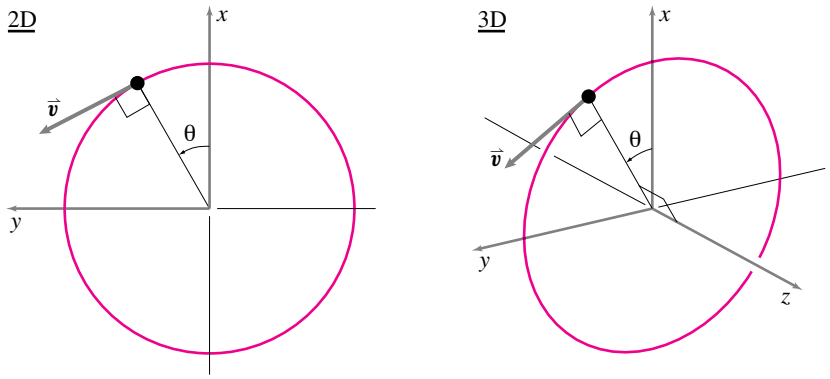
20.1 3-D description of circular motion

Let's first assume each particle is going in circles around the z axis, as in the previous chapter. The figure below shows this two-dimensional situation first two-dimensionally (left) and then as a two-dimensional motion in a three-dimensional world (right).



Either way, the velocity and acceleration are the same:

- the velocity is tangent to the circle it is going around and is proportional in magnitude to the radius of the circle and also to its angular speed. That is, the direction of the velocity is in the direction $\hat{\boldsymbol{e}}_\theta$ and has magnitude ωR, where ω is the angular rate of rotation and R is the radius of the circle that the particle is going around.
- the acceleration can be constructed as the sum of two vectors. One is pointed to the center of the circle and proportional in magnitude to both the square of the angular speed and to the radius. The other vector is tangent to the circle and equal in magnitude to the rate of increase of speed.

These two ideas are summarized by the following formulas:

$$\vec{\boldsymbol{v}} = \omega R \hat{\boldsymbol{e}}_\theta \qquad \text{and} \qquad \vec{\boldsymbol{a}} = -\underbrace{\omega^2 R}_{v^2/R}\,\hat{\boldsymbol{e}}_R + \underbrace{R\ddot{\theta}}_{\dot{v}}\,\hat{\boldsymbol{e}}_\theta \tag{20.1}$$

with

$$v = \omega R. \tag{20.2}$$

The axis of rotation might not be the z-axis of a convenient xyz coordinate system. So the xy plane of circles might not be the xy plane of the coordinate system you might want to use for some other reasons. Fortunately, we can write the formulas 20.1 in a way that rids us of these problems.

Here are some formulas which are equivalent to the formulas 20.1 but which do not make use of the polar coordinate base vectors.

$$\vec{\boldsymbol{v}} = \vec{\omega} \times \vec{\boldsymbol{r}}, \tag{20.3}$$

$$\vec{\boldsymbol{a}} = \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) + \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}. \tag{20.4}$$

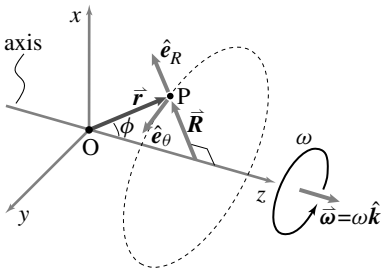


Figure 20.2: When a point P is going in circles about the z axis, we define the unit vector $\hat{\boldsymbol{e}}_R$ to be pointed from the axis to the point P. We define the unit vector $\hat{\boldsymbol{e}}_\theta$ to be tangent to the circle at P. Both of these vectors change in time as the point moves along its circular path.

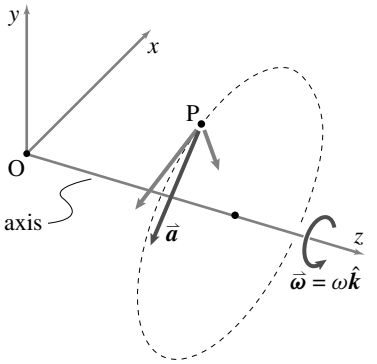


Figure 20.3: The acceleration is the sum of two components. One directed towards the center of the circle in the $-\hat{\boldsymbol{e}}_R$ direction, and one tangent to the circle in the $\hat{\boldsymbol{e}}_\theta$ direction.

To check that equations 20.3 and 20.4 are really equivalent to 20.1 we need to verify that the vector $\vec{\omega}\times\vec{r}$ is equal to $\omega R\hat{e}_\theta$, that the vector $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is equal to the vector $-\omega^2 R\hat{e}_R$, and that $\dot{\vec{\omega}}\times\vec{r}$ is equivalent to $R\ddot{\theta}\hat{e}_\theta$.

First, define $\vec{r}$ as the position of the point of interest relative to any point on the axis of rotation. If this point happens to be the center of the circle then $\vec{r}=\vec{R}$. But, in general, $\vec{r}\neq\vec{R}$.

Now look at $\vec{v}=\vec{\omega}\times\vec{r}$ with respect to fig. 20.2. Using the right hand rule, it is clear that the direction of the cross product $\vec{\omega}\times\vec{r}$ is in fact the $\hat{e}_\theta$ direction. What about the magnitude? The magnitude $|\vec{\omega}\times\vec{r}| = |\vec{\omega}||\vec{r}|\sin\phi$. But $|\vec{r}|\sin\phi = R$. So the magnitude of $\vec{\omega}\times\vec{r}$ is ωR. That is,

$$\begin{aligned}\vec{\omega}\times\vec{r} &= (|\vec{\omega}\times\vec{r}|)\cdot(\text{unit vector in the direction of }\vec{\omega}\times\vec{r}) && (20.5)\\ &= (\omega R)\hat{e}_\theta && (20.6)\\ &= \vec{v} && (20.7)\end{aligned}$$

So $\vec{v}=\vec{\omega}\times\vec{r}$ is correct. The check of the second term in the acceleration formula follows the same reasoning. But the check of the first term involves the triple cross product.

Triple cross product

The formula for acceleration of a point on a rigid body includes the centripetal term $\vec{\omega}\times(\vec{\omega}\times\vec{r})$. This expression is a special case of the general vector expression

$$\vec{A}\times(\vec{B}\times\vec{C})$$

which is sometimes called the 'vector triple product' because its value is a vector (as opposed to the scalar value of the 'scalar triple product'). The primary useful identity with vector triple products (see (17.29)) is:

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}. \qquad (20.8)$$

This formula may be remembered by the semi-mnemonic device 'cab minus bac' since $\vec{A}\cdot\vec{C}=\vec{C}\cdot\vec{A}$ and $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$. This formula is discussed in box 17.4 on page 865.

So now we can write

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{\omega}\times(\omega R)\cdot\hat{e}_\theta = -\omega^2\underbrace{R\hat{e}_R}_{\vec{R}} = -\omega^2\vec{R} = \vec{a}. \qquad (20.9)$$

Note that equations 20.3 and 20.4 are *vector* equations. They do not make use of any coordinate system. So, for example, we can use them even if $\vec{\omega}$ is not in the z direction.

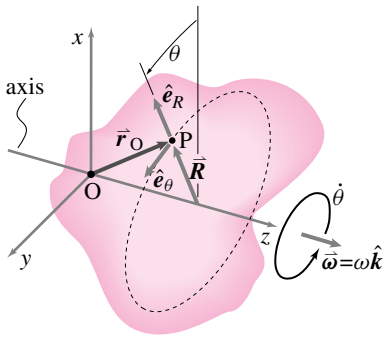


Figure 20.4: A rigid body spinning about the z axis. Every point on the body, like point P at $\vec{r}$, is going in circles. All of these circles have centers on the axis of rotation. All the points are going around at the same angular rate, $\dot{\theta}=\omega$.

Angular velocity of a rigid body in 3D

If a rigid body is constrained to rotate about an axis then all points on the body have the same angular rate about that axis. Hence one says that the

body has an angular velocity. So the measure of rotation rate of a three-dimensional rigid body is the body's angular velocity vector $\vec{\omega}$. At any instant in time a given body has one and only one angular velocity $\vec{\omega}$. Although we only discuss fixed axis rotation in this chapter, a given body has a unique angular velocity for general motion.

For rotations about the z-axis, $\vec{\omega} = \omega\hat{k}$. ω is the $\dot{\theta}$ shown in fig. 20.4. Since all points of a rigid body have the same $\dot{\theta}$, even if they have different θ's, the definition is not ambiguous. We would like to make this idea precise enough to be useful for calculations. Why, one may ask, do we talk about rotation rate ω or $\vec{\omega}$ instead of just using the derivative of an angle θ, namely $\dot{\theta}$? The answer is that for a rigid body one would have trouble deciding what angle θ to measure.

First recall the situation for a two-dimensional rigid body. Consider all possible θ_1, θ_2, θ_3, ... , the angles that all possible lines marked on a body could make with the positive x-axis, the positive y axis, or any other fixed line that does not rotate. As the body rotates all of these angles increment by the same amount. Therefore, each of these angles increases at the same rate. Because all these angular rates are the same, one need not define $\dot{\theta}_1 = \omega_1, \dot{\theta}_2 = \omega_2, \dot{\theta}_3 = \omega_3$, etc. for each of the lines. Every line attached to the body rotates at the same rate and we call this rate ω. So $\dot{\theta}_1 = \omega$, $\dot{\theta} = \omega, \dot{\theta}_3 = \omega$, etc. Rather than say the lengthy phrase 'the rate of rotation of every line attached to the rigid body is ω', we instead say 'the rigid body has angular velocity ω'. For use in vector equations, we define the angular velocity vector of a two-dimensional rigid body as $\vec{\omega} = \omega\hat{k}$ and for a 3-D body rotating about an axis in the $\hat{\lambda}$ direction as $\vec{\omega} = \omega\hat{\lambda}$.

What do we mean by these angles θ_i for crooked lines in a three-dimensional body? We simply look at shadows of lines drawn in or on the body of interest onto a plane perpendicular to the axis of rotation; *i.e.*, perpendicular to $\hat{\lambda}$. See fig. 20.5. The rate of change of their orientation ($\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3$) is ω, and $\vec{\omega}$ is therefore $\omega\hat{\lambda}$. This intuitive geometric definition of ω in terms of the rotation of shadows has run its course. It gives you a picture but is not very convenient for developing formulas.

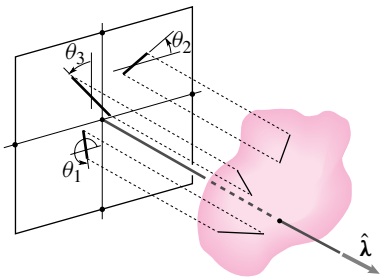


Figure 20.5: The shadows of lines marked in a 3-D rigid body are shown on a plane perpendicular to the axis of rotation. The shadows rotate on the plane at the rate $\dot{\theta}_1 = \dot{\theta}_2 = \omega$. The angular velocity vector is $\vec{\omega} = \omega\hat{\lambda}$.

Example: What are the velocity and acceleration of one corner of a cube that is spinning about a diagonal?

A one foot cube is spinning at 60 rpm about the diagonal OC. What are the velocity and acceleration of point B? First let's find the velocity using $\vec{v} = \vec{\omega} \times \vec{r}$:

$$\begin{aligned}
\vec{v} &= \vec{\omega} \times \vec{r} \\
&= (60\,\text{rpm}\,\hat{\lambda}_{OC}) \times \vec{r}_{OB} \\
&= \left(2\pi\,\text{s}^{-1}\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}\right) \times (1\,\text{ft}(\hat{\jmath}+\hat{k})) \\
&= (2\pi/\sqrt{3})(-\hat{\jmath}+\hat{k})\,\text{ft/s}.
\end{aligned}$$

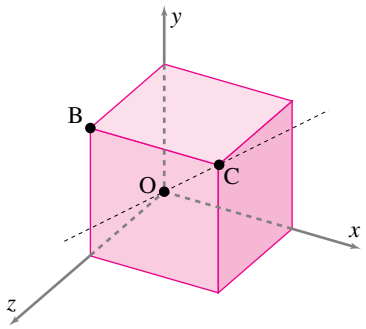


Figure 20.6: A spinning cube.

Now of course this equation could have been worked out with the first of equations 20.1 but it would have been quite tricky to find the vectors $\hat{e}_\theta$, $\vec{R}$, and $\hat{e}_R$! More simply, to find the acceleration we just plug in the formula $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$

as follows:

$$\begin{aligned}\vec{\boldsymbol{a}} &= \vec{\boldsymbol{\omega}}\times[\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}]\\ &= (60\,\text{rpm}\,\boldsymbol{\lambda}_{OC})\times[(60\,\text{rpm}\,\boldsymbol{\lambda}_{OC})\times\vec{\boldsymbol{r}}_{OB}]\\ &= (2\pi\,\text{s}^{-1}\frac{(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}})}{\sqrt{3}})\\ &\qquad\times\left[(\frac{2\pi\,\text{s}^{-1}(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}})}{\sqrt{3}})\times(1\,\text{ft}(\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}})\right]\\ &= (2\pi/\sqrt{3})^2(2\hat{\boldsymbol{\imath}}-\hat{\boldsymbol{\jmath}}-\hat{\boldsymbol{k}})\,\text{ft/s}^2.\end{aligned}$$

The last line of calculation is eased by the calculation of velocity above where the term in square brackets, the velocity, was already calculated.

Relative motion of points on a rigid body

The relative velocity of two points A and B is defined to be

$$\vec{\boldsymbol{v}}_{\text{B/A}} \equiv \vec{\boldsymbol{v}}_{\text{B}} - \vec{\boldsymbol{v}}_{\text{A}}$$

So, the relative velocity of two points glued to one rigid body, as observed from a Newtonian frame, is given by

$$\begin{aligned}\vec{\boldsymbol{v}}_{\text{B/A}} &\equiv \vec{\boldsymbol{v}}_{\text{B}} - \vec{\boldsymbol{v}}_{\text{A}} &(20.10)\\ &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\text{B/O}} - \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\text{A/O}} &(20.11)\\ &= \vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{r}}_{\text{B/O}} - \vec{\boldsymbol{r}}_{\text{A/O}}) &(20.12)\\ &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\text{B/A}}, &(20.13)\end{aligned}$$

where point O is a point in the Newtonian frame on the fixed axis of rotation. Clearly, since points A and B are fixed in the body $\mathcal{B}$ their velocities and hence their relative velocity as observed in a reference frame fixed to $\mathcal{B}$ is $\vec{\boldsymbol{0}}$. But, point A has some absolute velocity that is different from the absolute velocity of point B, as viewed from point O in the fixed frame. The relative velocity of points A and B, the difference in absolute velocity of the two points, is due to the difference in their positions relative to point O. Similarly, the relative acceleration of two points glued to one rigid body spinning about a fixed axis is

$$\vec{\boldsymbol{a}}_{\text{B/A}} \equiv \vec{\boldsymbol{a}}_{\text{B}} - \vec{\boldsymbol{a}}_{\text{A}} = \vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\text{B/A}}) + \dot{\vec{\boldsymbol{\omega}}}\times\vec{\boldsymbol{r}}_{\text{B/A}}. \qquad (20.14)$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point O fixed on the axis. Like their junior 2D cousins, these kinematics results, 20.13 and 20.14, are useful for calculating angular momentum relative to the center-of-mass as well as for the understanding of the motions of machines with moving connected parts.

To repeat, for two points on one rigid body we have that

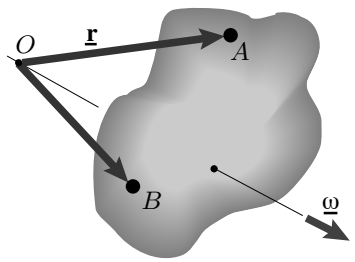


Figure 20.7: Two points, A and B on one body that has a fixed axis of rotation.

$$\dot{\vec{r}}_{B/A} = \vec{\omega} \times \vec{r}_{\mathrm{B/A}}. \tag{20.15}$$

Equation (20.15) is perhaps the most fundamental equation for those desiring an understanding of the motions of rigid bodies. Unless one desires to pursue matrix representations of rotation, equation (20.15) is *the* defining equation for $\vec{\omega}$. There is always exactly one vector $\vec{\omega}$ so that equation (20.15) is true for every pair of points on a rigid body.

Equation (20.15) is not so simple a defining equation as one would hope for such an intuitive concept as spinning. But, besides the pictorial definition with shadows, it's the simplest definition we have.

Relative velocity and acceleration using rotating frames

If we glued a coordinate system $x'y'$ to a rotating rigid body $\mathcal{C}$, we would have what is called a rotating frame as shown in fig. 20.8. The base vectors in this frame change in time the same way as did $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in section 15.1. That is

$$\frac{d}{dt}\hat{\boldsymbol{\imath}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}' \quad \text{and} \quad \frac{d}{dt}\hat{\boldsymbol{\jmath}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}'.$$

If we now write the relative position of B to A in terms of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, we have

$$\vec{r}_{\mathrm{B/A}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'.$$

Since the coordinates x' and y' rotate with the body to which A and B are attached, they are constant with respect to that body,

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

So

$$\begin{aligned}
\frac{d}{dt}(\vec{r}_{\mathrm{B/A}}) &= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right) \\
&= \underbrace{\dot{x}'}_{0}\hat{\boldsymbol{\imath}}' + x'\frac{d}{dt}\hat{\boldsymbol{\imath}}' + \underbrace{\dot{y}'}_{0}\hat{\boldsymbol{\jmath}}' + y'\frac{d}{dt}\hat{\boldsymbol{\jmath}}' \\
&= x'(\vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\omega}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}') \\
&= \vec{\omega}_{\mathcal{C}} \times \underbrace{(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')}_{\vec{r}_{B/A}} \\
&= \vec{\omega}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}}.
\end{aligned}$$

If we now try to calculate the rate of change of $\vec{v}_{B/A}$,

$$\begin{aligned}
\frac{d}{dt}(\vec{v}_{\mathrm{B/A}}) &= \frac{d}{dt}\left(\vec{\omega}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}}\right) \\
&= \frac{d\vec{\omega}_{\mathcal{C}}}{dt} \times \vec{r}_{\mathrm{B/A}} + \vec{\omega}_{\mathcal{C}} \times \frac{d\vec{r}_{B/A}}{dt} \\
\vec{a}_{\mathrm{B/A}} &= \dot{\vec{\omega}}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}} + \vec{\omega}_{\mathcal{C}} \times (\vec{\omega}_{\mathcal{C}} \times \vec{r}_{\mathrm{B/A}}).
\end{aligned}$$

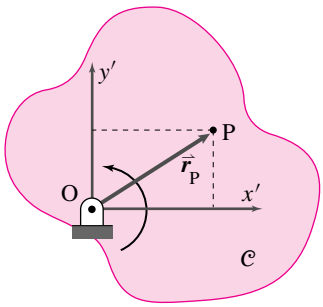


Figure 20.8: A rotating rigid body $\mathcal{C}$ with rotating frame $x'y'$ attached.

Mechanics

Now that we know the velocity and acceleration of every point in the system we are ready, in principle, to find $\dot{\vec{L}}$ and $\dot{\vec{H}}_{/O}$ in terms of the angular velocity vector $\vec{\omega}$, its rate of change $\dot{\vec{\omega}}$, and the position of all the mass in the system. This we do in the next section.

SAMPLE 20.1 Kinematics in 3-D—some basic questions: The following questions are about the velocity and acceleration formulae for the non-constant rate circular motion about a fixed axis:

$$\begin{aligned}\vec{\boldsymbol{v}} &= \vec{\omega}\times\vec{\boldsymbol{r}}\\ \vec{\boldsymbol{a}} &= \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}+\vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}})\end{aligned}$$

1. In the formulae above, what is $\vec{\boldsymbol{r}}$? How is it different from $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R$ used in the formulae $\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta$?
2. What is the difference between $\dot{\theta}$ and $\vec{\omega}$, and $\ddot{\theta}$ and $\dot{\vec{\omega}}$?
3. Are the parentheses around the $\vec{\omega}\times\vec{\boldsymbol{r}}$ term necessary in the acceleration formula?
4. Under what condition(s) can a particle have only tangential acceleration?

Solution

1. In the formulae for velocity and acceleration, $\vec{\boldsymbol{r}}$ refers to a vector from any point on the axis of rotation to the point of interest. Usually the origin of a coordinate system located on the axis of rotation is a convenient point to take as the base point for $\vec{\boldsymbol{r}}$. You can, however, choose any other point on the axis of rotation as the base point.

   The vector $\vec{\boldsymbol{r}}$ is different from $\vec{\boldsymbol{R}}$ in that $\vec{\boldsymbol{R}}$ is the position vector of the point of interest with respect to the center of the circular path that the point traces during its motion. See Fig. 5.2 of the text.

2. $\dot{\theta}$ and $\ddot{\theta}$ are the magnitudes of angular velocity and angular acceleration, respectively, in planar motion, *i.e.*, $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}} = \ddot{\theta}\hat{\boldsymbol{k}}$. We have introduced these notations to highlight the simple nature of planar circular motion. Of course, you are free to use $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}} = \alpha\hat{\boldsymbol{k}}$ if you wish.

3. Yes, the parentheses around $\vec{\omega}\times\vec{\boldsymbol{r}}$ in the acceleration formula are mandatory. The parentheses imply that this term has to be calculated before carrying out the cross product with $\vec{\omega}$ in the formula. Since the term in the parentheses is the velocity, you may also write the acceleration formula as

   $$\vec{\boldsymbol{a}} = \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}+\vec{\omega}\times\vec{\boldsymbol{v}}.$$

   Even if the formula is clear in your mind and you know which cross product to carry out first, it is a good idea to put the parentheses.

4. First of all let us identify the tangential and the normal (or radial) components of the acceleration:

   $$\vec{\boldsymbol{a}} = \overbrace{\dot{\vec{\omega}}\times\vec{\boldsymbol{r}}}^{\text{tangential}} + \overbrace{\vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}})}^{\text{radial/normal}}.$$

   Clearly, for a particle to have only tangential acceleration, the second term must be zero. For the second term to be zero we must have either $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$ or $\vec{\omega} = \vec{\boldsymbol{0}}$. But if $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$, then the tangential acceleration also becomes zero; the particle is on the axis of rotation and hence has no acceleration. Thus the condition that allows only tangential acceleration to survive is $\vec{\omega} = \vec{\boldsymbol{0}}$. Now remember that $\dot{\vec{\omega}}$ is *not* zero. Therefore, the condition we have found can be true only momentarily. This disappearance of the radial acceleration happens at start-up motions and in direction-reversing motions. ①

① In all start-up motions, the velocity is zero but the acceleration is not zero at the start up ($t = 0$). In direction-reversing motions, such as that of the washing machine drum during the wash-cycle, just at the moment when the direction of motion reverses, velocity becomes zero but the acceleration is non-zero.

SAMPLE 20.2 For a particle in circular motion, $\vec{\omega} = 2\,\text{rad/s}\hat{k}$, $\vec{\alpha} = 4\,\text{rad/s}^2\hat{k}$ and $\vec{r}_G = 3\,\text{m}(\cos 30^\circ \hat{\imath} + \sin 30^\circ \hat{\jmath})$, where $\vec{r}_G$ is the position vector of the particle. Find (i) $\vec{v}$, (ii) the tangential acceleration $\vec{a}_t$, and (iii) the radial acceleration $\vec{a}_r$, and show the resulting vectors.

Solution

$$\begin{aligned}\vec{\omega} &= \omega\hat{k} = 2\,\text{rad/s}\hat{k}\\ \vec{\alpha} &= \alpha\hat{k} = 4\,\text{rad/s}^2\hat{k}\\ \vec{r}_G &= r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath} = 3\,\text{m}(\cos 30^\circ\hat{\imath} + \sin 30^\circ\hat{\jmath}).\end{aligned}$$

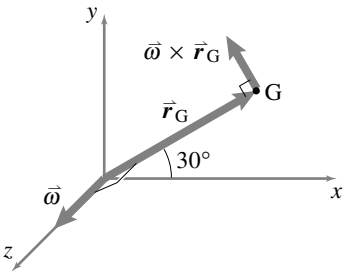


Figure 20.9: $\vec{v} = \vec{\omega} \times \vec{r}_G$.

(i) From the given formulae, the linear velocity

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r}_G\\ &= \omega\hat{k} \times (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) = \omega r_{G_x}\hat{\jmath} + \omega r_{G_y}(-\hat{\imath})\\ &= 6\,\text{m/s}(\cos 30^\circ\hat{\jmath} - \sin 30^\circ\hat{\imath})\\ &= 3\,\text{m/s}(-\hat{\imath} + \sqrt{3}\hat{\jmath}).\end{aligned}$$

The velocity vector $\vec{v}$ is perpendicular to both $\vec{\omega}$ and $\vec{r}_G$. These vectors are shown in Fig. 20.9. You should use the right hand rule to confirm the direction of $\vec{v}$.

$$\vec{v} = 3\,\text{m/s}(-\hat{\imath} + \sqrt{3}\hat{\jmath})$$

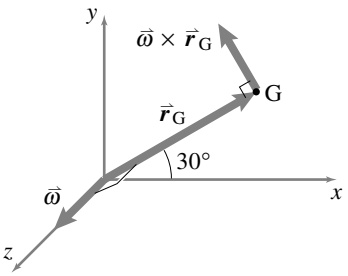


Figure 20.10: $\vec{a}_t = \vec{\alpha} \times \vec{r}_G$.

(ii) The tangential acceleration

$$\begin{aligned}\vec{a}_t &= \vec{\alpha} \times \vec{r}_G\\ &= \alpha\hat{k} \times (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) = \alpha r_{G_x}\hat{\jmath} - \alpha r_{G_y}\hat{\imath}\\ &= 8\,\text{m/s}^2(\cos 30^\circ\hat{\jmath} - \sin 30^\circ\hat{\imath}).\end{aligned}$$

Since $\vec{\omega}$ and $\vec{\alpha}$ are in the same direction, calculation of $\vec{a}_t$ is similar to that of $\vec{v}$ and $\vec{a}_t$ has to be in the same direction as $\vec{v}$. This vector is shown in Fig. 20.10. Once again, just as in the case of $\vec{v}$ we could easily check that $\vec{a}_t$ is perpendicular to both $\vec{\alpha}$ and $\vec{r}_G$.

$$\vec{a}_t = 4\,\text{m/s}^2(3\,\text{m/s}(-\hat{\imath} + \sqrt{3}\hat{\jmath})$$

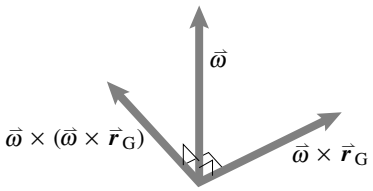


Figure 20.11: $\vec{a}_r = \vec{\omega} \times (\vec{\omega} \times \vec{r}_G)$

(iii) Finally, the radial acceleration

$$\begin{aligned}\vec{a}_r &= \vec{\omega} \times (\vec{\omega} \times \vec{r}_G)\\ &= \omega\hat{k} \times (\omega r_{G_x}\hat{\jmath} - \omega r_{G_y}\hat{\imath})\\ &= \omega^2 r_{G_x}(-\hat{\imath}) - \omega^2 r_{G_y}\hat{\jmath} = -\omega^2\vec{r}_G\\ &= 12\,\text{m/s}^2(-\cos 30^\circ\hat{\imath} - \sin 30^\circ\hat{\jmath}).\end{aligned}$$

This cross product is illustrated in Fig. 20.11. Both from the illustration as well as the calculation you should be able to see that $\vec{a}_r$ is in the direction of $-\vec{r}_G$. In fact, you could show that $\vec{a}_r = -\omega^2\vec{r}_G$.

$$\vec{a}_r = 6\,\text{m/s}^2(-\sqrt{3}\hat{\imath} - \hat{\jmath})$$

SAMPLE 20.3 Simple 3-D circular motion. A system with two point masses A and B is mounted on a rod OC which makes an angle $\theta = 45^\circ$ with the horizontal. The entire assembly rotates about the y-axis with constant angular speed $\omega = 3\,\text{rad/s}$, maintaining the angle θ. Find the velocity of point A. What is the radius of the circular path that A describes? Assume that at the instant shown, AB is in the xy plane.

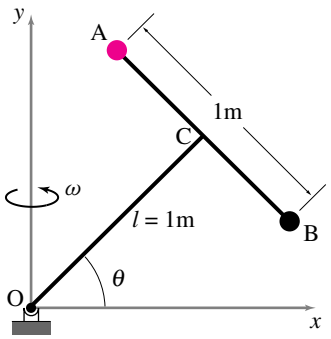


Figure 20.12

Solution The angular velocity of the system is

$$\vec{\omega} = \omega\hat{\boldsymbol{\jmath}} = 3\,\text{rad/s}\,\hat{\boldsymbol{\jmath}}.$$

Let $\vec{r}_A$ be the position vector of point A. Then the velocity of point A is

$$\begin{aligned}
\vec{v} &= \vec{\omega}\times\vec{r}_A \qquad (\text{and } \vec{r}_A = \vec{r}_C + \vec{r}_{A/C})\\
&= \omega\hat{\boldsymbol{\jmath}}\times(\underbrace{l\cos\theta\hat{\boldsymbol{\imath}} + l\sin\theta\hat{\boldsymbol{\jmath}}}_{\vec{r}_C} + \underbrace{d\cos\theta\hat{\boldsymbol{\jmath}} - d\sin\theta\hat{\boldsymbol{\imath}}}_{\vec{r}_{A/C}})\\
&= \omega\hat{\boldsymbol{\jmath}}\times[(l\cos\theta - d\sin\theta)\hat{\boldsymbol{\imath}} + (l\sin\theta + d\cos\theta)\hat{\boldsymbol{\jmath}}]\\
&= -(\omega l\cos\theta - \omega d\sin\theta)\hat{\boldsymbol{k}}\\
&= -[3\,\text{rad/s}(1\,\text{m}\cdot\cos 45^\circ - 0.5\,\text{m}\cdot\sin 45^\circ)]\\
&= -1.06\,\text{m/s}\hat{\boldsymbol{k}}
\end{aligned}$$

$$\vec{v} = -1.06\,\text{m/s}\hat{\boldsymbol{k}}.$$

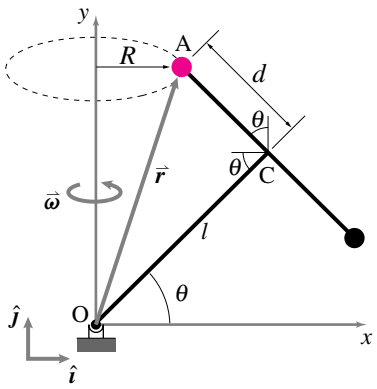


Figure 20.13

We can find the radius of the circular path of A by geometry. However, we know that the velocity of A is also given by

$$\vec{v} = \omega R\hat{\boldsymbol{e}}_\theta$$

where R is the radius of the circular path. At the instant of interest, $\hat{\boldsymbol{e}}_\theta = -\hat{\boldsymbol{k}}$ (see figure 20.14).

$$\text{Thus} \qquad \vec{v} = -\omega R\hat{\boldsymbol{k}}.$$

Comparing with the answer obtained above, we get

$$\begin{aligned}
-1.06\,\text{m/s}\hat{\boldsymbol{k}} &= -\omega R\hat{\boldsymbol{k}}\\
\Rightarrow \qquad R &= \frac{1.06\,\text{m/s}}{3\,\text{rad/s}}\\
&= 0.35\,\text{m}.
\end{aligned}$$

$$R = 0.35\,\text{m}$$

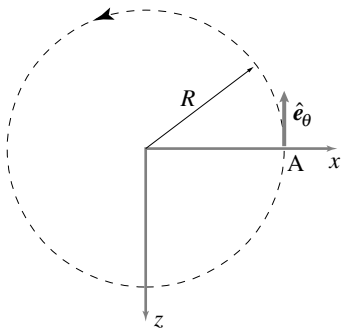


Figure 20.14: Circular trajectory of point A as seen by looking down along the y-axis. At the instant shown, $\hat{\boldsymbol{e}}_\theta = -\hat{\boldsymbol{k}}$.

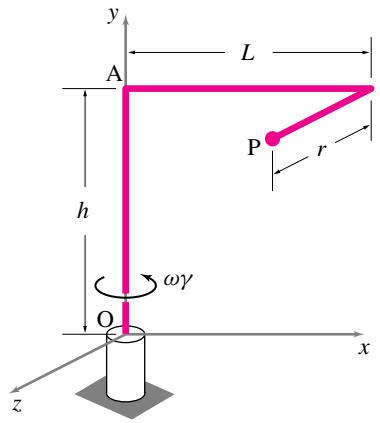
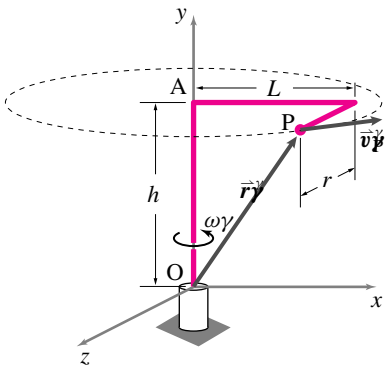


Figure 20.15

② An even better choice, perhaps, is the vector $\vec{r}_{P/A}$. Remember, the only requirement on $\vec{r}$ is that it must start at *some* point on the axis of rotation and must end at the point of interest.



Path of point P seen from the top:

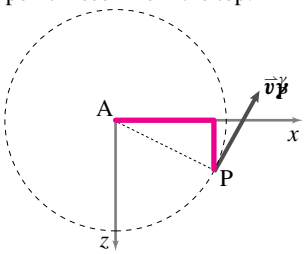


Figure 20.16: By drawing the velocity vector at point P (a vector tangent to the path) we see that $\vec{v}_P$ must have a positive x-component and a negative z-component.

SAMPLE 20.4 Velocity and acceleration in 3-D: The rod shown in the figure rotates about the y-axis at angular speed 10 rad/s and accelerates at the rate of 2 rad/s^2. The dimensions of the rod are $L = h = 2$ m and $r = 1$ m. There is a small mass P glued to the rod at its free end. At the instant shown, the three segments of the rod are parallel to the three axes.

1. Find the velocity of point P at the instant shown.
2. Find the acceleration of point P at the instant shown.

Solution We are given:

$$\vec{\omega} = \omega\hat{\jmath} = 10\,\text{rad/s}\,\hat{\jmath} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega}\hat{\jmath} = 2\,\text{rad/s}^2\,\hat{\jmath}.$$

1. The velocity of point P is

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

At the instant shown, the position vector of point P (the vector $\vec{r}_{P/O}$) seems to be a good choice for $\vec{r}$. ② Thus,

$$\vec{r} \equiv \vec{r}_{P/O} = L\hat{\imath} + h\hat{\jmath} + r\hat{k}.$$

Therefore,

$$\begin{aligned}\vec{v} &= \omega\hat{\jmath} \times (L\hat{\imath} + h\hat{\jmath} + r\hat{k}) \\ &= \omega(-L\hat{k} + r\hat{\imath}) \\ &= 10\,\text{rad/s} \cdot (-2\,\text{m}\hat{k} + 1\,\text{m}\hat{\imath}) \\ &= (20\hat{\imath} - 10\hat{k})\,\text{m/s}.\end{aligned}$$

As a check, we look down the y-axis and draw a velocity vector at point P (tangent to the circular path at point P) without paying attention to the answer we got. From the top view in Fig. 20.16 we see that at least the signs of the components of $\vec{v}$ seem to be correct.

$$\vec{v} = (20\hat{\imath} - 10\hat{k})\,\text{m/s}$$

2. The acceleration of point P is

$$\begin{aligned}\vec{a} &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ &= \dot{\omega}\hat{\jmath} \times (L\hat{\imath} + h\hat{\jmath} + r\hat{k}) + \omega\hat{\jmath} \times \underbrace{\omega(-L\hat{k} + r\hat{\imath})}_{\vec{\omega}\times\vec{r}} \\ &= \dot{\omega}(-L\hat{k} + r\hat{\imath}) + \omega^2(-L\hat{\imath} - r\hat{k}) \\ &= 2\,\text{rad/s}^2(-2\,\text{m}\hat{k} + 1\,\text{m}\hat{\imath}) - 100(\,\text{rad/s})^2(2\,\text{m}\hat{\imath} + 1\,\text{m}\hat{k}) \\ &= -(98\hat{\imath} + 104\hat{k})\,\text{m/s}^2.\end{aligned}$$

We can check the sign of the components of $\vec{a}$ also. Note that the tangential acceleration, $\dot{\vec{\omega}} \times \vec{r}$, is much smaller than the centripetal acceleration, $\vec{\omega} \times (\vec{\omega} \times \vec{r})$. Therefore, the total acceleration is almost in the same direction as the centripetal acceleration, that is, directed from point P to A. If you draw a vector from P to A, you should be able to see that it has negative components along both the x- and z-axes. Thus the answer we have got seems to be correct, at least in direction.

$$\vec{a} = -(98\hat{\imath} + 104\hat{k})\,\text{m/s}^2$$

20.1 3-D description of circular motion

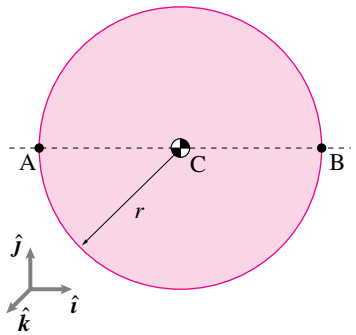
20.1.1 If $\vec{A}$ and $\vec{B}$ are perpendicular, what is $|\vec{A}\times(\vec{A}\times\vec{B})|$ in terms of $|\vec{A}|$ and $|\vec{B}|$? (Hint: make a neat 3-D sketch.)

20.1.2 A circular disk of radius $r = 100\,\text{mm}$ rotates at constant speed ω about a fixed but unknown axis. At the instant when the disk is in the xy-plane, the angular velocity of each point on the diameter AB (which is parallel to the x-axis) is $\vec{v} = -2\,\text{m/s}\hat{k}$. At the same instant, the magnitude of the acceleration of the center of mass C is $50\,\text{m/s}^2$.

a) Find the angular velocity of the disk.

b) Find the location of the axis of rotation.

c) Find the acceleration of points A and B.



Problem 20.1.2

20.1.3 In the expression for normal acceleration $\vec{a}_n = \vec{\omega}\times(\vec{\omega}\times\vec{r})$, the parenthesis around $\vec{\omega}\times\vec{r}$ is important because vector product is nonassociative. Showing $\vec{\omega}\times(\vec{\omega}\times\vec{r}) \neq (\vec{\omega}\times\vec{\omega})\times\vec{r}$ is almost trivial. (why?) Take $\vec{\omega}_1 = \omega_1\hat{k}$, $\vec{\omega}_2 = \omega_2\hat{i} + \omega_1\hat{k}$ and $\vec{r} = a\hat{i} - b\hat{j}$ and show that $\vec{\omega}_1\times(\vec{\omega}_2\times\vec{r}) \neq (\vec{\omega}_1\times\vec{\omega}_2)\times\vec{r}$.

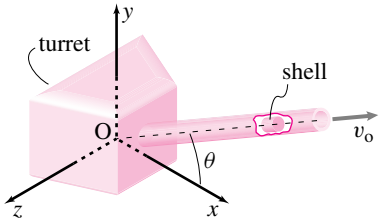
20.1.4 For the given angular velocity $\vec{\omega}$ and the position vector $\vec{r}$ below, Compute the velocity $\vec{v}$ in each of the three cases. Draw the circular orbit, and show the vectors $\vec{\omega}$, $\vec{r}$, and $\vec{v}$.

a) $\vec{\omega} = 2\,\text{rad/s}\hat{k}$, $\vec{r} = 3\,\text{m}\hat{i} + 4\,\text{m}\hat{j}$.

b) $\vec{\omega} = 4.33\,\text{rad/s}\hat{i} + 2.5\,\text{rad/s}\hat{j}$, $\vec{r} = 0.5\,\text{m}\hat{i} - 0.87\,\text{m}\hat{j}$.

c) $\vec{\omega} = 2\,\text{rad/s}\hat{i} + 2\,\text{rad/s}\hat{j} + 2.83\,\text{rad/s}\hat{k}$, $\vec{r} = 3.5\,\text{m}\hat{i} + 3.5\,\text{m}\hat{j} - 4.95\,\text{m}\hat{k}$.

20.1.5 A shell moves down a barrel. At the instant of interest the barrel is being raised at a constant angular speed ω_1. Simultaneously the gun turret (the structure which holds the barrel) rotates about the vertical axis at a constant rate ω_2. At the instant of interest, the barrel is tipped up an angle θ and the mass m is moving at speed v along the frictionless barrel. Neglecting gravity, find the force of the barrel on the shell.



Problem 20.1.5: A shell moves along a cannon barrel.

20.1.6 The following questions are about the velocity and acceleration formulae for the non-constant rate circular motion about a fixed axis:

$$\vec{v} = \vec{\omega}\times\vec{r}$$
$$\vec{a} = \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

a) For planar circular motion about the z-axis, take

$$\dot{\vec{\omega}} = \alpha\hat{k}, \quad \vec{\omega} = \omega\hat{k}, \text{ and } \vec{r} = R\hat{e}_R$$

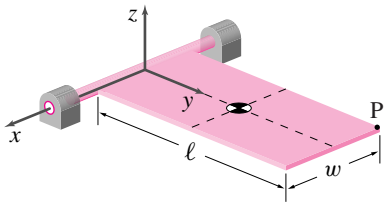
and show that the velocity and acceleration formulae reduce to the formulae from elementary physics: $\vec{v} = \omega R\hat{e}_\theta$ and $\vec{a} = -\omega^2 R\hat{e}_R + R\alpha\hat{e}_\theta$.

b) Draw a schematic picture of a particle (or a point on a rigid body) going in circles about a fixed axis in 3-D. Take two distinct points on the axis and draw vectors $\vec{r}_1$ and $\vec{r}_2$ from these two points to the particle of interest. Show, using the formulae for $\vec{v}$ and $\vec{a}$ above, that the acceleration and velocity of the particle are the same irrespective of which $\vec{r}$ is used and thus conclude that the vector $\vec{r}$ can be taken from any point on the axis of rotation.

20.1.7 A rectangular plate with width $w = 1\,\text{m}$ and length $\ell = 2\,\text{m}$ is rigidly attached to a shaft along one of its edges. The shaft rotates about the x-axis. The center of mass of the plate goes around at a constant linear speed $v = 2.5\,\text{m/s}$.

a) Find the angular velocity of the plate.

b) At the instant shown, find the velocity of point P.



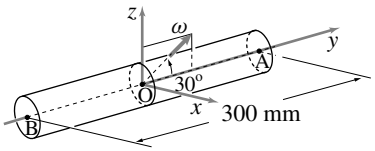
Problem 20.1.7

20.1.8 The solid cylinder shown in the figure has an angular velocity $\vec{\omega}$ of magnitude 40 rad/s. The vector $\vec{\omega}$ lies in the yz-plane. The origin O of the coordinate frame is at the center of the cylinder and is at rest. Find the following quantities:

a) The vector $\vec{\omega}$.

b) The velocity $\vec{v}_{A/B}$ of point A relative to point B.

c) The acceleration $\vec{a}_A$ of point A.



Problem 20.1.8

20.1.9 A rigid body rotates about a fixed axis with an angular acceleration $\vec{\alpha} = 3\pi^2(\hat{i} + 1.5\,\hat{j} + 1.5\,\hat{k})\,\text{rad/s}^2$. At a certain instant, the x and y components of the velocity of a point A of the body are $v_x = 3\,\text{m/s}$ and $v_y = -4\,\text{m/s}$, respectively. Find the magnitude v of the velocity of point A.

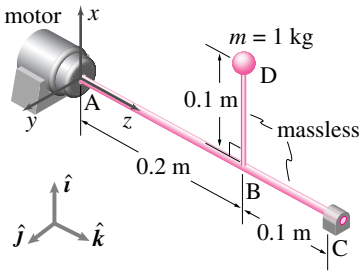
20.1.10 Particle attached to a shaft, no gravity. A shaft of negligible mass spins at a constant rate. The motor at A imposes this rate if needed. The bearing at A prevents translation of its end of the shaft in any direction but causes no torques other than that of the motor (which causes a torque in the $\hat{k}$ direction). The frictionless bearing at C

holds that end of the shaft from moving in the $\hat{\imath}$ and $\hat{\jmath}$ directions but allows slip in the $\hat{k}$ direction. A bar of negligible mass is welded at right angles to the shaft. A small sphere, considered as a particle, of mass 1 kg is attached to the free end of the bar. There is no gravity. The shaft is spinning counterclockwise (when viewed from the positive $\hat{k}$ direction) at 3 rad/s. In the configuration of interest the rod BD is parallel to the $\hat{\imath}$ direction.

a) What is the position vector of the mass in the given coordinate system at the instant of interest?

b) What is angular velocity vector of the shaft?

c) What are the velocity and acceleration of the particle at the instant of interest?



Problem 20.1.10

20.1.11 Consider the particle on a spinning shaft of problem 20.1.10 again.

a) What is the velocity of a point halfway down the rod BD relative to the velocity of the mass at D?

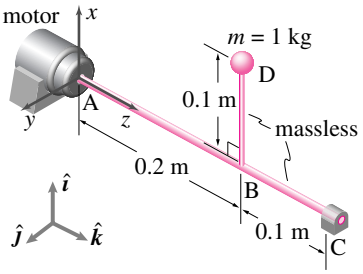
b) What is the acceleration of a point halfway down the rod BD relative to the acceleration of the mass at D?

20.1.12 Particle attached to a spinning and accelerating shaft, no gravity. Consider the particle on a spinning shaft of problem 20.1.10 again. At the instant of interest, the shaft is spinning counterclockwise (when viewed from the positive $\hat{k}$ direction) at $\omega = 3\,\text{rad/s}$ and $\dot{\omega} = 5\,\text{rad/s}^2$. In the configuration of interest the rod BD is parallel to the $\hat{\imath}$ direction. In the coordinate system shown,

a) What is the position vector of the particle at the instant of interest?

b) What are angular velocity and angular acceleration vectors of the shaft?

c) What are the velocity and acceleration of the particle at the instant of interest?



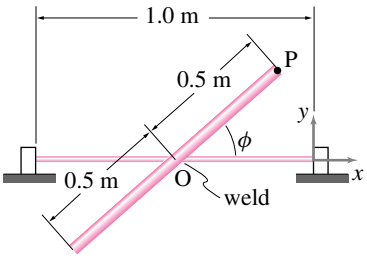
Problem 20.1.12

20.1.13 Particle attached to a spinning and accelerating shaft, still no gravity, relative motion. Reconsider the particle on the spinning and accelerating shaft of problem 20.1.12. What is the velocity and acceleration of a point halfway down the rod BD relative the velocity and acceleration of the particle?

20.1.14 A crooked rod spinning at a constant rate. A one meter long uniform rod is welded at its center to a shaft at an angle ϕ as shown. The shaft is supported by bearings at its ends and spins at a constant rate ω. At the instant of interest, the crooked rod lies in the xy-plane. In the coordinate system shown,

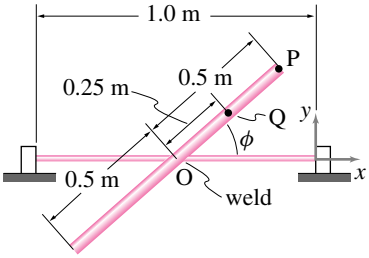
a) What is the position vector of the point P on the rod at the instant of interest? *

b) What are the velocity and acceleration of point P at the instant of interest? *



Problem 20.1.14: A crooked rod spins on a shaft.

20.1.15 The spinning crooked rod, again. Reconsider the crooked rod on a shaft in problem 20.1.14. What are the velocity and acceleration of point P relative to point Q?



Problem 20.1.15: A crooked rod spins on a shaft.

20.1.16 A crooked rod spinning at a variable rate. Consider the crooked rod shown in problem 20.1.15 again. The shaft is spinning counterclockwise (when viewed from the positive x direction) at a constant angular acceleration $\dot{\omega} = 5\,\text{rad/s}^2$. At the instant of interest, the crooked rod lies in the xy-plane and its angular speed is $\omega = 10\,\text{rad/s}$. In the coordinate system shown,

a) Find the position vector of the point P on the rod?

b) What are angular velocity and angular acceleration vectors of the rod and shaft?

c) What are the velocity and acceleration of point P?

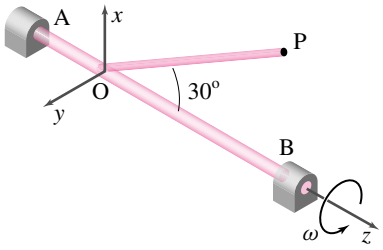
d) What is the velocity and acceleration of point P relative to point Q?

20.1.17 A rigid rod OP of length $\ell = 500\,\text{mm}$ is welded to a shaft AB at an angle $\theta = 30°$. The shaft rotates about its longitudinal axis at a constant rate $\omega = 120\,\text{rpm}$. At the instant shown, the rod OP is in the the xz-plane.

a) Neatly draw the path of point P. At the instant shown, draw the basis vectors $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$ and express them in terms of the basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.

b) Find the radius of the circular path of point P and calculate the velocity and acceleration of the point using planar circular motion formulae $\vec{\boldsymbol{v}} = \dot{\theta}\hat{\boldsymbol{e}}_\theta$ and $\vec{\boldsymbol{a}} = -\dot{\theta}^2 R\hat{\boldsymbol{e}}_r$.

c) Find the position vector $\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{r}}_{P/O}$ of point P. Use this position vector to compute the velocity and acceleration of point P using the general formulae $\vec{\boldsymbol{v}} = \vec{\omega} \times \vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{a}} = \vec{\omega} \times \vec{\omega} \times \vec{\boldsymbol{r}}$. Show that the answers obtained here are the same as those in (b).

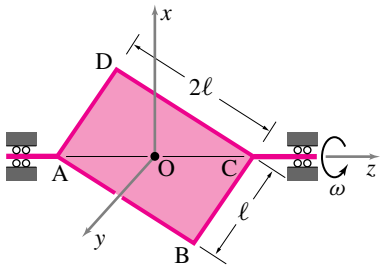


Problem 20.1.17

20.1.18 A uniform rectangular plate $ABCD$ of width ℓ and length 2ℓ rotates about its diagonal AC at a constant angular speed $\omega = 5$ rad/s. At the instant shown, the plate is in the xz-plane.

a) Find the velocity and acceleration of point B at the instant shown.

b) Find the velocity and acceleration of point B when the plate is in the yz-plane.



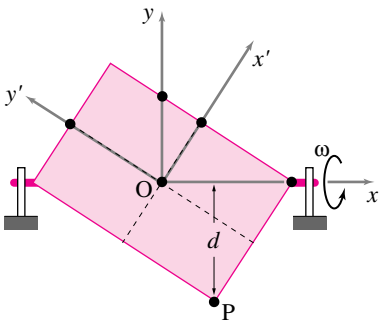
Problem 20.1.18

20.1.19 A rectangular plate of width $w = 100$ mm and length $\ell = 200$ mm rotates about one of its diagonals at a constant angular rate $\omega = 3$ rad/s. There is a fixed coordinate system xyz with the origin at the center of the plate and the x-axis aligned with the axis of rotation. Another set of axes $x'y'z'$, aligned with the principle axes of the plate, has the same origin but is attached to the plate and therefore, rotates with the plate. At the instant of interest, the plate is in the xy-plane and the two axes z and z' coincide. Find the velocity of the corner point P as follows:

a) Write the angular velocity $\vec{\omega}$ of the plate in terms of its components in the two coordinate systems.

b) Find the velocity of point P in both coordinate systems using the expressions for $\vec{\omega}$ found in part (a).

c) Find the perpendicular distance d of point P from the axis of rotation, and show that the linear speed of point P calculated from velocities found in part (b) is ωd in each case.

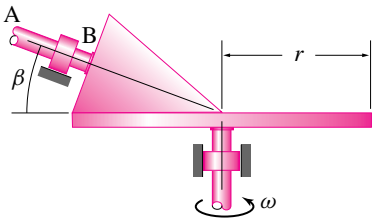


Problem 20.1.19

20.1.20 A cone of angle β sits on its side on a horizontal disk of radius $r = 100$ mm as shown in the figure. The cone is attached to shaft AB that turns in its bearing. The disk rotates at a constant angular speed $\omega = 120$ rpm. Assume that the cone rolls without slip on the disk. Thus, the disk drives the cone-and-shaft assembly causing it to turn in the bearing. Find the angular velocity of the cone as a function of the cone angle β. In particular, find the angular velocity of the cone for

a) $\beta = 30°$, and

b) $\beta = 90°$ (or 90° minus a hair if that makes you more comfortable).



Problem 20.1.20