has the same direction as $\vec{\omega}$ but $\vec{H}_2$ does not. $\vec{H}_1$ and $\vec{\omega}$ are both in the xy-plane but $\vec{H}_2$ is not; it is in 3-D.

Comments: Multiplying a vector by a scalar does not change the direction of the vector but multiplying by a matrix does change the direction, in general. Find the angles between $\vec{\omega}$ and $\vec{H}_1$ and between $\vec{\omega}$ and $\vec{H}_2$ to convince yourself.

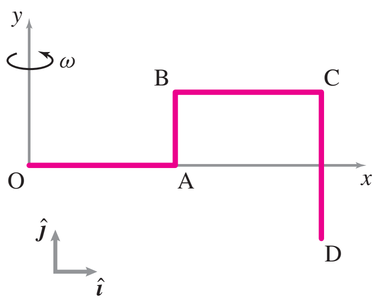

Figure 20.42

SAMPLE 20.12 The composite rod OABCD shown in Figure 20.42 is made up of three identical rods OA, BC, CD of mass 0.5 kg and length 20 cm each, and the rod AB which is half of rod CD. The composite rod goes in circles about the y-axis at a constant rate $\omega = 5\,\mathrm{rad/s}$. Find the angular momentum and the rate of change of angular momentum of the rod at the instant shown (*i.e.*, when the rod is in the xy-plane).

1. Are all the components of $[\boldsymbol{I}^{\mathrm{O}}]$ necessary to compute $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$? Find $[\boldsymbol{I}^{\mathrm{O}}]$ or the necessary components of $[\boldsymbol{I}^{\mathrm{O}}]$.
2. Find the angular momentum $\vec{\boldsymbol{H}}_{/\mathrm{O}}$.
3. Find the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$.
4. If the rod were rotating about the z-axis instead, (*i.e.*, if the motion were in the xy-plane) which components of $[\boldsymbol{I}^{\mathrm{O}}]$ would be required to find $\vec{\boldsymbol{H}}_{/\mathrm{O}}$? What would be the value of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ in that case?

Solution Since the rod rotates about the y-axis, and O is a fixed point on this axis,

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}.$$

1. Since

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{O}} &= \begin{bmatrix} I^{\mathrm{O}}_{xx} & I^{\mathrm{O}}_{xy} & I^{\mathrm{O}}_{xz} \\ I^{\mathrm{O}}_{xy} & I^{\mathrm{O}}_{yy} & I^{\mathrm{O}}_{yz} \\ I^{\mathrm{O}}_{xz} & I^{\mathrm{O}}_{yz} & I^{\mathrm{O}}_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\
&= I^{\mathrm{O}}_{xy}\omega\hat{\boldsymbol{\imath}} + I^{\mathrm{O}}_{yy}\omega\hat{\boldsymbol{\jmath}} + I^{\mathrm{O}}_{yz}\omega\hat{\boldsymbol{k}} \\
&= (I^{\mathrm{O}}_{xy}\hat{\boldsymbol{\imath}} + I^{\mathrm{O}}_{yy}\hat{\boldsymbol{\jmath}} + I^{\mathrm{O}}_{yz}\hat{\boldsymbol{k}})\omega, \qquad (20.46)
\end{aligned}$$

we only need to find three components of $[\boldsymbol{I}^{\mathrm{O}}]$ to compute $\vec{\boldsymbol{H}}_{/\mathrm{O}}$, namely I^{O}_{xy}, I^{O}_{yy} and I^{O}_{yz}.

We can compute these components by considering each rod individually. For any rod, the components can be calculated using the values of the components about the center-of-mass of the rod (see table IV in the back of the book) and then using the parallel axis theorem.

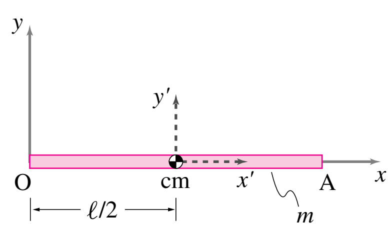

Figure 20.43

$$\begin{aligned}
\underline{\text{Rod OA}}: \quad I^{\mathrm{O}}_{yy} &= I^{\mathrm{cm}}_{y'y'} + m\frac{l^2}{4} = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 \\
I^{\mathrm{O}}_{xy} &= \underbrace{I^{\mathrm{cm}}_{x'y'}}_{0} + m(-x_{\mathrm{cm/O}}\underbrace{y_{\mathrm{cm/O}}}_{0}) = 0 \\
I^{\mathrm{O}}_{yz} &= \underbrace{I^{\mathrm{cm}}_{y'z'}}_{0} + m(-\underbrace{y_{\mathrm{cm/O}}}_{0}\underbrace{z_{\mathrm{cm/O}}}_{0}) = 0.
\end{aligned}$$

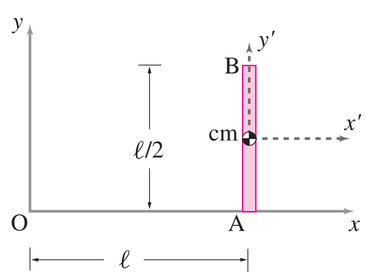

Figure 20.44

$$\begin{aligned}
\underline{\text{Rod AB}}: \quad I^{\mathrm{O}}_{yy} &= \underbrace{I^{\mathrm{cm}}_{y'y'}}_{0} + \frac{m}{2}(x^2_{\mathrm{cm/O}} + \underbrace{z^2_{\mathrm{cm/O}}}_{0}) = \frac{ml^2}{2} \\
I^{\mathrm{O}}_{xy} &= \underbrace{I^{\mathrm{cm}}_{x'y'}}_{0} + \frac{m}{2}(-\underbrace{x_{\mathrm{cm/O}}}_{l}\underbrace{y_{\mathrm{cm/O}}}_{\frac{l}{4}}) = -\frac{ml^2}{8} \\
I^{\mathrm{O}}_{yz} &= \underbrace{I^{\mathrm{O}}_{y'z'}}_{0} + m(-y_{\mathrm{cm/O}}\underbrace{z_{\mathrm{cm/O}}}_{0}) = 0.
\end{aligned}$$

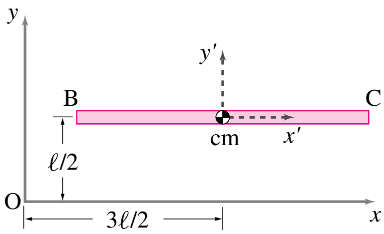

Figure 20.45

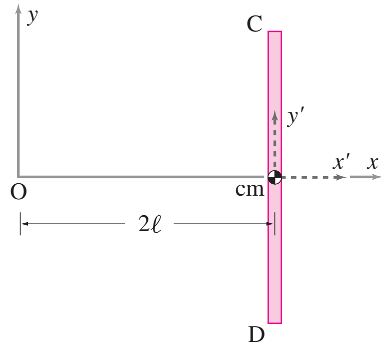

Figure 20.46

$$\underline{\text{Rod BC}}: \quad I^O_{yy} = I^{cm}_{y'y'} + m(x^2_{cm/O} + \underbrace{z^2_{cm/O}}_{0}) = \frac{ml^2}{12} + m\frac{9l^2}{4} = \frac{7}{3}ml^2$$

$$I^O_{xy} = \underbrace{I^{cm}_{x'y'}}_{0} + m(-x_{cm/O}y_{cm/O}) = -m\frac{3l}{2}\cdot\frac{l}{2} = -\frac{3ml^2}{4}$$

$$I^O_{yz} = 0.$$

$$\underline{\text{Rod CD}}: \quad I^O_{yy} = \underbrace{I^{cm}_{y'y'}}_{0} + m(x^2_{cm/O} + \underbrace{z^2_{cm/O}}_{0}) = m(2l)^2 = 4ml^2$$

$$I^O_{xy} = 0$$

$$I^O_{yz} = 0.$$

Thus for the entire rod,

$$I^O_{yy} = \frac{1}{3}ml^2 + \frac{1}{2}ml^2 + \frac{7}{3}ml^2 + 4ml^2 = \frac{43}{6}ml^2$$

$$I^O_{xy} = -\frac{1}{8}ml^2 - \frac{3}{4}ml^2 = -\frac{7}{8}ml^2$$

$$I^O_{yz} = 0.$$

$$\boxed{I^O_{yy} = \tfrac{43}{6}ml^2,\ I^O_{xy} = -\tfrac{7}{8}ml^2,\ I^O_{yz} = 0.}$$

2.

$$\vec{\boldsymbol{H}}_{/O} = (I^O_{yy}\hat{\boldsymbol{\imath}} + I^O_{xy}\hat{\boldsymbol{\jmath}} + I^O_{yz}\hat{\boldsymbol{k}})\omega \quad \text{(from Eqn 20.46)}$$

$$= ml^2\omega(\frac{43}{6}\hat{\boldsymbol{\jmath}} - \frac{7}{8}\hat{\boldsymbol{\imath}})$$

$$= 0.5\,\text{kg}(0.2\,\text{m})^2\cdot 5\,\text{rad/s}(7.17\hat{\boldsymbol{\jmath}} - 0.87)\hat{\boldsymbol{\imath}})$$

$$= (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\text{kg}\cdot\text{m}^2/\text{s}.$$

$$\boxed{\vec{\boldsymbol{H}}_{/O} = (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\text{kg}\cdot\text{m}^2/\text{s}.}$$

3.

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/O} = \omega\hat{\boldsymbol{\jmath}} \times ([\boldsymbol{I}^O]\cdot\vec{\boldsymbol{\omega}})$$

$$= 5\,\text{rad/s}\,\hat{\boldsymbol{\jmath}} \times (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\text{kg}\cdot\text{m}^2/\text{s}$$

$$= 0.435\,\text{N}\cdot\text{m}\hat{\boldsymbol{k}}.$$

$$\boxed{\dot{\vec{\boldsymbol{H}}}_{/O} = 0.435\,\text{N}\cdot\text{m}\hat{\boldsymbol{k}}.}$$

4. If the rod were rotating in the xy-plane, it would be planar circular motion. The only component of $[\boldsymbol{I}^O]$ required for the calculation of $\vec{\boldsymbol{H}}_{/O} = I^O_{zz}\vec{\boldsymbol{\omega}}$ will be I^O_{zz}. Also,

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/O} = \vec{\boldsymbol{\omega}} \times I^O_{zz}\vec{\boldsymbol{\omega}} = 0$$

since $\vec{\boldsymbol{\omega}}$ is parallel to $I^O_{zz}\vec{\boldsymbol{\omega}}$.

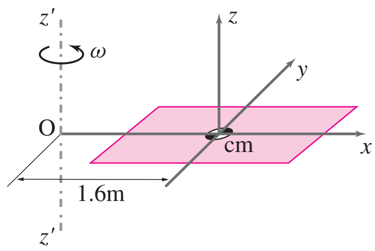

Figure 20.47

SAMPLE 20.13 A 0.5 kg uniform rectangular solar panel rotates about an off-centered axis $z'z'$ with constant angular speed $\omega = 1.5\,\mathrm{rad/s}$. Axis $z'z'$ is parallel to the transverse z-axis of the plate and is 1.6 m away from the center-of-mass of the plate. The in-plane moments of inertia I_{xx}^{cm} and I_{yy}^{cm} of the plate are given: $I_{xx}^{\mathrm{cm}} = 4.0\,\mathrm{kg\cdot m^2}$ and $I_{yy}^{\mathrm{cm}} = 12.4\,\mathrm{kg\cdot m^2}$. At the instant shown in Fig 20.47, calculate

1. the linear momentum of the panel,
2. the angular momentum of the panel, and
3. the kinetic energy of the panel.

Solution Since the plate rotates about the $z'z'$-axis which is parallel to the z-axis,

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} = 1.5\,\mathrm{rad/s}\hat{\boldsymbol{k}}.$$

1. **Linear momentum:**

$$\begin{aligned}\vec{\boldsymbol{L}} &= \mathrm{m_{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} = m(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\mathrm{cm/O}})\\ &= 0.5\,\mathrm{kg}\cdot(1.5\,\mathrm{rad/s}\hat{\boldsymbol{k}}\times 1.6\,\mathrm{m}\hat{\boldsymbol{\imath}})\\ &= 1.2\,\mathrm{kg\cdot m/s}\hat{\boldsymbol{\jmath}}\end{aligned}$$

$$\boxed{\vec{\boldsymbol{L}} = 1.2\,\mathrm{kg\cdot m/s}\hat{\boldsymbol{\jmath}}}$$

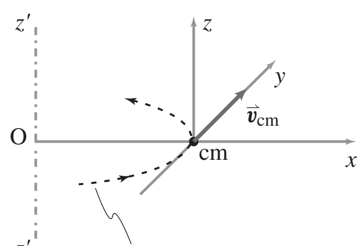

Figure 20.48

We can easily check the direction of $\vec{\boldsymbol{L}}$ since $\vec{\boldsymbol{L}} = m\vec{\boldsymbol{v}}_{\mathrm{cm}}$, it has to be in the same direction as $\vec{\boldsymbol{v}}_{\mathrm{cm}}$. The center-of-mass goes in circles about O, therefore, $\vec{\boldsymbol{v}}_{\mathrm{cm}}$ is tangential to the circular path, i.e. in the y-direction (see Fig 20.48).

2. **Angular momentum:**

$$\vec{\boldsymbol{H}} = I_{z'z'}^{\mathrm{O}}\,\omega\hat{\boldsymbol{k}}$$

Thus to find $\vec{\boldsymbol{H}}$, we need to find $I_{z'z'}^{\mathrm{O}}$. Since $z'z' \parallel zz$, we can use the parallel axis theorem to find $I_{z'z'}^{\mathrm{O}}$ if we know I_{zz}^{cm}. We are given the in-plane moments of inertia I_{xx}^{cm} and I_{yy}^{cm}. Therefore, from the perpendicular axis theorem:

$$I_{zz}^{\mathrm{cm}} = I_{xx}^{\mathrm{cm}} + I_{yy}^{\mathrm{cm}} = (4.0 + 12.4)\,\mathrm{kg\cdot m^2} = 16.4\,\mathrm{kg\cdot m^2}.$$

Now using the parallel axis theorem,

$$\begin{aligned}I_{z'z'}^{\mathrm{O}} &= I_{zz}^{\mathrm{cm}} + Mr_{\mathrm{cm/O}}^2\\ &= 16.4\,\mathrm{kg\cdot m^2} + 0.5\,\mathrm{kg}(1.6\,\mathrm{m})^2 = 17.68\,\mathrm{kg\cdot m^2}.\\ \text{Therefore,}\quad \vec{\boldsymbol{H}} &= 17.68\,\mathrm{kg\cdot m^2}\cdot(1.5\,\mathrm{rad/s}\hat{\boldsymbol{k}})\\ &= 26.52\,\mathrm{kg\cdot m^2/s}\hat{\boldsymbol{k}}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}} = 26.52\,\mathrm{kg\cdot m^2/s}\hat{\boldsymbol{k}}.}$$

3. **Kinetic energy:**

$$\begin{aligned}E_{\mathrm{K}} &= \frac{1}{2}I_{z'z'}^{\mathrm{O}}\omega^2 = \frac{1}{2}(17.68\,\mathrm{kg\cdot m^2})(1.5\,\mathrm{rad/s})^2\\ &= 19.89\frac{\mathrm{kg\cdot m^2}}{\mathrm{s^2}} == 19.89\,\mathrm{N\cdot m}\\ &= 19.89\ \mathrm{J}\end{aligned}$$

$$\boxed{E_{\mathrm{K}} = 19.89\ \mathrm{J}}$$

SAMPLE 20.14 **The rotating crooked plate again.** For the crooked plate considered in Sample 20.7 and shown again in Fig. 20.49,

1. compute the moment of inertia matrix $[\boldsymbol{I}]$ in the $x'y'z'$ coordinate system,
2. compute the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ using the moment of inertia $[\boldsymbol{I}]_{x'y'z'}$ computed above,
3. express $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ as a vector in the xyz coordinate system.

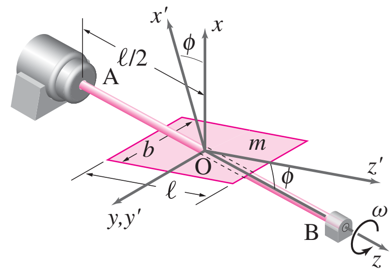

Figure 20.49: A rectangular plate of mass m rotates with shaft AB at a constant speed $\vec{\omega} = \omega\hat{\boldsymbol{k}}$. A coordinate system $x'y'z'$ is aligned with the *principle axes* of the plate.

Solution A line sketch of the plate and the two coordinate systems attached to it are shown in Fig. 20.50. The set of basis vectors $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$ and $(\hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}', \hat{\boldsymbol{k}}')$ associated with coordinate systems xyz and $x'y'z'$ respectively are shown separately for the sake of clarity. From the diagram of the two basis vector sets, we may write

$$\left.\begin{aligned} \hat{\boldsymbol{\imath}}' &= \cos\phi\,\hat{\boldsymbol{\imath}} - \sin\phi\,\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{\jmath}}' &= \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}}' &= \sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{k}} \end{aligned}\right\}. \tag{20.47}$$

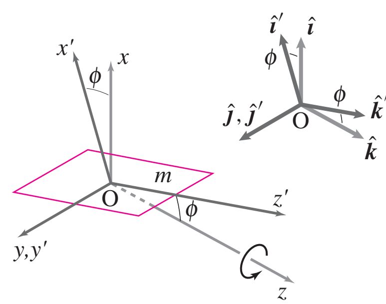

Figure 20.50: A line sketch of the rotating plate along with the two coordinate systems $x'y'z'$ and xyz. The basis vectors associated with the two systems are also shown.

1. **Calculation of** $[\boldsymbol{I}]_{x'y'z'}$**:**

$$[\boldsymbol{I}]_{x'y'z'} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{x'y'} & I_{y'y'} & I_{y'z'} \\ I_{x'z'} & I_{y'z'} & I_{z'z'} \end{bmatrix}.$$

Let us first consider the off diagonal terms of $[\boldsymbol{I}]_{x'y'z'}$.
Since $x' = 0$ on the entire plate (the plate is in the $y'z'$ plane and the origin is on the plate.),

$$I_{x'y'} = \int_m \underbrace{x'}_{0}\, y'\,dm = 0, \quad I_{x'z'} = \int_m \underbrace{x'}_{0}\, z'\,dm = 0.$$

How about $I_{y'z'}$? Well, you can calculate it two ways: (a) carry out the integration $I_{y'z'} = \int_m y'z'\,dm$ over the entire plate mass and find that $I_{y'z'} = 0$, or (b) realize that for every mass element $dm\,(=\frac{m}{\ell b}dA)$ with coordinates $(+y'z')$, there exists another element dm at $(-y'z')$ such that the sum of their contributions to the integral is zero. Therefore, $I_{y'z'} = \int_m y'z'\,dm = 0$ ① Now the other terms:

$$\begin{aligned} I_{z'z'} &= \int_m (\underbrace{x'^2}_{0} + y'^2)\,dm = \int_{-\ell/2}^{\ell/2}\int_{-b/2}^{b/2} y'^2 \cdot \frac{m}{\ell b}dy'\,dz' \\ &= \frac{m}{\ell b}\int_{-\ell/2}^{\ell/2}\left(\left.\frac{y'^3}{3}\right|_{-b/2}^{b/2}\right)dz' = \frac{m}{\ell b}\cdot\frac{b^3}{12}\int_{-\ell/2}^{\ell/2} dz' \\ &= \frac{m}{\ell b}\cdot\frac{b^3}{12}\cdot\ell = \frac{mb^2}{12}, \end{aligned}$$

and similarly,

$$\begin{aligned} I_{y'y'} &= \int_m (\underbrace{x'^2}_{0} + z'^2)\,dm = \frac{m\ell^2}{12}, \\ I_{x'x'} &= \int_m (y'^2 + z'^2)\,dm = I_{y'y'} + I_{z'z'} = \frac{m}{12}(\ell^2 + b^2). \end{aligned}$$

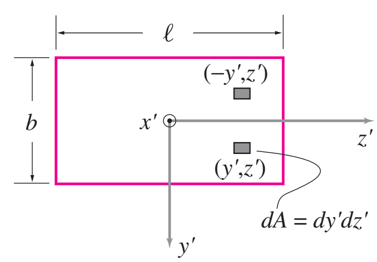

Figure 20.51: Symmetry about the z' axis implies that $I_{y'z'} = 0$. Same result holds if we consider the symmetry about the y' axis.

① You can use a similar argument to find the off-diagonal terms in $[\boldsymbol{I}]$ whenever there is such symmetry with respect to the coordinate axes.

Thus,

$$[\boldsymbol{I}]_{x'y'z'} = \frac{m}{12}\begin{bmatrix} \ell^2+b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}.$$

2. **Calculation of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$:**

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}} \quad \text{and} \quad \vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}]_{x'y'z'}\{\vec{\boldsymbol{\omega}}\}_{x'y'z'}.$$

The subscripts $x'y'z'$ in $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ have been used to denote that both $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ are expressed in $x'y'z'$ coordinate system. ② We have calculated $[I]_{x'y'z'}$ above. Now we need to find $\{\vec{\omega}\}_{x'y'z'}$.

② We can express $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ in any coordinate system of our choice but both of them must be in the same system for their product to be valid.

Let $\vec{\boldsymbol{\omega}} = \omega_{x'}\hat{\boldsymbol{\imath}}' + \omega_{y'}\hat{\boldsymbol{\jmath}}' + \omega_{z'}\hat{\boldsymbol{k}}'$. But, we can also write, $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. So,

$$\omega\hat{\boldsymbol{k}} = \omega_{x'}\hat{\boldsymbol{\imath}}' + \omega_{y'}\hat{\boldsymbol{\jmath}}' + \omega_{z'}\hat{\boldsymbol{k}}' \tag{20.48}$$

Dotting both sides of Eqn. (20.48) with $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, and $\hat{\boldsymbol{k}}'$ and using the relationships in (20.47) we get

$$\omega_{x'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\imath}}'}^{-\sin\phi}) = -\omega\sin\phi, \omega_{y'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\jmath}}'}^{0}) = 0, \omega_{k'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'}^{\cos\phi}) = \omega\cos\phi.$$

Thus,

$$\{\vec{\omega}\}_{x'y'z'} = -\omega\sin\phi\hat{\boldsymbol{\imath}}' + \omega\cos\phi\hat{\boldsymbol{k}}'$$

So,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{O}} &= \frac{m}{12}\begin{bmatrix} \ell^2+b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}\begin{Bmatrix} -\omega\sin\phi \\ 0 \\ \omega\cos\phi \end{Bmatrix} \\
&= \begin{Bmatrix} -\frac{m}{12}(\ell^2+b^2)\omega\sin\phi \\ 0 \\ \frac{m}{12}b^2\omega\cos\phi \end{Bmatrix} \\
\text{or} \quad \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \frac{m\omega}{12}[-(\ell^2+b^2)\sin\phi\hat{\boldsymbol{\imath}}' + b^2\cos\phi\hat{\boldsymbol{k}}'].
\end{aligned}$$

Now we can easily compute $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ as follows:

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/\mathrm{O}} \\
&= (-\omega\sin\phi\hat{\boldsymbol{\imath}}' + \omega\cos\phi\hat{\boldsymbol{k}}') \times \frac{m\omega}{12}[-(\ell^2+b^2)\sin\phi\hat{\boldsymbol{\imath}}' + b^2\cos\phi\hat{\boldsymbol{k}}'] \\
&= \frac{m\omega^2}{12}b^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}' - \frac{m\omega^2}{12}(\ell^2+b^2)\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}' \\
&= -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

3. **Now, back to the xyz coordinate system:** Since $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$ [Eqn (20.47)], we have

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}$$

which is the same result as obtained in Sample 20.7.

SAMPLE 20.15 The calculation of $\dot{\vec{H}}$ using the moment of inertia matrix. For the crooked plate considered in Sample 20.7, find the rate of change of angular momentum $\dot{\vec{H}}_{/O}$, using the moment of inertia $[I^O]$, calculated in the xyz coordinate system

Solution We calculate $\dot{\vec{H}}_{/O}$ using the following formula:

$$\dot{\vec{H}}_{/O} = [I^O] \cdot \underbrace{\dot{\vec{\omega}}}_{\vec{0}} + \vec{\omega} \times \overbrace{([I^O] \cdot \vec{\omega})}^{\vec{H}_{/O}}.$$

Note that point O is the center-of-mass of the plate. Let us write the expression for the angular momentum $\vec{H}_{/O}$ in matrix form.

$$\left\{ \begin{array}{c} H_{x/O} \\ H_{y/O} \\ H_{z/O} \end{array} \right\} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \left\{ \begin{array}{c} 0 \\ 0 \\ \omega \end{array} \right\}.$$

It is clear from the components of $\vec{\omega}$ that we only need the last column of $[I]$ matrix since other elements of $[I]$ will multiply with zeros of $\vec{\omega}$ vector. Carrying out the multiplication we get

$$\left\{ \begin{array}{c} H_{0x} \\ H_{0y} \\ H_{0z} \end{array} \right\} = \left\{ \begin{array}{c} I_{xz}\omega \\ I_{yz}\omega \\ I_{zz}\omega \end{array} \right\}$$

or, in vector form

$$\vec{H}_{/O} = I_{xz}\omega\hat{\imath} + I_{yz}\omega\hat{\jmath} + I_{zz}\omega\hat{k}.$$

Therefore,

$$\begin{aligned} \dot{\vec{H}}_{/O} &= \omega\hat{k} \times \omega(I_{xz}\hat{\imath} + I_{yz}\hat{\jmath} + I_{zz}\hat{k}) \\ &= -\omega^2 I_{yz}\hat{\imath} + \omega^2 I_{xz}\hat{\jmath}. \end{aligned}$$

But,

$$I_{xz} = -\int_m xz\,dm \quad \text{and} \quad I_{yz} = -\int_m yz\,dm$$

and $x = w\sin\phi,\ y = y,\ z = w\cos\phi$ (see Fig. 20.28), hence

$$\begin{aligned} \dot{\vec{H}}_{/O} &= -\omega^2 \int_m (-wy\cos\phi\hat{\imath} + w^2\sin\phi\cos\phi\hat{\jmath})\,dm \\ &= \int_{-b/2}^{b/2}\int_{-\ell/2}^{\ell/2} \omega^2(-w^2\sin\phi\cos\phi\hat{\jmath} + wy\cos\phi\hat{\imath})\frac{m}{\ell b}dw\,dy \end{aligned}$$

which is the same integral as obtained in Sample 20.7 for $\dot{\vec{H}}_{/O}$. Therefore, the result is also the same:

$$\dot{\vec{H}}_{/O} = -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\jmath}.$$

$$\boxed{\dot{\vec{H}}_{/O} = -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\jmath}}$$

SAMPLE 20.16 **Direct application of formula:** A rectangular plate is mounted on a massless shaft with the center-of-mass of the plate on the shaft axis. The shaft rotates about its axis with angular acceleration $\dot{\vec{\omega}} = 0.5\,\text{rpm/s}(\hat{\imath} + \hat{\jmath})$. At the instant of interest, the angular velocity is $\vec{\omega} = 100\,\text{rpm}(\hat{\imath} + \hat{\jmath})$ and the components of the moment of inertia matrix of the plate are $I_{xx} = 2\,\text{kg}{\cdot}\text{m}^2$, $I_{yy} = 4\,\text{kg}{\cdot}\text{m}^2$, $I_{zz} = 6\,\text{kg}{\cdot}\text{m}^2$ and $I_{xy} = I_{yz} = I_{xz} = 0$.

1. Find the angular momentum of the plate about its mass-center and show that it is not in the same direction as the angular velocity.
2. Find the net moment acting on the plate.

Solution We are given the angular velocity, the angular acceleration, and the moment of inertia matrix of the plate:

$$\begin{aligned}
\vec{\omega} &= 100\,\text{rpm}(\hat{\imath} + \hat{\jmath}) = 10.47\,\text{rad/s}(\hat{\imath} + \hat{\jmath}) \\
\dot{\vec{\omega}} &= 5\,\text{rpm/s}(\hat{\imath} + \hat{\jmath}) = 0.52\,\text{rad/s}^2(\hat{\imath} + \hat{\jmath}) \\
[\boldsymbol{I}^{\text{cm}}] &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{kg}{\cdot}\text{m}^2.
\end{aligned}$$

1. **The angular momentum:** The angular momentum of the plate is

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\text{cm}} &= [\boldsymbol{I}^{\text{cm}}] \cdot \vec{\omega} \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{kg}{\cdot}\text{m}^2 \cdot \begin{Bmatrix} 10.47 \\ 10.47 \\ 0 \end{Bmatrix} \text{rad/s} \\
&= (20.94\hat{\imath} + 41.88\hat{\jmath})\, \underbrace{\text{kg}{\cdot}\text{m}^2 \cdot \text{s}^{-1}}_{\text{N}\cdot\text{m}\cdot\text{s}} \\
&= (20.94\hat{\imath} + 41.88\hat{\jmath})\,\text{N}{\cdot}\text{m} \cdot \text{s}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}}_{\text{cm}} = (20.94\hat{\imath} + 41.88\hat{\jmath})\,\text{N}{\cdot}\text{m} \cdot \text{s}}$$

There are many ways of showing that $\vec{\boldsymbol{H}}_{\text{cm}}$ is not parallel to $\vec{\omega}$. We can simply draw the two vectors and show that they are not parallel We can, alternatively, take the cross product of the two vectors:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\text{cm}} \times \vec{\omega} &= (20.94\hat{\imath} + 41.88\hat{\jmath})\,\text{N}{\cdot}\text{m} \cdot \text{s} \times 10.47\,\text{rad/s}(\hat{\imath} + \hat{\jmath}) \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 20.94 & 41.88 & 0 \\ 10.47 & 10.47 & 0 \end{vmatrix} \text{N}{\cdot}\text{m} \\
&= \hat{\boldsymbol{k}}(219.24 - 438.48)\,\text{N}{\cdot}\text{m} = -20.94\,\text{N}{\cdot}\text{m}\hat{\boldsymbol{k}}
\end{aligned}$$

which is not zero, implying that the two vectors are not parallel.

2. **The net moment:** The net moment on the plate can be found by applying angular momentum balance:

$$\begin{aligned}
\vec{\boldsymbol{M}}_{/\text{cm}} &= \dot{\vec{\boldsymbol{H}}}_{\text{cm}} = \vec{\omega} \times \underbrace{[\boldsymbol{I}^{\text{cm}}] \cdot \vec{\omega}}_{\vec{H}_{\text{cm}}} + [\boldsymbol{I}^{\text{cm}}] \cdot \dot{\vec{\omega}} \\
&= \underbrace{20.94\,\text{N}{\cdot}\text{m}\hat{\boldsymbol{k}}}_{\vec{\omega}\times\vec{H}_{\text{cm}}} + \underbrace{1.04\,\text{N}{\cdot}\text{m}\hat{\imath} + 2.08\,\text{N}{\cdot}\text{m}\hat{\jmath}}_{[I^{\text{cm}}]\cdot\dot{\vec{\omega}}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{M}}_{/\text{cm}} = (1.04\hat{\imath} + 2.08\hat{\jmath} + 20.94\hat{\boldsymbol{k}})\,\text{N}{\cdot}\text{m}}$$

SAMPLE 20.17 **A rod swings around a tipped axis in 3-D.** A rigid shaft of negligible mass is connected to frictionless hinges at A and B. The shaft is tipped from the horizontal plane (xy-plane) such that the shaft axis makes an angle γ with the vertical axis. A uniform rod OC of mass m and length ℓ is welded to the shaft. At time $t=0$, the rod is tipped by a small angle ϕ from its position OC' in the yz (or $y'z'$) plane. Find the equation of motion of the rod.

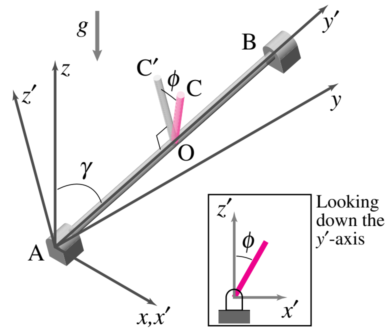

Figure 20.52: A rod, welded to a tipped shaft AB, swings around the shaft axis if it is tipped slightly from the vertical plane yz.

Solution Let $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ be the basis vectors associated with the $x'y'z'$ coordinate system. Since $y'z'$ axes can be obtained by rotating yz axes counterclockwise about the x-axis by an angle $90^o-\gamma$, we can relate the basis vectors of the two coordinate systems with the help of Fig. 20.53:

$$
\begin{aligned}
\hat{\imath}' &= \hat{\imath}, \\
\hat{\jmath}' &= \sin\gamma\,\hat{\jmath}+\cos\gamma\,\hat{k}, \qquad (20.49)\\
\hat{k}' &= -\cos\gamma\,\hat{\jmath}+\sin\gamma\,\hat{k}.
\end{aligned}
$$

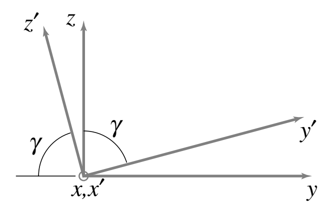

Figure 20.53: $y'z'$ axes are obtained by rotating yz axes counterclockwise about the x-axis by an angle $90^o-\gamma$.

The free-body diagram of the shaft with rod OC is shown in Fig. 20.54. We can write angular momentum balance for this system about any point on the axis of rotation AB. However, rather than writing angular momentum balance about a point, let us write angular momentum balance about axis AB. This 'trick' will eliminate reactions $\vec{R}_A$ and $\vec{R}_B$ from our equations. Angular Momentum Balance about axis AB is:

$$
\begin{aligned}
\hat{\lambda}_{AB}\cdot[\textstyle\sum\vec{M}_{\rm O} &= \dot{\vec{H}}_{/\rm O}]\\
\text{or}\qquad \hat{\jmath}'\cdot\textstyle\sum\vec{M}_{\rm O} &= \hat{\jmath}'\cdot\dot{\vec{H}}_{/\rm O}.
\end{aligned}
$$

Calculation of ($\hat{\jmath}'\cdot\sum\vec{M}_{\rm O}$): Since $\vec{R}_A$ and $\vec{R}_B$ pass through axis AB, they do not produce any moment about this axis. Therefore,

$$
\begin{aligned}
\hat{\jmath}'\cdot\textstyle\sum\vec{M}_{\rm O} &= \hat{\jmath}'\cdot[\vec{r}_{G/O}\times mg(-\hat{k})]\\
&= \hat{\jmath}'\cdot\left[\frac{L}{2}(\sin\phi\,\hat{\imath}'+\cos\phi\,\hat{k}')\times mg(-\hat{k})\right]\\
&= \overbrace{(\sin\gamma\,\hat{\jmath}+\cos\gamma\,\hat{k})}^{\hat{\jmath}'}\cdot[\frac{L}{2}mg\overbrace{(\sin\phi\,\hat{\jmath}+\cos\phi\,\cos\gamma\,\hat{\imath})}^{\text{using Eqn.(20.49)}}]\\
&= \frac{\ell}{2}mg\sin\phi\,\sin\gamma
\end{aligned}
$$

Calculation of $\hat{\jmath}'\cdot\dot{\vec{H}}_{/\rm O}$: Since the rod rotates about axis AB, we may write

$$
\begin{aligned}
\vec{\omega} &= \dot{\phi}\,\hat{\jmath}',\\
\dot{\vec{\omega}} &= \ddot{\phi}\,\hat{\jmath}'.
\end{aligned}
$$

Now,

$$\dot{\vec{H}}_{/\rm O}=[\boldsymbol{I}^{\rm O}]\{\dot{\vec{\omega}}\}+\vec{\omega}\times[\boldsymbol{I}^{\rm O}]\{\vec{\omega}\}$$

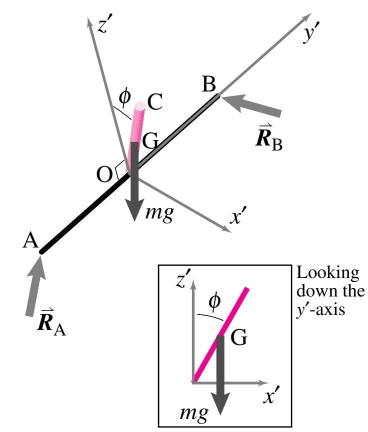

Figure 20.54: Free-body diagram of the bar and shaft system. In the inset, only the weight of the bar, mg, is shown for clarity of geometry.

where

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]\{\dot{\vec{\omega}}\} &= [\boldsymbol{I}^{\mathrm{O}}]_{x'y'z'}\,\{\dot{\vec{\omega}}\}_{x'y'z'} \\
&= \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{x'y'} & I_{y'y'} & I_{y'z'} \\ I_{x'z'} & I_{y'z'} & I_{z'z'} \end{bmatrix} \begin{Bmatrix} 0 \\ \ddot{\phi} \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} I_{x'y'}\ddot{\phi} \\ I_{y'y'}\ddot{\phi} \\ I_{y'z'}\ddot{\phi} \end{Bmatrix}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]\{\vec{\omega}\} &= \begin{Bmatrix} I_{x'y'}\dot{\phi} \\ I_{y'y'}\dot{\phi} \\ I_{y'z'}\dot{\phi} \end{Bmatrix}, \\
\Rightarrow \qquad \vec{\omega} \times [\boldsymbol{I}^{\mathrm{O}}]\{\vec{\omega}\} &= \dot{\phi}\hat{\boldsymbol{\jmath}}' \times [I_{x'y'}\dot{\phi}\hat{\boldsymbol{\imath}}' + I_{y'y'}\dot{\phi}\hat{\boldsymbol{\jmath}}' + I_{y'z'}\dot{\phi}\hat{\boldsymbol{k}}'] \\
&= -I_{x'y'}\dot{\phi}^2\hat{\boldsymbol{k}}' + I_{y'z'}\dot{\phi}^2\hat{\boldsymbol{\imath}}'.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\boldsymbol{\jmath}}' \cdot \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \hat{\boldsymbol{\jmath}}' \cdot [(I_{x'y'}\ddot{\phi}\hat{\boldsymbol{\imath}}' + I_{y'y'}\ddot{\phi}\hat{\boldsymbol{\jmath}}' + I_{y'z'}\ddot{\phi}\hat{\boldsymbol{k}}') + (-I_{x'y'}\dot{\phi}^2\hat{\boldsymbol{k}}' + I_{y'z'}\dot{\phi}^2\hat{\boldsymbol{\imath}}')] \\
&= I_{y'y'}\ddot{\phi}.
\end{aligned}$$

Now, setting $\hat{\boldsymbol{\jmath}}' \cdot \sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \hat{\boldsymbol{\jmath}}' \cdot \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$, we get

$$\begin{aligned}
\frac{L}{2} mg \sin\phi\, \sin\gamma &= I_{y'y'}\ddot{\phi} \\
\text{or} \qquad \ddot{\phi} &= \frac{mg\,(L/2)\, \sin\gamma}{I_{y'y'}} \sin\phi \\
\text{or} \qquad \ddot{\phi} &= C \sin\phi
\end{aligned}$$

where

$$C = \frac{mg\,(L/2)\, \sin\gamma}{I_{y'y'}}.$$

For rod OC,

$$I_{y'y'} = \int_m (x'^2 + z'^2)\, dm = \int_0^L l^2 \frac{m}{L}\, dl = \frac{1}{3} mL^2.$$

Therefore, the equation of motion of the rod is

$$\begin{aligned}
\ddot{\phi} - \frac{mg\,(L/2)\, \sin\gamma}{(1/3)mL^2} \sin\phi &= 0 \\
\ddot{\phi} - \frac{3g\, \sin\gamma}{2L} \sin\phi &= 0.
\end{aligned}$$

$$\ddot{\phi} - \frac{3g\, \sin\gamma}{2L} \sin\phi = 0$$

SAMPLE 20.18 Kinetic energy in 3-D rotation. A thin rod of mass 2 kg and length $L = \frac{1}{2}$ m is welded to a massless shaft at an angle $\theta = 45^{\circ}$. The shaft rotates about its longitudinal axis (y-axis) at 100 rpm. The moment of inertia matrix of the rod about the weld point O is

$$[\boldsymbol{I}^{\mathrm{O}}] = 0.08\,\mathrm{kg\cdot m^2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find the kinetic energy of the rod.

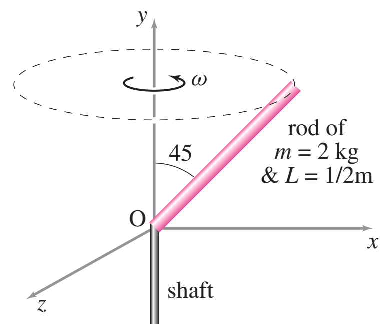

Figure 20.55

Solution The rod rotates about the fixed point O with angular velocity

$$\vec{\omega} = \omega\hat{\boldsymbol{\jmath}} \quad \text{where} \quad \omega = 100\,\mathrm{rpm} = 10.47\,\mathrm{rad/s}.$$

The kinetic energy of a rigid body rotating about a fixed point O is given by

$$E_{\mathrm{K}} = \frac{1}{2}\vec{\omega}\cdot[[\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\omega}]$$

For the given problem, let us write the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$ of the rod as ③

③ This step is simply to facilitate computation. We carry out the multiplication and at the end substitute the value of K_0.

$$[\boldsymbol{I}^{\mathrm{O}}] = K_0\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where $K_0 = 0.08\,\mathrm{kg\cdot m^2}$. Now

$$\vec{\omega} = \omega\hat{\boldsymbol{\jmath}} = \{0 \quad \omega \quad 0\}^{\mathrm{T}}.$$

Therefore,

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\omega} &= K_0\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\
&= K_0\begin{Bmatrix} -\omega \\ \omega \\ 0 \end{Bmatrix} \\
&= -K_0\omega\hat{\boldsymbol{\imath}} + K_0\omega\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

Substituting the expression in the formula for E_{K} we get

$$\begin{aligned}
E_{\mathrm{K}} &= \frac{1}{2}\omega\hat{\boldsymbol{\jmath}}\cdot(-K_0\omega\hat{\boldsymbol{\imath}} + K_0\omega\hat{\boldsymbol{\jmath}}) \\
&= \frac{1}{2}K_0\omega^2 \\
&= \frac{1}{2}\cdot 0.08\,\mathrm{kg\cdot m^2}\cdot(10.47\,\mathrm{rad/s})^2 \\
&= 4.38\,\mathrm{N\cdot m} = 4.38\mathrm{J}
\end{aligned}$$

$$\boxed{E_{\mathrm{K}} = 4.38\mathrm{J}}$$