19.3 Dynamics of multi-degree-of-freedom 2-D mechanisms

A typical machine has many parts. They may work in concert as a one degree-of-freedom system or they may be designed to move independently.

One degree-of-freedom machines. Some mechanisms are designed so the parts work together to do something, one thing, well. A car going straight down a road has dozens of moving parts: cylinders, connecting rods, crank shaft, cam shaft, transmission gears, drive shaft, differential, wheel axles, wheels and the car itself. All of these parts move, each in its own different way, approximately like a rigid object. But in the simplest description① they move as a one-degree-of freedom mechanism towards the end of fighting friction and moving the car down the road. If the crankshaft rotates a small amount, each part moves a corresponding amount; the amount of rotation of the crank shaft determines the position of all of the parts. And all of the velocities and accelerations of all of the bits of mass in the car can be determined by the rotation θ_{crank} and its derivatives ($dot\theta_{\text{crank}}$ and $ddot\theta_{\text{crank}}$). Thus a car, even with lots of pieces moving this way and that, might be well-modeled for some purposes as a one degree-of-freedom mechanism. For one-degree-of-freedom mechanisms the methods of Section 19.2 may be appropriate.

①**A car has one degree-of-freedom?** For studying the acceleration of a car going straight one might ignore the flexibility of the drive shaft and the bounce of the suspension, *etc.*thus leading to a 1 DOF model. But if you want to see how acceleration causes the car to tip back, or how the car oscillates when the engine lugs, then a 2, or more, DOF model would be appropriate.

Multi DOF systems In some modeling of machines it is important to keep track of multiple degrees of freedom (See fig. 18.51 on page 974 for examples of simple mechanisms). There are various reasons that a multi-DOF analysis is relevant:

- Some machines intentionally have various ways of moving that are controlled by various motors. Classic examples include:
  - Robots,
  - Animal bodies.
- Some systems intentionally have various degrees of freedom so as to respond smoothly to disturbances, for example the suspensions on the bottoms of cars and washing machines.
- Some systems have undesirable degrees of freedom due to the parts which are nominally rigid actually being elastic. Thus a machine which is intended to have one degree of freedom might have various undesirable vibration modes.

Mechanical analysis of multi-DOF systems. To solve problems with multiple degrees of freedom the basic strategy is, as described at the start of the chapter:

1. Draw FBDs of each object,
2. Pick configuration variables,
3. Write linear and angular momentum balance equations
4. Solve the equations for variables of interest (usually forces and second derivatives of the configuration variables).
5. Set up and solve the resulting differential equations (if you are trying to find the motion).

There are two basic approaches to these multi-object problems which, for lack of better language we call “brute-force” and “clever”.

A. In the **brute force** approach you write three times as many scalar balance equations as you have objects (limiting attention to 2D where each object has 3 DOFs and 3 independent momentum balance equations). That is, for example, for each free-body diagram you write linear momentum balance and angular momentum balance about the center of mass. Then you take this set of $3n$ equations and add and subtract them to solve for variables of interest.

   This approach is quite suitable for computers so most commercial general purpose dynamic simulators use a variant of this approach. For individual use with packaged software, the brute-force approach is generally both more reliable and more time consuming.

B. In the clever approach you write as many scalar momentum balance equations as you have unknowns. For example, if you have 2 degrees of freedom and you are concerned with motions and not reaction forces, you write 2 equations. You do this by finding momentum balance equations that do not include the variables you are not interested in. Usually this involves using angular momentum balance about hinge points, or linear momentum balance orthogonal to sliding contacts.

   The **clever** approach does not always work; the four-bar linkage is the classic problem case. However the desire to find such minimal sets of equations of motion it is historically important②.

② Attempts to automate the clever approach, that is to quickly find minimal equations of motion, led to Lagrange equations which led to Hamilton’s equations which led to quantum mechanics (but we won’t be that clever here).

At this point in the subject there are no quick simple problems. All problems are involved, especially if taken from start all the way to plotting solutions to the differential equations. The examples that follow emphasize getting to the equations of motion. The skills for numerically solving the differential equations and plotting the solutions are the same as from the start of dynamics so are not discussed until the sample problems which show all the work from setup to solution.

Example: Block sliding on sliding block: clever approach
Block 1 with mass m_1 rolls without friction on ideal massless wheels at A and B (see fig. 19.43). Block 2 with mass m_2 rolls down the tipped top of block 1 on

ideal massless rollers at C and D. The locations of G_1 relative to A and B, and of G_2 relative to C and D are known. How do blocks 1 and 2 move?

First look at the free-body diagram of the system and note that there are no unknown forces in the $\hat{\boldsymbol{\imath}}$ direction. So, for the system

$$\begin{aligned}
\left\{\sum \vec{\boldsymbol{F}}_{\rm i}\right. &= \left.\dot{\vec{\boldsymbol{L}}}\right\}\cdot\hat{\boldsymbol{\imath}} \\
\Rightarrow\quad 0 &= m_1\ddot{x} + m_2(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}'\hat{\boldsymbol{\jmath}}')\cdot\hat{\boldsymbol{\imath}} \\
&= (m_1+m_2)\ddot{x} - \ddot{y}' m_2\cos\theta. \qquad (19.47)
\end{aligned}$$

Looking at the free-body diagram of mass 2 note that there are no unknown forces in the $\hat{\boldsymbol{\jmath}}'$ direction, so

$$\begin{aligned}
\left\{\sum \vec{\boldsymbol{F}}_{\rm i}\right. &= \left.\dot{\vec{\boldsymbol{L}}}\right\}\cdot\hat{\boldsymbol{\jmath}}' \\
\Rightarrow\quad m_2 g\sin\theta &= m_2(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}'\hat{\boldsymbol{\jmath}}')\cdot\hat{\boldsymbol{\jmath}}' \\
m_2 g\sin\theta &= m_2(-\cos\theta\,\ddot{x} + \ddot{y}'). \qquad (19.48)
\end{aligned}$$

Eqns. 19.47 and 19.48 are a system of two equations in the two unknowns $\ddot{x}$ and $\ddot{y}'$

$$\begin{aligned}
(m_1+m_2)\,\ddot{x} \quad -m_2\cos\theta\,\ddot{y}' &= 0 \\
-m_2\cos\theta\,\ddot{x} \quad +m_2\,\ddot{y}' &= m_2 g\sin\theta
\end{aligned}$$

which can be solved for $\ddot{x}$ and $\ddot{y}'$ by hand or on the computer. Finding $x(t)$ and $y'(t)$ is then easy because both $\ddot{x}$ and $\ddot{y}'$ are constants.

Now we look at the same example, but proceed in a more naive manner.

Example: Block sliding on sliding block: brute force approach

Now we look at the free-body diagrams of the two separate blocks. We will use 3 balance equations from each free-body diagram, taking account of the kinematic constraints.

For the lower block we have

$$\text{AMB}_{G_1} \quad\Rightarrow\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_1} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_1} \qquad (19.49)$$

$$\begin{aligned}
\text{where}\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_1} = \;& \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{G}_1}\times(-F_D\hat{\boldsymbol{\imath}}') \\
&+\vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{G}_1}\times(-F_C\hat{\boldsymbol{\imath}}') \\
&+\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{G}_1}\times F_B\hat{\boldsymbol{\jmath}} \\
&+\vec{\boldsymbol{r}}_{\mathrm{A}/\mathrm{G}_1}\times F_A\hat{\boldsymbol{\jmath}}
\end{aligned}$$

$$\text{and}\quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_1} = \vec{\boldsymbol{0}} \qquad (\vec{\boldsymbol{\omega}}_1 = \vec{\boldsymbol{0}})$$

$$\text{and LMB} \quad\Rightarrow\quad \sum \vec{\boldsymbol{F}}_{\rm i} = \dot{\vec{\boldsymbol{L}}} \qquad (19.50)$$

$$\text{where}\quad \sum \vec{\boldsymbol{F}}_{\rm i} = -F_D\hat{\boldsymbol{\imath}}' - F_C\hat{\boldsymbol{\imath}}' + F_B\hat{\boldsymbol{\jmath}} + F_A\hat{\boldsymbol{\jmath}} - m_1 g\hat{\boldsymbol{\jmath}}$$

$$\text{and}\quad \dot{\vec{\boldsymbol{L}}} = m_1\ddot{x}\hat{\boldsymbol{\imath}}$$

Similarly for the upper block:

$$\text{AMB}_{G_2} \quad\Rightarrow\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_2} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_2} \qquad (19.51)$$

$$\begin{aligned}
\text{where}\quad \sum \vec{\boldsymbol{M}}_{/\mathrm{G}_1} = \;& \vec{\boldsymbol{r}}_{\mathrm{D}/\mathrm{G}_2}\times F_D\hat{\boldsymbol{\imath}}' \\
&+\vec{\boldsymbol{r}}_{\mathrm{C}/\mathrm{G}_2}\times F_C\hat{\boldsymbol{\imath}}'
\end{aligned}$$

$$\text{and}\quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{G}_2} = \vec{\boldsymbol{0}} \qquad (\vec{\boldsymbol{\omega}}_1 = \vec{\boldsymbol{0}})$$

$$\text{and LMB} \quad\Rightarrow\quad \sum \vec{\boldsymbol{F}}_{\rm i} = \dot{\vec{\boldsymbol{L}}} \qquad (19.52)$$

$$\text{where}\quad \sum \vec{\boldsymbol{F}}_{\rm i} = F_D\hat{\boldsymbol{\imath}}' + F_C\hat{\boldsymbol{\imath}}' - m_2 g\hat{\boldsymbol{\jmath}}'$$

$$\text{and}\quad \dot{\vec{\boldsymbol{L}}} = m_2(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}'\hat{\boldsymbol{\jmath}}')$$

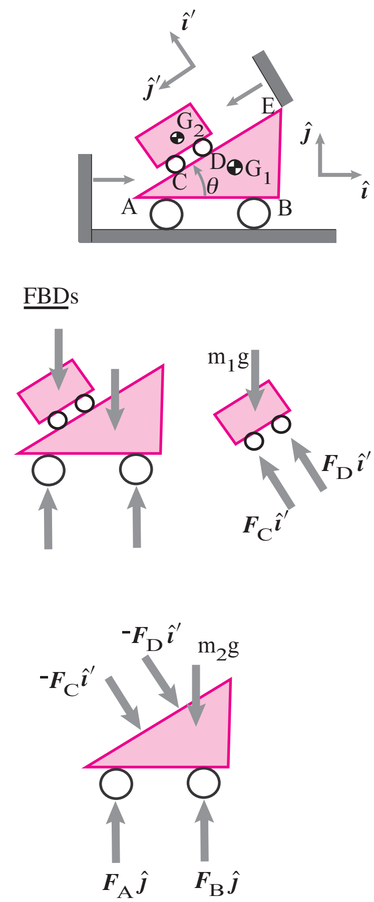


Figure 19.43: Block 2 rolls on block 1 which rolls on the ground. All rollers are ideal (frictionless and massless). The moving $\hat{\boldsymbol{\imath}}'\,\hat{\boldsymbol{\jmath}}'$ frame moves with the lower block and is oriented with the slope.

Eqns. 19.49-19.52 can be written as scalar equations by dotting the LMB equations with $\hat{\imath}$ and $\hat{\jmath}$ and the AMB equations with $\hat{k}$. All is known in these 6 equations but the six scalars: $F_A, F_B, F_C, F_D, \ddot{x}$, and $\ddot{y}'$. These could be set up as a matrix equation and solved on the computer, or you could try to find your way through by adding and subtracting equations. In any case you could solve for $\ddot{x}$ and $\ddot{y}'$ and thus have differential equations to solve to find the motions.

One quick inference one can make is from looking at the equations: no term in the coefficients of the unknowns depends on $x, \dot{x}, y$, or $\dot{y}'$. So all of the reactions F_A, F_B, F_C, and F_D as well as the accelerations $\ddot{x}$ and $\ddot{y}'$ are constants in time (until the upper mass hits the ground).

Example: Block sliding on sliding block: even more brute force

This "multi-body" problem can be solved in an even more naive and more brute force manner. The method is the same as shown in Section 14.1 on page 686 for a one dimensional problem.

We would use 6 configuration variables: the x and y coordinates of G_1 and G_2 and the rotations of the two bodies:

$$x_1, y_1, \theta_1, x_2, y_2, \quad \text{and} \quad \theta_2$$

That the two bodies don't rotate would be expressed indirectly by noting that the accelerations of points A and B on the lower mass must be in the $\hat{\imath}$ direction. These two equations would be added to the 6 linear and angular momentum balance equations. Similar constraint equations would be written for the interactions at C and D. Altogether there are now 6 configuration variables and 4 constraint forces. But there are 6 differential equations of motion and 4 constraint equations. Thus at one instant in time a set of 10 simultaneous equations needs to be solved. Then these are used to evaluate the right hand sides in the differential equations.

These 10 equations are an impractical mess for solving one simple problem like this. But these equations lend themselves to easy automation and this is closest to the approach used by general purpose dynamics simulators.

The example above is particularly simple because block 1 moves in a straight line without rotating and block 2 moves in a straight line without rotating relative to block 1. Even for a system of just two objects the situation could be much more complex if the first object had a complex motion and the second a complex motion relative to the first. But because of the preponderance of hinges in the world, circular motion, and motion relative to circular motion, is the most complex motion that need be considered by many engineers. Here is a version of the most common example of that class.

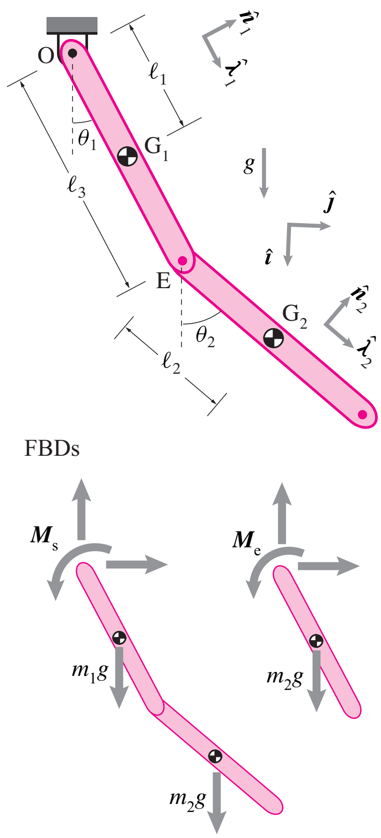


Figure 19.44: A two link robot arm. Free-body diagrams are shown of the whole system (including the motor torque at the shoulder M_s) and of the fore-arm (including the motor torque at the elbow M_e).

Example: A two link robot arm: finesse the finding of reactions

A robot arm has two links. There are motors that apply known torque M_s at the shoulder (reacted by the base and a torque M_e at the elbow (reacted by the upper arm). Dimensions are as marked. This system has 2 degrees of freedom. So we need 2 configuration variables and 2 independent balance equations to find the motion.

The natural configuration variables are the angles of the upper arm relative to a fixed reference and the angle of the lower arm relative to a fixed reference. It would also be natural to instead use the angle of the lower arm relative to the upper arm. This leads to simpler equations in the end, but more work in set up.

Angular momentum balance for the system about the shoulder contains no unknown reaction forces, nor does angular momentum balance of the fore-arm about the hinge. So we base our work on these two equations:

$$\text{System AMB}_{/O} \quad \Rightarrow \quad \sum \vec{\boldsymbol{M}}_{/\mathrm{O}} \quad = \quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \tag{19.53}$$

$$\text{Forearm AMB}_{/E} \quad \Rightarrow \quad \sum \vec{\boldsymbol{M}}_{/\mathrm{E}} \quad = \quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{E}}. \tag{19.54}$$

The goal, equations of motion, is reached by evaluating the left and right sides of these equations in terms of known geometric and mass quantities as well as the configuration variables When we write $\sum \vec{\boldsymbol{M}}_{/\mathrm{O}}$ and $\sum \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ we implicitly mean for the whole system. And $\sum \vec{\boldsymbol{M}}_{/\mathrm{E}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{E}}$ apply just to the forearm.

At each step in the calculations below imagine the results can be substituted into the later steps. We don't do that here because the expressions grow in size. Further, if the angles and their rates of change are known, as they are when doing most dynamics problems, the intermediate calculations will result in numbers, and these do not become more numerous as the calculation proceeds.

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_1 &= \cos\theta_1\hat{\boldsymbol{\imath}} + \sin\theta_1\hat{\boldsymbol{\jmath}} & \text{and} \quad & \hat{\boldsymbol{\lambda}}_2 = \cos\theta_2\hat{\boldsymbol{\imath}} + \sin\theta_2\hat{\boldsymbol{\jmath}}\\
\vec{\boldsymbol{r}}_{\mathrm{G_1/O}} &= \ell_1\hat{\boldsymbol{\lambda}}_1,\ \vec{\boldsymbol{r}}_{\mathrm{E/O}} = \ell_3\hat{\boldsymbol{\lambda}}_1 & \text{and} \quad & \vec{\boldsymbol{r}}_{\mathrm{G_2/E}} = \ell_2\hat{\boldsymbol{\lambda}}_2\\
& & \text{and} \quad & \vec{\boldsymbol{r}}_{\mathrm{G_2/O}} = \vec{\boldsymbol{r}}_{\mathrm{E/O}} + \vec{\boldsymbol{r}}_{\mathrm{G_2/E}}.
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{G_1/O}} &= -\dot{\theta}_1^2\vec{\boldsymbol{r}}_{\mathrm{G_1/O}} + \ddot{\theta}_1\hat{\boldsymbol{k}}\times\vec{\boldsymbol{r}}_{\mathrm{G_1/O}} & ,\\
\vec{\boldsymbol{a}}_{\mathrm{E/O}} &= -\dot{\theta}_1^2\vec{\boldsymbol{r}}_{\mathrm{E/O}} + \ddot{\theta}_1\hat{\boldsymbol{k}}\times\vec{\boldsymbol{r}}_{\mathrm{E/O}} & ,\\
\vec{\boldsymbol{a}}_{\mathrm{G_2/E}} &= -\dot{\theta}_2^2\vec{\boldsymbol{r}}_{\mathrm{G_2/E}} + \ddot{\theta}_2\hat{\boldsymbol{k}}\times\vec{\boldsymbol{r}}_{\mathrm{G_2/E}} & \text{and} \quad & \vec{\boldsymbol{a}}_{\mathrm{G_2/O}} = \vec{\boldsymbol{a}}_{\mathrm{E/O}} + \vec{\boldsymbol{a}}_{\mathrm{G_2/E}}.
\end{aligned}$$

These terms are all we need to evaluate the 4 terms in Eqns. 19.53 and 19.54.

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/\mathrm{O}} &= \vec{\boldsymbol{r}}_{\mathrm{G_1/O}}\times(m_1 g\hat{\boldsymbol{\imath}}) + \vec{\boldsymbol{r}}_{\mathrm{G_2/O}}\times(-m_2 g\hat{\boldsymbol{\jmath}}) + M_s\hat{\boldsymbol{k}},\\
\sum \vec{\boldsymbol{M}}_{/\mathrm{E}} &= \vec{\boldsymbol{r}}_{\mathrm{G_2/E}}\times(m_2 g\hat{\boldsymbol{\imath}}) + M_e\hat{\boldsymbol{k}},
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= m_1\vec{\boldsymbol{r}}_{\mathrm{G_1/O}}\times\vec{\boldsymbol{a}}_{\mathrm{G_1/O}} + \ddot{\theta}_1 I_1\hat{\boldsymbol{k}} + m_2\vec{\boldsymbol{r}}_{\mathrm{G_2/O}}\times\vec{\boldsymbol{a}}_{\mathrm{G_2/O}} + \ddot{\theta}_2 I_2\hat{\boldsymbol{k}},\\
\text{and} \quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{E}} &= m_2\vec{\boldsymbol{r}}_{\mathrm{G_2/E}}\times\vec{\boldsymbol{a}}_{\mathrm{G_2/O}} + \ddot{\theta}_2 I_2\hat{\boldsymbol{k}}.
\end{aligned}$$

Once these are substituted into Eqns. 19.53 and 19.54 one has 2 vector equations with only $\hat{\boldsymbol{k}}$ components. In other words we have two scalar equations in the two unknowns $\ddot{\theta}_1$ and $\ddot{\theta}_2$. One can go through the algebra and solve for them explicitly, but the expressions are quite complex, even when simplified. At given values of $\theta_1, \dot{\theta}_1, \theta_2$, and $\dot{\theta}_2$ however these are just two linear equations in two unknowns.

Closed kinematic chains

When a series of mechanical links is *open* you can not go from one link to the next successively and get back to your starting point. Such chains include a pendulum (1 link), a double pendulum (2 links), a 100 link pendulum, and a model of the human body (so long as only one foot is on the ground). A *closed* chain has at least one loop in it. You can go from link to next and get back to where you started. A slider-crank, a 4-bar linkage, and a person with two feet on the ground are closed chains.

Closed chains are kinematically difficult because they have fewer degrees of freedom than they have joints. So some of the joint angles depend on the others. The values of any minimal set of configuration variables, say some of the joint angles, determines all of the joint angles, but by geometry that is difficult or impossible to express with formulas.

Example: Four bar linkage.
It is impractically difficult to write the positions velocities and accelerations of a 4-bar linkage in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$ of any one of its joints. Why? Because finding all of the bar angles from one angle is a trigonometric mess. And differentiating that mess once (for velocities) and then once again (for accelerations) makes a huge mess.

SAMPLE 19.15 **Dynamics of sliding wedges.** A wedge shaped body of mass m_2 sits on a frictionless ground. Another wedge shaped body of mass m_1 is gently placed on the inclined face of the stationary wedge. The top wedge starts to slide down. The coefficient of friction between the two wedges is μ. Find the sliding acceleration of the top wedge along the incline (*i.e.,* the relative acceleration of m_1 with respect to m_2).

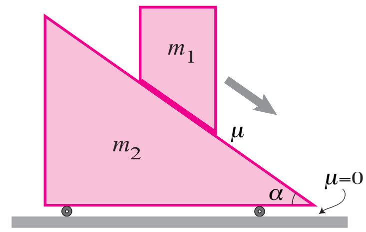


Figure 19.45

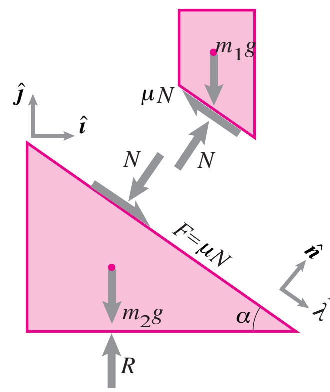


Figure 19.46

Solution The free-body diagrams of the two wedges are shown in fig. 19.46. Note that the friction force is μN since the wedges are sliding with respect to each other (if they were not sliding already then the friction force is an unknown force $F \leq \mu N$). Let the absolute acceleration of m_2 be $\vec{\boldsymbol{a}}_2 = a_2\hat{\boldsymbol{\imath}}$. Then, the absolute acceleration of m_1 is $\vec{\boldsymbol{a}}_1 = \vec{\boldsymbol{a}}_2 + \vec{\boldsymbol{a}}_{1/2} = a_2\hat{\boldsymbol{\imath}} + a_{\text{rel}}\hat{\boldsymbol{\lambda}}$. Now, we can write the linear momentum balance for m_1 and m_2 as follows.

$$N\hat{\boldsymbol{n}} - m_1 g\hat{\boldsymbol{\jmath}} - \mu N\hat{\boldsymbol{\lambda}} = m_1(a_2\hat{\boldsymbol{\imath}} + a_{\text{rel}}\hat{\boldsymbol{\lambda}}) \tag{19.55}$$

$$(R - m_2 g)\hat{\boldsymbol{\jmath}} - N\hat{\boldsymbol{n}} + \mu N\hat{\boldsymbol{\lambda}} = m_2 a_2\hat{\boldsymbol{\imath}} \tag{19.56}$$

where $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\boldsymbol{\imath}} - \sin\alpha\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{n}} = \sin\alpha\hat{\boldsymbol{\imath}} + \cos\alpha\hat{\boldsymbol{\jmath}}$. Here, we have 4 independent scalar equations (from the two 2-D vector equations) in four unknowns N, R, a_2, and a_{rel}. Thus, we can certainly solve for them. We are, however, only interested in a_{rel}. So, we should try to find the answer with fewer calculations. Dotting eqn. (19.55) with $\hat{\boldsymbol{\lambda}}$, we have

$$m_1 a_{\text{rel}} = -m_1 a_2 \cos\alpha - \mu N + m_1 g\sin\alpha \tag{19.57}$$

So, to find a_{rel}, we need a_2 and N. Dotting eqn. (19.55) with $\hat{\boldsymbol{n}}$, we have

$$m_1 a_2 \sin\alpha = N - m_1 g\cos\alpha, \tag{19.58}$$

and dotting eqn. (19.56) with $\hat{\boldsymbol{\imath}}$, we have

$$m_2 a_2 = -N\sin\alpha + \mu N\cos\alpha. \tag{19.59}$$

Solving eqn. (19.58) and (19.59) simultaneously, and using new variables (for convenience) $M = m_1/m_2$, $C = \cos\alpha$, and $S = \sin\alpha$, we get

$$a_2 = \frac{MC(\mu C - S)}{1 - MS(\mu C - S)}g, \quad N = m_1 g\frac{C}{1 - MS(\mu C - S)}$$

Substituting these expression in eqn. (19.57), we get

$$a_{\text{rel}} = gS - \frac{gC\left[MC(\mu C - S) - \mu\right]}{1 - MS(\mu C - S)}$$

$$\boxed{a_{\text{rel}} = g\sin\alpha - \frac{g\cos\alpha\left[\frac{m_1}{m_2}\cos\alpha(\mu\cos\alpha - \sin\alpha) - \mu\right]}{1 - \frac{m_1}{m_2}\sin\alpha(\mu\cos\alpha - \sin\alpha)}}$$

Note that when there is no friction ($\mu = 0$), the expression for a_{rel} reduces to

$$a_{\text{rel}} = gS + \frac{gMC^2S}{1 + MS^2}$$

and if we let $m_2 \to \infty$ (*i.e.,* m_2 represents fixed ramp) so that $M \to 0$, then $a_{\text{rel}} = g\sin\alpha$ which is the acceleration of a point mass down a frictionless ramp of slope $\tan\alpha$.

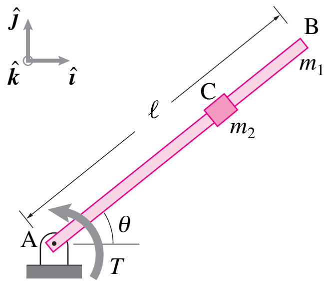


Figure 19.47

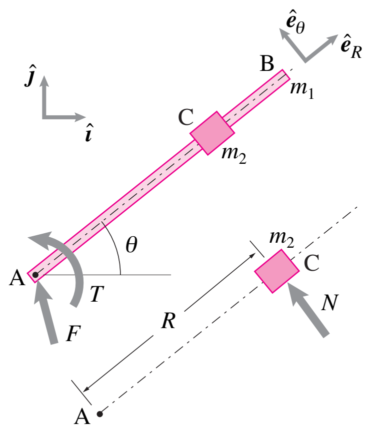


Figure 19.48

SAMPLE 19.16 Dynamics of a new gun. A new gun consists of a uniform rod AB of mass m_1 and a small collar C of mass m_2 that slides freely on the rod. A motor at A rotates the rod with constant torque T.

1. Find the equations of motion of the collar.
2. Show that if $T = 0$ then the equations of motion imply conservation of angular momentum about point A.

Solution

1. Let us denote the configuration of the collar with R, the radial distance from the fixed point A along the rod, and θ, the angular displacement of the rod. We need to find differential equations that determine R and θ as functions of time. The free-body diagram of the whole system (rod and collar together) and that of the collar is shown in fig. 19.48. We can write angular momentum balance for the whole system about point A so that the unknown reaction force F at A does not enter the equations. Noting that the acceleration of the collar is $\vec{a}_C = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$, and letting $I_1 \equiv I_z^A z$ be the moment of inertia of the rod about A, we have,

$$\begin{aligned} \sum \vec{M}_A &= \dot{\vec{H}}_A \\ T\hat{k} &= I_1\ddot{\theta}\hat{k} + R\hat{e}_R \times m_2\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta\right] \\ &= I_1\ddot{\theta}\hat{k} + m_2R^2\ddot{\theta}\hat{k} + 2m_2R\dot{R}\dot{\theta}\hat{k} \end{aligned}$$

Dotting this equation with $\hat{k}$, we have

$$\ddot{\theta} = \frac{T}{I_1 + m_2R^2} - \frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \quad (19.60)$$

Thus we have obtained the equation of motion for θ. Now we consider the free-body diagram of the collar alone and write the linear momentum balance for it in the $\hat{e}_R$ direction, *i.e.*, $\hat{e}_R \cdot (\sum \vec{F} = m\vec{a})$, so that we do not have to care about the unknown normal reaction $\vec{N}$. So, we have,

$$\begin{aligned} 0 &= \hat{e}_R \cdot m_2\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta\right] \\ &= m_2(\ddot{R} - R\dot{\theta}^2) \\ \Rightarrow \quad \ddot{R} &= R\dot{\theta}^2. \end{aligned} \quad (19.61)$$

Thus we have the required equations of motion. Note that eqn. (19.60) and (19.61) are coupled nonlinear differential equations. So, to find $\theta(t)$ and $R(t)$ we need to solve them numerically.

$$\boxed{\ddot{\theta} = \frac{T}{I_1+m_2R^2} - \frac{2m_2R}{I_1+m_2R^2}\dot{R}\dot{\theta}, \qquad \ddot{R} = R\dot{\theta}^2}$$

2. Now we set $T = 0$ in our equations of motion. Note that the equation for R is independent of T. The equation for θ becomes

$$\ddot{\theta} = -\frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \quad \Rightarrow \quad (I_1 + m_2R^2)\ddot{\theta} + 2m_2R\dot{R}\dot{\theta} = 0$$

But the last expression is simply $\dot{H}_A$ for the system. Thus we have $\dot{H}_A = 0$ which implies that H_A = constant. That is conservation of angular momentum about point A.

SAMPLE 19.17 **Numerical solutions of new gun equations.** Consider Sample 19.16 again. Set up the equations of motion for numerical solution. Take $T = 1\,\mathrm{N{\cdot}m}$, $\ell = 1\,\mathrm{m}$, $m_2 = 1\,\mathrm{kg}$, and $m_1 = m_2/3$. Carry out numerical solutions for the following cases.

1. Let the system start from rest at $\theta = 0$ and $R = 0.1\,\mathrm{m}$. Find the solution from $t = 0$ to $t = 1\,\mathrm{s}$. Plot $R(t)$, $\theta(t)$ and $R(\theta)$ (in polar coordinates).
2. Find the solution till the collar leaves the rod. What is the speed of the collar at this instant?
3. Compute and plot the total energy of the system as a function of time. Also, plot the work done by the torque as a function of time and show that the work done is equal to the total energy of the system at each instant.
4. Vary torque T and carry out solutions for several values of T. Find the terminal value of θ_f (when the collar leaves the rod) for each T. Justify your observation about θ_f by plotting $\dot{R}/\dot{\theta}$ as a function of T.

Solution We first need to write the equations of motion, eqn. (19.60) and (19.61), as a set of first order ODEs. We can easily do so by introducing new variables $\omega \equiv \dot{\theta}$ and $v_R \equiv \dot{R}$, so that we have,

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{R} \\ \dot{v}_R \end{pmatrix} = \begin{pmatrix} \omega \\ \frac{T}{I_1 + m_2 R^2} - \frac{2 m_2 R}{I_1 + m_2 R^2} v_R \omega \\ v_R \\ R\omega^2 \end{pmatrix}$$

Given the values of all constants, we only need to specify the initial conditions for θ, ω, R, and v_R for solving these equations numerically.

1. We use the following pseudocode to carry out the numerical solution.

```
Set T = 1, L = 1, m2 = 1, m1 = m2/3
Let I1 = m1*L^2/3, I2 = m2*R^2,
ODEs = {thetadot = w,
        wdot = (T-2*m2*R*vR*w)/(I1+I2),
        Rdot = vR,
        vRdot = R*w^2}
IC = {theta = 0, w = 0, R = 0.1, vR = 0}
Solve ODEs with IC for t=0 to t=1
Plot t vs R, Plot t vs theta,
Polarplot theta vs R
```

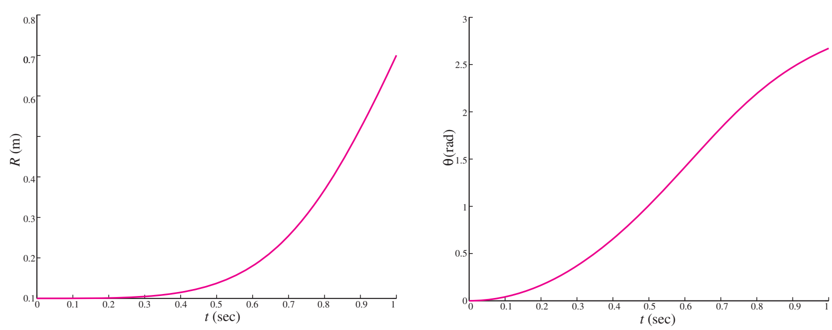


Figure 19.50:

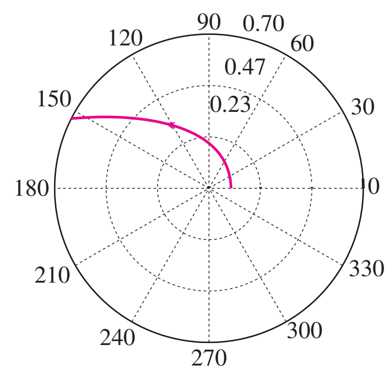


Figure 19.49

The $R(t)$ and $\theta(t)$ plots obtained from the numerical solution are shown in fig. 19.50 and the polar plot of $R(\theta)$ is shown in fig. 19.49.

2. We do not know a priori the value of t at which the collar leaves the rod. So, we have to carry out the solution for some assumed t_f which gives us $R(t_f) > \ell$ so that we know the collar has gone past the end of the rod. We then plot $R(t)$, including the unreal value of $R(t_f) > \ell$, and find the time t at which $R(t) = \ell$, either by zooming into the graph or by interpolation (although, there are various sophisticated algorithms to find this t). Following the method of zooming into the graph (see fig. 19.51) we find the terminal value of t to be 1.147 s. We carry out the numerical solution again from $t = 0$ to $t_f = 0.147$ s and find that $R(t_f) = 1$ m, $v_R(t_f) =$ 2.13 m/s, and $\omega(t_f) = 1.03$ rad/s, so that $v_f = \sqrt{\dot{R}^2 + (R\dot{\theta})^2} = 2.37$ m/s. This is the terminal speed of the collar.

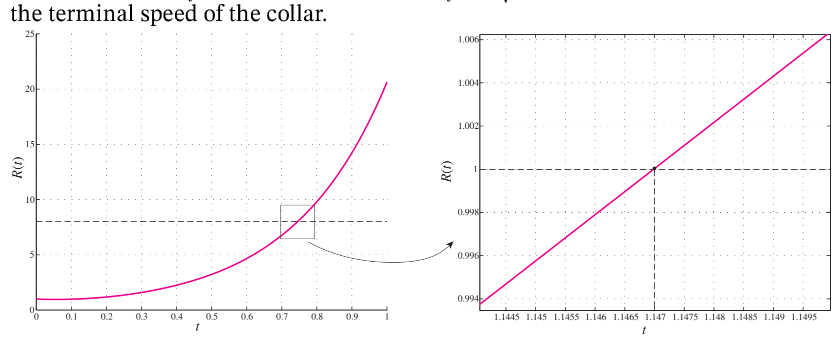


Figure 19.51: Finding the time t at which the collar leaves the rod from the graph of $R(t)$.

3. The work done by the torque is $W = T\theta$ at any instant. The system only possesses kinetic energy. So the energy of the system at any instant is $E = E_1 + E_2$ where $E_1 = \frac{1}{2}I_1\omega^2$ and $E_2 = \frac{1}{2}m_2(\dot{R}^2 + R^2\omega^2)$. Computing these quantities for the solution obtained above, we plot W and E vs t as shown in fig. 19.53. Clearly, $W = E$.

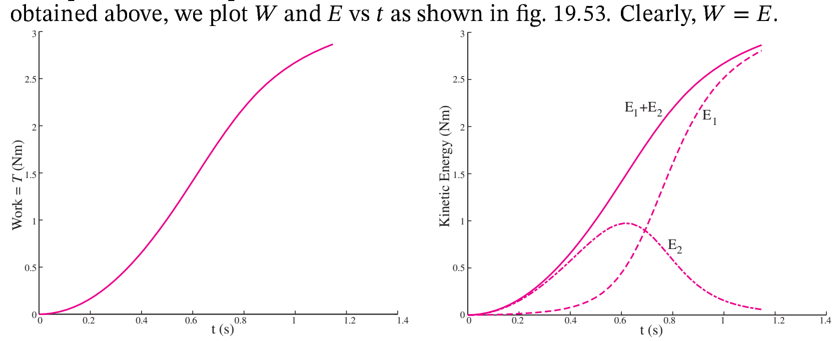


Figure 19.53: Work done by the torque and the kinetic energy of the system.

4. Now we take several values of T (0.1, 0.5, 1, 1.5, 2, 2.5, and 3) and carry out the numerical solutions for each T. We note the terminal values of $\theta, \omega(=\dot{\theta})$, and $v_R(=\dot{R})$. By plotting the terminal value of θ against T (fig. 19.52), we see that the collar leaves the rod at exactly the same $\theta = 2.86$ rad for each T! But this is possible only if $\dot{R}$ and $\dot{\theta}$ both change in the same proportion for each T. So, plot the ratio $\dot{R}/\dot{\theta}$ just for the terminal values against T and find that the ratio is indeed constant (see fig. 19.54).

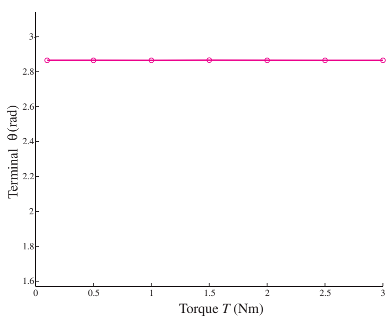


Figure 19.52: Angle θ at which the collar leaves the rod vs torque T.

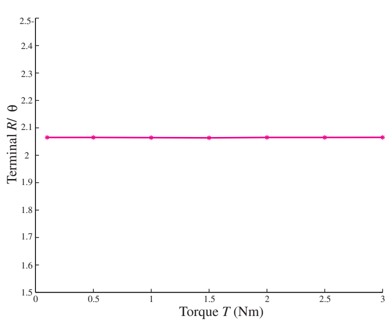


Figure 19.54: The ratio of terminal $\dot{R}/\dot{\theta}$ vs torque T.

SAMPLE 19.18 Dynamics of a sliding-base pendulum. A cart of mass M slides down a frictionless inclined plane as shown in the figure. A simple pendulum of mass m and length ℓ hangs from the center-of-mass of the cart. Find the equation of motion of the pendulum.

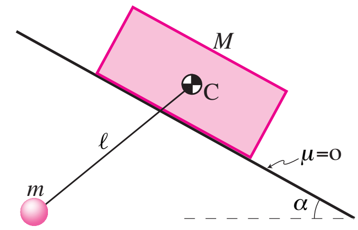


Figure 19.55

Solution Let us measure the angular displacement of the pendulum with respect to the cart with angle θ measured anticlockwise from the normal to the inclined plane. Let s be the position of the cart along the inclined plane from some reference point. Then, the acceleration of the cart can be written as $\vec{\boldsymbol{a}}_C = \ddot{s}\hat{\boldsymbol{\lambda}}$ and the acceleration of the pendulum mass as $\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}_C + \vec{\boldsymbol{a}}_{\text{rel}} = \ddot{s}\hat{\boldsymbol{\lambda}} + \ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R$.

The free-body diagram of the cart and the pendulum system is shown in fig. 19.56. Writing angular momentum balance of the system about point C, we get

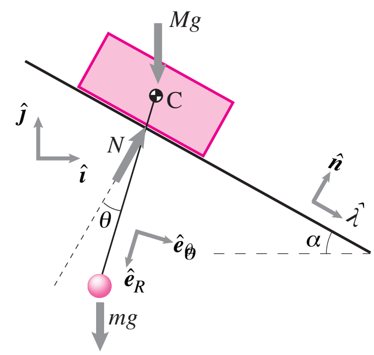


Figure 19.56

$$
\begin{aligned}
\vec{\boldsymbol{M}}_C &= \dot{\vec{\boldsymbol{H}}}_C \\
\ell\hat{\boldsymbol{e}}_R \times (-mg\hat{\boldsymbol{\jmath}}) &= \ell\hat{\boldsymbol{e}}_R \times m(\ddot{s}\hat{\boldsymbol{\lambda}} + \ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R) \\
-mgl\sin(\theta-\alpha)\hat{\boldsymbol{k}} &= m\ddot{s}\ell\cos\theta\hat{\boldsymbol{k}} + m\ell^2\ddot{\theta}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \ddot{\theta} &= -\frac{g}{\ell}\sin(\theta-\alpha) - \frac{\ddot{s}}{\ell}\cos\theta. \qquad (19.62)
\end{aligned}
$$

To find $\ddot{s}$, we write the linear momentum balance for the whole system in the $\hat{\boldsymbol{\lambda}}$ (so that we do not involve the unknown normal reaction N) direction.

$$
\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot (-Mg\hat{\boldsymbol{\jmath}} - mg\hat{\boldsymbol{\jmath}}) &= \hat{\boldsymbol{\lambda}} \cdot \left[M\ddot{s}\hat{\boldsymbol{\lambda}} + m(\ddot{s}\hat{\boldsymbol{\lambda}} + \ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R)\right] \\
\Rightarrow \quad (M+m)g\sin\sin\alpha &= (M+m)\ddot{s} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta \\
\Rightarrow \quad \ddot{s} &= g\sin\alpha - \frac{m\ell}{M+m}(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta) \qquad (19.63)
\end{aligned}
$$

Substituting eqn. (19.63) in eqn. (19.62) and rearranging terms, we get

$$
\ddot{\theta} = -\frac{g}{\ell}\sin\theta\frac{(1+\frac{m}{M})\cos\alpha}{1+\frac{m}{M}\sin^2\theta} + \frac{\frac{m}{M}\dot{\theta}^2\sin\theta\cos\theta}{1+\frac{m}{M}\sin^2\theta}
$$

$$
\boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta\frac{(1+\frac{m}{M})\cos\alpha}{1+\frac{m}{M}\sin^2\theta} + \frac{\frac{m}{M}\dot{\theta}^2\sin\theta\cos\theta}{1+\frac{m}{M}\sin^2\theta}}
$$

Note that if we set $\alpha = 0$ and let $M \to \infty$ so that the cart behaves like a fixed ground, then we recover the equation of simple pendulum, $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$, from the equation of motion above. It is a good practice to carry out such simple checks wherever possible.

Remarks: We could write the equations of motion, eqn. (19.62) and eqn. (19.63) in the coupled form as

$$
\begin{bmatrix} \ell & \cos\theta \\ m\ell\cos\theta & M+m \end{bmatrix}
\begin{pmatrix} \ddot{\theta} \\ \ddot{s} \end{pmatrix}
=
\begin{pmatrix} -\frac{g}{\ell}\sin(\theta-\alpha) \\ (M+m)g\sin\alpha - m\ell\dot{\theta}^2\sin\theta \end{pmatrix}
$$

and leave it at that, since for most computational purposes, it is enough. It is not so hard to find expressions for $\ddot{\theta}$ and $\ddot{s}$ from here by solving the matrix equation, even by hand.

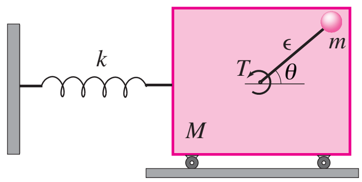


Figure 19.57

SAMPLE 19.19 **Resonant capture.** A slightly unbalanced motor mounted on an elastic machine part is modeled as a spring mass system with a simple pendulum of mass m and length ϵ driven by a constant torque T as shown in the figure. The spring has stiffness k and the motor has mass M. There is no friction between mass M and the horizontal surface.

1. Find the equation of motion of the system.
2. Take $M = m = 1\,\mathrm{kg}$, $k = 1\,\mathrm{N/m}$, $T = 15\times10^{-3}\,\mathrm{N{\cdot}m}$. Solve (numerically) the equations of motion with zero initial conditions and plot $x(t)$ and $\theta(t)$ for $t = 0$ to $100\,\mathrm{s}$.

Solution

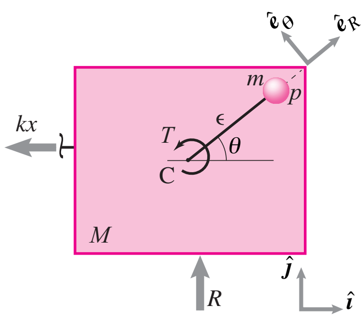


Figure 19.58

1. The free-body diagram of the system is shown in fig. 19.58. Let the angular displacement of the eccentric mass m at some instant t be θ. At the same instant, let the displacement of the motor be x from the relaxed state of the spring. Then we can write the acceleration of the motor as $\ddot{x}\hat{\imath}$ and that of the eccentric mass as $\vec{a}_P = \ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\theta^2\hat{e}_R$. Now, we can write the angular momentum balance for the system about point C (fixed in the stationary frame of reference but instantly coincident with the center-of-mass of motor M) as

$$
\begin{aligned}
T\hat{k} &= \epsilon\hat{e}_R \times m(\ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R) \\
&= m\epsilon\ddot{x}(\hat{e}_R \times \hat{\imath}) + m\epsilon^2\ddot{\theta}(\hat{e}_R \times \hat{e}_\theta) \\
&= m\epsilon(-\ddot{x}\sin\theta + \epsilon\ddot{\theta})\hat{k} \\
\Rightarrow \quad \epsilon\ddot{\theta} - \sin\theta\,\ddot{x} &= \frac{T}{m\epsilon} \qquad (19.64)
\end{aligned}
$$

This is just one scalar equation in $\ddot{\theta}$ and $\ddot{x}$. We need one more independent equation $\ddot{\theta}$ and $\ddot{x}$ without involving any other unknowns. So, we write the linear momentum balance for the system in the x-direction:

$$
\begin{aligned}
-kx &= M\ddot{x} + m(\ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R)\cdot\hat{\imath} \\
&= (M+m)\ddot{x} - m\epsilon\ddot{\theta}\sin\theta - m\epsilon\dot{\theta}^2\cos\theta \\
\Rightarrow \quad \epsilon\sin\theta\,\ddot{\theta} - \left(\frac{M+m}{m}\right)\ddot{x} &= \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta. \qquad (19.65)
\end{aligned}
$$

Thus we have the required equations of motion. We can write eqn. (19.64) and (19.65) compactly as

$$
\begin{bmatrix} \epsilon & -\sin\theta \\ \epsilon\sin\theta & -\frac{m+M}{M} \end{bmatrix}
\begin{pmatrix} \ddot{\theta} \\ \ddot{x} \end{pmatrix}
=
\begin{pmatrix} \frac{T}{m\epsilon} \\ \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta. \end{pmatrix}
$$

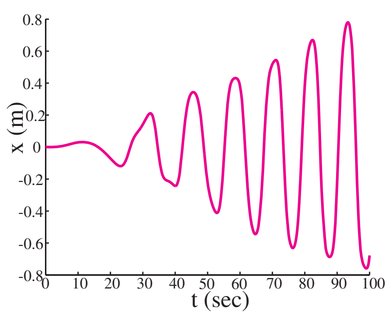


Figure 19.59

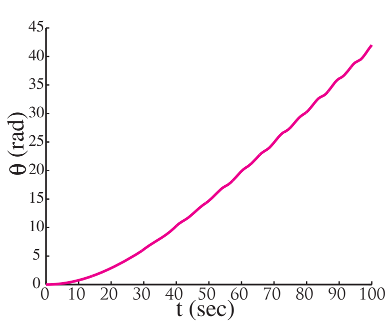


Figure 19.60

2. We use the following pseudocode to solve the equations of motion. Note that we first convert the two second order ODEs into four first order ODEs by introducing new variables $\omega = \dot{\theta}$ and $u = \dot{x}$.

```
Set T = 0.015, m = 1, M = 1, k = 1, e = 1
A=[e -sin(theta); e*sin(theta) -(1+M/m)];
b = [T/(m*e); k/m*x-e*omega^2*cos(theta)];
solve A*acln = b for acln  % acln = accelerations
ODEs = {omega = thetadot, u  = xdot,
        omegadot = acln(1), udot = acln(2)}
IC   = {theta = 0, x = 0, omega = 0, u = 0}
Solve ODEs with IC for t=0 to t=100
```

The plots of $x(t)$ and $\theta(t)$ obtained from the numerical solution are shown in fig. 19.59 and fig. 19.60, respectively. Note the resonance of M for the given values of the system.

SAMPLE 19.20 Dynamics using a rotating and translating coordinate system. Consider the rotating wheel of Sample 18.11 which is shown here again in Figure 19.61. At the instant shown in the figure find

1. the linear momentum of the mass P and
2. the net force on the mass P.

For calculations, use a frame $\mathcal{B}$ attached to the rod and a coordinate system in $\mathcal{B}$ with origin at point A of the rod OA.

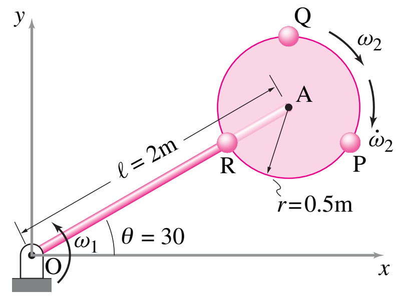


Figure 19.61

Solution We attach a frame $\mathcal{B}$ to the rod. We choose a coordinate system $x'y'z'$ in this frame with its origin O′ at point A. We also choose the orientation of the primed coordinate system to be parallel to the fixed coordinate system xyz (see Fig. 19.62), *i.e.*, $\hat{\imath}' = \hat{\imath}$, $\hat{\jmath}' = \hat{\jmath}$, and $\hat{k}' = \hat{k}$.

1. **Linear momentum of P:** The linear momentum of the mass P is given by

$$\vec{L} = m\vec{v}_P.$$

Clearly, we need to calculate the velocity of point P to find $\vec{L}$. Now,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\rm rel} = \underbrace{\vec{v}_{O'} + \vec{v}_{P'/O'}}_{\vec{v}_{P'}} + \vec{v}_{\rm rel}.$$

Note that O′ and P′ are two points on the same (imaginary) rigid body OAP′. Therefore, we can find $\vec{v}_{P'}$ as follows:

$$\begin{aligned}
\vec{v}_{P'} &= \overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}}^{\vec{v}_{O'}} + \overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}}^{\vec{v}_{P'/O'}} \\
&= \omega_1\hat{k} \times \ell(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + \omega_1\hat{k} \times r(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= \omega_1[(\ell + r)\cos\theta\hat{\jmath} - (\ell - r)\sin\theta\hat{\imath}] \\
&= 3\,\text{rad/s} \cdot [2.5\,\text{m} \cdot \cos 30^\circ\hat{\jmath} - 1.5\,\text{m} \cdot \sin 30^\circ\hat{\imath}] \\
&= (6.50\hat{\jmath} - 2.25\hat{\imath})\,\text{m/s} \quad \text{(same as in Sample 18.11.)},
\end{aligned}$$

$$\begin{aligned}
\vec{v}_{\rm rel} &= \vec{v}_{P/\mathcal{B}} \\
&= -\omega_2\hat{k}' \times r(\cos\theta\hat{\imath}' - \sin\theta\hat{\jmath}') \\
&= -\omega_2 r(\cos\theta\hat{\jmath}' + \sin\theta\hat{\imath}') \\
&= -(2.16\hat{\jmath}' + 1.25\hat{\imath}')\,\text{m/s} \\
&= -(2.16\hat{\jmath} + 1.25\hat{\imath})\,\text{m/s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{v}_P &= \vec{v}_{P'} + \vec{v}_{\rm rel} \\
&= (4.34\hat{\jmath} - 3.50\hat{\imath})\,\text{m/s} \qquad \text{and} \\
\vec{L} &= m\vec{v}_P \\
&= 0.5\,\text{kg} \cdot (4.34\hat{\jmath} - 3.50\hat{\imath})\,\text{m/s} \\
&= (-1.75\hat{\imath} + 2.17\hat{\jmath})\,\text{kg·m/s}.
\end{aligned}$$

$$\vec{L} = (-1.75\hat{\imath} + 2.17\hat{\jmath})\,\text{kg·m/s}$$

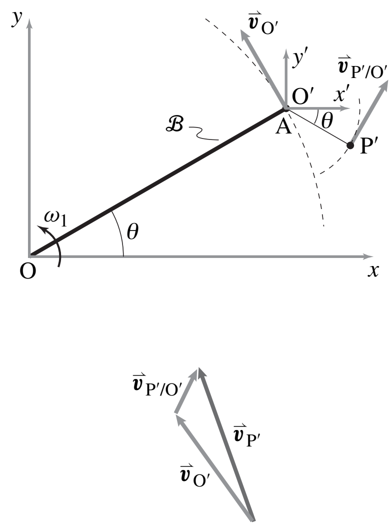


Figure 19.62: The velocity of point P' is the sum of two terms: the velocity of O' and the velocity of P' relative to O'.

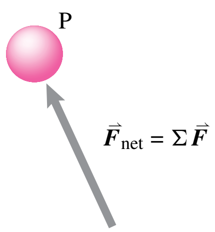


Figure 19.63

2. **Net force on P:** From the
$$\sum\vec{F} = m\vec{a}$$
for the mass P we get $\sum\vec{F} = m\vec{a}_P$. Thus to find the net force $\sum\vec{F}$ we need to find $\vec{a}_P$. The calculation of $\vec{a}_P$ is the same as in Sample 18.11 except that $\vec{a}_{P'}$ is now calculated from
$$\vec{a}_{P'} = \vec{a}_{O'} + \vec{a}_{P'/O'}$$
where
$$\begin{aligned}
\vec{a}_{O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}) \\
&= -\omega_1^2 \vec{r}_{O'/O} \\
&= -\omega_1^2 \ell(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
&= -(3\,\text{rad/s})^2 \cdot 2\,\text{m} \cdot (\cos 30^\circ\,\hat{\imath} + \sin 30^\circ\,\hat{\jmath}) \\
&= -(15.59\hat{\imath} + 9.00\hat{\jmath})\,\text{m/s}^2,
\end{aligned}$$
$$\begin{aligned}
\vec{a}_{P'/O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\
&= -\omega_1^2 \vec{r}_{P'/O'} \\
&= -\omega_1^2 r(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= -(3\,\text{rad/s})^2 \cdot 0.5\,\text{m} \cdot (\cos 30^\circ\,\hat{\imath} - \sin 30^\circ\,\hat{\jmath}) \\
&= -(3.90\hat{\imath} - 2.25\hat{\jmath})\,\text{m/s}^2.
\end{aligned}$$
Thus,
$$\vec{a}_{P'} = -(19.49\hat{\imath} + 6.75\hat{\jmath})\,\text{m/s}^2$$
which, of course, is the same as calculated in Sample 18.11. The other two terms, $\vec{a}_{\text{cor}}$ and $\vec{a}_{\text{rel}}$, are exactly the same as in Sample 18.11. Therefore, we get the same value for $\vec{a}_P$ by adding the three terms:
$$\vec{a}_P = -(17.83\hat{\imath} + 3.63\hat{\jmath})\,\text{m/s}^2.$$
The net force on P is
$$\begin{aligned}
\sum\vec{F} &= m\vec{a}_P \\
&= 0.5\,\text{kg} \cdot (-17.83\hat{\imath} - 3.63\hat{\jmath})\,\text{m/s}^2 \\
&= -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\text{N}.
\end{aligned}$$
$$\boxed{\sum\vec{F} = -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\text{N}}$$

SAMPLE 19.21 Inverse dynamics of a four bar mechanism. A four bar mechanism ABCD consists of three uniform bars AB, BC, and CD of length ℓ_1, ℓ_2, ℓ_3, and mass m_1, m_2, m_3, respectively. The mechanism is driven by a torque T applied at A such that bar AB rotates at constant angular speed. Write equations to find the torque T at some instant t.

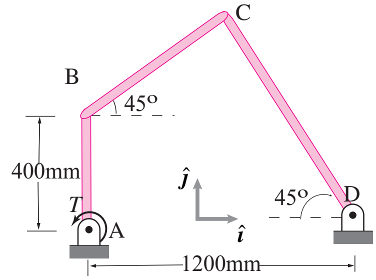


Figure 19.64

Solution This is an inverse dynamics problem, that is, we are given the motion and we are supposed to find the forces (torque T in this case) that cause that motion. We are given that rod AB rotates at constant angular speed, say $\dot{\theta}$. From kinematics, we can find out angular velocities and angular accelerations of the other two bars as well as the accelerations of center-of-mass of each rod. Then we can write the momentum balance equations and compute the forces and moments required to generate this motion. So, in contrast to what we usually do, let us do the kinematics first. Please see Sample 18.5 on page 990. We found the angular velocities, $\dot{\beta}$ (eqn. (18.71)) and $\dot{\phi}$ (eqn. (18.70)), of rods BC and CD, respectively, in terms of $\dot{\theta}$. We can rewrite those equations as

$$\begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix} \dot{\beta} \\ \dot{\phi} \end{pmatrix} = \dot{\theta}\begin{pmatrix} \ell_1\sin\theta \\ \ell_1\cos\theta \end{pmatrix} \tag{19.66}$$

We wrote this equation in matrix form to make it easier for us to find the angular accelerations which we do by simply differentiating this equation once:

$$\begin{aligned} \begin{bmatrix} -\ell_2\cos\beta\,\dot{\beta} & \ell_3\cos\phi\,\dot{\phi} \\ \ell_2\sin\beta\,\dot{\beta} & -\ell_3\sin\phi\,\dot{\phi} \end{bmatrix}\begin{pmatrix} \dot{\beta} \\ \dot{\phi} \end{pmatrix} &+ \begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix} \ddot{\beta} \\ \ddot{\phi} \end{pmatrix} \\ &= \ell_1\begin{pmatrix} \ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta \\ \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta \end{pmatrix} \end{aligned}$$

Rearranging terms we get

$$\begin{aligned} \begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix} \ddot{\beta} \\ \ddot{\phi} \end{pmatrix} &= -\begin{bmatrix} -\ell_2\cos\beta & \ell_3\cos\phi \\ \ell_2\sin\beta & \ell_3\sin\phi \end{bmatrix}\begin{pmatrix} \dot{\beta}^2 \\ \dot{\phi}^2 \end{pmatrix} \\ &\quad +\ell_1\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{pmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{pmatrix} \end{aligned} \tag{19.67}$$

Thus, we can find the angular accelerations of BC and CD, $\ddot{\beta}$ and $\ddot{\phi}$, because the quantities on the right hand side are known ($\ddot{\theta}(=0)$ and $\dot{\theta}$ are given, and $\dot{\beta}$ and $\dot{\phi}$ are determined by eqn. (19.66)). Now, we can find the accelerations of center-of-mass of each rod as follows.

$$\vec{\boldsymbol{a}}_{G_1} = -\frac{\ell_1}{2}\dot{\theta}^2\hat{\boldsymbol{\lambda}}_1 \tag{19.68}$$

$$\vec{\boldsymbol{a}}_{G_2} = \vec{\boldsymbol{a}}_B + \vec{\boldsymbol{a}}_{G_2/B} = -\ell_1\dot{\theta}^2\hat{\boldsymbol{\lambda}}_1 - \frac{\ell_2}{2}\dot{\beta}^2\hat{\boldsymbol{\lambda}}_2 + \ell_2\ddot{\beta}\hat{\boldsymbol{n}}_2 \tag{19.69}$$

$$\vec{\boldsymbol{a}}_{G_3} = -\frac{\ell_3}{2}\dot{\phi}^2\hat{\boldsymbol{\lambda}}_3 \tag{19.70}$$

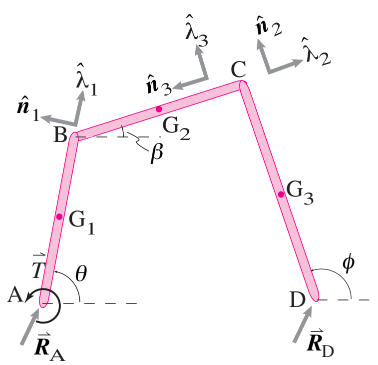


Figure 19.65

We are now ready to write momentum balance equations. Since we are only interested in finding the torque T, we should try to write equations involving minimum number of unknown forces. So, we draw free-body diagrams of the whole mechanism, of part BCD, and of bar CD alone; and write angular momentum balance equations about appropriate points so that we involve only the unknown torque T and the unknown reaction $\vec{\boldsymbol{R}}_D$ at D. Thus, we will have only three scalar unknowns T, R_{D_x} and R_{D_y} (since $\vec{\boldsymbol{R}}_D = R_{D_x}\hat{\boldsymbol{\imath}} + R_{D_y}\hat{\boldsymbol{\jmath}}$). So, we will need only three independent equations.

Consider the free-body diagram of the whole mechanism. We can write angular momentum balance about point A for the whole mechanism as

$$T\hat{\boldsymbol{k}} + \vec{\boldsymbol{r}}_{D/A}\times\vec{\boldsymbol{R}}_D = \dot{\vec{\boldsymbol{H}}}_A = \dot{\vec{\boldsymbol{H}}}_{1/A} + \dot{\vec{\boldsymbol{H}}}_{2/A} + \dot{\vec{\boldsymbol{H}}}_{3/A} \tag{19.71}$$

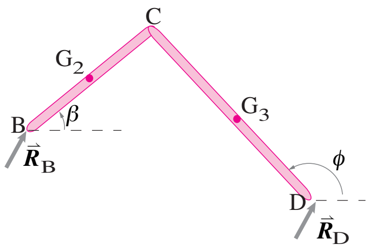


Figure 19.66

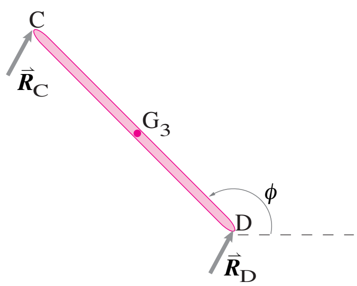


Figure 19.67

where

$$\begin{aligned}\dot{\vec{H}}_{1/A} &= I_1\ddot{\theta}\hat{k} + \vec{r}_{G_1} \times m_1\vec{a}_{G_1}\\ \dot{\vec{H}}_{2/A} &= I_2\ddot{\beta}\hat{k} + \vec{r}_{G_2} \times m_2\vec{a}_{G_2} = I_2\ddot{\beta}\hat{k} + (\vec{r}_B + \vec{r}_{G_2/B}) \times m_2\vec{a}_{G_2}\\ \dot{\vec{H}}_{3/A} &= I_3\ddot{\phi}\hat{k} + \vec{r}_{G_3} \times m_3\vec{a}_{G_3} = I_3\ddot{\phi}\hat{k} + (\vec{r}_D + \vec{r}_{G_3/D}) \times m_3\vec{a}_{G_3}.\end{aligned}$$

Similarly, the angular momentum balance about point B for BCD gives

$$\vec{r}_{D/B} \times \vec{R}_D = \dot{\vec{H}}_{2/B} + \dot{\vec{H}}_{3/B} \tag{19.72}$$

where

$$\begin{aligned}\dot{\vec{H}}_{2/B} &= I_2\ddot{\beta}\hat{k} + \vec{r}_{G_2/B} \times m_2\vec{a}_{G_2}\\ \dot{\vec{H}}_{3/B} &= I_3\ddot{\phi}\hat{k} + \vec{r}_{G_3/B} \times m_3\vec{a}_{G_3}\end{aligned}$$

and angular momentum balance of bar CD about point C gives

$$\vec{r}_{D/C} \times \vec{R}_D = \dot{\vec{H}}_{3/C} = I_3\ddot{\phi}\hat{k} + \vec{r}_{G_3/C} \times m_3\vec{a}_{G_3}. \tag{19.73}$$

Note that we can easily write the position vectors in terms of ℓ_1, ℓ_2, ℓ_3 and the unit vectors $(\hat{\lambda}_1, \hat{n}_1)$, $(\hat{\lambda}_2, \hat{n}_2)$ and $(\hat{\lambda}_3, \hat{n}_3)$ where

$$\begin{aligned}\hat{\lambda}_1 &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}, & \hat{n}_1 &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\\ \hat{\lambda}_2 &= \cos\beta\hat{\imath} + \sin\beta\hat{\jmath}, & \hat{n}_2 &= -\sin\beta\hat{\imath} + \cos\beta\hat{\jmath}\\ \hat{\lambda}_1 &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}, & \hat{n}_1 &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}.\end{aligned}$$

We can put all the three angular momentum balance equations, (19.71), (19.72), and (19.73), in one matrix equation by dotting both sides of the equations with $\hat{k}$ and assembling them as follows.

$$\begin{bmatrix} 1 & 0 & \ell_4\\ 0 & \hat{k}\cdot(\ell_2\hat{\lambda}_2 - \ell_3\hat{\lambda}_3)\times\hat{\imath} & \hat{k}\cdot(\ell_2\hat{\lambda}_2 - \ell_3\hat{\lambda}_3)\times\hat{\jmath}\\ 0 & \hat{k}\cdot(-\ell_3\hat{\lambda}_3\times\hat{\imath}) & \hat{k}\cdot(-\ell_3\hat{\lambda}_3\times\hat{\jmath})\end{bmatrix}\begin{pmatrix} T\\ R_{D_x}\\ R_{D_y}\end{pmatrix} = \begin{pmatrix}\dot{H}_{123/A}\\ \dot{H}_{23/B}\\ \dot{H}_{3/C}\end{pmatrix} \tag{19.74}$$

where $\dot{H}_{123/A} = \hat{k}\cdot(\dot{\vec{H}}_{1/A} + \dot{\vec{H}}_{2/A} + \dot{\vec{H}}_{3/A})$, $\dot{H}_{23/B} = \hat{k}\cdot(\dot{\vec{H}}_{2/B} + \dot{\vec{H}}_{3/B})$, and $\dot{H}_{3/C} = \hat{k}\cdot\dot{\vec{H}}_{3/C}$.

Note that we know the $\dot{\vec{H}}$'s on the right hand side and the matrix on the left side can be evaluated for any given (θ, β, ϕ). Thus we can solve for T, R_{D_x}, and R_{D_y}.

SAMPLE 19.22 **Numerical solution of the inverse dynamics problem.** Consider Sample 19.21 again. Using numerical solutions on a computer, find and plot torque T against θ for one complete cycle of the drive arm AB. Take $m_1 = m_2 = m_3 = 1\,\text{kg}$ and $\ell_1 = 400\,\text{mm}$, $\ell_2 = 400\sqrt{2}\,\text{mm}$, $\ell_3 = 800\sqrt{2}\,\text{mm}$, and $\ell_4 = 1200\,\text{mm}$.

Solution

Since we have to plot T against θ for one complete revolution, we need to find angular velocities, angular accelerations and center-of-mass accelerations for several values of θ and then solve for T for each of those θ's. We can do this several ways. One way would be to first solve kinematic equations to find $\theta(t), \beta(t)$, and $\phi(t)$ at discrete times over one complete cycle and then compute all other quantities at each $(\theta(t_i), \beta(t_i), \phi(t_i))$ where t_i represents a discrete time. So, let us follow this method step by step with pseudocodes. Here, we assume that we have vector functions called `dot` and `cross` that compute the dot product and the cross product of two vectors that are given as input arguments.

Step-1: solve for angular positions. Specify the given geometry

```
L1=0.4, L2=0.4*sqrt(2), L3=0.8*sqrt(2), L4=1.2
```

and use the pseudocode of Sample 18.5 to find $\theta(t_i), \beta(t_i), \phi(t_i)$ for, say, 100 values of t_i between 0 and 1 sec. Now, for each triad of $(\theta(t_i), \beta(t_i), \phi(t_i))$, follow all the steps below.

Step-2: solve for angular velocities. Since $\dot{\theta} = 2\pi\,\text{rad/s}$ is given, we only need to solve for $\dot{\beta}$ and $\dot{\phi}$. We use eqn. (18.71) and eqn. (18.70) to compute $\dot{\beta}$ and $\dot{\phi}$ as follows (or modify the pseudocode of Sample 18.5 to save $\dot{\beta}$ and $\dot{\phi}$ along with the values for β and ϕ.

```
define thdot=thetadot, bdot=betadot, pdot=phidot
thdot = 2*pi   % this is given
set th = theta(ti), b = beta(ti), p = phi(ti)
set unit vectors
  l1=[cos(th) sin(th) 0]', n1=[-sin(th) cos(th) 0]'
  l2=[cos(b) sin(b) 0]', n2=[-sin(b) cos(b) 0]'
  l3=[cos(p) sin(p) 0]', n3=[-sin(p) cos(p) 0]'
bdot = -(L1/L2)*(cross(n1,l3)/cross(n2,l3))*thdot
pdot = (L1/L3)*(cross(n1,l2)/cross(n3,l2))*thdot
```

Step-3: solve for angular accelerations. Now that we have (θ, β, ϕ) and the corresponding values of $(\dot{\theta}, \dot{\beta}, \dot{\phi})$, we can use eqn. (19.67) to calculate $\ddot{\beta}$ and $\ddot{\phi}$ (we are given $\ddot{\theta} = 0$).

```
define thddot=thetaddot, bddot=betaddot,
       pddot=phiddot
thddot = 0   % this is given
B = [-L2*sin(b) L3*sin(p); -L2*cos(b) L3*cos(p)]
C = L1*[sin(th) cos(th); cos(th) -sin(th)]
D = [-cos(b) cos(p); sin(b) -sin(p)]
c = [thddot thdot^2]',  d = [L2*bdot^2 L3*pdot^2]'
assume w = [bddot pddot]'
solve B*w = C*c + D*d for w
```

So, now we know $\ddot{\theta}, \ddot{\beta}, \ddot{\phi}$ also. We are now ready to compute $\dot{\boldsymbol{H}}$'s required for dynamic calculations.

Step-4: set up equations and solve for unknown forces. We need to set up and solve eqn. (19.74). Note that we need to compute several quantities for this equation but the vector computations are more or less straightforward.

```
% set mass and inertia properties
m1 = 1, m2 = 1, m3 = 1

I1 = m1*L1^2/12, I2 = m2*L2^2/12, I3 = m3*L3^2/12
% set fixed unit vectors
i = [1 0 0]', j = [0 1 0]', k = [0 0 1]'
% compute position vectors
rA = [0;0;0],  rB = rA+L1*l1
rC = rB+L2*l2, rD = L4*l4
rG1 = L1/2*l1
rG2 = rB+L2/2*l2
rG3 = rD+L3/2*l3
% compute center-of-mass accelerations
aG1 = 0.5*L1*(tddot*n1-tdot^2*l1)
aG2 = 2*aG1+0.5*L2*(bddot*n2-bdot^2*l2)    % aB = 2*aG1
aG3 = 0.5*L3*(pddot*n3-pdot^2*l3)

% compute Hdot_cms

Hdot_cm1 = I1*tddot*uk
Hdot_cm2 = I2*bddot*uk
Hdot_cm3 = I3*pddot*uk
% compute Hdots
Hdot_123_A = Hdot_cm1 + cross(rG1, m1*aG1)
              + Hdot_cm2 + cross(rG2, m2*aG2)
                + Hdot_cm3 + cross(rG3, m3*aG3)
Hdot_23_B = Hdot_cm2 + cross(rG2-rB, m2*aG2)
              + Hdot_cm3 + cross(rG3-rB, m3*aG3)
Hdot_3_C = Hdot_cm3 + cross(rG3-rC, m3*aG3)
% set up the linear eqns for torque and RD
b = [dot(Hdot_123_A,k) dot(Hdot_23_B,k) dot(Hdot_3_C,k)]
A = [1  dot(k,cross(rD,i))     dot(k,cross(rD,j))
     0  dot(k,cross(rD-rB,i))  dot(k,cross(rD-rB,j))
      0  dot(k,cross(rD-rC,i))  dot(k,cross(rD-rC,j))]
% let forces = [T  RDx  RDy]'
solve A*forces = b for forces
```

Step-5, repeat calculations. Now repeat Step-2 – Step-4 for each triad (θ, β, ϕ) obtained in Step-1 and save the corresponding values of T in a vector. Finally,

```
plot T vs theta
```

The plot thus obtained is shown in fig. 19.68. We can also plot T vs time (as shown in fig. 19.69), and, of course, expect to see the same graph of T since θ is just a linear function of t. Note that the area under the graph of T over one complete cycle must equal zero since the net impulse must be zero over one cycle.

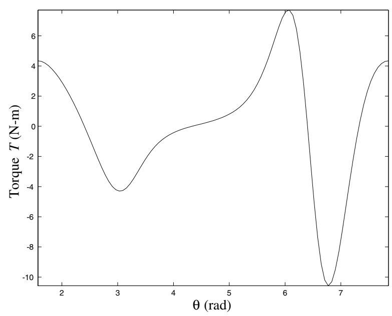


Figure 19.68: Torque T as a function of θ over one complete cycle of motion ($\theta(0) = \pi/2, \theta(1) = 5\pi/2$).

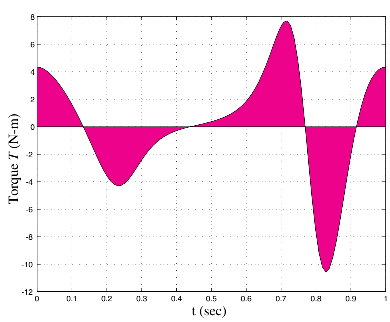


Figure 19.69: Torque T as a function of time over one complete cycle of motion.

19.3 Multi-degree-of-freedom 2-D mechanisms

19.3.1 Particle on a springy leash. A particle with mass m slides on a rigid horizontal frictionless plane. It is held by a string which is in turn connected to a linear elastic spring with constant k. The string length is such that the spring is relaxed when the mass is on top of the hole in the plane. The position of the particle is $\vec{r} = x\hat{\imath} + y\hat{\jmath}$. For each of the statements below, state the circumstances in which the statement is true (assuming the particle stays on the plane). Justify your answer with convincing explanation and/or calculation.

a) The force of the plane on the particle is $mg\hat{k}$.

b) $\ddot{x} + \frac{k}{m}x = 0$

c) $\ddot{y} + \frac{k}{m}y = 0$

d) $\ddot{r} + \frac{k}{m}r = 0$, where $r = |\vec{r}|$

e) $r =$ constant

f) $\dot{\theta}$ =constant

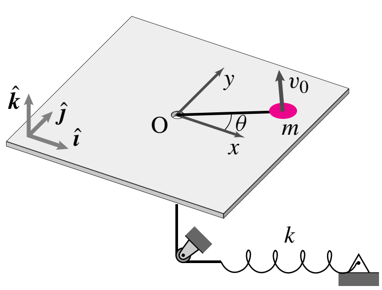
g) $r^2\dot{\theta} =$ constant.

h) $m(\dot{x}^2 + \dot{y}^2) + kr^2 =$ constant

i) The trajectory is a straight line segment.

j) The trajectory is a circle.

k) The trajectory is not a closed curve.



Problem 19.3.1: Particle on a springy leash.

19.3.2 "Yo-yo" mechanism of satellite de-spinning. A satellite, modeled as a uniform disk, is "de-spun" by the following mechanism. Before launch two equal length long strings are attached to the satellite at diametrically opposite points and then wound around the satellite with the same sense of rotation. At the end of the strings are placed 2 equal masses. At the start of de-spinning the two masses are released from their position wrapped against the satellite. Find the motion of the satellite as a function of time for the two cases: (a) the strings are wrapped in the same direction as the initial spin, and (b) the strings are wrapped in the opposite direction as the initial spin.

19.3.3 A particle with mass m is held by two long springs each with stiffness k so that the springs are relaxed when the mass is at the origin. Assume the motion is planar. Assume that the particle displacement is much smaller than the lengths of the springs.

a) Write the equations of motion in cartesian components. *

b) Write the equations of motion in polar coordinates. *

c) Express the conservation of angular momentum in cartesian coordinates. *

d) Express the conservation of angular momentum in polar coordinates. *

e) Show that (a) implies (c) and (b) implies (d) even if you didn't note them *a-priori.* *

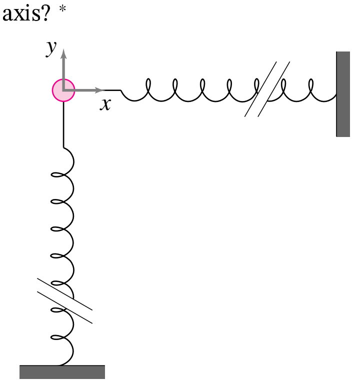
f) Express the conservation of energy in cartesian coordinates. *

g) Express the conservation of energy in polar coordinates. *

h) Show that (a) implies (f) and (b) implies (g) even if you didn't note them *a-priori.* *

i) Find the general motion by solving the equations in (a). Describe all possible paths of the mass. *

j) Can the mass move back and forth on a line which is not the x or y axis? *



Problem 19.3.3

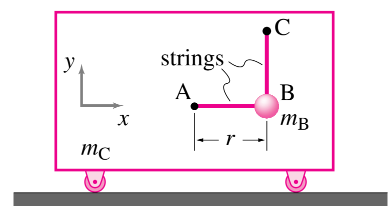
19.3.4 Cart and pendulum A mass $m_B = 6$ kg hangs by two strings from a cart with mass $m_C = 12$ kg. Before string BC is cut string AB is horizontal. The length of string AB is $r = 1$ m. At time $t = 0$ all masses are stationary and the string BC is cleanly and quietly cut. After some unknown time t_{vert} the string AB is vertical.

a) What is the net displacement of the cart x_C at $t = t_{\text{vert}}$? *

b) What is the velocity of the cart v_C at $t = t_{\text{vert}}$? *

c) What is the tension in the string at $t = t_{\text{vert}}$? *

d) (**Optional**) What is $t = t_{\text{vert}}$? [You will either have to leave your answer in the form of an integral you cannot evaluate analytically or you will have to get part of your solution from a computer.]

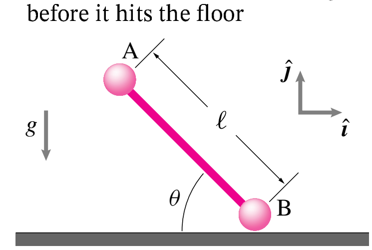


Problem 19.3.4

19.3.5 A dumbbell slides on a floor. Two point masses m at A and B are connected by a massless rigid rod with length ℓ. Mass B slides on a frictionless floor so that it only moves horizontally. Assume this dumbbell is released from rest in the configuration shown. *[Hint: What is the acceleration of A relative to B?]*

a) Find the acceleration of point B just after the dumbbell is released.

b) Find the velocity of point A just before it hits the floor



Problem 19.3.5

19.3.6 After a winning goal one second before the clock ran out a psychologically stunned hockey player (modeled as a uniform rod) stands nearly vertical, stationary and rigid. The player's perfectly slippery skates start to pop out from under her as she falls. Her height is ℓ, her mass m, her tip from the vertical θ, and the gravitational constant is g.

a) What is the path of her center of mass as she falls? (Show clearly with equations, sketches or words.)

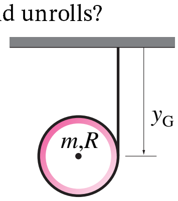
b) What is her angular velocity just before she hits the ice, a millisecond before she sticks out her hands and breaks her fall (first assume her skates remain in contact the whole time and then check the assumption)?

c) Find a differential equation that only involves θ, its time derivatives, m, g, and ℓ. (This equation could be solved to find θ as a function of time. It is a non-linear equation and you are not being asked to solve it numerically or otherwise.)

19.3.7 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R. G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

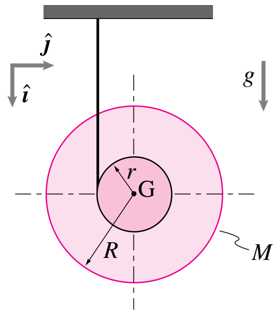
a) Find y_G at a later time t in terms of any or all of m, R, g, and t.

b) Does G move sideways as the hoop falls and unrolls?



Problem 19.3.7

19.3.8 A model for a yo-yo consists of a thin disk of mass M and radius R and a light drum of radius r, rigidly attached to the disk, around which a light inextensible cable is wound. Assuming that the cable unravels without slipping on the drum, determine the acceleration a_G of the center of mass.

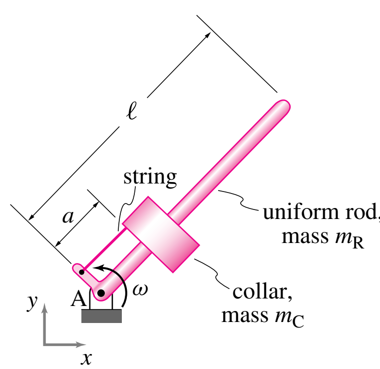


Problem 19.3.8

19.3.9 A uniform rod with mass m_R pivots without friction about point A in the xy-plane. A collar with mass m_C slides without friction on the rod after the string connecting it to point A is cut. There is no gravity. Before the string is cut, the rod has angular velocity ω_1.

a) What is the speed of the collar after it flies off the end of the rod? Use the following values for the constants and initial conditions: $m_R = 1\,\text{kg}$, $m_C = 3\,\text{kg}$, $a = 1\,\text{m}$, $\ell = 3\,\text{m}$, and $\omega_1 = 1\,\text{rad/s}$

b) Consider the special case $m_R = 0$. Sketch (approximately) the path of the motion of the collar from the time the string is cut until some time after it leaves the end of the rod.



Problem 19.3.9

19.3.10 Assume the rod in the figure for problem 18.1.5 has polar moment of inertia $Iozz$. Assume it is free to rotate. The bead is free to slide on the rod. Assume that at $t = 0$ the angular velocity of the rod is 1 rad/s, that the radius of the bead is one meter and that the radial velocity of the bead, dR/dt, is zero.

a) Draw separate free-body diagrams of the bead and rod.

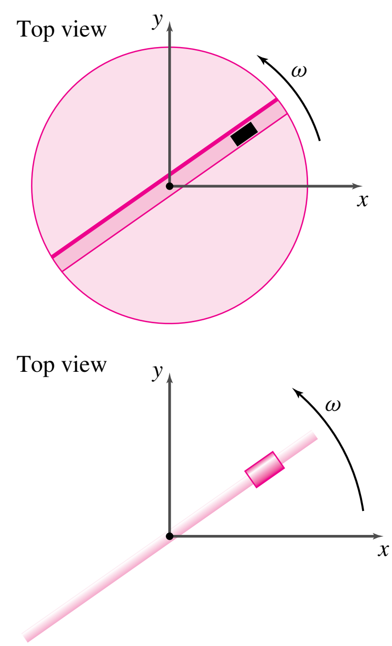
b) Write equations of motion for the system. *

c) Use the equations of motion to show that angular momentum is conserved. *

d) Find one equation of motion for the system using: (1) the equations of motion for the bead and rod and (2) conservation of angular momentum. *

e) Write an expression for conservation of energy. Let the initial total energy of the system be, say, E_0. *

f) As t goes to infinity does the bead's distance go to infinity? Its speed? The angular velocity of the turntable? The net angle of twist of the turntable? *



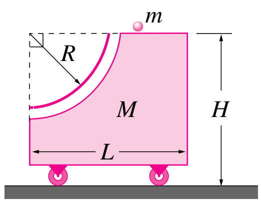
Problem 19.3.10: **Coupled motion of bead and rod and turntable.**

19.3.11 A primitive gun rides on a cart (mass M) and carries a cannon ball (of mass m) on a platform at a height H above the ground. The cannon ball is dropped through a frictionless tube shaped like a quarter circle of radius R.

a) If the system starts from rest, compute the horizontal speed (relative to the ground) that the cannon ball has as it leaves the bottom of the tube. Also find the cart's speed at the same instant.

b) Compute the velocity of the cannon ball relative to the cart.

c) If two balls are dropped simultaneously through the tube, what speed does the cart have when the balls reach the bottom? Is the same final speed also achieved if one ball is allowed to depart the system entirely before the second ball is released? Why/why not?



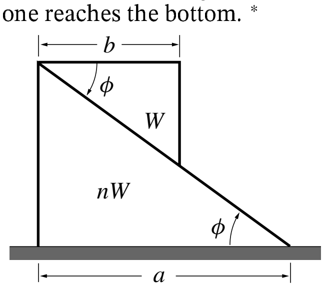
Problem 19.3.11

19.3.12 Numerically simulate the coupled system in problem 19.3.10. Use the simulation to show that the net angle of the turntable or rod is finite.

a) Write the equations of motion for the system from part (b) in problem 19.3.10 as a set of first order differential equations. *

b) Numerically integrate the equations of motion.

19.3.13 Two frictionless prisms of similar right triangular sections are placed on a frictionless horizontal plane. The top prism weighs W and the lower one nW. The prisms are held in the initial position shown and then released, so that the upper prism slides down along the lower one until it just touches the horizontal plane. The center of mass of a triangle is located at one-third of its height from the base. Compute the **velocities** of the two prisms at the moment just before the upper one reaches the bottom. *

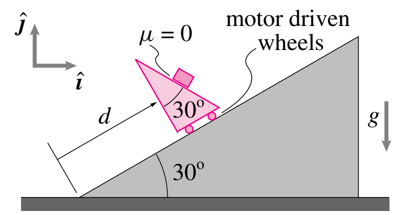


Problem 19.3.13

19.3.14 Mass slides on an accelerating cart. 2D. A cart is driven by a powerful motor to move along the 30° sloped ramp according to the formula: $d = d_0 + v_o t + a_0 t^2/2$ where d_0, v_0, and a_0 are given constants. The cart is held from tipping over. The cart itself has a 30° sloped upper surface on which rests a mass (given mass m). The surface on which the mass rests is frictionless. Initially the mass is at rest with regard to the cart.

a) What is the force of the cart on the mass? [in terms of $g, d_0, v_0, t, a_0, m, g, \hat{\imath}$, and $\hat{\jmath}$.] *

b) For what values of d_0, v_0, t and a_0 is the acceleration of the mass exactly vertical (i.e., in the $\hat{\jmath}$ direction)? *



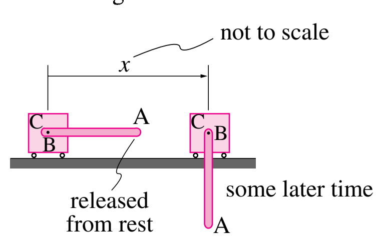
Problem 19.3.14

19.3.15 A thin rod AB of mass W_{AB} = 10 lbm and length L_{AB} = 2 ft is pinned to a cart C of mass W_C = 10 lbm, the latter of which is free to move along a frictionless horizontal surface, as shown in the figure. The system is released from rest with the rod in the horizontal position.

a) Determine the angular speed of the rod as it passes through the vertical position (at some later time). *

b) Determine the displacement x of the cart at the same instant. *

c) After the rod passes through vertical, it is momentarily horizontal but on the left side of the cart. How far has the cart moved when this configuration is reached?



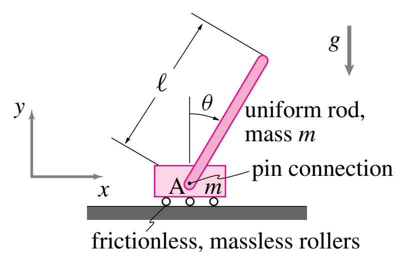
Problem 19.3.15

19.3.16 As shown in the figure, a block of mass m rolls without friction on a rigid surface and is at position x (measured from a fixed point). Attached to the block is a uniform rod of length ℓ which pivots about one end which is at the center of mass of the block. The rod and block have equal mass. The rod makes an angle θ with the vertical. Use the numbers below for the values of the constants and variables at the time of interest:

$$\begin{aligned}
\ell &= 1\,\text{m}\\
m &= 2\,\text{kg}\\
\theta &= \pi/2\\
d\theta/dt &= 1\,\text{rad/s}\\
d^2\theta/dt^2 &= 2\,\text{rad/s}^2\\
x &= 1\,\text{m}\\
dx/dt &= 2\,\text{m/s}\\
d^2x/dt^2 &= 3\,\text{m/s}^2
\end{aligned}$$

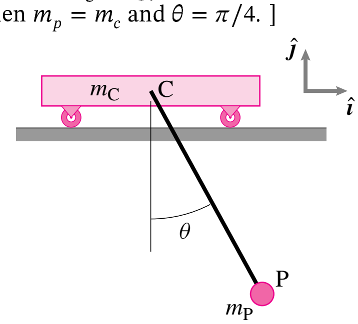
a) What is the kinetic energy of the system?

b) What is the linear momentum of the system (momentum is a vector)?



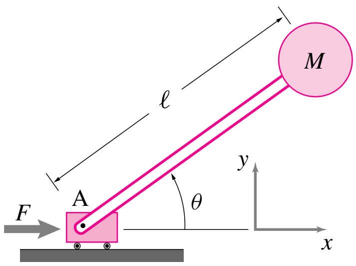
Problem 19.3.16

19.3.17 A pendulum of length ℓ hangs from a cart. The pendulum is massless except for a point mass of mass m_p at the end. The cart rolls without friction and has mass m_c. The cart is initially stationary and the pendulum is released from rest at an angle θ. What is the acceleration of the cart just after the mass is released? [Hints: $\mathbf{a}_P = \vec{\mathbf{a}}_C + \vec{\mathbf{a}}_P/C$. The answer is $\vec{\mathbf{a}}_C = (g/3)\mathbf{i}$ in the special case when $m_p = m_c$ and $\theta = \pi/4$.]



Problem 19.3.17

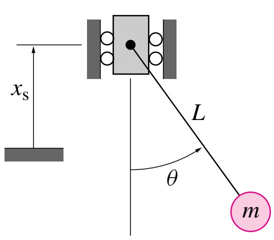
19.3.18 Due to the application of some unknown force F, the base of the pendulum A is accelerating with $\vec{\boldsymbol{a}}_A = a_A \boldsymbol{\imath}$. There is a frictionless hinge at A. The angle of the pendulum θ with the x axis and its rate of change $\dot{\theta}$ are assumed to be known. The length of the massless pendulum rod is ℓ. The mass of the pendulum bob M. There is no gravity. What is $\ddot{\theta}$? (Answer in terms of a_A, ℓ, M, θ and $\dot{\theta}$.)



Problem 19.3.18

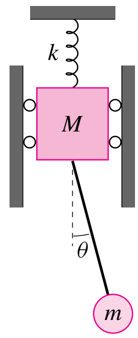
19.3.19 Pumping a Swing Can a swing be pumped by moving the support point up and down?

For simplicity, neglect gravity and consider the problem of swinging a rock in circles on a string. Let the rock be mass m attached to a string of fixed length ℓ. Can you speed it up by moving your hand up and down? How? Can you make a quantitative prediction? Let x_S be a function of time that you can specify to try to make the mass swing progressively faster.



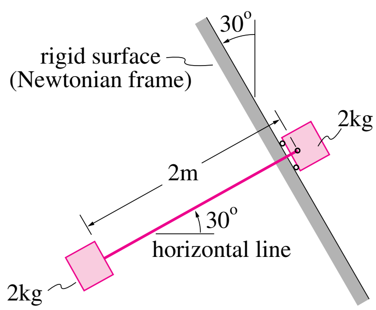
Problem 19.3.19

19.3.20 Using free-body diagrams and appropriate momentum balance equations, find differential equations that govern the angle θ and the vertical deflection y of the system shown. Be clear about your datum for y. Your equations should be in terms of θ, y and their time-derivatives, as well as M, m, ℓ, and g.



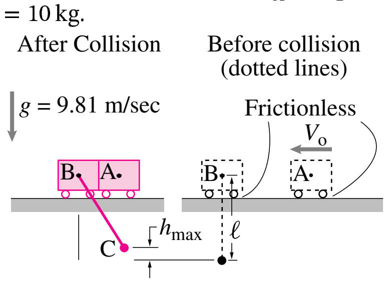
Problem 19.3.20

19.3.21 The two blocks shown are released from rest at $t = 0$. There is no friction and the cable is initially taut. (a) What is the tension in the cable immediately after release? (Use any reasonable value for the gravitational constant). (b) What is the tension after 5 s?



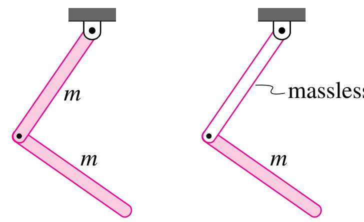
Problem 19.3.21

19.3.22 Carts A and B are free to move along a frictionless horizontal surface, and bob C is connected to cart B by a massless, inextensible cord of length ℓ, as shown in the figure. Cart A moves to the left at a constant speed $v_0 = 1\,\text{m/s}$ and makes a perfectly plastic collision ($e = 0$) with cart B which, together with bob C, is stationary prior to impact. Find the maximum vertical position of bob C, h_{max}, after impact. The masses of the carts and pendulum bob are $m_A = m_B = m_C = 10\,\text{kg}$.



Problem 19.3.22

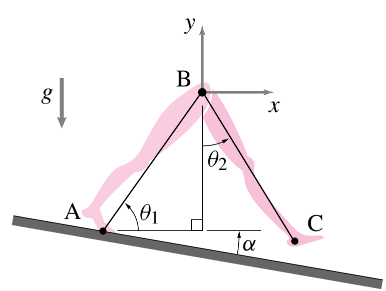
19.3.23 A double pendulum is made of two uniform rigid rods, each of length ℓ. The first rod is massless. Find equations of motion for the second rod. Define any variables you use in your solution.



Problem 19.3.23

19.3.24 A model of walking involves two straight legs. During the part of the motion when one foot is on the ground, the system looks like the picture in the figure, confined to motion in the $x-y$ plane. Write two equations from which one could find $\ddot{\theta}_1$, and $\ddot{\theta}_2$ given $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ and all mass and length quantities.

Hint:
$$\left[\begin{array}{lll} \sum \boldsymbol{M}_{/A} & = \dot{\boldsymbol{H}}_{/A} & \text{whole system} \\ \sum \boldsymbol{M}_{/B} & = \dot{\boldsymbol{H}}_{/B} & \text{for bar BC} \end{array}\right]$$



Problem 19.3.24

Advanced problems in 2D motion

19.3.25 A pendulum is hanging from a moving support in the xy-plane. The support moves in a known way given by $\vec{\boldsymbol{r}}(t) = r(t)\hat{\boldsymbol{\imath}}$. For the following cases, find a differential equation whose solution would determine $\theta(t)$, measured clockwise from vertical; then find an expression for $d^2\theta/dt^2$ in terms of θ and $r(t)$:

a) with no gravity and assuming the pendulum rod is massless with a point mass of mass m at the end,

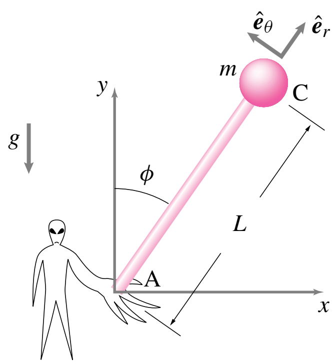
b) as above but with gravity,

c) assuming the pendulum is a uniform rod of mass m and length ℓ.

19.3.26 Robotics problem: balancing a broom stick by sideways motion. Try to balance a broom stick by moving your hand horizontally. Model your hand contact with the broom as a hinge. You can model the broom as a uniform stick or as a point mass at the end of a stick — your choice.

a) **Equation of motion.** Given the acceleration of your hand (horizontal only), the current tip, and the rate of tip of the broom, find the angular acceleration of the broom. *

b) **Control?** Can you find a hand acceleration in terms of the tip and the tip rate that will make the broom balance upright? *



Problem 19.3.26

19.3.27 Balancing the broom again: vertical shaking works too. You can balance a broom by holding it at the bottom and applying appropriate torques, as in problem 16.4.44 or by moving your hand back and forth in an appropriate manner, as in problem 19.3.26. In this problem, you will try to balance the broom differently. The lesson to be learned here is more subtle, and you should probably just wonder at it rather than try to understand it in detail.

In the previous balancing schemes, you used knowledge of the state of the broom (θ and $\dot{\theta}$) to determine what corrective action to apply. Now balance the broom by moving it in a way that ignores what the broom is doing. In the language of robotics, what you have been doing before is 'closed-loop feedback' control. The new strategy, which is simultaneously more simple minded and more subtle, is an 'open loop' control. Imagine that your hand connection to the broom is a hinge.

a) **Picture and model.** Assume your hand oscillates sinusoidally up and down with some frequency and some amplitude. The broom is instantaneously at some angle from vertical. Draw a picture which defines all the variables you will use. Use any mass distribution that you like.

b) **FBD.** Draw a FBD of the broom.

c) **Momentum balance.** Write the equation of angular momentum balance about the point instantaneously coinciding with the hinge.

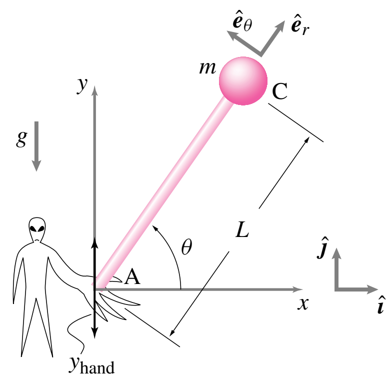
d) **Kinematics.** Use any geometry and kinematics that you need to evaluate the terms in the angular momentum balance equation in terms of the tip angle and its time derivatives and other known quantities (take the vertical motion of your hand as 'given'). [Hint: There are many approaches to this problem.]

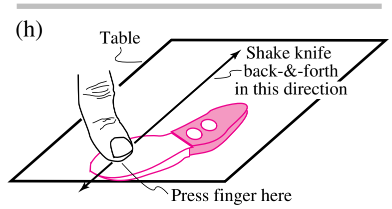
e) **Equation of motion.** Using the angular momentum balance equation, write a governing differential equation for the tip angle. *

f) **Simulation.** Taking the hand motion as given, simulate on the computer the system you have found.

g) **Stability?** Can you find an amplitude and frequency of shaking so that the broom stays upright if started from a near upright position? You probably *cannot* find linear equations to solve that will give you a control strategy. So, this problem might best be solved by guessing on the computer. Successful strategies require the hand acceleration to be quite a bit bigger than g, the gravitational constant. The stability obtained is like the stability of an undamped uninverted pendulum — oscillations persist. You improve the stability a little by including a little friction in the hinge.

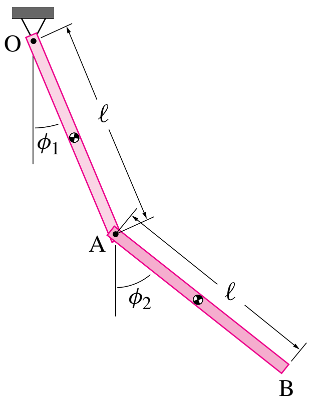
h) **Dinner table experiment for nerdy eaters.** If you put a table knife on a table and put your finger down on the tip of the blade, you can see that this experiment might work. Rapidly shake your hand back and forth, keeping the knife from sliding out from your finger but with the knife sliding rapidly on the table. Note that the knife aligns with the direction of shaking (use scratch-resistant surface). The knife is different from the broom in some important ways: there is no gravity trying to 'unalign' it and there is friction between the knife and table that is much more significant than the broom interaction with the air. Nonetheless, the experiment should convince you of the plausibility of the balancing mechanism. Because of the large accelerations required, you *cannot* do this experiment with a broom.





Problem 19.3.27: The sketch of the knife on the table goes with part (h).

19.3.28 Double pendulum. The double pendulum shown is made up of two uniform bars, each of length ℓ and mass m. The pendulum is released from rest at $\phi_1 = 0$ and $\phi_2 = \pi/2$. Just after release what are the values of $\ddot{\phi}_1$ and $\ddot{\phi}_2$? Answer in terms of other quantities. *

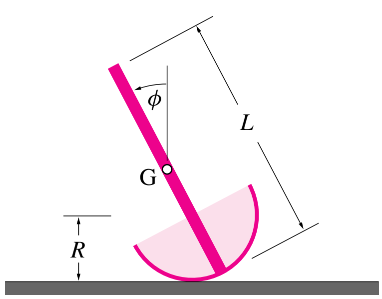


Problem 19.3.28

19.3.29 A rocker. A standing dummy is modeled as having massless rigid circular feet of radius R rigidly attached to their uniform rigid body of length L and mass m. The feet do not slip on the floor.

a) Given the tip angle ϕ, the tip rate $\dot{\phi}$ and the values of the various parameters (m, R, L, g) find $\ddot{\phi}$. [You may assume ϕ and $\dot{\phi}$ are small.] *

b) Using the result of (a) or any other clear reasoning find the conditions on the parameters (m, R, L, g) that make vertical passive dynamic standing stable. [Stable means that if the person is slightly perturbed from vertically up that their resulting motion will be such that they remain nearly vertically up for all future time.] *



Problem 19.3.29

19.3.30 Consider a rigid spoked wheel with no rim. Assume that when it rolls a spoke hits the ground and doesn't bounce. The body just swings around the contact point until the next spoke hits the ground. The uniform spokes have length R. Assume that the mass of the wheel is m, and that the polar moment of inertia about its center is I (use $I = mR^2/2$ if you want to get a better sense of the solution). Assume that just before collision number n, the angular velocity of the wheel is ω_{n-}, the kinetic energy is T_{n-}, the potential energy (you must clearly define your datum) is U_n-. Just after collision n the angular velocity of the wheel is ω_{n+}. The Kinetic Energy is T_{N+}, the potential energy (you must clearly define your datum) is U_{n+}. The wheel has k spokes (pick $k = 4$ if you have trouble with abstraction). This problem is not easy. It can be answered at a variety of levels. The deeper you get into it the more you will learn.

a) What is the relation between ω_{n-} and T_{n-}?

b) What is the relation between ω_{n-} and ω_{n+}?

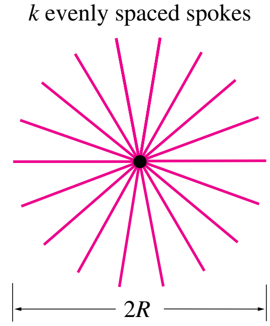
c) Assume 'rolling' on level ground. What is the relation between ω_n^+ and ω_{n+1}^+ (the angular velocities just after two successive collisions?

d) Assume rolling down hill at slope θ. What is the relation between ω_{n+} and ω_{n+1}?

e) Can it be true that $\omega_{n+} = \omega_{(n+1)+}$? About how fast is the wheel going in this situation?

f) As the number of spokes m goes to infinity, in what senses does this wheel become like an ordinary wheel?

k evenly spaced spokes



Problem 19.3.30