19.1 Mechanics of a constrained particle

The kinematics of time-varying base vectors help us deal with some more difficult particle motion mechanics problems. For one point mass it is easy to write balance of linear momentum. It is:

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}.$$

The mass of the particle m times its vector acceleration $\vec{\boldsymbol{a}}$ is equal to the total force on the particle $\vec{\boldsymbol{F}}$. No problem.

Now, however, we can write this equation in five somewhat distinct ways.

1. In general abstract vector form: $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.
2. In cartesian coordinates: $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}} = m[\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}]$.
3. In polar coordinates:
$$F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta + F_z\hat{\boldsymbol{k}} = m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}].$$
4. In path coordinates: $F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n = m[\dot{v}\hat{\boldsymbol{e}}_t + (v^2/\rho)\hat{\boldsymbol{e}}_n]$

All of these equations are always right. Additionally, for a given particle moving under the action of a given force there are many more correct equations that can be found by shifting the origin and orientation of the coordinate systems. For example for a moving frame $\mathcal{B}$ with origin $0'$, rotation rate $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ and angular acceleration $\vec{\boldsymbol{\alpha}}_{\mathcal{B}}$:

5. $\vec{\boldsymbol{F}} = m\left\{\vec{\boldsymbol{a}}_{0'} \quad - \omega_{\mathcal{B}}^2\vec{\boldsymbol{r}} \quad + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}} \quad + \vec{\boldsymbol{a}}_{/\mathcal{B}} \quad + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{/\mathcal{B}}\right\}$

where, to simplify the notation, all motions are relative to $\mathcal{F}$ and positions relative to 0 unless explicitly indicated by a $/\mathcal{B}$ or $/0'$. This is quite a collection of kinematic tools. In general we want to choose the best tools for the job. But to get a sense let's first look at a simple problem using each of these kinematic approaches, some of which are rather inappropriate.

A particle that moves with no net force

In the special case that a particle has no force on it we know intuitively, or from the verbal statement of Newton's First Law, that the particle travels in a straight line at constant speed. As a first example, let's try to find this result using the vector equations of motion five different ways: in the general abstract form, in cartesian coordinates, in polar coordinates, in path coordinates, and relative to a moving frame (see fig. 19.1).

General abstract form. The equation of linear momentum balance is $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ or, if there is no force, $\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$, which means that $d\vec{\boldsymbol{v}}/dt = \vec{\boldsymbol{0}}$. So $\vec{\boldsymbol{v}}$ is a constant. We can call this constant $\vec{\boldsymbol{v}}_0$. So after some time the particle is where it was at $t = 0$, say, $\vec{\boldsymbol{r}}_0$, plus its velocity $\vec{\boldsymbol{v}}_0$ times time. That is:

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{v}}_0 t. \tag{19.1}$$

This vector relation is a parametric equation for a straight line. The particle moves in a straight line, as expected.

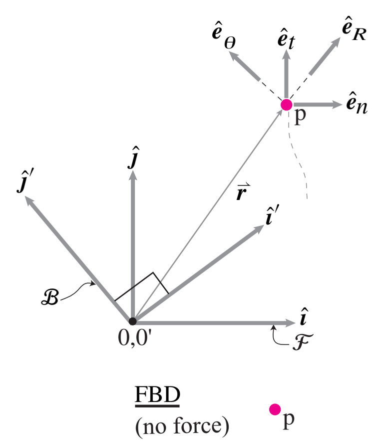

Figure 19.1: A particle P moves. One can track its motion using the general vector form $\vec{\boldsymbol{r}}$, Cartesian coordinates in the fixed frame $\mathcal{F} = 0\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}$, Polar coordinates using $\hat{\boldsymbol{e}}_R\&\hat{\boldsymbol{e}}_\theta$, path coordinates $\hat{\boldsymbol{e}}_t\&\hat{\boldsymbol{e}}_n$ and cartesian coordinates in a rotating frame $\mathcal{B} = 0'\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'$.

Cartesian coordinates. If instead we break the linear momentum balance equation into cartesian coordinates we get

$$F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}} = m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}).$$

Because the net force is zero and the net mass is not negligible,

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \text{ and } \qquad \ddot{z} = 0.$$

These equations imply that $\dot{x}$, $\dot{y}$, and $\dot{z}$ are all constants, let's call them v_{x0}, v_{y0}, v_{z0}. So x, y, and z are given by

$$x = x_0 + v_{x0}t, \quad y = y_0 + v_{y0}t, \quad \& \; z = z_0 + v_{z0}t.$$

We can put these components into their place in vector form to get:

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} = (x_0 + v_{x0}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{y0}t)\hat{\boldsymbol{\jmath}} + (z_0 + v_{z0}t)\hat{\boldsymbol{k}}. \quad (19.2)$$

Note that there are six free constants in this equation representing the initial position and velocity. Equation 19.2 is a cartesian representation of equation 19.1; it describes a straight line being traversed at constant rate.

Polar/cylindrical coordinates. When there is no force, in polar coordinates we have:

$$\underbrace{F_R}_{0}\,\hat{\boldsymbol{e}}_R + \underbrace{F_\theta}_{0}\,\hat{\boldsymbol{e}}_\theta + \underbrace{F_z}_{0}\,\hat{\boldsymbol{k}} = m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}].$$

This vector equation leads to the following three scalar differential equations, the first two of which are coupled non-linear equations (neither can be solved without the other).

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \\ \ddot{z} &= 0 \end{aligned}$$

A tedious calculation will show that these equations are solved by the following functions of time:

$$\begin{aligned} R &= \sqrt{d^2 + [v_0(t - t_0)]^2} \qquad (19.3) \\ \theta &= \theta_0 + \tan^{-1}[v_0(t - t_0)/d] \\ z &= z_0 + v_{z0}t, \end{aligned}$$

where θ_0, d, t_0, v_0, z_0, and v_{z0} are constants. Note that, though eqn. (19.4) looks different than eqn. (19.2), there are still 6 free constants. From the physical interpretation you know that eqn. (19.4) must be the parametric equation of a straight line. And, indeed, you can verify that picking arbitrary constants and using a computer to make a polar plot of eqn. (19.4) does in fact show a straight line. From eqn. (19.4) it seems that polar coordinates' main function is to obfuscate rather than clarify. For the simple case that a particle moves with no force at all, we have to solve non-linear differential equations whereas using cartesian coordinates we get linear equations which are easy to solve and where the solution is easy to interpret.

But, if we add a central force, a force like earth's gravity acting on an orbiting satellite (the force on the satellite is directed towards the center of the earth), the equations become almost intolerable in cartesian coordinates. But, in polar coordinates, the solution is almost as easy (which is not all that easy for most of us) as the solution 19.4. So the classic analytic solutions of celestial mechanics are usually expressed in terms of polar coordinates.

Path coordinates. When there is no force, $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ is expressed in path coordinates as

$$\underbrace{F_t}_{0}\,\hat{\boldsymbol{e}}_t + \underbrace{F_n}_{0}\,\hat{\boldsymbol{e}}_n = m(\dot{v}\hat{\boldsymbol{e}}_t + \underbrace{(v^2/\rho)\hat{\boldsymbol{e}}_n}_{v^2\vec{\boldsymbol{\kappa}}}).$$

That is,

$$\dot{v} = 0 \quad \text{and} \quad v^2/\rho = 0.$$

So the speed v must be constant and the radius of curvature ρ of the path infinite. That is, the particle moves at constant speed in a straight line.

Relative to a rotating reference frame Let's look at the equations using a frame $\mathcal{B}$ that shares an origin with $\mathcal{F}$ but is rotating at a constant rate $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}}$ relative to $\mathcal{F}$. Thus

$$\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \vec{\boldsymbol{0}} \quad \text{and} \quad \vec{\boldsymbol{a}}_{0'/0} = \vec{\boldsymbol{0}}$$

and we have that $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ is written as

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\vec{\boldsymbol{0}} &= m\{\underbrace{\vec{\boldsymbol{a}}_{0'}}_{\vec{\boldsymbol{0}}} \quad -\omega_{\mathcal{B}}^2\vec{\boldsymbol{r}} \quad + \underbrace{\vec{\boldsymbol{\alpha}}_{\mathcal{B}}}_{\vec{\boldsymbol{0}}}\times\vec{\boldsymbol{r}} \quad + \vec{\boldsymbol{a}}_{/\mathcal{B}} \quad + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}}\times\vec{\boldsymbol{v}}_{/\mathcal{B}}\} \\
&= -\omega^2\vec{\boldsymbol{r}} + \vec{\boldsymbol{a}}_{/\mathcal{B}} + 2\omega\hat{\boldsymbol{k}}\times\vec{\boldsymbol{v}}_{/\mathcal{B}}. \qquad (19.4)
\end{aligned}$$

Now, using the rotating base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, we have that $F_{x'}\hat{\boldsymbol{\imath}}' + F_{y'}\hat{\boldsymbol{\jmath}}' = 0\hat{\boldsymbol{\imath}}' + 0\hat{\boldsymbol{\jmath}}'$ and

$$\vec{\boldsymbol{r}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}', \quad \vec{\boldsymbol{v}}_{\mathcal{B}} = \dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}', \quad \text{and} \quad \vec{\boldsymbol{a}}_{\mathcal{B}} = \ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}'\hat{\boldsymbol{\jmath}}'$$

so eqn. (19.4) can be rewritten as

$$\vec{\boldsymbol{0}} = -\omega^2(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') + (\ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}'\hat{\boldsymbol{\jmath}}') + 2\omega\hat{\boldsymbol{k}}\times(\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}')$$

which in turn can be broken into components and written as:

$$\begin{aligned} \ddot{x}' &= \omega^2 x' + 2\omega \dot{y}' \\ \ddot{y}' &= \omega^2 y' - 2\omega \dot{x}' \end{aligned} \tag{19.5}$$

which makes up a pair of second order linear differential equations. With some work, someone good at ODEs can solve this with pencil and paper. But most of us would use a computer for such a system. If, in some consistent units, we had

$$y'(0) = 0, \dot{y}'(0) = 0, x'(0) = 0, \dot{x}'(0) = 1, \omega = 1$$

then the solution turns out to be

$$\begin{aligned} x' &= t\cos(\omega t) \\ y' &= -t\sin(\omega t) \end{aligned}$$

as you can check by substituting into eqn. (19.6). That is, a particle which goes in a straight line away from the origin goes, in spirals as seen in the rotating frame①.

①And the straight line motion could have been more complicated yet, as seen in the rotating frame, with variable rate rotation and acceleration.This is why we are lucky the earth is not spinning fast or at highly variable rates. Otherwise the motion we would see, us humans here on the rotating earth, would be particles with no force on them spiraling around all over the place. And that's what Isaac Newton would have seen. And he would have written "A particle in motion tends to go in crazy spirals, and a particle that is initially at rest also goes bonkers" and the equations we use through out this book would have been much harder to discover.

Constrained motion

A particle in a plane has 2 degrees of freedom. There is basically only one kind of constraint — to a path. When constrained to a path the particle has one remaining degree of freedom so its configuration can be described with one variable. For a given problem you must think about

- What force(s) constrain the motion to the path?
- What do you want to use for a configuration variable?

You use these ideas to

- draw an appropriate free-body diagram, and then
- calculate the velocity and acceleration in terms of the configuration variable and its derivatives.

After these key first steps you plug into equations of motion and solve for what you are interested in. Of course the needed math could be difficult or impossible, but the work is somewhat routine from a mechanics point of view.

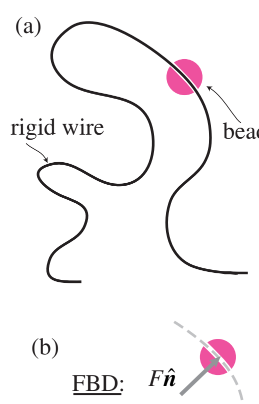

Figure 19.2: A point-mass bead slides on a rigid immobile frictionless wire. The free-body diagram shows that the only force on the bead is in the direction normal to the wire.

Example: Bead on frictionless wire

A bead slides on a frictionless wire with a crazy but smooth shape. No forces are applied to the bead besides the constraint force (see fig. 19.2).

Fig. 19.2b shows a free-body diagram where $\hat{\boldsymbol{n}}$ is the normal to the wire at the point of interest. It doesn't matter if you use for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_n$, or $\hat{\boldsymbol{n}} =$ a vector always, say, to the left, just so long as you know what you mean by $\hat{\boldsymbol{n}}$. The free-body diagram shows that you know that the constraint force is normal to the path (the frictionless wire) but that you don't know how big it is (F is an unknown scalar).

For some purposes, especially general problems like this where no specific path is given, the most appropriate configuration variable is s, the arc length along the path. If the path is given we assume we know the position at any given arc length by the functions

$$x(s) \quad \text{and} \quad y(s).$$

So

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}(s), \vec{\boldsymbol{v}} = \dot{s}\hat{\boldsymbol{e}}_t, \quad \text{and} \quad \vec{\boldsymbol{a}} = (\dot{s}^2/\rho)\hat{\boldsymbol{e}}_n + \ddot{s}\hat{\boldsymbol{e}}_t.$$

Now we can write linear momentum balance

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \qquad (19.6)\\
\{ F\hat{\boldsymbol{n}} &= m\left((\dot{s}^2/\rho)\hat{\boldsymbol{e}}_n + \ddot{s}\hat{\boldsymbol{e}}_t \right)\}\\
\{\}\cdot\hat{\boldsymbol{e}}_t \quad \Rightarrow \quad \dot{v} &= 0 \quad \text{and}\\
\{\}\cdot\hat{\boldsymbol{e}}_n \quad \Rightarrow \quad F &= mv^2/\rho.
\end{aligned}$$

Eqn. 19.6 tells us that the bead moves at constant speed, no matter what the shape of the wire. It also tells us that the more curved the wire, the bigger the constraint force needed to keep the bead on the wire.

Because this is a 1 DOF system, any one equation of motion should give us the result. Instead of linear momentum balance we could have used power balance to get the same result like this:

$$\begin{aligned}
P &= \dot{E}_{\mathrm{K}}\\
\Rightarrow \quad \vec{\boldsymbol{F}}_{\mathrm{tot}}\cdot\vec{\boldsymbol{v}} &= \frac{d}{dt}\left\{\frac{1}{2}mv^2\right\}\\
\Rightarrow \quad F\hat{\boldsymbol{e}}_n\cdot v\hat{\boldsymbol{e}}_t &= mv\dot{v}\\
\Rightarrow \quad 0 &= \dot{v}.
\end{aligned}$$

This is natural enough. The only force on the particle is perpendicular to its motion, so does no work. So the particle must have constant kinetic energy and its speed must be constant.

The example above doesn't seem that applicable; how often does one see beads on frictionless wires? It is a bit more useful than it appears. For example, if a car is coasting on an open road and tire resistance and sideslip can be ignored, the reasoning above shows that the car is neither speeded nor slowed by turning. Similarly, an idealization of an airplane wing is as something which only causes force perpendicular to motion. So in quick maneuvers where the gravity force is relatively small, the plane maintains its speed.

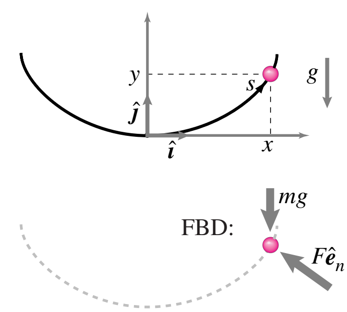

Figure 19.3: A bead slides on a frictionless wire on the curve implicitly defined by $y = cs^2/2$, where s is arc-length along the curve measured from the origin.

Example: The hard brachistochrone problem.
Here is a puzzle proposed by Johann Bernoulli in June 1696 2: given that a roller coaster has to coast from rest at one place to another place that is no higher, what shape should the track be to make the trip as quick as possible.

The solution. Finding the solution, or even verifying it, is a problem in the calculus of variations, *i.e.,* too advanced for this book. The solution turns out to be the brachistochrone curve that obeys the following relationship between arc-length s from the origin and vertical position y (drawn accurately in fig. 19.3):

$$y = \frac{1}{2}cs^2 \qquad (\text{for} |s| \leq 1/c). \qquad (19.7)$$

Starting at the origin this curve is close to $y = cx^2/2$ but gets a bit higher (bigger y) because for a given value of x, s is greater than x. The curve terminates at vertical tangents at $s = \pm 1/c$ where $y = 1/2c$. (see fig. 19.3). To solve the

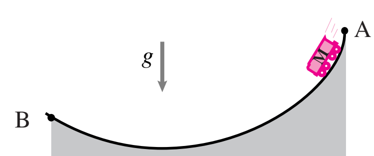

Figure 19.4: The roller coaster that gets from A to B the fastest is the one with a track in the shape of the brachistochrone $y = cs^2/2$.

② **The original brachistochrone (least time) puzzle:**
"I, Johann Bernoulli, greet the most clever mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem whose possible solution will bestow fame and remain as lasting monument. Following the example set by Pascal, Fermat, etc., I hope to earn the gratitude of the entire scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall then publicly declare him worthy of praise.

"*Let two points A and B be given in a vertical plane. Find the curve that a point M, moving on a path AMB must follow such that, starting from A, it reaches B in the shortest time under its own gravity.*"

Newton. Besides Bernoulli's brother Jakob, one of the people to solve the puzzle was 55 year old Isaac Newton. He attempted to keep his solution, said to have been worked out in one evening while he also invented the calculus of variations, anonymous. But Bernoulli supposedly saw through this deception, commenting "I recognize the lion by his paw print", which was presumably not a comment about Newton's handwriting.

③ Actually, the amplitude can't be arbitrarily large. The solution to the defining eqn. (19.7) only makes sense for $|s| < 1/c$. For $|s| > 1/c$ there is no curve satisfying eqn. (19.7).

puzzle this curve is scaled (by choosing a value of c) and displaced so that it has a vertical tangent at A and also so the curve goes through B. The idea that the hard math seems to be expressing, is that the particle should first build up as much speed as it can (by going straight down) and then head off in the right direction (see fig. 19.4).

On the other hand, here is an easier problem that is a virtual setup for the techniques now at hand.

Example: The easy brachistochrone problem
How long does it take for a particle to slide back and forth on a frictionless wire with $y = cs^2/2$ as driven by gravity? (see fig. 19.3)

Let's use s as our configuration variable. The power balance equation is:

$$\begin{aligned}
&& P &= \dot{E}_{\mathrm{K}} \\
(F\hat{\boldsymbol{e}}_n)\cdot(v\hat{\boldsymbol{e}}_t) = 0 \quad \Rightarrow && -mg\hat{\boldsymbol{\jmath}}\cdot(\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}) &= \frac{d}{dt}\left\{\frac{1}{2}mv^2\right\} \\
\Rightarrow && -mg\dot{y} &= m\frac{d}{dt}\frac{\dot{s}^2}{2} \\
\Rightarrow && -mgcs\dot{s} &= m\dot{s}\ddot{s} \\
\text{Assuming } \dot{s} \neq 0 \quad \Rightarrow && \ddot{s} + gcs &= 0.
\end{aligned}$$

This, remarkably, is the simple harmonic oscillator equation with general solution

$$s = A\cos(\sqrt{gc}\,t) + B\sin(\sqrt{gc}\,t).$$

Thus the period of oscillation (T such that $\sqrt{gc}T = 2\pi$) is

$$T = \frac{2\pi}{\sqrt{gc}}$$

which is independent of the amplitude of oscillation ③.

The key to this quick solution was using a configuration variable that made the expression for the velocity simple, and using an equation of motion that didn't involve the unknown reaction force F which we also didn't care about. We could have got the same equation of motion by writing $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and eliminated $F\hat{\boldsymbol{e}}_n$ by dotting both sides with a convenient vector orthogonal to $F\hat{\boldsymbol{e}}_n$, say $\vec{\boldsymbol{v}}$.

The brachistochrone is a famous curve that has various interesting properties (*e.g.,* Box 19.1 on page 7).

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Example: A collar on two rotating rods
Consider a pair of collars hinged together as a point mass m at P. Each slides frictionlessly on a rod about whose rotation everything is known (see fig. 19.5. What is the force of rod 1 on the mass? For this 2 degree of freedom system let's use configuration variables θ_1 and θ_2, and two sets of rotating base vectors: $\hat{\boldsymbol{\lambda}}_1\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{\lambda}}_2\hat{\boldsymbol{n}}_2$. These rotating base vectors can be written in terms of the θs, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in the standard manner. Assume we know ℓ_1 and ℓ_2 in this configuration. First find $\dot{\ell}_1$ and $\dot{\ell}_2$ by thinking of the velocity of the collar two different ways:

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\boldsymbol{v}} \qquad (19.8)\\
\{\,\dot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \dot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 &= \dot{\ell}_2\hat{\boldsymbol{\lambda}}_2 + \dot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\,\} \\
\{\}\cdot\hat{\boldsymbol{n}}_2 \quad &\Rightarrow \quad \dot{\ell}_1 = \frac{\dot{\theta}_2\ell_2 - \dot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2}{\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{n}}_2} \\
\{\}\cdot\hat{\boldsymbol{n}}_1 \quad &\Rightarrow \quad \dot{\ell}_2 = \frac{\dot{\theta}_1\ell_1 - \dot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_1}{\hat{\boldsymbol{\lambda}}_2\cdot\hat{\boldsymbol{n}}_1}.
\end{aligned}$$

Having found $\dot{\ell}_1$ and $\dot{\ell}_2$ we can find the velocity $\vec{\boldsymbol{v}}$ by evaluating either side of eqn. (19.8). Now we apply identical reasoning with the acceleration. The result looks messy, but the approach is straightforward:

$$\begin{aligned} \vec{\boldsymbol{a}} &= \vec{\boldsymbol{a}} \\ \{\, \ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1 &= \ddot{\ell}_2\hat{\boldsymbol{\lambda}}_2 + \ddot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2 + 2\dot{\ell}_2\dot{\theta}_2\hat{\boldsymbol{n}}_2 \,\} \end{aligned} \tag{19.9}$$

$$\{\}\cdot\hat{\boldsymbol{n}}_2 \quad\Rightarrow\quad \ddot{\ell}_1 = \frac{\ddot{\theta}_2\ell_2 + 2\dot{\ell}_2\dot{\theta}_2 - \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2 - 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2}{\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{n}}_2}$$

$$\{\}\cdot\hat{\boldsymbol{n}}_1 \quad\Rightarrow\quad \ddot{\ell}_2 = \frac{\ddot{\theta}_1\ell_1 + 2\dot{\ell}_1\dot{\theta}_1 - \ddot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_1 - 2\dot{\ell}_2\dot{\theta}_2\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_1}{\hat{\boldsymbol{\lambda}}_2\cdot\hat{\boldsymbol{n}}_1}$$

We use the results from eqn. (19.8) for $\dot{\ell}_1$ and $\dot{\ell}_2$ to evaluate the right hand sides of the expressions for $\ddot{\ell}_1$ and $\ddot{\ell}_2$ in eqn. (19.9). So now either the left hand side or the right hand side of the second of Eqns. 19.9 can be used to evaluate the acceleration $\vec{\boldsymbol{a}}$, all the terms in both expressions have been found.

To find the forces we use linear momentum balance and the free-body dia-

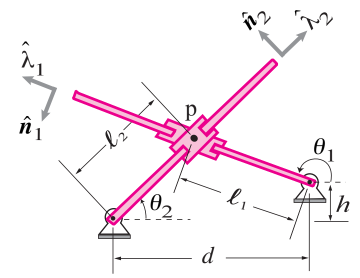
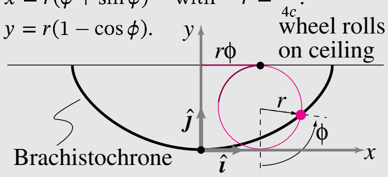


Figure 19.5: A point-mass collar slides simultaneously on 2 rods.

19.1 Some brachistochrone curiosities

The brachistochrone is a cycloid. There is no straightforward way to draw the curve $y = cs^2/2$ because the formula doesn't tell you the x coordinates of the points. You could find them by integrating $dx = \sqrt{ds^2 - dy^2}$ numerically or with calculus tricks. But it turns out (see below) that the curve with $y = cs^2/2$ is described by the parametric equations

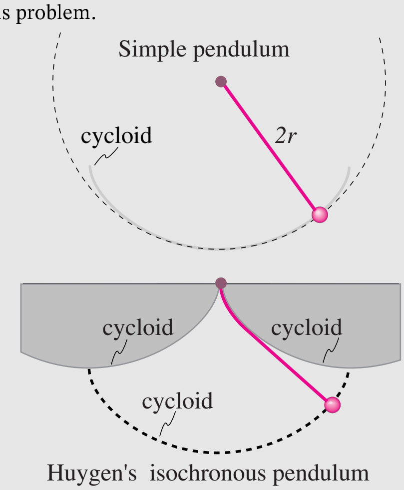
$$x = r(\phi + \sin\phi) \quad \text{with} \quad r = \frac{1}{4c}.$$
$$y = r(1 - \cos\phi).$$



This is the path of a particle on the perimeter of a wheel that rolls against a horizontal ceiling a distance $2r$ above the origin, as you can verify by adding up distances in the picture above (see page 937). We will show below that the upside down cycloid and the curve $y = cs^2/2$ are one and the same.

Note that the osculating circle of this cycloid at its lowest point has radius $4r = 1/c$, just the length of a simple pendulum that, for small oscillations, has the same frequency of oscillation as the bead on the brachistochrone. A point mass swinging on a string is like a bead on a frictionless circular wire and this, in turn, is close to the motion of a bead on a brachistochrone wire for small oscillations.

Galileo (1564-1642). Well before Bernoulli's challenge, Galileo was interested in things rolling and sliding on ramps. He knew that the shortest distance between two points is a straight line, and had noted that a ball rolling down an appropriately curved ramp gets to its destination faster than a ball traveling the shortest route. A ball going on a straight ramp just doesn't pick up much speed, and when it finally has its greatest speed the trip is over. Imagine sliding straight sideways; it takes forever on a straight-line route. Better, he must have reasoned, to get the ball rolling fast at the start and then go fast for most of its journey, possibly slowing at the end. Galileo thought the best shape was the bottom of a circle (or fraction thereof), which isn't far off either in shape or concept, but isn't quite right. Galileo was apparently obsessed with cycloids for other reasons but didn't see their connection to this problem.



A constant period pendulum. For clock time keeping, a pendulum is better than a bead on a wire because the friction of sliding is avoided. Unfortunately, a simple pendulum has a period which is longer if the amplitude is bigger. Not much longer, 18%

(continued...)

gram

$$
\begin{aligned}
\vec{\boldsymbol{F}}_{\text{tot}} &= m\vec{\boldsymbol{a}} \qquad (19.10)\\
\{\, F_1\hat{\boldsymbol{n}}_1 + F_2\hat{\boldsymbol{n}}_2 &= m\{\ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\}\,\}
\end{aligned}
$$

$$\{\}\cdot\hat{\boldsymbol{\lambda}}_2 \quad\Rightarrow\quad F_2 = \frac{\{\ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\}\cdot\hat{\boldsymbol{\lambda}}_2}{\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{\lambda}}_2}$$

$$\{\}\cdot\hat{\boldsymbol{\lambda}}_1 \quad\Rightarrow\quad F_1 = \frac{\{\ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\}\cdot\hat{\boldsymbol{\lambda}}_1}{\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{\lambda}}_1}$$

When actually evaluating the expressions above one can write the base vectors in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ or use geometry.

Often when working out a problem it is best to not substitute numbers until the end of a problem. This example shows the opposite. If we left the expressions for $\dot{\ell}_1$ and $\dot{\ell}_2$ with letters and substituted that into the expressions for $\ddot{\ell}_1$ and $\ddot{\ell}_2$ and left those expressions intact while substituting for the acceleration $\vec{\boldsymbol{a}}$ we would have large expressions for the force components F_1 and F_2. On the other hand, by using numbers as the calculation progresses the formulas do not grow so much in complexity.

19.1 Some brachistochrone curiosities (continued)

if the swinging is $\pm 90^\circ$ and only 1.7% longer if the amplitude is $\pm 30^\circ$, but enough to annoy clock designers. A bead sliding frictionlessly on the path $y = s^2/2$ has the nice property that the period does not depend on the amplitude. But any real bead sliding on any real wire has substantial friction. So, at first blush the brachistochrone curve, despite its nice constant-period property, cannot be used to keep time.

But Huygens, one of the smart old timers, looked for a curve that, when a string wraps around it, makes the end follow the brachistochrone curve. To this day you can see fancy old clocks with this wrapping device, a solid piece with two cuspoidal shapes at the hinge of the swinging ("isochronous" or "tautochrone") pendulum which wraps around it.

Geometry. The two key features discussed above, that the curve $y = cs^2/2$ is a cycloid, and that a cycloid can be generated by wrapping a string around another cycloid, can be found from the geometric construction below. Two cycloids are shown, one from wheel 1 rolling under line L_1 and another from wheel 2 rolling under line L_2 a distance $2r$ below. Both wheels have radius r. Imagine that the cycloids A_1M_1B and A_2M_2B are drawn by wheels always arranged with vertically aligned rolling contact points C_1 and C_2 and with points M_1 and M_2 initially aligned vertically a distance $4r$ apart at A_1 and A_2.

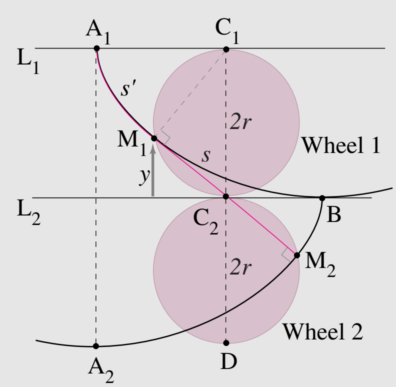
The two cycloids are thus the same shape but are displaced with one being $2r$ below and πr sideways from the other.

Because both wheels have rolled the same distance (A_1C_1) they have rotated the same amount and M_2 is as far forward of C_1C_2D as M_1 is behind. Similarly M_1 is as far above L_2 as M_2 is below. So the line M_1M_2 is bisected by the point C_2.

Because C_1C_2 is the diameter of a circle with M_1 on the perimeter, angle $C_1M_1C_2$ is a right angle. Because material point C_1 on the wheel has zero velocity the velocity of M_1 (and thus the tangent to the curve) is orthogonal to C_1M_1. Thus the line M_1M_2 is tangent to the upper cycloid.

The rolling of wheel 2 instantaneously rotating about C_2 makes the tangent to the lower cycloid orthogonal to M_1M_2, the condition for the motion of M_2 to be from the wrapping of an inextensible line around the curve A_1M_1B. This shows that cycloid A_2M_2B is generated by the wrapping of a line anchored at A_1 about the upper cycloid. And this is Huygen's wrapping mechanism for making a pendulum bob follow a cycloid. Because of this wrapping generation, the arc-length $s' + s$ of A_1M_1B must be $4r$ and the arc length s of M_1B is M_1M_2 so the length M_1C_2 is $s/2$. By the similarity of the two right triangles that share the length $s/2$ of M_1C_2:

$$\frac{y}{s/2} = \frac{s/2}{2r} \qquad\Rightarrow\qquad y = \frac{1}{4r}\frac{s^2}{2} = c\frac{s^2}{2}$$



which shows that the upper cycloid is the curve $y = cs^2/2$ if $c = 1/4r$, where s is measured from B. This was the equation used to show the constant period nature of the sliding motion of a bead on a frictionless cycloidal curve using power balance.

As is the case with most mechanism-mechanics problems, the hard work in getting the dynamics equations is in the kinematics. Generally there are no great short-cuts. There are alternative methods. In this case the location of the base points and the two angles determine the base and two angles of a triangle. This triangle can be solved for the location of the point P. Once that position is known in terms of θ_1 and θ_2 the velocity and acceleration can be found by differentiation.

As a robot manipulator, this design has the advantage that no motors need to be displaced. It has the disadvantage of requiring good sliding joints.

An alternative solution of the kinematics of this problem would be to use trigonometry to find the position of point P in terms of the angles θ_1 and θ_2. Then the acceleration of point P is found by taking two time derivatives. The result is approximately equal in the complexity of its appearance to the results used above. That method requires more cleverness at the start (solving an angle-side-angle triangle) and then just brute force differentiation using the chain rule and the product rule.

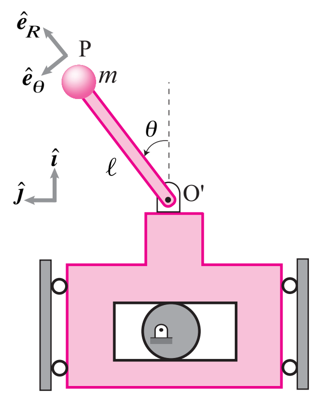


Figure 19.6: The base of a pendulum is vertically vibrated.

Example: Inverted pendulum with a vibrating base

Assume that the base 0′ of an inverted point-mass pendulum of length ℓ is vibrated according to (see fig. 19.6)

$$\vec{\boldsymbol{r}}_{0'} = d \sin \omega t \hat{\boldsymbol{\imath}}.$$

The point P thus has acceleration

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{P}} &= \vec{\boldsymbol{a}}_{0'} + \vec{\boldsymbol{a}}_{\mathrm{P}/0'} \\
&= -d\omega^2 \sin \omega t \hat{\boldsymbol{\imath}} + \ddot{\theta}\ell \hat{\boldsymbol{e}}_\theta - \dot{\theta}^2 \ell \hat{\boldsymbol{e}}_R
\end{aligned}$$

Now apply linear momentum balance as

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\mathrm{tot}} &= m\vec{\boldsymbol{a}}_{\mathrm{P}} \\
\{-mg\hat{\boldsymbol{\imath}} + -T\hat{\boldsymbol{e}}_R &= m\{-d\omega^2 \sin \omega t \hat{\boldsymbol{\imath}} + \ddot{\theta}\ell \hat{\boldsymbol{e}}_\theta - \dot{\theta}^2 \ell \hat{\boldsymbol{e}}_R\}\} \\
\{\} \cdot \hat{\boldsymbol{e}}_\theta \quad \Rightarrow \quad -g\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{e}}_\theta &= -d\omega^2 \sin \omega t \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{e}}_\theta + \ddot{\theta}\ell \\
\Rightarrow \quad g \sin \theta &= d\omega^2 \sin \omega t \sin \theta + \ddot{\theta}\ell
\end{aligned}$$

which you write as

$$0 = \ddot{\theta} + (d\omega^2 \sin \omega t - g) \sin \theta / \ell \quad \text{or} \quad \ddot{\theta} = (g - d\omega^2 \sin \omega t) \sin \theta / \ell$$

depending on whether you are analytically or numerically inclined. This is a second order non-linear ordinary differential equation. If $\omega = 0$ or $d = 0$ then this is the classic inverted pendulum equation and has solutions that show that the pendulum doesn't stay near upright. But, you can find by analytic cleverness or numerical integration that for some values of d and ω that the pendulum does not fall down! Just shaking the base keeps the pendulum up ($\omega^2 d > g$ for all cases where this is possible). This isn't just academic nonsense, the device can be built and the balancing demonstrated.

One alternative to using linear momentum balance in the equations above would be to use angular momentum balance about the point 0'. The resulting vector equation

$$\ell \hat{\boldsymbol{e}}_R \times (-mg\hat{\boldsymbol{\imath}}) = \vec{\boldsymbol{r}}_{\mathrm{P}/0'} \times (m\vec{\boldsymbol{a}})$$

yields the same second order scalar ODE.

The vibrating mechanism shown is a “Scotch yoke”. An eccentric disk is mounted to the shaft of a constant angular velocity motor. The rectangular slot moves up and down sinusoidally as the disk wobbles.

SAMPLE 19.1 A bead on a straight wire. A straight wire is hung between points A and B in the xy plane as shown in the figure. A bead slides down the wire from point A. Write the geometric constraint equation for the bead's motion and derive the conditions on velocity and acceleration components of the bead due to the constraint.

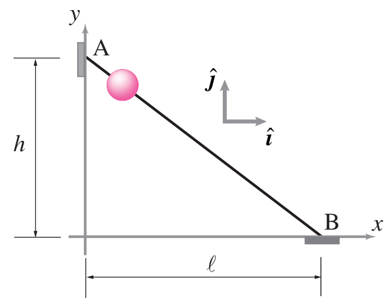


Figure 19.7

Solution The constraint on the bead's motion is that its path must be along the wire, *i.e.*, a straight line between points A and B. Thus the geometric constraint on the motion is expressed by the equation of the path which is

$$y = h - \frac{h}{\ell}x.$$

Since the bead is constrained to move on this path, its velocity and acceleration vectors are also constrained to be directed along AB. This imposes conditions on their x and y components that are easily derived by differentiating the geometric constraint equation with respect to t. Thus,

$$\begin{aligned} \dot{y} &= -\frac{h}{\ell}\dot{x}, \\ \ddot{y} &= -\frac{h}{\ell}\ddot{x}. \end{aligned}$$

$$y = h - \tfrac{h}{\ell}x, \quad \dot{y} = -\tfrac{h}{\ell}\dot{x}, \quad \ddot{y} = -\tfrac{h}{\ell}\ddot{x}$$

SAMPLE 19.2 A particle sliding on a parabolic path. A particle slides on a parabolic trough given by $y = ax^2$ where a is a constant. Write the geometric constraints of motion (on the path, velocity, and acceleration) of the particle. Write the velocity and acceleration of the particle at a generic location (x, y) on its path.

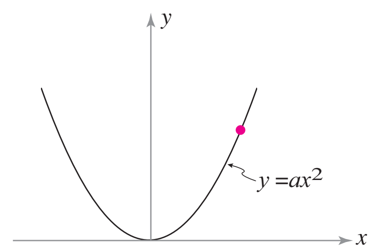


Figure 19.8

Solution The geometric constraint on the path of the particle is already given, $y = ax^2$. Differentiating the path constraint with respect to time, we get the constraint on velocity and acceleration components.

$$\begin{aligned} \dot{y} &= 2ax\dot{x}, \\ \ddot{y} &= 2ax\ddot{x} + 2a\dot{x}^2. \end{aligned}$$

Now, at a point (x, y), we can write the velocity and acceleration of the particle as

$$\begin{aligned} \vec{\boldsymbol{v}} &= \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\hat{\boldsymbol{\jmath}}, \\ \vec{\boldsymbol{a}} &= \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} = \ddot{x}\hat{\boldsymbol{\imath}} + (2ax\ddot{x} + 2a\dot{x}^2)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\hat{\boldsymbol{\jmath}}, \quad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + (2ax\ddot{x} + 2a\dot{x}^2)\hat{\boldsymbol{\jmath}}$$

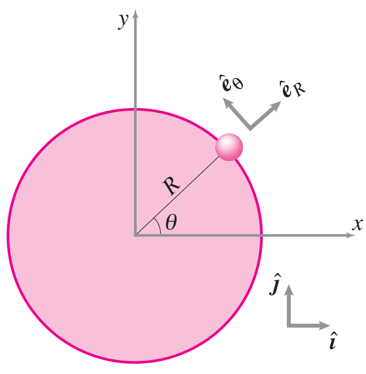


Figure 19.9

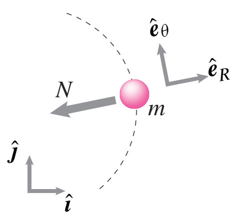


Figure 19.10

SAMPLE 19.3 Circular motion of a particle. A particle is constrained to move on a frictionless circular path of radius R_0 with constant angular speed $\dot{\theta}$. There is no gravity. Find the equation of motion of the particle in the x-direction and show that this motion is simple harmonic.

Solution This is a simple problem that you have solved before, probably a few times. Here, we do this problem again just to show how it works out with the constraint machinery in evidence. The geometric constraint on the path of the particle is $R = R_0$ (in polar coordinates). This constraint gives us $\dot{R} = 0$ and $\ddot{R} = 0$. Then the acceleration of the particle (in polar coordinates), $\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$ reduces to $\vec{a} = -R\dot{\theta}^2\hat{e}_R$, (of course).

The free-body diagram of the particle shows that there is only one force acting on the particle, the normal reaction N of the path acting in the $\hat{e}_R$ direction. Therefore, the linear momentum balance gives,

$$N\hat{e}_R = m\vec{a} = m(-R\dot{\theta}^2\hat{e}_R) \quad \Rightarrow \quad N = -mR\dot{\theta}^2.$$

But, to write the equation of motion in the x-direction, we need to write the linear momentum balance in the x-direction. We can write $\sum \vec{F} = m\vec{a}$ using mixed basis vectors as $N\hat{e}_R = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$. Dotting this equation with $\hat{\imath}$, we get

$$\ddot{x} = \frac{N}{m}(\hat{e}_R \cdot \hat{\imath}) = \frac{-mR\dot{\theta}^2}{m}\cos\theta = -\dot{\theta}^2(R\cos\theta) = -\dot{\theta}^2 x$$

or, $\ddot{x} + \dot{\theta}^2 x = 0$, which is the equation of simple harmonic motion in x. You can easily show that the motion in the y-direction is also simple harmonic ($\ddot{y} + \dot{\theta}^2 y = 0$).

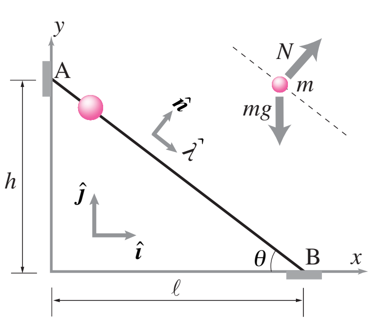


Figure 19.11

SAMPLE 19.4 A bead slides on a straight wire. Consider the problem of the bead sliding on a straight, inclined, frictionless wire of Sample 19.1 again. Find the position of the bead $x(t)$ and $y(t)$ assuming it slides under gravity starting from rest at A.

Solution To find the position of the bead, we need to write the equation of motion and solve it. This is a single DOF system and, therefore, one scalar equation of motion should suffice.

The free-body diagram of the bead is shown in fig. 19.11. Using basis vectors $(\hat{\lambda}, \hat{n})$ and $(\hat{\imath}, \hat{\jmath})$ we write the LMB for the bead as

$$-mg\hat{\jmath} + N\hat{n} = m\vec{a} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}).$$

We can easily eliminate the constraint force N from this equation by dotting this equation with $\hat{\lambda}$, which gives

$$\begin{aligned} -\cancel{m}g(\hat{\jmath}\cdot\hat{\lambda}) &= \cancel{m}[\ddot{x}(\hat{\imath}\cdot\hat{\lambda}) + \ddot{y}(\hat{\jmath}\cdot\hat{\lambda})] \\ \Rightarrow \quad g\sin\theta &= \ddot{x}\cos\theta - \ddot{y}\sin\theta. \end{aligned}$$

But, from the geometric constraint $y = h - (h/\ell)x$, we have $\ddot{y} = -(h/\ell)\ddot{x} = -(\tan\theta)\ddot{x}$. Therefore,

$$g\sin\theta = \ddot{x}\cos\theta + \ddot{x}\tan\theta\sin\theta \quad \Rightarrow \quad \ddot{x} = g\sin\theta\cos\theta.$$

Since $g\sin\theta\cos\theta$ is constant, we integrate the equation of motion easily to find $x(t) = \frac{1}{2}g\sin\theta\cos\theta t^2$ since $x(0) = 0, \dot{x}(0) = 0$. And, since $y = h - x\tan\theta$, we have $y(t) = h - \frac{1}{2}g\sin^2\theta t^2$.

$$\boxed{x(t) = \tfrac{1}{2}ght^2\sin\theta\cos\theta, \quad y(t) = h - \tfrac{1}{2}gt^2\sin^2\theta}$$

SAMPLE 19.5 **A bead sliding down a parabolic trough.** Consider the problem of Sample 19.2 again. Find the equation of motion of the bead.

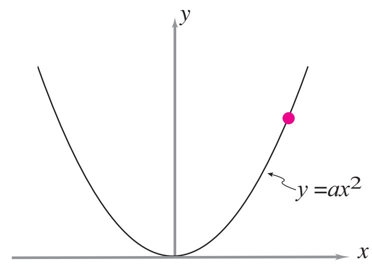


Figure 19.12

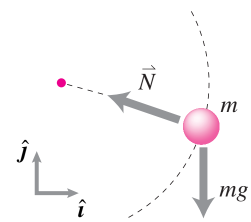


Figure 19.13

Solution This is, again, a one DOF system. Therefore, we will get a single scalar equation of motion. The free-body diagram shown in fig. 19.13 shows two forces acting on the bead. The constraint force N acts normal to the path. Let $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$ be unit vectors tangential and normal to the path, respectively. Then the linear momentum balance gives

$$-mg\,\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{e}}_n = m\vec{\boldsymbol{a}} = m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\,\hat{\boldsymbol{\jmath}}).$$

To eliminate the unknown constraint force N from this equation, we can take a dot product of this equation with $\hat{\boldsymbol{e}}_t$. However, we must first find $\hat{\boldsymbol{e}}_t$. Now $\hat{\boldsymbol{e}}_t$ is the unit tangent vector. So, we can find it by finding a tangent vector to the path (remember gradient of a function ∇f?) and then dividing it by the length of the vector. That is doable but a little complicated. All we need here is the dot product with a vector *normal* to $\hat{\boldsymbol{e}}_n$. Why not use the velocity vector $\vec{\boldsymbol{v}} = \dot{x}\hat{\imath} + \dot{y}\,\hat{\boldsymbol{\jmath}}$? The velocity vector is always tangential to the path. Furthermore, we know that from the geometric constraint ($y = ax^2$), $\dot{y} = 2ax\dot{x}$ and $\ddot{y} = 2ax\ddot{x} + 2a\dot{x}^2$. Therefore, $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\,\hat{\boldsymbol{\jmath}}$. Now, dotting the LMB equation with $\vec{\boldsymbol{v}}$ we get,

$$\begin{aligned}
-\cancel{m}g\overbrace{(\hat{\boldsymbol{\jmath}}\cdot\vec{\boldsymbol{v}})}^{2ax\dot{x}} &= \cancel{m}\left[\,\ddot{x}\overbrace{(\hat{\boldsymbol{\imath}}\cdot\vec{\boldsymbol{v}})}^{\dot{x}} + \ddot{y}\overbrace{(\hat{\boldsymbol{\jmath}}\cdot\vec{\boldsymbol{v}})}^{2ax\dot{x}}\,\right] \\
-2gax\cancel{\dot{x}} &= \ddot{x}\cancel{\dot{x}} + 2ax\ddot{y}\cancel{\dot{x}} \\
&= \ddot{x} + 2ax\underbrace{(2ax\ddot{x} + 2a\dot{x}^2)}_{\ddot{y}} \\
&= \ddot{x}(1 + 4a^2x^2) + 4a^2x\dot{x}^2.
\end{aligned}$$

Rearranging the terms above, we get the required equation of motion:

$$\ddot{x} + \frac{4a^2x}{1 + 4a^2x^2}\dot{x}^2 + \frac{2gax}{1 + 4a^2x^2} = 0.$$

As you can see, this is a nonlinear ODE. Analytical solution of this equation is rather difficult. We can, however, always solve it numerically. Note that a solution of this equation only gives you $x(t)$, *i.e.,* the x coordinate of the position of the bead. But, you can always find the y coordinate since $y = ax^2$.

$$\ddot{x} + \frac{4a^2x}{1+4a^2x^2}\dot{x}^2 + \frac{2gax}{1+4a^2x^2} = 0$$

Comment: Note that if we consider x and $\dot{x}$ to be very small so that we can ignore the $\dot{x}^2$ terms completely, and take $1 + 4a^2x^2 \approx 1$, then the equation of motion becomes

$$\ddot{x} + (2ga)\,x = 0$$

which is the equation of simple harmonic motion with frequency $\sqrt{2ga}$. Thus, if we consider a shallow parabola, and release the bead close to the origin, it executes simple harmonic motion, much like a simple pendulum. This is an intuitively realizable motion.

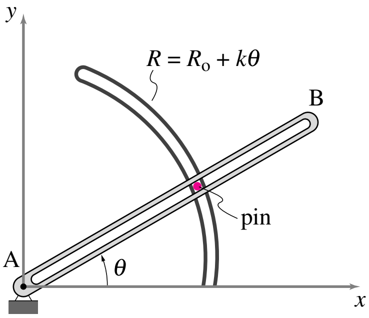


Figure 19.14: A pin is constrained to move in a groove and a slotted arm.

1 Note that R is a function of θ and θ is a function of time, therefore R is a function of time. Although we are interested in finding $\dot{R}$ and $\ddot{R}$ at $\theta = 60^\circ$, we cannot first substitute $\theta = 60^\circ$ in the expression for R and then take its time derivatives (which will be zero).

SAMPLE 19.6 Constrained motion of a pin. During a small interval of its motion, a pin of 100 grams is constrained to move in a groove described by the equation $R = R_0 + k\theta$ where $R_0 = 0.3\,\mathrm{m}$ and $k = 0.05\,\mathrm{m}$. The pin is driven by a slotted arm AB and is free to slide along the arm in the slot. The arm rotates at a constant speed $\omega = 6\,\mathrm{rad/s}$. Find the magnitude of the force on the pin at $\theta = 60^\circ$.

Solution Let $\vec{F}$ denote the net force on the pin. Then from the linear momentum balance

$$\vec{F} = m\vec{a}$$

where $\vec{a}$ is the acceleration of the pin. Therefore, to find the force at $\theta = 60^\circ$ we need to find the acceleration at that position.

From the given figure, we assume that the pin is in the groove at $\theta = 60^\circ$. Since the equation of the groove (and hence the path of the pin) is given in polar coordinates, it seems natural to use polar coordinate formula for the acceleration. For planar motion, the acceleration is

$$a = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta.$$

We are given that $\dot{\theta} \equiv \omega = 6\,\mathrm{rad/s}$ and the radial position of the pin $R = R_0 + k\,\theta$. Therefore, 1

$$\begin{aligned} \ddot{\theta} &= \frac{d\dot{\theta}}{dt} = 0 \quad (\text{ since } \dot{\theta} = \text{constant}) \\ \dot{R} &= \frac{d}{dt}(R_0 + k\theta) = k\dot{\theta} \quad \text{and} \\ \ddot{R} &= k\ddot{\theta} = 0. \end{aligned}$$

Substituting these expressions in the acceleration formula and then substituting the numerical values at $\theta = 60^\circ$, (remember, θ must be in radians!), we get

$$\begin{aligned} \vec{a} &= -\overbrace{(R_0 + k\theta)\dot{\theta}^2}^{R\dot{\theta}^2}\,\hat{\boldsymbol{e}}_R + \overbrace{2k\dot{\theta}^2}^{2\dot{R}\dot{\theta}}\,\hat{\boldsymbol{e}}_\theta \\ &= -(0.3\,\mathrm{m} + 0.05\,\mathrm{m}\cdot\frac{\pi}{3})\cdot(6\,\mathrm{rad/s})^2\,\hat{\boldsymbol{e}}_R + 2\cdot 0.05\,\mathrm{m}\cdot(6\,\mathrm{rad/s})^2\,\hat{\boldsymbol{e}}_\theta \\ &= -13.63\,\mathrm{m/s^2}\hat{\boldsymbol{e}}_R + 3.60\,\mathrm{m/s^2}\hat{\boldsymbol{e}}_\theta. \end{aligned}$$

Therefore the net force on the pin is

$$\begin{aligned} \vec{F} &= m\vec{a} \\ &= 0.1\,\mathrm{kg}\cdot(-13.63\hat{\boldsymbol{e}}_R + 3.60\hat{\boldsymbol{e}}_\theta)\,\mathrm{m/s^2} \\ &= (-1.36\hat{\boldsymbol{e}}_R + 0.36\hat{\boldsymbol{e}}_\theta)\,\mathrm{N} \end{aligned}$$

and the magnitude of the net force is

$$F = |\vec{F}| = \sqrt{(1.36\,\mathrm{N})^2 + (0.36\,\mathrm{N})^2} = 1.41\,\mathrm{N}.$$

$F = 1.41\,\mathrm{N}$

SAMPLE 19.7 **A puck sliding on a frictional rotating table.** A horizontal turntable rotates with constant angular speed $\omega = 100\,\text{rpm}$. A puck of mass $m = 0.1\,\text{kg}$ is gently placed on the rotating turntable. The puck begins to slide. The coefficient of friction between the puck and the turntable is 0.25. Find the equation of motion of the puck using

1. A fixed reference frame with cartesian coordinates
2. A rotating reference frame with cartesian coordinates

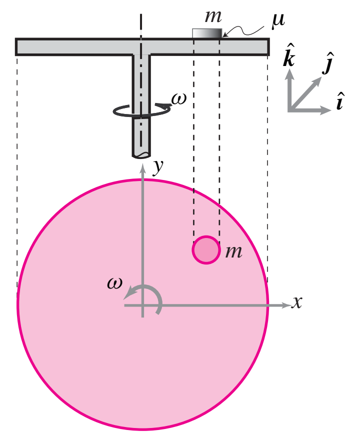


Figure 19.15

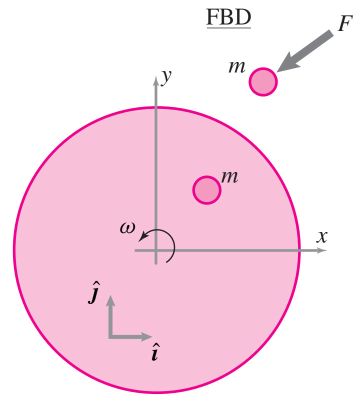


Figure 19.16: A partial free-body diagram of the puck. For linear momentum balance we need to consider only the forces acting in the plane of motion.

Solution

1. **Equation of motion using a fixed reference frame:** The puck has two DOF on the turntable. So, we will need two configuration variables, say x and y, and we will have to find equation of motion for each variable.

   Let us use a fixed cartesian coordinate system with the origin at the center of the turntable. Let $\vec{r}_P = x\hat{\imath} + y\hat{\jmath}$ be the position of the puck at some instant t, so that its velocity is $\vec{v}_P = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ and acceleration is $\vec{a}_P = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$.

   The free-body diagram of the puck should show three forces — the force of gravity (in $-\hat{k}$ direction), the normal reaction of the turntable (in $\hat{k}$ direction) and the friction force $\vec{F}$. Since, there is no motion in the vertical direction, we know that $N = mg$ and that $F = |\vec{F}| = \mu N = \mu mg$. But, what is the direction of the friction force? Well, we know that it acts in the opposite direction of the *relative slip*, so that

   $$\vec{F} = -\mu N \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|}.$$

   So, we need to find $\vec{v}_{\text{rel}}$. Now $\vec{v}_{\text{rel}}$ is the velocity of the puck relative to the turntable, or more precisely, relative to the point on the turntable just underneath the puck. Let us denote that point by P'. Clearly, P' goes in circles with constant speed, so that its velocity is

   $$\vec{v}_{P'} = \vec{\omega} \times \vec{r}_{P'} = \dot{\theta}\hat{k} \times (x\hat{\imath} + y\hat{\jmath}) = \dot{\theta}x\hat{\jmath} - \dot{\theta}y\hat{\imath}.$$

   Therefore, the relative velocity, $\vec{v}_{\text{rel}}$ is

   $$\vec{v}_{\text{rel}} = \vec{v}_P - \vec{v}_{P'} = (\dot{x} + \dot{\theta}y)\hat{\imath} + (\dot{y} - \dot{\theta}x)\hat{\jmath}$$

   Now the linear momentum balance for the puck in the xy plane gives

   $$\begin{aligned} -\mu \cancel{m} g \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|} &= \cancel{m}(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \\ \Rightarrow \quad \ddot{x} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|}(\vec{v}_{\text{rel}} \cdot \hat{\imath}) = -\frac{\mu g(\dot{x} + \dot{\theta}y)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}} \\ \text{and} \quad \ddot{y} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|}(\vec{v}_{\text{rel}} \cdot \hat{\jmath}) = -\frac{\mu g(\dot{y} - \dot{\theta}x)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}}. \end{aligned}$$

   These are coupled nonlinear ODEs that represent the equations of motion of the puck.

   $$\ddot{x} = -\frac{\mu g(\dot{x}+\dot{\theta}y)}{\sqrt{(\dot{x}+\dot{\theta}y)^2+(\dot{y}-\dot{\theta}x)^2}}, \quad \ddot{y} = -\frac{\mu g(\dot{y}-\dot{\theta}x)}{\sqrt{(\dot{x}+\dot{\theta}y)^2+(\dot{y}-\dot{\theta}x)^2}}$$

   Note that these equations are valid only as long as there is relative slip between the puck and the turntable. If the puck stops sliding due to friction, it simply goes in circles with the turntable and, therefore, its equations of motion then are $\ddot{x} = -\dot{\theta}^2 x$, $\ddot{y} = -\dot{\theta}^2 y$.

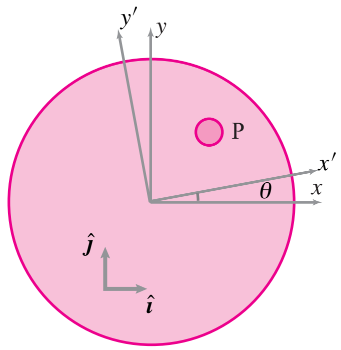


Figure 19.17: Axes (x', y') represent the rotating frame $\mathcal{B}$ fixed to the rotating turntable; ie (x', y') rotate with angular velocity $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{k}}$.

2. **Equation of motion using a rotating reference frame:** Now we derive the equations of motion using a rotating reference frame, $\mathcal{B}$, with (x', y') coordinate axes, fixed to the rotating turntable. Let the position of the puck in the rotating frame be $\vec{\boldsymbol{r}}_{\mathrm{P/O'}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'$. Note that the velocity of the puck in the rotating reference frame is $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} = \dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}'$, and acceleration is $\vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} = \ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}\hat{\boldsymbol{\jmath}}'$.

   Now, from the linear momentum balance ($\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) for the puck, we get

$$-\mu \cancel{m} g \frac{\vec{\boldsymbol{v}}_{\mathrm{rel}}}{|\vec{\boldsymbol{v}}_{\mathrm{rel}}|} = \cancel{m}(\vec{\boldsymbol{a}}_{\mathrm{P'}} + \vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}})$$

   where we have used the three term acceleration formula for $\vec{\boldsymbol{a}}_{\mathrm{P}}$. Here,

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{P'}} &= -\dot{\theta}^2(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') \\
\vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{B}} &= \ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}'\hat{\boldsymbol{\jmath}}' \\
2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} &= -2\dot{\theta}\dot{y}'\hat{\boldsymbol{\imath}}' + 2\dot{\theta}\dot{x}'\hat{\boldsymbol{\jmath}}'
\end{aligned}$$

   Note that the point P', coincident with P and fixed on the turntable, is stationary with respect to the rotating frame. Therefore, the relative velocity of P as observed in the rotating frame is $\vec{\boldsymbol{v}}_{\mathrm{rel}} = \vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} = \dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}'$. Substituting these terms in the LMB equation above, we have

$$-\mu g \frac{\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} = (-\dot{\theta}^2 x' + \ddot{x}' - 2\dot{\theta}\dot{y}')\hat{\boldsymbol{\imath}}' + (-\dot{\theta}^2 y' + \ddot{y}' + 2\dot{\theta}\dot{x}')\hat{\boldsymbol{\jmath}}'$$

   Dotting this equation with $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, respectively, we get

$$\begin{aligned}
\ddot{x}' &= \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} + 2\dot{\theta}\,\dot{y}' \\
\ddot{y}' &= \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} - 2\dot{\theta}\,\dot{x}'
\end{aligned}$$

   These are the required equations of motion for the puck in the rotating frame.

$$\boxed{\ddot{x}' = \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2+\dot{y}'^2}} + 2\dot{\theta}\,\dot{y}', \quad \ddot{y}' = \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2+\dot{y}'^2}} - 2\dot{\theta}\,\dot{x}'}$$

   Once we find a solution $x'(t)$ and $y'(t)$ of these equations, we can find the solution in the fixed frame by transforming (x', y') to (x, y) through

$$\left\{ \begin{array}{c} x(t) \\ y(t) \end{array} \right\} = \left[\begin{array}{cc} \cos(\dot{\theta}t) & -\sin(\dot{\theta}t) \\ \sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{array} \right] \left\{ \begin{array}{c} x'(t) \\ y'(t) \end{array} \right\}$$

   Also, note that when the solution of the equations of motion in the rotating reference frame brings the puck to halt, the puck stops with respect to the rotating turntable. To an observer in the fixed frame, the puck will be going in circles with a constant $\dot{\theta}$.

SAMPLE 19.8 A collar sliding on a frictional rod. A collar of mass $m = 0.5\,\text{lbm}$ slides on a massless rigid rod OA of length $\ell = 8\,\text{ft}$. The rod rotates counterclockwise with a constant angular speed $\dot{\theta} = 5\,\text{rad/s}$. The coefficient of friction between the rod and the collar is $\mu = 0.3$. At time $t = 0\,\text{s}$, the bar is horizontal, the collar is at $R_0 = 1\,\text{ft}$ and has radial speed $\dot{R}_0 = \mu\dot{\theta}R_0$ towards the pivot O. Ignore gravity.

1. How does the position of the collar change with time (*i.e.,* what is the equation of motion of the collar)?
2. Plot the path of the collar starting from $t = 0\,\text{s}$ till the collar shoots off the end of the bar.
3. How long does it take for the collar to leave the bar?

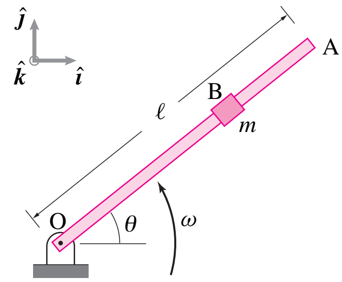


Figure 19.18: A collar slides on a frictional bar and finally shoots off the end. The bar rotates with constant angular speed.

Solution

1. A Free-Body Diagram of the collar at a general position (R, θ) is shown in Fig. 19.19. The geometry of the position vector and basis vectors is shown in Fig. 19.20. In vector notation, the forces on the collar are $\vec{\boldsymbol{N}} = N\hat{\boldsymbol{e}}_\theta$ acting normal to the rod and the force of friction $\vec{\boldsymbol{F}}_s = -\mu N\hat{\boldsymbol{e}}_R$ acting along the rod. Thus linear momentum balance for the collar is:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -\mu N\hat{\boldsymbol{e}}_R + N\hat{\boldsymbol{e}}_\theta &= m[(\ddot{R} - R\dot{\theta}^2)\,\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\,\underbrace{\ddot{\theta}}_{0}\,)\,\hat{\boldsymbol{e}}_\theta \end{aligned} \tag{19.11}$$

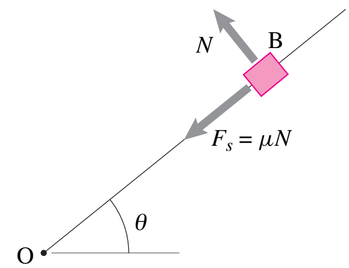


Figure 19.19: Free-Body Diagram of the collar. The only forces on the collar are from the bar: the normal force N and the friction force $F_s = \mu N$.

Note that $\ddot{\theta} = 0$ because the rod is rotating at a constant rate. Dotting both sides of eqn. (19.11) with $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ we get

$$\begin{aligned} [\text{Eqn. (19.11)}]\cdot\hat{\boldsymbol{e}}_R \quad \Rightarrow \quad -\mu N &= m(\ddot{R} - R\dot{\theta}^2) \\ \text{or} \quad \ddot{R} - R\dot{\theta}^2 &= -\frac{\mu N}{m} \\ [\text{Eqn. (19.11)}]\cdot\hat{\boldsymbol{e}}_\theta \quad \Rightarrow \quad N &= 2m\dot{R}\dot{\theta}. \end{aligned}$$

Eliminating N from the last two equations we get

$$\ddot{R} + 2\mu\dot{\theta}\,\dot{R} - \dot{\theta}^2\,R = 0.$$

Since $\dot{\theta} = \omega$ is constant, the above equation is of the form

$$\ddot{R} + C\,\dot{R} - \omega^2\,R = 0 \tag{19.12}$$

where $C = 2\mu\omega$ and $\omega = \dot{\theta}$.

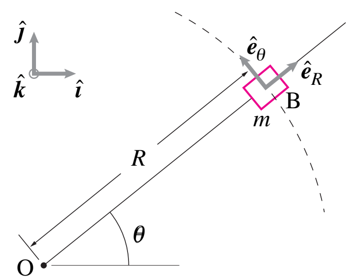


Figure 19.20: Geometry of the collar position at an arbitrary time during its slide on the rod.

Note that we could derive eqn. (19.12) from eqn. (19.11) in a single step by taking a dot product of eqn. (19.11) with $\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta$. Why? Look at the left hand side of eqn. (19.11). The net reaction force is $N(-\mu\hat{\boldsymbol{e}}_R + \hat{\boldsymbol{e}}_\theta)$ acting in the direction $(-\mu\hat{\boldsymbol{e}}_R + \hat{\boldsymbol{e}}_\theta)$. If we dot the reaction force with a vector normal to its direction, we get rid of the reaction force. A vector normal to $(-\mu\hat{\boldsymbol{e}}_R + \hat{\boldsymbol{e}}_\theta)$ is $\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta$.

$$\begin{aligned} [\text{Eqn. (19.11)}]\cdot(\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta) \quad \Rightarrow \quad 0 &= m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + 2\dot{R}\dot{\theta}\hat{\boldsymbol{e}}_\theta]\cdot(\hat{\boldsymbol{e}}_R + \mu\hat{\boldsymbol{e}}_\theta) \\ \Rightarrow \quad 0 &= \ddot{R} + 2\mu\dot{\theta}\,\dot{R} - \dot{\theta}^2\,R \end{aligned}$$

which is the same equation as eqn. (19.12).

Solution of equation (19.12): The characteristic equation(2) associated with

(2) If we assume a solution of the form $R(t) = e^{\lambda t}$ with some unknown λ and plug back into the given differential equation, we get an algebraic equation in λ which is called the characteristic equation. For more information, see box 11.3 on page 563, eqn. (11.9).

Eqn. (19.12) is

$$\begin{aligned}\lambda^2 + C\lambda - \omega^2 &= 0\\ \Rightarrow \quad \lambda &= \frac{-C \pm \sqrt{C^2+4\omega^2}}{2}\\ &= \omega(-\mu \pm \sqrt{\mu^2+1}).\end{aligned}$$

Therefore, the solution of Eqn. (19.12) is

$$\begin{aligned}R(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t}\\ &= Ae^{(-\mu+\sqrt{\mu^2+1})\omega t} + Be^{(-\mu-\sqrt{\mu^2+1})\omega t}.\end{aligned}$$

Substituting the given initial conditions: $R(0) = R^0 = 1\,\mathrm{ft}$ and $\dot{R}(0) = -\mu\dot{\theta}R^0$, we get3

3From the general solution and the given initial conditions, we have:

$$\begin{aligned}R_0 &= A + B\\ -\mu\omega R_0 &= A\lambda_1 + B\lambda_2\end{aligned}$$

Solving these two equations simultaneously, we get $A = \frac{\lambda_2+\mu\omega}{\lambda_2-\lambda_1}R_0$, and $B = -\frac{\lambda_1+\mu\omega}{\lambda_2-\lambda_1}R_0$. Substituting the values of λ_1 and λ_2, we get $A = B = \frac{R_0}{2} = \frac{1\,\mathrm{ft}}{2}$.

$$R(t) = \frac{1\,\mathrm{ft}}{2}\left[e^{(-\mu+\sqrt{\mu^2+1})\omega t} + e^{(-\mu-\sqrt{\mu^2+1})\omega t}\right]. \qquad (19.13)$$

$$\boxed{R(t) = \frac{1\,\mathrm{ft}}{2}\left[e^{(-\mu+\sqrt{\mu^2+1})\omega t} + e^{(-\mu-\sqrt{\mu^2+1})\omega t}\right].}$$

2. To draw the path of the collar we need both R and θ. Because $\dot{\theta} = 5\,\mathrm{rad/s} =$ constant,

$$\theta = \dot{\theta}\,t = (5\,\mathrm{rad/s})\,t.$$

Now we can take various values of t from 0 s to, say, 1 s, and calculate values of θ and R. Plotting all these values of R and θ, however, does not give us the correct path after the collar reaches the end of the bar, at R = length of the bar = 8 ft. So we need to find the time t_f when $R(t_f) = 8\,\mathrm{ft}$. Equation (19.13) is a nonlinear algebraic equation for t which we can solve iteratively on a computer, or with some patience, even on a calculator. One way to find t_f would be to simply plot $R(t)$ and find the intersection with $R = 8$ (see fig. 19.21) and read the corresponding value of t. Following either of these methods we find that $t_f \approx 0.74518$ s here. Now, we can plot the path of the collar by computing R and θ from $t = 0$ to $t = t_f$ and making a polar plot on a computer as follows (pseudocode).

```
tf = 0.74518                    % final value of t
t = 0:tf/100:tf;                % take 101 points in [0 tf]
R0 = 1; w = 5; mu = .3;         % initialize variables
f1 = -mu + sqrt(mu^2 +1);       % first partial exponent
f2 = -mu - sqrt(mu^2 +1);       % second partial exponent
R = 0.5*R0*(exp(f1*w*t) + exp(f2*w*t));  % calculate R
theta = w*t;                    % calculate theta
polarplot(theta,r)
```

The plot produced thus is shown in Fig. 19.22.

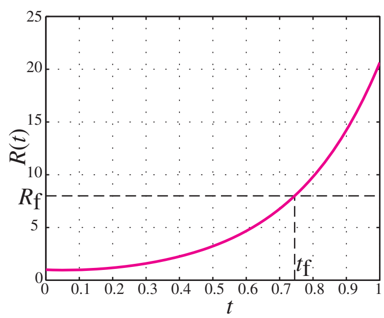


Figure 19.21: Finding the final time t_f from the graph of $R(t)$ such that $R(t_f) = R_f = 8\,\mathrm{ft}$.

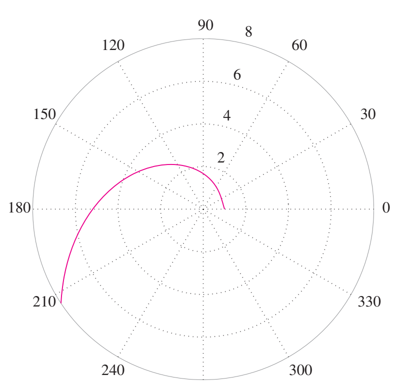


Figure 19.22: Plot of the path of the collar until it leaves the rod.

3. The time t_f computed above was

$$t_f = 0.7452\,\mathrm{s}.$$

By plugging this value in the expression for $R(t)$ (Eqn. (19.13) we get, indeed,

$$R = 8\,\mathrm{ft}.$$

$$\boxed{t_f = 0.7452\,\mathrm{s}}$$

Note, although we have not checked it explicitly, the increase in kinetic energy of the particle comes from the work of the force from the rod on the particle:

$$\Delta E_K = \int \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\,dt$$

SAMPLE 19.9 A collar sliding on a rotating rod in 3-D. A massless and frictionless rod AB is rotating about the vertical axis through point A. The rod is bent at an angle ϕ from the vertical and is rotating with a constant angular speed ω about the vertical axis. A small collar of mass m slides on the rod. Assume that at time $t = 0$ the collar is released from a rest position with respect to the rod at a distance R_0 from the axis of rotation. There is no gravity.

1. Find the equation of motion for the collar (a differential equation for the position of the collar).
2. How does the distance of the collar from the vertical axis change with time?
3. For $\phi = \pi/2$, show that the solution obtained in (ii) above is the same as that obtained in Sample 19.8 for $\mu = 0$.

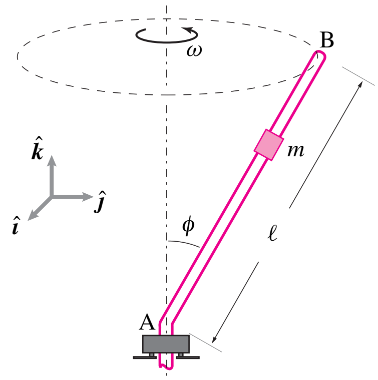


Figure 19.23: A collar of mass m slides on a massless and frictionless bar, bent at an angle ϕ with the vertical axis. The bar is rotating at a constant angular speed ω about the z-axis.

Solution Since the bar is bent and it rotates about the vertical axis, it sweeps a conical surface about the axis of rotation. As the collar slides on the rod, it traces a path on this surface. Let (R, θ, z) be the cylindrical coordinates of the collar at any general time t (Fig. 19.24(a)). Since the rod is frictionless and there is no gravity, the only force acting on the collar is the normal reaction from the rod. This force is shown in the free-body diagram of the collar in Fig. 19.24(b). Note that there are a lot of possible directions for a normal to the rod at the collar. In fact, any vector in the plane perpendicular to the rod at the collar is normal to the rod. So, at this point let us write

$$\vec{\boldsymbol{N}} = N\hat{\boldsymbol{n}}$$

where $\hat{\boldsymbol{n}}$ is a unit normal to the rod. Thus, if $\hat{\boldsymbol{\lambda}}$ is a unit vector along the rod, then

$$\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{n}} = 0 \tag{19.14}$$

where

$$\hat{\boldsymbol{\lambda}} = \frac{\vec{\boldsymbol{r}}_{\mathrm{AB}}}{|\vec{\boldsymbol{r}}_{\mathrm{AB}}|} = \sin\phi\hat{\boldsymbol{e}}_R + \cos\phi\hat{\boldsymbol{k}}.$$

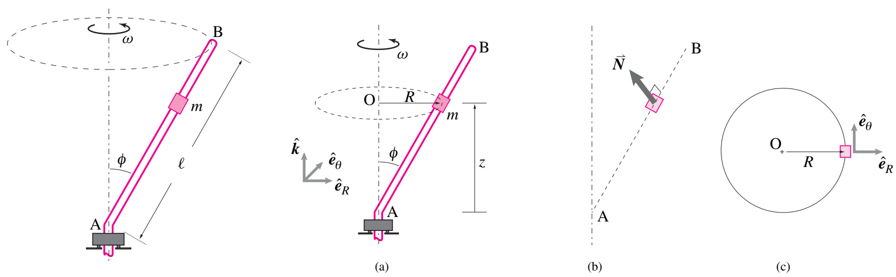


Figure 19.24: (a) Cylindrical coordinates R and z of the collar (θ is not shown) and the orientation of the cylindrical basis vectors. (b) Free-body diagram of the collar (there is no gravity). (c) Instantaneous R-$theta$ plane of the collar.

1. **Equation of Motion:** We now write the linear momentum balance for the collar:

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

where

$$\begin{aligned}\sum \vec{F} &= N\hat{\boldsymbol{n}}\\ \vec{\boldsymbol{a}} &= (\ddot{R}-R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta}+R\underbrace{\ddot{\theta}}_{0})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}\end{aligned}$$

Dotting both sides of the equation $\sum \vec{F} = m\vec{\boldsymbol{a}}$ by $\hat{\boldsymbol{\lambda}}$ we get

$$\begin{aligned}N\underbrace{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}}}_{0} &= m[(\ddot{R}-R\dot{\theta}^2)\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{\lambda}}}_{\sin\phi}+2\dot{R}\dot{\theta}\underbrace{\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{\lambda}}}_{0}+\ddot{z}\underbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\lambda}}}_{\cos\phi}]\\ \Rightarrow\quad & (\ddot{R}-R\dot{\theta}^2)\sin\phi+\ddot{z}\cos\phi = 0.\end{aligned}$$

This expression is *an* equation of motion of the collar. However, it is only a single equation in terms of derivatives of two variables R and z. Now, from geometry

$$R = z\tan\phi \qquad \Rightarrow \qquad \ddot{z} = \ddot{R}/\tan\phi.$$

Substituting this relationship in the equation of motion we get

$$\ddot{R}(\sin\phi + \frac{\cos\phi}{\tan\phi}) - R\dot{\theta}^2 = 0.$$

Noting that $\dot{\theta} = \omega =$ a constant, we may write the above equation, with some trigonometric simplifications, as

$$\ddot{R} + (\omega\ \sin\phi)^2 R = 0 \tag{19.15}$$

which is an equation of motion of the collar in terms of its distance R from the vertical axis.

2. **Solution of the equation of motion:** The solution of Eqn. (19.15) is given by (4)

$$R(t) = C_1 e^{(\omega\ \sin\phi)t} + C_2 e^{-(\omega\ \sin\phi)t}$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions. Substituting the given initial conditions: $R(0) = R_0$ and $\dot{R}(0) = 0$ we get

$$C_1 = C_2 = \frac{R_0}{2}.$$

Therefore, the solution may be written as

$$R(t) = \frac{R_0}{2}\left[e^{(\omega\ \sin\phi)t} + e^{-(\omega\ \sin\phi)t}\right]. \tag{19.16}$$

3. **Special case, $\phi = \pi/2$:** Substituting $\phi = \pi/2$ in Eqn. (19.16) we get

$$R(t) = \frac{R_0}{2}[e^{\omega t} + e^{-\omega t}]$$

which is the same solution as obtained in Eqn. (19.16) in Sample 19.8 for $\mu = 0$ and $R_0 = 1$ ft.

(4) You may find the solution by solving the corresponding *characteristic equation* (see the solution of Eqn. (19.15) in Sample 19.8, for example), or by looking up the table of "The Simplest ODEs and Their Solutions" on page 22 of the text, or by guessing a solution yourself if you have some experience with ODEs.

SAMPLE 19.10 **Osculating circle in 3-D.** A small bead is driven down a a wire-frame bent in the shape of a conical helix by a tiny motor imbedded in the bead. The combined mass of the bead and the motor is $m = 200$ gm. The shape of the helix is given: $R = R_0\theta$, $z = 2R_0\theta$ and where $R_0 = 0.3$ m. At the instant when $\theta = 2$ radians, the angular speed and the angular acceleration of the bead are $\dot\theta = 1$ rad/s and $\ddot\theta = 2$ rad/s^2. Find

1. the *net* normal force on the bead and
2. the radius of the osculating circle at the instant given.

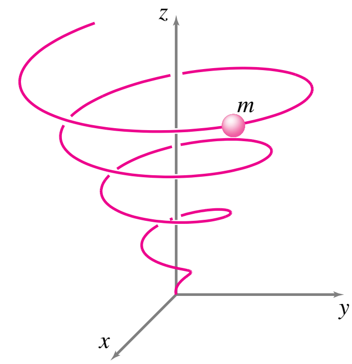


Figure 19.25: A motor driven bead slides down a helical wire-frame described by the equations $R = R_0\theta$, $z = 2R_0\theta$.

Solution

1. We can find the net normal force on the bead from the linear momentum balance of the bead (see the free-body diagram of the bead):

$$\begin{aligned} \sum \vec{F} &= m\vec{a} \\ N\hat{e}_n + T\hat{e}_t - mg\hat{k} &= m(a_t\hat{e}_t + a_n\hat{e}_n) \\ \Rightarrow \quad F_n &\equiv \sum \vec{F}\cdot\hat{e}_n = ma_n. \end{aligned}$$

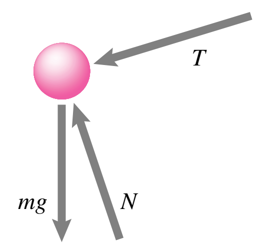


Figure 19.26: Free-body diagram of the bead. Note that the *net* normal force is $F_n = \sum \vec{F}\cdot\hat{e}_n$ where $\sum\vec{F} = N\hat{e}_n + T\hat{e}_t - mg\hat{k}$.

Thus we need to find the normal acceleration of the bead. We can write the position of the bead as

$$\begin{aligned} \vec{r} &= \underbrace{R}_{R_0\theta}\hat{e}_R + \underbrace{z}_{2R_0\theta}\hat{k} \\ \Rightarrow \quad \vec{v} &\equiv \dot{R}\hat{e}_R + R\dot\theta\hat{e}_\theta + \dot{z}\hat{k} \\ &= R_0\dot\theta\hat{e}_R + R_0\theta\dot\theta\hat{e}_\theta + 2R_0\dot\theta\hat{k} \\ &= R_0\dot\theta(\hat{e}_R + \theta\hat{e}_\theta + 2\hat{k}), \\ \text{and} \quad \vec{a} &\equiv (\ddot{R} - R\dot\theta^2)\hat{e}_R + (2\dot{R}\dot\theta + R\ddot\theta)\hat{e}_\theta + \ddot{z}\hat{k} \\ &= (R_0\ddot\theta - R_0\theta\dot\theta^2)\hat{e}_R + (2R_0\dot\theta^2 + R_0\theta\ddot\theta)\hat{e}_\theta + 2R_0\ddot\theta\hat{k} \\ &= R_0[(\ddot\theta - \theta\dot\theta^2)\hat{e}_R + (\dot\theta^2 + \theta\ddot\theta)\hat{e}_\theta + 2\ddot\theta\hat{k}] \end{aligned}$$

Substituting the given numerical values for $\theta, \dot\theta, R_0$ and $\ddot\theta$ in the above expressions for $\vec{v}$ and $\vec{a}$ we get the velocity and the acceleration of the bead at the instant of interest:

$$\begin{aligned} \vec{v} &= 0.3\,\text{m/s}(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k}) \\ \vec{a} &= 1.2\,\text{m/s}^2(2\hat{e}_\theta + \hat{k}). \end{aligned}$$

In path coordinates,

$$\vec{a} = \vec{a}_t + \vec{a}_n = a_t\hat{e}_t + a_n\hat{e}_n$$

where

$$\hat{e}_t = \frac{\vec{v}}{|\vec{v}|} \tag{19.17}$$

$$= \frac{0.3\,\text{m/s}(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k})}{0.3\sqrt{1+4+4}\ \text{m/s}} \tag{19.18}$$

$$= \frac{1}{3}(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k}). \tag{19.19}$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{a}}_t &= (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t \\
&= (\frac{4.8 + 2.4}{3}\ \mathrm{m/s^2})\hat{\boldsymbol{e}}_t \\
&= 0.8\,\mathrm{m/s^2}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta + 2\hat{\boldsymbol{k}}) \\
\vec{\boldsymbol{a}}_n &= \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_t \\
&= [2.4\hat{\boldsymbol{e}}_\theta + 1.2\hat{\boldsymbol{k}} - 0.8\,\mathrm{m/s^2}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta + 2\hat{\boldsymbol{k}})] \\
&= (-0.8\hat{\boldsymbol{e}}_R + 0.8\hat{\boldsymbol{e}}_\theta - 0.4\hat{\boldsymbol{k}})\,\mathrm{m/s^2} \\
a_n &= |\vec{\boldsymbol{a}}_n| = 1.2\,\mathrm{m/s^2} \\
\hat{\boldsymbol{e}}_n &= \frac{\vec{\boldsymbol{a}}_n}{a_n} \\
&= -\frac{2}{3}\hat{\boldsymbol{e}}_R + \frac{2}{3}\hat{\boldsymbol{e}}_\theta - \frac{1}{3}\hat{\boldsymbol{k}}.
\end{aligned}$$

Hence, the net normal force on the bead

$$\begin{aligned}
\vec{\boldsymbol{F}}_n &= m a_n \hat{\boldsymbol{e}}_n \\
&= 0.2\,\mathrm{kg} \cdot 1.2\,\mathrm{m/s^2} \cdot \frac{1}{3}(-2\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta - \hat{\boldsymbol{k}}) \\
&= (-0.16\hat{\boldsymbol{e}}_R + 0.16\hat{\boldsymbol{e}}_\theta - 0.08\hat{\boldsymbol{k}})\,\mathrm{N}.
\end{aligned}$$

$$\vec{\boldsymbol{F}}_n = 0.08\,\mathrm{N}(-2\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{k}})$$

2. For calculating the radius ρ of the osculating circle, we note that

$$\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
\Rightarrow \quad \rho &= \frac{v^2}{a_n} = \frac{(0.9\,\mathrm{m/s})^2}{1.2\,\mathrm{m/s^2}} = 0.675\,\mathrm{m}.
\end{aligned}$$

$$\rho = 0.675\,\mathrm{m}$$

19.1 Mechanics of a constrained particle

19.1.1 A bead slides on a frictionless circular hoop . The mass of the bead $m_{\text{bead}} = 2\,\text{grams}$, mass of hoop $m_{\text{hoop}} = 1\,\text{kg}$ and radius of hoop $R_{\text{hoop}} = 3\,\text{m}$. Neglect gravity. The center of the circular hoop is the origin O of a fixed (Newtonian) coordinate system $Oxyz$. The hoop is on the xy plane. The hoop is kept from moving by little angels who let the bead slide by unimpeded. At $t = 0$, the speed of the bead is $4\,\text{m/s}$, it is traveling counter-clockwise (looking down the z-axis), and it is on the $+x$-axis. There are no other external forces applied.

a) At $t = 0$ what is the bead's kinetic energy ?

b) At $t = 0$ what is the bead's linear momentum ?

c) At $t = 0$ what is the bead's angular momentum about the origin?

d) At $t = 0$ what is the bead's acceleration?

e) At $t = 0$ what is the radius of the osculating circle of the bead's path ρ?

f) At $t = 0$ what is the force of the hoop on the bead?

g) At $t = 0$ what is the net force of the angel's hands on the hoop?

h) At $t = 27s$ what is the x-component of the bead's linear momentum?

19.1.2 A warehouse operator wants to move a crate of weight $W = 100\,\text{lb}$ from a 6 ft high platform to ground level by means of a roller conveyer as shown. The rollers on the inclined plane are well lubricated and thus assumed frictionless; the rollers on the horizontal conveyer are frictional, thus providing an effective friction coefficient μ. Assume the rollers are massless.

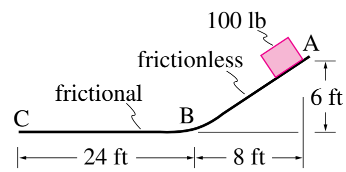
a) What are the kinetic and potential energies(pick a suitable datum) of the crate at the elevated platform, point A?

b) What are the kinetic and potential energies of the crate at the end of the inclined plane just before it moves onto the horizontal conveyor, point B?

c) Using conservation of energy, calculate the maximum speed attained by the crate at point B, assuming it starts moving from rest at point A.

d) If the crate slides a distance x, say, on the horizontal conveyor, what is the energy lost to sliding in terms of μ, W, and x?

e) Calculate the value of the coefficient of friction, μ, such that the crate comes to rest at point C.



Problem 19.1.2

19.1.3 On a wintry evening, a student of mass m starts down the steepest street in town, *Steep Street*, which is of height h and slope θ. At the top of the slope she starts to slide (coefficient of dynamic friction μ).

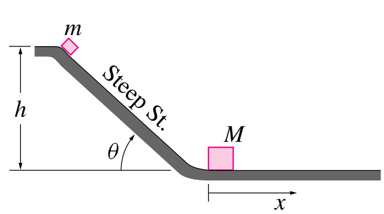
a) What is her initial kinetic and potential energy?

b) What is the energy lost to friction as she slides down the hill?

c) What is her velocity on reaching the bottom of the hill? Ignore air resistance and all cross streets (i.e. assume the hill is of constant slope).

d) If, upon reaching the bottom of the street, she collides with another student of mass M and they embrace, what is their instantaneous mutual velocity just after the embrace?

e) Assuming that friction still acts on the level flats on the bottom, how much time will it take before they come to rest?



Problem 19.1.3

19.1.4 Reconsider the system of blocks in problem 2.3.14, this time with equal mass, $m_1 = m_2 = m$. Also, now, both blocks are frictional and sitting on a frictional surface. Assume that both blocks are sliding to the right with the top block moving faster. The coefficient of friction between the two blocks is μ_1. The coefficient of friction between the lower block and the floor is μ_2. It is known that $\mu_1 > 2\mu_2$.

a) Draw free-body diagrams of the blocks together and separately.

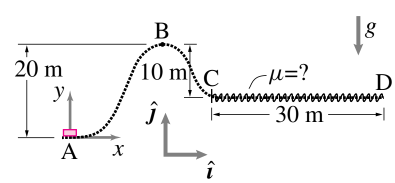
b) Write the equations of linear momentum balance for each block.

c) Find the acceleration of each block for the case of frictionless blocks. For the frictional blocks, find the acceleration of each block. What happens if $\mu_1 >> \mu_2$?

19.1.5 An initially motionless roller-coaster car of mass 20 kg is given a horizontal impulse $\int \vec{F}\,dt = P\hat{\imath}$ at position A, causing it to move along the track, as shown below.

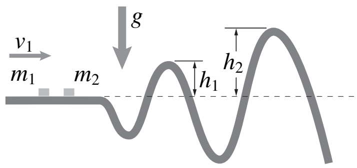
a) Assuming that the track is perfectly frictionless from point A to C and that the car never leaves the track, determine the magnitude of the impulse I so that the car "just makes it" over the hill at B.

b) In the ensuing motion, assume that the horizontal track C-D is frictional and determine the coefficient of friction required to bring the car to a stop at D.



Problem 19.1.5

19.1.6 Masses $m_1 = 1\,\text{kg}$ and m_2 kg move on the frictionless varied terrain shown. Initially, m_1 has speed $v_1 = 12\,\text{m/s}$ and m_2 is at rest. The two masses collide on the flat section. The coefficient of restitution in the collision is $e = 0.5$. Find the speed of m_2 at the top of the first hill of elevation $h_1 = 1\,\text{m}$. Does m_2 make it over the second hill of elevation $h_2 = 4\,\text{m}$?



Problem 19.1.6

19.1.7 A **Scotch yoke** is a device for converting rotary motion into linear motion. In this case it is used to make a horizontal platform go up and down. The motion of the platform is $y = A\sin(\omega t)$ with constant ω. A dead bug with mass m is standing on the platform. Her feet have no glue on them. There is gravity.

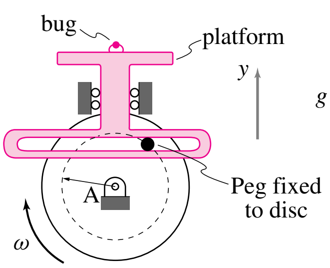
a) Draw a free-body diagram of the bug.

b) What is the acceleration of the platform and, hence, the bug?

c) Write the equation of linear momentum balance for the bug.

d) What condition must be met so that the bug does not bounce the platform?

e) What is the maximum spin rate ω that can be used if the bug is not to bounce off the platform?



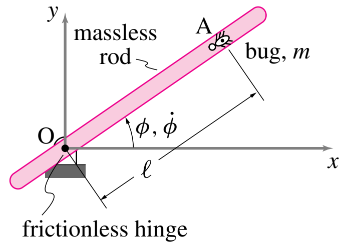
Problem 19.1.7

19.1.8 A bug of mass m walks straight-forwards with speed v_A and rate of change of speed $\dot{v}_A$ on a straight light (assumed to be massless) stick. The stick is hinged at the origin so that it is always horizontal but is free to rotate about the z axis. Assume that the distance the bug is from the origin, ℓ, the angle the stick makes with the x axis, ϕ, and its rate of change, $\dot{\phi}$, are known at the instant of interest. Ignore gravity. Answer the following questions in terms of ϕ, $\dot{\phi}$, ℓ, m, v_A, and $\dot{v}_A$.

a) What is $\ddot{\phi}$?

b) What is the force exerted by the rod on the support?

c) What is the acceleration of the bug?



Problem 19.1.8

19.1.9 A new kind of gun. Assume $\omega = \omega_0$ is a constant for the rod in the figure. Assume the mass is free to slide. At $t = 0$, the rod is aligned with the x-axis and the bead is one foot from the origin and has no radial velocity ($dR/dt = 0$).

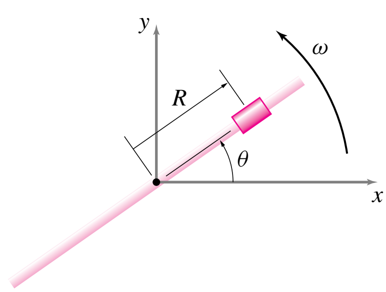
a) Find a differential equation for $R(t)$. *

b) Turn this equation into a differential equation for $R(\theta)$. *

c) How far will the bead have moved after one revolution of the rod? How far after two? *

d) What is the speed of the bead after one revolution of the rod (use $\omega_0 = 2\pi\,\mathrm{rad/s} = 1\,\mathrm{rev/sec}$)? *

e) How much kinetic energy does the bead have after one revolution and where did it come from? *



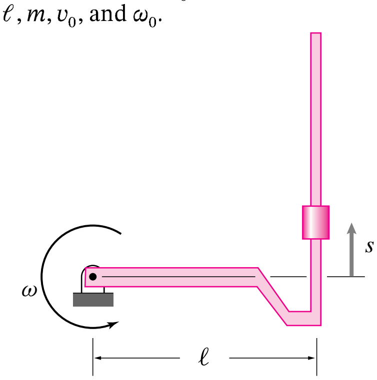
Problem 19.1.9

19.1.10 The new gun gets old and rusty. Reconsider the bead on a rod in problem 19.1.9. This time, friction cannot be neglected. The friction coefficient is μ. At the instant of interest the bead with mass m has radius R_0 with rate of change $\dot{R}_0$. The angle θ is zero and ω is a constant. Neglect gravity.

a) What is $\ddot{R}$ at this instant? Give your answer in terms of any or all of R_0, $\dot{R}_0$, ω, m, μ, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_R$, $\hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$. *

b) After a very long time it is observed that the angle ϕ between the path of the bead and the rod/trough is nearly constant. What, in terms of μ, is this ϕ? *

19.1.11 A newer kind of gun. As an attempt to make an improvement on the 'new gun' demonstrated in problem 19.1.9, a person adds a length ℓ to the shaft on which the bead slides. Assume there is no friction between the bead (mass m) and the wire. Assume the bead starts at $s = 0$ with $\dot{s} = v_0$. The rigid rod, on which the bead slides, rotates at a constant rate $\omega = \omega_0$. Find $s(t)$ in terms of ℓ, m, v_0, and ω_0.



Problem 19.1.11

19.1.12 Slippery bead on straight rotating stick. A long stick with mass m_s rotates at a constant rate and a bead, modeled as a point mass, slides on the stick. The stick rotates at constant rate $\vec{\omega} = \omega\hat{\boldsymbol{k}}$. Neglect gravity. The bead has mass m_b. Initially the mass is at a distance R_0 from the hinge point on the stick and has no radial velocity ($\dot{R} = 0$). The initial angle of the stick is $\theta = 0$, measured counterclockwise from the positive x axis.

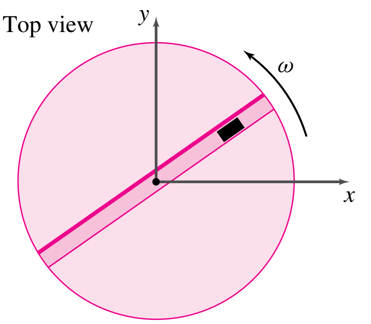
a) What is the torque, as a function of the net angle the stick has rotated, θ, required in order to keep the stick rotating at a constant rate ?

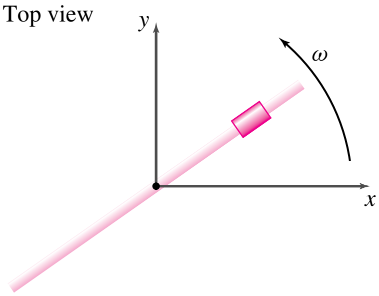
b) What is the path of the bead in the xy-plane? (Draw an accurate picture showing about one half of one revolution.)

c) How long should the stick be if the bead is to fly in the negative x-direction when it gets to the end of the stick?

d) **Add friction.** How does the speed of the bead over one revolution depend on μ, the coefficient of friction between the bead and the wire? Make a plot of $|\vec{v}_{\text{one revolution}}|$ versus μ. [Note, you have been working with $\mu = 0$ in the problems above.]

19.1.13 Mass on a lightly greased slotted turntable or spinning uniform rod. Assume that the rod/turntable in the figure is massless and also free to rotate. Assume that at $t = 0$, the angular velocity of the rod/turntable is 1 rad/s, that the radius of the bead is one meter, and that the radial velocity of the bead, dR/dt, is zero. The bead is free to slide on the rod. Where is the bead at $t = 5$ sec?

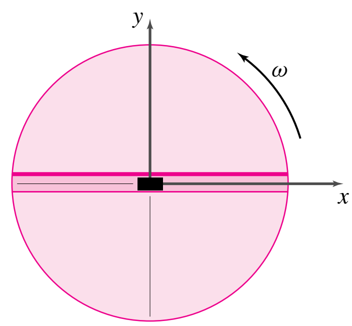




Problem 19.1.13

19.1.14 A bead slides in a frictionless slot in a turntable. The turntable spins a constant rate ω. The slot is straight and goes through the center of the turntable. If the bead is at radius R_o with $\dot{R} = 0$ at $t = 0$, what are the components of the acceleration vector in the directions normal and tangential to the path of the bead after one revolution? Neglect gravity.

19.1.15 A bead of mass $m = 1$ gm is constrained to slide in a straight frictionless slot in a disk which is spinning counterclockwise at constant rate $\omega = 3$ rad/s. At time $t = 0$, the slot is parallel to the x-axis and the bead is in the center of the disk moving out (in the plus x direction) at a rate $\dot{R}_o = .5$ m/s. After a net rotation θ of one and one eighth (1.125) revolutions, what is the force $\vec{F}$ of the disk on the bead? Express this answer in terms of $\hat{\imath}$ and $\hat{\jmath}$. Make the unreasonable assumption that the slot is long enough to contain the bead for this motion.

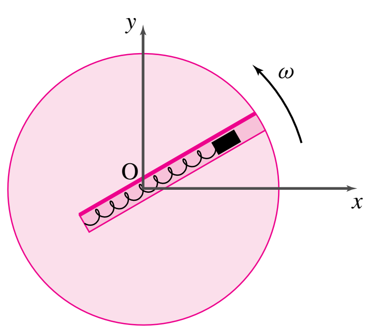


Problem 19.1.15

19.1.16 Bead on springy leash in a slot on a turntable. The bead in the figure is held by a spring that is relaxed when the bead is at the origin. The constant of the spring is k. The turntable speed is controlled by a strong stiff motor.

a) Assume $\omega = 0$ for all time. What are possible motions of the bead?

b) Assume $\omega = \omega_0$ is a constant. What are possible motions of the bead? Notice there are two cases depending on the value of ω_0. What is going on here?



Problem 19.1.16: Mass in a slot.

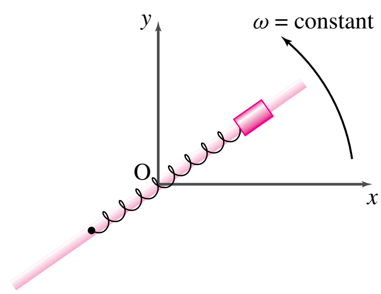
19.1.17 A small bead with mass m slides without friction on a rigid rod which rotates about the z axis with constant ω (maintained by a stiff motor not shown in the figure). The bead is also attached to a spring with constant K, the other end of which is attached to the rod. The spring is relaxed when the bead is at the center position. Assume the bead is pulled to a distance d from the center of the rod and then released with an initial $\dot{R} = 0$. If needed, you may assume that $K > m\omega^2$.

a) Derive the equation of motion for the position of the bead $R(t)$.

b) How is the motion affected by large versus small values of K?

c) What is the magnitude of the force of the rod on the bead as the bead passes through the center position? Neglect gravity. Answer in terms of m, ω, d, K.

d) Write an expression for the Coriolis acceleration. Give an example of a situation in which this acceleration is important and explain why it arises.



Problem 19.1.17

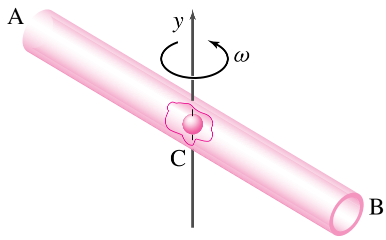
19.1.18 A small ball of mass m_b = 500 grams may slide in a slender tube of length $l = 1.2$ m. and of mass m_t = 1.5 kg. The tube rotates freely about a vertical axis passing through its center C. (Hint: Treat the tube as a slender rod.)

a) If the angular velocity of the tube is $\omega = 8$ rad/s as the ball passes through C with speed relative to the tube, v_0, calculate the angular velocity of the tube just before the ball leaves the tube. *

b) Calculate the angular velocity of the tube just after the ball leaves the tube. *

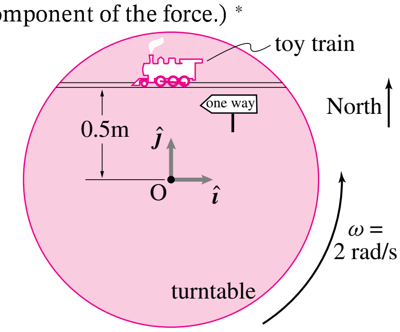
c) If the speed of the ball as it passes C is $v = 1.8$ m/s, determine the transverse and radial components of velocity of the ball as it leaves the tube. *

d) After the ball leaves the tube, what constant torque must be applied to the tube (about its axis of rotation) to bring it to rest in 10 s? *



Problem 19.1.18

19.1.19 Toy train car on a turn-around. The 0.1 kg toy train's speedometer reads a constant $1 m/s$ when, heading west, it passes due north of point O. The train is on a level straight track which is mounted on a spinning turntable whose center is 'pinned' to the ground. The turntable spins at the constant rate of 2 rad/s. What is the force of the turntable on the train? (Don't worry about the z-component of the force.) *



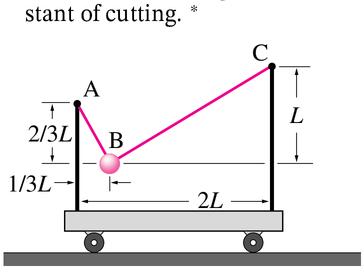
Problem 19.1.19

19.1.20 Due to forces not shown, the cart moves to the right with constant acceleration a_x. The ball B has mass m_B. At time $t = 0$, the string AB is cut. Find

a) the tension in string BC before cutting, *

b) the absolute acceleration of the mass at the instant of cutting, *

c) the tension in string BC at the instant of cutting. *

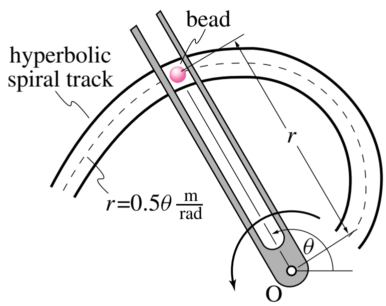


Problem 19.1.20

19.1.21 The forked arm mechanism pushes the bead of mass 1 kg along a frictionless hyperbolic spiral track given by $r = 0.5\theta(\,\mathrm{m/\,rad})$. The arm rotates about its pivot point at O with constant angular acceleration $\ddot{\theta} = 1\,\mathrm{rad/s^2}$ driven by a motor (not shown). The arm starts from rest at $\theta = 0^\circ$.

a) Determine the radial and transverse components of the acceleration of the bead after 2 s have elapsed from the start of its motion.

b) Determine the magnitude of the net force on the mass at the same instant in time.



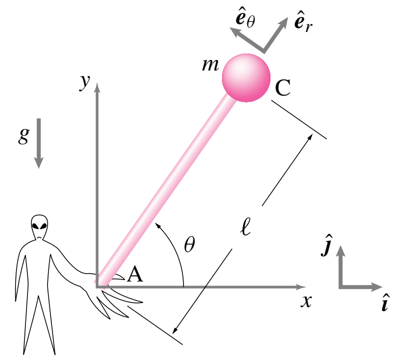
Problem 19.1.21

19.1.22 A rod is on the palm of your hand at point A. Its length is ℓ. Its mass m is assumed to be concentrated at its end at C. Assume that you know θ and $\dot{\theta}$ at the instant of interest. Also assume that your hand is accelerating both vertically and horizontally with $\vec{\boldsymbol{a}}_{\text{hand}} = a_{hx}\hat{\boldsymbol{\imath}} + a_{hy}\hat{\boldsymbol{\jmath}}$. Coordinates and directions are as marked in the figure.

a) Draw a Free-Body Diagram of the rod.

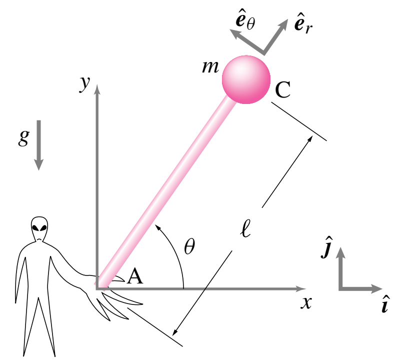
b) Assume that the hand is stationary. Solve for $\ddot{\theta}$ in terms of g, ℓ, m, θ, and $\dot{\theta}$.

c) If the hand is not stationary but $\ddot{\theta}$ has been determined somehow, find the vertical force of the hand on the rod in terms of $\ddot{\theta}$, $\dot{\theta}$, θ, g, ℓ, m, a_{hx}, and a_{hy}.



Problem 19.1.22: Rod on a Moving Support(2-D).

19.1.23 Balancing a broom. Assume the hand is accelerating to the right with acceleration $\boldsymbol{a} = a\hat{\boldsymbol{\imath}}$. What is the force of the hand on the broom in terms of m, ℓ, θ, $\dot{\theta}$, a, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and g? *(You may not have any $\hat{\boldsymbol{e}}_R$ or $\hat{\boldsymbol{e}}_\theta$ in your answer.)*



Problem 19.1.23

19.1.1 A bead slides on a frictionless circular hoop . The mass of the bead $m_{\text{bead}} = 2$ grams, mass of hoop $m_{\text{hoop}} = 1$ kg and radius of hoop $R_{\text{hoop}} = 3$ m. Neglect gravity. The center of the circular hoop is the origin O of a fixed (Newtonian) coordinate system $Oxyz$. The hoop is on the xy plane. The hoop is kept from moving by little angels who let the bead slide by unimpeded. At $t = 0$, the speed of the bead is 4 m/s, it is traveling counter-clockwise (looking down the z-axis), and it is on the $+x$-axis. There are no other external forces applied.

a) At $t = 0$ what is the bead's kinetic energy ?

b) At $t = 0$ what is the bead's linear momentum ?

c) At $t = 0$ what is the bead's angular momentum about the origin?

d) At $t = 0$ what is the bead's acceleration?

e) At $t = 0$ what is the radius of the osculating circle of the bead's path ρ?

f) At $t = 0$ what is the force of the hoop on the bead?

g) At $t = 0$ what is the net force of the angel's hands on the hoop?

h) At $t = 27s$ what is the x-component of the bead's linear momentum?

18.1.1

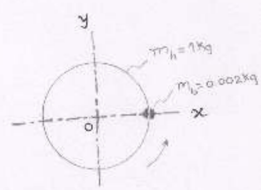
Given mass of bead, $m_b = 2$ gram

mass of the hoop, $m_h = 1$ Kg

radius of the hoop, $r_h = 3$ m

at $t=0$, bead speed, $V_b = 4$ m/s

Neglect gravity



a) Bead's Kinetic energy, $E_K = \frac{1}{2} m_b \cdot V_b^2$ at $t=0$

$\Rightarrow E_K = \frac{1}{2} \times 0.002 \times 4$ J $= 0.016$ J

b) Beads' Linear momentum at $t=0$, $L = mV = 0.008$ N.s

c) Beads' angular momentum at $t=0$, $\underset{\sim}{H}_{/o} = \underset{\sim}{r} \times m\underset{\sim}{v}$

$\Rightarrow \underset{\sim}{H}_{/o} = 3\hat{\imath} \times m_b 4\hat{\jmath}$ Kg m²/s $= 0.024\hat{k}$ N.m.s.

d) Beads' acceleration at $t=0$, $\underset{\sim}{a} = \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})$ [given $\dot{\underset{\sim}{\omega}} = 0$]

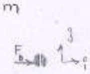
$\Rightarrow \underset{\sim}{a} = -\omega^2 r_h \hat{\imath}$ as $\underset{\sim}{v} = \underset{\sim}{\omega} \times \underset{\sim}{r}$, $\omega = \frac{4}{3}$ rad/s

$\Rightarrow \underset{\sim}{a} = -\frac{16}{3}\hat{\imath}$

e) Radius of osculating circle, $\rho = \frac{1}{|\kappa|}$

As $\underset{\sim}{\kappa} = \frac{d\hat{e}_t}{ds} \Rightarrow |\kappa| = \left|\frac{d\theta \cdot \hat{e}_t}{R d\theta}\right| = \frac{1}{R} \Rightarrow \rho = R = 3$m

f) Applying L.M.B on bead, $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}$

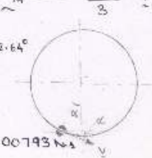


$\Rightarrow \{F_b \hat{\imath} = m\underset{\sim}{a}\}; F_b\hat{\imath} = 0.002 \frac{16}{3}$ N$\hat{\imath}$ $\Rightarrow \{\ \}\cdot\hat{\imath} \Rightarrow F_b = -\frac{0.032}{3}$ N

$\therefore F_b$ is acting towards the center

g) Two forces act on the hoop, one from bead, other from angels' hands(F_A). As acceleration of hoop $= 0$, $\therefore \underset{\sim}{F}_A = -\frac{0.032}{3}\hat{\imath}$

h) At $t = 27$s, $\theta(t) = \omega.t\big|_{t=27s} \Rightarrow \theta = 36$ rad. $= 2062.64°$

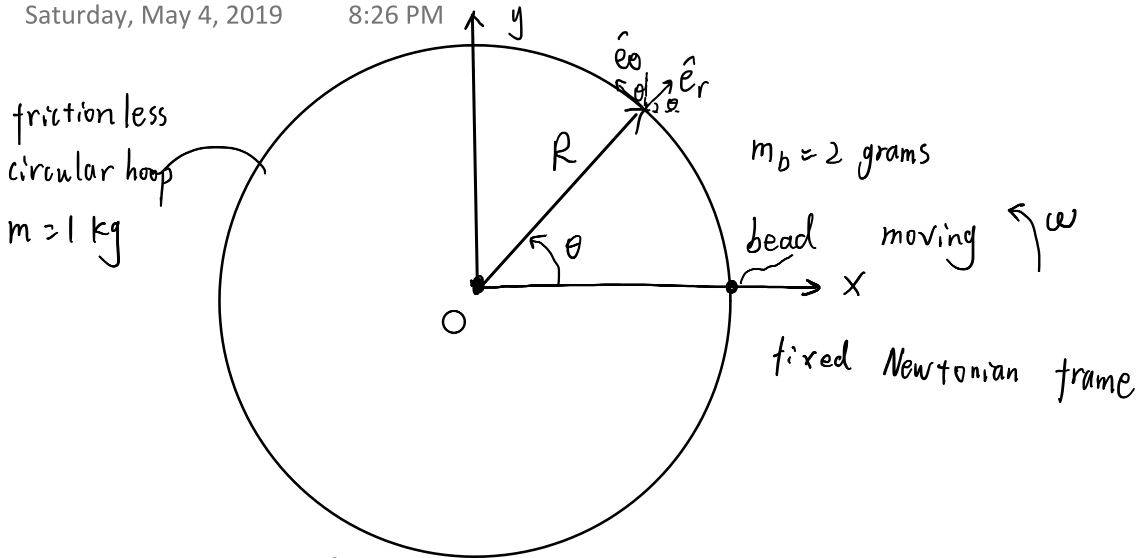


As 6 cycles correspond to 2160°, $\alpha = 7.35°$

x component of $\underset{\sim}{L} \Rightarrow \{\underset{\sim}{L} = m\underset{\sim}{v}\}\cdot\hat{\imath} \Rightarrow L_x = mv\cos 7.35° = 0.00793$ N.s

18.1.1

Saturday, May 4, 2019 8:26 PM



Given: $R = 3m$

$V_b(t=0) = 4\ m/s = V_{b0}$

$t=0$, bead is on the $+x$ axis

At $t=0$

a) Find the bead's kinetic energy

$$E_{k0} = \frac{1}{2} m_b V_b^2(t=0) = \frac{1}{2} \times 2g \times (4m/s)^2 = 0.016\ kg\, m^2/s^2$$

b) Find the bead's linear momentum

$$\vec{L} = m\vec{V}_{b0} = 2g \times 4m/s\ \hat{e}_\theta = 0.008\ \hat{e}_\theta\ kg\, m/s$$

c) Find the bead's angular momentum /o.

$$\vec{H}_{/o} = \vec{r}_{/o} \times m\vec{v} = 3\hat{i} \times 2g \cdot 4\ m/s\ \hat{j} = 0.024\ \hat{k}\ [kg\, m/s]$$

d) Find the bead's acceleration

$$\vec{a}_0 = -\frac{V_{b0}^2}{R}\hat{i} = -\frac{(4m/s)^2}{3m}\hat{i} = -5.33\ \hat{i}\ [m/s^2]$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

e) Find the radius ρ of the osculating circle of bead's path

$$\rho = \frac{V_{bo}^2}{a_o} = \frac{(4m/s)^2}{5.33\, m/s^2} = 3m = \text{radius of the circle}$$

Circle kisses itself

f) Find the force on the bead

LMB: $\Sigma \vec{F}_o = m\vec{a}_o = 2g\ (-5.33\hat{i}\ m/s^2) = -0.0106\,\hat{i}\ [kg\ m/s^2]$

g) Find the net force on the hoop.

LMB: $\Sigma \vec{F}'_o = m\vec{a}_o = \Sigma \vec{F}_o = -0.0106\,\hat{i}\ [kg\ m/s]$

h) At $t = 27s$ Find the x-component of the bead's Linear momentum

No friction, and the force on the bead is perpendicular to its velocity. Thus the bead is moving at const speed (acceleration is perpendicular to velocity all the time)

Angular distance $\theta = \frac{V_b t}{R} = \frac{4m/s \cdot 27s}{3m} = 36$ rad

$$x = R \cdot \cos\theta = 3m \cdot \cos 36 = -0.5739\,[m]$$

$$L_x = \vec{L} \cdot \hat{i} = m\vec{V} \cdot \hat{i} = mV\hat{e}_\theta \cdot \hat{i} = mV(-\sin\theta)$$

$$= 0.008\ kg\,m/s \cdot (-0.99) = -0.0079\ kg\,m/s$$

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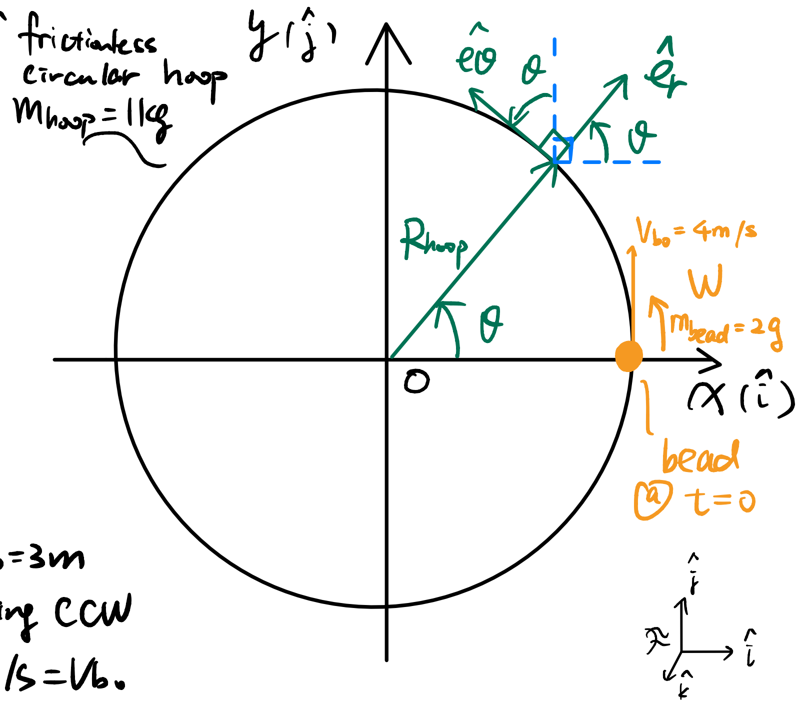
Katherine Cao edited from Shuwen Lu



Given: no gravity
hoop frictionless
$m_{hoop} = 1\,kg$ $R_{hoop} = 3m$
$m_{bead} = 2g$ traveling CCW
$V_{bead}(t=0) = 4m/s = V_{b0}$
$t=0$ bead on $+x$ axis
fixed newtonian coord Oxyz (z out of page) unit vectors $\hat{i}\ \hat{j}\ \hat{k}$

Find:

$$E_{k0} = \frac{1}{2} m_{bead} V_{b0}^2 = \frac{1}{2}(2g)(4m/s)^2 = \boxed{0.016[J]}$$

$$\vec{L}_o = m\vec{V}_{b0} = mV_{b0}\hat{j} = (2g)(4m/s)\hat{j} = \boxed{0.008\,\hat{j}\,[kg\ m/s]}$$

$$\vec{H}_{/O} = \vec{r}_{/O} \times \underbrace{(m\vec{V}_{b0})}_{\vec{L}_o} = (R_{hoop}\,\hat{i}) \times (mV_{b0}\,\hat{j})$$

$$= (3m\,\hat{i}) \times (0.008\,\hat{j}\ kg\ m/s) = \boxed{0.024\,\hat{k}\,[kg\ m^2/s]}$$

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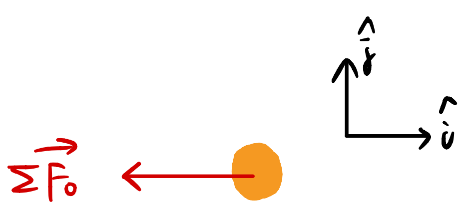
$$\vec{a}_o = -\frac{V_{bo}^2}{R_{hoop}}\hat{\imath} = -\frac{(4m/s)^2}{3m}\hat{\imath} = \boxed{-5.33\,\hat{\imath}\ [m/s^2]}$$

$$\rho = \frac{V_{bo}^2}{|\vec{a}_o|} = \frac{(4m/s)^2}{5.33m/s^2} = \boxed{3m = R_{hoop}}$$

"backwards" of d) circle kisses itself



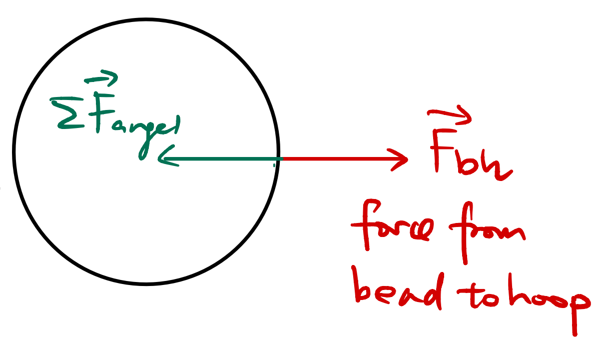
LMB: $\sum\vec{F}_o = m\vec{a}_o$

$$= (2g)(-5.33\,\hat{\imath}\ m/s^2)$$

$$\boxed{\sum\vec{F}_o = -0.0107\ \hat{\imath}\ [kg\ m/s^2]}$$

$$\sum\vec{F}_{hoop} = \sum\vec{F}_{angel} + \vec{F}_{bh}$$

$$= m\vec{a}_{hoop} = \vec{0} \text{ (not moving)}$$



$$\sum\vec{F}_{angel} = -\vec{F}_{bh}$$

$$\vec{F}_{bh} = -\sum\vec{F}_o \quad \text{(action and reaction forces btw bead \& hoop)}$$

$$\Rightarrow \sum\vec{F}_{angel} = -\vec{F}_{bh} = -(-\sum\vec{F}_o) = \sum\vec{F}_o = \boxed{-0.0107\ \hat{\imath}\ [kg\ m/s^2]}$$

No friction

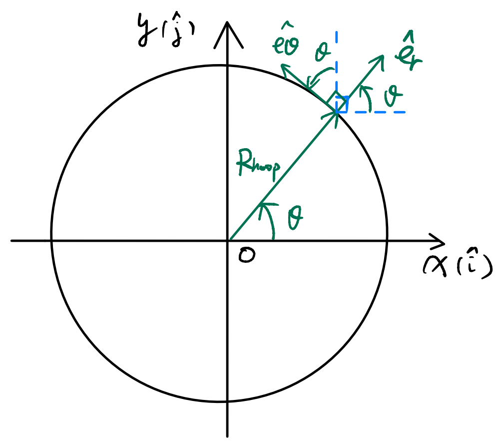
$\Rightarrow$ force on the bead perpendicular to $\vec{V}_{bead}$ all the time

$\Rightarrow$ constant $V_{bead} = |\vec{V}_{bead}| = 4\,m/s$

Angular distance traveled:

$$\theta = \frac{V_{bead} \cdot t}{R_{hoop}} = \frac{(4\,m/s)(27\,s)}{3\,m} = 36\,rad$$

$$\vec{L} = m\vec{V}_{bead} = m V_{bead}\,\hat{e}_\theta$$



From figure on the right,

$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

$$\vec{L}_x = (\vec{L} \cdot \hat{\imath})\,\hat{\imath}$$

$$= mV_{bead}(\hat{e}_\theta \cdot \hat{\imath})\,\hat{\imath}$$

$$= mV_{bead}(-\sin\theta)\,\hat{\imath}$$

$$= (2g)(4\,m/s)(-\sin(36\,rad))\,\hat{\imath}$$

$$= \boxed{0.00793\ [kg\ m/s]}$$

19.1.2 A warehouse operator wants to move a crate of weight $W = 100\,\text{lb}$ from a $6\,\text{ft}$ high platform to ground level by means of a roller conveyer as shown. The rollers on the inclined plane are well lubricated and thus assumed frictionless; the rollers on the horizontal conveyer are frictional, thus providing an effective friction coefficient μ. Assume the rollers are massless.

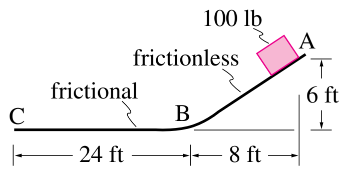
a) What are the kinetic and potential energies(pick a suitable datum) of the crate at the elevated platform, point A?

b) What are the kinetic and potential energies of the crate at the end of the inclined plane just before it moves onto the horizontal conveyor, point B?

c) Using conservation of energy, calculate the maximum speed attained by the crate at point B, assuming it starts moving from rest at point A.

d) If the crate slides a distance x, say, on the horizontal conveyor, what is the energy lost to sliding in terms of μ, W, and x?

e) Calculate the value of the coefficient of friction, μ, such that the crate comes to rest at point C.



Problem 19.2

19.1.2

Weight of the crate, W = 100 lb

a) Kinetic and Potential Energies:

Let BC be the datum

Potential Energy, $E_p^A = mgh$

$m = 45.36\,kg$ $h = 1.83\,m$ $\therefore E_p^A = 45.36 \times 1.83 \times 9.81$

$\Rightarrow \boxed{E_p^A = 814.32\,J}$

As the mass is stationary, Kinetic energy $\boxed{E_k^A = 0\,J}$

b) Potential energy at point B, $\boxed{E_p^B = 0\,J}$ as $h = 0$

As total energy remains constant, $E_{tot}^A = E_{tot}^B$

$E_p^A + E_k^A = E_p^B + E_k^B$

$814.32 + 0 = 0 + \frac{1}{2} m v_B^2$

$\therefore$ Kinetic energy at B, $\boxed{E_k^B = 814.32\,J}$

c) $v_B^2 = \frac{2E_{tot}^A}{m}$ from conservation of Energy $E_{tot}^A = E_{tot}^B$

$\Rightarrow$ Velocity of the crate at B, $\boxed{v_B = \sqrt{\frac{2 \times 814.32}{45.36}} = 5.99\,\frac{m}{s}}$

d) Energy lost to sliding,

$E_s = \int_0^x F_f\,dx$

where F_f = frictional force, $= \mu N = \mu mg$

$\Rightarrow E_s = \int_0^x \mu mg\,dx \Rightarrow \boxed{E_s = \mu mgx}$

e) Using the principle of work & energy from B to C

$\Sigma U = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$

(or) $\frac{1}{2} m v_i^2 + \Sigma U = \frac{1}{2} m v_f^2$

Final velocity, $v_f = 0$

$\Rightarrow \frac{1}{2} m v^2 + [-E_s] = 0$

$\Rightarrow \frac{1}{2} m v_B^2 - \mu mg x = 0$

$\Rightarrow \mu = \frac{m v_B^2}{2 mg x}$ (or) $\mu = \frac{v_B^2}{2 g x}$

In the given configuration $x = 24\,ft = 7.31\,m$

$\boxed{\mu = 0.25}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.1.3 On a wintry evening, a student of mass m starts down the steepest street in town, *Steep Street*, which is of height h and slope θ. At the top of the slope she starts to slide (coefficient of dynamic friction μ).

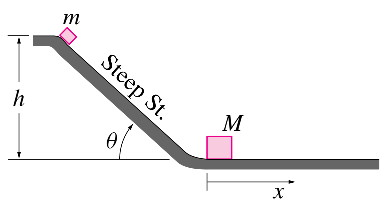
a) What is her initial kinetic and potential energy?

b) What is the energy lost to friction as she slides down the hill?

c) What is her velocity on reaching the bottom of the hill? Ignore air resistance and all cross streets (i.e. assume the hill is of constant slope).

d) If, upon reaching the bottom of the street, she collides with another student of mass M and they embrace, what is their instantaneous mutual velocity just after the embrace?

e) Assuming that friction still acts on the level flats on the bottom, how much time will it take before they come to rest?



Problem 19.3

19.1.3

a) Initial Kinetic energy of m

$E_k^A = 0$

Initial Potential energy of m

$E_p^A = mgh$

b) LMB of m

$\{\sum \underline{F} = m\underline{a}\} \cdot \hat{n}$

$N - mg\cos\theta = m\underline{a} \cdot \hat{n} = 0$

∴ Frictional force, $F_f = \mu N = \mu mg\cos\theta$

Energy lost to friction, $E_s = F_f\, d$

$\Rightarrow E_s = \mu mg\cos\theta \dfrac{h}{\sin\theta}$

∴ Energy lost in sliding, $E_s = \dfrac{\mu mgh}{\tan\theta}$

c) Using Principle of work and energy,

$\sum U = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$

$\sum U$ = work done by the forces mg and F_f

$\sum U = +mgh - E_s$

As $v_i = 0$

$\frac{1}{2}mv_f^2 = mgh - \dfrac{\mu mgh}{\tan\theta} = mgh\left(1 - \dfrac{\mu}{\tan\theta}\right)$

$\Rightarrow v_f = \sqrt{2gh\left(1 - \dfrac{\mu}{\tan\theta}\right)}$

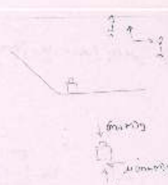
d) If m and M embrace after collision, from linear momentum balance, we have

$L = L'$

$mv_f + M(0) = (m+M)v$

$\Rightarrow v = \dfrac{m}{m+M}\sqrt{2gh\left(1 - \dfrac{\mu}{\tan\theta}\right)}$

e) LMB on (m+M)



$\{\sum \underline{F} = m\underline{a}\} \cdot \hat{i}$

$-\mu(m+M)g = (m+M)a_x$

$\Rightarrow a_x = -\mu g$

Using $v_f = v_i + at$

to let $v_f = 0$, $t = \dfrac{-v_i}{a}$

$\Rightarrow t = \dfrac{v}{\mu g}$ ∴ $t = \dfrac{m}{\mu g(m+M)}\sqrt{2gh\left(1 - \dfrac{\mu}{\tan\theta}\right)}$

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19.1.4 Reconsider the system of blocks in problem **??**, this time with equal mass, $m_1 = m_2 = m$. Also, now, both blocks are frictional and sitting on a frictional surface. Assume that both blocks are sliding to the right with the top block moving faster. The coefficient of friction between the two blocks is μ_1. The coefficient of friction between the lower block and the floor is μ_2. It is known that $\mu_1 > 2\mu_2$.

a) Draw free-body diagrams of the blocks together and separately.

b) Write the equations of linear momentum balance for each block.

c) Find the acceleration of each block for the case of frictionless blocks. For the frictional blocks, find the acceleration of each block. What happens if $\mu_1 >> \mu_2$?

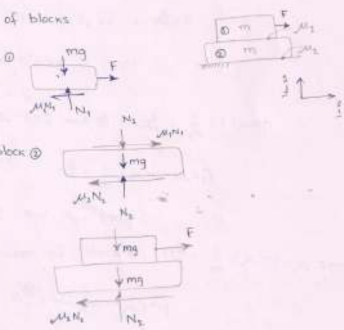
18.1.4

a) FBDs of blocks

top block ①

Bottom block ②

Together



b) $\{\sum \underline{F}_1 = m_1 \underline{a}_1\} \Rightarrow \underline{F} + \underline{N}_1 - mg\hat{\jmath} - \mu_1 N_1 \hat{\imath} = m_1 \underline{a}$

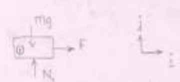
$\{\} \cdot \hat{\jmath} \Rightarrow -mg + N_1 = 0 \Rightarrow N_1 = mg$

For block ② $\sum \underline{F} = \dot{\underline{L}} \Rightarrow \underline{N}_1 - mg\hat{\jmath} + \underline{N}_2 + \mu_1 N_1 \hat{\imath} - \mu_2 N_2 \hat{\imath} = m_2 \underline{a}$

$\Rightarrow \{\} \cdot \hat{\imath}$

$\Rightarrow \mu_1 N_1 - \mu_2 N_2 = m a_2$

c) Frictionless block



① $\sum \underline{F} = \dot{\underline{L}}$

$\{F\hat{\imath} + N_1\hat{\jmath} - mg\hat{\jmath} = m\underline{a}\}$

$\{\} \cdot \hat{\imath} \Rightarrow a_x^{①} = \frac{F}{m}$

② LMB on block ②

$\sum \underline{F} = \dot{\underline{L}}$

$\{-N_1\hat{\jmath} - mg\hat{\jmath} + N_2\hat{\jmath} = m\underline{a}\}$

$\{\} \cdot \hat{\imath} \Rightarrow m a_x = 0 \Rightarrow a_x^{②} = 0$

Frictional blocks

① $\sum \underline{F} = \dot{\underline{L}}$

From the FBD and ⓑ $a_x^{①} = \frac{1}{m}(F - \mu_1 N_1)$

for block ②. $a_x^{②} = \frac{1}{m}(\mu_1 N_1 - \mu_2 N_2)$

$N_1 = mg$ and $N_2 = 2mg$

Acceleration of block ②, $a_x^{②} = \frac{1}{m}(\mu_1 mg - \mu_2\, 2mg)$

$a_x^{②} = \frac{1}{m}(\mu_1 - 2\mu_2) mg$

(or) $a_x^{②} = (\mu_1 - 2\mu_2) g$

If $\mu_1 >> \mu_2$, the relative acceleration between the

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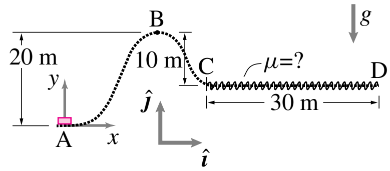
19.1.5 An initially motionless roller-coaster car of mass 20 kg is given a horizontal impulse $\int \vec{F}\,dt = P\hat{\imath}$ at position A, causing it to move along the track, as shown below.

a) Assuming that the track is perfectly frictionless from point A to C and that the car never leaves the track, determine the magnitude of the impulse I so that the car "just makes it" over the hill at B.

b) In the ensuing motion, assume that the horizontal track C-D is frictional and determine the coefficient of friction required to bring the car to a stop at D.

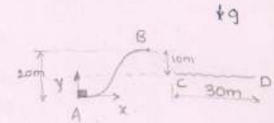


Problem 19.5

18.1.5

Mass of roller-coaster car, $m = 20\text{Kg}$

Impulse, $\int \underset{\sim}{F}\,dt = P\underset{\sim}{i}$



a) Find impulse, I, so that the car "just makes it" over the hill at B

Using the Linear impulse - momentum relationship, at A

$$P\underset{\sim}{i} = \int \underset{\sim}{F}\,dt = \underset{\sim}{L}^{+} - \underset{\sim}{L}^{-} = m(\underset{\sim}{v}^{+} - \underset{\sim}{v}^{-}) \quad \left[\text{given } \underset{\sim}{v}^{-} = \underset{\sim}{0}\right]$$

$$\Rightarrow |P\underset{\sim}{i}| = I = mv^{+}$$

track from A through C is frictionless, hence, using energy balance,

$$E_{tot}^{A} = E_{tot}^{B}$$

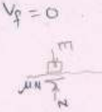
$$\frac{1}{2}mv^{+2} = mgh \Rightarrow v^{+} = \sqrt{2gh}$$

$$\therefore I = m\sqrt{2gh} \Rightarrow \boxed{I = 20\sqrt{2 \cdot 9.81 \cdot 20} = 396.18\ \text{N}\cdot\text{s}}$$

b) Using the principle of work & energy

$$\Sigma U = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \qquad v_f = 0$$

$$-\mu N \cdot d = -\frac{1}{2}mv_i^2$$



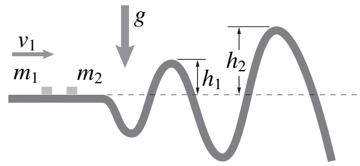
$$\mu = \frac{mv_i^2}{Nd}$$, v_i is obtained from energy balance, between B and C

$$E_{tot}^{B} = E_{tot}^{C} \Rightarrow mgh = mg\frac{h}{2} + \frac{1}{2}mv_i^2 \Rightarrow v_i = \sqrt{gh}$$

$$\therefore \mu = \frac{mgh}{Nd}$$ $\therefore$ Required $\boxed{\mu \text{ to stop Car at D} = 0.67}$

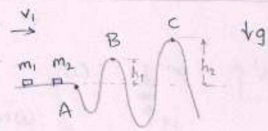
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.1.6 Masses $m_1 = 1\,\text{kg}$ and $m_2\,\text{kg}$ move on the frictionless varied terrain shown. Initially, m_1 has speed $v_1 = 12\,\text{m/s}$ and m_2 is at rest. The two masses collide on the flat section. The coefficient of restitution in the collision is $e = 0.5$. Find the speed of m_2 at the top of the first hill of elevation $h_1 = 1\,\text{m}$. Does m_2 make it over the second hill of elevation $h_2 = 4\,\text{m}$?



Problem 19.6

18.1.6



Speed of $m_1 = v_1^- = 12\,\text{m/s}$

Speed of m_2, $v_2^- = 0$.

Coefficient of restitution, $e = 0.5$

From definition, $$e = \frac{v_2^+ - v_1^+}{v_1^- - v_2^-}$$

$$v_2^+ - v_1^+ = 0.5(12-0)\tfrac{m}{s} = 6\,\text{m/s}$$

$$m_1 v_1^- + m_2 v_2^- = m_1 v_1^+ + m_2 v_2^+ \quad \text{(Conservation of Momentum)}$$

$$12\tfrac{m}{s} m_1 + 0\tfrac{m}{s} = m_1 v_1^+ + m_2(v_1^+ + 6\tfrac{m}{s})$$

$$\Rightarrow v_1^+ = \frac{12m_1 - 6m_2}{(m_1+m_2)} \tfrac{m}{s}$$

If $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$, $v_1^+ = 0$ & $v_2^+ = 6\,\text{m/s}$

Using energy balance, $E^A_{tot} = E^B_{tot} = E^C_{tot}$

$$\tfrac{1}{2} m_2 v_A^2 + 0 = m_2 g h_1 + \tfrac{1}{2} m_2 v_B^2$$

$$\Rightarrow v_B^2 = v_A^2 - 2gh_1$$

$$\Rightarrow v_B = \sqrt{v_2^{+2} - 2gh_1} \Rightarrow v_B = 4.05\,\text{m/s}$$

Consider $E^A_{tot} = E^C_{tot}$

$$\tfrac{1}{2} m_2 v_2^{+2} = m_2 g h_2 + \tfrac{1}{2} m_2 v_C^2 \Rightarrow v_C^2 = v_2^{+2} - 2gh_2$$

$$v_C^2 = [36 - 78.48](\tfrac{m}{s})^2 \Rightarrow v_C^2 < 0$$

$\therefore$ m_2 can not reach the second hill of elevation 'C'.

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19.1.7 A **Scotch yoke** is a device for converting rotary motion into linear motion. In this case it is used to make a horizontal platform go up and down. The motion of the platform is $y = A\sin(\omega t)$ with constant ω. A dead bug with mass m is standing on the platform. Her feet have no glue on them. There is gravity.

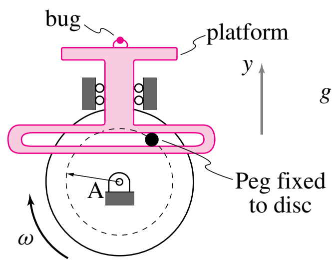
a) Draw a free-body diagram of the bug.

b) What is the acceleration of the platform and, hence, the bug?

c) Write the equation of linear momentum balance for the bug.

d) What condition must be met so that the bug does not bounce the platform?

e) What is the maximum spin rate ω that can be used if the bug is not to bounce off the platform?



Problem 19.7

18.1.7 Bug on a Scotch Yoke

(a) $\downarrow mg$, $\uparrow N$; $y\,(\hat{j})$, $x\,(\hat{i})$

(b) Motion of the platform, $y = A\sin\omega t$

Acceleration of the platform, $\frac{d^2y}{dt^2} = \ddot{y} = -A\omega^2\sin\omega t$

(c) L.M.B. for the bug.

$$\left\{\sum \underset{\sim}{F} = m\underset{\sim}{a}\right\}$$

$$\{\ \}\cdot\underset{\sim}{j} \Rightarrow N - mg = m\ddot{y}$$

(d) Bug looses contact with the platform if the downward acceleration is greater than 'g'. (or) if $N=0$

$$-\ddot{y} > g$$

For the bug not to bounce, $-\ddot{y} < g$

(e) Spin rate limit

From (b) and (d) $\quad -[-A\omega^2\sin\omega t] < g$

$\therefore$ Maximum Spin rate, $\boxed{\omega < \sqrt{\frac{g}{A}}}$

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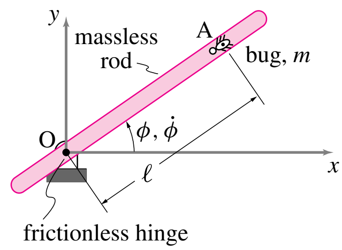
19.1.8 A bug of mass m walks straightforwards with speed v_A and rate of change of speed $\dot{v}_A$ on a straight light (assumed to be massless) stick. The stick is hinged at the origin so that it is always horizontal but is free to rotate about the z axis. Assume that the distance the bug is from the origin, ℓ, the angle the stick makes with the x axis, ϕ, and its rate of change, $\dot{\phi}$, are known at the instant of interest. Ignore gravity. Answer the following questions in terms of ϕ, $\dot{\phi}$, ℓ, m, v_A, and $\dot{v}_A$.

a) What is $\ddot{\phi}$?

b) What is the force exerted by the rod on the support?

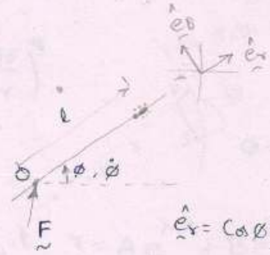
c) What is the acceleration of the bug?



Problem 19.8

18.1.8

Given $\phi, \dot{\phi}, l, m, v_A$ and $\dot{v}_A$



$\hat{e}_r = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{e}_\theta = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

a) $\ddot{\phi}$ is the angular acceleration.

$$\ddot{\phi} = \frac{d\dot{\phi}}{dt} \quad (\text{or } \ddot{\phi} = \frac{d^2\phi}{dt^2})$$

b) L.M.B for the stick, $\sum \underset{\sim}{F} = m\underset{\sim}{a}$

$$\underset{\sim}{F} = m\{\cancel{\underset{\sim}{a}_0}^{0} + -\dot{\phi}^2 l \hat{e}_r + \ddot{\phi} \times l \hat{e}_r + \dot{v}_A \hat{e}_r + 2\dot{\phi}\hat{k} \times v_A \hat{e}_r\}$$

$$\Rightarrow \underset{\sim}{F} = m\{-\dot{\phi}^2 l \hat{e}_r + \ddot{\phi} l \hat{e}_\theta + \dot{v}_A \hat{e}_r + 2\dot{\phi} v_A \hat{e}_\theta\}$$

$$\underset{\sim}{F} = m\{(\dot{v}_A - \dot{\phi}^2 l)\hat{e}_r + (\ddot{\phi} l + 2\dot{\phi} v_A)\hat{e}_\theta\}$$

Force exerted by stick on the support $= -F$

$\therefore$ Force on support $= m\{(\dot{\phi}^2 l - \dot{v}_A)(\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$

$$- (\ddot{\phi} l + 2\dot{\phi} v_A)(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})\}$$

(c) Acceleration of the bug

$$\underset{\sim}{a} = \cancel{\underset{\sim}{a}_0}^{0} - \dot{\phi}^2 l \hat{e}_r + \ddot{\phi} l \hat{e}_\theta + \dot{v}_A \hat{e}_r + 2\dot{\phi} v_A \hat{e}_\theta$$

$$\underset{\sim}{a} = (\dot{v}_A - \dot{\phi}^2 l)\hat{e}_r + (\ddot{\phi} l + 2\dot{\phi} v_A)\hat{e}_\theta$$

(or)

$$\underset{\sim}{a} = (\dot{v}_A - \dot{\phi}^2 l)(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + (\ddot{\phi} l + 2\dot{\phi} v_A)(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

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19.1.9 A new kind of gun. Assume $\omega = \omega_0$ is a constant for the rod in the figure. Assume the mass is free to slide. At $t = 0$, the rod is aligned with the x-axis and the bead is one foot from the origin and has no radial velocity ($dR/dt = 0$).

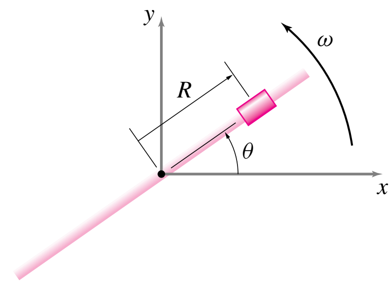
a) Find a differential equation for $R(t)$.

b) Turn this equation into a differential equation for $R(\theta)$.

c) How far will the bead have moved after one revolution of the rod? How far after two?

d) What is the speed of the bead after one revolution of the rod (use $\omega_0 = 2\pi$ rad/s $= 1$ rev/sec)?

e) How much kinetic energy does the bead have after one revolution and where did it come from?



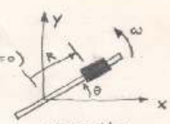
Problem 19.9

Answer:
(a) $\ddot{R}(t) - \omega_0^2 R(t) = 0$.
(b) $\frac{d^2R(\theta)}{d\theta^2} - R(\theta) = 0$.
(c) $R(\theta = 2\pi) = 267.7\,\text{ft}$, $R(\theta = 4\pi) = 1.43 \times 10^5\,\text{ft}$.
(d) $v = 1682.3\,\text{ft/s}$
(e) $E_K = (2.83 \times 10^6) \times (mass)(\text{ft/s})^2$.

\# 6.38

FBD

$\omega = \omega_0$ constant
bead has mass m
$R = 1\,ft = r(t=0)$
$\dot{r}(t=0) = 0$
no friction



no gravity

A

LMB: $N\hat{e}_\theta = m\underline{a}$

$N\hat{e}_\theta = m[(\ddot{r} - r\omega^2)\hat{e}_r + (2\dot{r}\omega + r\dot{\omega})\hat{e}_\theta]$

dotting with $\hat{e}_r$

$0 = m(\ddot{r} - r\omega^2)$ $\boxed{\ddot{R}(t) - R(t)\omega_0^2 = 0}$

B

we can rewrite $\frac{d^2R(t)}{dt^2} = \frac{d}{d\theta}\left(\frac{dR(\theta)}{d\theta}\frac{d\theta}{dt}\right)\frac{d\theta}{dt} = \frac{d^2R(\theta)}{d\theta^2}\omega_0^2$

so $\frac{d^2R(\theta)}{d\theta^2}\omega_0^2 - R(\theta)\omega_0^2 = 0$ or

$\boxed{\frac{d^2R(\theta)}{d\theta^2} - R(\theta) = 0}$ solve this equation for Part C

C

from ODE's we know $R(\theta) = Ae^{\theta} + Be^{-\theta}$

$R(\theta=0) = R(t=0) = 1\,ft$ $\Rightarrow A + B = 1\,ft$

$\left.\frac{dR}{d\theta}\right|_{\theta=0} = 0$ $\Rightarrow A - B = 0$

so $A = \frac{1}{2}\,ft = B$

$R(\theta) = \cosh(\theta)\,ft$ or $R(\theta) = (\frac{1}{2}ft)(e^{\theta} + e^{-\theta})$ if your calculator doesn't do cosh!

$\boxed{\begin{array}{l} R(\theta = 2\pi) = 267.7\,ft \\ R(\theta = 4\pi) = 1.43 \times 10^5\,ft \end{array}}$

D $\underline{v}_{B/\mathcal{N}} = \underline{v}_{B/\mathcal{B}} + \underline{\omega}_{\mathcal{B}/\mathcal{N}} \times \underline{r}_{B/\mathcal{B}}$ velocity of bead in Newtonian Frame
speed is magnitude of $\underline{v}_{\mathcal{N}}$

$\underline{v}_{B/\mathcal{B}}$ = velocity of bead along rod $\frac{d(R(\theta))}{dt} = \sinh(\theta)\dot{\theta}\,(ft)$

but $\dot{\theta} = \omega_0 = 2\pi$ rad/sec so $\underline{v}_B = 1682$ ft/sec $\hat{e}_r$

$\underline{r}_B = R(\theta = 2\pi) = 267.7$ ft $\hat{e}_r$ from above

$\underline{\omega}_B = 2\pi$ rad/sec $\hat{k}$

$\underline{v}_{B/\mathcal{N}} = 1682$ ft/sec $\hat{e}_r$ + 1682 ft/sec $\hat{e}_\theta$

$\boxed{v_{B/\mathcal{N}} = 2378.7\ \text{ft/sec}}$

E

$E_K = \frac{1}{2}m(v_{B/\mathcal{N}})^2 = \frac{1}{2}(mass)(2378.7\ \text{ft/sec})^2$

$\boxed{E_K = 2.83 \times 10^6\,(\text{ft/sec})^2\,(mass)}$

In order to keep the rod spinning at a constant speed ω_0, there must be a motor attached to it. The torque (moment) required to maintain constant angular velocity ω_0 is increasing exponentially. I calculate the moment required at any angle θ to be
$M(\theta) = (mass)(ft^2)\,2\cosh(\theta)\sinh(\theta)\,\omega_0^2$ Hence, motor provides E_K.

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19.1.10 The new gun gets old and rusty. Reconsider the bead on a rod in problem 19.1.9. This time, friction cannot be neglected. The friction coefficient is μ. At the instant of interest the bead with mass m has radius R_0 with rate of change $\dot{R}_0$. The angle θ is zero and ω is a constant. Neglect gravity.

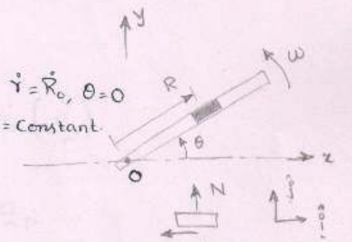
a) What is $\ddot{R}$ at this instant? Give your answer in terms of any or all of R_0, $\dot{R}_0$, ω, m, μ, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_R$, $\hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$.

b) After a very long time it is observed that the angle ϕ between the path of the bead and the rod/trough is nearly constant. What, in terms of μ, is this ϕ?

Answer:
(a) $\ddot{R} = R_0\omega^2 - 2\mu\dot{R}_0\omega$.
(b) $\phi = \sin^{-1}\left[\frac{1}{\sqrt{(\sqrt{1+\mu^2}-\mu)^2+1}}\right]$.

18.1.10 a)

At the instant of interest, $r = R_0$; $\dot{r} = \dot{R}_0$, $\theta = 0$ and ω = Constant.



Applying LMB on the bead.

$$\Sigma \underline{F} = m\underline{a}$$

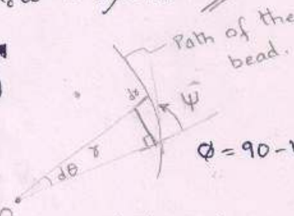
$$\Rightarrow \left\{ N\hat{\underline{j}} - \mu N \hat{\underline{i}} = m\left[\underline{a}_0 + -\dot{\theta}^2 r \hat{\underline{i}} + \ddot{\underline{\theta}} \times r\hat{\underline{i}} + \ddot{r}\hat{\underline{i}} + 2\dot{\underline{\theta}} \times \dot{r}\hat{\underline{i}}\right]\right\}$$

$$\{\}\cdot\hat{\underline{j}} \Rightarrow N = 2m\dot{r}\dot{\theta}$$

$$\{\}\cdot\hat{\underline{i}} \Rightarrow -\mu N = -m\dot{\theta}^2 r + m\ddot{r}$$

$\Rightarrow$ at the instant of interest $\ddot{r} = R_0\omega^2 - 2\mu\dot{R}_0\omega$

b) From the figure, $\cot\phi \approx \left(\frac{r\,d\theta}{dr}\right)$



$$\Rightarrow \tan\psi = \frac{r}{dr/d\theta}$$

$\phi = 90 - \psi$

Solving the differential equation, we get $r = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

where $\lambda_{1,2} = -\mu \pm \sqrt{1-\mu^2}\,\omega$

After a very long time, $r = Ae^{(-\mu+\sqrt{1-\mu^2})\theta}$

$$\therefore \tan\psi = \frac{1}{-\mu+\sqrt{1-\mu^2}} \quad (\text{or}) \quad \psi = \sin^{-1}\left[\frac{1}{\sqrt{1+(-\mu+\sqrt{1-\mu^2})^2}}\right]$$

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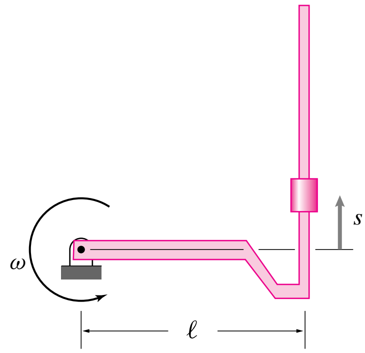
19.1.11 A newer kind of gun. As an attempt to make an improvement on the 'new gun' demonstrated in problem 19.1.9, a person adds a length ℓ to the shaft on which the bead slides. Assume there is no friction between the bead (mass m) and the wire. Assume the bead starts at $s = 0$ with $\dot{s} = v_0$. The rigid rod, on which the bead slides, rotates at a constant rate $\omega = \omega_0$. Find $s(t)$ in terms of ℓ, m, v_0, and ω_0.

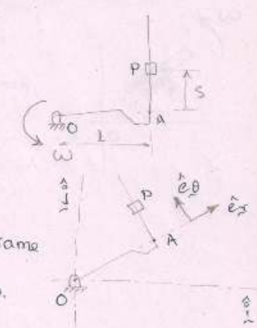


Problem 19.11

18.1.11

FBD of the bead

$N\hat{e}_r$



Let the body fixed coordinate frame be at 'A' with $\hat{e}_r$ & $\hat{e}_\theta$ as shown.

Acceleration of the bead:

$$\underset{\sim}{a}_P = \underset{\sim}{a}_A + \underset{\sim}{a}_{P/A}$$

Given ω = constant $= \omega_0$, we have

$$\underset{\sim}{a}_A = -\omega_0^2 \ell \, \hat{e}_r$$

$$\underset{\sim}{a}_{P/A} = +\ddot{s}\,\hat{e}_\theta - \omega_0^2 s\,\hat{e}_\theta + 2\omega_0 \dot{s}\,(-\hat{e}_r)$$

L.M.B for bead: $\quad \Sigma \underline{F} = m\underset{\sim}{a}_P$

$$N\hat{e}_r = -(m\ell\omega_0^2 + 2m\omega_0\dot{s})\,\hat{e}_r + (m\ddot{s} - ms\omega_0^2)\,\hat{e}_\theta$$

$\{\ \}\cdot\hat{e}_\theta \Rightarrow \quad 0 = m\ddot{s} - ms\omega_0^2$

$\Rightarrow \quad \ddot{s} = \omega_0^2 s$

$\Rightarrow \quad s = Ae^{\omega_0 t} + Be^{-\omega_0 t}$

Given that at $t=0$, $s=0$ and $\dot{s} = v_0$

$\Rightarrow A = -B$ & $v_0 = \omega_0 A - \omega_0 B \Rightarrow B = \dfrac{-v_0}{2\omega_0}$

$$\therefore \quad s = \frac{v_0}{2\omega_0} e^{\omega_0 t} - \frac{v_0}{2\omega_0} e^{-\omega_0 t}$$

$$(\text{or}) \quad s = \frac{v_0}{\omega_0} \sinh \omega_0 t$$

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19.1.12 Slippery bead on straight rotating stick. A long stick with mass m_s rotates at a constant rate and a bead, modeled as a point mass, slides on the stick. The stick rotates at constant rate $\vec{\omega} = \omega\hat{\boldsymbol{k}}$. Neglect gravity. The bead has mass m_b. Initially the mass is at a distance R_0 from the hinge point on the stick and has no radial velocity ($\dot{R} = 0$). The initial angle of the stick is $\theta = 0$, measured counterclockwise from the positive x axis.

a) What is the torque, as a function of the net angle the stick has rotated, θ, required in order to keep the stick rotating at a constant rate ?

b) What is the path of the bead in the xy-plane? (Draw an accurate picture showing about one half of one revolution.)

c) How long should the stick be if the bead is to fly in the negative x-direction when it gets to the end of the stick?

d) **Add friction.** How does the speed of the bead over one revolution depend on μ, the coefficient of friction between the bead and the wire? Make a plot of $|\vec{\boldsymbol{v}}_{\text{one revolution}}|$ versus μ. [Note, you have been working with $\mu = 0$ in the problems above.]

TAM 203 · Summer 96 · HW8 Solutions ·
TA: Erich Schneider · July 21, 1996

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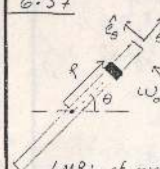
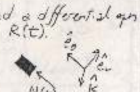


At $t=0$,
$\theta = 0$ and $\dot{R} = 0$
$\underset{\sim}{\omega} = \omega_0\hat{k} = \text{const}$

a) Find a differential eqn for $R(t)$.

FBD:



LMB: of mass

$\sum \underset{\sim}{F} = N(t)\hat{e}_\theta = m\left((-R\dot{\theta}^2 + \ddot{R})\hat{e}_r - (R\underbrace{\ddot{\theta}}_{=0 \text{ b/c } \omega = \dot{\theta} = \text{const}} + 2\dot{R}\dot{\theta})\hat{e}_\theta\right)$

dot LMB w/ $\hat{e}_r, \hat{e}_\theta$, get 2 scalar eqns:

$\hat{e}_r \Rightarrow \ddot{R} - R\dot{\theta}^2 = 0 \xRightarrow{\dot{\theta} = \text{const} = \omega_0} \boxed{\ddot{R} - \omega_0^2 R = 0}$

$\hat{e}_\theta \Rightarrow 2\dot{R}\dot{\theta} = \dfrac{N(t)}{m}$

b) Turn the above into a differential eqn for $R(\theta)$.

$\dfrac{d^2R}{dt^2} - \omega_0^2 R = 0 \Rightarrow \dfrac{d}{dt}\left(\dfrac{dR}{d\theta}\underbrace{\dfrac{d\theta}{dt}}_{=\omega_0}\right) - \omega_0^2 R = 0$

$\Rightarrow \omega_0 \dfrac{d^2R}{dt\,d\theta} - \omega_0^2 R = 0 \Rightarrow \omega_0 \dfrac{d}{d\theta}\left(\dfrac{dR}{d\theta}\dfrac{d\theta}{dt}\right) - \omega_0^2 R = 0$

$\Rightarrow \omega_0^2 \dfrac{d^2R}{d\theta^2} - \omega_0^2 R = 0 \Rightarrow \boxed{\dfrac{d^2R}{d\theta^2} - R = 0}$

c) what is $R(\theta = 2\pi)$ & $R(\theta = 4\pi)$?

Solve the above dif. eq. for $R(\theta)$:

soln of form $R = Ae^{\theta} + Be^{-\theta}$

$\Rightarrow R(\theta = 0) = 1\text{ft} \Rightarrow 1\text{ft} = A + B$ *

to use second IC, $\frac{dR}{d\theta}(\theta = 0) = 0$, differentiate:

$\dfrac{dR}{d\theta} = Ae^{\theta} - Be^{-\theta}$

$\Rightarrow \quad 0 = A - B$ **

From * and ** we find $A = B = 0.5\text{ft}$

thus $R(\theta) = (0.5\text{ft})(e^{\theta} + e^{-\theta})$

Putting in $\theta = 2\pi$ rad & 4π rad to the above gives:

$\boxed{\begin{aligned} R(\theta = 2\pi\text{ rad}) &= 268\text{ft} \\ R(\theta = 4\pi\text{ rad}) &= 143{,}000\text{ft}\ ! \end{aligned}}$

d. What is $|\underset{\sim}{v}|$ after 1 rev.? ($\omega_0 = 2\pi$ rad/s)

$\Rightarrow |\underset{\sim}{v}| = |R\dot{\theta}\hat{e}_\theta + \dot{R}\hat{e}_r|$

continued

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where $\dot{R} = \dfrac{dR}{dt} = \dfrac{dR}{d\theta}\dfrac{d\theta}{dt} = \omega\dfrac{dR}{d\theta}$

at $\theta = 2\pi$ rad, $\omega\dfrac{dR}{d\theta} = (2\pi\tfrac{1}{s})(0.5\text{ft})(e^{2\pi} - e^{-2\pi}) = 1682\text{ft/s}$

thus $|\underset{\sim}{v}| = |(268\text{ft})(2\pi\text{ rad/s})\hat{e}_\theta + 1682\text{ ft/s}\ \hat{e}_r| =$

$\boxed{|\underset{\sim}{v}| = 2380\text{ ft/s}}$

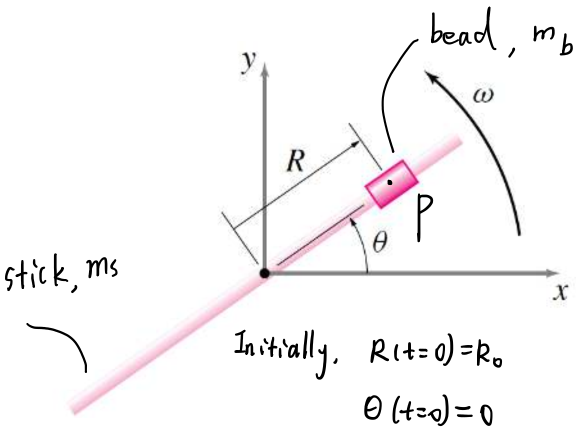
e. $KE = \frac{1}{2}m|\underset{\sim}{v}|^2 = (\text{mass})(2.83E6)\ \text{ft}^2/\text{s}^2$

The angular speed of the disk would not normally stay constant - try it yourself in lab by standing on the disk, spinning around and putting your arms out (increasing your moment of inertia). You will slow down. Thus, there has to be a motor of some sort attached to the rod that applies the power that keeps ω const (and thus accelerates the mass).

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

18.1.12

Saturday, April 27, 2019 6:22 PM



$\vec{\omega} = \omega \hat{k}$

Initially, $R(t=0) = R_0$

$\theta(t=0) = 0$

$\dot{R}(t=0) = 0$

a) Find the torque required to keep the stick rotating as const rate ω (in terms of θ)

AMB/0 :

$$\vec{M}_{/0} = \dot{\vec{H}}_{/0}$$

$$\vec{H}_{/0} = I^c \vec{\omega} + \vec{r}_{P/0} \times m \vec{v}_{P/0}$$

$$= I^c \vec{\omega} + \vec{r}_{P/0} \times m (\vec{\omega} \times \vec{r}_{P/0})$$

$$= I^c \vec{\omega} + m r_{P/0}^2 \vec{\omega}$$

$$= \left[I^c + m |\vec{R}(t)|^2 \right] \vec{\omega}$$

Since we need to take derivative of $\vec{H}_{/0}$ to get $\vec{M}_{/0}$ in terms of θ.
We need to find $R(\theta)$.
Firstly we find $R(t)$ in ode.

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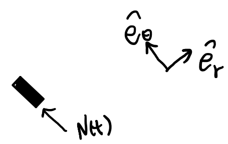
FBD:



LMB: $\sum \vec{F} = m\vec{a}$ in polar coordinates:

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_r + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta$$

$$\Rightarrow \left\{ \vec{N}(t) = m(\ddot{R} - R\dot{\theta}^2)\hat{e}_r + m(R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta \right\}$$

$$\{\ \}\cdot\hat{e}_r \Rightarrow \quad 0 = \ddot{R} - R\dot{\theta}^2 = \ddot{R} - R\omega^2 \quad (*)$$

Then turn the above into an ode for $R(\theta)$

$$\frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \omega\frac{dR}{d\theta}$$

$$\frac{d^2R}{dt^2} = \frac{d}{dt}\left(\omega_0\frac{dR}{d\theta}\right) = \omega_0\frac{d}{d\theta}\left(\frac{dR}{d\theta}\right)\cdot\frac{d\theta}{dt} = \omega^2\frac{d^2R}{d\theta^2}$$

plug into $(*)$ we get

$$\frac{d^2R}{d\theta^2} - R = 0$$

Soln: $R(\theta) = Ae^{\theta} + Be^{-\theta}$

plus initial conditions: $R(\theta=0) = R_0$

$\dot{R}(\theta=0) = 0$

$$\Rightarrow \begin{cases} A + B = R_0 \\ A - B = 0 \end{cases} \Rightarrow \quad A = B = \frac{R_0}{2}$$

$$\therefore \quad R(\theta) = \frac{R_0}{2}(e^{\theta} + e^{-\theta}) \qquad \frac{dR}{d\theta} = \frac{R_0}{2}(e^{\theta} - e^{-\theta})$$

Thus $\vec{M}_{/0} = \dot{\vec{H}}_{/0} = m2R\dot{R}\omega = 2m\omega^2 R\frac{dR}{d\theta}$

$$= 2m\omega^2\frac{R_0}{2}(e^{\theta} + e^{-\theta})\cdot\frac{R_0}{2}(e^{\theta} - e^{-\theta})$$

$$= \frac{m\omega^2 R_0^2}{2}(e^{2\theta} - e^{-2\theta})$$

b) Find path in x-y coordinates in $\frac{1}{2}$ revolution

$$\begin{cases} x = R(\theta)\cdot\cos\theta \\ y = R(\theta)\cdot\sin\theta \end{cases} \qquad 0 \le \theta \le \pi.$$

Code & path see attached

c) Find the length of the stick l if the bead is to fly in $-x$ direction when it gets to the end

$\vec{v}$ is in $-\hat{i}$ direction

Thus $\theta_f = 2k\pi + \pi \qquad k = 0, 1, 2 \cdots$

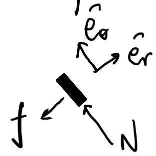
$= (2k+1)\pi$

$$\therefore \quad l = R(\theta_f) = \frac{m\,\omega^2 R_0}{2}\left(e^{(2k+1)\pi} + e^{-(2k+1)\pi}\right)$$

d) Same setup as before but with friction. coefficient is μ.

Find speed $v = |\vec{v}|$ in terms of μ

FBD



LMB: $\Sigma \vec{F} = \dot{\vec{L}} \Rightarrow$

$$\left\{ N\hat{e}_\theta - \mu N \hat{e}_r = m\left((-R\dot{\theta}^2 + \ddot{R})\hat{e}_r + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta\right) \right\}$$

$\{\}\cdot\hat{e}_r \Rightarrow \quad -\frac{\mu N}{m} = \ddot{R} - R\dot{\theta}^2 \qquad ①$

$\{\}\cdot\hat{e}_\theta \Rightarrow \quad \frac{N}{m} = 2\dot{R}\dot{\theta} \qquad ②$

plug ② into ① to eliminate N:

$\ddot{R} - R\dot{\theta} = -\mu(2\dot{R}\dot{\theta})$

$\dot{\theta} = \omega \Rightarrow \ddot{R} - \omega R = -2\mu\omega\dot{R} \quad *$

Also we have from part A:

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$$\Rightarrow \quad \dot{R} = \omega \frac{dR}{d\theta}$$

$$\ddot{R} = \omega^2 \frac{d^2R}{d\theta^2}$$

plug into $*$ $\quad \dfrac{d^2R}{d\theta^2} - R = -2\mu \dfrac{dR}{d\theta}$

$$\therefore \quad R = Ae^{c_1\theta} + Be^{c_2\theta}$$

where c_1, c_2 satisfy the characteristic equation

$$x^2 + 2\mu x - 1 = 0$$

$$c_1 = -\mu + \sqrt{\mu^2+1} \qquad c_2 = -\mu - \sqrt{\mu^2+1}$$

with initial conditions, $R(\theta=0) = R_0$

$\dot{R}(\theta=0) = 0$

$$\Rightarrow \begin{cases} A + B = R_0 \\ Ac_1 + Bc_2 = 0 \end{cases} \Rightarrow \begin{cases} A = -\dfrac{c_2}{c_1-c_2} R_0 \\ B = \dfrac{c_1}{c_1-c_2} R_0 \end{cases}$$

$$\therefore \quad \frac{dR}{d\theta} = \frac{c_1c_2}{c_1-c_2} R_0 \left(e^{c_2\theta} - e^{-c_1\theta}\right)$$

$$\vec{v} = R\dot{\theta}\hat{e}_\theta + \dot{R}\hat{e}_r$$

$$\therefore \quad |\vec{v}| = \sqrt{R^2\dot{\theta}^2 + \dot{R}^2} = \sqrt{R^2\omega^2 + \dot{R}^2} = \omega\sqrt{R^2 + \left(\frac{dR}{d\theta}\right)^2}$$

$\boxed{\text{Code \& plots see attached}}$

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```
% Problem 18.1.12 bead on straight rotating stick
clc;
clear;
close all;
% Parameters
mb = 1;       % mass of the bead
ms = 2;       % mass of the stick
R0 = 1;       % initially the bead is at a distance R0 from the origin
omega = 1;    % const rotating angular velocity
% the circular path of the initial point of the stick
angle = 0:pi/80:2*pi;
circlex = R0*cos(angle);
circley = R0*sin(angle);
plot(circlex,circley);
text(-2,0,'the rotating circle of the stick');
hold on
axis equal
% Path of the bead
theta = 0:pi/80:pi;    % half of the revolution
R = R0/2*(exp(theta)+exp(-theta));     % R in terms of theta
% in x-y coordinates
x = R.*cos(theta);
y = R.*sin(theta);
plot(x,y)
text(-4,4,'the actual path of the bead');
title('Actual path of the bead on rotating stick without friction');

% speed vs mu with friction
mu = 0:0.02:1;
c1 = -mu + sqrt(mu.^2+1);
c2 = -mu - sqrt(mu.^2+1);
A = - R0 * c2./(c1-c2);
B =   R0 * c1./(c1-c2);
figure
for theta = 0:pi/4:pi    % make theta different for each plot
    R = A.*exp(c1*theta) + B.*exp(c2*theta);
    dRdtheta = R0.*c1.*c2/(c1-c2).*(exp(c2*theta) - exp(c1*theta));
    v = omega*sqrt(R.^2+dRdtheta.^2);
    plot(mu,v)
    hold on
end
title('speed of the bead vs mu with different theta');
xlabel('mu');
ylabel('sped of the bead');
legend('theta = 0','theta = pi/4','theta = pi/2','theta =
 3*pi/4','theta = pi');
```


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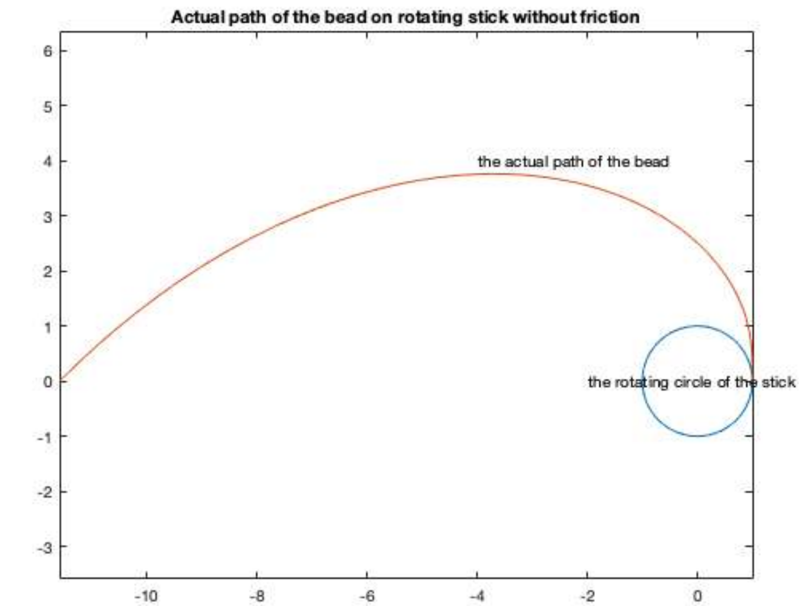
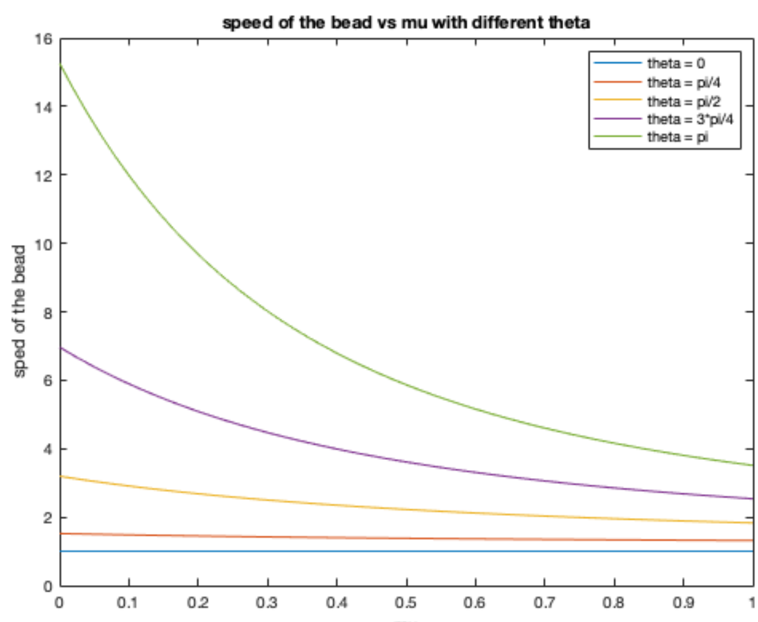

Actual path of the bead on rotating stick without friction
the actual path of the bead
the rotating circle of the stick


speed of the bead vs mu with different theta
theta = 0
theta = pi/4
theta = pi/2
theta = 3*pi/4
theta = pi
sped of the bead
mu

2

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#67 Solution Michelle deSouza

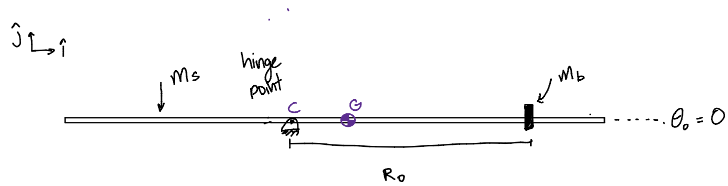
19.1.12 Slippery bead on straight rotating stick. A long stick with mass m_s rotates at a constant rate and a bead, modeled as a point mass, slides on the stick. The stick rotates at constant rate $\vec{\omega} = \omega\hat{k}$. Neglect gravity. The bead has mass m_b. Initially the mass is at a distance R_0 from the hinge point on the stick and has no radial velocity ($\dot{R} = 0$). The initial angle of the stick is $\theta = 0$, measured counterclockwise from the positive x axis.

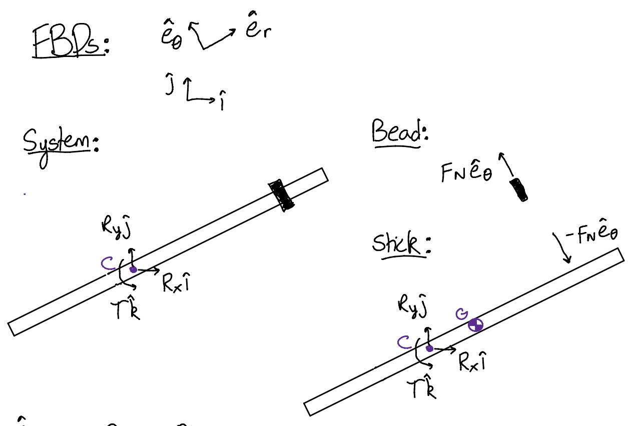
a) What is the torque, as a function of the net angle the stick has rotated, θ, required in order to keep the stick rotating at a constant rate ?

b) What is the path of the bead in the xy-plane? (Draw an accurate picture showing about one half of one revolution.)

c) How long should the stick be if the bead is to fly in the negative x-direction when it gets to the end of the stick?

d) **Add friction.** How does the speed of the bead over one revolution depend on μ, the coefficient of friction between the bead and the wire? Make a plot of $|\vec{v}_{\text{one revolution}}|$ versus μ. [Note, you have been working with $\mu = 0$ in the problems above.]





$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ $\qquad \theta = \omega t + \cancel{\theta_o}\ 0$

What is $r(\theta)$ for the bead?

Bead LMB: $\sum \vec{F} = F_N\hat{e}_\theta = m_b\vec{a}_b$

$$\vec{a}_b = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta \qquad \dot{\theta} = \omega,\ \ddot{\theta} = 0$$

$$= (\ddot{r} - r\omega^2)\hat{e}_r + 2\dot{r}\omega\hat{e}_\theta$$

$$F_N\hat{e}_\theta = m_b\left[(\ddot{r} - r\omega^2)\hat{e}_r + 2\dot{r}\omega\hat{e}_\theta\right]$$

$\{\}\cdot\hat{e}_\theta$ $\quad F_N = 2m_b\dot{r}\omega$

$\{\}\cdot\hat{e}_r$ $\quad 0 = m_b(\ddot{r} - r\omega^2)$

$0 = \ddot{r} - r\omega^2 \longleftarrow$ ODE for $r(t)$, use chain rule to get $r(\theta)$

$0 = \omega^2\frac{d^2r}{d\theta^2} - \omega^2 r \qquad \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \omega\frac{dr}{d\theta}$

$0 = \frac{d^2r}{d\theta^2} - r \qquad \frac{d^2r}{dt^2} = \frac{d}{dt}\left(\omega\frac{dr}{d\theta}\right) = \cancel{\dot{\omega}\frac{dr}{d\theta}} + \omega\frac{d}{d\theta}\left(\omega\frac{dr}{d\theta}\right) = \omega^2\frac{d^2r}{d\theta^2}$

Assume solution of the form $r = Ae^{\lambda\theta}$

$0 = \lambda^2\cancel{Ae^{\lambda\theta}} - \cancel{Ae^{\lambda\theta}}$

$\lambda^2 = 1 \rightarrow \lambda = \pm 1$ $\qquad$ (compare with frictional case below)

$r = Ae^{\theta} + Be^{-\theta}$ $\quad$ Use I.C.s to find A, B

① $r(0) = R_0 = A + B$

② $\frac{dr}{d\theta}(0) = 0 = A - B$

$A = B = \frac{R_0}{2}$

$$\boxed{\vec{r}_{b/c} = \frac{R_0}{2}(e^{\theta} + e^{-\theta})\,\hat{e}_r}$$

Given: ① $r(t)|_{t=0} = R_0 \rightarrow r(\theta)|_{\theta=0} = R_0$

② $\dot{r}(t)|_{t=0} = 0 \rightarrow \omega \frac{dr(\theta)}{d\theta}|_{\theta=0} = 0$

③ $\theta(0)|_{t=0} = 0$ plug into ①, ②

a) Torque required to maintain constant ω

Stick AMB: $\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$

$$\vec{r}_b \times -F_N\hat{e}_\theta + T\hat{k} = I^G\ddot{\theta}\hat{k} + \underbrace{\vec{r}_{G/c} \times M_s\vec{a}_G}$$

$\rightarrow r_{G/c}\hat{e}_r \times M_s(-r_{G/c}\dot{\theta}^2\hat{e}_r + r_{G/c}\cancel{\ddot{\theta}}\hat{e}_\theta)$

$= r_{G/c}\hat{e}_r \times -M_s r_{G/c}\omega^2\hat{e}_r$

$= \vec{0}$

$$T\hat{k} = \vec{r}_{b/c} \times F_N\hat{e}_\theta \,^*$$

$$\vec{r} = \frac{R_0}{2}(e^{\theta} + e^{-\theta})\,\hat{e}_r$$

$$F_N = 2m_b\omega\dot{r} = 2m_b\omega^2\frac{dr}{d\theta} = \cancel{2}m_b\omega^2\frac{R_0}{\cancel{2}}(e^{\theta} - e^{-\theta})$$

$$T\hat{k} = \frac{R_0}{2}(e^{\theta} + e^{-\theta})\hat{e}_r \times m_b R_0\omega^2(e^{\theta} - e^{-\theta})\,\hat{e}_\theta$$

$\{\}\cdot\hat{k}$ $\quad T = \frac{1}{2}m_b R_0^2\omega^2(e^{2\theta} - e^{-2\theta})$

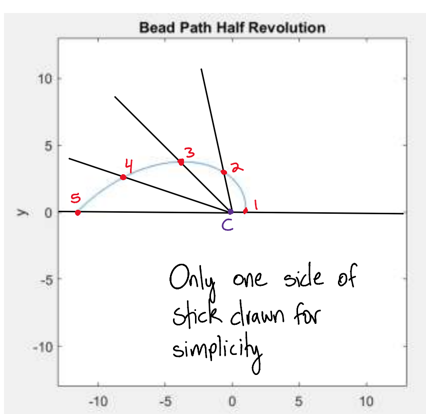
b) Path in x-y plane

$$r = \frac{R_0}{2}(e^{\theta} + e^{-\theta})$$

$x = r\cos\theta \qquad y = r\sin\theta$

plot for half a rotation $\rightarrow \theta = [0, \pi]$

See Matlab



Bead rotates with the stick but also moves radially outward (Would require a centripetal force to keep circular)

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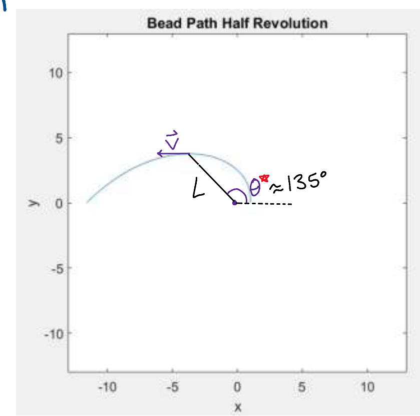
c) Stick length for bead to fly off in $-\hat{\imath}$ direction

- To move in $-\hat{\imath}$, $\vec{v}\cdot\hat{\jmath} = 0 \rightarrow$ Find θ^* to make this true
- To fly off stick, stick length must be equal to $r(\theta^*)$
- Can see from part b) graph that θ^* occurs at point 3



$$\vec{v} = \underbrace{\dot{r}}_{\omega \frac{dr}{d\theta}}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad \hat{e}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

$$= \omega\frac{R_0}{2}(e^{\theta} - e^{-\theta})\hat{e}_r + \omega\frac{R_0}{2}(e^{\theta} + e^{-\theta})\hat{e}_\theta \qquad \hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

$$\vec{v}\cdot\hat{\jmath} = \omega\frac{R_0}{2}\underbrace{\left[\sin\theta(e^{\theta} - e^{-\theta}) + \cos\theta(e^{\theta} + e^{-\theta})\right]}_{=0}$$

Find zero using Matlab $\rightarrow \theta \approx 2.35$ rad $(\sim 135°)$ matches graph ✓

$\theta = 2.35 + 2\pi n$ for n=integer (once per rev)

$L = \frac{R_0}{2}\left(e^{2.35+2\pi n} + e^{-2.35-2\pi n}\right)$ For $R_0 = 1$ m, $n = 0$, $L = 5.3$ m, matches graph ✓

d) Add friction, find $\vec{v}$ vs. μ after one revolution

FBD:

$F_f = \mu F_N$

$\vec{v} = \dot{r}\hat{e}_r + r\omega\hat{e}_\theta$ solve bead LMB for $r, \dot{r}$

Bead LMB: $\sum\vec{F} = F_N\hat{e}_\theta - \mu F_N\hat{e}_r = m_b\left[(\ddot{r} - r\omega^2)\hat{e}_r + 2\dot{r}\omega\hat{e}_\theta\right]$

$\{\}\cdot\hat{e}_\theta \quad F_N = 2m_b\dot{r}\omega$

$\{\}\cdot\hat{e}_r \quad -\mu F_N = m_b(\ddot{r} - r\omega^2)$

$$-\mu(2\cancel{m_b}\dot{r}\omega) = \cancel{m_b}(\ddot{r} - r\omega^2)$$

$$0 = \ddot{r} + 2\mu\omega\dot{r} - r\omega^2 \longleftarrow \text{Looking for speed over 1 revolution, use chain rule for } r(\theta)$$

$$\dot{r} = \frac{dr}{dt} = \omega\frac{dr}{d\theta} \qquad \ddot{r} = \frac{d^2r}{dt^2} = \omega^2\frac{d^2r}{d\theta^2}$$

$$0 = \frac{d^2r}{d\theta^2}\omega^2 + 2\mu\omega^2\frac{dr}{d\theta} - r\omega^2$$

$$0 = \frac{d^2r}{d\theta^2} + 2\mu\frac{dr}{d\theta} - r$$

Assume solution of the form $r = Ae^{\lambda\theta}$

$$0 = \lambda^2\cancel{Ae^{\lambda\theta}} + 2\mu\lambda\cancel{Ae^{\lambda\theta}} - \cancel{Ae^{\lambda\theta}}$$

$$0 = \lambda^2 + 2\mu\lambda - 1$$

$$\lambda = \frac{-2\mu \pm \sqrt{4\mu^2 + 4}}{2} = \underline{-\mu \pm \sqrt{\mu^2 + 1}} \quad \text{call this } \lambda_1, \lambda_2$$

$$r = Ae^{\lambda_1\theta} + Be^{\lambda_2\theta}$$

Note: If $\mu = 0$, λ reduces to ± 1, like solution above

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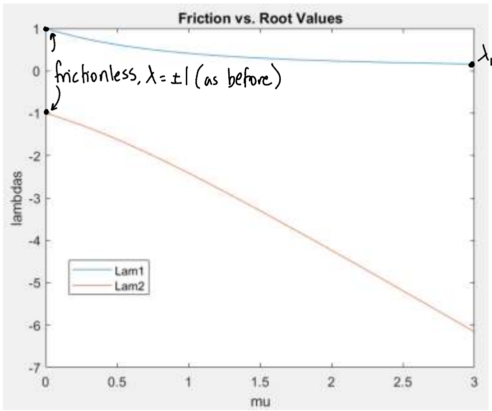


$\lambda_1 \approx 0.1$ (slow exponential decay)

– $\lambda_1 = -\mu + \sqrt{\mu^2+1}$ $\quad A = \frac{\mu + \sqrt{\mu^2+1}}{2\sqrt{\mu^2+1}} R_0$ (found below)

As μ increases, $\lambda_1 \to 0$, $Ae^{\lambda_1\theta} \to A \to R_0$

– $\lambda_2 = -\mu - \sqrt{\mu^2+1}$

As μ increases, $\lambda_2 \to -\infty$, $Be^{\lambda_2\theta} \to 0$

$$r = Ae^{\lambda_1\theta} + Be^{\lambda_2\theta}$$

Same I.C.s as frictionless case

① $r(0) = R_0 = A + B$

② $\frac{dr}{d\theta}(0) = 0 = \lambda_1 A + \lambda_2 B$

$$0 = \lambda_1 A + \lambda_2 (R_0 - A)$$

$A' = \frac{-\lambda_2}{\lambda_1 - \lambda_2} = \frac{\mu + \sqrt{\mu^2+1}}{2\sqrt{\mu^2+1}}$ $\quad A = A'R_0$ $\quad$ Result scales with R_0

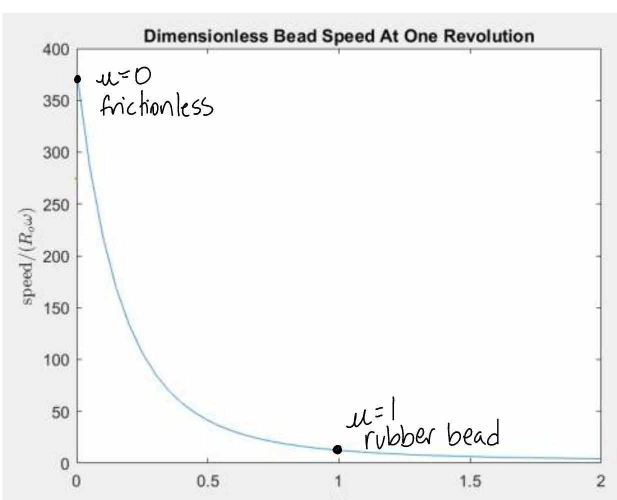
$B' = \frac{-\mu + \sqrt{\mu^2+1}}{2\sqrt{\mu^2+1}}$ $\quad B = B'R_0$ $\quad$ Define A' and B' independent of R_0

$$r = R_0\left(A'e^{\lambda_1\theta} + B'e^{\lambda_2\theta}\right)$$

$$\dot{r} = \omega R_0\left(\lambda_1 A'e^{\lambda_1\theta} + \lambda_2 B'e^{\lambda_2\theta}\right)$$

$$\vec{v} = \dot{r}\hat{e}_r + r\omega\hat{e}_\theta$$

$$\frac{\vec{v}}{R_0\omega} = \left[A'e^{\lambda_1\theta} + B'e^{\lambda_2\theta}\right]\hat{e}_r + \left[\lambda_1 A'e^{\lambda_1\theta} + \lambda_2 B'e^{\lambda_2\theta}\right]$$ ⟵ dimensionless, use in Matlab



As μ increases, speed decreases

Frictionless: $V = 379 R_0\omega$

Rubber bead: $V = 12.5 R_0\omega$

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Table of Contents

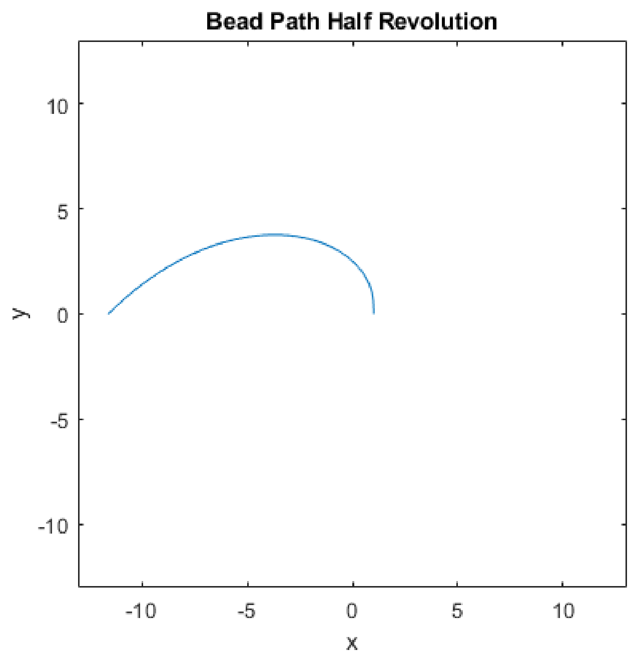

```
% MAE 2030
% HW 12 #67 (19.1.12)
% Michelle deSouza
clear variables; close all; clc;

Ro = 1; %hinge is at the origin
omega = 0.1;
```

b) bead path over 1/2 revolution

```
th = linspace(0,pi,201); % half a circle
r = 0.5*Ro*(exp(th) + exp(-th)); %derived formula for r
%to Cartesian
x = r.*cos(th);
y = r.*sin(th);

plot(x,y)
axis equal
title("Bead Path Half Revolution")
xlabel("x")
ylabel("y")
axis([-13 13 -13 13])
```



c) Find length for bead to fly in -i direction

```
syms x; %correct angle
vdotj = sin(x)*(exp(x)-exp(-x))+cos(x)*(exp(x)+exp(-x));
assume(x,'positive')
theta = double(solve(vdotj));
```

```
Warning: Unable to solve symbolically. Returning a numeric solution
 using <a
href="matlab:web(fullfile(docroot, 'symbolic/
vpasolve.html'))">vpasolve</a>.
```

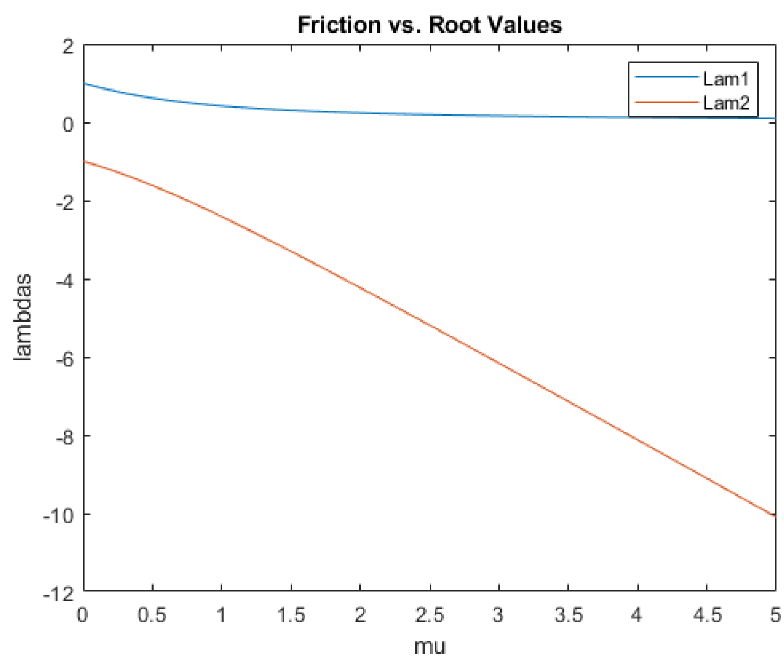
d) Add friction - speed vs. mu after one revolution

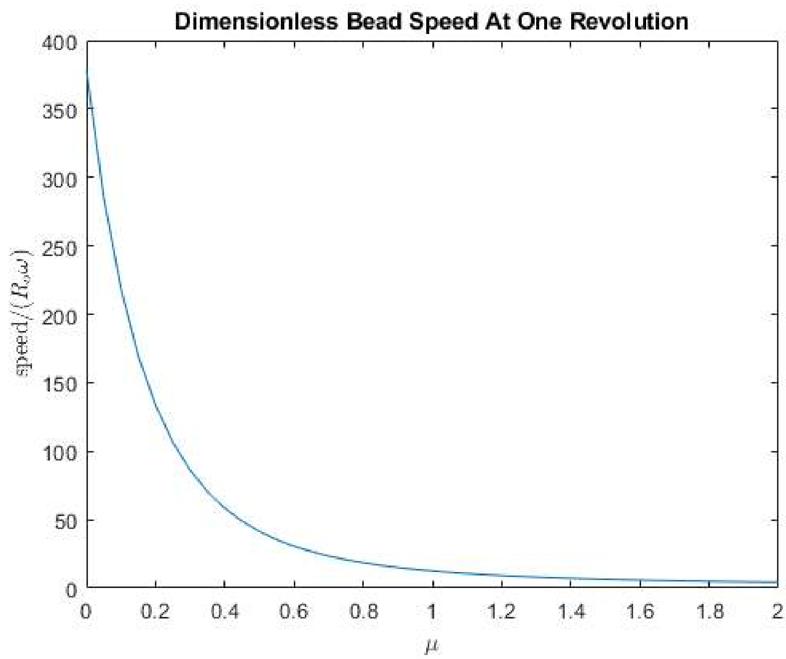
```
th = 2*pi; % one circle
mu = linspace(0,5,101);

%roots of characteristic equation
lam1 = -mu + sqrt(mu.^2+1);
lam2 = -mu - sqrt(mu.^2+1);
figure
plot(mu,lam1,mu,lam2)
title("Friction vs. Root Values")
```

```
xlabel("mu")
ylabel("lambdas")
legend("Lam1", "Lam2")
%coefficients
Ap = -lam2./(lam1-lam2); %Ap = A/Ro
Bp = 1-Ap; %Bp = B/Ro
%r, rdot
rp = Ap.*exp(lam1*th) + Bp.*exp(lam2*th); %rp = r/Ro
rpdot = Ap.*lam1.*exp(lam1.*th) + Bp.*lam2.*exp(lam2.*th); %rpdot =
 rdot/(Ro*omega)
%find dimensionless speed and plot
vp= [rpdot; rp];
vpmag = sqrt(vp(1,:).^2 + vp(2,:).^2);

figure
plot(mu, vpmag)
xlabel('\mu')
xlim([0 2])
ylabel('speed/$(R_{o}\omega)$', 'Interpreter', 'latex')
title("Dimensionless Bead Speed At One Revolution")
hold off
```





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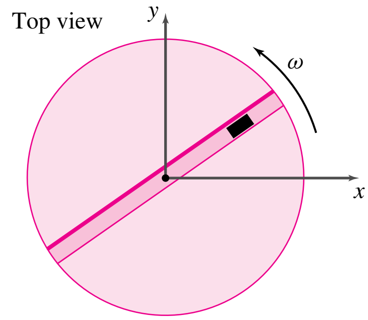
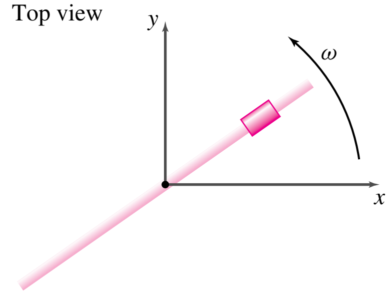
19.1.13 Mass on a lightly greased slotted turntable or spinning uniform rod. Assume that the rod/turntable in the figure is massless and also free to rotate. Assume that at $t = 0$, the angular velocity of the rod/turntable is 1 rad/s, that the radius of the bead is one meter, and that the radial velocity of the bead, dR/dt, is zero. The bead is free to slide on the rod. Where is the bead at $t = 5$ sec?





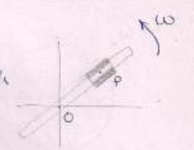
Problem 19.13

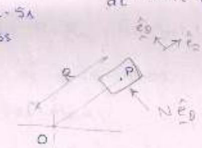
18.1.13

At $t = 0s$, $R = 1\,m$, $\frac{dR}{dt} = 0\,m/s$ & $\omega = 1\,rad/s$

Find R at $t = 5s$

FBD of mass





Acceleration of mass:

$\underset{\sim}{a}_P = \underset{\sim}{a}_O + \underset{\sim}{a}_{P/O}$

$= 0 + -R\omega^2\hat{e}_r + \dot{\omega}R\hat{e}_\theta + \ddot{R}\hat{e}_r + 2\omega\dot{R}\hat{e}_\theta$

$\underset{\sim}{a}_P = (\ddot{R} - R\omega^2)\hat{e}_r + (\dot{\omega}R + 2\omega\dot{R})\hat{e}_\theta$

L.M.B for mass: $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}_P$

$\{ N\hat{e}_\theta = m(\ddot{R} - R\omega^2)\hat{e}_r + m(\dot{\omega}R + 2\omega\dot{R})\hat{e}_\theta \}$

$\{\}\cdot\hat{e}_r \Rightarrow 0 = m(\ddot{R} - R\omega^2)$

$\Rightarrow \ddot{R} = R\omega^2$ is the equation of motion

$(D^2 - \omega^2)R = 0$

$\Rightarrow R = Ae^{\omega t} + Be^{-\omega t} \Rightarrow \dot{R} = A\omega e^{\omega t} - B\omega e^{-\omega t}$

At $t = 0$, $R = 1$ & $\dot{R} = 0 \Rightarrow A = B$ & $1 = A + B \Rightarrow A = B = \frac{1}{2}$

$R = \frac{1}{2}e^{\omega t} + \frac{1}{2}e^{-\omega t}$ (or) $R = \cosh\omega t$

Given. $\omega = 1\,rad/s$

$\therefore$ After 5 sec, $R = \cosh(1 \times 5)$

$\Rightarrow R = 74.209\,m$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

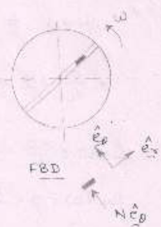
19.1.14 A bead slides in a frictionless slot in a turntable. The turntable spins a constant rate ω. The slot is straight and goes through the center of the turntable. If the bead is at radius R_o with $\dot{R} = 0$ at $t = 0$, what are the components of the acceleration vector in the directions normal and tangential to the path of the bead after one revolution? Neglect gravity.

18.1.14

Given that ω = constant.

Bead radius, $R = R_o$ & $\dot{R} = 0$ at $t = 0$s.

Components of acceleration.



Acceleration of mass;

$$\underset{\sim}{a} = (\ddot{R} - R\omega^2)\hat{e}_r + (\overset{0}{\dot{\omega}} R + 2\omega\dot{R})\hat{e}_\theta$$

L.M.B for mass: $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}$

$$\left\{ N\hat{e}_\theta = (m\ddot{R} - mR\omega^2)\hat{e}_r + (mR\dot{\omega} + 2m\omega\dot{R})\hat{e}_\theta \right\}$$

$\{\ \} \cdot \hat{e}_r \Rightarrow \ddot{R} - \omega^2 R = 0$

$\Rightarrow R = A\cosh\omega t + B\sinh\omega t$

Using the initial conditions, $B = 0$ & $A = R_o$

$\therefore R = R_o\cosh\omega t$

$\Rightarrow \dot{R} = R_o\omega\sinh\omega t$ and $\ddot{R} = R_o\omega^2\cosh\omega t$

Normal acceleration after one revolution, $a_r = (\ddot{R} - R\omega^2)\hat{e}_r$

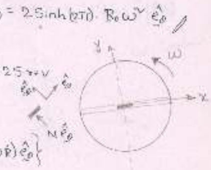
$\Rightarrow \underset{\sim}{a_r} = \underset{\sim}{0}$ //

Tangential acceleration after one revolution, $a_\theta = 2\omega\dot{R}\hat{e}_\theta$

$\Rightarrow \underset{\sim}{a_\theta} = 2\omega R_o\omega\sinh(2\pi)\hat{e}_\theta \Rightarrow \underset{\sim}{a_\theta} = 2\sinh(2\pi) R_o\omega^2\hat{e}_\theta$ //

18.1.15

Find the force on the bead at $\theta = 1.125$ rev



L.M.B for mass $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}$

$$\left\{ N\hat{e}_\theta = (m\ddot{R} - mR\omega^2)\hat{e}_r + (mR\overset{0}{\dot{\omega}} + 2m\omega\dot{R})\hat{e}_\theta \right\}$$

$\{\ \} \cdot \hat{e}_\theta \Rightarrow N = 2m\omega\dot{R}$

To find $\dot{R}$ at $\theta = 1.125$ rev

$\{\ \} \cdot \hat{e}_r \Rightarrow \ddot{R} - \omega^2 R = 0 \Rightarrow R = A\cosh\omega t + B\sinh\omega t$

Given that at $t = 0$; $R = 0$m and $\dot{R} = 0.5$ m/s

$\Rightarrow A = 0$ & $0.5 = B\omega(1) \Rightarrow B = \frac{0.5}{3}$ m

$\Rightarrow R = \frac{0.5}{3}\sinh\omega t \Rightarrow \dot{R} = \frac{0.5}{3}3\cosh\omega t$

When $\theta = 1.125$ rev $\Rightarrow \dot{R} = 0.5\cosh\omega t$

$\omega t = 1.125 \times 2\pi$ rad

$\dot{R} = 0.5 \times \cosh(1.125 \times 2\pi) = 293.62$ m/s

$N\hat{e}_\theta = 2 \times m \times \omega \times \dot{R}\ \hat{e}_\theta$

$= 2 \cdot 0.001(\text{kg}) \cdot 3(\text{rad/s}) \cdot 293.62(\text{m/s})\hat{e}_\theta$

$N\hat{e}_\theta = 1.762\ (\text{N})\ \hat{e}_\theta$

$\hat{e}_\theta = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$

$\phi = 0.125$ rev $= \frac{1}{8} \times 2\pi = \pi/4$ rad

$\therefore \hat{e}_\theta = -\sin\frac{\pi}{4}\hat{\imath} + \cos\frac{\pi}{4}\hat{\jmath} \Rightarrow \hat{e}_\theta = -\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$

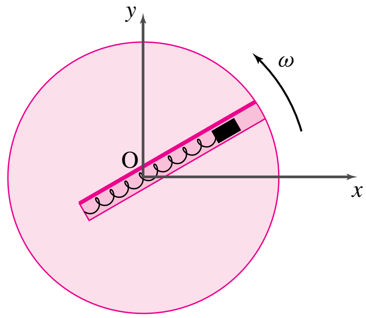
$\therefore$ Force in $\hat{\imath}, \hat{\jmath}$ is $\underset{\sim}{N} = \{-1.246\hat{\imath} + 1.246\hat{\jmath}\}$ (N)

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.1.16 Bead on springy leash in a slot on a turntable. The bead in the figure is held by a spring that is relaxed when the bead is at the origin. The constant of the spring is k. The turntable speed is controlled by a strong stiff motor.

a) Assume $\omega = 0$ for all time. What are possible motions of the bead?

b) Assume $\omega = \omega_0$ is a constant. What are possible motions of the bead? Notice there are two cases depending on the value of ω_0. What is going on here?

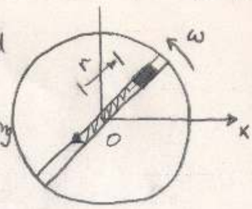


Problem 19.16: Mass in a slot.

6.42

bead has mass, m

r distance of bead from the origin



FBD

no friction

force of spring

$\underset{\sim}{T} = -kr\,\hat{e}_r$

LMB $\quad -kr\,\hat{e}_r + N\,\hat{e}_\theta = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right]$

$\boxed{A}$ if $\omega = 0$ then $\dot{\theta} = 0$, $\ddot{\theta} = 0$ dotting LMB with $\hat{e}_r$

$m\ddot{r} + kr = 0$ or $\ddot{r} + \frac{k}{m}r = 0$

motion of bead is oscillatory $\lambda = \sqrt{\frac{k}{m}}$

$r(t) = A\cos\lambda t + B\sin\lambda t$ simple harmonic oscillator!

$\boxed{B}$ if $\omega = \omega_0$ is a constant dotting LMB with $\hat{e}_r$

$-kr = m\ddot{r} - mr\omega_0^2$ or $\ddot{r} + \left(\frac{k}{m} - \omega_0^2\right)r = 0$

suppose $\lambda = \frac{k}{m} - \omega_0^2$, λ either greater than, less than, or equal to 0

case 1 $\quad \lambda > 0 \quad \omega_0 < \sqrt{\frac{k}{m}}$

motion is oscillatory, $\ddot{r} + \lambda r = 0$

case 2 $\quad \lambda < 0 \quad \omega_0 > \sqrt{\frac{k}{m}}$

radial position of bead increases exponentially $\ddot{r} - |\lambda| r = 0$

case 3 $\quad \lambda = 0 \quad \omega = \sqrt{\frac{k}{m}}$

radial position of bead doesn't change

for $\omega_0 < \sqrt{\frac{k}{m}}$ spring is strong enough to supply centripetal acceleration of bead (k is large enough).

for $\omega_0 > \sqrt{\frac{k}{m}}$ spring is not strong enough to supply centripetal acceleration of bead (k is small, spring is wimpy).

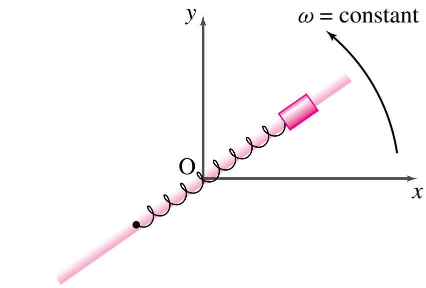
19.1.17 A small bead with mass m slides without friction on a rigid rod which rotates about the z axis with constant ω (maintained by a stiff motor not shown in the figure). The bead is also attached to a spring with constant K, the other end of which is attached to the rod. The spring is relaxed when the bead is at the center position. Assume the bead is pulled to a distance d from the center of the rod and then released with an initial $\dot{R} = 0$. If needed, you may assume that $K > m\omega^2$.

a) Derive the equation of motion for the position of the bead $R(t)$.

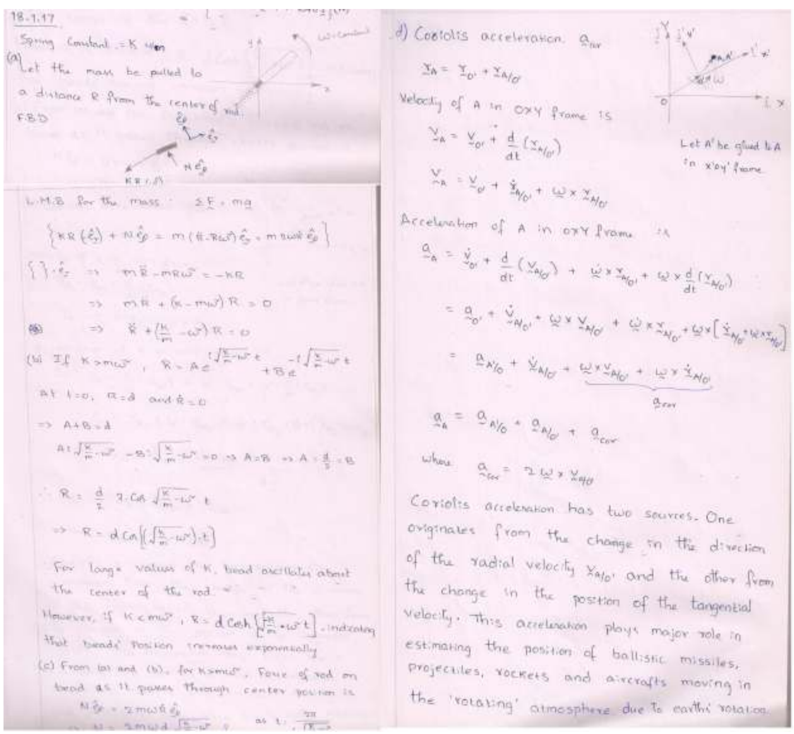
b) How is the motion affected by large versus small values of K?

c) What is the magnitude of the force of the rod on the bead as the bead passes through the center position? Neglect gravity. Answer in terms of m, ω, d, K.

d) Write an expression for the Coriolis acceleration. Give an example of a situation in which this acceleration is important and explain why it arises.



Problem 19.17



18.1.17

Spring Constant = K N/m

(a) Let the mass be pulled to a distance R from the center of rod.

FBD

L.M.B for the mass : $\sum \underline{F} = m\underline{a}$

$$\left\{ KR(\hat{e}_r) + N\hat{e}_\theta = m(\ddot{R} - R\omega^2)\hat{e}_r + m2\omega\dot{R}\hat{e}_\theta \right\}$$

$$\{\}\cdot\hat{e}_r \Rightarrow m\ddot{R} - mR\omega^2 = -KR$$

$$\Rightarrow m\ddot{R} + (K - m\omega^2)R = 0$$

$$\Rightarrow \ddot{R} + \left(\frac{K}{m} - \omega^2\right)R = 0$$

(b) If $K > m\omega^2$, $R = Ae^{i\sqrt{\frac{K}{m}-\omega^2}\,t} + Be^{-i\sqrt{\frac{K}{m}-\omega^2}\,t}$

At $t=0$, $R=d$ and $\dot{R}=0$

$\Rightarrow A + B = d$

$Ai\sqrt{\frac{K}{m}-\omega^2} - Bi\sqrt{\frac{K}{m}-\omega^2} = 0 \Rightarrow A = B \Rightarrow A = \frac{d}{2} = B$

$\therefore R = \frac{d}{2}\, 2\cos\sqrt{\frac{K}{m}-\omega^2}\, t$

$\Rightarrow R = d\cos\left\{\left(\sqrt{\frac{K}{m}-\omega^2}\right)t\right\}$

For large values of K, bead oscillates about the center of the rod.

However, if $K < m\omega^2$, $R = d\cosh\left[\sqrt{-\frac{K}{m}+\omega^2}\,t\right]$, indicating that beads' position increases exponentially

(c) From (a) and (b), for $K > m\omega^2$, Force of rod on bead as it passes through center position is

$N\hat{e}_\theta = 2m\omega\dot{R}\hat{e}_\theta$

d) Coriolis acceleration $\underline{a}_{cor}$

Let A' be glued to A in x'oy' frame

$\underline{r}_A = \underline{r}_{O'} + \underline{r}_{A/O'}$

Velocity of A in OXY frame is

$\underline{v}_A = \underline{v}_{O'} + \frac{d}{dt}(\underline{r}_{A/O'})$

$\underline{v}_A = \underline{v}_{O'} + \dot{\underline{r}}_{A/O'} + \underline{\omega}\times\underline{r}_{A/O'}$

Acceleration of A in OXY frame is

$\underline{a}_A = \dot{\underline{v}}_{O'} + \frac{d}{dt}(\underline{v}_{A/O'}) + \dot{\underline{\omega}}\times\underline{r}_{A/O'} + \underline{\omega}\times\frac{d}{dt}(\underline{r}_{A/O'})$

$= \underline{a}_{O'} + \dot{\underline{v}}_{A/O'} + \underline{\omega}\times\underline{v}_{A/O'} + \dot{\underline{\omega}}\times\underline{r}_{A/O'} + \underline{\omega}\times\left[\dot{\underline{r}}_{A/O'} + \underline{\omega}\times\underline{r}_{A/O'}\right]$

$= \underline{a}_{A'/O} + \dot{\underline{v}}_{A/O'} + \underbrace{\underline{\omega}\times\underline{v}_{A/O'} + \underline{\omega}\times\dot{\underline{r}}_{A/O'}}_{\underline{a}_{cor}}$

$\underline{a}_A = \underline{a}_{A'/O} + \underline{a}_{A/O'} + \underline{a}_{cor}$

where $\underline{a}_{cor} = 2\underline{\omega}\times\underline{v}_{A/O'}$

Coriolis acceleration has two sources. One originates from the change in the direction of the radial velocity $v_{A/O'}$ and the other from the change in the position of the tangential velocity. This acceleration plays major role in estimating the position of ballistic missiles, projectiles, rockets and aircrafts moving in the 'rotating' atmosphere due to earth's rotation.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

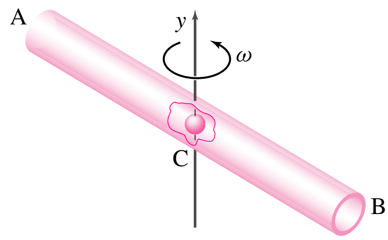
19.1.18 A small ball of mass m_b = 500 grams may slide in a slender tube of length $l = 1.2\,\text{m}$. and of mass $m_t = 1.5\,\text{kg}$. The tube rotates freely about a vertical axis passing through its center C. (Hint: Treat the tube as a slender rod.)

a) If the angular velocity of the tube is $\omega = 8\,\text{rad/s}$ as the ball passes through C with speed relative to the tube, v_0, calculate the angular velocity of the tube just before the ball leaves the tube.

b) Calculate the angular velocity of the tube just after the ball leaves the tube.

c) If the speed of the ball as it passes $C$is $v = 1.8\,\text{m/s}$, determine the transverse and radial components of velocity of the ball as it leaves the tube.

d) After the ball leaves the tube, what constant torque must be applied to the tube (about its axis of rotation) to bring it to rest in 10 s?



Problem 19.18

Answer:
(a) $\omega^- = 4\,\text{rad/s}$, where the minus sign '-' means 'just before leaving'.
(b) $\omega^+ = 4\,\text{rad/s}$, where the plus sign '+' means 'just after leaving'.
(c) $v_r = 3.841\,\text{m/s}$, $v_t = 2.4\,\text{m/s}$.
(d) torque=0.072 N·m

18.1.18

Given: ball mass, $m_b = 0.5\,\text{Kg}$
mass of the tube, $m_t = 1.5\,\text{Kg}$
Length of the tube, $l = 1.2\,\text{m}$

a) Angular velocity of the tube, $\omega^- = 8\,\text{rad/s}$

Angular velocity of the tube just before the ball leaves the tube be ω^+

Conservation of angular momentum applied when ball is at C & B:

$\Rightarrow \quad \underline{H}_{/o}^- = \underline{H}_{/o}^+$

$\Rightarrow \left\{ m_t \frac{l^2}{12} \omega^- \hat{\underline{k}} = \frac{m_t l^2}{12} \omega^+ \hat{\underline{k}} + \underline{r} \times m_b \underline{v} \right\}$

$\{\ \} \cdot \hat{\underline{k}} \Rightarrow \quad \omega^+ = \dfrac{m_t \frac{l^2}{12}}{m_t \frac{l^2}{12} + m_b \left(\frac{l}{2}\right)^2} \omega^- = \dfrac{1}{1 + \frac{3m_b}{m_t}} \omega^- = 4\,\text{rad/s}.$

b) As there is no torque acting on the tube during the balls' motion, angular velocity does not change even after the ball leaves the tube.

c) Speed of the ball as it passes 'C' is $V = 1.8\,\text{m/s}$.

Energy balance when ball is at C and B:

$\frac{1}{2} m_b v^{-2} + \frac{1}{2} I \omega^{-2} = \frac{1}{2} m_b v^{+2} + \frac{1}{2} I \omega^{+2} \qquad I = \frac{m_t l^2}{12}$

$\Rightarrow \quad v^+ = \left(\dfrac{m_b v^{-2} + I(\omega^{-2} - \omega^{+2})}{m_b} \right)^{1/2} = 4.53\,\text{m/s}$

As $v^+ = \left| \omega^+ \times \frac{l}{2} \hat{\underline{e}}_\theta + v_r \hat{\underline{e}}_r \right|$, $V_t = 2.40\,\text{m/s}$ and $V_r = 3.84\,\text{m/s}$

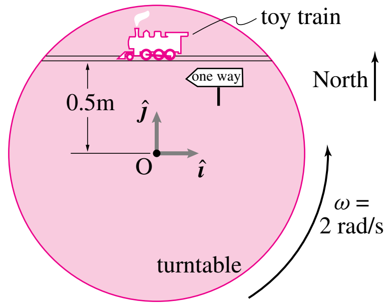
d) Apply A.M.B about O: $\sum \underline{M}_{/o} = \dot{\underline{H}}_{/o}$ (or) $\left\{ T \hat{\underline{k}} = \frac{m_t l^2}{12} \dot{\omega} \hat{\underline{k}} \right\}$

As $\omega_f = \omega_i + \dot{\omega} t$, for $\omega_f = 0$, & $t = 10\,\text{s}$, $\dot{\omega} = \frac{-\omega_i}{t}$ (or) $\dot{\omega} = -0.4\,\text{rad/s}^2$

$\{\ \} \cdot \hat{\underline{k}}$
$\Rightarrow \quad T = -0.072\,\text{Nm}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.1.19 Toy train car on a turn-around. The 0.1 kg toy train's speedometer reads a constant $1m/s$ when, heading west, it passes due north of point O. The train is on a level straight track which is mounted on a spinning turntable whose center is 'pinned' to the ground. The turntable spins at the constant rate of 2 rad/s. What is the force of the turntable on the train? (Don't worry about the z-component of the force.)



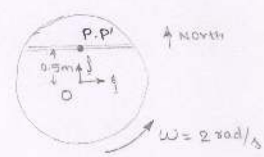
Problem 19.19

Answer:
$\vec{F} = -0.6\,\mathrm{N}\,\hat{\jmath}$.

mass of the toy train, $m = 0.1$ Kg

Velocity, $\underset{\sim}{v} = -1\hat{\imath}$ m/s

Spin rate, $\underset{\sim}{\omega} = 2\hat{k}$ rad/s



Applying LMB on toy train, $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}$

$$\underset{\sim}{a} = (-r\omega^2\hat{\jmath} + \cancel{(\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r})}^{0}) + \cancel{(\underset{\sim}{a}_{P'/O})}^{0} + 2\underset{\sim}{\omega} \times \underset{\sim}{v}$$

$$\Rightarrow \underset{\sim}{a} = -2\hat{\jmath} - 2(2\hat{k} \times -1\hat{\imath}) = -2\hat{\jmath} - 4\hat{\jmath} = -6\hat{\jmath}\ \mathrm{m/s^2}$$

$$\therefore \underset{\sim}{N} = m\underset{\sim}{a} \Rightarrow \underset{\sim}{N} = -0.6\hat{\jmath}\ \mathrm{N}$$

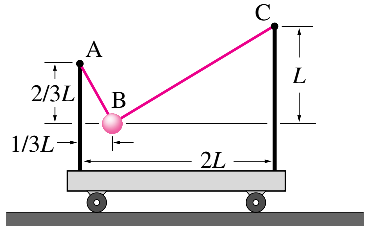
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.1.20 Due to forces not shown, the cart moves to the right with constant acceleration a_x. The ball B has mass m_B. At time $t = 0$, the string AB is cut. Find

a) the tension in string BC before cutting,

b) the absolute acceleration of the mass at the instant of cutting,

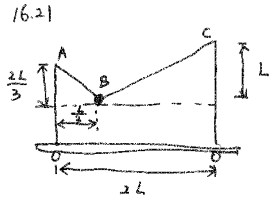
c) the tension in string BC at the instant of cutting.



Problem 19.20

Answer:
(a) $T_{BC} = m\frac{\sqrt{34}}{13}(2a_x + g)$.
(b) $\vec{\boldsymbol{a}} = \frac{1}{34}[(25a_x+15g)\hat{\boldsymbol{\imath}}+(15a_x-25g)\hat{\boldsymbol{\jmath}}]$
(c) $T_{BC} = m\frac{1}{\sqrt{34}}(5a_x + 3g)$.

16.21



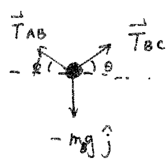
Cart moving forward with $\vec{a} = a_x\hat{\imath}$

At time $t=0$, cut AB.

i) Before cutting, tension in BC ?

The ball B moves with the cart, $\vec{a}_B = a_x\hat{\imath}$

FBD :



$\vec{T}_{BC} = T_{BC}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$

$\vec{T}_{AB} = T_{AB}(-\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$

LMB : $\{\vec{T}_{AB} + \vec{T}_{BC} - m_Bg\hat{\jmath} = m_B a_x\hat{\imath}\}$

$\{\}\cdot(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$

$\Rightarrow \quad T_{BC}(\cos\theta\sin\phi + \sin\theta\cos\phi) = m_Bg\cos\phi + m_B a_x\sin\phi$

$\therefore \quad \cos\theta = \dfrac{\frac{5}{3}L}{\sqrt{(\frac{5}{3}L)^2 + L^2}} = \dfrac{5}{\sqrt{34}}, \quad \sin\theta = \dfrac{3}{\sqrt{34}}$

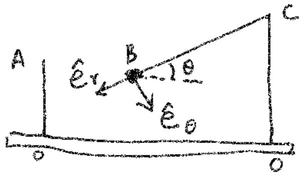
$\cos\phi = \dfrac{1}{\sqrt{5}}, \quad \sin\phi = \dfrac{2}{\sqrt{5}}$

$\Rightarrow \boxed{T_{BC} = \frac{\sqrt{34}}{13} m_B(g + 2a_x)}$

~~BC~~ before cutting AB

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

After cutting AB, the tension in BC is not that before cutting.



Since BC is inextensible, ball B can only swing about C relative to the cart.

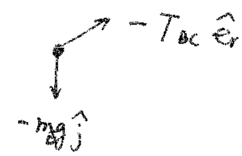
$$\therefore \vec{a}_B = \vec{a}_{cart} + \vec{a}_{B/cart}$$

$$= a_x \hat{i} + \ddot{\theta} d \hat{e}_\theta - d \dot{\theta}^2 \hat{e}_r$$

where $d = |\vec{r}_{B/C}|$: distance between B and C.

Just after cutting, $\dot{\theta} = 0$

$$\therefore \vec{a}_B = a_x \hat{i} + \ddot{\theta} d \hat{e}_\theta$$



LMB: $-T_{BC}\hat{e}_r - mg\hat{j} = m\vec{a}_B$

$$\Rightarrow \{-T_{BC}\hat{e}_r - mg\hat{j} = m a_x \hat{i} + m\ddot{\theta} d \hat{e}_\theta\}$$

$\{\}\cdot\hat{e}_r \Rightarrow \quad -T_{BC} = mg(\hat{j}\cdot\hat{e}_r) + m a_x(\hat{i}\cdot\hat{e}_r)$

$\{\}\cdot\hat{e}_\theta \Rightarrow \quad m\ddot{\theta} d = -mg(\hat{j}\cdot\hat{e}_\theta) - m a_x(\hat{i}\cdot\hat{e}_\theta)$

$\hat{j}\cdot\hat{e}_r = -\sin\theta$, $\hat{i}\cdot\hat{e}_r = -\cos\theta$, $\hat{j}\cdot\hat{e}_\theta = -\cos\theta$, $\hat{i}\cdot\hat{e}_\theta = \sin\theta$

$$\Rightarrow T_{BC} = mg\sin\theta + m a_x\cos\theta = \frac{m}{\sqrt{34}}(3g + 5a_x)$$

and $m\ddot{\theta} d = mg\cos\theta - m a_x\sin\theta = \frac{m}{\sqrt{34}}(5g - 3a_x)$

$\therefore$ Absolute acceleration of B : $\cancel{\vec{a}_B = a_x} \hat{i}$

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$$\vec{a}_B = a_x \hat{\imath} + \ddot{\theta} d \, \hat{e}_\theta$$

$$= a_x \hat{\imath} + \ddot{\theta} d \sin\theta \, \hat{\imath} + \ddot{\theta} d \, (-\cos\theta) \, \hat{\jmath}$$

$$= a_x \hat{\imath} + \frac{1}{\sqrt{34}} (5g - 3a_x) \frac{3}{\sqrt{34}} \hat{\imath} + \frac{1}{\sqrt{34}} (5g - 3a_x)\left(-\frac{5}{\sqrt{34}}\right) \hat{\jmath}$$

$$= \frac{1}{34} (25 a_x + 15 g) \hat{\imath} + \frac{1}{34} (15 a_x - 25 g) \hat{\jmath}$$

$$\therefore \boxed{\vec{a}_B = \frac{1}{34}(25 a_x + 15 g) \hat{\imath} + \frac{1}{34} (15 a_x - 25 g) \hat{\jmath}}$$

Tension in BC after cutting :

$$\boxed{T_{BC} = \frac{m}{\sqrt{34}} (3g + 5a_x)}$$

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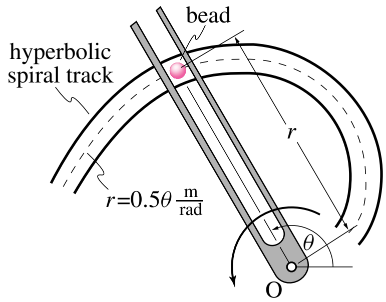
19.1.21 The forked arm mechanism pushes the bead of mass 1 kg along a frictionless hyperbolic spiral track given by $r = 0.5\theta(\,\mathrm{m}/\,\mathrm{rad})$. The arm rotates about its pivot point at O with constant angular acceleration $\ddot{\theta} = 1\,\mathrm{rad}/\mathrm{s}^2$ driven by a motor (not shown). The arm starts from rest at $\theta = 0^\circ$.

a) Determine the radial and transverse components of the acceleration of the bead after 2 s have elapsed from the start of its motion.

b) Determine the magnitude of the net force on the mass at the same instant in time.



Problem 19.21

6.44

$\hat{e}_\theta$ $\hat{e}_r$

θ

Spiral track: particle position given by

$r = 0.5\theta$ [m/rad]

$\ddot{\theta} = 1$ rad/s²

a) determine $\underset{\sim}{a}\cdot\hat{e}_r$ & $\underset{\sim}{a}\cdot\hat{e}_\theta$

continued

6.44 continued

p. 4

if $\ddot{\theta} = 1\,\mathrm{rad/s^2}$, then:

$\dot{\theta} = \int \ddot{\theta}\,dt = t$ rad/s

$\theta = \int \dot{\theta}\,dt = t^2/2$ rad.

Thus, since $R = 0.5\theta$ (m/rad)

$\dot{R} = 0.5\dot{\theta}$ (m/s·rad) and $\ddot{R} = 0.5\ddot{\theta}$ (m/s²·rad)

So, at $t = 2$ sec, we have

$\ddot{\theta} = 1\,\mathrm{rad/s^2}$, $\dot{\theta} = 2$ rad/s, $\theta = 2$ rad,

$\ddot{R} = 0.5\,\mathrm{m/s^2}$, $\dot{R} = 1$ m/s, $R = 1$ m

Then $\underset{\sim}{a} = (-R\dot{\theta}^2 + \ddot{R})\hat{e}_r + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$

$\underset{\sim}{a} = [(-4 + 0.5)\hat{e}_r + (4 + 1)\hat{e}_\theta]$ m/s²

$\boxed{\underset{\sim}{a}\cdot\hat{e}_r = -3.5\,\mathrm{m/s^2},\ \underset{\sim}{a}\cdot\hat{e}_\theta = 5\,\mathrm{m/s^2}}$

b) Net force: $\Sigma \underset{\sim}{F} = m\underset{\sim}{a}$

magnitude: $|\Sigma \underset{\sim}{F}| = m|\underset{\sim}{a}| = 1\mathrm{kg}\left((-3.5\,\mathrm{m/s^2})^2 + (5\,\mathrm{m/s^2})^2\right)^{1/2}$

$\Rightarrow \boxed{|\Sigma \underset{\sim}{F}| = 6.10\ \mathrm{N}}$

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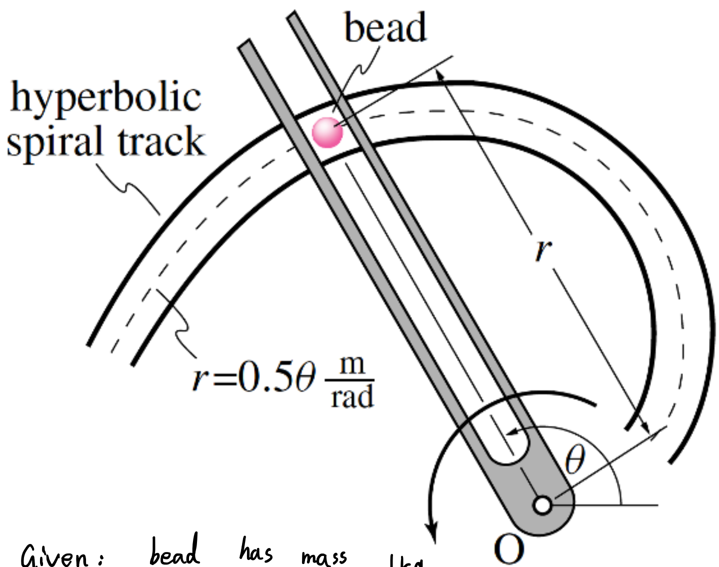
18.1.21

Saturday, May 4, 2019 7:50 PM



Given: bead has mass 1kg

hyperbolic spiral track is frictionless

$r = 0.5\theta$ [m/rad]

rotates about O with const angular acceleration $\ddot{\theta} = 1$[rad/s²]

Starts from rest at $\theta = 0°$.

a) Find $\vec{a} \cdot \hat{e}_r$ & $\vec{a} \cdot \hat{e}_\theta$ of the bead

$\dot{\theta} = \int_0^t \ddot{\theta}\, dt = t$ [rad/s²]

$\ddot{\theta} = \int_0^t \dot{\theta}\, dt = \frac{1}{2}t^2$ [rad/s²]

Since $R = 0.5\theta \frac{m}{rad}$

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Take derivative on both sides

$$\dot{R} = 0.5\,\dot{\theta}\,\frac{m}{rad}$$

$$\ddot{R} = 0.5\,\ddot{\theta}\,\frac{m}{rad}$$

At $t = 2s$, we have

$$\ddot{\theta} = 1 rad/s^2 \qquad \dot{\theta} = 2 rad/s \qquad \theta = 2 rad$$

$$\ddot{R} = 0.5\ m/s^2 \qquad \dot{R} = 1 m/s \qquad R = 1m$$

Thus $\vec{a} = (-R\dot{\theta}^2 + \ddot{R})\hat{e}_r + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$

$$= -3.5\ \hat{e}_r + 5\ \hat{e}_\theta \quad [m/s^2]$$

$\therefore$ $\vec{a}\cdot\hat{e}_r = -3.5\ [m/s^2]$

$\vec{a}\cdot\hat{e}_\theta = 5\ [m/s^2]$

b) Find the magnitude of the net force at $t=2s$

LMB $\Sigma\vec{F} = m\vec{a}$

magnitude $|\Sigma\vec{F}| = m|\vec{a}| = 1kg \cdot \sqrt{3.5^2 + 5^2}\ m/s^2 = 6.10 N$

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16.21 Given; $\ddot{\theta} = 1$ (constant); $\dot{\theta}(0) = \theta(0) = 0$

$r = 0.5\theta \Rightarrow r(t) = 0.5\theta(t)$

Find $a_r, a_t, |\vec{F}|$ at $t = 2s$

Lets solve for $r(t)$ and $\theta(t)$

$\ddot{\theta} = 1$

Integrate twice $\theta(t) = \frac{t^2}{2} + C_1 t + C_2$

But $\dot{\theta}(0) = \theta(0) = 0 \Rightarrow \boxed{\theta(t) = 0.5t^2}$

Also $r(t) = 0.5\theta(t) \Rightarrow \boxed{r(t) = 0.25t^2}$

(a) $\vec{a} = (\ddot{r} - \dot{\theta}^2 r)\hat{e}_r + (2\dot{r}\dot{\theta} + \ddot{\theta} r)\hat{e}_\theta$

$r(2) = 1$ $\quad \dot{r}(t) = 0.5t \Rightarrow \dot{r}(2) = 1$

$\ddot{r}(t) = 0.5 \Rightarrow \ddot{r}(2) = 0.5$

$\dot{\theta}(t) = t \Rightarrow \dot{\theta}(2) = 2$

$\ddot{\theta}(t) = 1 \Rightarrow \ddot{\theta}(2) = 1$

Thus $\vec{a} = (0.5 - 2^2 \cdot 1)\hat{e}_r + (2 \times 1 \times 2 + 1 \times 1)\hat{e}_\theta$

$\boxed{\vec{a} = -3.5\hat{e}_r + 5\hat{e}_\theta}$ (Answer)

(b) $\vec{F} = m\vec{a} = -3.5\hat{e}_r + 5\hat{e}_\theta$ $\{$As $m = 1$ kg$\}$

$|\vec{F}| = \sqrt{(-3.5)^2 + 5^2} = 6.1$

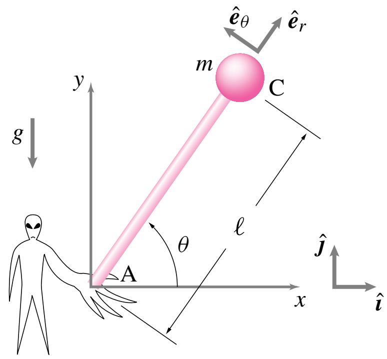
$\boxed{|\vec{F}| = 6.1 \text{ N}}$

19.1.22 A rod is on the palm of your hand at point A. Its length is ℓ. Its mass m is assumed to be concentrated at its end at C. Assume that you know θ and $\dot{\theta}$ at the instant of interest. Also assume that your hand is accelerating both vertically and horizontally with $\vec{a}_{\text{hand}} = a_{hx}\hat{\imath} + a_{hy}\hat{\jmath}$. Coordinates and directions are as marked in the figure.

a) Draw a Free-Body Diagram of the rod.

b) Assume that the hand is stationary. Solve for $\ddot{\theta}$ in terms of g, ℓ, m, θ, and $\dot{\theta}$.

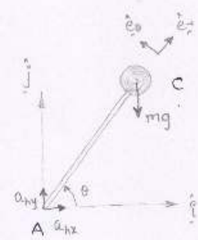
c) If the hand is not stationary but $\ddot{\theta}$ has been determined somehow, find the vertical force of the hand on the rod in terms of $\ddot{\theta}$, $\dot{\theta}$, θ, g, ℓ, m, a_{hx}, and a_{hy}.



Problem 19.22: Rod on a Moving Support(2-D).

18.1.22

a) Free body diagram:



b) Assume $\underline{a}_{hand} = \underline{0}$, $\ddot{\theta} = ?$

Applying AMB about A : $\sum \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

$\Rightarrow \; l\hat{e}_r \times mg(-\hat{j}) = l\hat{e}_r \times m\underline{a}$

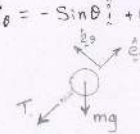
$\underline{a} = \underline{a}_{hand} + \ddot{\theta}\hat{k} \times l\hat{e}_r + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times l\hat{e}_r)$

$\Rightarrow \; \underline{a} = l\ddot{\theta}\hat{e}_\theta - l\omega^2\hat{e}_r$

$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\therefore \{-mgl\cos\theta\,\hat{k} = ml^2\ddot{\theta}\,\hat{k}\}$



$\{\ \}\cdot\hat{k} \;\Rightarrow\; \ddot{\theta} = -\frac{g}{l}\cos\theta$ //

c) Applying LMB on mass : $\sum \underline{F} = m\underline{a}$

$\Rightarrow \{-T\hat{e}_r - mg\hat{j} = m(a_{hx}\hat{i} + a_{hy}\hat{j} + \ddot{\theta}l\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r)\}\cdot\hat{e}_r$

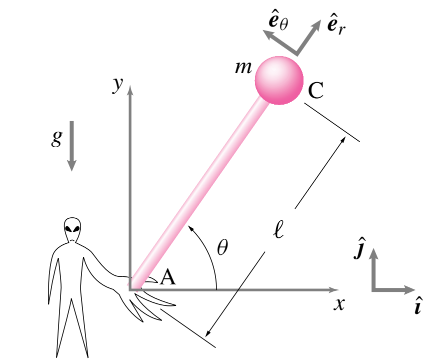
$\{\ \}\cdot\hat{e}_r \;\Rightarrow\; -T - mg\sin\theta = m(a_{hx}\cos\theta + a_{hy}\sin\theta - l\dot{\theta}^2)$

$\Rightarrow\; T = m[l\dot{\theta}^2 - a_{hx}\cos\theta - (a_{hy}+g)\sin\theta]$

$\therefore$ Force on rod. $\underline{F} = -T\hat{e}_r = m[a_{hx}\cos\theta + (a_{hy}+g)\sin\theta - l\dot{\theta}^2]\,\hat{e}_r$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

19.1.23 Balancing a broom. Assume the hand is accelerating to the right with acceleration $\boldsymbol{a} = a\hat{\boldsymbol{\imath}}$. What is the force of the hand on the broom in terms of m, ℓ, θ, $\dot{\theta}$, a, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and g? *(You may not have any $\hat{\boldsymbol{e}}_R$ or $\hat{\boldsymbol{e}}_\theta$ in your answer.)*



Problem 19.23

18.1.23

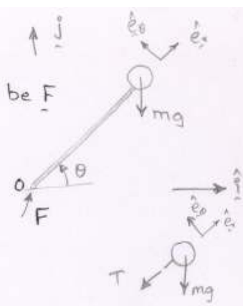
Let the force of the hand on the broom be $\underline{F}$

Applying L.M.B : $\Sigma \underline{F} = m\underline{a}$

$mg(-\hat{\jmath}) - T\hat{e}_r = m\left[\underline{a}_0 + \ddot{\theta}\ell\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_r\right]$

$\{\}\cdot\hat{e}_r \Rightarrow -T - mg\sin\theta = m\left[a\cos\theta - \ell\dot{\theta}^2\right]$

$\Rightarrow T = m\left[\ell\dot{\theta}^2 - g\sin\theta - a\cos\theta\right](\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$



$\underline{a}_0 = a\hat{\imath}$

$\Rightarrow$ Force on broom, $F = m\cos\theta\left[g\sin\theta + a\cos\theta - \ell\dot{\theta}^2\right]\hat{\imath} + m\sin\theta\left[g\sin\theta + a\cos\theta - \ell\dot{\theta}^2\right]\hat{\jmath}$

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