17.5 Rigid-object collision mechanics

This section extends the particle collisions discussion in Section 13.2 that starts on page 665.

2D collisions

For collisions between rigid bodies with more general motions before and after the collisions we depend on the three key ideas

I. Collision forces are big,

II. Collisions are quick, and

III. The laws of mechanics apply during the collision.

There are two extra assumptions that are needed in simple analysis:

IV. Collision forces are few. For a given rigid body there is one, or at most two non-negligible collision forces. This is the real import of idea (I) above. Because collision forces are big most other forces can be neglected.

V. The collision force(s) act at a well defined point which does not move during the collision.

Based on these assumptions one then uses linear and angular momentum balance in their time-integrated form.

Example: Two bodies in space
Two bodies collide at point C. The impulse acting on body 2 is $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}_{\text{coll}}\, dt$. If the mass and inertia properties of both bodies is known, as are the velocities and rotation rates before the collision we have the following linear and angular momentum balance equations for the two bodies:

$$
\begin{aligned}
-\vec{\boldsymbol{P}} &= m_1\left(\vec{\boldsymbol{v}}^+_{\text{G1}} - \vec{\boldsymbol{v}}^-_{\text{G1}}\right)\\
\vec{\boldsymbol{P}} &= m_2\left(\vec{\boldsymbol{v}}^+_{\text{G2}} - \vec{\boldsymbol{v}}^-_{\text{G2}}\right)\\
\vec{\boldsymbol{r}}_{\text{C/G1}} \times (-\vec{\boldsymbol{P}}) &= I^{\text{cm}\,1}_{zz}\left(\omega^+_1 - \omega^-_1\right)\hat{\boldsymbol{k}}\\
\vec{\boldsymbol{r}}_{\text{C/G2}} \times \vec{\boldsymbol{P}} &= I^{\text{cm}\,2}_{zz}\left(\omega^+_2 - \omega^-_2\right)\hat{\boldsymbol{k}}.
\end{aligned}
\qquad (17.57)
$$

These make up 6 scalar equations (2 for each momentum equation, 1 for each angular momentum equation). There are 8 scalar unknowns: $\vec{\boldsymbol{v}}^+_{\text{G1}}$ (2), $\vec{\boldsymbol{v}}^+_{\text{G2}}$ (2), ω^+_1 (1), ω^+_2 (1), and $\vec{\boldsymbol{P}}$ (2). Thus the motion after the collision cannot be determined.

[Note that linear and angular momentum balance for the system would give equations which could be obtained by adding and subtracting combinations of the equations above. So adding system momentum balance equations does not add information (*i.e.,* adds linearly dependent equations).]

So, as for 1-D collisions, momentum balance is not enough to determine the outcome of the collision. Eqns. 17.57 aren't enough. A thousand different models and assumptions could be added to make the system solvable. But there are only two cases that are non-controversial and also relatively simple: 1) sticking collisions, and 2) frictionless collisions.

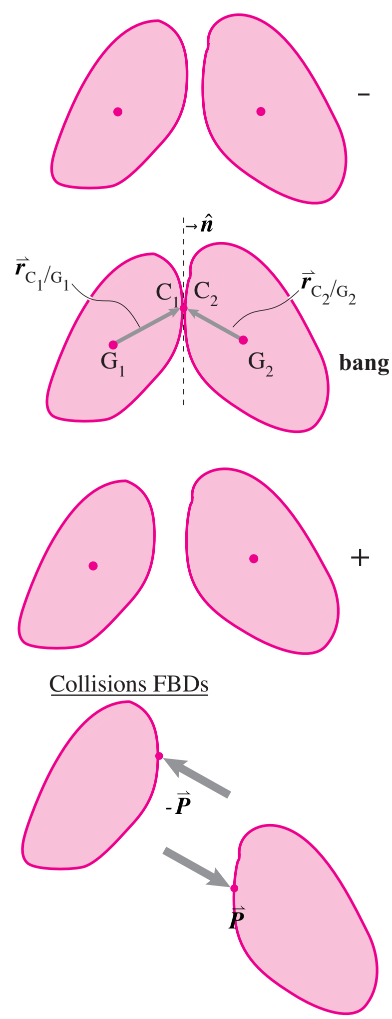


Figure 17.75: Two bodies collide at point C. The only non-negligible collision impulse is $\vec{\boldsymbol{P}}$ acting on body 2 (and $-\vec{\boldsymbol{P}}$ on body 1) at point C. The material points on the contacting bodies are C_1 and C_2. The outward normal to body 1 at C_1 is $\hat{\boldsymbol{n}}$.

Sticking collisions

A ‘perfectly-plastic’ sticking collision is one where the relative velocities of the two contacting points are assumed to go suddenly to zero. That is

$$\vec{\boldsymbol{v}}^{+}_{C1} = \vec{\boldsymbol{v}}^{+}_{C2}$$

Writing $\vec{\boldsymbol{v}}^{+}_{\mathrm{C1}} = \vec{\boldsymbol{v}}^{+}_{\mathrm{G1}} + (\omega^{+}_{1}\hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{C/G1}}$ and similarly for $\vec{\boldsymbol{v}}_{\mathrm{C2}}$ thus adds a vector equation (2 scalar equations) to the equation set 17.57. This gives 8 equations in 8 unknowns.

A little cleverness can reduce the problem to one of solving only 4 equations in 4 unknowns. Linear momentum balance for the system, angular momentum balance for the system and angular momentum balance for object 2 make up 4 scalar equations. None of these equations includes the impulse $\vec{\boldsymbol{P}}$. Because the system moves as if hinged at C_1 after the collision, the state of motion after the system is fully characterized by $\vec{\boldsymbol{v}}^{+}_{\mathrm{G1}}$, ω^{+}_{1}, and ω^{+}_{2}. Thus we have 4 equations in 4 unknowns.

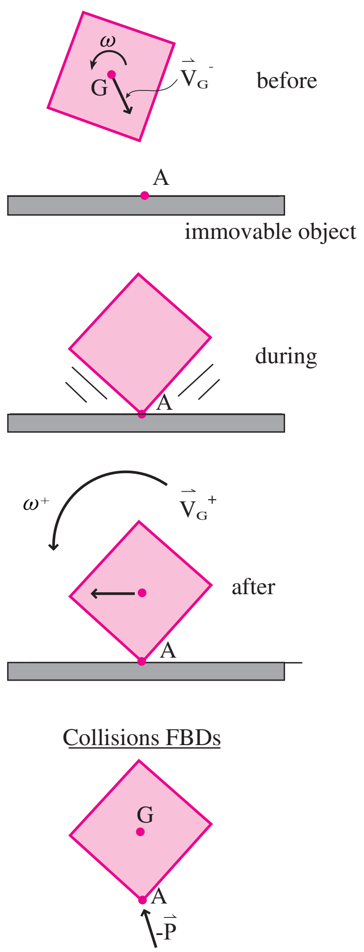


Figure 17.76: Sticking collision with an immovable object. The box sticks at A and then rotates about A. Angular momentum about point A is conserved in the collision.

Example: One body is hugely massive: collision with an immovable object

If body 2, say, is huge compared to body 1 then it can be taken to be immovable and collision problems can be solved by only considering body 1 (see fig. 17.76). In the case of a sticking collision the full state of the system after the collision is determined by ω^{+}_{1}. This can be found from the single scalar equation obtained from angular momentum balance about the collision point.

$$\begin{aligned} \vec{\boldsymbol{H}}^{-}_{\mathrm{A}} &= \vec{\boldsymbol{H}}^{+}_{\mathrm{A}} \\ \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}^{-}_{\mathrm{G}} + I^{\mathrm{cm}}_{zz}\omega^{-}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}^{+}_{\mathrm{G}} + I^{\mathrm{cm}}_{zz}\omega^{+}\hat{\boldsymbol{k}} \end{aligned}$$

Because the state of the system before the collision is assumed known (the left “-” side of the equation, and because the post-collision (+) state is a rotation about A, this equation is one scalar equation in the one unknown ω^{+}. Note that $\vec{\boldsymbol{H}}^{+}_{\mathrm{A}}$ could also be evaluated as $\vec{\boldsymbol{H}}^{+}_{\mathrm{A}} = \omega^{+} I^{A}_{zz}\hat{\boldsymbol{k}}$. So one way of expressing the post-collision state is as

$$\omega^{+} = \frac{\left(\vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}^{-}_{\mathrm{G}} + I^{\mathrm{cm}}_{zz}\omega^{-}\hat{\boldsymbol{k}}\right)\cdot\hat{\boldsymbol{k}}}{I^{A}_{zz}} \quad \text{and} \quad \vec{\boldsymbol{v}}^{+}_{\mathrm{G}} = \omega^{+}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/A}}.$$

Note also that the same $\vec{\boldsymbol{r}}_{\mathrm{G/A}}$ is used on the right and left sides of the equation because only the velocity and not the position is assumed to jump during the collision.

The collision impulse $\vec{\boldsymbol{P}}$ can then be found from linear momentum balance as

$$\vec{\boldsymbol{P}} = m\left(\vec{\boldsymbol{v}}^{+}_{\mathrm{G}} - \vec{\boldsymbol{v}}^{-}_{\mathrm{G}}\right).$$

Sticking collisions are used as models of projectiles hitting targets, of robot and animal limbs making contact with the ground, of monkeys and acrobats grabbing hand holds, and of some particularly dead and frictional collisions between solids (such as when a car trips on a curb).

Frictionless collisions

The second special case is that of a frictionless collision. Here we add two assumptions:

1. There is no friction so $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$. The number of unknowns is thus reduced from 8 to 7.
2. There is a coefficient of (normal) restitution e.

The normal restitution coefficient is taken as a property of the colliding bodies. It is a given number with $0 < e < 1$ with this defining equation:

$$(\vec{\boldsymbol{v}}^{+}_{C2} - \vec{\boldsymbol{v}}^{+}_{C1}) \cdot \hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}^{-}_{C2} - \vec{\boldsymbol{v}}^{-}_{C1}) \cdot \hat{\boldsymbol{n}}.$$

This says that the normal part of the relative velocity of the contacting points reverses sign and its magnitude is attenuated by e. This adds a scalar equation to the set Eqns. 17.57 thus giving 7 scalar equations (4 momentum, 2 angular momentum, 1 restitution) for 7 unknowns (4 velocity components, 2 angular velocities and the normal impulse).

The most popular application of the frictionless collision model is for billiard or pool balls, or carom pucks. These things have relatively small coefficients of friction.

We state without proof that a frictionless collision with $e = 1$ conserves energy.

Example: Pool balls
Assume one ball approaches the other with initial velocity $\vec{\boldsymbol{v}}^{+}_{G1} = v\hat{\boldsymbol{\imath}}$ and has an elastic frictionless collision with the other ball at a collision angle of θ as shown in fig. 17.77. Defining $\hat{\boldsymbol{n}} \equiv \cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$ we have that $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$. To determine the outcome of the equation we have the angular momentum balance equations (about the center-of-mass) which trivially tell us that

$$\omega^{+}_{1} = \omega^{+}_{2} = 0$$

because the balls start with no spin and the frictionless collision impulses $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ and $-\vec{\boldsymbol{P}} = -P\hat{\boldsymbol{n}}$ have no moment about the center-of-mass. Linear momentum balance for each of the balls

$$\begin{aligned} -P\hat{\boldsymbol{n}} &= m\vec{\boldsymbol{v}}^{+}_{G1} - mv\hat{\boldsymbol{\imath}} \\ P\hat{\boldsymbol{n}} &= m\vec{\boldsymbol{v}}^{+}_{G2} - \vec{\boldsymbol{0}} \end{aligned}$$

gives 4 scalar equations which are supplemented by the restitution equation (using $e = 1$)

$$(\Delta\vec{\boldsymbol{v}}^{+}) \cdot \hat{\boldsymbol{n}} = -e\,(\Delta\vec{\boldsymbol{v}}^{-}) \cdot \hat{\boldsymbol{n}}$$
$$\Rightarrow -v\cos\theta = \vec{\boldsymbol{v}}^{+}_{G2} \cdot \hat{\boldsymbol{n}} - \vec{\boldsymbol{v}}^{+}_{G1} \cdot \hat{\boldsymbol{n}}$$

which together make 5 scalar equations in the 5 scalar unknowns $\vec{\boldsymbol{v}}^{+}_{G1}$, $\vec{\boldsymbol{v}}^{+}_{G2}$, and P (each vector has 2 unknown components). These have the solution

$$\begin{aligned} \vec{\boldsymbol{v}}^{+}_{G1} &= v\sin\theta(\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}), \\ \vec{\boldsymbol{v}}^{+}_{G2} &= v\cos\theta(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}), \quad \text{and} \\ P &= mv\cos\theta. \end{aligned}$$

The solution can be checked by plugging back into the momentum and restitution equations. Also, as promised, this $e = 1$ solution conserves kinetic energy. The solution has the interesting property that the outgoing trajectories of the

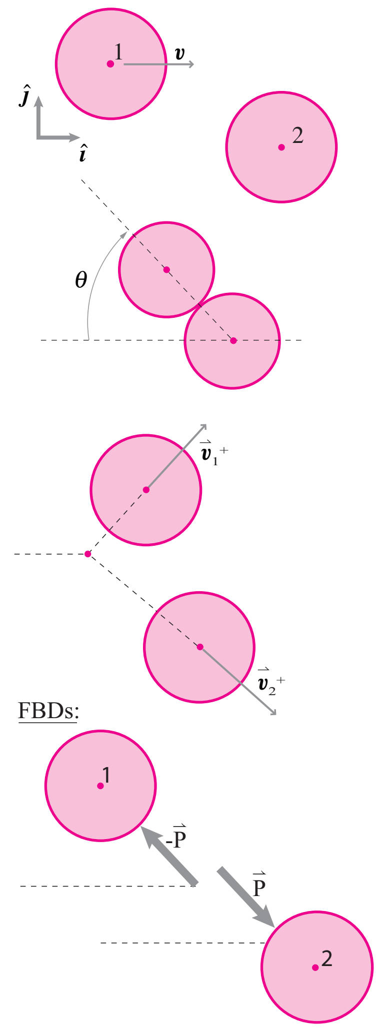


Figure 17.77: Frictionless collision between two identical round objects. Ball one is initially moving to the right, ball 2 is initially stationary. The impulse of ball 1 *on* ball 2 is $\vec{\boldsymbol{P}}$.

two balls are orthogonal for all θ but $\theta = 0$ in which case ball 1 comes to rest in the collision. [The solution can be found graphically by looking for two outgoing vectors which add to the original velocity of mass 1, where the sum of the squares of the outgoing speeds must add to the square of the incoming speed.]

Frictional collisions

For a collision with friction, but not so much that total sticking is accurate, the modeling is complex and subtle. As of this writing there are no standard acceptable ways of dealing with such situations. Commercial simulation packages should be used for such with skeptical caution. They are generally defective in that they can predict only a limited range of phenomena and/or they can create energy even with innocent input parameters.

Why is it hard to find a good collision law

Ideally one would like a rule to determine how bodies move after a collision from how they move before the collision. Such a rule would be called a collision law or a constitutive relation for collisions. That accurate collision laws are rare at best might be surmised from the basic problem that the phrase *rigid body collisions* is in some sense a contradiction in terms, an oxymoron. The force generated in the contact comes from material deformation, and deformation is just what we generally try to neglect when doing rigid body mechanics.

There is a temptation to say that one wants to continue to neglect deformation during the collision, but for an infinitesimal contact region. And some collision laws are formulated with this approach. Even then, there are no reliable models for the deformation in that small region, and such laws are doomed to inaccuracy in situations where the deformation is not so limited.

For complex shaped bodies touching at various points that are generally not known *a priori*, no collision law is reliably accurate.

SAMPLE 17.27 For two masses conservation of momentum is expressed by the vector equation $m_1\vec{\boldsymbol{v}}_1 + \mathrm{m}_2\vec{\boldsymbol{v}}_2 = m_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+$. Suppose $\vec{\boldsymbol{v}}_1 = \vec{\mathbf{0}}$, $\vec{\boldsymbol{v}}_2 = -v_0\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}}_1^+ = v_1^+\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{v}}_2^+ = v_{2_t}^+\hat{\boldsymbol{e}}_t + v_{2_n}^+\hat{\boldsymbol{e}}_n$, where $\hat{\boldsymbol{e}}_t = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_n = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

1. Obtain two independent scalar equations from the momentum equation corresponding to projections in the $\hat{\boldsymbol{e}}_n$ and $\hat{\boldsymbol{e}}_t$ directions.
2. Assume that you are given another equation $v^+{}_{2_t} = -v_0\sin\theta$. Set up a matrix equation to solve for v_1^+, $v_{2_t}^+$, and $v_{2_n}^+$ from the three equations.

Solution

1. The given equation of conservation of linear momentum is

$$\begin{aligned} m_1 \underbrace{\vec{\boldsymbol{v}}_1}_{\mathbf{0}} + \mathrm{m}_2\vec{\boldsymbol{v}}_2 &= m_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+ \\ \text{or} \qquad -m_2v_0\hat{\boldsymbol{\jmath}} &= m_1v_1^+\hat{\boldsymbol{\imath}} + m_2(v_{2_t}^+\hat{\boldsymbol{e}}_t + v_{2_n}^+\hat{\boldsymbol{e}}_n). \end{aligned} \tag{17.58}$$

Dotting both sides of eqn. (17.58) with $\hat{\boldsymbol{e}}_n$ gives

$$-m_2v_0\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\jmath}})}^{\cos\theta} = m_1v_1^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\imath}})}^{-\sin\theta} + m_2v_{2_t}^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_t)}^{0} + m_2v_{2_n}^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_n)}^{1}$$

$$\text{or} \qquad -m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+. \tag{17.59}$$

Dotting both sides of eqn. (17.58) with $\hat{\boldsymbol{e}}_t$ gives

$$-m_2v_0\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\jmath}})}^{\sin\theta} = m_1v_1^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\imath}})}^{\cos\theta} + m_2v_{2_t}^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_t)}^{1} + m_2v_{2_n}^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_n)}^{0}$$

$$\text{or} \qquad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+. \tag{17.60}$$

$$-m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+, \quad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+$$

2. Now, we rearrange eqn. (17.59) and 17.60 along with the third given equation, $v^+{}_{2_t} = -v_0\sin\theta$, so that all unknowns are on the left hand side and the known quantities are on the right hand side of the equal sign. These equations, in matrix form, are as follows.

$$\begin{bmatrix} -m_1\sin\theta & 0 & m_2 \\ -m_1\cos\theta & m_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} v_1^+ \\ v_{2_t}^+ \\ v_{2_n}^+ \end{Bmatrix} = \begin{Bmatrix} -m_2v_0\cos\theta \\ -m_2v_0\sin\theta \\ -v_0\sin\theta \end{Bmatrix}.$$

This equation can be easily solved on a computer for the unknowns.

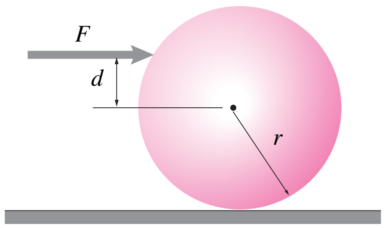


Figure 17.78

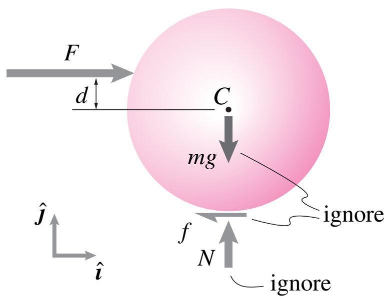


Figure 17.79: FBD of the ball during the strike. The nonimpulsive forces mg, N, and f can be ignored in comparison to the strike force F.

SAMPLE 17.28 Cueing a billiard ball. A billiard ball is cued by striking it horizontally at a distance $d = 10\,\mathrm{mm}$ above the center of the ball. The ball has mass $m = 0.2\,\mathrm{kg}$ and radius $r = 30\,\mathrm{mm}$. Immediately after the strike, the center-of-mass of the ball moves with linear speed $v = 1\,\mathrm{m/s}$. Find the angular speed of the ball immediately after the strike. Ignore friction between the ball and the table during the strike.

Solution Let the force imparted during the strike be F. Since the ball is cued by giving a blow with the cue, F is an impulsive force. Impulsive forces, such as F, are in general so large that all non-impulsive forces are negligible in comparison during the time such forces act. Therefore, we can ignore all other forces (mg, N, f) acting on the ball from its free-body diagram during the strike.

Now, from the linear momentum balance of the ball we get

$$F\hat{\boldsymbol{\imath}} = \dot{\vec{\boldsymbol{L}}} \quad \text{or} \quad (F\hat{\boldsymbol{\imath}})\mathrm{d}t = \mathrm{d}\vec{\boldsymbol{L}} \quad \Rightarrow \quad \int (F\hat{\boldsymbol{\imath}})\mathrm{d}t = \vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1$$

where $L_2 - L_1 = \Delta\vec{\boldsymbol{L}}$ is the net change in the linear momentum of the ball during the strike. Since the ball is at rest before the strike, $\vec{\boldsymbol{L}}_1 = m \underbrace{\vec{\boldsymbol{v}}_1}_{0} = \vec{\boldsymbol{0}}$. Immediately after the strike, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}} = 1\,\mathrm{m/s}$.

$$\text{Thus} \quad \vec{\boldsymbol{L}}_2 = m\vec{\boldsymbol{v}} = 0.2\,\mathrm{kg}\cdot 1\,\mathrm{m/s}\hat{\boldsymbol{\imath}} = 0.2\,\mathrm{N\cdot s}\hat{\boldsymbol{\imath}}.$$

$$\text{Hence} \quad \int (F\hat{\boldsymbol{\imath}})\mathrm{d}t = 0.2\,\mathrm{N\cdot s}\hat{\boldsymbol{\imath}} \quad \text{or} \quad \int F\mathrm{d}t = 0.2\,\mathrm{N\cdot s}. \tag{17.61}$$

To find the angular speed we apply the angular momentum balance. Let ω be the angular speed immediately after the strike and $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. Now,

$$\sum \vec{\boldsymbol{M}}_{\mathrm{cm}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}} \quad \Rightarrow \quad \int \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\,\mathrm{d}t = \int \mathrm{d}\vec{\boldsymbol{H}}_{\mathrm{cm}} = (\vec{\boldsymbol{H}}_{\mathrm{cm}})_2 - (\vec{\boldsymbol{H}}_{\mathrm{cm}})_1.$$

$$\begin{aligned}
\text{Because} \quad \vec{\boldsymbol{H}}_{\mathrm{cm}} &= I_{cm}^{zz}\vec{\boldsymbol{\omega}} \quad \text{and just before the strike, } \vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}, \\
(\vec{\boldsymbol{H}}_{\mathrm{cm}})_1 &\equiv \text{angular momentum just before the strike } = \vec{\boldsymbol{0}} \\
(\vec{\boldsymbol{H}}_{\mathrm{cm}})_2 &\equiv \text{angular momentum just after the strike } = I_{cm}^{zz}\omega\hat{\boldsymbol{k}}, \\
\int \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\,\mathrm{d}t &= I_{cm}^{zz}\omega\hat{\boldsymbol{k}} = \frac{2}{5}mr^2\omega\hat{\boldsymbol{k}} \text{ (since for a sphere, } I_{cm}^{zz} = \tfrac{2}{5}mr^2).
\end{aligned}$$

$$\begin{aligned}
\text{But} \quad & \sum \vec{\boldsymbol{M}}_{\mathrm{cm}} = -Fd\hat{\boldsymbol{k}}, \\
\text{therefore} \quad & -\int (Fd)\mathrm{d}t\hat{\boldsymbol{k}} = \frac{2}{5}mr^2\omega\hat{\boldsymbol{k}} \\
\text{or} \quad & -\underbrace{d}_{\text{constant}} \int F\mathrm{d}t = \frac{2}{5}mr^2\omega \quad \Rightarrow \quad \omega = -\frac{5d}{2mr^2}\int F\mathrm{d}t.
\end{aligned}$$

Substituting the given values and $\int F\mathrm{d}t = 0.2\,\mathrm{N\cdot s}$ from equation 17.61 we get

$$\omega = -\frac{5(0.01\,\mathrm{m})}{2\cdot 0.2\,\mathrm{kg}\cdot(0.03\,\mathrm{m})^2}\cdot 0.2\,\mathrm{N\cdot s} = -27.78\,\mathrm{rad/s}.$$

The negative value makes sense because the ball will spin clockwise after the strike, but we assumed that ω was anticlockwise.

$$\boxed{\omega = -27.78\,\mathrm{rad/s}.}$$

SAMPLE 17.29 **Falling stick.** A uniform bar of length ℓ and mass m falls on the ground at an angle θ as shown in the figure. Just before impact at point C, the entire bar has the same velocity v directed vertically downwards. Assume that the collision at C is plastic, *i.e.*, end C of the bar gets stuck to the ground upon impact.

1. Find the angular velocity of the bar just after impact.
2. Assuming θ to be small, find the velocity of end B of the bar just after impact.

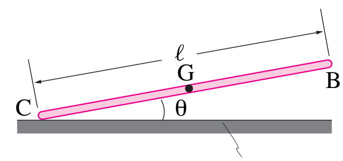


Figure 17.80

Solution We are given that the impact at point C is plastic. That is, end C of the bar has zero velocity after impact. Thus end C gets stuck to the ground. Then we expect the rod to rotate about point C as the rest of the bar moves (perhaps faster) to touch the ground. The free-body diagram of the bar is shown in fig. 17.81 during the impact at point C. Note that we can ignore the force of gravity in comparison to the large impulsive force F_c due to impact at C.

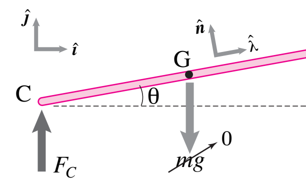


Figure 17.81: The free-body diagram of the bar during collision. The impulsive force at the point of impact C is so large that the force of gravity can be completely ignored in comparison.

1. Now, if we carry out angular momentum balance about point C, there will be no net moment acting on the bar, and therefore, angular momentum about the impact point C is conserved. Distinguishing the kinematic quantities before and after impact with superscripts '-' and '+', respectively, we get from the conservation of angular momentum about point C,

$$\begin{aligned}\vec{\boldsymbol{H}}_C^- &= \vec{\boldsymbol{H}}_C^+\\ I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^- + \vec{\boldsymbol{r}}_{\text{G/C}}\times m\vec{\boldsymbol{v}}_G^- &= I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{\boldsymbol{r}}_{\text{G/C}}\times m\vec{\boldsymbol{v}}_G^+.\end{aligned}$$

Now, we know that $\vec{\boldsymbol{\omega}}^- = \vec{\mathbf{0}}$ since every point on the bar has the same vertical velocity $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{j}}$, and that just after impact, $\vec{\boldsymbol{v}}_G^+ = \vec{\boldsymbol{\omega}}^+\times\vec{\boldsymbol{r}}_{\text{G/C}}$ where we can take $\vec{\boldsymbol{\omega}}^+ = \omega(-\hat{\boldsymbol{k}})$. Thus, ①

$$\begin{aligned}\vec{\boldsymbol{H}}_C^- &= \vec{\boldsymbol{r}}_{\text{G/C}}\times m\vec{\boldsymbol{v}}_G^- = (\ell/2)\hat{\boldsymbol{\lambda}}\times mv(-\hat{\boldsymbol{j}})\\ &= -\frac{mv\ell}{2}\cos\theta\hat{\boldsymbol{k}} \qquad (\text{since } \hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})\\ \vec{\boldsymbol{H}}_C^+ &= I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{\boldsymbol{r}}_{\text{G/C}}\times m(\vec{\boldsymbol{\omega}}^+\times\vec{\boldsymbol{r}}_{\text{G/C}})\\ &= -I_{zz}^{\text{cm}}\omega\hat{\boldsymbol{k}} + (\ell/2)\hat{\boldsymbol{\lambda}}\times m\underbrace{(-\omega\hat{\boldsymbol{k}}\times\ell/2\hat{\boldsymbol{\lambda}})}_{\omega\ell/2(-\hat{\boldsymbol{n}})}\\ &= -\frac{1}{12}m\ell^2\omega\hat{\boldsymbol{k}} - \frac{1}{4}m\ell^2\omega\hat{\boldsymbol{k}} = -\frac{1}{3}m\ell^2\omega\hat{\boldsymbol{k}}.\end{aligned}$$

Now, equating $\vec{\boldsymbol{H}}_C^-$ and $\vec{\boldsymbol{H}}_C^+$ we get

$$\omega = \frac{3v}{2\ell}\cos\theta, \qquad \Rightarrow \qquad \vec{\boldsymbol{\omega}} = -\frac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}.$$

$$\boxed{\vec{\boldsymbol{\omega}} = -\tfrac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}}$$

2. The velocity of the end B is now easily found using $\vec{\boldsymbol{v}}_{\text{B}} = \vec{\boldsymbol{v}}_{\text{C}} + \vec{\boldsymbol{v}}_{\text{B/C}} = \vec{\boldsymbol{v}}_{\text{B/C}}$ and $\vec{\boldsymbol{v}}_{\text{B/C}} = \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\text{B/C}}$. Thus,

$$\begin{aligned}\vec{\boldsymbol{v}}_{\text{B/C}} &= \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{\text{B/C}} = -\omega\hat{\boldsymbol{k}}\times\ell\hat{\boldsymbol{\lambda}}\\ &= -\omega\ell\hat{\boldsymbol{n}} = -\frac{3v}{2}\cos\theta(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})\end{aligned}$$

but, for small θ, $\cos\theta \approx 1$, and $\sin\theta \approx 0$. Therefore, $\vec{\boldsymbol{v}}_{\text{B/C}} = -\frac{3v}{2}\hat{\boldsymbol{j}}$. Thus, end B of the bar speeds up by one and a half times its original speed due to the plastic impact at C.

$$\boxed{\vec{\boldsymbol{v}}_{\text{B/C}} = -(3/2)v\hat{\boldsymbol{\jmath}}}$$

①Since C is a fixed point for the motion of the bar after impact, we could calculate $\vec{\boldsymbol{H}}_C^+$ as follows.

$$\vec{\boldsymbol{H}}_C^+ = I_{zz}^C\vec{\boldsymbol{\omega}} = \underbrace{\frac{1}{3}m\ell^2}_{I_{zz}^C}\omega(-\hat{\boldsymbol{k}}).$$

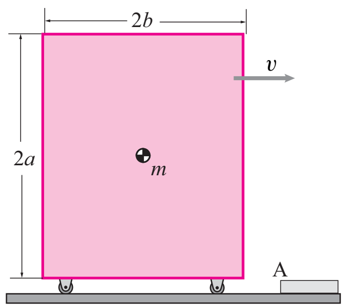


Figure 17.82

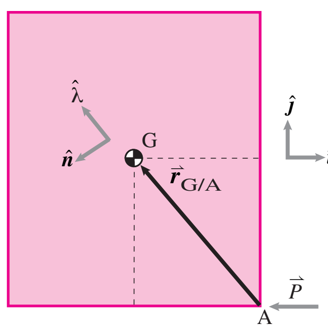


Figure 17.83: The free-body diagram of the box during collision.

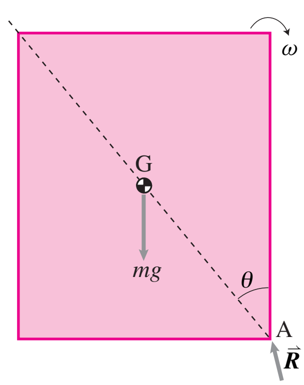


Figure 17.84: The free-body diagram of the box just after the collision is over.

SAMPLE 17.30 **Tipping box.** A box of mass m and dimensions $2a$ and $2b$ moves along a horizontal surface with uniform speed v. Suddenly, it bumps into an obstacle at A. Assume that the impact is plastic and point A is at the lowest level of the box. What is the maximum v the box can have so that it does not tip over after the impact.

Solution Whether the box can tip or not depends on whether it gets sufficient initial angular speed just after collision to overcome the restoring moment due to gravity about the point of rotation A. So, first we need to find the angular velocity of the box immediately following the collision. The free-body diagram of the box during collision is shown in fig. 17.83. There is an impulse $\vec{\boldsymbol{P}}$ acting at the point of impact. If we carry out the angular momentum balance about point A, we see that the impulse at A produces no moment impulse about A, and therefore, the angular momentum about point A has to be conserved. That is, $\vec{\boldsymbol{H}}_{\mathrm{A}}{}^{+} = \vec{\boldsymbol{H}}_{\mathrm{A}}{}^{-}$. Now,

$$\vec{\boldsymbol{H}}_{\mathrm{A}}{}^{-} = \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}_{\mathrm{G}}{}^{-} = (-b\hat{\boldsymbol{\imath}} + a\,\hat{\boldsymbol{\jmath}}) \times mv\hat{\boldsymbol{\imath}} = -mav\hat{\boldsymbol{k}}$$

Let the box have angular velocity $\vec{\boldsymbol{\omega}}^{+} = \omega\hat{\boldsymbol{k}}$ just after impact. Then,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\mathrm{A}}{}^{+} &= I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}^{+} + \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}_{\mathrm{G}}{}^{+} = I_{zz}^{\mathrm{cm}}\omega\hat{\boldsymbol{k}} + r\hat{\boldsymbol{\lambda}} \times m(\omega\hat{\boldsymbol{k}} \times r\hat{\boldsymbol{\lambda}}) \\
&= I_{zz}^{\mathrm{cm}}\omega\hat{\boldsymbol{k}} + mr^2\omega\hat{\boldsymbol{k}} = \frac{1}{12}(4a^2 + 4b^2)m\omega\hat{\boldsymbol{k}} + m(a^2 + b^2)\omega\hat{\boldsymbol{k}} \\
&= \frac{4}{3}(a^2 + b^2)m\omega\hat{\boldsymbol{k}}.
\end{aligned}$$

Now equating the two momenta, we get

$$\omega^{+} = -\frac{3a}{4(a^2 + b^2)}\,v \quad \Rightarrow \quad \vec{\boldsymbol{\omega}}^{+} = -\frac{3a}{4(a^2 + b^2)}\,v\hat{\boldsymbol{k}}. \tag{17.62}$$

Thus we know the angular velocity immediately after impact. Now let us find out if it is enough to get over the hill, so to speak. Once the impact is over (in a few milliseconds), the usual forces show up on the free-body diagram (see fig. 17.84).

In this post-collision part of the motion energy is conserved. Calling just after collision '+' and the top position 't' (where G is directly above A), using that $I_{zz}^{A} = \frac{4}{3}m(a^2+b^2)$ and solving for the critical condition, that the block has just enough energy to get to the top position, we have,

$$\begin{aligned}
E_{\mathrm{P}}{}^{+} + E_{\mathrm{K}}{}^{+} &= E_{\mathrm{K}}{}^{t} + E_{\mathrm{P}}{}^{t} \\
E_{\mathrm{P}}{}^{t} - E_{\mathrm{P}}{}^{+} &= -(E_{\mathrm{K}}{}^{t} - E_{\mathrm{K}}{}^{+}) \\
mg(\sqrt{(b^2 + a^2)} - b) &= -\left(0 - \frac{1}{2}(\omega^{+})^2 I_{zz}^{A}\right) \\
(\omega^{+})^2 &= \frac{3}{2}g\frac{\left(1 - \frac{b}{\sqrt{(b^2+a^2)}}\right)}{\sqrt{(b^2 + a^2)}}.
\end{aligned}$$

This tells us how big ω^{+} has to be. Substituting this ω^{+} into eqn. (17.62) we find how big v has to be. We find that the box does not tip over so long as v is small enough, as given by

$$\boxed{v \le \frac{2}{a}\sqrt{2g(b^2 + a^2)^{3/2}\left(1 - \frac{b}{\sqrt{(b^2+a^2)}}\right)/3}.}$$

SAMPLE 17.31 Ball hits the bat. A uniform bar of mass $m_2 = 1\,\text{kg}$ and length $2\ell = 1\,\text{m}$ hangs vertically from a hinge at A. A ball of mass $m_1 = 0.25\,\text{kg}$ comes and hits the bar horizontally at point D with speed $v = 5\,\text{m/s}$. The point of impact D is located at $d = 0.75\,\text{m}$ from the hinge point A. Assume that the collision between the ball and the bar is plastic.

1. Find the velocity of point D on the bar immediately after impact.
2. Find the impulse on the bar at D due to the impact.
3. Find and plot the impulsive reaction at the hinge point A as a function of d, the distance of the point of impact from the hinge point. What is the value of d which makes the impulse at A zero?

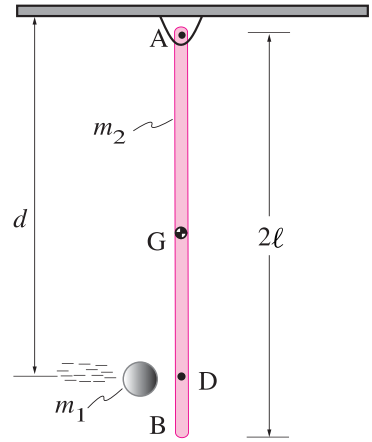


Figure 17.85

Solution The free-body diagram of the ball and the bar as a single system is shown in fig. 17.86 during impact. There is only one external impulsive force $\vec{F}_A$ acting at the hinge point A. We take the ball and the bar together here so that the impulsive force acting between the ball and the bar becomes internal to the system and we are left with only one external force at A. Then, the angular momentum balance about point A yields $\dot{\vec{H}}_A = \vec{0}$ since there is no net moment about A. Thus the angular momentum about A is conserved during the impact.

1. Let us distinguish the kinematic quantities just before impact and immediately after impact with superscripts '-' and '+', respectively. Then, from the conservation of angular momentum about point A, we get $\vec{H}_A^- = \vec{H}_A^+$. Now,

$$\begin{aligned}\vec{H}_A^- &= (\vec{H}_A^-)_{\text{ball}} + (\vec{H}_A^-)_{\text{bar}}\\ &= \vec{r}_{D/A} \times m_1\vec{v}^- + I_{zz}^A\vec{\omega}^-\\ &= d\hat{\jmath} \times m_1 v(-\hat{\imath}) + \vec{0} = m_1 dv\hat{k}.\end{aligned}$$

Similarly,

$$\vec{H}_A^+ = \vec{r}_{D/A} \times m_1\vec{v}^+ + I_{zz}^A\vec{\omega}^+$$

but, $\vec{v}^+ = \vec{\omega}^+ \times \vec{r}_{D/A} = -\omega d\hat{\imath}$, where $\vec{\omega}^+ = \omega\hat{k}$ (let). Hence,

$$\begin{aligned}\vec{H}_A^+ &= d\hat{\jmath} \times m_1(-\omega d\hat{\imath}) + \frac{1}{3}m_2(2\ell)^2\omega\hat{k}\\ &= (m_1 d^2 + \frac{4}{3}m_2\ell^2)\omega\hat{k}.\end{aligned}$$

Equating the two momenta, we get

$$\begin{aligned}\omega &= \frac{m_1 dv}{m_1 d^2 + (4/3)m_2\ell^2}\\ &= \frac{v}{d\left(1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2\right)}\\ \Rightarrow \quad \vec{v}_D &= \vec{\omega}^+ \times \vec{r}_{D/A} = \omega d(-\hat{\imath})\\ &= -\frac{v}{1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\imath}.\end{aligned}$$

Now, substituting the given numerical values, $v = 5\,\text{m/s}$, $m_1 = 0.25\,\text{kg}$, $m_2 = 1\,\text{kg}$, $\ell = 0.5\,\text{m}$, and $d = 0.75\,\text{m}$, we get $\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}$

$$\boxed{\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}}$$

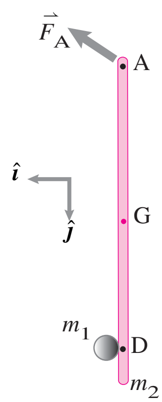


Figure 17.86: The free-body diagram of the ball and the bar together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram.

2. To find the impulse at D due to the impact, we can consider either the ball or the bar separately, and find the impulse by evaluating the change in the linear momentum of the body. Let us consider the ball since it has only one impulse acting on it. The

free-body diagram of the ball during impact is shown in fig. 17.87. From the linear impulse-momentum relationship we get,

$$
\begin{aligned}
\vec{P}_D = \int \vec{F}_D\,dt &= \vec{L}^+ - \vec{L}^- = m_1(\vec{v}^+ - \vec{v}^-)\\
&= m_1\left(-\frac{v}{1+\frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\imath} + v\hat{\imath}\right)\\
&= m_1 v\left(1 - \frac{1}{1+\frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\right)\hat{\imath}.
\end{aligned}
$$

Substituting the given numerical values, we get $\vec{P}_D = 0.73\,\text{kg·m/s}\hat{\imath}$. The impulse on the bar is equal and opposite. Therefore, the impulse on the bar is $-\vec{P}_D = -0.73\,\text{kg·m/s}\hat{\imath}$.

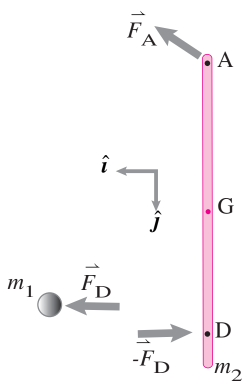


Figure 17.87: Separate free-body diagrams of the ball and the bar during collision.

Impulse at D $= -0.73\,\text{kg·m/s}\hat{\imath}$

3. Now that we know the impulse at D, we can easily find the impulse at A by applying impulse-momentum relationship to the bar. Since, the bar is stationary just before impact, its initial momentum is zero. Thus, for the bar,

$$\int(\vec{F}_A - \vec{F}_D)dt = \vec{L}^+ - \vec{L}^- = \vec{L}^+ = m_2\vec{v}^+_{cm}.$$

Denoting the impulse at A with $\vec{P}_A$, the mass ratio m_2/m_1 by m_r, and the length ratio ℓ/d by q, and noting that $\vec{v}^+_{cm} = \omega\hat{k}\times\ell\hat{\jmath} = -\omega\ell\hat{\imath}$, we get

$$
\begin{aligned}
\vec{P}_A &\equiv \int \vec{F}_A\,dt = \int \vec{F}_D\,dt + m_2(-\omega\ell\hat{\imath})\\
&= m_1 v\left(1 - \frac{1}{1+\frac{4}{3}m_r q^2}\right)\hat{\imath} - m_2\ell\frac{v}{d\left(1+\frac{4}{3}m_r q^2\right)}\hat{\imath}\\
&= m_1 v\left(\frac{\frac{4}{3}m_r q^2}{1+\frac{4}{3}m_r q^2}\right)\hat{\imath} - m_2 v\left(\frac{q}{1+\frac{4}{3}m_r q^2}\right)\hat{\imath}\\
&= \frac{(4/3)m_2 q^2 - m_2 q}{1+\frac{4}{3}m_r q^2}v\hat{\imath} = \frac{q(4q-3)}{3\left(1+\frac{4}{3}m_r q^2\right)}m_2 v\hat{\imath}.
\end{aligned}
$$

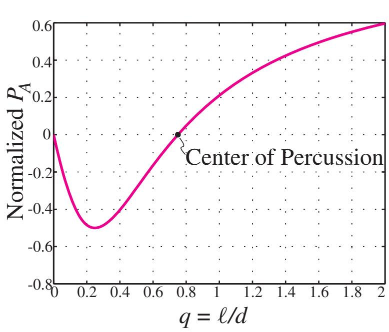


Figure 17.88: Plot of normalized impulse at A as a function of $q = \ell/d$.

Now, we are ready to graph the impulse at A as a function of $q \equiv \ell/d$. However, note that a better quantity to graph will be $P_A/(m_1 v)$, that is, the nondimensional impulse at A, normalized with respect to the initial linear momentum $m_1 v$ of the ball. The plot is shown in fig. 17.88. It is clear from the plot, as well as from the expression for $\vec{P}_A$, that the impulse at A is zero when $q = 3/4$ or $d = 4\ell/3 = 2/3(2\ell)$, that is, when the ball strikes at two thirds the length of the bar. Note that this location of the impact point is independent of the mass ratio m_r.

$d = 2/3(2\ell)$ for $\vec{P}_A = \vec{0}$

Comment: This particular point of impact D (when $d = 2/3(2\ell)$) which induces no impulse at the support point A is called the *center of percussion.* If you imagine the bar to be a bat or a racquet and point A to be the location of your grip, then hitting a ball at D gives you an impulse-free shot. In sports, point D is called a *sweet spot.*

SAMPLE 17.32 **Flying dish collides with solar panel.** A uniform rectangular plate of dimensions $2a = 2\,\mathrm{m}$ and $2b = 1\,\mathrm{m}$ and mass $m_P = 2\,\mathrm{kg}$ drifts in space at a uniform speed $v_p = 10\,\mathrm{m/s}$ (relative to some local Newtonian reference frame) in the direction shown. A circular disk of radius $R = 0.25\,\mathrm{m}$ and mass $m_D = 1\,\mathrm{kg}$ is heading towards the plate at a linear speed $v_0 = v_D = 1\,\mathrm{m/s}$ directed normal to the facing edge of the plate. In addition, the disk is spinning at $\omega_D = 5\,\mathrm{rad/s}$ in the clockwise direction. The plate and the disk collide at point A of the plate, located at $d = 0.8\,\mathrm{m}$ from the center of the long edge. Assume that the collision is frictionless and purely elastic. Find the linear and angular velocities of the plate and the disk immediately after the collision.

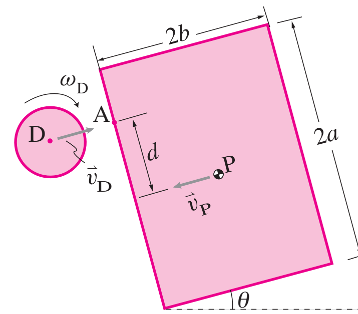


Figure 17.89

Solution We are given the linear and angular velocities of the two bodies before their collision. We have to find their linear and angular velocities after the collision. Let the linear velocities of the plate and the disk after the collision be $\vec{v}_P^+$ and $\vec{v}_D^+$, respectively. We assume that the collision takes place in the plane of the two bodies (that is, all motions before and after collision are planar). Then the unknown angular velocities of the plate and the disk after the collision can be assumed as $\vec{\omega}_P^+ = \omega_P^+\hat{k}$ and $\vec{\omega}_D^+ = \omega_D^+\hat{k}$. Thus, in total, we have 6 scalar unknowns here — 4 for linear velocities of the disk and the plate (each velocity has two components) and 2 for the two angular velocities.

Since we are trying to find velocities after a collision, given the velocities before the collision, we can use linear and angular momentum-impulse relations. Let $\vec{P}$ be the impulse acting during the collision. Note that the collision is frictionless and hence $\vec{P}$ must act normal to the two colliding surfaces. Therefore, the direction of the impulse is known, only its magnitude is not known. Thus, in total, we have 7 unknowns now. To solve for them, we need 7 independent equations.

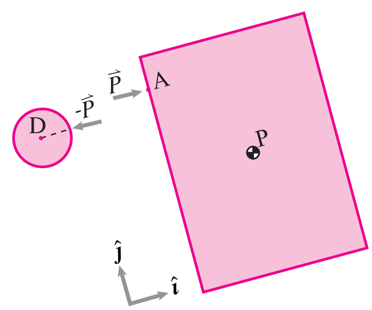


Figure 17.90: Separate free-body diagrams of the disk and the plate during collision. The impulse $\vec{P}$ acts normal to the colliding surfaces and hence we can assume that on the plate, $\vec{P} = P\hat{\imath}$. Therefore, the impulse on the disk is $-P\hat{\imath}$.

We have 6 independent equations from the linear and angular impulse-momentum balance for the two bodies (3 each). We need one more equation. That equation is the relationship between the normal components of the relative velocities of approach and departure with the coefficient of restitution e (=1 for elastic collision). Thus we have 7 equations for 7 unknowns. First we set them up, then we solve them with a calculator or computer.

The free-body diagrams of the disk and the plate together and the two separately are shown in fig. 17.91 and 17.90, respectively. Using an xy coordinate system oriented as shown in fig. 17.91, we can write

$$\begin{array}{lrcl}
\text{LMB for disk:} & m_D(\vec{v}_D^+ - \vec{v}_D^-) & = & -P\hat{\imath} \\
\text{LMB for plate:} & m_P(\vec{v}_P^+ - \vec{v}_P^-) & = & P\hat{\imath} \\
\text{AMB for disk:} & I_D^{\mathrm{cm}}(\vec{\omega}_D^+ - \vec{\omega}_D^-) & = & \vec{0} \\
\text{AMB for plate:} & I_P^{\mathrm{cm}}(\vec{\omega}_P^+ - \vec{\omega}_P^-) & = & \vec{r}_{\mathrm{A/P}} \times P\hat{\imath} \\
\text{kinematics:} & \vec{v}_{A_D}^+ - \vec{v}_{A_P}^+ & = & e(\vec{v}_{A_P}^- - \vec{v}_{A_D}^-)
\end{array}$$

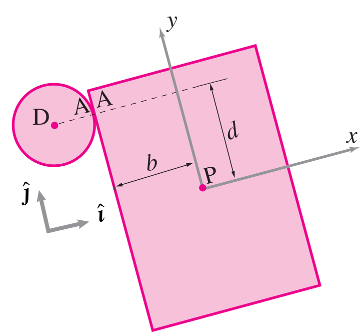


Figure 17.91: The free-body diagram of the disk and the plate together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram .

where, in the last equation $\vec{v}_{A_D}$ and $\vec{v}_{A_P}$ refer to the velocities of the material points located at A on the disk and on the plate, respectively. Other linear velocities in the equations above refer to the velocities at the center-of-mass of the corresponding bodies. We are given that $\vec{v}_D^- = v_D\hat{\imath}$, $\vec{v}_P^- = -v_P\hat{\imath}$, $\vec{\omega}_D^- = -\omega_D\hat{k}$, and $\vec{\omega}_P^- = \vec{0}$. Let us assume that $\vec{\omega}_D^+ = \omega_D^+\hat{k}$, $\vec{\omega}_P^+ = \omega_P^+\hat{k}$, $\vec{v}_D^+ = v_{D_x}^+\hat{\imath} + v_{D_y}^+\hat{\jmath}$, and similarly, $\vec{v}_P^+ = v_{P_x}^+\hat{\imath} + v_{P_y}^+\hat{\jmath}$. Then,

$$\begin{array}{rcl}
\vec{v}_{A_D}^- & = & \vec{v}_D^- + \vec{\omega}_D^- \times \vec{r}_{\mathrm{A/O}} = v_D\hat{\imath} - \omega_D R\hat{\jmath} \\
\vec{v}_{A_D}^+ & = & \vec{v}_D^+ + \vec{\omega}_D^+ \times \vec{r}_{\mathrm{A/O}} = v_{D_x}^+\hat{\imath} + (v_{D_y}^+ + \omega_D^+ R)\hat{\jmath} \\
\vec{v}_{A_P}^- & = & \vec{v}_P^- = -v_P\hat{\imath} \\
\vec{v}_{A_P}^+ & = & \vec{v}_P^+ + \vec{\omega}_P^+ \times \vec{r}_{\mathrm{A/P}} = (v_{P_x}^+ - \omega_P^+ d)\hat{\imath} + (v_{P_y}^+ - \omega_P^+ b)\hat{\jmath}.
\end{array}$$

Substituting these quantities in the kinematics equation above and dotting with $\hat{\imath}$, the unit normal at A, we get

$$v^+_{D_x} - v^+_{P_x} + \omega^+_P d = \underbrace{e}_{1}(-v_P - v_D) = -v_P - v_D. \tag{17.63}$$

Now, let us extract the scalar equations from the impulse-momentum equations for the disk and the plate by dotting with appropriate unit vectors.

Dotting LMB for the disk with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_D(v^+_{D_x} - v_D) = -P \tag{17.64}$$

$$m_D v^+_{D_y} = 0. \tag{17.65}$$

Dotting LMB for the plate with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_P(v^+_{P_x} + v_P) = P \tag{17.66}$$

$$m_P v^+_{P_y} = 0. \tag{17.67}$$

Dotting AMB for the disk and the plate with $\hat{\boldsymbol{k}}$, we get

$$I^{\text{cm}}_D(\omega^+_D + \omega_D) = 0 \tag{17.68}$$

$$I^{\text{cm}}_P \omega^+_P = -Pd. \tag{17.69}$$

We have all the equations we need. Let us rearrange these equations in a matrix form, taking the known quantities to the right and putting all unknowns to the left side. We then, write eqns. (17.64)–(17.69), and then eqn. (17.63) as

$$\begin{bmatrix} m_D & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & m_D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_P & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & m_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{\text{cm}}_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{\text{cm}}_P & d \\ 1 & 0 & -1 & 0 & 0 & d & 0 \end{bmatrix} \begin{Bmatrix} v^+_{D_x} \\ v^+_{D_y} \\ v^+_{P_x} \\ v^+_{P_y} \\ \omega^+_D \\ \omega^+_P \\ P \end{Bmatrix} = \begin{Bmatrix} m_D v_D \\ 0 \\ -m_P v_P \\ 0 \\ -I^{\text{cm}}_D \omega_D \\ 0 \\ -v_P - v_D \end{Bmatrix}.$$

Substituting the given numerical values for the masses and the pre-collision velocities, and the moments of inertia, $I^{\text{cm}}_D = (1/2)m_D R^2$ and $I^{\text{cm}}_P = (1/12)m_P(4a^2 + 4b^2)$, and then solving the matrix equation on a computer, we get,

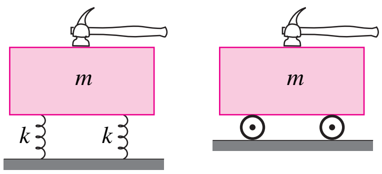
$$\vec{\boldsymbol{v}}^+_D = -8.70\,\text{m/s}\hat{\boldsymbol{\imath}}, \quad \vec{\boldsymbol{v}}^+_P = -5.15\,\text{m/s}\hat{\boldsymbol{\imath}}$$
$$\vec{\boldsymbol{\omega}}^+_D = -5\,\text{rad/s}\hat{\boldsymbol{k}}, \quad \vec{\boldsymbol{\omega}}^+_P = -9.31\,\text{rad/s}\hat{\boldsymbol{k}}$$
$$P = 9.70\,\text{kg}\cdot\text{m/s}.$$

You can easily check that the results obtained satisfy the conservation of linear momentum for the plate and the disk taken together as one system.

Comments: In this particular problem, the equations are simple enough to be solved by hand. For example, eqns. (17.65), (17.67), and (17.68) are trivial to solve and immediately give, $v^+_{D_y} = 0, v^+_{P_y} = 0$, and $\omega^+_D = \omega^D = -5\,\text{rad/s}$. The rest of the equations can be solved by usual eliminations and substitutions, etc. The key is that you have 7 linearly independent equations for the 7 unknowns.

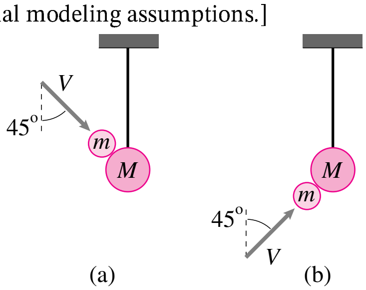
17.5 Collisions

17.5.1 The two blocks shown in the figure are identical except that one rests on two springs while the other one sits on two massless wheels. Draw free-body diagrams of each mass as each is struck by a hammer. Here we are interested in the free-body diagrams only during collision. Therefore, ignore all forces that are much smaller than the *impulsive* forces. State in words why the forces you choose to show should not be ignored during the collision.



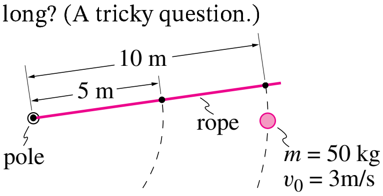
Problem 17.5.1

17.5.2 These problems concern two colliding masses. In (a) the smaller mass hits the hanging mass from above at an angle 45° with the vertical. In (b) the second case the smaller mass hits the hanging mass from below at the same angle. Assuming perfectly elastic impact between the balls, find the velocity of the hanging mass just after the collision. [Note, these problems are *not* well posed and can only be solved if you make additional modeling assumptions.]



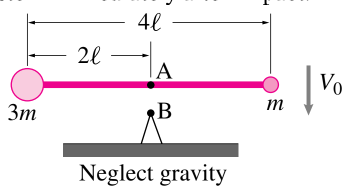
Problem 17.5.2

17.5.3 A narrow pole is in the middle of a pond with a 10 m rope tied to it. A frictionless ice skater of mass 50 kg and speed 3 m/s grabs the rope. The rope slowly wraps around the pole. What is the speed of the skater when the rope is 5 m long? (A tricky question.)



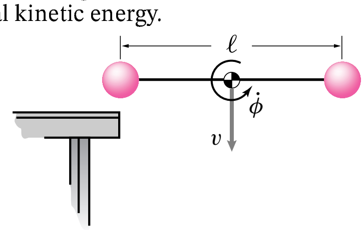
Problem 17.5.3

17.5.4 The masses m and $3m$ are joined by a light-weight bar of length 4ℓ. If point A in the center of the bar strikes fixed point B vertically with velocity V_0, and is not permitted to rebound, find $\dot{\theta}$ of the system immediately after impact.



Problem 17.5.4: Neglect gravity!

17.5.5 Two equal masses each of mass m are joined by a massless rigid rod of length ℓ. The assembly strikes the edge of a table as shown in the figure, when the center of mass is moving downward with a linear velocity v and the system is rotating with angular velocity $\dot{\phi}$ in the counter-clockwise sense. The impact is 'elastic'. Find the immediate subsequent motion of the system, assuming that no energy is lost during the impact and that there is no gravity. Show that there is an interchange of translational and rotational kinetic energy.



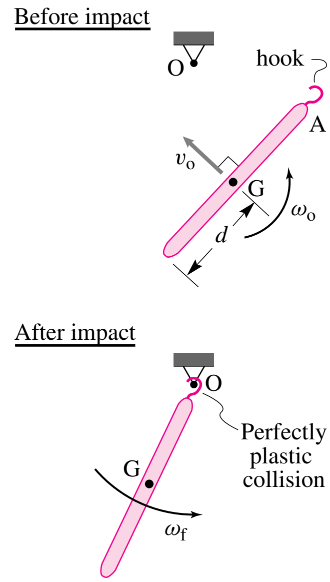
Problem 17.5.5

17.5.6 In the *absence of gravity*, a thin rod of mass m and length ℓ is initially tumbling with constant angular speed ω_o, in the counter-clockwise direction, while its mass center has constant speed v_o, directed as shown below. The end A then makes a perfectly plastic collision with a rigid peg O (via a hook). The velocity of the mass center happens to be perpendicular to the rod just before impact.

a) What is the angular speed ω_f immediately after impact?

b) What is the angular speed 10 seconds after impact? Why?

c) What is the loss in energy in the plastic collision?



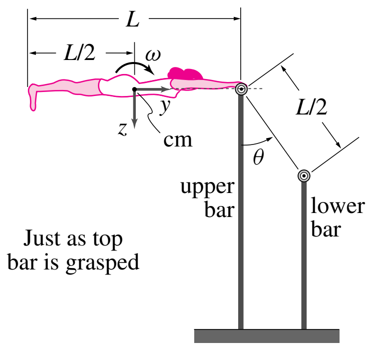
Problem 17.5.6

17.5.7 A gymnast of mass m and extended height L is performing on the uneven parallel bars. About the x, y, z axes which pass through her center of mass, her radii of gyration are k_x, k_y, and k_z, respectively. Just before she grasps the top bar, her fully extended body is horizontal and rotating with angular rate ω; her center of mass is then stationary. Neglect any friction between the bar and her hands and assume that she remains rigid throughout the entire stunt.

a) What is the gymnast's rotation rate just *after* she grasps the bar? State clearly any approximations/assumptions that you make.

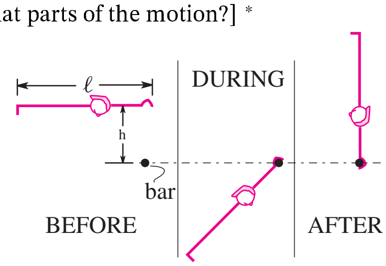
b) Calculate the linear speed with which her hips (CM) strike the lower bar. State all assumptions/approximations.

c) Describe (in words, no equations please) her motion immediately after her hips strike the lower bar if she releases her hands just prior to this impact.



Problem 17.5.7

17.5.8 An acrobat modeled as a rigid body with uniform rigid mass m of length l. She falls without rotation in the position shown from height h where she was stationary. She then grabs a bar with a firm but slippery grip. What is h so that after the subsequent motion the acrobat ends up in a stationary handstand? [Hint: What quantities are preserved in what parts of the motion?] *

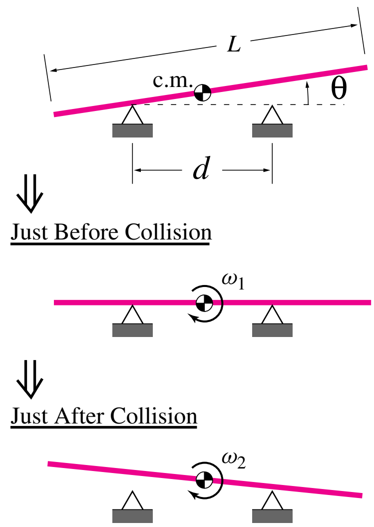


Problem 17.5.8

17.5.9 A crude see-saw is built with two supports separated by distance d about which a rigid plank (mass m, length L) can pivot smoothly. The plank is placed symmetrically, so that its center of mass is midway between the supports when the plank is at rest.

a) While the left end is resting on the left support, the right end of the plank is lifted to an angle θ and released. At what angular velocity ω_1 will the plank strike the right hand support?

b) Following the impact, the left end of the plank can pivot purely about the right end if d/L is properly chosen and the right end does not bounce. Find ω_2 under these circumstances.



Problem 17.5.9

17.5.10 Baseball bat. In order to convey the ideas without making the calculation too complicated, some of the simplifying assumptions here are highly approximate. Assume that a bat is a uniform rigid stick with length L and mass m_s. The motion of the bat is a pivoting about one end held firmly in place with hands that rotate but do not move. The swinging of the bat occurs by the application of a constant torque M_s at the hands over an angle of $\theta = \pi/2$ until the point of impact with the ball. The ball has mass m_b and arrives perpendicular to the bat at an absolute speed v_b at a point a distance ℓ from the hands. The collision between the bat and the ball is completely elastic.

a) To maximize the speed v_{hit} of the hit ball *How heavy should a baseball bat be? Where should the ball hit the bat?* Here are some hints for one way to approach the problem.

   - Find the angular velocity of the bat just before collision by drawing a FBD of the bat etc.
   - Find the total energy of the ball and bat system just before the collision.
   - Draw a FBD of the ball and of the bat during the collision (with this model there is an impulse at the hands on the bat). Call the magnitude of the impulse of the ball on the bat (and vice versa) $\int F dt$.
   - Use various momentum equations to find the angular velocity of the bat and velocity of the ball just after the collision in terms of $\int F dt$ and other quantities above. Use these to find the energy of the system just after collision.
   - Solve for the value of $\int F dt$ that conserves energy. As a check you should see if this also predicts that the relative separation speed of the ball and bat (at the impact point) is the same as the relative approach speed (it should be).
   - You now can calculate v_{hit} in terms of m_b, m_s, M_s, L, ℓ, and v_b.
   - Find the maximum of the above expression by varying m_s and ℓ. Pick numbers for the fixed quantities if you like.

b) Can you explain in words what is wrong with a bat that is too light or too heavy?

c) Which aspects of the model above do you think lead to the biggest errors in predicting what a real ball player should pick for a bat and place on the bat to hit the ball?

d) Describe as clearly as possible a different model of a baseball swing that you think would give a more accurate prediction. (You need not do the calculation).