SAMPLE 20.9 For the dumbbell shown in Figure 20.32, take $m = 0.5\,\text{kg}$ and $\ell = 0.4\,\text{m}$. Given that at the instant shown $\theta = 30^\circ$ and the dumbbell is in the yz-plane, find the moment of inertia matrix $[\boldsymbol{I}^{\text{O}}]$, where O is the midpoint of the dumbbell.

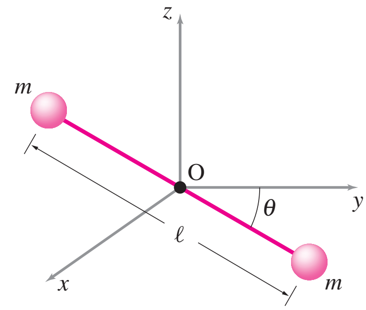

Figure 20.32

Solution The dumbbell is made up of two point masses. Therefore we can calculate $[\boldsymbol{I}^{\text{O}}]$ for each mass using the formula from the table on the inside back cover of the text and then adding the two matrices to get $[\boldsymbol{I}^{\text{O}}]$ for the dumbbell.
Now, from Table 4.9 of the text,

$$[\boldsymbol{I}^{\text{O}}] = m \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

For mass 1 (shown in Figure 20.33)

$$x = 0, \quad y = -\frac{\ell}{2}\cos\theta, \quad z = \frac{\ell}{2}\sin\theta.$$

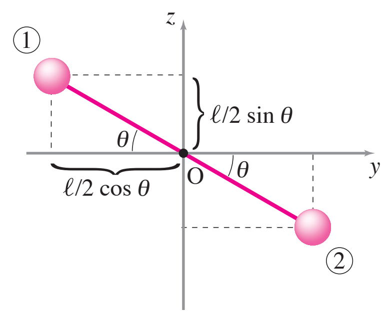

Figure 20.33

Therefore,

$$\begin{aligned}
[\boldsymbol{I}^{\text{O}}]_{\text{mass1}} &= m \begin{bmatrix} \frac{\ell^2}{4} & 0 & 0 \\ 0 & \frac{\ell^2}{4}\sin^2\theta & \frac{\ell^2}{4}\cos\theta\sin\theta \\ 0 & \frac{\ell^2}{4}\cos\theta\sin\theta & \frac{\ell^2}{4}\cos^2\theta \end{bmatrix} \\
&= 0.5\,\text{kg} \begin{bmatrix} 0.04\,\text{m}^2 & 0 & 0 \\ 0 & 0.04\,\text{m}^2\cdot\frac{1}{4} & 0.04\,\text{m}^2\cdot\frac{\sqrt{3}}{4} \\ 0 & 0.04\,\text{m}^2\cdot\frac{\sqrt{3}}{4} & 0.04\,\text{m}^2\cdot\frac{3}{4} \end{bmatrix} \\
&= 0.02\,\text{kg}\cdot\text{m}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}
\end{aligned}$$

Similarly for mass 2,

$$x = 0, \quad y = \frac{\ell}{2}\cos\theta, \quad z = -\frac{\ell}{2}\sin\theta$$

$$\Rightarrow \quad [\boldsymbol{I}^{\text{O}}]_{\text{mass2}} = 0.02\,\text{kg}\cdot\text{m}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
[\boldsymbol{I}^{\text{O}}] &= [\boldsymbol{I}^{\text{O}}]_{\text{mass1}} + [\boldsymbol{I}^{\text{O}}]_{\text{mass2}} \\
&= 0.04\,\text{kg}\cdot\text{m}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}.
\end{aligned}$$

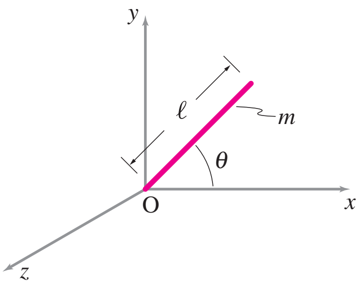

Figure 20.34

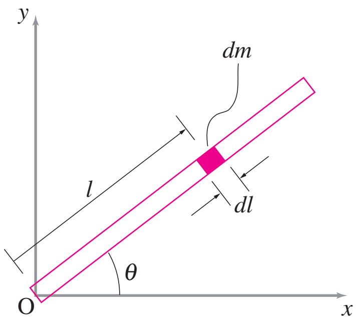

Figure 20.35

SAMPLE 20.10 A uniform rod of mass $m = 2\,\text{kg}$ and length $\ell = \frac{1}{2}\,\text{m}$ is pivoted at one of its ends. At the instant shown, the rod is in the xy-plane and makes an angle $\theta = 45^\circ$ with the x-axis. Find the moment of inertia matrix $[\boldsymbol{I}^{\text{O}}]$ for the rod.

Solution The moment of inertia matrix $[\boldsymbol{I}^{\text{O}}]$ for a continuous system is given by

$$[\boldsymbol{I}^{\text{O}}] = \int_{\text{over all mass}} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm.$$

Thus we need to carry out the integrals for the rod to find each component of the inertia matrix $[\boldsymbol{I}^{\text{O}}]$. Let us consider an infinitesimal length element dl of the rod at distance l from O (see Fig 20.35). The mass of this element is $dm = \underbrace{\frac{m}{\ell}}_{\text{mass/length}} \cdot dl$, where m is the total mass.

The coordinates of this element are

$$x = l\cos\theta, \quad y = l\sin\theta, \quad z = 0 \quad \text{(since the rod is in the } xy\text{-plane).}$$

Therefore,

$$\begin{aligned} I_{xx} &= \int_m (y^2 + \underbrace{z^2}_{0})dm = \int_0^\ell \underbrace{l^2\sin^2\theta}_{y^2} \cdot \underbrace{\frac{m}{\ell}dl}_{dm} \\ &= \frac{m}{\ell}\sin^2\theta \int_0^\ell l^2 dl = \frac{m}{\ell}\sin^2\theta \cdot \frac{\ell^3}{3} = \frac{m\ell^2}{3}\sin^2\theta. \end{aligned}$$

Similarly,

$$\begin{aligned} I_{yy} &= \int_m (x^2 + \underbrace{z^2}_{0})dm = \int_0^\ell l^2\cos^2\theta \frac{m}{\ell}dl = \frac{m\ell^2}{3}\cos^2\theta. \\ I_{zz} &= \int_m (x^2+y^2)dm = \int_0^\ell l^2 \frac{m}{\ell}dl = \frac{m\ell^2}{3}. \\ I_{xy} &= -\int_m xy\,dm = -\int_0^\ell l^2\cos\theta\sin\theta \cdot \frac{m}{\ell}dl = -\frac{m\ell^2}{3}\cos\theta\sin\theta. \\ I_{xz} &= -\int_m x\underbrace{z}_{0}\,dm = 0. \quad I_{yz} = -\int_m y\underbrace{z}_{0}\,dm = 0. \end{aligned}$$

Thus,

$$[\boldsymbol{I}^{\text{O}}] = \frac{m\ell^2}{3}\begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting the values of m, ℓ and θ, we get

$$[\boldsymbol{I}^{\text{O}}] = 0.083\,\text{kg}\cdot\text{m}^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$