15.1 Kinematics of a particle in circular motion

Preparatory Problems

15.1.1 A particle goes on a circular path with radius R making the angle $\theta = ct$ measured counter clockwise from the positive x axis. Assume $R = 5\,\text{cm}$ and $c = 2\pi\,\text{s}^{-1}$.

a) Plot the path.

b) What is the angular rate in revolutions per second?

c) Put a dot on the path for the location of the particle at $t = t^* = 1/6$ s.

d) What are the x and y coordinates of the particle position at $t = t^*$? Mark them on your plot.

e) Draw the vectors $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_R$ at $t = t^*$.

f) What are the x and y components of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ at $t = t^*$?

g) What are the R and θ components of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ at $t = t^*$?

h) Draw an arrow representing both the velocity and the acceleration at $t = t^*$.

i) Find the $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ components of position $\vec{\boldsymbol{r}}$, velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ at $t = t^*$.

j) Find the x and y components of position $\vec{\boldsymbol{r}}$, velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ at $t = t^*$. Find the velocity and acceleration two ways:

1. Differentiate the position given as $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$.
2. Differentiate the position give as $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{e}}_r$ and then convert the results to Cartesian coordinates.

15.1.2 A bead goes around a circular track of radius 1 ft at a constant speed. It makes it around the track in exactly 1 s.

a) Find the speed of the bead. Does this vary in time?

b) Find the magnitude of acceleration of the bead. Does this vary in time?

c) Is the magnitude of the acceleration the derivative of the speed (*i.e.*, $|\vec{\boldsymbol{a}}| \overset{?}{=} \frac{d}{dt}|\vec{\boldsymbol{v}}|$)

15.1.3 If a particle moves along a circle at constant rate (constant $\dot{\theta}$) following the equation

$$\vec{\boldsymbol{r}}(t) = R\cos(\dot{\theta}t)\hat{\boldsymbol{\imath}} + R\sin(\dot{\theta}t)\hat{\boldsymbol{\jmath}}$$

which of these things are true and why? If not true, explain why.

a) $\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$

b) $\vec{\boldsymbol{v}} =$ constant

c) $|\vec{\boldsymbol{v}}| =$ constant

d) $\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$

e) $\vec{\boldsymbol{a}} =$ constant

f) $|\vec{\boldsymbol{a}}| =$ constant

g) $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{a}}$

15.1.4 A particle moves according to:

$$\begin{aligned} x(t) &= R\cos(ct), \\ y(t) &= R\sin(ct). \end{aligned}$$

where $R = 1\,\text{m}$ and $c = 5\,\text{rad/s}$.

a) Show that the speed of the particle is constant.

b) How much time does the particle take to go from P at $(0, 1\,\text{m})$ to Q at $(1\,\text{m}, 0)$?

c) What is the acceleration of the particle at point Q?

More-Involved Problems

15.1.5 A 200 mm diameter gear rotates at a constant speed of 100 rpm.

a) What is the speed of a peripheral point on the gear?

b) If no point on the gear is to exceed the centripetal acceleration of $25\,\text{m/s}^2$, find the maximum allowable angular speed (in rpm) of the gear.

15.1.6 A particle is in circular motion in the xy-plane at the constant angular speed of $\dot{\theta} = 2\,\text{rad/s}$ at radius 0.5 m. At $t = 0$ the particle is at $\theta = 0$.

a) Draw the path and mark the position of the particle at $t = 0.5$ s and $t = 15$ s.

b) Find the velocity and acceleration of the particle at $t = 0.5$ s and $t = 15$ s.

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15.1.7 A particle undergoes constant rate circular motion in the xy-plane. At some instant t_0, its velocity is $\vec{\boldsymbol{v}}(t_0) = -3\,\text{m/s}\hat{\boldsymbol{\imath}} + 4\,\text{m/s}\hat{\boldsymbol{\jmath}}$ and after 5 s the velocity is $v(t_0 + 5\,\text{s}) = (5/\sqrt{2})\,\text{m/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$. If the particle has not yet completed one revolution between the two instants, find

a) the angular speed of the particle,

b) the distance traveled by the particle in 5 s, and

c) the acceleration of the particle at the two instants.

15.1.8 A bead on a circular path of radius R in the xy-plane has rate of change of angular speed $\alpha = bt^2$. The bead starts from rest at $\theta = 0$.

a) What is the bead's angular position θ (measured from the positive x-axis) and angular speed $\dot{\theta}$ as a function of time ?

b) What is the angular speed as function of angular position?

15.1.9 A bead on a circular wire has an angular speed given by $\dot{\theta} = c\theta^{1/2}$. The bead starts from rest at $\theta = 0$. What is the angular position and speed of the bead as a function of time? [This problem is subtle because it has multiple solutions. One answer you can find with a quick guess. Another you can find by separation of variables. The full general solution is an appropriate mixture of these two.)] *

15.1.10 Solve $\ddot{\theta} = C$, given $\dot{\theta}(0) = \dot{\theta}_0$, $\theta(0) = \theta_0$ and that C is a constant. That is, find θ in terms of some or all of C, $\dot{\theta}_0$, θ_0 and t.

15.1.11 Given that $\ddot{\theta} - \lambda^2\theta = 0$, $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$, and $\lambda = 3/\,\text{s}$ find the value of θ at $t = 1$ s.

15.1.12 Two runners run on a circular track side-by-side at the same constant angular rate $\dot{\theta} = 0.25\,\text{rad/s}$ about the center of the track. The inside runner is in a lane of radius $r_i = 35\,\text{m}$ and the outside runner is in a lane of radius $r_o = 37\,\text{m}$. What is the velocity of the outside runner relative to the inside runner?

15.1.13 A particle oscillates on the arc of a circle with radius R according to the equation $\theta = \theta_0 \cos(\lambda t)$. What are the conditions on R, θ_0, and λ so that the maximum acceleration in this motion occurs at $\theta = 0$. "Acceleration" here means the magnitude of the acceleration vector.

15.1.14 A particle moves on a circular arc starting from rest at $\theta_0 = 0$. As θ increases, the magnitude of the acceleration is constant. Assume, all in consistent units, that $R = 1$ and $|\vec{\boldsymbol{a}}| = 1$.

a) Write the statement 'the magnitude of acceleration is constant' as an equation in terms of $\dot{\theta}$ and $\ddot{\theta}$.

b) Find a solution to the equation with the given initial conditions (analytically or numerically).

c) Find and plot $\dot{\theta}$ vs t and θ vs t.

d) In circular motion does $|\vec{\boldsymbol{a}}|$=constant necessarily mean that the motion is at or is gradually approaching constant rate circular motion? Is so, why? If not show a counter-example.

15.1.15 A particle moves in a circle so that its acceleration $\vec{\boldsymbol{a}}$ always makes a fixed angle ϕ with the position vector $-\vec{\boldsymbol{r}}$, with $0 \le \phi \le \pi/2$. For example, $\phi = 0$ would be constant rate circular motion. Assume $\phi = \pi/4$, $R = 1\,\text{m}$ and $\dot{\theta}_0 = 1\,\text{rad/s}$. How long does it take the particle to reach

a) the speed of sound ($\approx 300\,\text{m/s}$)?

b) the speed of light ($\approx 3 \cdot 10^8\,\text{m/s}$)?

c) ∞?