20.3 Moment of inertia matrices $[\boldsymbol{I}^{\rm cm}]$ and $[\boldsymbol{I}^{\rm O}]$

We now know how to find the velocity and acceleration of every bit of mass on a rigid body as it spins about a fixed axis. It is just a matter of doing integrals or sums to calculate the various motion quantities (momenta, energy) of interest. As the body moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid body one has to evaluate a different integral or sum at every different configuration. But there is a shortcut. A big sum (over all atoms, say), or a difficult integral is reduced to a simple multiplication using the moment of inertia. In three-dimensions this multiplication is a matrix multiplication.

[**I**], the moment of inertia matrix①, is defined for the purpose of simplifying the expressions for the angular momentum, the rate of change of angular momentum, and the energy of *a system which moves like a rigid body*.

①In fact the moment of inertia matrix for a given object depends on what reference point is used. Most commonly when people say 'the' moment of inertia they mean to use the center-of-mass as the reference point. For clarity this moment of inertia matrix is often written as $[\boldsymbol{I}^{\rm cm}]$ in this book. If a different reference point, say point O is used, the matrix is notated as $[\boldsymbol{I}^{\rm O}]$.

First review the situation for flat objects in planar motion. A flat object spinning with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ in the xy plane has a mass distribution which gives a polar moment of inertia I_{zz}^{cm} or just 'I' so that:

$$\vec{\boldsymbol{H}}_{\rm cm} = I\omega\hat{\boldsymbol{k}} \tag{20.26}$$

$$\dot{\vec{\boldsymbol{H}}}_{\rm cm} = \vec{\boldsymbol{0}} \tag{20.27}$$

$$E_{{\rm K}/cm} = \frac{1}{2}\omega^2 I. \tag{20.28}$$

Now, for a rigid body spinning in 3-D about a fixed axis with the angular velocity $\vec{\boldsymbol{\omega}}$ we need matrix multiplication, where the determination of the needed matrix is the central topic of this section.

$$\vec{\boldsymbol{H}}_{\rm cm} = [I^{cm}]\cdot\vec{\boldsymbol{\omega}} \tag{20.29}$$

$$\dot{\vec{\boldsymbol{H}}}_{\rm cm} = \vec{\boldsymbol{\omega}}\times\underbrace{[I^{cm}]\cdot\vec{\boldsymbol{\omega}}}_{\vec{\boldsymbol{H}}_{\rm cm}}+[I^{cm}]\cdot\dot{\vec{\boldsymbol{\omega}}} \tag{20.30}$$

$$E_{{\rm K}/cm} = \frac{1}{2}\vec{\boldsymbol{\omega}}\cdot([I^{cm}]\cdot\vec{\boldsymbol{\omega}}) = \frac{1}{2}\vec{\boldsymbol{\omega}}\cdot\vec{\boldsymbol{H}}_{\rm cm}. \tag{20.31}$$

In detail, for example,

$$\begin{bmatrix} H_{x/cm} \\ H_{y/cm} \\ H_{z/cm} \end{bmatrix} = \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix}_{xyz} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{20.32}$$

where $\vec{\boldsymbol{H}}_{\rm cm} = H_{x/cm}\hat{\boldsymbol{\imath}} + H_{y/cm}\hat{\boldsymbol{\jmath}} + H_{z/cm}\hat{\boldsymbol{k}}$. Note that the 2-D results are a special case of the 3-D results because, as you will soon see, for 2-D objects $I_{xz} = I_{yz} = 0$. We postpone the use of these equations until section 20.4.

The moment of inertias in 3-D: $[\boldsymbol{I}^{\mathrm{cm}}]$ and $[\boldsymbol{I}^{\mathrm{O}}]$

For the study of three-dimensional mechanics, including the simple case of constant rate rotation about a fixed axis, one often makes use of the moment of inertia matrix, defined below and motivated by the box 20.1 on page 6.

The distances x, y, z in the formulas below are the x, y, z components of the position of mass relative to a coordinate system which has either the center-of-mass (cm) or the point O as its origin. ②

②**Caution:** While we have $I_{xy} = -\int xy\,dm$, some old books define $I_{xy} = \int xy\,dm$. They then have minus signs in front of the off-diagonal terms in the moment of inertia matrix. They would say $I_{12} = -I_{xy}$. The numerical values in the matrix they write is the same as in the one we write. They just have a different sign convention in the definition of the components.

$$[\boldsymbol{I}] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \tag{20.33}$$

$$= \begin{bmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{bmatrix} \tag{20.34}$$

If all mass is on the xy plane then it is clear that $I_{xz} = I_{yz} = 0$ since $z = 0$ for the whole xy plane. If rotation is also about the z-axis then $\vec{\omega} = \omega\hat{\boldsymbol{k}}$.

Applying the formulas above we find that, if all of the mass is in the xy-plane and rotation is about the z-axis, the only relevant non-zero term in $[\boldsymbol{I}]$ is $I_{zz} = \int (x^2+y^2)\,dm$. And, I_{xx}, I_{yy}, and I_{xy} don't contribute to $\vec{\boldsymbol{H}}$, $\dot{\vec{\boldsymbol{H}}}$, or E_{K}. In this manner you can check that the three dimensional equations, when applied to two-dimensional bodies, give the same results that we found directly for two-dimensional bodies.

Example: Moment of inertia matrix for a uniform sphere

A sphere is a special shape which is, naturally enough, spherically symmetric. Therefore,

$$I_{xx}^{cm} = I_{yy}^{cm} = I_{zz}^{cm}$$

and

$$I_{xy}^{cm} = I_{xz}^{cm} = I_{yz}^{cm} = 0.$$

So, all we need is I_{xx}^{cm} or I_{yy}^{cm} or I_{zz}^{cm}. Here is the trick:

$$\begin{aligned}
I_{xx}^{cm} &= \frac{1}{3}(I_{xx}^{cm} + I_{yy}^{cm} + I_{zz}^{cm}) \\
&= \frac{1}{3}\left[\int (y^2+z^2)dm + \int (x^2+z^2)dm + \int (x^2+y^2)dm\right] \\
&= \frac{2}{3}\int (x^2+y^2+z^2)dm \\
&= \frac{2}{3}\int r^2\,dm \\
&= \frac{2}{3}\int_0^R r^2(4\rho\pi r^2\,dr) \\
&= \frac{8}{3}\rho\pi\int_0^R r^4\,dr \\
&= \frac{8}{15}\rho\pi R^5 \\
&= \frac{2}{5}mR^2 \qquad (m = \frac{4}{3}\rho\pi R^3).
\end{aligned}$$

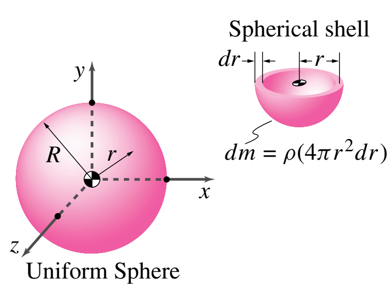


Figure 20.31

So,

$$[\boldsymbol{I}^{\mathrm{cm}}] = \frac{2}{5}mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The parallel axis theorem for rigid bodies in three dimensions

The 3-D parallel axis theorem is stated below and in the table on the inside back cover. It is derived in box 20.2 on page 9. The parallel axis theorem for rigid bodies in three dimensions is the equation

$$[\boldsymbol{I}^{\mathrm{O}}] = [\boldsymbol{I}^{\mathrm{cm}}] + m \begin{bmatrix} y_{cm/o}^2 + z_{cm/o}^2 & -x_{cm/o}y_{cm/o} & -x_{cm/o}z_{cm/o} \\ -x_{cm/o}y_{cm/o} & x_{cm/o}^2 + z_{cm/o}^2 & -y_{cm/o}z_{cm/o} \\ -x_{cm/o}z_{cm/o} & -y_{cm/o}z_{cm/o} & x_{cm/o}^2 + y_{cm/o}^2 \end{bmatrix} \tag{20.35}$$

In this equation, $x_{cm/o}$, $y_{cm/o}$, and $z_{cm/o}$ are the x, y, and z coordinates, respectively, of the center-of-mass defined with respect to a coordinate system whose origin is located at some point O not at the center-of-mass cm. That is, if you know $[\boldsymbol{I}^{\mathrm{cm}}]$, you can find $[\boldsymbol{I}^{\mathrm{O}}]$ without doing any more integrals or sums. Like the 2-D parallel axis theorem, the primary utility of the 3-D parallel axis theorem is for the determination of $[\mathbf{I}]$ for an object that is a composite of simpler objects. Such are not beyond the scope of this book in principle. But in fact, given the finite time available for calculation, we do not leave much time for practice of this tedious but routine calculation.

Matrices and tensors

We have just introduced the 3 by 3 moment of inertia matrix $[\boldsymbol{I}]$. We will find it in expressions having to do with angular momentum sitting next to either a vector $\vec{\boldsymbol{\omega}}$ or a vector $\vec{\boldsymbol{\alpha}}$: $\quad [\boldsymbol{I}]\cdot\vec{\boldsymbol{\omega}} \quad$ or $\quad [\boldsymbol{I}]\cdot\vec{\boldsymbol{\alpha}}$. What we mean by this expression is the three element column vector that comes from matrix multiplication of the matrix $[\boldsymbol{I}]$ and the column vector for $\vec{\boldsymbol{\omega}}$, $\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$, an expression that only makes sense if everyone knows what bases are being used.

More formally, and usually only in more advanced treatments, people like to define a coordinate-free quantity called the tensor $\underline{\underline{\boldsymbol{I}}}$. Then we would have

$$\underline{\underline{\boldsymbol{I}}}\cdot\vec{\boldsymbol{\omega}}$$

by which we would mean the vector whose components would be found by $[\boldsymbol{I}] \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$.

Eigenvectors and Eigenvalues

A square matrix $[\boldsymbol{A}]$ when multiplied by a column vector $[\boldsymbol{v}]$ yields a new vector $[\boldsymbol{w}]$. A given matrix has a few special vectors, somehow characteristic of that matrix, called *eigenvectors*. The vector $\vec{\boldsymbol{v}}$ is an eigenvector of $[\boldsymbol{A}]$ if

$$[\boldsymbol{A}] \cdot [\boldsymbol{v}] \text{ is parallel to } [\boldsymbol{v}].$$

In other words, if

$$[\boldsymbol{A}] \cdot [\boldsymbol{v}] = \lambda \, [\boldsymbol{v}]$$

for some λ.

The scalar λ is called the *eigenvalue* associated with the eigenvector $[\boldsymbol{v}]$ of the matrix $[\boldsymbol{A}]$. The eigen-values and eigen-vectors of a matrix are found with a single command in many computer math programs. In statics you had little or no use for eigen-values and eigen-vectors. In dynamics, eigenvectors and eigenvalues are useful for understanding dynamic balance, 3-D rigid body rotations, and normal mode vibrations.

Eigenvectors of $[\boldsymbol{I}]$

The moment of inertia matrix is always a symmetric matrix. This symmetry means that $[\boldsymbol{I}]$ always has a set of three mutually orthogonal eigenvectors. The importance of the eigenvectors of $[\boldsymbol{I}]$ will be discussed in section 20.5 on dynamic balance. Sometimes a pair of the eigenvalues are equal to each other implying that any vector in the plane of the corresponding eigenvectors is also an eigenvector.

If the physical object has any natural symmetry directions these directions will usually manifest themselves in the dynamics of the body as being in the directions of the eigenvectors of the object's moment of inertia matrix. For example, the dotted lines on fig. 2.20 are all in directions of eigenvectors for the objects shown. But even if an object is wildly asymmetric in shape, its moment of inertia matrix is always symmetric and thus all objects have moment of inertia matrices with at least three different eigenvectors at least three of which are mutually orthogonal.

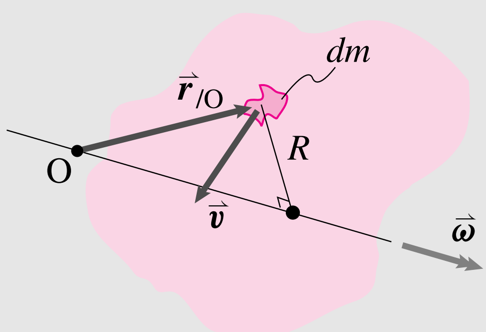
Properties of $[\boldsymbol{I}]$

For those with experience with linear algebra various properties of the moment of inertia matrix $[\boldsymbol{I}]$ are worth noting (although not worth proving here). Unless all mass is distributed on one straight line, the moment of inertia matrix is invertible (it is non-singular and has rank 3). Further, when invertible it is positive definite. In the special case that all the mass is on some straight line, the moment of inertia matrix is non-invertible and only positive semi-definite. The positive (semi) definiteness of the moment of inertia matrix is equivalent to the statement that the rotational kinetic energy of a body is always equal to or greater than zero. Finally, the eigenvalues of the moment of inertia matrix are all positive and have the property that no one can be greater than the sum of the other two (the same inequalities are satisfied by the lengths of the sides of a triangle, the "triangle inequality").

20.1 Discovering the moment of inertia matrix

Derivation 1

Here we present a direct derivation of the moment of inertia matrix; that is, a derivation in which the moment of inertia matrix arises as a convenient short hand. Assume a rigid body is moving in such a way that point O is fixed (i.e., it is either on the line of a hinge or a ball-and-socket joint).



The most basic kinematic relation for a rigid body is that

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where $\vec{r}_{/O} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is the position of a point on the body relative to O and $\vec{\omega}$, the angular velocity of the body.

Now, we tediously calculate and arrange the terms in the angular momentum about point O,

$$\begin{aligned}
\vec{H}_O &= \int \vec{r}_{/O} \times \vec{v} dm \\
&= \int \vec{r}_{/O} \times (\vec{\omega} \times \vec{r}_{/O}) dm \\
&= \int (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \times \\
&\quad \left[(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}) \times (x\hat{\imath} + y\hat{\jmath} + z\hat{k})\right] dm \\
&= \int (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \times \\
&\quad \left[(\omega_y z - \omega_z y)\hat{\imath} + (\omega_z x - \omega_x z)\hat{\jmath} + (\omega_x y - \omega_y x)\hat{k}\right] dm \\
&= \int \Big[\Big(y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z)\Big)\hat{\imath} \\
&\qquad + \Big(z(\omega_y z - \omega_z y) - x(\omega_x y - \omega_y x)\Big)\hat{\jmath} \\
&\qquad + \Big(x(\omega_z x - \omega_x z) - y(\omega_y z - \omega_z y)\Big)\hat{k}\Big] dm \\
&= \int \Big[\Big((y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z\Big)\hat{\imath} \\
&\qquad + \Big(-yx\omega_x + (z^2 + x^2)\omega_y - yz\omega_z\Big)\hat{\jmath} \\
&\qquad + \Big(-zx\omega_x - xy\omega_y + (x^2 + y^2)\omega_z\Big)\hat{k}\Big] dm
\end{aligned}$$

Since the integral is over the mass and $\vec{\omega}$ is constant over the body, we can pull $\vec{\omega}$ out of the integral so that we may write the equation in matrix form. Writing $\vec{H}_O$ as a column vector, we can rewrite the last equation as

$$\begin{bmatrix} H_{O_x} \\ H_{O_y} \\ H_{O_z} \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} \int (y^2 + z^2)\, dm & -\int xy\, dm & -\int xz\, dm \\ -\int xy\, dm & \int (x^2 + z^2)\, dm & -\int yz\, dm \\ -\int xz\, dm & -\int yz\, dm & \int (x^2 + y^2)\, dm \end{bmatrix}}_{[\boldsymbol{I}^O]} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

Finally, defining $[\boldsymbol{I}^O]$ by the matrix above, we can compactly write

$$\vec{H}_{/O} = [\boldsymbol{I}^O] \cdot \vec{\omega}$$

assuming O is a fixed point on the body where we represent $\vec{H}_{/O}$ and $\vec{\omega}$ in terms of x, y, and z components.

Center-of-mass inertia matrix

For any system moving, distorting, and rotating any crazy way, we have the general result that

$$\vec{H}_{/O} = \overbrace{\vec{r}_{cm/O} \times m_{tot} \vec{v}_{cm}}^{\text{Contribution of the system to } \vec{H}_{/O} \text{ if treated as a particle at the system center-of-mass}} + \overbrace{\int (\vec{r}_{/cm} \times \underbrace{\vec{v}_{/cm}}_{\boxed{\vec{v}_{/cm} = (\vec{v} - \vec{v}_{cm})}}) dm}^{\vec{H}_{cm}}$$

as you can verify by substituting $\vec{v} = \vec{v}_{cm} + \vec{v}_{/cm}$ and $\vec{r} = \vec{r}_{cm} + \vec{r}_{/cm}$ into the general definition of $\vec{H}_{/O} = \int \vec{r}_{/O} \times \vec{v} dm$. For a rigid body, we have

$$\vec{v}_{/cm} = \vec{\omega} \times \vec{r}_{/cm}.$$

So, by a derivation essentially identical to that for $\vec{H}_{/O}$, we get

$$\vec{H}_{cm} = [\boldsymbol{I}^{cm}] \cdot \vec{\omega}$$

with $[\boldsymbol{I}^{cm}]$ being defined using x, y, and z, as the distances from the center of mass rather than from point O. So, for a rigid body in general motion, we can find the angular momentum by

$$\vec{H}_{/O} = \vec{r}_{cm/O} \times \vec{v}_{cm} m_{tot} + [\boldsymbol{I}^{cm}] \cdot \vec{\omega} \qquad (20.36)$$

(continued...)

20.1 Discovering the moment of inertia matrix (continued)

Comment (aside)

In the special case that the body is rotating about point O, we also have

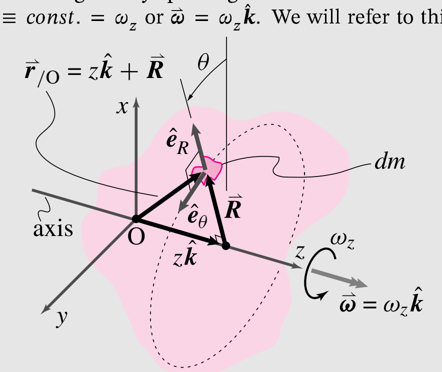
$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}}. \tag{20.37}$$

You will see, if you look at the parallel axis theorem, that these two expressions 20.36 and 20.37 do in fact agree.

Derivation 2

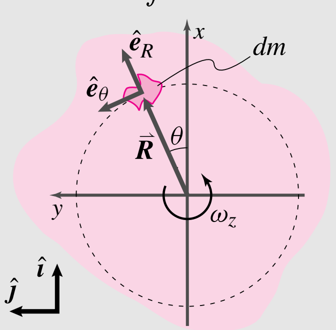
Here, we present a less direct but perhaps more intuitive derivation of the moment of inertia matrix. We start with the special case of a 3-D rigid body spinning in circles at constant rate about a fixed axis.

To see from where the moment of inertia matrix comes, we will first calculate the angular momentum about point O of a general 3-D rigid body spinning about the z-axis with constant rate $\dot\theta \equiv const. = \omega_z$ or $\vec{\boldsymbol{\omega}} = \omega_z \hat{\boldsymbol{k}}$. We will refer to this case as (1).



Starting with the definition of angular momentum, we get

$$\begin{aligned}\vec{\boldsymbol{H}}_O &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}} dm \\ &= \int (z\hat{\boldsymbol{k}} + \vec{\boldsymbol{R}}) \times (\dot\theta R \hat{\boldsymbol{e}}_\theta) dm.\end{aligned}$$



Looking down the z-axis in the figure, we see that

$$\begin{aligned}\vec{\boldsymbol{R}} &= R\hat{\boldsymbol{e}}_R = \overbrace{R\cos\theta}^{x}\,\hat{\boldsymbol{\imath}} + \overbrace{R\sin\theta}^{y}\,\hat{\boldsymbol{\jmath}}, \quad \text{or} \\ \vec{\boldsymbol{R}} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}.\end{aligned}$$

where $R = \sqrt{x^2 + y^2}$. To compute the cross product in the integrand, we need

$$\begin{aligned}\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_\theta &= \hat{\boldsymbol{k}} \times (-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) = -\hat{\boldsymbol{e}}_R \quad \text{and} \\ \vec{\boldsymbol{R}} \times \hat{\boldsymbol{e}}_\theta &= R\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = R\hat{\boldsymbol{k}}.\end{aligned}$$

Therefore, we now have

$$\begin{aligned}\vec{\boldsymbol{H}}_O &= \textstyle\int -z\dot\theta R\hat{\boldsymbol{e}}_R dm + \int \dot\theta R^2 \hat{\boldsymbol{k}} dm \\ &= \dot\theta \left[-\textstyle\int z(\underbrace{R\cos\theta}_{x}\,\hat{\boldsymbol{\imath}} + \underbrace{R\sin\theta}_{y}\,\hat{\boldsymbol{\jmath}})dm + \int (x^2+y^2)\hat{\boldsymbol{k}} dm \right] \\ &= \dot\theta \left[(-\textstyle\int zx\, dm)\hat{\boldsymbol{\imath}} + (-\int zy\, dm)\hat{\boldsymbol{\jmath}} + \left(\int (x^2+y^2) dm\right)\hat{\boldsymbol{k}} \right].\end{aligned}$$

To 'un-clutter' this expression, let's define the following:

$$\begin{aligned}I^O_{xz} &= -\int xz\, dm \\ I^O_{yz} &= -\int yz\, dm \\ I^O_{zz} &= \int (x^2+y^2)\, dm\end{aligned}$$

So, now, we have for case (1)

$$(\vec{\boldsymbol{H}}_O)_1 = I^O_{xz}\omega_z \hat{\boldsymbol{\imath}} + I^O_{yz}\omega_z \hat{\boldsymbol{\jmath}} + I^O_{zz}\omega_z \hat{\boldsymbol{k}}.$$

The substitutions we have defined form the elements of the third column of the inertia matrix, as we will see below in the general case. Let's now move on to general 3-D rigid body motion and infer the first and second columns of the inertia matrix.

In general, the angular velocity of a rigid body is given by

$$\vec{\boldsymbol{\omega}} = \omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{k}}$$

So far, we have considered the special case above, $\vec{\boldsymbol{\omega}} = \omega_z \hat{\boldsymbol{k}}$. But, we could have looked at $\vec{\boldsymbol{\omega}} = \omega_x \hat{\boldsymbol{\imath}}$, case (2), and, similarly, would obtain instead the following angular momentum about point O

$$(\vec{\boldsymbol{H}}_O)_2 = I^O_{xx}\omega_x \hat{\boldsymbol{\imath}} + I^O_{yx}\omega_x \hat{\boldsymbol{\jmath}} + I^O_{zx}\omega_x \hat{\boldsymbol{k}}.$$

Likewise, for $\vec{\boldsymbol{\omega}} = \omega_y \hat{\boldsymbol{\jmath}}$, case (3), we would obtain

$$(\vec{\boldsymbol{H}}_O)_3 = I^O_{xy}\omega_y \hat{\boldsymbol{\imath}} + I^O_{yy}\omega_y \hat{\boldsymbol{\jmath}} + I^O_{zy}\omega_y \hat{\boldsymbol{k}}.$$

Finally, for $\vec{\boldsymbol{\omega}} = \omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{k}}$, we obtain

$$\begin{aligned}\vec{\boldsymbol{H}}_O &= (\vec{\boldsymbol{H}}_O)_1 + (\vec{\boldsymbol{H}}_O)_2 + (\vec{\boldsymbol{H}}_O)_3 \\ &= I^O_{xz}\omega_z \hat{\boldsymbol{\imath}} + I^O_{yz}\omega_z \hat{\boldsymbol{\jmath}} + I^O_{zz}\omega_z \hat{\boldsymbol{k}} \\ &\quad + I^O_{xx}\omega_x \hat{\boldsymbol{\imath}} + I^O_{yx}\omega_x \hat{\boldsymbol{\jmath}} + I^O_{zx}\omega_x \hat{\boldsymbol{k}} \\ &\quad + I^O_{xy}\omega_y \hat{\boldsymbol{\imath}} + I^O_{yy}\omega_y \hat{\boldsymbol{\jmath}} + I^O_{zy}\omega_y \hat{\boldsymbol{k}}.\end{aligned}$$

Collecting components, we get

$$\vec{\boldsymbol{H}}_O = H_{O_x}\hat{\boldsymbol{\imath}} + H_{O_y}\hat{\boldsymbol{\jmath}} + H_{O_z}\hat{\boldsymbol{k}}$$

where

$$\begin{aligned}H_{O_x} &= I^O_{xx}\omega_x + I^O_{xy}\omega_y + I^O_{xz}\omega_z \\ H_{O_y} &= I^O_{yx}\omega_x + I^O_{yy}\omega_y + I^O_{yz}\omega_z \\ H_{O_z} &= I^O_{zx}\omega_x + I^O_{zy}\omega_y + I^O_{zz}\omega_z.\end{aligned}$$

(continued...)

20.1 Discovering the moment of inertia matrix (continued)

We can combine the above results into a matrix representation. Representing $\vec{\boldsymbol{H}}_O$ as a column vector, we can re-write the above set of three equations as a product of a matrix and the angular velocity written as a column vector.

$$\begin{bmatrix} H_{O_x} \\ H_{O_y} \\ H_{O_z} \end{bmatrix} = \begin{bmatrix} I^O_{xx} & I^O_{xy} & I^O_{xz} \\ I^O_{yx} & I^O_{yy} & I^O_{yz} \\ I^O_{zx} & I^O_{zy} & I^O_{zz} \end{bmatrix} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

We define the coefficient matrix above to be the *moment of inertia matrix* about point O

$$[\boldsymbol{I}^{\mathrm{O}}] = \begin{bmatrix} I^O_{xx} & I^O_{xy} & I^O_{xz} \\ I^O_{yx} & I^O_{yy} & I^O_{yz} \\ I^O_{zx} & I^O_{zy} & I^O_{zz} \end{bmatrix}.$$

whose components are

$$\begin{array}{lll} I^O_{xx} = \int(y^2+z^2)dm & I^O_{xy} = -\int xy\,dm & I^O_{xz} = -\int xz\,dm \\ I^O_{yx} = -\int yx\,dm & I^O_{yy} = \int(x^2+z^2)dm & I^O_{yz} = -\int yz\,dm \\ I^O_{zx} = -\int zx\,dm & I^O_{zy} = -\int zy\,dm & I^O_{zz} = \int(x^2+y^2)dm. \end{array}$$

By inspection, one can see that $I^O_{xy} = I^O_{yx}$, $I^O_{xz} = I^O_{zx}$, and $I^O_{yz} = I^O_{zy}$. Thus, the inertia matrix is symmetric; i.e., $[\boldsymbol{I}^{\mathrm{O}}]^T = [\boldsymbol{I}^{\mathrm{O}}]$. So, there are always at most only six, not nine, independent components in the inertia matrix to compute.

20.2 3-D parallel axis theorem

In three dimensions, the two matrices $[\boldsymbol{I}^{\mathrm{O}}]$ and $[\boldsymbol{I}^{\mathrm{cm}}]$ are related to each other in a way similar to the two-dimensional case. Since the inertia matrix has six independent entries in it, the derivation involves six integrals. Let's look at a typical term on the diagonal, say, I_{zz}^{O} and a typical off-diagonal term, say, I_{xy}^{O}. The calculation for the other terms is similar with a simple change of letters in the subscript notation.

First, $I_{zz}^{O} = \int (x_{/O}^2 + y_{/O}^2) dm = I_{zz}^{cm} + m(x_{cm/O}^2 + y_{cm/O}^2)$ by exactly the same reasoning used to derive the 2-D parallel axis theorem. We *cannot* do the last line in that derivation, however, since $r_{cm/O}^2 = x_{cm/O}^2 + y_{cm/O}^2 + z_{cm/O}^2 \neq x_{cm/O}^2 + y_{cm/O}^2$, because now, for three-dimensional objects, $z_{cm/O} \neq 0$.

Now, let's look at an off-diagonal term.

$$
\begin{aligned}
I_{xy}^{O} &= -\int x_{/O} y_{/O} dm \\
&= -\int \overbrace{(x_{cm/O} + x_{/cm})}^{x_{/O}} \overbrace{(y_{cm/O} + y_{/cm})}^{y_{/O}} dm \\
&= -x_{cm/O} y_{cm/O} \underbrace{\int dm}_{m} - y_{cm/O} \underbrace{\int x_{/cm} dm}_{0} \\
&\quad - x_{cm/O} \underbrace{\int y_{/cm} dm}_{0} - \underbrace{\int x_{/cm} y_{/cm} dm}_{-I_{xy}^{cm}} \\
&= -m x_{cm/O} y_{cm/O} + I_{xy}^{cm}
\end{aligned}
$$

Similarly, we can calculate the other terms to get the whole 3-D parallel axis theorem.

$$
[\boldsymbol{I}^{\mathrm{O}}] = [\boldsymbol{I}^{\mathrm{cm}}] +
m \begin{bmatrix}
y_{cm/o}^2 + z_{cm/o}^2 & -x_{cm/o} y_{cm/o} & -x_{cm/o} z_{cm/o} \\
-x_{cm/o} y_{cm/o} & x_{cm/o}^2 + z_{cm/o}^2 & -y_{cm/o} z_{cm/o} \\
-x_{cm/o} z_{cm/o} & -y_{cm/o} z_{cm/o} & x_{cm/o}^2 + y_{cm/o}^2
\end{bmatrix}.
$$

Again, one can think of this result as follows. The moment of inertia matrix about point O is the same as that for parallel axes through the center-of-mass *plus* the moment of inertia matrix for a point mass at the center-of-mass.

Relation between 2-D and 3-D parallel axis theorems

The (3,3) (lower right corner) element in the matrix of the 3-D parallel axis theorem is the 2-D parallel axis theorem.