SAMPLE 17.1 Velocity of a point on a rigid body in planar motion. An equilateral triangular plate ABC is in motion in the x-y plane. At the instant shown in the figure, point B has velocity $\vec{v}_B = 0.3\,\mathrm{m/s}\hat{\imath} + 0.6\,\mathrm{m/s}\hat{\jmath}$ and the plate has angular velocity $\vec{\omega} = 2\,\mathrm{rad/s}\hat{k}$.

1. Find the velocity of point A using vector calculations.
2. Find the velocity of point C using matrix calculations.

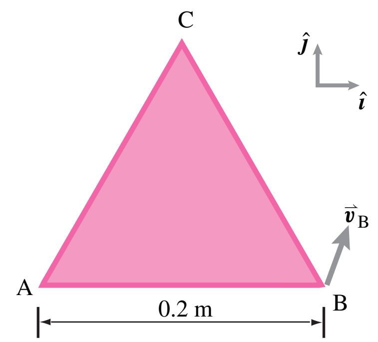


Figure 17.12: The rigid triangular plate ABC rotates counterclockwise at 5 rad/s. Velocity of point B is given. We have to find the velocity of points A and C.

Solution

1. We are given $\vec{v}_B$ and ω, and we need to find $\vec{v}_A$, the velocity of point A on the same rigid body. We know that,
$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$
Thus, to find $\vec{v}_A$, we need to find $\vec{r}_{A/B}$. Let us take an x-y coordinate system whose origin coincides with point A of the plate at the instant of interest and the x-axis is along AB. Then,
$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\mathrm{m}\hat{\imath}) = -0.2\,\mathrm{m}\hat{\imath}$$
Thus,
$$\begin{aligned} \vec{v}_A &= \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} \\ &= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\mathrm{m/s} + 2\,\mathrm{rad/s}\hat{k} \times (-0.2\hat{\imath})\,\mathrm{m} \\ &= (0.3\hat{\imath} + 0.2\hat{\jmath})\,\mathrm{m/s}. \end{aligned}$$
$$\boxed{\vec{v}_A = (0.3\hat{\imath} + 0.2\hat{\jmath})\,\mathrm{m/s}}$$

2. Let xy axes with origin at A be fixed in space with x-axis along AB at the instant of interest. Let $x'y'$ axes be fixed to the plate with origin at O and the x'-axis along AC as shown in fig. 17.13. Thus the coordinates of point C in the rotating axes are
$$\begin{bmatrix} x'_C \\ y'_C \end{bmatrix} = \begin{bmatrix} 0.2\,\mathrm{m} \\ 0 \end{bmatrix}$$
Clearly, at the instant of interest, the rotating axes make an angle of $\theta = \pi/3$ with the fixed axes. Now, we can calculate the velocity of point C as follows.
$$[\vec{v}_C]_{xy} = [\vec{v}_A]_{xy} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'_{C/A} \\ y'_{C/A} \end{bmatrix}.$$
Using $\theta = \pi/3$, the angle that the rotating axes make with the fixed axes at the instant of interest, and $\omega = 2\,\mathrm{rad/s}$, we get
$$\begin{aligned} [\vec{v}_C]_{xy} &= \begin{bmatrix} 0.3\,\mathrm{m/s} \\ 0.2\,\mathrm{m/s} \end{bmatrix} + \begin{bmatrix} 0 & -2\,\mathrm{rad/s} \\ 2\,\mathrm{rad/s} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.2\,\mathrm{m} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.3\,\mathrm{m/s} \\ 0.2\,\mathrm{m/s} \end{bmatrix} + \begin{bmatrix} -0.35\,\mathrm{m/s} \\ 0.2\,\mathrm{m/s} \end{bmatrix} = \begin{bmatrix} -0.05\,\mathrm{m/s} \\ 0.4\,\mathrm{m/s} \end{bmatrix}. \end{aligned}$$
Thus, $\vec{v}_C = -0.05\,\mathrm{m/s}\hat{\imath} + 0.4\,\mathrm{m/s}\hat{\jmath}$. We can also verify this answer by computing $\vec{v}_C = \vec{v}_B + \vec{\omega} \times \vec{r}_{C/B}$.
$$\boxed{\vec{v}_C = -0.05\,\mathrm{m/s}\hat{\imath} + 0.4\,\mathrm{m/s}\hat{\jmath}}$$

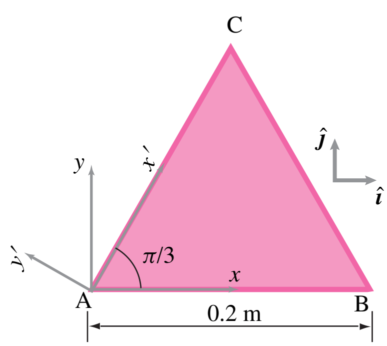


Figure 17.13: Velocity of point A is known. The axes xy are fixed in space and the axes $x'y'$ are fixed to the rotating plate in the orientation shown. Thus in the $x'y'$ axes, the coordinates of point C will always be $x'_C = 0.2\,\mathrm{m}$, $y'_C = 0$.

SAMPLE 17.2 **The instantaneous center of rotation.** A rigid body is in planar motion. At some instant t, the angular velocity of the body is $\vec{\omega} = 5\,\text{rad/s}\hat{\boldsymbol{k}}$ and the linear velocity of a point C on the body is $\vec{\boldsymbol{v}}_C = 2\,\text{m/s}\hat{\boldsymbol{\imath}} - 5\,\text{m/s}\hat{\boldsymbol{\jmath}}$. Find the instantaneous center of rotation.

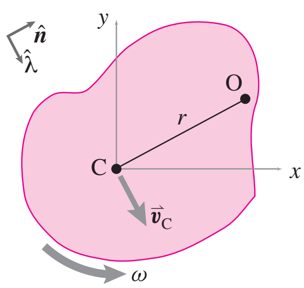


Figure 17.14

Solution Let a point O be the instantaneous center of rotation. We need to find the location of O using the given information. Whether O lies inside the object or outside is irrelevant here since the object boundary is not specified. So, let us consider an abstract rigid object as shown in fig. 17.14. What we know about O is that it has zero velocity (since it is the center of rotation). Let point O be a distance r away from C in some unknown direction $\hat{\boldsymbol{n}}$ (to be found). Thus, $\vec{\boldsymbol{r}}_{O/C} = r\hat{\boldsymbol{n}}$ is the position vector of point O with respect to point C. Now,

$$\begin{aligned}
\vec{\boldsymbol{v}}_C &= \underbrace{\vec{\boldsymbol{v}}_O}_{\vec{\boldsymbol{0}}} + \vec{\omega} \times \underbrace{\vec{\boldsymbol{r}}_{C/O}}_{-\vec{\boldsymbol{r}}_{O/C}} \\
&= -\omega\hat{\boldsymbol{k}} \times r\hat{\boldsymbol{n}} = -\omega r(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}).
\end{aligned}$$

Thus, $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}$ must be parallel to and in the opposite direction of $\vec{\boldsymbol{v}}_C$. Let $\vec{\boldsymbol{v}}_C = v_C\hat{\boldsymbol{\lambda}}$. Then, we must have $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = -\hat{\boldsymbol{\lambda}}$, which says that $\hat{\boldsymbol{n}}$ must be normal to $\hat{\boldsymbol{\lambda}}$. Since $\hat{\boldsymbol{\lambda}} = \vec{\boldsymbol{r}}_C/|\vec{\boldsymbol{r}}_C| = (2\,\text{m/s}\hat{\boldsymbol{\imath}} - 5\,\text{m/s}\hat{\boldsymbol{\jmath}})/(\sqrt{2^2+5^2}\,\text{m/s}) = 0.37\hat{\boldsymbol{\imath}} - 0.93\hat{\boldsymbol{\jmath}}$, we can immediately write $\hat{\boldsymbol{n}} = 0.93\hat{\boldsymbol{\imath}} + 0.37\hat{\boldsymbol{\jmath}}$. So, now,

$$\begin{aligned}
v_C\hat{\boldsymbol{\lambda}} &= -\omega r\overbrace{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}})}^{-\hat{\lambda}} \\
\Rightarrow \quad v_C &= \omega r \\
\Rightarrow \quad r &= v_C/\omega \\
&= \sqrt{29}\,\text{m/s}/(5\,\text{rad/s}) = 1.08m.
\end{aligned}$$

Thus

$$\vec{\boldsymbol{r}}_{O/C} = r\hat{\boldsymbol{n}} = 1.08\,\text{m}(0.93\hat{\boldsymbol{\imath}} + 0.37\hat{\boldsymbol{\jmath}}) = 1\,\text{m}\hat{\boldsymbol{\imath}} + 0.4\,\text{m}\hat{\boldsymbol{\jmath}}.$$

$$\boxed{\vec{\boldsymbol{r}}_{O/C} = 1\,\text{m}\hat{\boldsymbol{\imath}} + 0.4\,\text{m}\hat{\boldsymbol{\jmath}}}$$

Alternatively,

We can find $\vec{\boldsymbol{r}}_{O/C}$ purely by vector algebra. Let $\vec{\boldsymbol{r}}_{O/C} = (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})\,\text{m}$. So,

$$\begin{aligned}
\vec{\boldsymbol{v}}_O &= \vec{\boldsymbol{v}}_C + \vec{\omega} \times \vec{\boldsymbol{r}}_{O/C} \\
\text{or} \qquad \vec{\boldsymbol{0}} &= 2\,\text{m/s}\hat{\boldsymbol{\imath}} - 5\,\text{m/s}\hat{\boldsymbol{\jmath}} + (5\,\text{rad/s})\hat{\boldsymbol{k}} \times (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})\,\text{m} \\
&= (2 - 5y)\,\text{m/s}\hat{\boldsymbol{\imath}} + (-5 + 5x)\,\text{m/s}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ respectively, we get

$$\begin{aligned}
2 - 5y &= 0 \\
-5 + 5x &= 0.
\end{aligned}$$

Solving these two equations simultaneously, we get, $x = 1$ and $y = 0.4$. Thus,

$$\vec{\boldsymbol{r}}_{O/C} = 1\,\text{m}\hat{\boldsymbol{\imath}} + 0.4\,\text{m}\hat{\boldsymbol{\jmath}}$$

as obtained above.

SAMPLE 17.3 A cheerleader throws her baton up in the air in the vertical xy-plane. At an instant when the baton axis is at $\theta = 60^\circ$ from the horizontal, the velocity of end A of the baton is $\vec{v}_A = 2\,\mathrm{m/s}\,\hat{\imath} + \sqrt{3}\,\mathrm{m/s}\,\hat{\jmath}$. At the same instant, end B of the baton has velocity in the negative x-direction (but $|\vec{v}_B|$ is not known). If the length of the baton is $\ell = \frac{1}{2}$ m and the center-of-mass is in the middle of the baton, find the velocity of the center-of-mass.

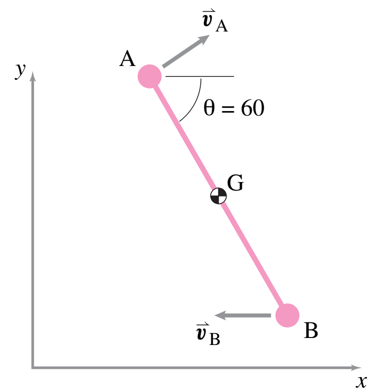


Figure 17.15

Solution

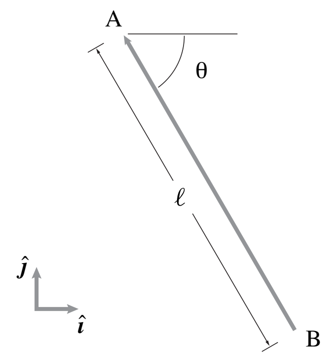
$$\begin{aligned} \text{We are given:} \quad \vec{v}_A &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\mathrm{m/s} \\ \text{and} \quad \vec{v}_B &= -v_B\hat{\imath} \end{aligned}$$

where $v_B = |\vec{v}_B|$ is unknown. We need to find $\vec{v}_G$. Using the relative velocity formula for two points on a rigid body, we can write:

$$\vec{v}_G = \vec{v}_A + \vec{\omega} \times \vec{r}_{G/A} \tag{17.9}$$

Here, $\vec{v}_A$ and $\vec{r}_{G/A}$ are known. Thus, to find $\vec{v}_G$, we need to find $\vec{\omega}$, the angular velocity of the baton. Since the motion is in the vertical xy-plane, let $\vec{\omega} = \omega\hat{k}$. Then,

$$\begin{aligned} \vec{v}_B &= \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} = \vec{v}_A + \omega\hat{k} \times \underbrace{\ell(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})}_{\vec{r}_{B/A}} \\ \text{or} \quad -v_B\hat{\imath} &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\mathrm{m/s} + \omega\ell(\cos\theta\hat{\jmath} + \sin\theta\hat{\imath}) \\ &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\mathrm{m/s} + \omega\cdot\frac{1}{2}\,\mathrm{m}\cdot\left(\frac{1}{2}\hat{\jmath} + \frac{\sqrt{3}}{2}\hat{\imath}\right) \end{aligned}$$



$\vec{r}_{A/B} = \ell\,(-\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath})$

Figure 17.16

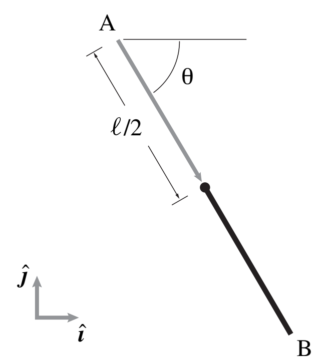
Dotting both sides of this equation with $\hat{\jmath}$ we get:

$$\begin{aligned} 0 &= \sqrt{3}\,\mathrm{m/s} + \frac{\omega}{2}\,\mathrm{m}\cdot\frac{1}{2} \\ \Rightarrow \quad \omega &= -\sqrt{3}\frac{\cancel{\mathrm{m}}}{\mathrm{s}}\cdot\frac{4}{1\,\cancel{\mathrm{m}}} = -4\sqrt{3}\,\mathrm{rad/s}. \end{aligned}$$

Now substituting the appropriate values in Eqn (17.9) we get:

$$\begin{aligned} \vec{v}_G &= \vec{v}_A + \omega\hat{k} \times \underbrace{\frac{\ell}{2}(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})}_{\vec{r}_{G/A}} \\ &= \vec{v}_A + \frac{\omega\ell}{2}(\cos\theta\hat{\jmath} + \sin\theta\hat{\imath}) \\ &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\mathrm{m/s} - \sqrt{3}\,\mathrm{m/s}\cdot\left(\frac{1}{2}\hat{\jmath} + \frac{\sqrt{3}}{2}\hat{\imath}\right) \\ &= \left(2 - \frac{3}{2}\right)\mathrm{m/s}\,\hat{\imath} + \left(\sqrt{3} - \frac{\sqrt{3}}{2}\right)\mathrm{m/s}\,\hat{\jmath} \\ &= \frac{1}{2}\,\mathrm{m/s}\,\hat{\imath} + \frac{\sqrt{3}}{2}\,\mathrm{m/s}\,\hat{\jmath} \end{aligned}$$

$$\boxed{\vec{v}_G = (0.5\hat{\imath} + 0.87\hat{\jmath})\,\mathrm{m/s}}$$



$\vec{r}_{G/A} = \ell/2(\cos\theta\hat{\imath} - \sin\theta\,\hat{\jmath})$

Figure 17.17

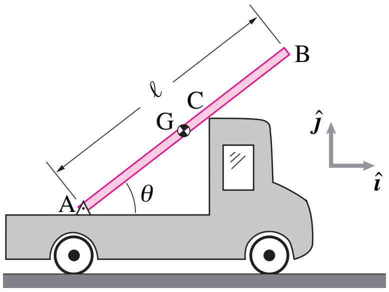


Figure 17.18: A board in the back of an accelerating truck.

SAMPLE 17.4 **A board in the back of an accelerating truck.** A 10 ft long uniform board AB rests in the back of a flat-bed truck as shown in Fig. 17.18. End A of the board is hinged to the bed of the truck. The truck is going on a level road at 55 mph. In preparation for overtaking a vehicle in the front, the trucker accelerates at a rate of 3 mph/s. At the instant when the speed of the truck is 60 mph, the magnitude of the relative velocity and relative acceleration of end B with respect to the bed of the truck are 10 ft/s and 12 ft/s^2, respectively. There is wind and at this instant, the board has lost contact with point C. If the angle θ between the board and the bed is 45° at the instant mentioned, find

1. the angular velocity and angular acceleration of the board,
2. the absolute velocity and absolute acceleration of point B, and
3. the acceleration of the center-of-mass G of the board.

Solution At the instant of interest, the quantities given are:

$$\begin{aligned}
\vec{v}_A &= \text{velocity of the truck } = 60\,\text{mph}\,\hat{\imath} = 88\,\text{ft/s}\,\hat{\imath} \\
\vec{a}_A &= \text{acceleration of the truck } = 3\,\text{mph}/s = 4.4\,\text{ft/s}^2\,\hat{\imath} \\
|\vec{v}_{B/A}| &= v_{B/A} = \text{magnitude of relative velocity of B } = 10\,\text{ft/s} \\
|\vec{a}_{B/A}| &= a_{B/A} = \text{magnitude of relative acceleration of B } = 12\,\text{ft/s}^2.
\end{aligned}$$

Let $\vec{\omega} = \omega\hat{k}$ be the angular velocity and $\dot{\vec{\omega}} = \dot{\omega}\hat{k}$ be the angular acceleration of the board.

1. The relative velocity of end B of the board with respect to end A is

$$\begin{aligned}
\vec{v}_{B/A} &= \vec{\omega}\times\vec{r}_{B/A} = \omega\hat{k}\times\ell(\cos\theta\hat{\imath}+\sin\theta\hat{\jmath}) \\
&= \omega\ell(\cos\theta\hat{\jmath}-\sin\theta\hat{\imath}) \\
\Rightarrow \quad \omega &= \frac{|\vec{v}_{B/A}|}{\ell} = \frac{v_{B/A}}{\ell} = \frac{10\,\text{ft/s}}{10\,\text{ft}} = 1\,\text{rad/s}.
\end{aligned}$$

Note that we have taken the positive value for ω because the board is rotating counterclockwise at the instant of interest (since it is given that the board has lost contact with point C).

Similarly, we can compute the angular acceleration:

$$\begin{aligned}
\vec{a}_{B/A} &= \dot{\vec{\omega}}\times\vec{r}_{B/A} - \omega^2\vec{r}_{B/A} \\
&= \dot{\omega}\hat{k}\times\ell(\cos\theta\hat{\imath}+\sin\theta\hat{\jmath}) - \omega^2\ell(\cos\theta\hat{\imath}+\sin\theta\hat{\jmath}) \\
&= \dot{\omega}\ell(\cos\theta\hat{\jmath}-\sin\theta\hat{\imath}) - \omega^2\ell(\cos\theta\hat{\imath}+\sin\theta\hat{\jmath}) \\
\Rightarrow \quad |\vec{a}_{B/A}| &= \sqrt{(\dot{\omega}\ell)^2+(\omega^2\ell)^2} = a_{B/A}\ \text{(given)} \\
\Rightarrow \quad a^2_{B/A} &= (\dot{\omega}\ell)^2+(\omega^2\ell)^2 \\
\Rightarrow \quad \dot{\omega} &= \sqrt{\frac{a^2_{B/A}}{\ell^2}-\omega^4} = \sqrt{\left(\frac{12\,\text{ft/s}^2}{10\,\text{ft}}\right)^2-(1\,\text{rad/s})^4} \\
&= \pm 0.663\,\text{rad/s}^2.
\end{aligned}$$

Once again, we select the positive value for $\dot{\omega}$ since we assume that the board accelerates counterclockwise.

$$\boxed{\vec{\omega} = 1\,\text{rad/s}\hat{k},\ \dot{\vec{\omega}} = 0.663\,\text{rad/s}^2\hat{k}}$$

2. The absolute velocity and the absolute acceleration of the end point B can be found as follows.

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\
&= v_A\hat{\boldsymbol{\imath}} + v_{B/A}(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) \\
&= 88\,\mathrm{ft/s}\hat{\boldsymbol{\imath}} + 10\,\mathrm{ft/s}(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}}) \\
&= 80.93\,\mathrm{ft/s}\hat{\boldsymbol{\imath}} + 7.07\,\mathrm{ft/s}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_B &= \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A} \\
&= \vec{\boldsymbol{a}}_A + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{B/A} - \omega^2\vec{\boldsymbol{r}}_{B/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\hat{\boldsymbol{k}} \times \ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) - \omega^2\ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
&= (a_A - \dot{\omega}\ell\sin\theta - \omega^2\ell\cos\theta)\hat{\boldsymbol{\imath}} + (\dot{\omega}\ell\cos\theta - \omega^2\ell\sin\theta)\hat{\boldsymbol{\jmath}} \\
&= \left(4.4\,\mathrm{ft/s^2} - \frac{0.66}{\mathrm{s^2}}\cdot 10\,\mathrm{ft}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\mathrm{s^2}}\cdot 10\,\mathrm{ft}\cdot\frac{1}{\sqrt{2}}\right)\hat{\boldsymbol{\imath}} \\
&\quad + \left(\frac{0.66}{\mathrm{s^2}}\cdot 10\,\mathrm{ft}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\mathrm{s^2}}\cdot 10\,\mathrm{ft}\cdot\frac{1}{\sqrt{2}}\right)\hat{\boldsymbol{\jmath}} \\
&= -7.34\,\mathrm{ft/s^2}\hat{\boldsymbol{\imath}} - 2.40\,\mathrm{ft/s^2}\,\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_B = (80.93\hat{\boldsymbol{\imath}} + 7.07\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s}, \quad \vec{\boldsymbol{a}}_B = -(7.34\hat{\boldsymbol{\imath}} + 2.40\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}.}$$

3. Now, we can easily calculate the acceleration of the center-of-mass as follows.

$$\begin{aligned}
\vec{\boldsymbol{a}}_G &= \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{G/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{G/A} - \omega^2\vec{\boldsymbol{r}}_{G/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\hat{\boldsymbol{k}} \times \frac{\ell}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) - \omega^2\frac{\ell}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\frac{\ell}{2}(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) - \omega^2\frac{\ell}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
&= 4.4\,\mathrm{ft/s^2}\hat{\boldsymbol{\imath}} + 0.663\,\mathrm{rad/s^2}\cdot\frac{10\,\mathrm{ft}}{2}\cdot(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}}) \\
&\quad -(1\,\mathrm{rad/s})^2\cdot\frac{10\,\mathrm{ft}}{2}\cdot(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}) \\
&= -1.48\,\mathrm{ft/s^2}\hat{\boldsymbol{\imath}} - 1.19\,\mathrm{ft/s^2}\,\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_G = -(1.48\hat{\boldsymbol{\imath}} + 1.19\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}}$$

Comments: This problem is admittedly artificial. We, however, solve this problem to show kinematic calculations.

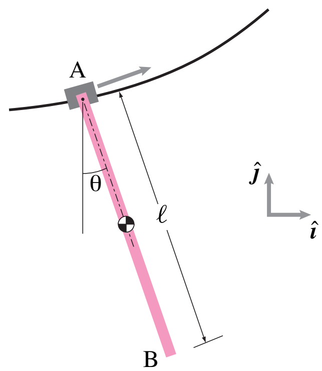


Figure 17.19

SAMPLE 17.5 **Tracking motion.** A cart moves along a suspended curved path. A rod AB of length $\ell = 1$ m hangs from the cart. End A of the rod is attached to a motor on the cart. The other end, B, hangs freely. The motor rotates the rod such that $\theta(t) = \theta_0 \sin(\lambda t)$ (θ is measured with respect to a fixed vertical line) while the cart moves along the path such that $\vec{\boldsymbol{r}}_A = t\hat{\boldsymbol{\imath}} + \frac{t^3}{18}\hat{\boldsymbol{\jmath}}$, where all variables ($r$, t, etc.) are nondimensional.

1. Find the velocity and acceleration of point B as a function of nondimensional time t.
2. Take $\theta_0 = \pi/3$ and $\lambda = 6$. Find and plot the position of the bar at $t = 0, 0.1, 0.3, 0.9, 1, 1.1, 1.2,$ and 1.5. Find and draw $\vec{\boldsymbol{v}}_B$ and $\vec{\boldsymbol{a}}_B$ at the specified t.

Solution

1. The velocity and acceleration of point B are given by

$$\begin{aligned}\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \\ \vec{\boldsymbol{a}}_B &= \vec{\boldsymbol{a}}_A + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{B/A} - \omega^2 \vec{\boldsymbol{r}}_{B/A}.\end{aligned}$$

Thus, in order to find the velocity and acceleration of point B, we need to find the velocity and acceleration of point A and the angular velocity and angular acceleration of the bar. We are given the position of point A and the angular position of the bar as functions of t. We can, therefore, find $\vec{\boldsymbol{v}}_A$, $\vec{\boldsymbol{a}}_A$, $\vec{\boldsymbol{\omega}}$, and $\dot{\vec{\boldsymbol{\omega}}}$ by differentiating the given functions with respect to t.

$$\begin{aligned} & \vec{\boldsymbol{r}}_A &=& \; t\hat{\boldsymbol{\imath}} + \frac{t^3}{18}\hat{\boldsymbol{\jmath}} \\ \Rightarrow & \vec{\boldsymbol{v}}_A &\equiv& \; \dot{\vec{\boldsymbol{r}}}_A = \hat{\boldsymbol{\imath}} + (t^2/6)\hat{\boldsymbol{\jmath}} \quad (17.10) \\ \Rightarrow & \vec{\boldsymbol{a}}_A &\equiv& \; \dot{\vec{\boldsymbol{v}}}_A = (t/3)\hat{\boldsymbol{\jmath}} \quad (17.11)\end{aligned}$$

and

$$\begin{aligned} & \theta\hat{\boldsymbol{k}} &=& \; \theta_0 \sin(\lambda t)\hat{\boldsymbol{k}} \\ \Rightarrow & \vec{\boldsymbol{\omega}} &\equiv& \; \dot{\theta}\hat{\boldsymbol{k}} = \theta_0\lambda\cos(\lambda t)\hat{\boldsymbol{k}} \quad (17.12) \\ \Rightarrow & \dot{\vec{\boldsymbol{\omega}}} &\equiv& \; \ddot{\theta}\hat{\boldsymbol{k}} = -\theta_0\lambda^2\sin(\lambda t)\hat{\boldsymbol{k}}. \quad (17.13)\end{aligned}$$

So,

$$\begin{aligned}\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \ell(\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) \\ &= \hat{\boldsymbol{\imath}} + (t^2/6)\hat{\boldsymbol{\jmath}} + \ell\dot{\theta}(\sin\theta\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\boldsymbol{\imath}}) \\ &= (1 + \ell\dot{\theta}\cos\theta)\hat{\boldsymbol{\imath}} + (t^2/6 + \ell\dot{\theta}\sin\theta)\hat{\boldsymbol{\jmath}} \quad (17.14) \\ \vec{\boldsymbol{a}}_B &= \vec{\boldsymbol{a}}_A + \ddot{\theta}\hat{\boldsymbol{k}} \times \ell(\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) - \dot{\theta}^2\ell(\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) \\ &= (t/3)\hat{\boldsymbol{\jmath}} + \ell\ddot{\theta}\sin\theta\hat{\boldsymbol{\jmath}} + \ell\ddot{\theta}\cos\theta\hat{\boldsymbol{\imath}} - \ell\dot{\theta}^2\sin\theta\hat{\boldsymbol{\imath}} + \ell\dot{\theta}^2\cos\theta\hat{\boldsymbol{\jmath}} \\ &= \ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\boldsymbol{\imath}} + [t/3 + \ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)]\hat{\boldsymbol{\jmath}} \\ & \quad (17.15)\end{aligned}$$

where $\theta = \theta_0\sin(\lambda t)$, $\dot{\theta} = \theta_0\lambda\cos(\lambda t)$, and $\ddot{\theta} = -\theta_0\lambda^2\sin(\lambda t) = -\lambda^2\theta$. Thus $\vec{\boldsymbol{v}}_B$ and $\vec{\boldsymbol{a}}_B$ are functions of t.

2. The position of the rod at any time t is specified by the position of the two end points A and B (or alternatively, the position of A and the angle of the rod θ). The

position of point A is easily determined by substituting the value of t in the given expression for $\vec{r}_A$. The position of end B is given by

$$\begin{aligned}\vec{r}_B &= \vec{r}_A + \vec{r}_{B/A} = t\hat{\imath} + (t^3/18)\hat{\jmath} + \ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \\ &= (t + \ell\sin\theta)\hat{\imath} + (t^3/18 - \ell\cos\theta)\hat{\jmath}.\end{aligned}$$

To compute the positions, velocities, and accelerations of end points A and B at the given instants, we first compute θ, $\dot{\theta}$, and $\ddot{\theta}$, and then substitute them in the expressions for $\vec{r}_A, \vec{r}_B, \vec{v}_A, \vec{v}_B, \vec{a}_A$, and $\vec{a}_B$. A pseudocode for computer calculation is given below.

```
t = [0 0.1 0.3 0.9 1.0 1.1 1.2 1.5]
theta0=pi/3, L=.5, lam=6
for each t, compute
  theta = theta0*sin(lam*t)
  w = lam*theta0*cos(lam*t)
  wdot = -lam^2*theta
       % Position of A and B
  xA=t, yA=t^3/18
  xB = xA + L*sin(theta)
  yB = yA - L*cos(theta)
       % Velocity of A and B
  uA = 1,  vA = t^2/6
  uB = uA + L*w.*cos(theta)
  vB = vA + L*w.*sin(theta)
       % Acceleration of A and B
  axA = 0,  ayA = t/3
  axB = L*wdot*cos(theta) - L*w^2*sin(theta)
  ayB = ayA + L*wdot*sin(theta) + L*w^2*cos(theta)
```

From the above calculation, we get the desired quantities at each t. For example, at $t = 0$ we get,

```
xA = 0, yA = 0, xB = 0, yB = -0.5
uA = 1, vA = 0, uB = 4.14, vB = 0, axB = 0, ayB = 19.74
```

which means,

$$\vec{r}_A = \vec{0}, \quad \vec{r}_B = -0.5\hat{\jmath}, \quad \vec{v}_A = \hat{\imath}, \quad \vec{v}_B = 4.14\hat{\imath}, \quad \vec{a}_B = 19.74\hat{\jmath}.$$

The position of the bar, the velocity vectors at points A and B, and the acceleration vector at B, thus obtained, are shown in fig. 17.20 graphically. ①

① We can take several values of t, say 400 equally spaced values between $t = 0$ and $t = 4$, and draw the bar at each t to see its motion and the trajectory of its end points. fig. 17.21 shows such a graph.

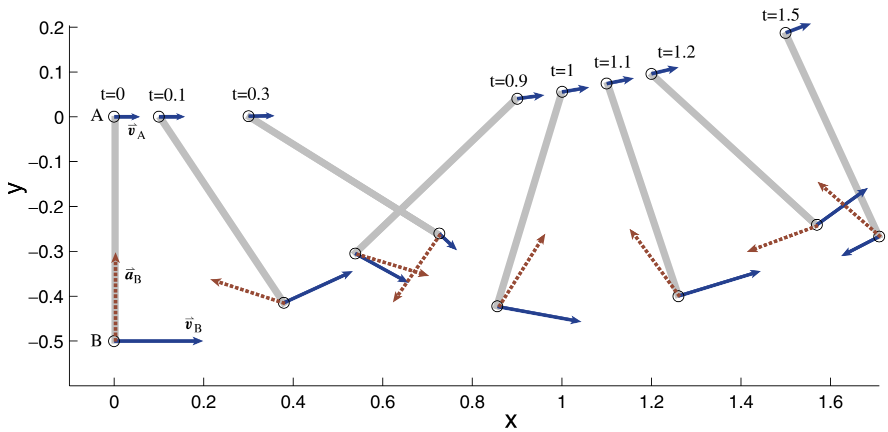


Figure 17.20: Position, velocity of the end points A and B, and acceleration of point B at various time instants.

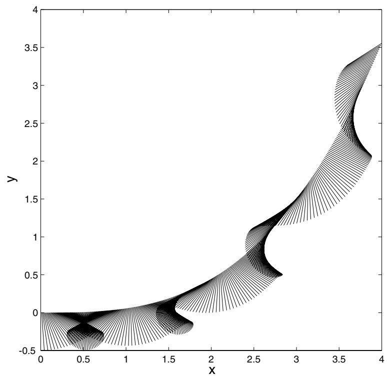


Figure 17.21: Graph of closely spaced configuration of the bar between $t = 0$ to $t = 4$.