16.1 Rotation of a rigid object in planar circular motion

When two parts are glued together or attached by welding, gluing, several tight screws, bolts, rivets or the like we call the connection a 'rigid attachment'. And, for the purposes of mechanics analysis, the two connected parts make up one bigger object. But most machines have various parts that are connected to each other, but not welded to each other. The most common such non-rigid attachment in engineering is a hinge. In 2D,

> a *hinge* attachment between two objects keeps two points, one from each object, on top of each other while freely allowing relative rotation of the two objects about the hinge point.

In 3D a hinge keeps two lines, one on each body, coincident and allows relative rotation about that line. The common line, or in 2D the line orthogonal to the plane through the points, is called the hinge, the hinge axis, or the axis of relative rotation.

One example of a hinge is a car axle which allows rotation of a wheel relative to the car suspension. The hinge axis is the wheel axle[1]. Physically, hinges are made various ways, sometimes by poking a cylindrical pin through the two objects and sometimes with ball bearings (see box 5.7 on page 273). So hinges are also called *pin connections* or *bearings* (fig. 16.3). In this chapter we limit our attention to a simple use of a hinge: one rigid part is hinged to a second fixed part that doesn't move at all. Such a non-moving part can be thought of as connected to, or an extension of, the ground or 'fixed frame'. In the simple case of interest in this chapter, one point on the moving part does not move and the rest of the part rotates about that point.

For definiteness and simplicity let's call the hinge location 0 and the hinge axis through 0 the z axis. One function of the hinge is to make the part's only possible motion to be rotation about O. Thus to understand the dynamics of a hinged part we need to understand the position, velocity and acceleration of points on a rigid object which rotates. This whole section is about the kinematics (the geometry of motion) for this rotation. We will measure the amount of rotation by the angle θ, and the rate of change of θ by the *angular velocity* $\vec{\omega}$ ('omega'), and the rate of change of this angular velocity $\vec{\omega}$ by the *angular acceleration* $\vec{\alpha}$ ('alpha').

As simple as this topic seems at first glance, you should pay close attention to the meanings and uses of these quantities. The rest of the book completely depends on the material in this section.

Rotation of an object counterclockwise by θ

We start by imagining the object in a distinguished configuration which we call the *reference configuration*, *reference state* or *reference position*. For

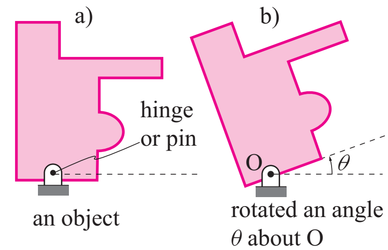


Figure 16.3: a) An object, b) rotated counterclockwise an angle θ about 0.

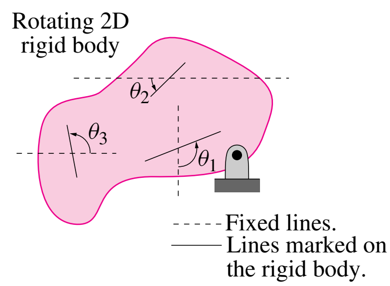


Figure 16.4: **Rotation of lines on a rotating rigid object.** Some real or imagined lines marked on the rigid object are shown. They make the angles θ_1, θ_2, θ_3, ... with respect to various fixed lines which do not rotate. As the object rotates, each of these angles increases by the same amount.

[1] The word axis is obviously related to the word axle. More generally the word axis means 'line'. For example the x and y axes are generally not axles about which anything rotates, they are just lines.

example we could take the left figure in fig. 16.3 as the reference configuration. If possible it's usually best to pick the reference state to be one in which a prominent feature of the object is aligned with the x or y axes. The reference state may or may not be the start of the motion of interest. Even if not, we measure an object's rotation by the change, relative to the reference state, in the counterclockwise angle θ of a reference line marked in the object relative to a fixed line outside. Which reference line? Fortunately,

> *All real or imagined lines marked on a rotating rigid object rotate by the same angle, the* rotation angle*, θ. (See box 16.1).*

In three dimensions things are more complicated. General rotation of a rigid object is then represented not with a single angle θ, but rather with 3 angles, or with a unit vector and an angle, or a 3×3 matrix. So we wait to discuss which of the 2D ideas here generalize to 3D and which do not.

16.1 Rotation is uniquely defined for a rigid object (2D)

Most people will find it self-evident that, starting with a rigid object at a reference orientation, all lines marked on the object rotate by the same angle θ. Here, for the doubting, we demonstrate this fact.

A rigid object is defined this way:

> For every pair of material points A and B on a rigid object the distance |AB| between them does not change as the object moves.

In particular, when a rigid object rotates all distances between all pairs of points are preserved. Thus, by the "side-side-side" similar triangle theorem of elementary geometry, all relative angles between marked line segments are preserved by the rotation. For example, for a triangle ABC the angle at B is constant as the object rotates. Now consider any pair of line segments on the object.

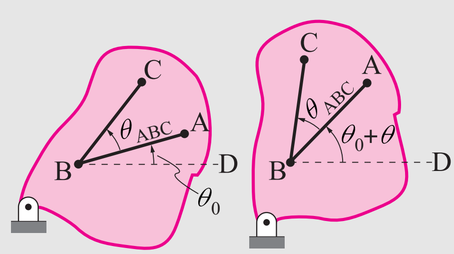
[By 'on' the object we don't mean a projected image drawn with a light pen that can move around on the object relative to the atoms. Rather, by 'on the object' we mean something defined by a particular set of atoms that make up the object.]

If the segments do not cross we can extend them to a point of intersection B. Such a pair of intersecting lines is shown here before and after rotation.Initially BA makes an angle θ_0 with a horizontal reference line. BC then makes an angle of $\theta_{ABC} + \theta_0$. After rotation we measure the angle to the line BD. BA now makes an angle of $\theta_0 + \theta$. By the addition of angles in the rotated configuration line BC now makes an angle of $\theta_{ABC} + \theta_0 + \theta$ which makes an increase by θ of the angle made by BC with the horizontal reference line. So both BA and BC rotate by the same angle θ.

We could use one of these two lines and compare it with an arbitrary third line through B and show that the third line also has equal rotation, and then a fourth, and so on. So all lines on the object through a point B rotate by the same angle θ. The demonstration for a pair of parallel lines, one of them through B, is easy, they stay parallel so always make a common angle with any reference line.

Because any line on the object either goes through B or is parallel to a line through B, all lines marked on a rigid object rotate by the same angle θ.

> The rotation of a rigid object in 2D is thus unambiguously defined as *the* angle through which all lines on the object rotate.



Rotated coordinates and base vectors $\hat{\imath}'$ and $\hat{\jmath}'$

We pick two orthogonal lines on the rotating object and give them distinguished status as *object-fixed* (or body-fixed) rotating coordinate axes x' and y'. Think of these axes as $x'y'$ coordinate axes on a piece of graph paper that is glued to the object (see fig. 16.5a&b). It's easiest if we start by assuming that the $x'y'$ axes have the same origin 0 as the xy axes and are parallel with the fixed xy axes when the object is in the reference configuration (when $\theta = 0$).

These rotating coordinate axes, x' and y', have associated rotating base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ (fig. 16.6 and 16.7). So $\hat{\boldsymbol{\imath}}'$ is always in the x' direction and $\hat{\boldsymbol{\jmath}}'$ always in the y' direction. We will use these rotating coordinates and base vectors to keep track of some particle of interest P that is 'glued' to the object. To start, note that particle P which is glued to the object has x' and y' coordinates that don't change as the rotation progresses.

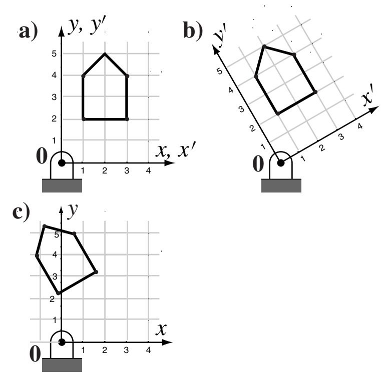


Figure 16.5: a) A house is drawn by connecting lines between 6 points, b) the house and coordinate system are rotated, thus its coordinates in the rotating system do not change c) But the coordinates in the original system do change

Example: A particle on the x' axis
If a particle P is fixed on the x'-axis at position $x' = 3\,\text{cm}$, then we have.

$$\vec{\boldsymbol{r}}_{\text{P}} \quad = \quad 3\,\text{cm}\,\hat{\boldsymbol{\imath}}'$$

for all time, even as the object rotates.

The position vector of a point P fixed to a rigid object hinged at O remains, as the rotation progresses,

$$\vec{\boldsymbol{r}}_{\text{P}} \quad = \quad x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}', \tag{16.1}$$
$$\tag{16.2}$$

with x' and y' both constant. These rotating coordinate system components, $[\vec{\boldsymbol{r}}]_{x'y'} = [x', y']$, are sometimes written as $[\vec{\boldsymbol{r}}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

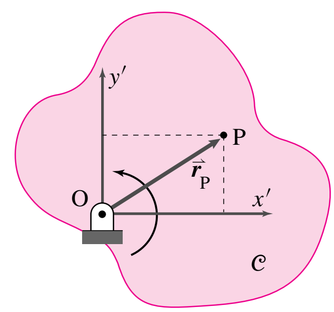


Figure 16.6: A rotating rigid object $\mathcal{C}$ with rotating coordinates $x'y'$ rigidly attached.

You will see that much of the math for rotating $x'y'$ coordinates is reminiscent of that for polar coordinates. However, the spirit is a bit different. In polar coordinates the $\hat{\boldsymbol{e}}_R$ axis was picked to track a particular particle of interest. Here we pick axes that rotate with an extended object and use that one set of axes to track any and all particles of interest.

Note, even though neither x' nor y' change as θ changes, the point P they describe moves, in circles actually. How can the particle's position change if its coordinates don't change? Well, in eqn. (16.3) the change in position is represented by the base vectors changing as the object rotates. Thus we could write more explicitly that

$$\vec{\boldsymbol{r}}_{\text{P}} \quad = \quad x'\hat{\boldsymbol{\imath}}'(\theta) + y'\hat{\boldsymbol{\jmath}}'(\theta). \tag{16.3}$$

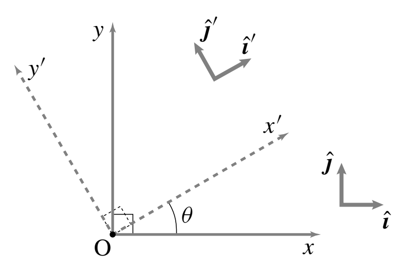


Figure 16.7: Fixed coordinate axes and rotating coordinate axes.

Here we show more explicitly that the base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ depend on θ. Just like for polar base vectors (see eqn. (15.5) on page 727) we can express the rotating base vectors in terms of the fixed base vectors and θ.

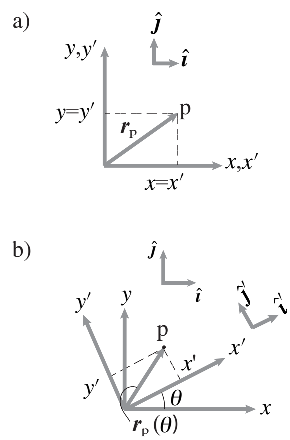


Figure 16.8: The x' and y' coordinates of a point fixed on a rotating object stay constant while the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ rotate with the object.

②Advanced aside. Sometimes a *reference frame* is defined as the set of all coordinate systems that could be attached to a rigid object. Two coordinate systems, even if rotated with respect to each other, then represent the same frame so long as they rotate together. Some of the results we will develop only depend on this more slack definition of frame, that the coordinates are glued to the object with no mind of their orientation in the reference configuration.

$$\begin{aligned} \hat{\imath}' &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \\ \hat{\jmath}' &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \end{aligned} \tag{16.4}$$

Also we can express the fixed basis vectors in terms of the rotating vectors like this:

$$\begin{aligned} \hat{\imath} &= \cos\theta\,\hat{\imath}' - \sin\theta\,\hat{\jmath}' \\ \hat{\jmath} &= \sin\theta\,\hat{\imath}' + \cos\theta\,\hat{\jmath}'. \end{aligned} \tag{16.5}$$

Please review the section on dot products, 1.2, to see one derivation of these formulae.

We will use the phrase *reference frame* or just *frame* to mean "a coordinate system attached to a rigid object". One can think of the coordinate grid as like an invisible metal framework (hence the word 'frame') that rotates with the object. We refer to a calculation based on the rotating coordinates in fig. 16.6 variously as "in the frame $\mathcal{C}$" or "using the $x'y'$ frame" or "in the $\hat{\imath}'\,\hat{\jmath}'$ frame②.

In computer calculations we usually manipulate lists and arrays of numbers and not geometric vectors. So on a computer we keep track of vectors by keeping track of their lists of components. Let's look at a point fixed to the object and whose coordinates we know in the reference configuration:

$$\left[\vec{r}_{\mathrm{P}}^{\ \mathrm{ref}}\right]_{xy} = \begin{bmatrix} x^{\mathrm{ref}} \\ y^{\mathrm{ref}} \end{bmatrix}.$$

Assuming the object axes and fixed axes coincide in the reference configuration, the object coordinates of a point $[\vec{r}_{\mathrm{P}}]_{x'y'}$ are equal to the space fixed coordinates of the point in the reference configuration $\left[\vec{r}_{\mathrm{P}}^{\ \mathrm{ref}}\right]_{xy}$. We can think of the point as defined either way, so

$$[\vec{r}_{\mathrm{P}}]_{x'y'} = \left[\vec{r}_{\mathrm{P}}^{\ \mathrm{ref}}\right]_{xy}.$$

The rotation matrix $[R]$

Here is a question we often need to answer: What are the fixed basis coordinates of a point that has the rotating-frame coordinates $[\vec{r}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$? Here is one way to find the answer:

$$\begin{aligned} \vec{r}_{\mathrm{P}} &= x'\hat{\imath}' + y'\hat{\jmath}' \\ &= x'(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + y'(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \\ &= \underbrace{\big((\cos\theta)x' - (\sin\theta)y'\big)}_{x}\hat{\imath} + \underbrace{\big((\sin\theta)x' + (\cos\theta)y'\big)}_{y}\hat{\jmath} \end{aligned} \tag{16.6}$$

so we can pull out the x and y coordinates compactly as,

$$[\vec{r}_{\mathrm{P}}]_{xy} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta\, x' + \sin\theta(-y') \\ \sin\theta\, x' + \cos\theta(y') \end{bmatrix}. \tag{16.7}$$

But this can, in turn be written in matrix notation as

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{[R]} \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \text{or} \\ [\vec{r}_{\mathrm{P}}]_{xy} &= [R]\,[\vec{r}_{\mathrm{P}}]_{x'y'}, \quad \text{or} \\ [\vec{r}_{\mathrm{P}}]_{xy} &= [R]\left[\vec{r}_{\mathrm{P}}^{\ \mathrm{ref}}\right]_{xy}. \end{aligned} \tag{16.8}$$

The matrix $[R]$ or $[R(\theta)]$ is the *rotation matrix* for counterclockwise rotations by θ. As shown above, if you know the coordinates of a point fixed on an object before rotation, you can find its coordinates after rotation by multiplying the coordinate column vector by the matrix $[R]$. You can remember what $[R]$ is by remembering its components or by remembering that

the first and second column of $[R]$ are the components of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, respectively, in the fixed coordinate system.

For example, the first column of $[R]$ consists of the x and y components of $\hat{\boldsymbol{\imath}}'$. A feature of eqn. (16.8) is that the same matrix $[R]$ prescribes the coordinate change for every different point on the object. Thus for points called 1, 2 and 3 we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [R]\begin{bmatrix} x_1' \\ y_1' \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = [R]\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} \qquad \& \\ \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = [R]\begin{bmatrix} x_3' \\ y_3' \end{bmatrix}.$$

A more compact way to write a matrix times a list of column vectors is to arrange the column vectors one next to the other in a matrix. By multiplying this matrix by $[R]$ we get a new matrix whose columns are the new coordinates of various points. For example,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = [R]\begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix}. \tag{16.9}$$

Eqn. 16.9 is useful for computer animation of rotating things in video games (and in dynamics simulations too) where points 1,2, and 3 are points on an object.

Example: Rotate a picture

If a simple picture of a house is drawn by connecting the six points (fig. 16.5a) with the first point at $(x, y) = (1, 2)$, the second at $(x, y) = (3, 2)$, etc., and the sixth point on top of the first, we have,

$$[xy \text{ points BEFORE}] \equiv \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix}.$$

After a 30° counter-clockwise rotation about O, the coordinates of the house, in a coordinate system that rotates with the house, are unchanged (fig. 16.5b). But in the fixed (non-rotating, Newtonian) coordinate system the new coordinates of the rotated house points are,

$$\begin{aligned} [xy \text{ points AFTER}] &= [R]\,[xy \text{ points BEFORE}] = [R]\,[x'y' \text{ points}] \\ &= \begin{bmatrix} \sqrt{3}/2 & -.5 \\ .5 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix} \\ &\approx \begin{bmatrix} -0.1 & 1.6 & 0.6 & -0.8 & -1.1 & -0.1 \\ 2.2 & 3.2 & 5.0 & 5.3 & 4.0 & 2.2 \end{bmatrix} \end{aligned}$$

as shown in fig. 16.5c.

SAMPLE 16.1 Computing rotated position of an object: A rigid object AOB is pinned at point O and is free to rotate about this point. Using the rotation matrix, find the coordinates of points A and B when $\theta = 30°$ and $110°$ respectively.

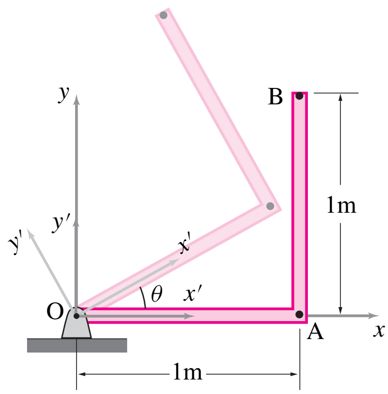


Figure 16.9

Solution Let $x'y'$ be a set of axes glued to the object with origin at O which is also the origin of the space-fixed coordinate axes xy. We first write the coordinates of points A and B in the body-fixed axes.

$$\vec{r}_A|_{x'y'} = \begin{bmatrix} 1\,\text{m} \\ 0 \end{bmatrix}, \qquad \vec{r}_B|_{x'y'} = \begin{bmatrix} 1\,\text{m} \\ 1\,\text{m} \end{bmatrix}.$$

The rotation matrix for the $x'y'$ axes rotating counterclockwise along with the object is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We can find the coordinates of A and B in the rotated position by multiplying their $x'y'$ coordinates with the rotation matrix. We first write the coordinates of both points A and B in a single matrix, one column for each, and multiply with R to get the new coordinates:

$$[\ \vec{r}_A \quad \vec{r}_B\ ]_{xy} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'_A & x'_B \\ y'_A & y'_B \end{bmatrix}.$$

Now, we calculate the new coordinates for the given values of θ.

- For $\theta = 30°$, we have,

$$\begin{aligned} [\ \vec{r}_A \quad \vec{r}_B\ ]_{xy} &= \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 1\,\text{m} & 1\,\text{m} \\ 0 & 1\,\text{m} \end{bmatrix} \\ &= \begin{bmatrix} 0.866\,\text{m} & 0.366\,\text{m} \\ 0.5\,\text{m} & 1.366\,\text{m} \end{bmatrix}. \end{aligned}$$

  Thus,

$$[\vec{r}_A]_{xy} = \begin{bmatrix} 0.866\,\text{m} \\ 0.5\,\text{m} \end{bmatrix}, \qquad [\vec{r}_B]_{xy} = \begin{bmatrix} 0.366\,\text{m} \\ 1.366\,\text{m} \end{bmatrix}.$$

- Similarly, for $\theta = 110°$, we get,

$$\begin{aligned} [\ \vec{r}_A \quad \vec{r}_B\ ]_{xy} &= \begin{bmatrix} -0.342 & -0.94 \\ 0.94 & -0.342 \end{bmatrix} \begin{bmatrix} 1\,\text{m} & 1\,\text{m} \\ 0 & 1\,\text{m} \end{bmatrix} \\ &= \begin{bmatrix} -0.342\,\text{m} & -1.282\,\text{m} \\ 0.94\,\text{m} & 0.598\,\text{m} \end{bmatrix}. \end{aligned}$$

$$[\vec{r}_A]_{xy} = \begin{bmatrix} -0.342\,\text{m} \\ 0.94\,\text{m} \end{bmatrix}, \qquad [\vec{r}_B]_{xy} = \begin{bmatrix} -1.282\,\text{m} \\ 0.598\,\text{m} \end{bmatrix}.$$

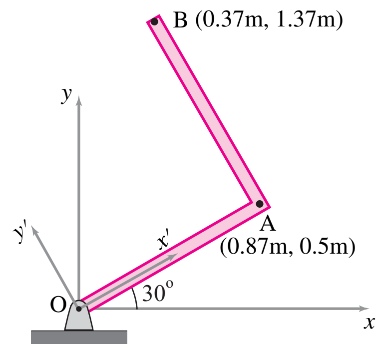


Figure 16.10: Coordinates of points A and B after a rotation of 30°.

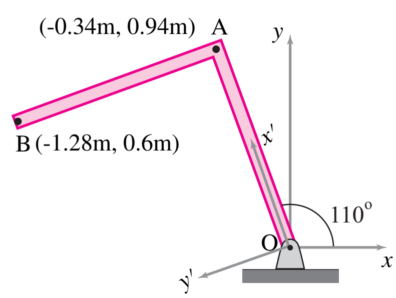


Figure 16.11: Coordinates of points A and B after a rotation of 110°.

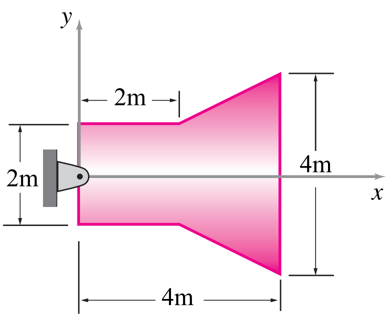


Figure 16.12

SAMPLE 16.2 **Computer program for object rotation:** The object shown in the figure rotates counterclockwise with its angular position θ given by $\theta(t) = \dot{\theta}_0 t$ where $\dot{\theta}_0 = 10°/s$. Write a computer program to animate the motion of the object by plotting its position at specified time instants. Use your program to plot the position of the object every two seconds for a total of 18 seconds.

Solution In order to draw the object on the computer screen, we need to define it by taking a sufficient number of points on its boundary so that plotting all those points with connected line segments represents the object as closely as possible. In this case, we can select all the corner points on its boundary and define the object with the coordinates of these points. With the given geometry, it is fairly easy to find these coordinates. Denoting the coordinates of the kth point by (x_k, y_k), we can define this object by the following set of coordinates:

$$\text{object} = \begin{bmatrix} x_1 & x_2 & \dots & x_6 & x_1 \\ y_1 & y_2 & \dots & y_6 & y_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 4 & 2 & 0 & 0 \\ -1 & -1 & -2 & 2 & 1 & 1 & -1 \end{bmatrix}.$$

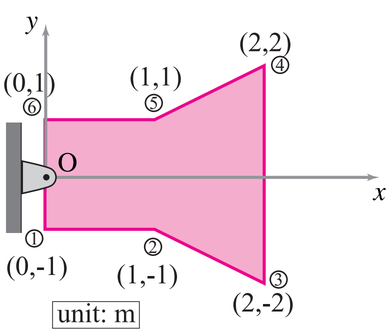


Figure 16.13: Coordinates of the six corner points are sufficient to define this object.

Note that the last column is a repeat of the first column. This is essential so that when we plot the object using the x and y coordinates listed here, we get a closed boundary.

Animation of the angular motion of this object basically requires determining the rotated position of the object and plotting it at sufficiently small time intervals. To do this, we need to define the initial orientation (position), define the time increment, find the angle of rotation θ corresponding to the new time, compute the corresponding rotation matrix, multiply the object coordinates with the rotation matrix, separate out the x and y coordinates, and plot x vs y. We then repeat the whole sequence until we reach the final time.

The following pseudocode can be easily adapted to a computer program to do the required animation.

```
object = [ 0  2  4  4  2  0  0
          -1 -1 -2  2  1  1 -1]     % coordinates of points
t = 0                               % specify initial time
t_final = 36                        % specify final time
delta_t = 2                         % specify time increment
thetadot0 = 10*pi/180               % specify thetadot0 in rad/s

while t < t_final
      t = t + delta_t               % increment the time
      theta = thetadot0*t           % compute current theta
      R = [ cos(theta) -sin(theta)  % find the rotation matrix
            sin(theta)  cos(theta)]
      new_position = R*object       % find new coordinates
      x = new_position(1st row)     % extract all x-coordinates
      y = new_position(2nd row)     % extract all y-coordinates
      plot y vs x                   % plot the new position
end
```

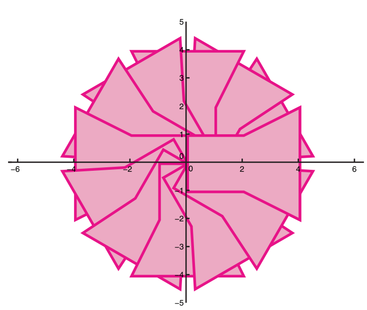


Figure 16.14: Rotated position of the object plotted at 2 second intervals.

The plot obtained by implementing this code is shown in fig. 16.14.

16.1 Rotation of a rigid body

Preparatory Problems

16.1.1 Give the definition of each term below in words and, if possible, with an equation.

a) rotation angle

b) (x', y') *

c) (x, y) *

d) rotation matrix *

16.1.2 Assume that in the reference configuration, when $\theta = 0$, that the $x'y'$ axes attached to an object are aligned with the xy axes. Consider a point P attached to the moving frame (object) that has coordinates (x', y'). Find each of the quantities below in terms of some or all of $x', y', \hat{\imath}, \hat{\jmath}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

a) $\vec{r}_{\mathrm{P}}$ *

b) The rotation matrix $[R]$. *

16.1.3 Assume that in the reference configuration, when $\theta = 0$, that the $x'y'$ axes are aligned with the xy axes. Consider two points P_1 and P_2 attached to the moving frame. P_1 and P_2 have coordinates (x_1', y_1') (x_2', y_2'). Find each of the quantities below in terms of some or all of $x_1', y_1', x_2', y_2', \hat{\imath}, \hat{\jmath}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

a) $\vec{r}_{\mathrm{P}_1/\mathrm{P}_2}$ *

b) The rotation matrix $[R]$. *

16.1.4 The square plate OABC shown in the figure measures $\ell = 1\,\mathrm{m}$ on a side. The coordinate axes $x'y'$ are fixed to the plate and the axes xy are fixed in space. The plate is rotated by an angle $\theta = 60°$ as shown.

a) Find the coordinates x' and y' of the corner point B in the rotated frame.

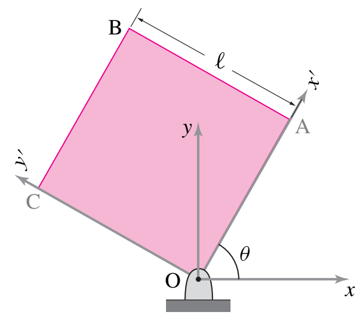
b) Find the coordinates x and y of the same point in the fixed frame without using the rotation matrix.

c) Write the rotation matrix R for the given rotation.

d) Find the coordinates of point B in the fixed frame using its (x', y') coordinates and the rotation matrix R.

e) Find the coordinates of point B in the rotated frame using its (x, y) coordinates and the rotation matrix R in its appropriate form.

*



Problem 16.1.4

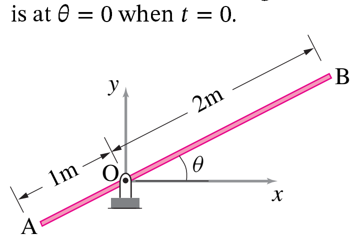
16.1.5 The rod shown in the figure rotates counterclockwise, starting from $\theta = 0$. Find the following quantities.

a) Write the position vector of points A and B as a list of numbers in an array, *e.g.*, $[\vec{r}_{\mathrm{A}}]_{xy} = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$, at $\theta = 0$. *

b) Write the rotation matrix R for $\theta = 45°$. *

c) Find the coordinates of points A and B using the rotation matrix R for $\theta = 45°$. *

d) If the rod rotates at a constant angular speed $\omega = \pi/3\,\mathrm{rad/s}$, find the coordinates of points A and B after 2 seconds, assuming the rod is at $\theta = 0$ when $t = 0$.



Problem 16.1.5

16.1.6 Find the angle of rotation corresponding to the following rotation matrices:

a) $\begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$. *

b) $\begin{bmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{bmatrix}$. *

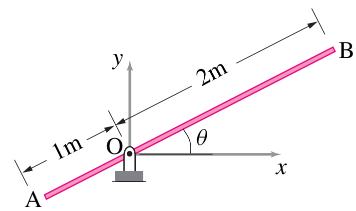
c) $\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$. *

More-Involved Problems

16.1.7 The rod shown in the figure rotates with constant angular speed $\omega = 10\,\mathrm{rad/s}$. $\theta = 0$ at $t = 0\,\mathrm{s}$.

a) Find the position vector $\vec{r}_{\mathrm{B}}|_{t_1}$ and the velocity $\vec{v}_{\mathrm{B}}|_{t_1}$ of point B at $t_1 = 1\,\mathrm{s}$. *

b) Find the position vector $\vec{r}_{\mathrm{B}}|_{t_2}$ of point B at $t_2 = 1.2\,\mathrm{s}$. What is the net displacement of point B during the time interval $\Delta t = t_2 - t1$? Is this net displacement equal to $\vec{v}_{\mathrm{B}}|_{t_1}\Delta t$? Why not? *



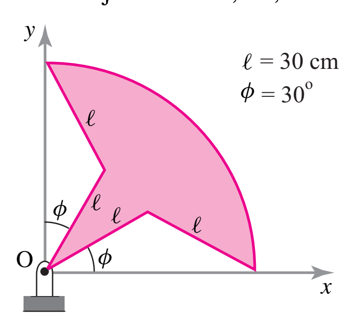
Problem 16.1.7

16.1.8 Write a computer program to animate the rotation of an object. Your input should be a set of x and y coordinates defining the object (such that `plot y vs x` draws the object on the screen) and the rotation angle θ. The output should be the rotated coordinates of the object.

a) From the geometric information given in the figure, generate coordinates of enough points to define the given object. *

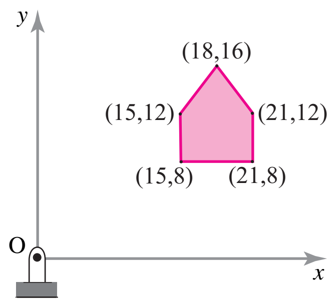
b) Using your program, plot the object at $\theta = 20°, 60°, 100°, 160°$, and $270°$.

c) Assume that the object rotates with constant angular speed $\omega = 2\,\mathrm{rad/s}$. Find and plot the position of the object at $t = 1\,\mathrm{s}$, $2\,\mathrm{s}$, and $3\,\mathrm{s}$.



Problem 16.1.8

16.1.9 The arrow shaped object shown in the figure is defined by the five points whose coordinates are given in some normalized units. The position shown is at $t = 0$. The time dependent angular position of the object is given by $\theta(t) = C_1 t^2 + C_2 t$ where $C_1 = 0.25\,\mathrm{rad/s^2}$ and $C_2 = 0.1\,\mathrm{rad/s}$. Animate the motion of the object and show its position from $t = 0$ to $t = 5\,\mathrm{s}$ at every second.



Problem 16.1.9