SAMPLE 16.13 **A rod in a constant rate circular motion:** A uniform rod of mass m and length ℓ is connected to a motor at end O. A ball of mass m is attached to the rod at end B. The motor turns the rod in the counterclockwise direction at a constant angular speed ω. There is gravity pointing in the $-\hat{\jmath}$ direction. Find the torque applied by the motor (i) at the instant shown and (ii) when $\theta = 0°,\ 90°,\ 180°$. How does the torque change if the angular speed is doubled?

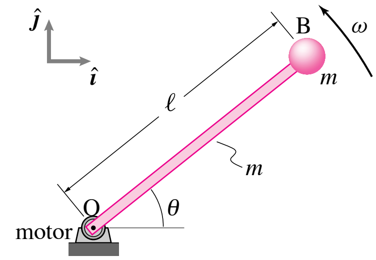


Figure 16.51: A rod goes in circles at a constant rate.

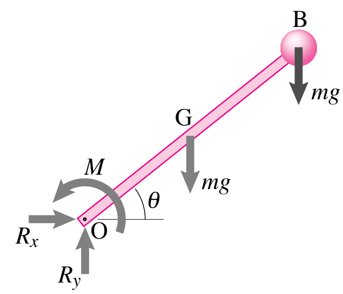
Solution The FBD of the rod and ball system is shown in Fig. 16.52(a). Since the system is undergoing circular motion at a constant speed, the acceleration of the ball as well as every point on the rod is just radial (pointing towards the center of rotation O) and is given by $\vec{a} = -\omega^2 r\hat{\lambda}$ where r is the radial distance from the center O to the point of interest and $\hat{\lambda}$ is a unit vector along OB pointing away from O (Fig. 16.52(b)).

Angular Momentum Balance about point O gives

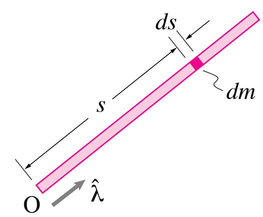
$$\sum \vec{M}_O = \dot{\vec{H}}_{/O}$$

$$\begin{aligned}
\sum \vec{M}_O &= \vec{r}_{G/O} \times (-mg\hat{\jmath}) + \vec{r}_{B/O} \times (-mg\hat{\jmath}) + M\hat{k} \\
&= -\frac{\ell}{2}\cos\theta mg\hat{k} - \ell\cos\theta mg\hat{k} + M\hat{k} \\
&= (M - \frac{3\ell}{2} mg\cos\theta)\hat{k} \qquad (16.48)
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{H}}_{/O} &= \overbrace{\vec{r}_{B/O} \times m\vec{a}_B}^{(\dot{\vec{H}}_{/O})_{\text{ball}}} + \overbrace{\int_m \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm}^{(\dot{\vec{H}}_{/O})_{\text{rod}}} \\
&= \ell\hat{\lambda} \times (-m\omega^2\ell\hat{\lambda}) + \int_m \overbrace{s\hat{\lambda}}^{\vec{r}_{dm/O}} \times \overbrace{(-\omega^2 s\hat{\lambda})}^{\vec{a}_{dm}}\, dm \\
&= \vec{0} \qquad \text{(since } \hat{\lambda} \times \hat{\lambda} = \vec{0}\text{)} \qquad (16.49)
\end{aligned}$$



(a) FBD of rod+ball system



(b) Calculation of $\dot{\vec{H}}_O$ of the rod

Figure 16.52: A rod goes in circles at a constant rate.

(i) Equating (16.48) and (16.49) we get

$$M = \frac{3}{2} mg\ell\cos\theta.$$

$$\boxed{M = \tfrac{3}{2} mg\ell\cos\theta}$$

(ii) Substituting the given values of θ in the above expression we get

$$M(\theta = 0°) = \frac{3}{2}mg\ell, \quad M(\theta = 90°) = 0 \quad M(\theta = 180°) = -\frac{3}{2}mg\ell$$

$$\boxed{M(0°) = \tfrac{3}{2}mg\ell, \quad M(90°) = 0 \quad M(180°) = -\tfrac{3}{2}mg\ell}$$

①

①The values obtained above make sense (at least qualitatively). To make the rod and the ball go up from the 0° position, the motor has to apply some torque in the counterclockwise direction. In the 90° position no torque is required for the *dynamic balance*. In the 180° position the system is accelerating downwards under gravity; therefore, the motor has to apply a clockwise torque to make the system maintain a uniform speed.

It is clear from the expression of the torque that it does not depend on the value of the angular speed ω! Therefore, the torque will not change if the speed is doubled. In fact, as long as the speed remains constant at any value, the only torque required to maintain the motion is the torque to counteract the moments due to gravity at O.

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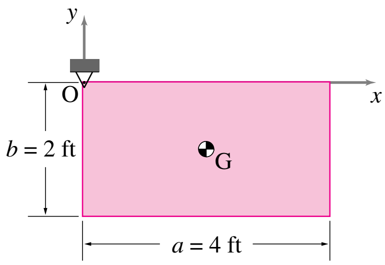

Figure 16.53: A rectangular plate is released from rest from the position shown.

SAMPLE 16.14 At the onset of circular motion: A $2' \times 4'$ rectangular plate of mass 20 lbm is pivoted at one of its corners as shown in the figure. The plate is released from rest in the position shown. Find the force on the support immediately after release.

Solution The free-body diagram of the plate is shown in Fig. 16.54. The force $\vec{F}$ applied on the plate by the support is unknown.

The linear momentum balance for the plate gives

$$\begin{aligned}
\sum \vec{F} &= m\vec{a}_G \\
\vec{F} - mg\hat{\jmath} &= m(\ddot{\theta}\, r_{G/O}\hat{e}_\theta - \dot{\theta}^2 r_{G/O}\hat{e}_R) \\
&= m\,\ddot{\theta}\, r_{G/O}\hat{e}_\theta \quad (\text{since } \dot{\theta} = 0 \text{ at } t = 0).
\end{aligned} \tag{16.50}$$

Thus to find $\vec{F}$ we need to find $\ddot{\theta}$.

The angular momentum balance for the plate about the fixed support point O gives

$$\vec{M}_O = \dot{\vec{H}}_O$$

where

$$\begin{aligned}
\vec{M}_O &= \vec{r}_{G/O} \times mg(-\hat{\jmath}) \\
&= \underbrace{\left(\frac{a}{2}\hat{\imath} - \frac{b}{2}\hat{\jmath}\right)}_{\vec{r}_{G/O}} \times mg(-\hat{\jmath}) = -mg\frac{a}{2}\hat{k},
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{H}}_{/O} &= \dot{\omega}\hat{k}\int_m r^2 dm = \ddot{\theta}\hat{k}\int_0^b\int_0^a \overbrace{(x^2+y^2)}^{r^2}\overbrace{\frac{m}{ab}dxdy}^{dm} \\
&= \frac{m(a^2+b^2)}{3}\ddot{\theta}\hat{k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
-mg\frac{a}{2}\hat{k} &= \frac{m(a^2+b^2)}{3}\ddot{\theta}\hat{k} \\
\Rightarrow \quad \ddot{\theta} &= -\frac{3ga}{2(a^2+b^2)} \\
&= -\frac{3 \cdot 32.2\,\mathrm{ft/s^2} \cdot 4\,\mathrm{ft}}{2(16+4)\,\mathrm{ft}^2} = -9.66\,\mathrm{rad/s^2}.
\end{aligned}$$

From eqn. (16.50), the support force is now readily calculated:

$$\begin{aligned}
\vec{F} &= mg\hat{\jmath} + m\ddot{\theta}\, r_{G/O}\hat{e}_\theta \\
&= mg\hat{\jmath} + m\ddot{\theta}\frac{\sqrt{a^2+b^2}}{2}\,\frac{b\hat{\imath} + a\hat{\jmath}}{\sqrt{(a^2+b^2)}} \\
&= \frac{1}{2}m\ddot{\theta}b\hat{\imath} + (mg + \frac{1}{2}m\ddot{\theta}a)\hat{\jmath}
\end{aligned}$$

Using the given numerical values of m, a, and b, $\ddot{\theta} = -9.66\,\mathrm{rad/s^2}$, and $g = 32.2\,\mathrm{ft/s^2}$, we get

$$\vec{F} = (-6\hat{\imath} + 8\hat{\jmath})\,\mathrm{lbf}.$$

$\boxed{\vec{F} = (-6\hat{\imath} + 8\hat{\jmath})\,\mathrm{lbf}}$

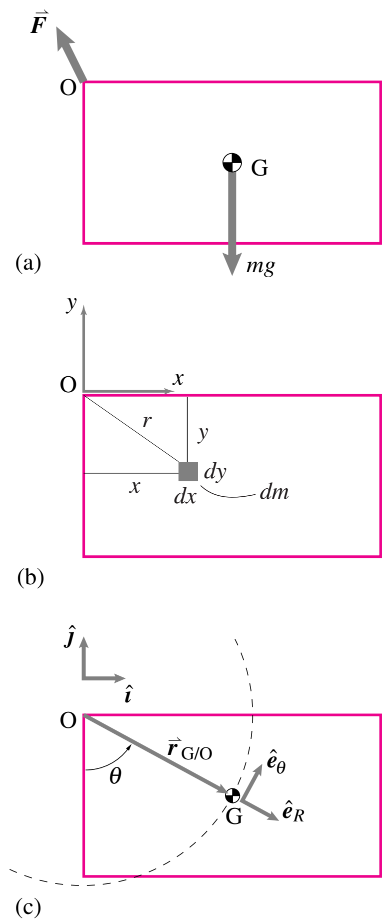

Figure 16.54: (a) The free-body diagram of the plate. (b) Computation of the integral in $\dot{\vec{H}}_{/O} = \dot{\omega}\hat{k}\int_m r^2 dm$. (c) The geometry of motion. From the given dimensions, $\hat{e}_R = \frac{a\hat{\imath} - b\hat{\jmath}}{\sqrt{(a^2+b^2)}}$, $\hat{e}_\theta = \frac{b\hat{\imath} + a\hat{\jmath}}{\sqrt{(a^2+b^2)}}$, and $r_{G/O} = \frac{\sqrt{a^2+b^2}}{2}$.

SAMPLE 16.15 **A compound gear train.** When the gear of an input shaft, often called the *driver* or the *pinion*, is directly meshed in with the gear of an output shaft, the motion of the output shaft is opposite to that of the input shaft. To get the output motion in the same direction as that of the input motion, an *idler* gear is used. If the idler shaft has more than one gear in mesh, then the gear train is called a *compound gear train*.

In the gear train shown in Fig. 16.55, the input shaft is rotating at 2000 rpm and the input torque is 200 N-m. The efficiency (defined as the ratio of output power to input power) of the train is 0.96 and the various radii of the gears are: $R_A = 5\,\text{cm}$, $R_B = 8\,\text{cm}$, $R_C = 4\,\text{cm}$, and $R_D = 10\,\text{cm}$. Find

1. the input power P_{in} and the output power P_{out},
2. the output speed ω_{out}, and
3. the output torque.

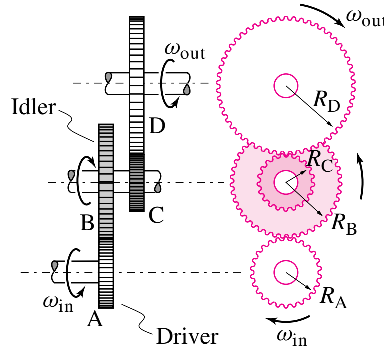

Figure 16.55: A compound gear train.

Solution

1. The power:

$$\begin{aligned} P_{\text{in}} &= M_{\text{in}}\omega_{\text{in}} = 200\,\text{N}\cdot\text{m}\cdot 2000\,\text{rpm} \\ &= 400000\,\text{N}\cdot\text{m}\cdot\frac{\cancel{\text{rev}}}{\cancel{\text{min}}}\cdot\frac{2\pi}{1\,\cancel{\text{rev}}}\cdot\frac{1\,\cancel{\text{min}}}{60\,\text{s}} \\ &= 41887.9\,\text{N m/s} \approx 42\,\text{kW}. \\ \Rightarrow\quad p_{\text{out}} &= \text{efficiency}\cdot P_{\text{in}} = 0.96\cdot 42\,\text{kW} \approx 40\,\text{kW} \end{aligned}$$

$$\boxed{P_{\text{in}} = 42\,\text{kW}, \quad p_{\text{out}} = 40\,\text{kW}}$$

2. The angular speed of meshing gears can be easily calculated by realizing that the linear speed of the point of contact has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus,

$$\begin{aligned} v_P &= \omega_{\text{in}} R_A = \omega_B R_B \\ \Rightarrow\quad \omega_B &= \omega_{\text{in}}\cdot\frac{R_A}{R_B} \\ \text{and}\quad v_R &= \omega_C R_C = \omega_{\text{out}} R_D \\ \Rightarrow\quad \omega_{\text{out}} &= \omega_C\cdot\frac{R_C}{R_D} \\ \text{But}\quad \omega_C &= \omega_B \\ \Rightarrow\quad \omega_{\text{out}} &= \omega_{\text{in}}\cdot\frac{R_A}{R_B}\cdot\frac{R_C}{R_D} \\ &= 2000\,\text{rpm}\cdot\frac{5}{8}\cdot\frac{4}{10} = 500\,\text{rpm}. \end{aligned}$$

$$\boxed{\omega_{\text{out}} = 500\,\text{rpm}}$$

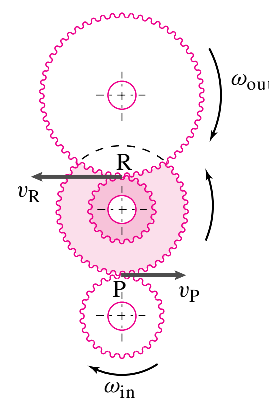

Figure 16.56:

3. The output torque,

$$\begin{aligned} M_{\text{out}} = \frac{P_{\text{out}}}{\omega_{\text{out}}} &= \frac{40\,\text{kW}}{500\,\text{rpm}} = \frac{40}{500}\cdot 1000\frac{\text{N}\cdot\text{m}}{\cancel{\text{s}}}\cdot\frac{\cancel{\text{min}}}{\cancel{\text{rev}}}\cdot\frac{1\,\cancel{\text{rev}}}{2\pi}\cdot\frac{60\,\cancel{\text{s}}}{1\,\cancel{\text{min}}} \\ &= 764\,\text{N}\cdot\text{m}. \end{aligned}$$

$$\boxed{M_{\text{out}} = 764\,\text{N}\cdot\text{m}}$$

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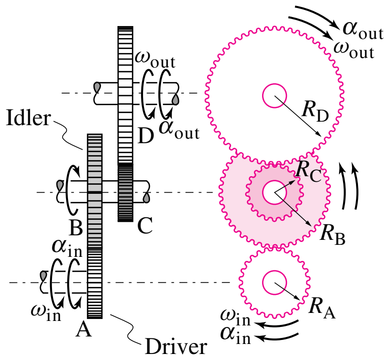

Figure 16.57: An accelerating compound gear train.

SAMPLE 16.16 An accelerating gear train. In the gear train shown in Fig. 16.57, the torque at the input shaft is $M_{in} = 200\,\text{N·m}$ and the angular acceleration is $\alpha_{in} = 50\,\text{rad/s}^2$. The radii of the various gears are: $R_A = 5\,\text{cm}$, $R_B = 8\,\text{cm}$, $R_C = 4\,\text{cm}$, and $R_D = 10\,\text{cm}$ and the moments of inertia about the shaft axis passing through their respective centers are: $I_A = 0.1\,\text{kg m}^2$, $I_{BC} = 5I_A$, $I_D = 4I_A$. Find the output torque M_{out} of the gear train.

Solution Since the difference between the input power and the output power is used in accelerating the gears, we may write

$$P_{in} - P_{out} = \dot{E}_K$$

Let M_{out} be the output torque of the gear train. Then,

$$P_{in} - P_{out} = M_{in}\,\omega_{in} - M_{out}\,\omega_{out}. \tag{16.51}$$

Now,

$$\begin{aligned}
\dot{E}_K &= \frac{d}{dt}(E_K) && (16.52)\\
&= \frac{d}{dt}\left(\frac{1}{2}I_A\,\omega_{in}^2 + \frac{1}{2}I_{BC}\,\omega_{BC}^2 + \frac{1}{2}I_D\,\omega_{out}^2\right)\\
&= I_A\,\omega_{in}\,\dot{\omega}_{in} + I_{BC}\,\omega_{BC}\,\dot{\omega}_{BC} + I_D\,\omega_{out}\,\dot{\omega}_{out}\\
&= I_A\,\omega_{in}\,\alpha_{in} + 5I_A\,\omega_{BC}\,\alpha_{BC} + 4I_A\,\omega_{out}\,\alpha_{out}. && (16.53)
\end{aligned}$$

The different ω's and the α's can be related by realizing that the linear speed or the tangential acceleration of the point of contact between any two meshing gears has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus, using the linear speed and tangential acceleration calculations for points P and R in Fig. 16.58, we can find ω_B, α_B and ω_{out}, α_{out}. Considering the linear speed of point P, we find

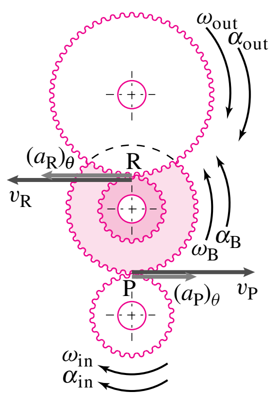

Figure 16.58: The velocity or acceleration of the point of contact between two meshing gears has to be the same irrespective of which meshing gear's geometry and motion is used to compute them.

$$\begin{aligned}
v_P &= \omega_{in}\,R_A = \omega_B\,R_B\\
\Rightarrow\quad \omega_B &= \omega_{in}\cdot\frac{R_A}{R_B},
\end{aligned}$$

and considering the tangential acceleration of point P, $(a_P)_\theta$, we find

$$\begin{aligned}
(a_P)_\theta &= \alpha_{in}\,R_A = \alpha_B\,R_B\\
\Rightarrow\quad \alpha_B &= \alpha_{in}\cdot\frac{R_A}{R_B}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
v_R &= \omega_C\,R_C = \omega_{out}\,R_D\\
\Rightarrow\quad \omega_{out} &= \omega_C\cdot\frac{R_C}{R_D},
\end{aligned}$$

and

$$\begin{aligned}
(a_R)_\theta &= \alpha_C\,R_C = \alpha_{out}\,R_D\\
\Rightarrow\quad \alpha_{out} &= \alpha_C\cdot\frac{R_C}{R_D}.
\end{aligned}$$

But

$$\begin{aligned}
\omega_C &= \omega_B = \omega_{BC}\\
\Rightarrow\quad \omega_{out} &= \omega_{in}\cdot\frac{R_A}{R_B}\cdot\frac{R_C}{R_D}
\end{aligned}$$

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and

$$
\begin{aligned}
\alpha_C &= \alpha_B = \alpha_{BC} \\
\Rightarrow \quad \alpha_{\text{out}} &= \alpha_{\text{in}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D}.
\end{aligned}
$$

Substituting these expressions for ω_{out}, α_{out}, ω_{BC} and α_{BC} in equations (16.51) and (16.53), we get

$$
\begin{aligned}
P_{\text{in}} - P_{\text{out}} &= M_{\text{in}}\,\omega_{\text{in}} - M_{\text{out}}\omega_{\text{in}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \\
&= \omega_{\text{in}} \left(M_{\text{in}} - M_{\text{out}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right). \\
\dot{E}_{\text{K}} &= I_A \left[\omega_{\text{in}}\alpha_{\text{in}} + 5\omega_{\text{in}}\alpha_{\text{in}} \left(\frac{R_A}{R_B} \right)^2 + 4\omega_{\text{in}}\alpha_{\text{in}} \left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right] \\
&= I_A\omega_{\text{in}} \left[\alpha_{\text{in}} + 5\alpha_{\text{in}} \left(\frac{R_A}{R_B} \right)^2 + 4\alpha_{\text{in}} \left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right].
\end{aligned}
$$

Now equating the two quantities, $P_{\text{in}} - P_{\text{out}}$ and $\dot{E}_{\text{K}}$, and canceling ω_{in} from both sides, we obtain

$$
\begin{aligned}
M_{\text{out}} \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} &= M_{\text{in}} - I_A\alpha_{\text{in}} \left[1 + 5\left(\frac{R_A}{R_B} \right)^2 + 4\left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right] \\
M_{\text{out}} \frac{5}{8} \cdot \frac{4}{10} &= 200\,\text{N}\cdot\text{m} - 5\,\text{kg m}^2 \cdot \text{rad/s}^2 \left[1 + 5\left(\frac{5}{8} \right)^2 + 4\left(\frac{5}{8} \cdot \frac{4}{10} \right)^2 \right] \\
M_{\text{out}} &= 735.94\,\text{N}\cdot\text{m} \\
&\approx 736\,\text{N}\cdot\text{m}.
\end{aligned}
$$

$$M_{\text{out}} = 736\,\text{N}\cdot\text{m}$$

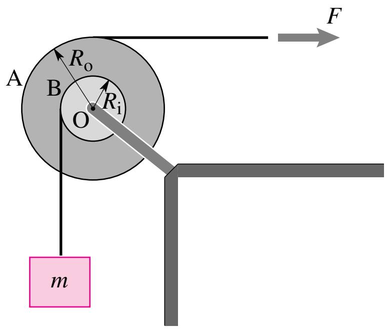

Figure 16.59: Two drums with strings wrapped around are used to pull up a mass m.

SAMPLE 16.17 Drums used as pulleys. Two drums, A and B of radii $R_o = 200\,\mathrm{mm}$ and $R_i = 100\,\mathrm{mm}$ are welded together. The combined mass of the drums is $m_D = 20\,\mathrm{kg}$ and the combined moment of inertia about the z-axis passing through their common center O is $I_{zz/O} = 1.6\,\mathrm{kg\ m^2}$. A string attached to and wrapped around drum B supports a mass $m = 2\,\mathrm{kg}$. The string wrapped around drum A is pulled with a force $F = 20\,\mathrm{N}$ as shown in Fig. 16.59. Assume there is no slip between the strings and the drums. Find

1. the angular acceleration of the drums,
2. the tension in the string supporting mass m, and
3. the acceleration of mass m.

Solution The free-body diagram of the drums and the mass are shown in Fig. 16.60 separately where T is the tension in the string supporting mass m and O_x and O_y are the support reactions at O. Since the drums can only rotate about the z-axis, let

$$\vec{\omega} = \omega\hat{\boldsymbol{k}} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{k}}.$$

Now, let us do angular momentum balance about the center of rotation O:

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$$

$$\begin{aligned}\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= TR_i\hat{\boldsymbol{k}} - FR_o\hat{\boldsymbol{k}} \\ &= (TR_i - FR_o)\hat{\boldsymbol{k}}.\end{aligned}$$

Since the motion is restricted to the xy-plane (*i.e.,* 2-D motion), the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ may be computed as

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= I_{zz/cm}\,\dot{\omega}\hat{\boldsymbol{k}} + \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{a}}_{cm}\ m_D \\ &= I_{zz/O}\,\dot{\omega}\hat{\boldsymbol{k}} + \underbrace{\vec{\boldsymbol{r}}_{O/O}}_{0} \times \underbrace{\vec{\boldsymbol{a}}_{cm}}_{0}\ m_D \\ &= I_{zz/O}\,\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

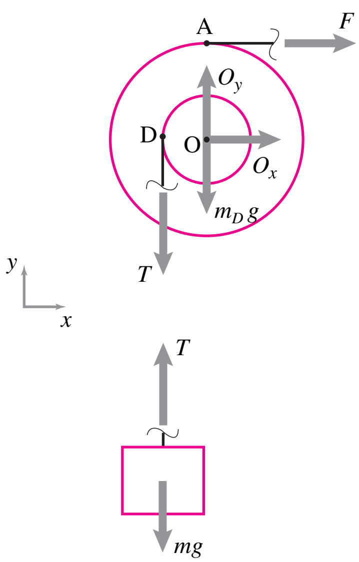

Figure 16.60: Free-body diagram of the drums and the mass m. T is the tension in the string supporting mass m and O_x and O_y are the reactions of the support at O.

Setting $\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ we get

$$TR_i - FR_o = I_{zz/O}\,\dot{\omega}. \tag{16.54}$$

Now, let us write linear momentum balance, $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, for mass m:

$$\underbrace{(T - mg)\hat{\boldsymbol{\jmath}}}_{\sum \vec{\boldsymbol{F}}} = m\vec{\boldsymbol{a}}.$$

Do we know anything about acceleration $\vec{\boldsymbol{a}}$ of the mass? Yes, we know its direction ($\pm\hat{\boldsymbol{\jmath}}$) and we also know that it has to be the same as the tangential acceleration $(\vec{\boldsymbol{a}}_D)_\theta$ of point D on drum B (why?). Thus,

$$\begin{aligned}\vec{\boldsymbol{a}} &= (\vec{\boldsymbol{a}}_D)_\theta \\ &= \dot{\omega}\hat{\boldsymbol{k}} \times (-R_i\hat{\boldsymbol{\imath}}) \\ &= -\dot{\omega}R_i\hat{\boldsymbol{\jmath}}.\end{aligned} \tag{16.55}$$

Therefore,

$$T - mg = -m\dot{\omega}R_i. \tag{16.56}$$

1. **Calculation of $\dot{\omega}$:** We now have two equations, (16.54) and (16.56), and two unknowns, $\dot{\omega}$ and T. Subtracting R_i times Eqn.(16.56) from Eqn. (16.54) we get

$$\begin{aligned}
-FR_o + mgR_i &= (I_{zz/O} + mR_i^2)\dot{\omega} \\
\Rightarrow \quad \dot{\omega} &= \frac{-FR_o + mgR_i}{(I_{zz/O} + mR_i^2)} \\
&= \frac{-20\,\text{N}\cdot 0.2\,\text{m} + 2\,\text{kg}\cdot 9.81\,\text{m/s}^2\cdot 0.1m}{1.6\,\text{kg m}^2 + 2\,\text{kg}\cdot(0.1\,\text{m})^2} \\
&= \frac{-2.038\,\text{kg m}^2/\text{s}^2}{1.62\,\text{kg m}^2} \\
&= -1.258\frac{1}{\text{s}^2}
\end{aligned}$$

$$\boxed{\dot{\vec{\omega}} = -1.26\,\text{rad/s}^2\hat{\boldsymbol{k}}}$$

2. **Calculation of tension T:** From equation (16.56):

$$\begin{aligned}
T &= mg - m\dot{\omega}R_i \\
&= 2\,\text{kg}\cdot 9.81\,\text{m/s}^2 - 2\,\text{kg}\cdot(-1.26\,\text{s}^{-2})\cdot 0.1\,\text{m} \\
&= 19.87\,\text{N}
\end{aligned}$$

$$\boxed{T = 19.87\,\text{N}}$$

3. **Calculation of acceleration of the mass:** Since the acceleration of the mass is the same as the tangential acceleration of point D on the drum, we get (from eqn. (16.55))

$$\begin{aligned}
\vec{\boldsymbol{a}} &= (\vec{\boldsymbol{a}}_D)_\theta = -\dot{\omega}R_i\hat{\boldsymbol{j}} \\
&= -(-1.26\,\text{s}^{-2})\cdot 0.1\,\text{m} \\
&= 0.126\,\text{m/s}^2\,\hat{\boldsymbol{j}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = 0.13\,\text{m/s}^2\,\hat{\boldsymbol{j}}}$$

Comments: It is important to understand why the acceleration of the mass is the same as the tangential acceleration of point D on the drum. We have assumed (as is common practice) that the string is massless and inextensible. Therefore each point of the string supporting the mass must have the same linear displacement, velocity, and acceleration as the mass. Now think about the point on the string which is momentarily in contact with point D of the drum. Since there is no relative slip between the drum and the string, the two points must have the same vertical acceleration. This vertical acceleration for point D on the drum is the tangential acceleration $(\vec{\boldsymbol{a}}_D)_\theta$.

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SAMPLE 16.18 Energy method for pulley dynamics: Consider the pulley problem of Sample 16.17 again. Use energy method to

1. find the angular acceleration of the pulley, and
2. the acceleration of the mass.

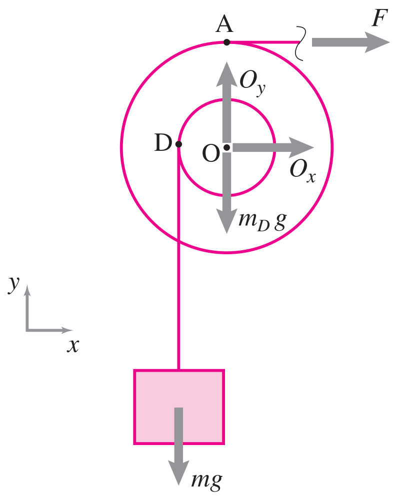

Figure 16.61: Free-body-diagram of the pulley-mass system together. Note that the forces acting at the support point O do no work since point O is stationary.

Solution In energy method we use speeds, not velocities. Therefore, we have to be careful in our thinking about the direction of motion. In the present problem, let us assume that the pulley rotates and accelerates clockwise. Consequently, the mass moves up against gravity.

1. The energy equation we want to use is

$$P = \dot{E}_K.$$

The power P is given by $P = \sum \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}_i$ where the sum is carried out over all external forces. For the mass and pulley system the external forces that do work are F and mg. The other external forces on the system—the reaction force of the support point O and the weight of the pulley—are acting at point O (see fig. 16.61). But, since point O is stationary, these forces do no work. Therefore,

$$\begin{aligned} P &= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_A + m\vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{v}}_m \\ &= F\hat{\boldsymbol{\imath}} \cdot v_A\hat{\boldsymbol{\imath}} + (-mg\hat{\boldsymbol{\jmath}}) \cdot \underbrace{v_D\hat{\boldsymbol{\jmath}}}_{\vec{\boldsymbol{v}}_m} \\ &= Fv_A - mgv_D. \end{aligned} \tag{16.57}$$

Now we need to calculate the rate of change of kinetic energy $\dot{E}_K$. There are two objects here that have kinetic energy—the hanging mass and the pulley. Hence,

$$\dot{E}_K = (\dot{E}_K)_m + (\dot{E}_K)_{\text{pulley}}.$$

The hanging mass has pure translational motion and hence its kinetic energy is

$$(E_K)_m = \frac{1}{2}mv^2$$

where v is the linear speed of the mass. If we assume the string to be inextensible, then the linear speed v of the mass has to be the same as the tangential speed of point D of the pulley. Thus $v = v_D$ and

$$(E_K)_m = \frac{1}{2}mv_D^2.$$

The pulley, on the other hand, has pure rotational motion about point O, and hence its kinetic energy is given by

$$(E_K)_{\text{pulley}} = \frac{1}{2}I_{zz}^{\text{O}}\omega^2.$$

Summing the two kinetic energies and differentiating with respect to time t, we get

$$\dot{E}_K = \frac{d}{dt}\left(\frac{1}{2}\, m\, v_D^2 + \frac{1}{2}I_{zz}^{\text{O}}\omega^2\right) \tag{16.58}$$

$$= m\, v_D\, \dot{v}_D + I_{zz}^{\text{O}}\, \omega\, \dot{\omega}. \tag{16.59}$$

Now equating the power and the rate of change of kinetic energy from eqns. eqn. (16.57) and eqn. (16.59), we get

$$F\, v_A - mg\, v_D = m\, v_D\, \dot{v}_D + I_{zz}^{\text{O}}\, \omega\, \dot{\omega}.$$

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From kinematics of circular motion,

$$\begin{aligned} v_A &= \omega R_o, \\ v_D &= \omega R_i \\ \text{and} \qquad \dot{v}_D \equiv (a_D)_\theta &= \dot{\omega} R_i. \end{aligned}$$

Substituting these values in the power balance equation above, we get

$$\begin{aligned} \omega(FR_o - mgR_i) &= \omega\,\dot{\omega}(mR_i^2 + I_{zz}^{\text{O}}) \\ \Rightarrow \quad \dot{\omega} &= \frac{FR_o - mgR_i}{(I_{zz}^{\text{O}} + mR_i^2)} \\ &= \frac{20\,\text{N}\cdot 0.2\,\text{m} - 2\,\text{kg}\cdot 9.81\,\text{m/s}^2\cdot 0.1m}{1.6\,\text{kg m}^2 + 2\,\text{kg}\cdot(0.1\,\text{m})^2} \\ &= 1.258\frac{1}{\text{s}^2}. \qquad \text{(same as the answer before.)} \end{aligned}$$

Since the sign of $\dot{\omega}$ is positive, our initial assumption of clockwise acceleration of the pulley is correct.

$$\boxed{\dot{\omega} = 1.26\,\text{rad/s}^2}$$

2. From kinematics of circular motion,

$$\begin{aligned} a_m &= (a_D)_\theta \\ &= \dot{\omega} R_i \\ &= 0.126\,\text{m/s}^2. \end{aligned}$$

$$\boxed{a_m = 0.13\,\text{m/s}^2}$$

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SAMPLE 16.19 Energy Accounting: Consider the pulley problem of Sample 16.17 again.

1. What percentage of the input energy (work done by the applied force F) is used in raising the mass by 1 m?
2. Where does the rest of the energy go? Provide an energy-balance sheet.

Solution

1. Let W_i and W_h be the input energy and the energy used in raising the mass by 1 m, respectively. Then the percentage of energy used in raising the mass is

$$\%\text{ of input energy used} = \frac{W_h}{W_i} \times 100.$$

Thus we need to calculate W_i and W_h to find the answer. W_i is the work done by the force F on the system during the interval in which the mass moves up by 1 m. Let s be the displacement of the force F during this interval. Since the displacement is in the same direction as the force (we know it is from Sample 16.17), the input-energy is

$$W_i = F\,s.$$

So to find W_i we need to find s.
For the mass to move up by 1 m the inner drum B must rotate by an angle θ where

$$1\,\text{m} = \theta\,R_i \qquad \Rightarrow \qquad \theta = \frac{1\,\text{m}}{0.1\,\text{m}} = 10\,\text{rad}.$$

Since the two drums, A and B, are welded together, drum A must rotate by θ as well. Therefore the displacement of force F is

$$s = \theta\,R_o = 10\,\text{rad} \cdot 0.2\,\text{m} = 2\,\text{m},$$

and the energy input is

$$W_i = F\,s = 20\,\text{N} \cdot 2\,\text{m} = 40\text{J}.$$

Now, the work done in raising the mass by 1 m is

$$W_h = mgh = 2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot 1\,\text{m} = 19.62\text{J}.$$

Therefore, the percentage of input-energy used in raising the mass

$$= \frac{19.62\,\text{N·m}}{40} \times 100 = 49.05\% \approx 49\%.$$

2. The rest of the energy ($= 51\%$) goes in accelerating the mass and the pulley. Let us find out how much energy goes into each of these activities. Since the initial state of the system from which we begin energy accounting is not prescribed (that is, we are not given the height of the mass from which it is to be raised 1 m, nor do we know the velocities of the mass or the pulley at that initial height), let us assume that at the initial state, the angular speed of the pulley is ω_o and the linear speed of the mass is v_o. At the end of raising the mass by 1 m from this state, let the angular speed of the pulley be ω_f and the linear speed of the mass be v_f. Then, the energy

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used in accelerating the pulley is

$$
\begin{aligned}
(\Delta E_K)_{\text{pulley}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
&= \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_o^2 \\
&= \frac{1}{2} I \underbrace{(\omega_f^2 - \omega_o^2)}
\end{aligned}
$$

assuming constant acceleration, $\omega_f^2 = \omega_o^2 + 2\alpha\theta$, or $\omega_f^2 - \omega_o^2 = 2\alpha\theta$.

$$
\begin{aligned}
&= I\alpha\,\theta \qquad (\text{from Sample 16.18, } \alpha = 1.258\,\text{rad/s}^2.\ ) \\
&= 1.6\,\text{kg m}^2 \cdot 1.258\,\text{rad/s}^2 \cdot 10\,\text{rad} \\
&= 20.13\,\text{N·m} = 20.13\,\text{J}.
\end{aligned}
$$

Similarly, the energy used in accelerating the mass is

$$
\begin{aligned}
(\Delta E_K)_{\text{mass}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
&= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_o^2 \\
&= \frac{1}{2} m \underbrace{(v_f^2 - v_o^2)}_{2ah} \\
&= mah \\
&= 2\,\text{kg} \cdot 0.126\,\text{m/s}^2 \cdot 1\,\text{m} \\
&= 0.25\,\text{J}.
\end{aligned}
$$

We can calculate the percentage of input energy used in these activities to get a better idea of energy allocation. Here is the summary table:

Activities	**Energy Spent**	
	in Joule	as % of input energy
In raising the mass by 1 m	19.62	49.05%
In accelerating the mass	0.25	0.62 %
In accelerating the pulley	20.13	50.33 %
Total	40.00	100 %

So, what would you change in the set-up so that more of the input energy is used in raising the mass? Think about what aspects of the motion would change due to your proposed design.

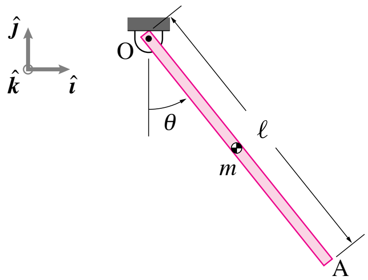

Figure 16.62: A uniform rod swings in the plane about its pinned end O.

SAMPLE 16.20 Equation of motion of a swinging stick: A uniform bar of mass m and length ℓ is pinned at one of its ends O. The bar is displaced from its vertical position by an angle θ and released (Fig. 16.62).

1. Find the equation of motion using momentum balance.
2. Find the reaction at O as a function of $(\theta,\ \dot{\theta},\ g,\ m,\ \ell)$.

Solution First we draw a simple sketch of the given problem showing relevant geometry (Fig. 16.62(a)), and then a free-body diagram of the bar (Fig. 16.62(b)).

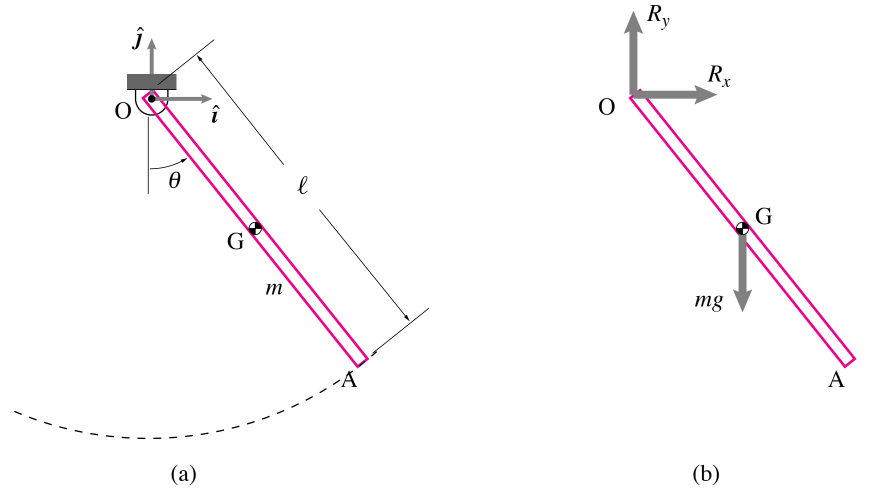

Figure 16.63: (a) A line sketch of the swinging rod and (b) free-body diagram of the rod.

We should note for future reference that

$$\begin{aligned} \vec{\omega} &= \omega\hat{\boldsymbol{k}} \equiv \dot{\theta}\hat{\boldsymbol{k}} \\ \dot{\vec{\omega}} &= \dot{\omega}\hat{\boldsymbol{k}} \equiv \ddot{\theta}\hat{\boldsymbol{k}} \end{aligned}$$

1. **Equation of motion using momentum balance:** We can write angular momentum balance about point O as

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}.$$

Let us now calculate both sides of this equation:

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \vec{\boldsymbol{r}}_{G/\mathrm{O}} \times mg(-\hat{\boldsymbol{\jmath}}) \\ &= \frac{\ell}{2}(\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\,\hat{\boldsymbol{\jmath}}) \times mg(-\hat{\boldsymbol{\jmath}}) \\ &= -\frac{\ell}{2}mg\,\sin\theta\hat{\boldsymbol{k}}. \qquad (16.60) \\ \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \dot{\omega}\hat{\boldsymbol{k}}\int_m r^2 dm \\ &= \ddot{\theta}\hat{\boldsymbol{k}}\int_0^{\ell} s^2\frac{m}{\ell}ds \quad (dm = m/\ell \cdot ds) \\ &= \frac{m\ddot{\theta}}{\ell}\left[\left.\frac{s^3}{3}\right|_0^{\ell}\right] = \frac{m\ell^2}{3}\ddot{\theta}\hat{\boldsymbol{k}} \qquad (16.61) \end{aligned}$$

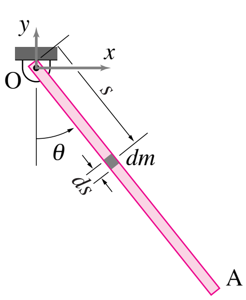

Figure 16.64: Computation of $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ by integration over the rod.

Equating (16.60) and (16.61) we get

$$\begin{aligned} -\frac{\ell}{2}\cancel{m}\, g\, \sin\theta &= \cancel{m}\frac{\ell^2}{3}\ddot{\theta} \\ \text{or} \qquad \dot{\omega} + \frac{3g}{2\ell}\sin\theta &= 0 \\ \text{or} \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \end{aligned} \tag{16.62}$$

$$\boxed{\ddot{\theta} + \tfrac{3g}{2\ell}\sin\theta = 0}$$

2. **Reaction at O:** Using linear momentum balance

$$\begin{aligned} \sum \vec{F} &= m\,\vec{a}_G, \\ \text{where} \quad \sum \vec{F} &= R_x\hat{\imath} + (R_y - mg)\hat{\jmath}, \\ \text{and} \quad \vec{a}_G &= \frac{\ell}{2}\dot{\omega}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + \frac{\ell}{2}\omega^2(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) \\ &= \frac{\ell}{2}[(\dot{\omega}\cos\theta - \omega^2\sin\theta)\hat{\imath} + (\dot{\omega}\sin\theta + \omega^2\cos\theta)\hat{\jmath}]. \end{aligned}$$

Dotting both sides of $\sum \vec{F} = m\,\vec{a}_G$ with $\hat{\imath}$ and $\hat{\jmath}$ and rearranging, we get

$$\begin{aligned} R_x &= m\frac{\ell}{2}(\dot{\omega}\cos\theta - \omega^2\sin\theta) \\ &\equiv m\frac{\ell}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \\ R_y &= mg + m\frac{\ell}{2}(\dot{\omega}\sin\theta + \omega^2\cos\theta) \\ &\equiv mg + m\frac{\ell}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \end{aligned}$$

Now substituting the expression for $\ddot{\theta}$ from (16.62) in R_x and R_y, we get

$$R_x = -m\sin\theta\left(\frac{3}{4}g\cos\theta + \frac{\ell}{2}\dot{\theta}^2\right), \tag{16.63}$$

$$R_y = mg\left(1 - \frac{3}{4}\sin^2\theta\right) + m\frac{\ell}{2}\dot{\theta}^2\cos\theta. \tag{16.64}$$

$$\boxed{\vec{R} = -m(\tfrac{3}{4}g\cos\theta + \tfrac{\ell}{2}\dot{\theta}^2)\sin\theta\hat{\imath} + [mg(1 - \tfrac{3}{4}\sin^2\theta) + m\tfrac{\ell}{2}\dot{\theta}^2\cos\theta]\hat{\jmath}}$$

Check: We can check the reaction force in the special case when the rod does not swing but just hangs from point O. The forces on the bar in this case have to satisfy static equilibrium. Therefore, the reaction at O must be equal to mg and directed vertically upwards. Plugging $\theta = 0$ and $\dot{\theta} = 0$ (no motion) in Eqn. (16.63) and (16.64) we get $R_x = 0$ and $R_y = mg$, the values we expect.

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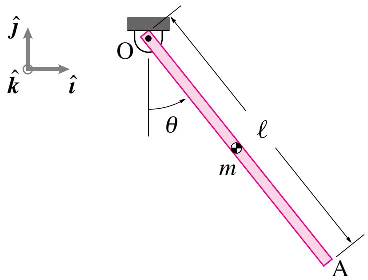

Figure 16.65: A uniform rod swings in the plane about its pinned end O.

SAMPLE 16.21 Swinging stick dynamics using moment of inertia: A uniform bar of mass m and length ℓ is pinned at one of its ends O. The bar is displaced from its vertical position by an angle θ and released (Fig. 16.65). Find the equation of motion of the stick.

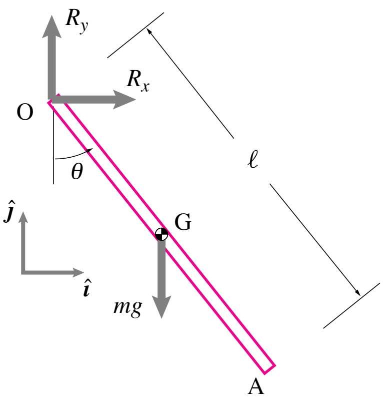

Figure 16.66: The free-body diagram of the rod.

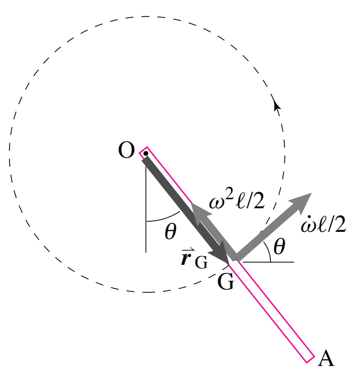

Figure 16.67: Radial and tangential components of $\vec{a}_G$. Since the radial component is parallel to $\vec{r}_G$, $\vec{r}_G \times \vec{a}_G = \frac{\ell^2}{4}\dot{\omega}\hat{k}$.

Solution We repeat the problem solved in Sample 16.20 here with just one different step of finding the rate of change of angular momentum with the help of moment of inertia formula. As usual, we first draw a free-body diagram of the bar (Fig. 16.66). We assume, $\vec{\omega} = \omega\hat{k} \equiv \dot{\theta}\hat{k}$, and $\dot{\vec{\omega}} = \dot{\omega}\hat{k} \equiv \ddot{\theta}\hat{k}$ We can write angular momentum balance about point O as

$$\sum \vec{M}_O = \dot{\vec{H}}_{/O}.$$

Let us now calculate both sides of this equation:

$$\begin{aligned}
\sum \vec{M}_O &= \vec{r}_{G/O} \times mg(-\hat{\jmath}) \\
&= \frac{\ell}{2}(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \times mg(-\hat{\jmath}) \\
&= -\frac{\ell}{2}mg\sin\theta\hat{k}. && (16.65) \\
\dot{\vec{H}}_{/O} &= I_{zz/G}\,\dot{\vec{\omega}} + \vec{r}_G \times m\vec{a}_G \\
&= \frac{m\ell^2}{12}\dot{\omega}\hat{k} + \vec{r}_G \times m\overbrace{(\dot{\omega}\hat{k} \times \vec{r}_G - \omega^2\vec{r}_G)}^{\vec{a}_G} \\
&= \frac{m\ell^2}{12}\dot{\omega}\hat{k} + \frac{m\ell^2}{4}\dot{\omega}\hat{k} && (16.66) \\
&= \frac{m\ell^2}{3}\dot{\omega}\hat{k} && (16.67)
\end{aligned}$$

where the last step, $\vec{r}_G \times m\vec{a}_G = \frac{m\ell^2}{4}\dot{\omega}\hat{k}$, should be clear from Fig. 16.67. Equating (16.60) and (16.61) we get

$$\begin{aligned}
-\frac{\ell}{2}\cancel{m}g\sin\theta &= \cancel{m}\frac{\ell^2}{3}\dot{\omega} \\
\text{or}\quad \dot{\omega} + \frac{3g}{2\ell}\sin\theta &= 0 \\
\text{or}\quad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. && (16.68)
\end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0}$$

SAMPLE 16.22 Numerical solution of the swinging stick motion: For the swinging stick considered in Samples 16.20 or 16.23, find the time that the rod takes to fall from $\theta = \pi/2$ to $\theta = 0$ if it is released from rest at $\theta = \pi/2$?

Solution The given initial angle $\pi/2$ is a big value of θ – big in that we cannot assume $\sin\theta \approx \theta$ (obviously $1 \neq 1.5708$). Therefore we may not use the linearized equation (16.71) to solve for t explicitly. We have to solve the full nonlinear equation (16.68) to find the required time. Unfortunately, we cannot get a *closed form* solution of this equation using mathematical skills you have at this level. Therefore, we resort to numerical integration of this equation.

For numerical integration, we need to first write the given differential equation as a set of first order ordinary differential equations. To do so, we introduce ω as a new variable and rewrite eqn. (16.68) as

$$\dot{\theta} = \omega \quad (16.69)$$

$$\dot{\omega} = -\frac{3g}{2\ell}\sin\theta \quad (16.70)$$

Now we need to specify the initial conditions and the time duration for integration, and solve the equations using some ODE solver program. Here is a pseudo-code that lists the steps:

```
g = 9.81,   L = 1      % define constants
ODES = { thetadot = omega
         omegadot = -3*g/(2*L) * sin(theta) }
 ICs = { theta_0 = pi/2
         omega_0 = 0 }
solve ODES with ICs for t = 0 to 4 s
plot theta vs t and plot omega vs t
```

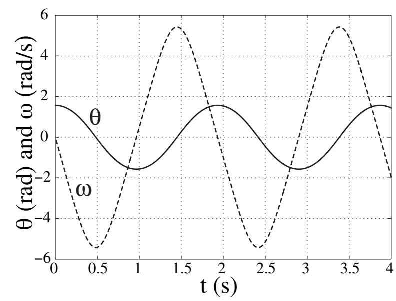

Figure 16.68: Numerical solution is shown by plotting θ and ω against time.

The results obtained from the numerical solution are shown in Fig. 16.68.

The problem of finding the time taken by the bar to fall from $\theta = \pi/2$ to $\theta = 0$ numerically is nontrivial. It is called a *boundary value problem*. We have only illustrated how to solve *initial value problems*. However, we can get a fairly good estimate of the time from the solution obtained.

We first plot θ against time t as shown in fig. 16.69. We find the values of t and the corresponding values of θ that bracket $\theta = 0$. Now, we can use linear interpolation to find the value of t at $\theta = 0$. Proceeding this way, we get $t = 0.48$ (seconds), a little more than we get from the linear ODE in sample 16.23 ($t = 0.41$).

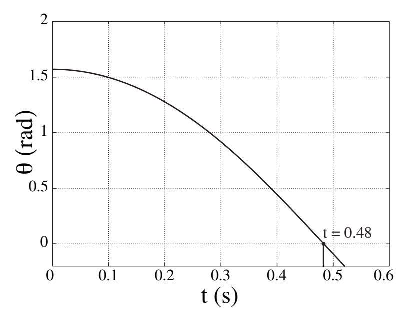

Figure 16.69: Time t corresponding to $\theta = 0$.

Comments: Additionally, we can also get the value of $\dot{\theta} \equiv \omega$ when $\theta = 0$ using similar interpolation. In fact, from the ω vs t plot, we find that at $t = 0.48\,\text{s}$, $\omega = -5.42\,\text{rad/s}$. How does this result compare with the analytical value of ω from sample 16.23 (which did not depend on the small angle approximation)? Well, we found that

$$\omega = -\sqrt{\frac{3g}{\ell}} = -\sqrt{\frac{3 \cdot 9.81\,\text{m/s}^2}{1\,\text{m}}} = -5.4249\,\text{s}^{-1}.$$

Thus, we get a fairly accurate value from numerical integration.

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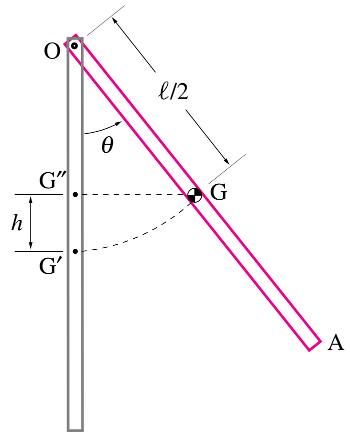


Figure 16.70: Work done by the force of gravity in moving from G′ to G $\int \vec{F} \cdot d\vec{r} = -mg\hat{\boldsymbol{\jmath}} \cdot h\hat{\boldsymbol{\jmath}} = -mgh$.

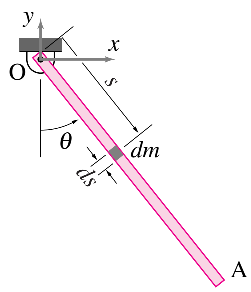


Figure 16.71: The infinitesimal mass dm considered in the calculation of E_K.

SAMPLE 16.23 The swinging stick dynamics with energy balance: Consider the same swinging stick as in Sample 16.20. The stick is, again, displaced from its vertical position by an angle θ and released (See Fig. 16.62).

1. Find the equation of motion using energy balance.
2. What is $\dot{\theta}$ at $\theta = 0$ if $\theta(t = 0) = \pi/2$?
3. Find the period of small oscillations about $\theta = 0$.

Solution

1. **Equation of motion using energy balance:** We use the power equation, $\dot{E}_K = P$, to derive the equation of motion of the bar. Now, the kinetic energy is given by

$$E_K = \frac{1}{2}\int_m v^2 dm$$

where v is the speed of the infinitesimal mass element dm. Referring to fig. 16.71, we can write, $dm = (m/\ell)ds$, and $v = \omega s \equiv \dot{\theta}s$. Thus,

$$\begin{aligned} E_K &= \frac{1}{2}\int_0^\ell \dot{\theta}^2 s^2 \frac{m}{\ell} ds \\ &= \frac{m\dot{\theta}^2}{2\ell}\int_0^\ell s^2 ds \\ &= \frac{1}{6} m\ell^2\dot{\theta}^2 \end{aligned}$$

and, therefore,

$$\dot{E}_K = \frac{d}{dt}\left(\frac{1}{6}m\ell^2\omega^2\right) = \frac{1}{3}m\ell^2\omega\,\dot{\omega} = \frac{1}{3}m\ell^2\dot{\theta}\ddot{\theta}.$$

Calculation of power (P): There are only two forces acting on the bar, the reaction force, $\vec{\boldsymbol{R}}(= R_x\hat{\boldsymbol{\imath}} + R_y\hat{\boldsymbol{\jmath}})$ and the force due to gravity, $-mg\hat{\boldsymbol{\jmath}}$. Since the support point O does not move, no work is done by $\vec{\boldsymbol{R}}$. Therefore,

$$W = \text{Work done by gravity force in moving from G}' \text{ to G.} = -mgh$$

Note that the negative sign stands for the work done *against* gravity. Now,

$$h = OG' - OG'' = \frac{\ell}{2} - \frac{\ell}{2}\cos\theta = \frac{\ell}{2}(1 - \cos\theta).$$

Therefore,

$$\begin{aligned} W &= -mg\frac{\ell}{2}(1 - \cos\theta) \\ \text{and} \quad P &= \dot{W} = \frac{dW}{dt} = -mg\frac{\ell}{2}\sin\theta\dot{\theta}. \end{aligned}$$

Equating $\dot{E}_K$ and P we get

$$\begin{aligned} -\cancel{m}g\frac{\ell}{2}\sin\theta\cancel{\dot{\theta}} &= \frac{1}{3}\cancel{m}\ell^2\cancel{\dot{\theta}}\ddot{\theta} \\ \text{or} \quad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0}$$

This equation is, of course, the same as we obtained using balance of angular momentum in Sample 16.20.

2. **Find ω at $\theta = 0$:** We are given that at $t = 0$, $\theta = \pi/2$ and $\dot{\theta} \equiv \omega = 0$ (released from rest). This position is (1) shown in Fig. 16.72. In position (2) $\theta = 0$, i.e., the rod is vertical. Since there are no dissipative forces, the total energy of the system remains constant. Therefore, taking datum for potential energy as shown in Fig. 16.72, we may write

$$
\begin{aligned}
\underbrace{E_{K_1}}_{0} + V_1 &= E_{K_2} + \underbrace{V_2}_{0} \\
\text{or} \qquad mg\frac{\ell}{2} &= \frac{1}{2}\int_m v^2 dm \\
&= \frac{1}{6} m\ell^2\omega^2 \quad \text{(see part (a))} \\
\Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}}
\end{aligned}
$$

$$\boxed{\omega = \pm\sqrt{\frac{3g}{\ell}}}$$

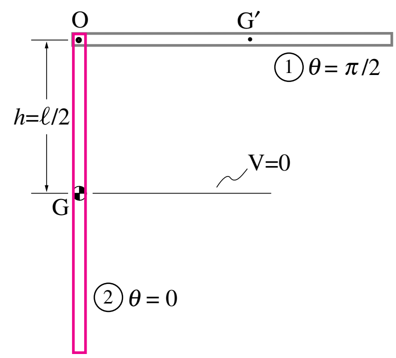


Figure 16.72: The total energy between positions (1) and (2) is constant.

3. **Period of small oscillations:** The equation of motion is

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0.$$

For small θ, $\sin\theta \approx \theta$

$$
\begin{aligned}
\Rightarrow \qquad & \ddot{\theta} + \tfrac{3g}{2\ell}\theta = 0 \qquad (16.71) \\
\text{or} \qquad & \ddot{\theta} + \lambda^2\theta = 0 \\
\text{where} \qquad & \lambda^2 = \tfrac{3g}{2\ell}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{the circular frequency} &= \lambda = \sqrt{\frac{3g}{2\ell}}, \\
\text{and the time period T} &= \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{2\ell}{3g}}.
\end{aligned}
$$

$$\boxed{T = 2\pi\sqrt{\frac{2\ell}{3g}}}$$

[Say for $g = 9.81\,\text{m/s}^2, \ell = 1\,\text{m}$ we get $\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{2}{3}\frac{1}{9.81}}\,\text{s} = 0.4097\,\text{s}$]

SAMPLE 16.24 The swinging stick with a destabilizing torque. Consider the swinging stick of Sample 16.20 once again.

1. Find the equation of motion of the stick, if a torque $\vec{M} = M\hat{k}$ is applied at end O and a force $\vec{F} = F\hat{\imath}$ is applied at the other end A.
2. Take $F = 0$ and $M = C\theta$. For $C = 0$ you get the equation of free oscillations obtained in Sample 16.20 or 16.23. For small C, does the period of the pendulum increase or decrease?
3. What happens if C is big?

Solution

1. A free-body diagram of the bar is shown in Fig. 16.73. Once again, we can use $\sum \vec{M}_O = \dot{\vec{H}}_{/O}$ to derive the equation of motion as in Sample 16.20. We calculated $\sum \vec{M}_O$ and $\dot{\vec{H}}_{/O}$ in Sample 16.20. Calculation of $\dot{\vec{H}}_{/O}$ remains the same in the present problem. We only need to recalculate $\sum \vec{M}_O$.

$$\begin{aligned}\sum \vec{M}_O &= M\hat{k} + \vec{r}_{G/O} \times mg(-\hat{\jmath}) + \vec{r}_{A/O} \times \vec{F} \\ &= M\hat{k} - \frac{\ell}{2} mg\sin\theta\hat{k} + F\ell\cos\theta\hat{k} \\ &= (M + F\ell\cos\theta - \frac{\ell}{2}mg\sin\theta)\hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_{/O} = m\ddot{\theta}\frac{\ell^2}{3}\hat{k} \quad \text{(see Sample 16.20)}$$

Therefore, from $\sum \vec{M}_O = \dot{\vec{H}}_{/O}$

$$\begin{aligned} M + F\ell\cos\theta - \frac{\ell}{2}mg\sin\theta &= m\ddot{\theta}\frac{\ell^2}{3} \\ \Rightarrow \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3F}{m\ell}\cos\theta - \frac{3M}{m\ell^2} &= 0.\end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3F}{m\ell}\cos\theta - \frac{3M}{m\ell^2} = 0}$$

2. Now, setting $F = 0$ and $M = C\theta$ we get

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3C\theta}{m\ell^2} = 0 \tag{16.72}$$

Numerical Solution: We can numerically integrate (16.72) just as in the previous Sample to find $\theta(t)$. Here is the pseudo-code that can be used for this purpose.

```
g = 9.81, L = 1      % specify parameters
m = 1, C = 4
ODES = { thetadot = omega
         omegadot = -(3*g/(2*L)) * sin(theta)
                    + 3*C/(m*L^2) * theta      }
ICs =  { thetazero = pi/20
         omegazero = 0  }
solve ODES with ICs until t = 10
```

Using this pseudo-code, we find the response of the pendulum. Figure 16.74 shows different responses for various values of C. Note that for $C = 0$, it is the same case as unforced bar pendulum considered above. From Fig. 16.74 it is clear that the bar has periodic motion for small C, with the period of motion increasing with

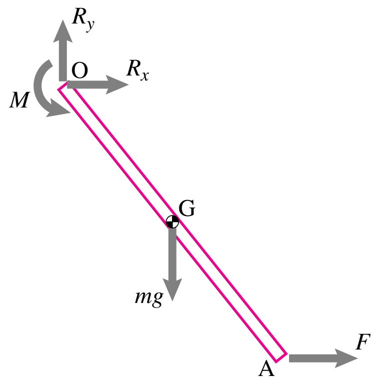


Figure 16.73: Free-body diagram of the bar with applied torque $\vec{M}$ and force $\vec{F}$

increasing values of C. It makes sense if you look at Eqn. (16.72) carefully. Gravity acts as a restoring force while the applied torque acts as a destabilizing force. Thus, with the resistance of the applied torque, the stick swings more sluggishly making its period of oscillation bigger.

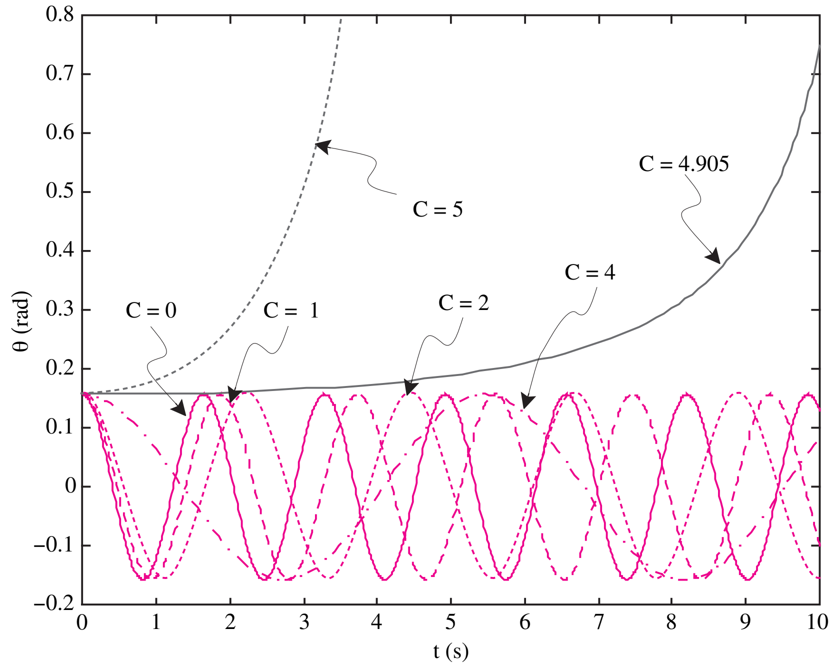


Figure 16.74: $\theta(t)$ with applied torque $M = C\theta$ for $C = 0,\ 1,\ 2,\ 4,\ 4.905,\ 5.$ Note that for small C the motion is periodic but for large C ($C \geq 4.4$) the motion becomes aperiodic.

3. From Fig. 16.74, we see that at about $C \approx 4.9$ the stability of the system changes completely. $\theta(t)$ is not periodic anymore. It keeps on increasing at a faster and faster rate, that is, the bar makes complete loops about point O with ever increasing speed. Does it make physical sense? Yes, it does. As the value of C is increased beyond a certain value (can you guess the value?), the applied torque overcomes any restoring torque due to gravity. Consequently, the bar is forced to rotate continuously in the direction of the applied force.

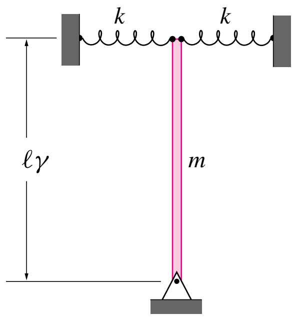
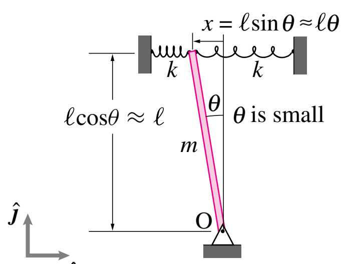


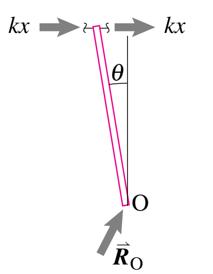
Figure 16.75

SAMPLE 16.25 **A torsional pendulum with linear springs:** A uniform rigid bar of mass $m = 2\,\mathrm{kg}$ and length $\ell = 1\,\mathrm{m}$ is pinned at one end and connected to two springs, each with spring constant k, at the other end. The bar is tweaked slightly from its vertical position. It then oscillates about its original position. The bar is timed for 20 full oscillations which take 12.5 seconds. Ignore gravity.

1. Find the equation of motion of the rod.
2. Find the spring constant k.
3. What should be the spring constant of a torsional spring if the bar is attached to one at the bottom and has the same oscillating motion characteristics?



(a) Geometry



(b) Free body diagram

Figure 16.76

Solution

1. Refer to the free-body diagram in figure 16.76. Angular momentum balance for the rod about point O gives

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$$

$$\begin{aligned} \text{where}\quad \vec{\boldsymbol{M}}_{\mathrm{O}} &= -2k\overbrace{x}^{\ell\sin\theta}\cdot\ell\cos\theta\hat{\boldsymbol{k}} \\ &= -2k\ell^2\sin\theta\cos\theta\hat{\boldsymbol{k}}, \\ \text{and}\quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= I_{zz}^{O}\ddot{\theta}\hat{\boldsymbol{k}} = \underbrace{\frac{1}{3}m\ell^2}_{I_{zz}^{O}}\ddot{\theta}\hat{\boldsymbol{k}}. \end{aligned}$$

Thus

$$\frac{1}{3}m\ell^2\ddot{\theta} = -2k\ell^2\sin\theta\cos\theta.$$

However, for small θ, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$,

$$\Rightarrow \quad \ddot{\theta} + \frac{6k\ell^2}{m\ell^2}\theta = 0. \tag{16.73}$$

$$\boxed{\ddot{\theta} + \frac{6k}{m}\theta = 0}$$

2. Comparing Eqn. (16.73) with the standard harmonic oscillator equation $\ddot{x} + \lambda^2 x = 0$, we get

$$\begin{aligned} \text{angular frequency}\quad \lambda &= \sqrt{\frac{6k}{m}}, \\ \text{and the time period}\quad T &= \frac{2\pi}{\lambda} \\ &= 2\pi\sqrt{\frac{m}{6k}}. \end{aligned}$$

From the measured time for 20 oscillations, the time period (time for one oscillation) is

$$T = \frac{12.5}{20}\,\mathrm{s} = 0.625\,\mathrm{s}$$

Now equating the measured T with the derived expression for T we get

$$
\begin{aligned}
2\pi\sqrt{\frac{m}{6k}} &= 0.625\,\mathrm{s} \\
\Rightarrow \quad k &= 4\pi^2 \cdot \frac{m}{6(0.625\,\mathrm{s})^2} \\
&= \frac{4\pi^2 \cdot 2\,\mathrm{kg}}{6(0.625\,\mathrm{s})^2} \\
&= 33.7\,\mathrm{N/m}.
\end{aligned}
$$

$$\boxed{k = 33.7\,\mathrm{N/m}}$$

3. If the two linear springs are to be replaced by a torsional spring at the bottom, we can find the spring constant of the torsional spring by comparison. Let k_{tor} be the spring constant of the torsional spring. Then, as shown in the free-body diagram (see figure 16.77), the restoring torque applied by the spring at an angular displacement θ is $k_{\mathrm{tor}}\theta$. Now, writing the angular momentum balance about point O, we get

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \\
-k_{\mathrm{tor}}\theta\hat{\boldsymbol{k}} &= I_{zz}^{O}(\ddot{\theta}\hat{\boldsymbol{k}}) \\
\Rightarrow \quad \ddot{\theta} + \frac{k_{\mathrm{tor}}}{I_{zz}^{O}}\theta &= 0.
\end{aligned}
$$

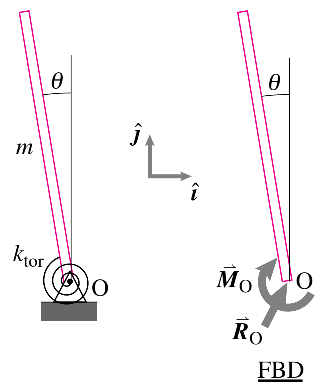


Figure 16.77

Comparing with the standard harmonic equation, we find the angular frequency

$$\lambda = \sqrt{\frac{k_{\mathrm{tor}}}{I_{zz}^{O}}} = \sqrt{\frac{k_{\mathrm{tor}}}{\frac{1}{3}m\ell^2}}.$$

If this system has to have the same period of oscillation as the first system, the two angular frequencies must be equal, *i.e.,*

$$
\begin{aligned}
\sqrt{\frac{k_{\mathrm{tor}}}{\frac{1}{3}m\ell^2}} &= \sqrt{\frac{6k}{m}} \\
\Rightarrow \quad k_{\mathrm{tor}} &= 6k \cdot \frac{1}{3}\ell^2 = 2k\ell^2 \\
&= 2 \cdot (33.7\,\mathrm{N/m}) \cdot (1\,\mathrm{m})^2 \\
&= 67.4\,\mathrm{N{\cdot}m}.
\end{aligned}
$$

$$\boxed{k_{\mathrm{tor}} = 67.4\,\mathrm{N{\cdot}m}}$$

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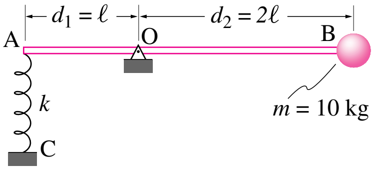

Figure 16.78

SAMPLE 16.26 **A spring loaded seesaw:** A kid, modelled as a point mass with $m = 10\,\text{kg}$, is sitting at end B of a rigid rod AB of negligible mass. The rod is supported by a spring at end A and a pin at point O. The system is in static equilibrium when the rod is horizontal. Someone pushes the kid vertically downwards by a small distance y and lets go. Given that $AB = 3\,\text{m}$, $AC = 0.5\,\text{m}$, $k = 1\,\text{kN/m}$; find

1. the unstretched (relaxed) length of the spring,
2. the equation of motion (a differential equation relating the position of the mass to its acceleration) of the system, and
3. the natural frequency of the system.

If the rod is pinned at the midpoint instead of at O, what is the natural frequency of the system? How does the new natural frequency compare with that of a mass m simply suspended by a spring with the same spring constant?

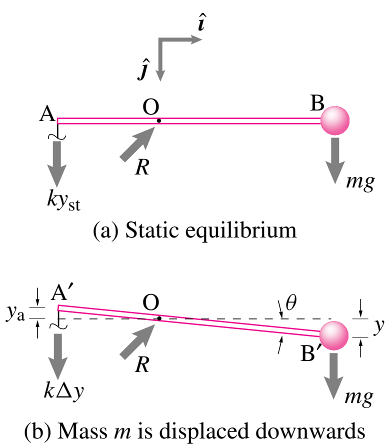

Figure 16.79: Free-body diagrams

Solution

1. **Static Equilibrium:** The FBD of the (rod + mass) system is shown in Fig. 16.79. Let the stretch in the spring in this position be y_{st} and the relaxed length of the spring be ℓ_0. The balance of angular momentum about point O gives:

$$
\begin{aligned}
\sum \vec{M}_{/o} = \dot{\vec{H}}_{/o} &= \vec{0} \quad \text{(no motion)}\\
\Rightarrow \quad (ky_{st})d_1 - (mg)d_2 &= 0\\
\Rightarrow \quad y_{st} &= \frac{mg}{k}\cdot\frac{d_2}{d_1}\\
&= \frac{10\,\text{kg}\cdot 9.8\,\text{m/s}^2\cdot 2\ell}{1000\,\text{N/m}\cdot\ell} = 0.196\,\text{m}\\
\text{Therefore,} \quad \ell_0 &= AC - y_{st}\\
&= 0.5\,\text{m} - 0.196\,\text{m} = 0.304\,\text{m}.
\end{aligned}
$$

$\boxed{\ell_0 = 30.4\,\text{cm}}$

2. **Equation of motion:** As point B gets displaced downwards by a distance y, point A moves up by a proportionate distance y_a. From geometry, ②

② Here, we are considering a very small y so that we can ignore the arc the point mass B moves on and take its motion to be just vertical (i.e., $\sin\theta \approx \theta$ for small θ).

$$
\begin{aligned}
y &\approx d_2\theta \quad \Rightarrow \quad \theta = \frac{y}{d_2}\\
y_a &\approx d_1\theta = \frac{d_1}{d_2}y
\end{aligned}
$$

Therefore, the total stretch in the spring, in this position,

$$
\Delta y = y_a + y_{st} = \frac{d_1}{d_2}y + \frac{d_2}{d_1}\frac{mg}{k}
$$

Now, Angular Momentum Balance about point O gives:

$$
\begin{aligned}
\sum \vec{M}_{/o} &= \dot{\vec{H}}_{/o}\\
\sum \vec{M}_{/o} &= \vec{r}_B \times mg\hat{\jmath} + \vec{r}_A \times k\Delta y\,\hat{\jmath}\\
&= (d_2 mg - d_1 k\Delta y)\hat{k} && (16.74)\\
\dot{\vec{H}}_{/o} &= \vec{r}_B \times m\vec{a} = \vec{r}_B \times m\ddot{y}\,\hat{\jmath} && (16.75)\\
&= d_2 m\ddot{y}\hat{k} && (16.76)
\end{aligned}
$$

Equating (16.74) and (16.76) we get

$$
\begin{aligned}
d_2 mg - d_1 k\Delta y &= d_2 m\ddot{y} \\
\text{or} \qquad d_2 mg - d_1 k\left(\frac{d_1}{d_2}y + \frac{d_2 mg}{d_1 k}\right) &= d_2 m\ddot{y} \\
\text{or} \qquad \cancel{d_2 mg} - k\frac{d_1^2}{d_2}y - \cancel{d_2 mg} &= d_2 m\ddot{y} \\
\text{or} \qquad \ddot{y} + \frac{k}{m}\left(\frac{d_1}{d_2}\right)^2 y &= 0
\end{aligned}
$$

$$\boxed{\ddot{y} + \frac{k}{m}\left(\frac{d_1}{d_2}\right)^2 y = 0}$$

3. **The natural frequency of the system:** We may also write the previous equation as

$$\ddot{y} + \lambda^2 y = 0 \qquad \text{where} \qquad \lambda^2 = \frac{k}{m}\frac{d_1^2}{d_2^2}. \tag{16.77}$$

Substituting $d_1 = \ell$ and $d_2 = 2\ell$ in the expression for λ we get the natural frequency of the system

$$\lambda = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}\sqrt{\frac{1000\,\mathrm{N/m}}{10\,\mathrm{kg}}} = 5\,\mathrm{s}^{-1}$$

$$\boxed{\lambda = 5\,\mathrm{s}^{-1}}$$

4. **Comparison with a simple spring mass system:**

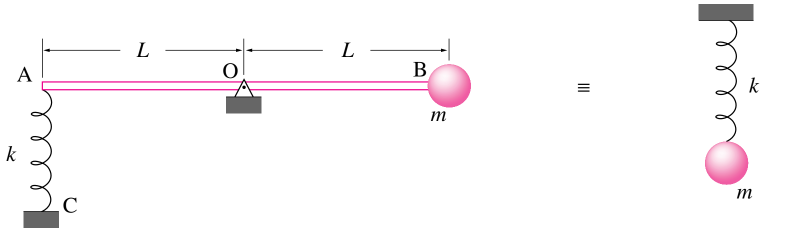

Figure 16.80:

When $d_1 = d_2$, the equation of motion (16.77) becomes

$$\ddot{y} + \frac{k}{m}y = 0$$

and the natural frequency of the system is simply

$$\lambda = \sqrt{\frac{k}{m}}$$

which corresponds to the natural frequency of a simple spring mass system shown in Fig. 16.80.

In our system (with $d_1 = d_2$) any vertical displacement of the mass at B induces an equal amount of stretch or compression in the spring which is exactly the case in the simple spring-mass system. Therefore, the two systems are mechanically equivalent. Such equivalences are widely used in modeling complex physical systems with simpler mechanical models.

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